MONOTONE ITERATIVE METHODS FOR SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

A thesis presented in partial fulfilment of the requirements for the degree of
Doctor of Philosophy
in
Mathematics
at Massey University, Palmerston North,
New Zealand.

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2020

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Abstract

A key aspect of the simulation process is the formulation of proper mathematical models. The model must be able to emulate the physical phenomena under investigation. Partial differential equations play a major role in the modelling of many processes which arise in physics, chemistry and engineering. Most of these partial differential equations cannot be solved analytically and classical numerical methods are not always applicable. Thus, efficient and stable numerical approaches are needed. A fruitful method for solving the nonlinear difference schemes, which discretize the continuous problems, is the method of upper and lower solutions and its associated monotone iterations. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below to a solution of the problem. This monotone property ensures the theorem on existence and uniqueness of a solution. This method can be applied to a wide number of applied problems such as the enzyme-substrate reaction diffusion models, the chemical reactor models, the logistic model, the reactor dynamics of gasses, the Volterra-Lotka competition models in ecology and the Belousov-Zhabotinskii reaction diffusion models.

In this thesis, for solving coupled systems of elliptic and parabolic equations with quasi-monotone reaction functions, we construct and investigate block monotone iterative methods incorporated with Jacobi and Gauss–Seidel methods, based on the method of upper and lower solutions . The idea of these methods is the decomposition technique which reduces a computational domain into a series of nonoverlapping one dimensional intervals by slicing the domain into a finite number of thin strips, and then solving a two-point boundary-value problem for each strip by a standard computational method such as the Thomas algorithm.

We construct block monotone Jacobi and Gauss-Seidel iterative methods with quasi-monotone reaction functions and investigate their monotone properties. We prove theorems on existence and uniqueness of a solution, based on the monotone properties of iterative sequences. Comparison theorems on the rate of convergence for the block Jacobi and Gauss-Seidel methods are presented. We prove that the numerical solutions converge to the unique solutions of the corresponding continuous problems. We estimate the errors between the numerical and exact solutions of the nonlinear difference

schemes, and the errors between the numerical solutions and the exact solutions of the corresponding continuous problems. The methods of construction of initial upper and lower solutions to start the block monotone iterative methods are given.

Acknowledgements

Firstly, I would like to express my sincere gratitude to my Supervisor Prof. Igor Boglaev for the continuous support of my doctoral study and related research, for his patience, motivation, and immense knowledge. His guidance helped me all the time in my research and writing of this thesis. I could not have imagined having a better advisor and mentor for my doctoral study.

Besides my supervisor, I would like to thank my co-supervisor Dr Tammy Lynch for her comments and encouragement.

I would like to thank my fellow doctoral students for their feedback, cooperation and of course friendship. In addition, I would like to express my gratefulness to the staff of School of Fundamental Sciences for the last minute favors.

My sincere thanks also go to the soul of my father, to my mother who always supplied me with her blessed supplications and remembrance through her prayers, to my wife who always stood by me and encouraged me spiritually and would like to thank my brothers, sisters and kids for supporting me spiritually throughout writing this thesis and my life in general.

Chapter 1

Introduction

1.1 Overview of the method of upper and lower solutions

The monotone method and its associated upper-lower solutions for nonlinear ordinary and partial differential equations have been given extensive attention in recent years. The method is popular because not only does it give constructive proof for existence theorems but it also leads to various comparison results which are effective tools for the study of qualitative properties of solutions. The monotone behaviour of the sequence of iterations is also useful in the treatment of numerical solutions of various boundary value and initial-boundary value problems.

1.1.1 The monotone method of upper and lower solutions for continuous problems

The first steps in the theory of lower and upper solutions were given by Picard in 1890 [67] for partial differential equations, and in [68] he extended his theory for ordinary differential equations. In both cases, the existence of a solution is guaranteed from a monotone iterative technique. Existence of solutions for Cauchy equations was proved by Perron in 1915 [66]. Müller extended Perron's results to initial value systems in [52]. Dragoni [36], [35] introduced the notion of the method of lower and upper solutions for ordinary differential equations with Dirichlet boundary conditions.

In the classical books of Bernfeld and Lakshmikantham [12] and Ladde et al. [46], the classical theory of the method of lower and upper solutions and the monotone iterative technique is presented. This theory treats the solution as the limit of a monotone sequence formed by solutions of linear problems related to nonlinear equations.

To illustrate the basic idea of the monotone method, let us consider a typical elliptic boundary value problem in the form

$$-Lu(x) + f(x, u) = 0, \quad x \in \omega, \quad u(x) = g(x), \quad x \in \partial\omega,$$

where L is a uniformly elliptic operator in a bounded domain $\omega \in \mathbb{R}^\kappa$ ($\kappa = 1, 2, \ldots$) and $\partial\omega$ is a boundary. Uniform elliptic operator means that the matrix $(a_{i,j})$, $i, j = 1, \ldots \kappa$ of the coefficients of the second derivatives is positive definite and bounded from above and below, that is,

$$Lu(x) = \sum_{i,j=1}^{\kappa} a_{i,j} \partial^2 u / \partial x_i \partial x_j + \sum_{j=1}^{\kappa} b_j(x) \partial u / \partial x_j, \tag{1.1}$$

$$d_0 \|\xi\|^2 \leq \sum_{i,j=1}^{\kappa} a_{ij}(x) \xi_i \xi_j \leq d_1 \|\xi\|^2, \quad \|\xi\| = \left(\sum_{i=1}^{\kappa} \xi_i^2 \right)^{\frac{1}{2}},$$

where d_0 and d_1 are positive constants. Suppose there exists an ordered pair of upper and lower solutions $\widetilde{u}$ and $\widehat{u}$, that is, $\widetilde{u}$ and $\widehat{u}$ are smooth functions with $\widetilde{u} \geq \widehat{u}$ such that

$$-L\widetilde{u}(x) + f(x, \widetilde{u}) \geq 0, \quad x \in \omega, \quad \widetilde{u}(x) \geq g(x), \quad x \in \partial\omega,$$

and $\widehat{u}$ satisfies the reversed inequalities. Then by using $\widetilde{u}$ and $\widehat{u}$ as two distinct initial iterations one can construct two sequences $\{\overline{u}^{(n)}\}$ and $\{\underline{u}^{(n)}\}$ from the iteration process

$$\begin{aligned} &- Lu^{(n)}(x) + c(x)\left(u^{(n)}(x) - u^{(n-1)}(x)\right) = -f\left(x, u^{(n-1)}\right), \quad x \in \omega, \\ &u^{(n)}(x) = g(x), \quad x \in \partial\omega, \end{aligned}$$

where $u^{(n)}$ stands for $\overline{u}^{(n)}$ or $\underline{u}^{(n)}$, and the function $c(x)$ is taken as any upper bound of $\partial f/\partial u$ for $\widehat{u} \leq u \leq \widetilde{u}$. Based on the property of upper and lower solutions, one establishes that the sequence $\left\{\overline{u}^{(n)}\right\}$ is monotone nonincreasing and the sequence $\left\{\underline{u}^{(n)}\right\}$ is monotone nondecreasing, and both sequences converge, respectively, to solutions $\overline{u}$ and $\underline{u}$ of the problem. The monotone property of these sequences leads to the relation

$$\widehat{u} \leq \underline{u}^{(n-1)} \leq \underline{u}^{(n)} \leq \underline{u} \leq \overline{u} \leq \overline{u}^{(n)} \leq \overline{u}^{(n-1)} \leq \widetilde{u}, \quad \text{in } \overline{\omega}, \quad n \geq 1.$$

When $\overline{u} = \underline{u}$, there is a unique solution in the sector $\langle \widehat{u}, \widetilde{u} \rangle$ between $\widehat{u}$ and $\widetilde{u}$; otherwise the problem has multiple solutions.

A major advance of this technique is the extension of the idea of upper-lower solutions to coupled systems of a finite number of parabolic and elliptic equations [46], [59]. For coupled systems of equations, whether parabolic or elliptic, the definition of upper-lower solutions depends on the quasi-monotone property of the vector reaction function $\mathbf{f}$ in the system. Based on the quasi-monotone property of the reaction functions one can also construct two sequences which are monotone. Although these two sequences converge to some limits $\underline{\mathbf{u}}$ and $\overline{\mathbf{u}}$, it is not certain that $\underline{\mathbf{u}}$ or $\overline{\mathbf{u}}$ is a solution of the problem except in the special cases where every component of the reaction function

$\mathbf{f}$ is quasi-monotone nondecreasing and for systems of two equations with the quasi-monotone nonincreasing property of the reaction functions. The method of upper and lower solutions has been developed for continuous systems of partial differential equations with the focus on comparison results and qualitative behavior of the solutions [8], [10], [45], [47], [53], [54], [55], [70].

1.1.2 The monotone method of upper and lower solutions for difference schemes

Various reaction-diffusion-convection-type problems in the chemical, physical and engineering sciences are described by nonlinear elliptic and parabolic equations. In order to treat such nonlinear problems numerically, the nonlinear problems are approximated by using the finite difference or finite element methods, which lead to nonlinear systems of algebraic equations. The main mathematical concern is to investigate whether these systems have a solution and to find efficient, stable and computationally effective methods for solving these discrete systems.

The idea of upper and lower solutions was employed by Parter [64] and Greenspan and Parter [43] for solving finite difference schemes which approximate elliptic problems. Under the condition that the nonlinear function is bounded, they constructed explicitly initial upper and lower solutions. Russell and Shampine [69] used a similar approach for a singular boundary value problem. The method of upper and lower solutions was applied for treating scalar elliptic problems in [16], [28], [29], [31], [48], [59], [70] and for scalar parabolic problems in [8], [15], [21], [23], [24], [26], [40], [50], [54], [55].

This method gains more complexity when it is applied to coupled systems. A great deal of research has been done on investigating the method for systems of elliptic problems [17], [19], [20], [47], [49], [56], [57] and for systems of parabolic problems [27], [38], [44], [55], [60], [58], [72].

The idea of block monotone methods is based on the decomposition technique which reduces a domain into a series of nonoverlapping one dimensional intervals by slicing the domain into a finite number of thin strips, and then solving a two-point boundary-value problem for each strip by a standard computational method such as the Thomas algorithm [51]. Block monotone iterative methods, based on the method of upper and lower solutions, were developed in [13], [18], [22], [25], [30], [61], [62],[73], [74] for solving scalar elliptic equations and in [14], [63] for solving scalar parabolic equations.

In [61], block Jacobi and block Gauss-Seidel monotone iterative schemes were presented for solving second-order nonlinear elliptic equations. Theorems on existence and uniqueness theorems of the solution were proved. These block monotone iterative schemes have been extended for the fourth-order elliptic equations in [62]. In [63], block

Jacobi and block Gauss-Seidel monotone iterative methods were constructed for treating nonlinear scalar parabolic equations. In [73], the block monotone method, suitable for parallel computers, was developed for numerical solutions of nonlinear scalar elliptic boundary value problems. This block method is based on the block monotone Jacobi method. In [13], [14], [18], [30], block monotone domain decomposition methods, based on a Schwarz alternating method and a block successive underrelaxation method, were developed for numerical solutions of nonlinear scalar elliptic and parabolic problems with interior and boundary layers.

The method of upper and lower solutions can be successfully applied to many applied problems. Some of the models which are governed by elliptic boundary value problems, where the numerical methods of upper and lower solutions can be applicable, are i) the steady-state enzyme-substrate reaction model [9], where the effect of inhibition is taken into consideration; ii) the logistic model [32] which describes population growth; iii) reactor dynamics and the subsonic motion of gasses [7].

Some models which are governed by parabolic boundary problems, where the numerical methods of upper and lower solutions can be applicable, are i) the time-dependent enzyme-substrate reaction model [9], where the effect of inhibition is neglected; ii) the chemical reactor method [42], when the isothermal reaction is irreversible.

Models governed by systems of nonlinear elliptic equations, where the numerical methods of upper and lower solutions can be applicable, are i) the gas-liquid interaction model [34], where a dissolved gas and a dissolved reactant interact in a bounded diffusion medium; ii) the Volterra-Lotka competition model in ecology [33] which describes the coexistence of competing species in ecology; iii) the Belousov-Zhabotinskii reaction diffusion model [11], [59] which includes the metal-ion-catalyzed oxidation by bromate ion of organic materials.

Models governed by systems of nonlinear parabolic equations, where the numerical methods of upper and lower solutions can be applicable, are i) the time-dependent gas-liquid interaction model [34]; ii) the time-dependent Belousov-Zhabotinskii reaction diffusion model [11], [59]; iii) the time-dependent Volterra-Lotka competition model [33].

In thesis, for solving coupled systems of elliptic and parabolic equations with quasi-monotone reaction functions, we construct and investigate block monotone Jacobi and Gauss-Seidel iterative methods. We estimate the errors between the numerical and exact solutions of the nonlinear difference schemes, and the errors between the numerical solutions and the exact solutions of the corresponding continuous problems. The methods of construction of initial upper and lower solutions to start the block monotone iterative methods are given. The block monotone iterative methods are applied to the gas-liquid interaction model [34], the Volterra-Lotka competition model in ecology

[33] and the Belousov-Zhabotinskii reaction diffusion model [11], [59] in the case of elliptic systems, and applied to the time dependent version of the Volterra-Lotka co-operation model [33], the Belousov-Zhabotinskii reaction diffusion model [11] and the Volterra-Lotka competition model in ecology [33] in the case of parabolic systems.

1.2 Monotone iterative method for elliptic equations

Elliptic differential equations are used to characterize a wide family of problems in chemistry, physics and engineering sciences. The elliptic problem under consideration in this section is in the form

$$-\,\mathrm{L}u(x) + f(x,u) = 0, \quad x \in \omega, \quad u(x) = g(x), \quad x \in \partial\omega, \tag{1.2}$$

where the domain ω is bounded and connected in $\mathbb{R}^{\kappa}$ ($\kappa = 1, 2, \ldots$), and $\partial\omega$ is the boundary. The differential operator $\mathrm{L}(x)$ is given by

$$\mathrm{L}u = \sum_{\nu=1}^{\kappa} \frac{\partial}{\partial x_\nu}\left(D_\nu(x)\frac{\partial u}{\partial x_\nu}\right) + \sum_{\nu=1}^{\kappa} \upsilon_\nu(x)\frac{\partial u}{\partial x_\nu},$$

where the coefficients of the differential operator are assumed to be smooth and $D(x) > 0$ in $\overline{\omega}$. The functions f and g are also assumed smooth in their corresponding domains.

1.2.1 Nonlinear difference scheme

On the domain $\overline{\omega}$, we introduce a mesh $\overline{\Lambda}^h = \Lambda^h \cup \partial\Lambda^h$, where Λ^h and $\partial\Lambda^h$, are, respectively, a set of interior mesh points and a set of boundary mesh points. For solving the nonlinear problem (1.2), we consider the nonlinear difference scheme

$$\mathcal{A}(p)U(p) + f(p,U) = 0, \quad p \in \Lambda^h, \quad U(p) = g(p), \quad p \in \partial\Lambda^h, \tag{1.3}$$

where $U(p)$, $p \in \overline{\Lambda}^h$ is an unknown mesh function. The difference operator $\mathcal{A}(p)$ is defined by

$$\mathcal{A}(p)U(p) = d(p)U(p) - \sum_{p' \in \sigma'(p)} a(p')U(p'), \tag{1.4}$$

where $\sigma'(p) = \sigma(p) \setminus \{p\}$, $\sigma(p)$ is a stencil of the scheme at an interior mesh point $p \in \Lambda^h$. The five-point stencil of a point in the grid is a stencil made up of the point itself together with its four neighbors. The coefficients of the difference operator $\mathcal{A}(p)$ are assumed to satisfy the assumptions

$$d(p) > 0, \quad a(p') \geq 0, \quad p' \in \sigma'(p), \quad d(p) - \sum_{p' \in \sigma'(p)} a(p') \geq 0, \quad p \in \Lambda^h. \tag{1.5}$$

We assume that the mesh domain $\overline{\Lambda}^h$ is connected, that is, for two interior mesh points $\widetilde{p}$ and $\widehat{p}$, there exists a finite set of mesh points $\{p_1, p_2, \dots, p_r\}$ such that

$$p_1 \in \sigma'(\widetilde{p}),\ p_2 \in \sigma'(p_1), \dots,\ p_r \in \sigma'(p_{r-1}),\ \widehat{p} \in \sigma'(p_r). \tag{1.6}$$

We introduce the linear difference problem

$$\mathcal{A}(p)W(p) + c^*(p)W(p) = \phi(p), \quad p \in \Lambda^h, \quad W(p) = g(p), \quad p \in \partial\Lambda^h, \tag{1.7}$$

where $c^*(p)$ is a positive bounded mesh function. We now consider the maximum principle for the difference operator $\mathcal{A}(p) + c^*(p)$ and give a bound on the magnitude of the solution to (1.7).

Lemma 1.2.1. *Let the coefficients of the difference operator $\mathcal{A}(p)$ satisfy (1.5) and the mesh domain $\overline{\Lambda}^h$ be connected (1.6).*

(i) If a mesh function $W(p)$ satisfies the conditions

$$(\mathcal{A}(p) + c^*(p))\, W(p) \geq 0 \quad (\leq 0), \quad p \in \Lambda^h, \quad W(p) \geq 0\ (\leq 0), \quad p \in \partial\Lambda^h, \tag{1.8}$$

then $W(p) \geq 0$ (≤ 0), $\quad p \in \overline{\Lambda}^h$.

(ii) The following bound on the magnitude of the solution to (1.7) holds

$$\|W\|_{\overline{\Lambda}^h} \leq \max\left\{ \|g\|_{\partial\Lambda^h}, \frac{\|\phi\|_{\Lambda^h}}{\|c^*\|_{\overline{\Lambda}^h}} \right\}, \tag{1.9}$$

where

$$\|W\|_{\overline{\Lambda}^h} \equiv \max_{p \in \overline{\Lambda}^h} |W(p)|, \quad \|g\|_{\partial\Lambda^h} \equiv \max_{p \in \partial\Lambda^h} |g(p)|.$$

Proof. We prove part (i) of the lemma by the contradiction argument. From condition (1.8) and the definition of the difference operator (1.4), we have

$$d(p) + c^*(p) - \sum_{p' \in \sigma'(p)} a(p') \geq 0. \tag{1.10}$$

Assume by contradiction that there exist mesh points in Λ^h such that

$$\min_{p \in \Lambda^h} W(p) = W(p_*) < 0. \tag{1.11}$$

From condition (1.8) of the lemma, we have at p_*

$$W(p_*) \geq \frac{\sum_{p'_* \in \sigma'(p_*)} a(p'_*)W(p'_*)}{d(p_*) + c^*(p_*)}.$$

From here and (1.11), it follows that

$$W(p_*) \geq \lambda W(p_*), \quad \lambda \equiv \frac{\sum_{p'_* \in \sigma'(p_*)} a(p'_*)}{d(p_*) + c^*(p_*)}.$$

With (1.10), we conclude that

$$W(p_*)(1 - \lambda) \geq 0, \quad \lambda < 1.$$

Since $(1 - \lambda) > 0$ and $W(p_*) < 0$, we get the contradiction with our assumption.

Now we prove part (ii) of the lemma. We consider the problem

$$(\mathcal{A}(p) + c^*(p))\, V(p) = |\phi(p)|, \quad p \in \Lambda^h, \quad V(p) = \|g\|_{\partial\Lambda^h}, \quad p \in \partial\Lambda^h. \tag{1.12}$$

Denoting $S(p) \equiv V(p) - W(p)$, $p \in \overline{\Lambda}^h$, from (1.7) and (1.12), we have

$$(\mathcal{A}(p) + c^*(p))\, S(p) = |\phi(p)| - \phi(p) \geq 0, \quad S(p) \geq 0, \quad p \in \partial\Lambda^h.$$

From here, by using the maximum principle (i) of the lemma, we conclude that

$$S(p) = V(p) - W(p) \geq 0, \quad p \in \overline{\Lambda}^h.$$

Similarly, we can prove that

$$V(p) + W(p) \geq 0, \quad p \in \overline{\Lambda}^h.$$

Thus, we prove that

$$|W(p)| \leq V(p), \quad p \in \overline{\Lambda}^h.$$

We now prove that

$$V(p) \leq k, \quad k \equiv \max\left\{ \|g\|_{\partial\Lambda^h}, \frac{\|\phi\|_{\Lambda^h}}{\|c^*\|_{\overline{\Lambda}^h}} \right\}. \tag{1.13}$$

Case 1. Assume that in (1.13)

$$k = \|g\|_{\partial\Lambda^h} \geq \frac{\|\phi\|_{\Lambda^h}}{\|c^*\|_{\overline{\Lambda}^h}}.$$

By contradiction, suppose that for some mesh points in Λ^h, the following inequality holds

$$V(p_*) = \max_{p \in \Lambda^h} V(p) > \|g\|_{\partial\Lambda^h}. \tag{1.14}$$

From (1.12), we have

$$(\mathcal{A}(p_*) + c^*(p_*))\, V(p_*) = |\phi(p_*)|, \quad p_* \in \Lambda^h. \tag{1.15}$$

From the definition of the difference operator $\mathcal{A}(p)$ in (1.8) and (1.14), for the left hand side of (1.15), we have

$$(d(p_*) + c^*(p_*))\, V(p_*) - \sum_{p'_* \in \sigma'(p_*)} a(p'_*) V(p'_*) \geq q(p_*) V(p_*),$$
$$q(p_*) = d(p_*) + c^*(p_*) - \sum_{p'_* \in \sigma'(p_*)} a(p'_*).$$

From here and (1.15), we conclude that

$$V(p_*) \leq \frac{|\phi(p_*)|}{q(p_*)}.$$

From here and assumption (1.5), we conclude that

$$V(p_*) \leq \frac{|\phi(p_*)|}{c^*(p_*)} \leq \frac{\|\phi\|_{\Lambda^h}}{c^*(p_*)} \leq \|g\|_{\partial\Lambda^h}.$$

We have the contradiction with our assumption.

Case 2. Assume that in (1.13)

$$k = \frac{\|\phi\|_{\Lambda^h}}{\|c^*\|} \geq \|g\|_{\partial\Lambda^h}.$$

We consider the same argument as in Case 1. By contradiction, we suppose that for some mesh points in Λ^h, the following inequality holds

$$V(p_*) = \max_{p \in \Lambda^h} V(p) > \frac{\|\phi\|_{\Lambda^h}}{c^*(p)}. \tag{1.16}$$

From (1.12), similar to (1.15), we have

$$(\mathcal{A}(p_*) + c^*(p_*))\, V(p_*) = |\phi(p_*)|, \quad p \in \Lambda^h.$$

From here, (1.4) and (1.16), we conclude that

$$V(p_*) \leq \frac{|\phi(p_*)|}{c^*(p_*)} \leq \frac{\|\phi\|_{\Lambda^h}}{c^*(p_*)}.$$

We have the contradiction with our assumption. □

Remark 1.2.2. *A difference scheme which satisfies the maximum principle from Lemma*

1.2.1 is said to be monotone. The monotonicity condition guarantees that systems of algebraic equations based on such methods are well-posed.

1.2.2 The method of upper and lower solutions

Two mesh functions $\widetilde{U}(p)$ and $\widehat{U}(p)$, $p \in \overline{\Lambda}^h$, are called ordered upper and lower solutions of the difference scheme (1.3), if they satisfy inequalities

$$\widehat{U}(p) \leq \widetilde{U}(p), \quad p \in \overline{\Lambda}^h, \tag{1.17a}$$
$$\mathcal{A}(p)\widehat{U}(p) + f(p, \widehat{U}) \leq 0 \leq \mathcal{A}(p)\widetilde{U}(p) + f(p, \widetilde{U}), \quad p \in \Lambda^h, \tag{1.17b}$$
$$\widehat{U}(p) \leq g(p) \leq \widetilde{U}(p), \quad p \in \partial\Lambda^h. \tag{1.17c}$$

For given upper and lower solutions $\widetilde{U}(p)$, $\widehat{U}(p)$, $p \in \overline{\Lambda}^h$, we define the sector

$$\langle \widehat{U}, \widetilde{U} \rangle = \left\{ U(p) : \widehat{U}(p) \leq U(p) \leq \widetilde{U}(p), \quad p \in \overline{\Lambda}^h \right\}.$$

We assume that $f(p, U)$ satisfies the constraint

$$f_u(p, U) \leq c(p), \quad U \in \langle \widehat{U}, \widetilde{U} \rangle, \quad p \in \overline{\Lambda}^h, \quad f_u \equiv \frac{\partial f}{\partial u}, \tag{1.18}$$

where $c(p)$ is a positive bounded function in $\overline{\Lambda}^h$.

To solve the nonlinear difference scheme (1.3), we construct an iterative method which satisfies the monotone convergence property. The sequence of solutions $\{U^{(n)}(p)\}$, $p \in \overline{\Lambda}^h$, is calculated by the following iterative method:

$$\begin{aligned} &(\mathcal{A}(p) + c(p))\, Z^{(n)}(p) = -\mathcal{K}(p, U^{(n-1)}), \quad p \in \Lambda^h, \\ &\mathcal{K}(p, U^{(n-1)}) = \mathcal{A}(p)U^{(n-1)}(p) + f(p, U^{(n-1)}), \\ &Z^{(1)}(p) = g(p) - U^{(0)}(p), \quad Z^{(n)}(p) = 0, \quad n \geq 2, \quad p \in \partial\Lambda^h, \\ &Z^{(n)}(p) \equiv U^{(n)}(p) - U^{(n-1)}(p), \quad p \in \overline{\Lambda}^h, \end{aligned} \tag{1.19}$$

where $\mathcal{K}(p, U^{(n-1)})$, $p \in \Lambda^h$ is the residual of the nonlinear difference scheme (1.3) and $c(p)$ is defined in (1.18).

We introduce the notation

$$\Gamma(p, U) = c(p)U(p) - f(p, U), \quad p \in \overline{\Lambda}^h, \tag{1.20}$$

and prove the monotone property of Γ.

Lemma 1.2.3. *Assume that $U_1(p)$ and $U_2(p)$, $p \in \overline{\Lambda}^h$ are functions in $\langle \widehat{U}, \widetilde{U} \rangle$, such* *that $U_1(p) \ge U_2(p)$ and (1.6), (1.18) are satisfied. Then*

$$\Gamma(p, U_1) \ge \Gamma(p, U_2), \quad p \in \overline{\Lambda}^h. \tag{1.21}$$

Proof. From (1.20), we have

$$\Gamma(p, U_1) - \Gamma(p, U_2) = c(p)[U_1(p) - U_2(p)] - [f(p, U_1) - f(p, U_2)].$$

By using the mean-value theorem, we have

$$f(p, U_1) - f(p, U_2) = f_u(p, Q)\left(U_1(p) - U_2(p)\right),$$

where $U_2(p) \le Q(p) \le U_1(p)$, $p \in \overline{\Lambda}^h$. From here, using the assumption of the lemma and (1.18), we conclude (1.21). □

In the following theorem, we prove the monotone convergence of upper and lower sequences generated by (1.19).

Theorem 1.2.4. *Suppose that the coefficients of the difference operator $\mathcal{A}(p)$ in (1.3) satisfy (1.5) and $f(p, U)$ satisfies (1.18). Let $\widetilde{U}(p)$ and $\widehat{U}(p)$, $p \in \overline{\Lambda}^h$, be upper and lower solutions (1.17). Then upper $\left\{\overline{U}^{(n)}(p)\right\}$ and lower $\left\{\underline{U}^{(n)}(p)\right\}$, $p \in \overline{\Lambda}^h$ sequences generated by (1.19) with, respectively, $\overline{U}^{(0)}(p) = \widetilde{U}(p)$ and $\underline{U}^{(0)}(p) = \widehat{U}(p)$, $p \in \overline{\Lambda}^h$, converge monotonically, such that,*

$$\underline{U}^{(n-1)}(p) \le \underline{U}^{(n)}(p) \le \overline{U}^{(n)}(p) \le \overline{U}^{(n-1)}(p), \quad p \in \overline{\Lambda}^h. \tag{1.22}$$

Proof. Since $\underline{U}^{(0)}(p) = \widehat{U}(p)$, $p \in \overline{\Lambda}^h$, is a lower solution, it follows that $\mathcal{K}\left(p, \underline{U}^{(0)}\right) \le 0$. From here and (1.19), we obtain

$$\left(\mathcal{A}(p) + c(p)\right)\underline{Z}^{(1)}(p) \ge 0, \quad p \in \Lambda^h, \quad \underline{Z}^{(1)}(p) \ge 0, \quad p \in \partial\Lambda^h.$$

By using the maximum principle in Lemma 1.2.1, we conclude that

$$\underline{Z}^{(1)}(p) \ge 0, \quad p \in \overline{\Lambda}^h. \tag{1.23}$$

Similarly, for the upper solution $\overline{U}^{(0)}(p) = \widetilde{U}(p)$, $p \in \overline{\Lambda}^h$, we have

$$\overline{Z}^{(1)}(p) \le 0, \quad p \in \overline{\Lambda}^h. \tag{1.24}$$

We now prove that $\overline{U}^{(1)}(p)$, and $\underline{U}^{(1)}(p)$, $p \in \overline{\Lambda}^h$ are ordered upper and lower solutions (1.17). Letting $W^{(n)}(p) = \overline{U}^{(n)}(p) - \underline{U}^{(n)}(p)$, $p \in \overline{\Lambda}^h$, using notation (1.20), from (1.19), we obtain

$$(\mathcal{A}(p) + c(p))\, W^{(1)}(p) = \Gamma(p, \overline{U}^{(0)}) - \Gamma(p, \underline{U}^{(0)}), \quad p \in \Lambda^h, \quad W^{(1)}(p) = 0, \quad p \in \partial\Lambda^h.$$

From here, (1.21) and taking into account that $\underline{U}^{(0)}(p) \le \overline{U}^{(0)}(p)$, $p \in \overline{\Lambda}^h$, by using Lemmas 1.2.1 and 1.2.3 , we conclude that

$$W^{(1)}(p) \ge 0, \quad p \in \overline{\Lambda}^h. \tag{1.25}$$

Thus, we prove (1.17a). From (1.19) and using notation (1.20), we obtain that

$$\mathcal{K}(p, \underline{U}^{(1)}) = \Gamma(p, \underline{U}^{(0)}) - \Gamma(p, \underline{U}^{(1)}), \quad p \in \Lambda^h. \tag{1.26}$$

From here, (1.21) and (1.25), it follows that

$$\mathcal{K}(p, \underline{U}^{(1)}) \le 0, \quad p \in \Lambda^h. \tag{1.27}$$

Similarly, we can prove that

$$\mathcal{K}(p, \overline{U}^{(1)}) \ge 0, \quad p \in \Lambda^h. \tag{1.28}$$

From the boundary condition in (1.19), it follows that $\underline{U}^{(1)}(p)$ and $\overline{U}^{(1)}(p)$, $p \in \partial\Lambda^h$, satisfy (1.17c). From here, (1.25), (1.27) and (1.28), we conclude that $\overline{U}^{(1)}(p)$ and $\underline{U}^{(1)}(p)$, $p \in \overline{\Lambda}^h$ are ordered upper and lower solutions (1.17).

By induction on n, we can prove that $\overline{U}^{(n)}(p)$ and $\underline{U}^{(n)}(p)$, $p \in \overline{\Lambda}^h$ are ordered upper and lower solutions (1.17) which satisfies (1.22). □

1.2.3 Existence and uniqueness of a solution of the nonlinear difference scheme

We now prove the existence of a solution of the nonlinear difference scheme (1.3).

Theorem 1.2.5. *Let the assumptions in Theorem 1.2.4 be satisfied. Then the nonlinear difference scheme (1.3) has maximal $\overline{U}(p)$ and minimal $\underline{U}(p)$, $p \in \overline{\Lambda}^h$ solutions in the sector $\langle \widehat{U}, \widetilde{U} \rangle$. If $V(p)$, $p \in \overline{\Lambda}^h$ is any solution in $\langle \widehat{U}, \widetilde{U} \rangle$, then*

$$\underline{U}(p) \le V(p) \le \overline{U}(p), \quad p \in \overline{\Lambda}^h. \tag{1.29}$$

Proof. From (1.22), we conclude that $\lim \underline{U}^{(n)}(p) = \underline{U}(p)$, $p \in \overline{\Lambda}^h$ as $n \to \infty$ exists, and

$$\lim_{n\to\infty} \underline{Z}^{(n)}(p) = 0, \quad p \in \overline{\Lambda}^h. \tag{1.30}$$

From (1.19), by using the mean-value theorem, we conclude that

$$\mathcal{K}(p,\underline{U}^{(n)}) = -\left(c(p) - f_u(p,\underline{Q}^{(n)})\right)\underline{Z}^{(n)}(p), \quad p \in \Lambda^h, \tag{1.31}$$

where $\underline{U}^{(n-1)}(p) \le \underline{Q}^{(n)}(p) \le \underline{U}^{(n)}(p)$, $p \in \overline{\Lambda}^h$.

By taking the limit of both sides and using (1.30), it follows that

$$\mathcal{K}(p,\underline{U}) = 0, \quad p \in \Lambda^h. \tag{1.32}$$

Similarly, we can prove that

$$\mathcal{K}(p,\overline{U}) = 0, \quad p \in \Lambda^h,$$

where $\overline{U}(p) = \lim_{n\to\infty}\overline{U}^{(n)}(p)$, $p \in \overline{\Lambda}^h$. Thus, from here and (1.32), we conclude that $\underline{U}(p)$ and $\overline{U}(p)$, $p \in \overline{\Lambda}^h$, are, respectively, minimal and maximal solutions of the nonlinear difference scheme (1.3) in the sector $\langle \widehat{U}, \widetilde{U}\rangle$.

Now we prove (1.29). Using $V(p)$ and $\widehat{U}(p)$, $p \in \overline{\Lambda}^h$ as initial upper and lower iterations, the sequence $\left\{\underline{U}^{(n)}(p)\right\}$, $p \in \overline{\Lambda}^h$ remains unchanged and converges to the solution $\underline{U}(p)$, $p \in \overline{\Lambda}^h$. Taking into account that the sequence $\left\{\overline{U}^{(n)}(p)\right\}$, $p \in \overline{\Lambda}^h$ with

$$\overline{U}^{(0)}(p) = V(p), \quad p \in \overline{\Lambda}^h,$$

consists of the single element $V(p)$, $p \in \overline{\Lambda}^h$, from (1.22), it follows that

$$V(p) \ge \underline{U}(p), \quad p \in \overline{\Lambda}^h. \tag{1.33}$$

Similarly, by using $\widetilde{U}(p)$ and $V(p)$, $p \in \overline{\Lambda}^h$ as initial upper and lower iterations, the sequence $\left\{\overline{U}^{(n)}(p)\right\}$, $p \in \overline{\Lambda}^h$ remains unchanged and converges to the solution $\overline{U}(p)$, $p \in \overline{\Lambda}^h$. Taking into account that the sequence $\{\underline{U}(p)\}$, $p \in \overline{\Lambda}^h$, with

$$\underline{U}^{(0)}(p) = V(p), \quad p \in \overline{\Lambda}^h,$$

consists of the single element $V(p)$, $p \in \overline{\Lambda}^h$, from (1.22), it follows that

$$V(p) \le \overline{U}(p), \quad p \in \overline{\Lambda}^h.$$

From here and (1.33), we conclude (1.29). □

For uniqueness of a solution of (1.3), we assume that $f(p,U)$, satisfies the two sided inequalities

$$\underline{c}(p) \le f_u(p,U) \le c(p), \quad U(p) \in \langle \widehat{U}, \widetilde{U}\rangle, \quad p \in \overline{\Lambda}^h, \tag{1.34}$$

where $\underline{c}(p)$ is a bounded function and $c(p)$ is defined in (1.18).

Theorem 1.2.6. *Suppose that the coefficients of the difference operator $\mathcal{A}(p)$ in (1.3) satisfy (1.5) and $f(p,U)$ satisfies (1.34). Then the nonlinear difference scheme (1.3) has a unique solution.*

Proof. From Theorem 1.2.5, it follows that $\underline{U}(p)$ and $\overline{U}(p)$, $p \in \overline{\Lambda}^h$, are two solutions to the nonlinear difference scheme (1.3). For uniqueness of a solution, it suffices to prove that $\underline{U}(p) = \overline{U}(p)$, $p \in \overline{\Lambda}^h$. From (1.22), we conclude that

$$\underline{U}^{(n)}(p) \leq \underline{U}(p) \leq \overline{U}(p) \leq \overline{U}^{(n)}(p), \quad p \in \overline{\Lambda}^h. \tag{1.35}$$

Letting $W(p) = \overline{U}(p) - \underline{U}(p)$, $p \in \overline{\Lambda}^h$, from (1.3), it follows that

$$\mathcal{A}(p)W(p) + f(p,\overline{U}) - f(p,\underline{U}) = 0, \quad p \in \Lambda^h \quad W(p) = 0, \quad p \in \partial\Lambda^h.$$

By using the mean-value theorem, we conclude that

$$\left(\mathcal{A}(p) + f_u(p,Q)\right) W(p) = 0, \quad p \in \Lambda^h, \quad W(p) = 0, \quad p \in \partial\Lambda^h, \tag{1.36}$$

where $\underline{U}(p) \leq Q(p) \leq \overline{U}(p)$, $p \in \overline{\Lambda}^h$. From (1.35), we conclude that the partial derivative $f_u(p,Q)$ satisfies (1.34). From here, (1.34) and (1.36), by using (1.9), we conclude that $W(p) = 0$, $p \in \overline{\Lambda}^h$. □

1.2.4 Convergence analysis of the point monotone iterative method

We now investigate convergence properties of the monotone iterative method (1.19).

Linear rate of convergence

We modify the monotone iterative method (1.19) by replacing $c(p)$ by the constant $\widetilde{c}$ as follows:

$$\widetilde{c} = \max_{p \in \overline{\Lambda}^h} c(p). \tag{1.37}$$

Theorem 1.2.4 still holds if we replace $c(p)$ by $\widetilde{c}$.

Theorem 1.2.7. *Suppose that the coefficients of the difference operator $\mathcal{A}(p)$ in (1.3) satisfy (1.5) and $f(p,U)$ satisfies (1.34). Let $\widetilde{U}(p)$ and $\widehat{U}(p)$, $p \in \overline{\Lambda}^h$, be ordered upper and lower solutions (1.17). Then for the sequence $\left\{U^{(n)}(p)\right\}$, $p \in \overline{\Lambda}^h$ generated by (1.19), the following estimate holds:*

$$\left\|Z^{(n)}\right\|_{\overline{\Lambda}^h} \leq q^{n-1} \left\|Z^{(1)}\right\|_{\overline{\Lambda}^h}, \quad q = 1 - \frac{\widehat{c}}{\widetilde{c}}, \quad \widehat{c} = \min_{p \in \overline{\Lambda}^h} \underline{c}(p), \quad n \geq 2, \tag{1.38}$$

where $\underline{c}(p)$ is defined in (1.34), $\widetilde{c}$ is defined in (1.37), and q, $0 < q < 1$ is the linear rate of convergence.

Proof. Similar to (1.31) with the assumption (1.37), we conclude that

$$\mathcal{K}\left(p, \underline{U}^{(n-1)}\right) = -\left(\widetilde{c} - f_u\left(p, \underline{Q}^{(n-1)}\right)\right)\underline{Z}^{(n-1)}(p), \tag{1.39}$$

where $\underline{U}^{(n-2)}(p) \leq \underline{Q}^{(n-1)}(p) \leq \underline{U}^{(n-1)}(p)$. From (1.24) and (1.25), it follows that $\langle \underline{U}^{(n-2)}, \underline{U}^{(n-1)} \rangle \in \langle \widehat{U}, \widetilde{U} \rangle$, which leads to $f_u(p, \underline{Q}^{(n-1)})$ satisfies (1.34). From here, (1.19), (1.37) and (1.39), we obtain that

$$(\mathcal{A}(p) + \tilde{c})\,\underline{Z}^{(n)}(p) = (\tilde{c} - f_u(p, \underline{Q}^{(n-1)})\underline{Z}^{(n-1)}(p).$$

By using (1.9), it follows that

$$\left\|\underline{Z}^{(n)}\right\|_{\overline{\Lambda}^h} \leq q \left\|\underline{Z}^{(n-1)}\right\|_{\overline{\Lambda}^h},$$

where $q < 1$, since $\widehat{c} < \widetilde{c}$. If $\widehat{c} = \widetilde{c}$, it means that problem (1.2) is linear. By induction on n, we can prove (1.38) for a lower sequence $\left\{\underline{U}^{(n)}(p)\right\}$, $p \in \overline{\Lambda}^h$. By a similar argument, we can prove (1.38) for an upper sequence $\left\{\overline{U}^{(n)}(p)\right\}$, $p \in \overline{\Lambda}^h$. □

Quadratic rate of convergence

We modify the monotone iterative method (1.19) by replacing $c(p)$ by $c^{(n-1)}(p)$ and calculating the sequence $\{U^{(n)}\}$, $p \in \overline{\Lambda}^h$, as follows:

$$\begin{aligned}
&\left(\mathcal{A}(p) + c^{(n-1)}\right) Z^{(n)}(p) = -\mathcal{K}\left(p, U^{(n-1)}\right), \quad p \in \Lambda^h, \\
&\mathcal{K}(p, U^{(n-1)}) \equiv \mathcal{A}(p)U^{(n-1)}(p) + f\left(p, U^{(n-1)}\right), \\
&Z^{(1)}(p) = g(p) - U^{(0)}(p), \quad Z^{(n)}(p) = 0, \quad n \geq 2, \quad p \in \partial\Lambda^h, \\
&Z^{(n)}(p) = U^{(n)}(p) - U^{(n-1)}(p), \quad p \in \overline{\Lambda}^h,
\end{aligned} \tag{1.40}$$

where the mesh function $c^{(n-1)}(p)$ is given by

$$c^{(n-1)}(p) = \max_U \{f_u(p, U)\}, \; \underline{U}^{(n-1)}(p) \leq U(p) \leq \overline{U}^{(n-1)}(p). \tag{1.41}$$

Two sequences $\left\{\underline{U}^{(n)}(p)\right\}$ and $\left\{\overline{U}^{(n)}(p)\right\}$, $p \in \overline{\Lambda}^h$ are in use for calculating $c^{(n-1)}(p)$.

Introduce the notation

$$\xi = \max_{p \in \overline{\Lambda}^h} \left[\max_U \left\{|f_{uu}(p, U)|, U(p) \in \langle \widehat{U}, \widetilde{U} \rangle\right\}\right]. \tag{1.42}$$

We now prove the quadratic convergence of the monotone iterative method (1.40), (1.41) in the following theorem.

Theorem 1.2.8. *Suppose that the coefficients of the difference operator $\mathcal{A}(p)$ in (1.3) satisfy (1.5). Assume that f satisfies (1.18). Then for the sequences $\{\overline{U}^{(n)}(p)\}$ and $\{\underline{U}^{(n)}(p)\}$, $p \in \overline{\Lambda}^h$, generated by (1.40), the following estimate holds:*

$$\left\| W^{(n)} \right\|_{\overline{\Lambda}^h} \leq \frac{\xi}{\widehat{c}} \left\| W^{(n-1)} \right\|^2_{\overline{\Lambda}^h}, \tag{1.43}$$

where $W^{(n)}(p) = \overline{U}^{(n)}(p) - \underline{U}^{(n)}(p)$, $p \in \overline{\Lambda}^h$, $\widehat{c}$ and ξ are, respectively, defined in (1.38) and (1.42).

Proof. From (1.40) with the modification (1.41), we obtain

$$\begin{aligned} &\left(\mathcal{A}(p) + c^{(n-1)}(p)\right) W^{(n)}(p) = G^{(n-1)}(p), \quad p \in \Lambda^h, \\ &G^{(n-1)}(p) = c^{(n-1)}(p) W^{(n-1)}(p) - [f(p, \overline{U}^{(n-1)}) - f(p, \underline{U}^{(n-1)})], \\ &W^{(n)}(p) = 0, \quad p \in \partial\Lambda^h. \end{aligned} \tag{1.44}$$

By using the mean-value theorem, we have

$$f(p, \overline{U}^{(n-1)}) - f(p, \underline{U}^{(n-1)}) = f_u(p, Q^{(n-1)}) W^{(n-1)}(p),$$

where

$$Q^{(n-1)}(p) \in \langle \underline{U}^{(n-1)}, \overline{U}^{(n-1)} \rangle.$$

From (1.41), we have

$$c^{(n-1)}(p) = f_u(p, Y^{(n-1)}),$$

where $Y^{(n-1)}(p) \in \langle \underline{U}^{(n-1)}, \overline{U}^{(n-1)} \rangle$. We now present the right hand side $G^{(n-1)}(p)$ of (1.44) in the form

$$G^{(n-1)}(p) = \left(f_u(p, Y^{(n-1)}) - f_u(p, Q^{(n-1)}) \right) W^{(n-1)}(p).$$

By using the mean-value theorem, it follows that

$$f_u(p, Y^{(n-1)}) - f_u(p, Q^{(n-1)}) = f_{uu}(p, H^{(n-1)}) \left(Y^{(n-1)}(p) - Q^{(n-1)}(p) \right),$$

where $H^{(n-1)}(p)$ lies between $Y^{(n-1)}$ and $Q^{(n-1)}$. Taking into account that

$$\left| Y^{(n-1)}(p) - Q^{(n-1)}(p) \right| \leq \overline{U}^{(n-1)}(p) - \underline{U}^{(n-1)}(p).$$

In the notation (1.42), we can estimate $G^{(n-1)}(p)$ as follows:

$$\left\| G^{(n-1)} \right\|_{\Lambda^h} \leq \xi \left\| W^{(n-1)} \right\|^2_{\overline{\Lambda}^h}.$$

From here, (1.44) and using (1.9), we conclude (1.43). □

1.3 Monotone iterative method for parabolic equations

Parabolic differential equations are used to characterize a wide family of problems in chemistry, physics and engineering sciences. Here, we study monotone iterative methods for solving the parabolic problem in the form

$$\frac{\partial u}{\partial t} - \mathrm{L}u(x,t) + f(x,t,u) = 0, \quad (x,t) \in Q_T = \omega \times (0,T], \tag{1.45}$$
$$u(x,t) = g(x,t), \quad (x,t) \in \partial Q_T = \partial\omega \times (0,T], \quad u(x,0) = \psi(x), \quad x \in \overline{\omega},$$

where the domain ω is bounded and connected in $\mathbb{R}^k$ ($k = 1,2,\ldots$), and $\partial\omega$ is the boundary. The differential operator $\mathrm{L}(x,t)$ is given by

$$\mathrm{L}u = \sum_{\nu=1}^{k} \frac{\partial}{\partial x_\nu}\left(D(x,t)\frac{\partial u}{\partial x_\nu}\right) + \sum_{\nu=1}^{k} v_\nu(x,t)\frac{\partial u}{\partial x_\nu},$$

where the coefficients of the differential operator $\mathrm{L}(x,t)$ are assumed to be smooth and $D(x,t) > 0$ in $\overline{\omega} \times [0,T]$. The functions f, g and $\psi(x)$ are also assumed smooth in their corresponding domains.

1.3.1 Nonlinear implicit difference scheme

On the domains $\overline{\omega}$ and $[0,T]$, we introduce, respectively, meshes $\overline{\Lambda}^h = \Lambda^h \cup \partial\Lambda^h$ and $\overline{\Lambda}^\tau = \Lambda^\tau \cup \partial\Lambda^\tau$, where Λ^h and $\partial\Lambda^h$ are sets of interior and boundary spatial points and

$$\Lambda^\tau = \{t_m : t_1 < t_2 < \ldots < t_{N_\tau} = T\}, \quad \partial\Lambda^\tau = \{t_0 = 0\}.$$

For solving the nonlinear problem (1.45), we consider the nonlinear implicit difference scheme

$$(\mathcal{A}(p,t_m) + \tau_m^{-1}I)U(p,t_m) + f(p,t_m,U) - \tau_m^{-1}U(p,t_{m-1}) = 0, \quad p \in \Lambda^h, \tag{1.46}$$
$$U(p,t_m) = g(p,t_m), \quad p \in \partial\Lambda^h, \quad m \geq 1, \quad U(p,0) = \psi(p), \quad p \in \overline{\Lambda}^h,$$

where I is the identity operator and the time step $\tau_m = t_m - t_{m-1}$, $m \geq 1$, $t_0 = 0$. On each time level t_m, $m \geq 1$, the difference operator $\mathcal{A}(p,t_m)$ is defined by

$$\mathcal{A}(p,t_m)U(p,t_m) = d(p,t_m)U(p,t_m) - \sum_{p' \in \sigma'(p)} a(p',t_m)U(p',t_m), \tag{1.47}$$

where $\sigma'(p) = \sigma(p) \backslash \{p\}$, $\sigma(p)$ is a stencil of the scheme at an interior mesh point $p \in \Lambda^h$. The coefficients of the difference operator are assumed to satisfy the assumptions

$$d(p, t_m) > 0, \quad a(p', t_m) \geq 0, \quad p' \in \sigma'(p), \tag{1.48}$$
$$d(p, t_m) - \sum_{p' \in \sigma'(p)} a(p', t_m) \geq 0, \quad p \in \Lambda^h.$$

It is assumed that the mesh domain $\overline{\Lambda}^h$ is connected (1.6).

On each time level t_m, $m \geq 1$, we introduce the linear difference problem

$$\left(\mathcal{A}(p, t_m) + (\tau_m^{-1} + c^*(p, t_m))I\right) W(p, t_m) = \phi(p, t_m), \quad p \in \Lambda^h, \tag{1.49}$$
$$W(p, t_m) = g(p, t_m), \quad p \in \partial\Lambda^h, \quad c^*(p, t_m) \geq 0, \quad p \in \overline{\Lambda}^h.$$

We now consider the maximum principle for the difference operator

$$\mathcal{A}(p, t_m) + (\tau_m^{-1} + c^*(p, t_m))I,$$

and give a bound on the magnitude of the solution to (1.49).

Lemma 1.3.1. *Let the coefficients of the difference operator $\mathcal{A}(p, t_m)$ satisfy (1.48) and $\overline{\Lambda}^h$ be connected (1.6).*

(i) If a mesh function $W(p, t_m)$ satisfies the conditions

$$\left(\mathcal{A}(p, t_m) + (\tau_m^{-1} + c^*(p, t_m)I\right) W(p, t_m) \geq 0 \quad (\leq 0), \quad p \in \Lambda^h, \tag{1.50}$$
$$W(p, t_m) \geq 0 \ (\leq 0), \quad p \in \partial\Lambda^h,$$

then $W(p, t_m) \geq 0$ (≤ 0), $\quad p \in \overline{\Lambda}^h$.

(ii) The following bound on the magnitude of the solution to (1.49) holds

$$\|W(\cdot, t_m)\|_{\overline{\Lambda}^h} \leq \max\left\{ \|g(\cdot, t_m)\|_{\partial\Lambda^h}, \ \frac{\|\phi(\cdot, t_m)\|_{\Lambda^h}}{\|c^*(\cdot, t_m)\|_{\overline{\Lambda}^h} + \tau_m^{-1}} \right\}, \tag{1.51}$$

where

$$\|g(\cdot, t_m)\|_{\partial\Lambda^h} \equiv \max_{p\partial\Lambda^h} |g(p, t_m)|, \quad \|\phi(\cdot, t_m)\|_{\Lambda^h} \equiv \max_{p\in\Lambda^h} |\phi(p, t_m)|.$$

Proof. The proof of the lemma on each time level t_m, $m \geq 1$, repeats the proof of Lemma 1.2.1 for the case of the elliptic problem with the following modifications. In (1.10) and (1.12), we have now, respectively,

$$d(p, t_m) + c^*(p, t_m) + \tau_m^{-1} - \sum_{p' \in \sigma'(p)} a(p', t_m) \geq 0,$$

and

$$\big(\mathcal{A}(p,t_m)+(c^*(p,t_m)+\tau_m^{-1})I\big)\,V(p,t_m)=|\phi(p,t_m)|,\quad p\in\Lambda^h,$$
$$V(p,t_m)=\|g(\cdot,t_m)\|_{\partial\Lambda^h},\quad p\in\partial\Lambda^h.$$

□

1.3.2 The method of upper and lower solutions

On each time level t_m, $m\geq 1$, two mesh functions $\widetilde{U}(p,t_m)$ and $\widehat{U}(p,t_m)$, $p\in\overline{\Lambda}^h$ are called ordered upper and lower solutions of the difference scheme (1.46), if they satisfy the inequalities

$$\widehat{U}(p,t_m)\leq\widetilde{U}(p,t_m),\quad p\in\overline{\Lambda}^h, \tag{1.52a}$$
$$\big(\mathcal{A}(p,t_m)+\tau_m^{-1}I\big)\,\widetilde{U}p+f(p,t_m,\widetilde{U})-\tau_m^{-1}\widetilde{U}(p,t_{m-1})\geq 0,\quad p\in\Lambda^h, \tag{1.52b}$$
$$\big(\mathcal{A}(p,t_m)+\tau_m^{-1}I\big)\,\widehat{U}(p,t_m)+f(p,t_m,\widehat{U})-\tau_m^{-1}\widehat{U}(p,t_{m-1})\leq 0,\quad p\in\Lambda^h,$$
$$\widehat{U}(p,t_m)\leq g(p,t_m)\leq\widetilde{U}(p,t_m),\quad p\in\partial\Lambda^h, \tag{1.52c}$$
$$\widehat{U}(p,0)\leq\psi(p)\leq\widetilde{U}(p,0),\quad p\in\overline{\Lambda}^h.$$

For given upper and lower solutions $\widetilde{U}(p,t_m)$, $\widehat{U}(p,t_m)$ and t_m fixed, we define the sector

$$\langle\widehat{U}(t_m),\widetilde{U}(t_m)\rangle=\Big\{U(p,t_m):\widehat{U}(p,t_m)\leq U(p,t_m)\leq\widetilde{U}(p,t_m),\quad p\in\overline{\Lambda}^h,\quad m\geq 1\Big\}.$$

We assume that $f(p,t_m,U)$ satisfies the constraint

$$f_u(p,t_m,U)\leq c(p,t_m)\quad U(p,t_m)\in\langle\widehat{U}(t_m),\widetilde{U}(t_m)\rangle,\quad p\in\overline{\Lambda}^h,\quad f_u\equiv\frac{\partial f}{\partial u}, \tag{1.53}$$

where $c(p,t_m)$ is a nonnegative bounded mesh function.

To solve the nonlinear difference scheme (1.46), we construct an iterative method which satisfies the monotone convergence property. On each time level t_m, $m\geq 1$, the sequence of solutions $\{U^{(n)}(p,t_m)\}$, $p\in\overline{\Lambda}^h$ is calculated by the following iterative method:

$$\big(\mathcal{A}(p,t_m)+(\tau_m^{-1}+c(p,t_m))I\big)\,Z^{(n)}(p,t_m)=-\mathcal{K}(p,t_m,U^{(n-1)}),\quad p\in\Lambda^h, \tag{1.54}$$
$$Z^{(1)}(p,t_m)=g(p,t_m)-U^{(0)}(p,t_m),\quad Z^{(n)}(p,t_m)=0,\quad m\geq 2,\quad p\in\partial\Lambda^h,$$
$$U(p,0)=\psi(p),\quad p\in\overline{\Lambda}^h,\quad U(p,t_m)=U^{(n_m)}(p,t_m),$$
$$\mathcal{K}(p,t_m,U^{(n-1)})=(\mathcal{A}(p,t_m)+\tau_m^{-1}I)U^{(n-1)}(p,t_m)+f(p,t_m,U^{(n-1)})-\tau_m^{-1}U(p,t_{m-1}),$$
$$Z^{(n)}(p,t_m)=U^{(n)}(p,t_m)-U^{(n-1)}(p,t_m),\quad p\in\overline{\Lambda}^h,\quad m\geq 1,$$

where $\mathcal{K}(p,t_m,U^{(n-1)})$, $p\in\Lambda^h$ is the residual of the implicit difference scheme (1.46), $c(p,t_m)$ is defined in (1.53), $U(p,t_m)$ is the approximate solution on each time level t_m and n_m is the number of iterates on time level t_m.

We introduce the notation

$$\Gamma(p,t_m,U)=c(p,t_m)U(p,t_m)-f(p,t_m,U),\quad p\in\overline{\Lambda}^h, \tag{1.55}$$

and prove the monotone property of $\Gamma(p,t_m,U)$.

Lemma 1.3.2. *Assume that $U_1(p,t_m)$ and $U_2(p,t_m)$, $p\in\overline{\Lambda}^h$, $m\geq 1$, are functions in $\langle\widehat{U}(t_m),\widetilde{U}(t_m)\rangle$, such that $U_1(p,t_m)\leq U_2(p,t_m)$, and (1.6), (1.53) are satisfied. Then*

$$\Gamma(p,t_m,U_1)\leq\Gamma(p,t_m,U_2),\quad p\in\overline{\Lambda}^h,\quad m\geq 1. \tag{1.56}$$

Proof. From (1.55), we obtain

$$\begin{aligned}\Gamma(p,t_m,U_2)-\Gamma(p,t_m,U_1) &= c(p,t_m)[U_2(p,t_m)-U_1(p,t_m)]\\ &\quad -[f(p,t_m,U_2)-f(p,t_m,U_1)].\end{aligned}$$

By using the mean-value theorem, we have

$$f(p,t_m,U_2)-f(p,t_m,U_1)=f_u(p,t_m,Q)\left(U_2(p,t_m)-U_1(p,t_m)\right),$$

where $U_1(p,t_m)\leq Q(p,t_m)\leq U_2(p,t_m)$, $p\in\overline{\Lambda}^h$. From here, using the assumption of the lemma and (1.53), we conclude (1.56). □

In the following theorem, we prove the monotone convergence of upper and lower sequences generated by (1.54).

Theorem 1.3.3. *Suppose that the coefficients of the difference operator $\mathcal{A}(p,t_m)$ in (1.46) satisfy (1.48), $f(p,t_m,U)$ satisfies (1.53) and $\overline{\Lambda}^h$ is connected (1.6). Let $\widetilde{U}(p,t_m)$ and $\widehat{U}(p,t_m)$, $p\in\overline{\Lambda}^h$, $m\geq 1$, be ordered upper and lower solutions (1.52). Then upper $\{\overline{U}^{(n)}(p,t_m)\}$ and lower $\{\underline{U}^{(n)}(p,t_m)\}$, $p\in\overline{\Lambda}^h$, $m\geq 1$, sequences generated by (1.54) with, respectively, $\overline{U}^{(0)}(p,t_m)=\widetilde{U}(p,t_m)$ and $\underline{U}^{(0)}(p,t_m)=\widehat{U}(p,t_m)$, $p\in\overline{\Lambda}^h$, $m\geq 1$, converge monotonically, such that,*

$$\underline{U}^{(n-1)}(p,t_m)\leq\underline{U}^{(n)}(p,t_m)\leq\overline{U}^{(n)}(p,t_m)\leq\overline{U}^{(n-1)}(p,t_m),\quad p\in\overline{\Lambda}^h,\quad m\geq 1. \tag{1.57}$$

Proof. Since $\underline{U}^{(0)}(p,t_m)=\widehat{U}(p,t_m)$, $p\in\overline{\Lambda}^h$ is a lower solution, it follows that the residual $\mathcal{K}(p,t_1,\underline{U}^{(0)})\leq 0$. From here and (1.54), on the first time level t_1, we obtain

$$\left(\mathcal{A}(p,t_1)+(\tau_1^{-1}+c(p,t_1))I\right)\underline{Z}^{(1)}(p,t_1)\geq 0,\quad p\in\Lambda^h,\quad \underline{Z}^{(1)}(p,t_1)\geq 0,\quad p\in\partial\Lambda^h.$$

By using the maximum principle in Lemma 1.3.1, we conclude that

$$\underline{Z}^{(1)}(p,t_1) \geq 0, \quad p \in \overline{\Lambda}^h. \tag{1.58}$$

Similarly, for the upper solution $\overline{U}^{(0)}(p,t_1) = \widetilde{U}(p,t_1)$, $p \in \overline{\Lambda}^h$, we have

$$\overline{Z}^{(1)}(p,t_1) \leq 0, \quad p \in \overline{\Lambda}^h. \tag{1.59}$$

We now prove that $\overline{U}^{(1)}(p,t_1)$ and $\underline{U}^{(1)}(p,t_1)$, $p \in \overline{\Lambda}^h$ are ordered upper and lower solutions. Denoting $W^{(1)}(p,t_1) = \overline{U}^{(1)}(p,t_1) - \underline{U}^{(1)}(p,t_1)$, $p \in \overline{\Lambda}^h$, using notation (1.55), from (1.54), we obtain that

$$\begin{aligned}
&\left(\mathcal{A}(p,t_1) + (\tau_1^{-1} + c(p,t_1))I\right) W^{(1)}(p,t_1) = \Gamma(p,t_1,\overline{U}^{(0)}) - \Gamma(p,t_1,\underline{U}^{(0)}), \quad p \in \Lambda^h,\\
&W^{(1)}(p,t_1) = 0, \quad p \in \partial\Lambda^h.
\end{aligned}$$

From here, (1.56) and taking into account that $\underline{U}^{(0)}(p,t_1) \leq \overline{U}^{(0)}(p,t_1)$, $p \in \overline{\Lambda}^h$, by Lemma 1.3.1, we conclude that

$$W^{(1)}(p,t_1) \geq 0, \quad p \in \overline{\Lambda}^h. \tag{1.60}$$

Thus, we prove (1.52a). From (1.54) and using notation (1.55), we have

$$\mathcal{K}(p,t_1,\underline{U}^{(1)}) = \Gamma(p,t_1,\underline{U}^{(0)}) - \Gamma(p,t_1,\underline{U}^{(1)}), \quad p \in \Lambda^h, \tag{1.61}$$

From here, (1.56) and (1.60), it follows that

$$\mathcal{K}(p,t_1,\underline{U}^{(1)}) \leq 0, \quad p \in \Lambda^h. \tag{1.62}$$

Similarly, we can prove that

$$\mathcal{K}(p,t_1,\overline{U}^{(1)}) \geq 0, \quad p \in \Lambda^h. \tag{1.63}$$

Thus, we conclude (1.52b). From the boundary and initial conditions in (1.54), it follows that $\underline{U}^{(1)}(p,t_1)$ and $\overline{U}^{(1)}(p,t_1)$, $p \in \overline{\Lambda}^h$ satisfy (1.52c). From here, (1.60), (1.62) and (1.63), we conclude that $\overline{U}^{(1)}(p,t_1)$ and $\underline{U}^{(1)}(p,t_1)$, $p \in \overline{\Lambda}^h$ are ordered upper and lower solutions (1.52). By induction on $n \geq 1$, we can prove that $\overline{U}^{(n)}(p,t_1)$ and $\underline{U}^{(n)}(p,t_1)$, $p \in \overline{\Lambda}^h$ are ordered upper and lower solutions (1.52) which satisfy the monotone property (1.57) on the first time level t_1.

On the time level t_1, from (1.57), we have

$$\widehat{U}(p,t_1) \leq \underline{U}^{(n_1)}(p,t_1) \leq \overline{U}_1^{(n_1)}(p,t_1) \leq \widetilde{U}(p,t_1), \quad p \in \overline{\Lambda}^h.$$

From the assumption of the theorem that $\widetilde{U}(p,t_2)$ and $\widehat{U}(p,t_2)$, $p\in\overline{\Lambda}^h$ are upper and lower solutions (1.52), we obtain that $\widetilde{U}(p,t_2)$ and $\widehat{U}(p,t_2)$, $p\in\overline{\Lambda}^h$ are upper and lower solutions with respect to $\overline{U}^{(n_1)}(p,t_1)$ and $\underline{U}^{(n_1)}(p,t_1)$, $p\in\overline{\Lambda}^h$, that is,

$$(\mathcal{A}(p,t_2)+\tau_2^{-1}I)\widetilde{U}(p,t_2)+f(p,t_2,\widetilde{U})-\tau_2^{-1}\overline{U}_1^{(n_1)}(p,t_1)\geq 0,\quad p\in\Lambda^h,$$

$$(\mathcal{A}(p,t_2)+\tau_2^{-1}I)\widehat{U}(p,t_2)+f(p,t_2,\widehat{U})-\tau_2^{-1}\underline{U}^{(n_1)}(p,t_1)\leq 0,\quad p\in\Lambda^h.$$

On the second time level t_2, from (1.54), we have

$$\begin{aligned}\big(\mathcal{A}(p,t_2)+(\tau_2^{-1}+c(p,t_2))I\big)\,\underline{U}^{(1)}(p,t_2)&=c(p,t_2)\underline{U}^{(0)}(p,t_2)-f(p,t_2,\underline{U}^{(0)})\\&\quad+\tau_2^{-1}\underline{U}^{(n_1)}(p,t_1),\quad p\in\Lambda^h,\\ \underline{U}^{(1)}(p,t_2)=g(p,t_2),\quad p\in\partial\Lambda^h.&\end{aligned}$$

From here and using notation (1.55), for $W^{(1)}(p,t_2)=\overline{U}^{(1)}(p,t_2)-\underline{U}^{(1)}(p,t_2)$, $p\in\overline{\Lambda}^h$, we have the following difference problem

$$\begin{aligned}\big(\mathcal{A}(p,t_2)+(\tau_2^{-1}+c(p,t_2))I\big)\,W^{(1)}(p,t_2)&=\Gamma(p,t_2,\overline{U}^{(0)})-\Gamma(p,t_2,\underline{U}^{(0)})\\&\quad+\tau_2^{-1}\left[\overline{U}^{(n_1)}(p,t_1)-\underline{U}^{(n_1)}(p,t_1)\right].\end{aligned}$$

Taking into account that $\underline{U}^{(0)}(p,t_2)\leq\overline{U}^{(0)}(p,t_2)$ and $\underline{U}^{(n_1)}(p,t_2)\leq\overline{U}^{(n_1)}(p,t_2)$, $p\in\overline{\Lambda}^h$, (1.56) and using Lemma 1.3.1, it follows that $W^{(1)}(p,t_2)\geq 0$, that is,

$$\underline{U}^{(1)}(p,t_2)\leq\overline{U}^{(1)}(p,t_2),\quad p\in\overline{\Lambda}^h.$$

The proof that $\overline{U}^{(1)}(p,t_2)$ and $\underline{U}^{(1)}(p,t_2)$, $p\in\overline{\Lambda}^h$ are ordered upper and lower solutions (1.52) repeats the proof on the first time level t_1. By induction on $n\geq 1$, we can prove that $\overline{U}^{(n)}(p,t_2)$ and $\underline{U}^{(n)}(p,t_2)$, $p\in\overline{\Lambda}^h$ are ordered upper and lower solutions (1.52), which satisfy the monotone property (1.57) on the second time level t_2. By induction on $m\geq 1$, we can prove (1.57) for $m\geq 1$. □

1.3.3 Existence and uniqueness of a solution to the nonlinear difference scheme

Theorem 1.3.4. *Let the assumptions in Theorem 1.3.3 be satisfied. Then the nonlinear difference scheme (1.46) has maximal* $\overline{U}(p,t_m)$ *and minimal* $\underline{U}(p,t_m)$, $p\in\overline{\Lambda}^h$, $m\geq 1$, *solutions in the sector* $\langle\widehat{U}(t_m),\widetilde{U}(t_m)\rangle$. *If* $V(p,t_m)$, $p\in\overline{\Lambda}^h$, $m\geq 1$, *is any other solution in* $\langle\widehat{U}(t_m),\widetilde{U}(t_m)\rangle$, *then*

$$\underline{U}(p,t_m)\leq V(p,t_m)\leq\overline{U}(p,t_m),\quad p\in\overline{\Lambda}^h,\quad m\geq 1. \tag{1.64}$$

Proof. From (1.57), we conclude that $\lim \underline{U}^{(n)}(p,t_1) = \underline{U}(p,t_1)$, $p \in \overline{\Lambda}^h$ as $n \to \infty$ exists, and

$$\widehat{U}(p,t_1) \le \underline{U}^{(n-1)}(p,t_1) \le \underline{U}^{(n)}(p,t_1), \quad \lim_{n\to\infty} \underline{Z}^{(n)}(p,t_1) = 0, \quad p \in \overline{\Lambda}^h, \tag{1.65}$$

where $\underline{U}^{(0)}(p,t_1) = \widehat{U}(p,t_1)$, $p \in \overline{\Lambda}^h$. From (1.54) and using the mean-value theorem, we conclude that

$$\mathcal{K}(p,t_1,\underline{U}^{(n)}) = -\left(c(p,t_1) - f_u(p,t_1,\underline{Q}^{(n)})\right)\underline{Z}^{(n)}(p,t_1), \quad p \in \Lambda^h, \tag{1.66}$$

where $\underline{U}^{(n-1)}(p,t_1) \le \underline{Q}^{(n-1)}(p,t_1) \le \underline{U}^{(n)}(p,t_1)$, $p \in \overline{\Lambda}^h$.

By taking limit of the both sides and using (1.65), it follows that

$$\mathcal{K}\left(p,t_1,\underline{U}\right) = 0, \quad p \in \Lambda^h. \tag{1.67}$$

Similarly, we can prove that

$$\mathcal{K}\left(p,t_1,\overline{U}\right) = 0, \quad p \in \Lambda^h.$$

where $\overline{U}(p,t_1) = \lim_{n\to\infty} \overline{U}^{(n)}$, $p \in \overline{\Lambda}^h$. Thus, from here and (1.67), we conclude that $\underline{U}(p,t_1)$ and $\overline{U}(p,t_1)$, $p \in \overline{\Lambda}^h$, are, respectively, minimal and maximal solutions of the nonlinear difference scheme (1.46) in the sector $\langle \widehat{U}(t_1), \widetilde{U}(t_1)\rangle$. By the assumption of Theorem 1.3.3 that $\widehat{U}(p,t_2)$ is a lower solution and from (1.65), on the second time level t_2, we obtain that

$$\mathcal{K}(p,t_2,\widehat{U}) = (\mathcal{A}(p,t_2) + \tau_2^{-1}I)\widehat{U}(p,t_2) + f(p,t_2,\widehat{U}) - \tau_2^{-1}\underline{U}(p,t_1),$$

where $\underline{U}(p,t_1)$, $p \in \overline{\Lambda}^h$ is the approximate solution on the first time level t_1, which is defined in (1.54). From here and taking into account that from (1.57), $\underline{U}(p,t_1) \ge \widehat{U}(p,t_1)$, $p \in \overline{\Lambda}^h$, it follows that

$$\mathcal{K}(p,t_2,\widehat{U}) \le (\mathcal{A}(p,t_2) + \tau_2^{-1}I)\widehat{U}(p,t_2) + f(p,t_2,\widehat{U}) - \tau_2^{-1}\widehat{U}(p,t_1) \le 0,$$

which means that $\widehat{U}(p,t_2)$ is a lower solution with respect to $\underline{U}(p,t_1)$, $p \in \overline{\Lambda}^h$. By a similar argument as on the first time level t_1, we can prove that

$$\lim_{n\to\infty} \underline{U}^{(n)}(p,t_2) = \underline{U}(p,t_2), \quad p \in \overline{\Lambda}^h,$$

exists and solves (1.46) on the second time level t_2. By induction on $m \ge 1$, we can prove that

$$\underline{U}(p,t_m) = \lim_{n\to\infty} \underline{U}^{(n)}(p,t_m), \quad p \in \overline{\Lambda}^h,$$

is a solution of the nonlinear difference scheme (1.46).

Similarly, we can prove that

$$\overline{U}(p,t_m) = \lim_{n\to\infty} \overline{U}^{(n)}(p,t_m), \quad p \in \overline{\Lambda}^h,$$

is another solution to the nonlinear difference scheme (1.46).

On each time level t_m, $m \geq 1$, the proof of (1.64) repeats the proof of (1.29) from Theorem 1.2.5 for elliptic problems. $\square$

For uniqueness of a solution of (1.46), we assume that $f(p,t_m,U)$ satisfies the two sided inequalities

$$\underline{c}(p,t_m) \leq f_u(p,t_m,U) \leq c(p,t_m), \quad U(p,t_m) \in \langle \widehat{U}(t_m), \widetilde{U}(t_m)\rangle, \quad p \in \overline{\Lambda}^h, \quad m \geq 1, \tag{1.68}$$

where $\widetilde{U}(p,t_m)$, $\widehat{U}(p,t_m)$, $p \in \overline{\Lambda}^h$, $m \geq 1$, are given ordered upper and lower solutions of (1.46), $\underline{c}(p,t_m)$ and $c(p,t_m)$ are, respectively, bounded and nonnegative bounded mesh functions. It is assumed that the time step τ_m satisfies the assumption

$$\tau_m < \frac{1}{|\gamma_m|}, \quad \gamma_m = \min(0, \underline{c}_m), \quad \underline{c}_m = \min_{p\in\overline{\Lambda}^h} \underline{c}(p,t_m), \quad m \geq 1, \tag{1.69}$$

where $\underline{c}(p,t_m)$ is defined in (1.68). If $\gamma_m = 0$, then no restrictions on time exist.

In the following theorem, we prove the uniqueness of a solution of the nonlinear difference scheme (1.46).

Theorem 1.3.5. *Let the mesh $\overline{\Lambda}^h$ be connected (1.6), and τ_m, $m \geq 1$, satisfy (1.69). Assume that the coefficients of the difference operator $\mathcal{A}(p,t_m)$ in (1.46) satisfy (1.48) and $f(p,t_m,U)$ satisfies (1.68). Then the nonlinear difference scheme (1.46) has a unique solution.*

Proof. On each time level t_m, $m \geq 1$, from Theorem 1.3.4, it follows that $\underline{U}(p,t_m)$ and $\overline{U}(p,t_m)$, $p \in \overline{\Lambda}^h$, $m \geq 1$, are two solutions of the nonlinear difference scheme (1.46). For uniqueness of a solution, it is sufficient to prove that $\underline{U}(p,t_m) = \overline{U}(p,t_m)$, $p \in \overline{\Lambda}^h$, $m \geq 1$. On the first time level t_1, in the notation $W(p,t_1) = \overline{U}(p,t_1) - \underline{U}(p,t_1)$, $p \in \overline{\Lambda}^h$, from (1.46), it follows that

$$(\mathcal{A}(p,t_1) + \tau_1^{-1} I) W(p,t_1) + f(p,t_1,\overline{U}) - f(p,t_1,\underline{U}) = 0, \quad p \in \Lambda^h,$$
$$W(p,t_1) = 0, \quad p \in \partial\Lambda^h.$$

From here, by using the mean-value theorem, we conclude that

$$\big(\mathcal{A}(p,t_1)+(\tau_1^{-1}+f_u(p,t_1,Q))I\big)\,W(p,t_1)=0,\quad p\in\Lambda^h,\quad W(p,t_1)=0,\quad p\in\partial\Lambda^h, \tag{1.70}$$

where $\underline{U}(p,t_1) \le Q(p,t_1) \le \overline{U}(p,t_1)$. From (1.57) and (1.64), we conclude that $f_u(p,t_1,Q)$ satisfies (1.68). From (1.68) and (1.69), we obtain

$$\tau_1^{-1}+f_u(p,t_1,Q)>0.$$

From here and (1.70), by using Lemma 1.3.1, we conclude that $W(p,t_1)=0,\quad p\in\Lambda^h$.

On the second time level t_2, we have

$$\big(\mathcal{A}(p,t_2)+(\tau_2^{-1}+f_u(p,t_2,Q))I\big)\,W(p,t_2)=0,\quad p\in\Lambda^h,\quad W(p,t_2)=0,\quad p\in\partial\Lambda^h,$$

where $\underline{U}(p,t_2)\le Q(p,t_2)\le\overline{U}(p,t_2)$, $p\in\overline{\Lambda}^h$. By the same argument as for $W(p,t_1)=0$, $p\in\overline{\Lambda}^h$, we obtain $W(p,t_2)=0$, $p\in\overline{\Lambda}^h$. By induction on m, $m\ge1$, we can prove that $W(p,t_m)=0$, $p\in\overline{\Lambda}^h$, $m\ge1$. Thus, we prove the theorem. □

1.3.4 Convergence analysis of the monotone iterative method

Convergence analysis of the monotone iterative method on $[\mathbf{0},\mathbf{T}]$

Here, we investigate convergence of the monotone iterative method of the whole time interval $[0,T]$. We now choose a stopping criterion for the monotone iterative method (1.54) as follows:

$$\left\|\mathcal{K}\left(\cdot,t_m,U^{(n)}\right)\right\|_{\Lambda^h}\le\delta,\quad m\ge1, \tag{1.71}$$

where $U^{(n)}(p,t_m)$, is generated by (1.54), and δ is a prescribed accuracy. We set up $U(p,t_m)=U^{(n_m)}(p,t_m)$, $p\in\overline{\Lambda}^h$, $m\ge1$, such that n_m is minimal subject to (1.71). We now prove the following theorem for the convergence of the iterative method (1.54), (1.71).

Theorem 1.3.6. *Let the mesh $\overline{\Lambda}^h$ be connected (1.6), and τ_m, $m\ge1$, satisfy (1.69). Assume that the coefficients of the difference operator $\mathcal{A}(p,t_m)$ in (1.46) satisfy (1.48) and $f(p,t_m,U)$ satisfies (1.68). Then the following estimate holds:*

$$\max_{m\ge1}\|U(\cdot,t_m)-U^*(\cdot,t_m)\|_{\overline{\Lambda}^h}\le T\delta, \tag{1.72}$$

where $U(p,t_m)$, $p\in\overline{\Lambda}^h$, $m\ge1$ is the approximate solution generated by (1.54), (1.71) and $U^*(p,t_m)$, $p\in\overline{\Lambda}^h$, $m\ge1$, is the unique solution of the nonlinear difference scheme (1.46). Furthermore, on each time level $m\ge1$, the sequences converge monotonically

(1.57).

Proof. Theorem 1.3.3 gives the monotone convergence of the sequence $\{U^{(m)}(p,t_m)\}$, $p \in \overline{\Lambda}^h$, $m \geq 1$. The existence and uniqueness of a solution of the nonlinear difference scheme (1.46) are proved in Theorems 1.3.4 and 1.3.5. We present the difference problem for $\overline{U}(p,t_m) = \overline{U}^{(n_m)}(p,t_m)$, $p \in \overline{\Lambda}^h$, $m \geq 1$ in the form

$$\begin{aligned}
&(\mathcal{A}(p,t_m) + \tau_m^{-1} I)\overline{U}(p,t_m) + f(p,t_m,\overline{U}) - \tau_m^{-1}\overline{U}(p,t_{m-1}) = \mathcal{K}(p,t_m,\overline{U}), \quad p \in \Lambda^h, \\
&U_1(p,t_m) = g(p,t_m), \quad p \in \partial\Lambda^h.
\end{aligned}$$

From (1.46), for $U^*(p,t_m)$, we have

$$(\mathcal{A}(p,t_m) + \tau_m^{-1} I)U^*(p,t_m) + f(p,t_m,U^*) - \tau_m^{-1}U^*(p,t_{m-1}) = 0, \quad p \in \Lambda^h.$$

From here, for $\overline{W}(p,t_m) = \overline{U}(p,t_m) - U^*(p,t_m)$, $p \in \overline{\Lambda}^h$ and using the mean-value theorem, it follows that

$$\begin{aligned}
&(\mathcal{A}(p,t_m) + \tau_m^{-1} I)\overline{W}(p,t_m) + f_u(p,t_m,\overline{Q})\overline{W}(p,t_m) = \mathcal{K}(p,t_m,\overline{U}^{(n_m)}) \\
&\qquad\qquad\qquad\qquad + \tau_m^{-1}\overline{W}(p,t_{m-1}), \\
&p \in \Lambda^h, \quad \overline{W}(p,t_m) = 0, \quad p \in \partial\Lambda^h,
\end{aligned}$$

where $U^*(p,t_m) \leq \overline{Q}(p,t_m) \leq \overline{U}(p,t_m)$, $p \in \overline{\Lambda}^h$, $m \geq 1$. From here, (1.68), (1.69) and (1.71), by using (1.51), we obtain

$$\left\|\overline{W}(\cdot,t_m)\right\|_{\overline{\Lambda}^h} \leq \tau_m \delta + \left\|\overline{W}(\cdot,t_{m-1})\right\|_{\overline{\Lambda}^h}.$$

Taking into account that $\left\|\overline{W}(\cdot,t_0)\right\| = 0$, by induction on $m \geq 1$, we conclude that

$$\|\overline{W}(\cdot,t_m)\|_{\overline{\Lambda}^h} \leq \delta \sum_{s=1}^{m} \tau_s \leq \delta T.$$

Thus, we prove the theorem. □

We now investigate convergence properties of the monotone iterative method (1.54) on each time level.

Linear rate of convergence

We modify the monotone iterative method (1.54) by replacing $c(p,t_m)$ by the constant $\widetilde{c}$ as follows:

$$\widetilde{c} = \max_{(p,t_m)\in\overline{\Lambda}^h\times\overline{\Lambda}^\tau} c(p,t_m). \tag{1.73}$$

Theorem 1.3.3 still holds if we replace $c(p,t_m)$ by $\widetilde{c}$.

Theorem 1.3.7. *Suppose that the coefficients of the difference operator $\mathcal{A}(p,t_m)$ in (1.46) satisfy (1.48), $f(p,t_m,U)$ satisfies (1.53) and $\overline{\Lambda}^h$ is connected (1.6). Then for the sequence $\{U^{(n)}(p,t_m)\}$, $p \in \overline{\Lambda}^h$, $m \geq 1$, generated by (1.54), (1.73), the following estimate holds:*

$$\|Z^{(n)}(\cdot,t_m)\|_{\overline{\Lambda}^h} \leq q_m^{n-1}\|Z^{(1)}(\cdot,t_m)\|_{\overline{\Lambda}^h}, \quad q_m = \frac{\widetilde{c}}{\widetilde{c}+\tau_m^{-1}}, \tag{1.74}$$

where $q_m < 1$ is the linear rate of convergence.

Proof. We consider the case of lower solution. Similar to (1.66), with assumption (1.73), we conclude that

$$\mathcal{K}(p,t_m,\underline{U}^{(n-1)}) = -\left(\widetilde{c} - f_u(p,t_m,\underline{Q}^{(n-1)})\right)\underline{Z}^{(n-1)}(p,t_m), \quad p \in \Lambda^h, \tag{1.75}$$

where $\underline{U}^{(n-1)}(p,t_m) \leq \underline{Q}^{(n-1)}(p,t_m) \leq \underline{U}^{(n-1)}(p,t_m)$, $p \in \overline{\Lambda}^h$, $m \geq 1$. From (1.59) and (1.60), it follows that the partial derivative $f_u(p,t_n,\overline{Q}^{(n-1)})$ satisfies (1.53). From here, (1.54), (1.73) and (1.75), we obtain that

$$\left(\mathcal{A}(p,t_m) + (\tau_m^{-1}+\widetilde{c})I\right)\underline{Z}^{(n)}(p,t_m) = \left(\widetilde{c} - f_u(p,t_m,\underline{Q}^{(n-1)})\right)\underline{Z}^{(n-1)}(p,t_m), \quad p \in \Lambda^h.$$

By using Lemma 1.3.1, it follows that

$$\left\|\underline{Z}^{(n)}(\cdot,t_m)\right\|_{\overline{\Lambda}^h} \leq q_m \left\|\underline{Z}^{(n-1)}(\cdot,t_m)\right\|_{\overline{\Lambda}^h}.$$

By induction on n, we can prove (1.74) for a lower sequence $\{\underline{U}^{(n)}(p,t_m)\}$, $p \in \overline{\Lambda}^h$, $m \geq 1$. By a similar argument, we can prove (1.74) for $\{\overline{U}^{(n)}(p,t_m)\}$, $p \in \overline{\Lambda}^h$, $m \geq 1$. □

Quadratic rate of convergence

On each time level t_m, $m \geq 1$, we modify the monotone iterative method (1.54) by replacing $c(p,t_m)$ by $c^{(n-1)}(p,t_m)$, $n \geq 1$, and calculating the sequence $\{U^{(n)}(p,t_m)\}$, $p \in \overline{\Lambda}^h$, $m \geq 1$, as follows:

$$\left(\mathcal{A}(p,t_m) + (\tau_m^{-1} + c^{(n-1)}(p,t_m))I\right)Z^{(n)}(p,t_m) = -\mathcal{K}(p,t_m,U^{(n-1)}), \quad p \in \Lambda^h, \tag{1.76}$$
$$Z^{(1)}(p,t_m) = g(p,t_m) - U^{(0)}(p,t_m), \quad Z^{(n)}(p,t_m) = 0, \quad n \geq 2, \quad p \in \partial\Lambda^h,$$
$$U(p,0) = \psi(p), \quad p \in \overline{\Lambda}^h, \quad U(p,t_m) = U^{(n_m)}(p,t_m),$$
$$\mathcal{K}(p,t_m,U^{(n-1)}) = (\mathcal{A}(p,t_m)+\tau_m^{-1}I)U^{(n-1)}(p,t_m) + f(p,t_m,U^{(n-1)}) - \tau_m^{-1}U(p,t_{m-1}),$$
$$Z^{(n)}(p,t_m) = U^{(n)}(p,t_m) - U^{(n-1)}(p,t_m), \quad p \in \overline{\Lambda}^h, \quad m \geq 1,$$

where the mesh function $c^{(n-1)}(p,t_m)$ is given by

$$c^{(n-1)}(p,t_m)=\max_{U}\{f_u(p,t_m,U)\},\quad \underline{U}^{(n-1)}(p,t_m)\le U(p,t_m)\le \overline{U}^{(n-1)}(p,t_m). \quad (1.77)$$

On each time level $m\ge 1$, two sequences $\{\overline{U}^{(n)}(p,t_m)\}$ and $\{\underline{U}^{(n)}(p,t_m)\}$, $p\in\overline{\Lambda}^h$, $m\ge 1$, are in use for calculating $c^{(n-1)}(p,t_m)$.

We introduce the notation

$$\xi_m=\max_{p\in\overline{\Lambda}^h}\left[\max_{U}\left\{|f_{uu}(p,t_m,U)|\,,U(p,t_m)\in\langle\widehat{U}(t_m),\widetilde{U}(t_m)\rangle,\quad p\in\overline{\Lambda}^h\right\}\right]. \quad (1.78)$$

We now prove the quadratic convergence of the monotone iterative method (1.76), (1.77) in the following theorem.

Theorem 1.3.8. *Suppose that the coefficients of the difference operator $\mathcal{A}(p,t_m)$ in (1.46) satisfy (1.48), and mesh $\overline{\Lambda}^h$ is connected (1.6). Assume that f satisfies (1.53). Then for the sequences $\{\overline{U}^{(m)}(p,t_m)\}$ and $\{\overline{U}^{(m)}(p,t_m)\}$, $p\in\overline{\Lambda}^h$, $m\ge 1$, generated by (1.76), (1.77), the following estimate holds:*

$$\left\|W^{(n)}(\cdot,t_m)\right\|_{\overline{\Lambda}^h}\le\tau_m\xi_m\left\|W^{(n-1)}(\cdot,t_m)\right\|^2_{\overline{\Lambda}^h}, \quad (1.79)$$

where $W^{(n)}(p,t_m)=\overline{U}^{(n)}(p,t_m)-\underline{U}^{(n)}(p,t_m)$, $p\in\overline{\Lambda}^h$, $m\ge 1$, and ξ_m is defined in (1.78).

Proof. From (1.76) and (1.77), we obtain

$$\left(\mathcal{A}(p,t_m)+(\tau_m^{-1}+c^{(n-1)}(p,t_m))I\right)W^{(n)}(p,t_m)=G^{(n-1)}(p,t_m),\quad p\in\Lambda^h, \quad (1.80)$$
$$G^{(n-1)}(p,t_m)=c^{(n-1)}(p,t_m)W^{(n-1)}(p,t_m)-\left(f(p,t_m,\overline{U}^{(n-1)})-f(p,t_m,\underline{U}^{(n-1)})\right),$$
$$W^{(n)}(p,t_m)=0,\quad p\in\partial\Lambda^h.$$

By using the mean-value theorem, we have

$$f(p,t_m,\overline{U}^{(n-1)})-f(p,t_m,\underline{U}^{(n-1)})=f_u(p,t_m,Q^{(n-1)})W^{(n-1)}(p,t_m),$$

where $\underline{U}^{(n-1)}(p,t_m)\le Q^{(n-1)}(p,t_m)\le\overline{U}^{(n-1)}(p,t_m)$. From (1.77), we have

$$c^{(n-1)}(p,t_m)=f_u(p,t_m,Y^{(n-1)}),$$

where $\underline{U}^{(n-1)}(p,t_m)\le Y^{(n-1)}(p,t_m)\le\overline{U}^{(n-1)}(p,t_m)$. We now present the right hand side of $G^{(n-1)}(p,t_m)$ in (1.80) as follows:

$$G^{(n-1)}(p,t_m) = \left(f_u(p,t_m,Y^{(n-1)}) - f_u(p,t_m,Q^{(n-1)})\right) W^{(n-1)}(p,t_m).$$

By applying the mean-value theorem, we have

$$f_u(p,t_m,Y^{(n-1)}) - f_u(p,t_m,Q^{(n-1)}) = \\ f_{uu}(p,t_m,H^{(n-1)}) \left(Y^{(n-1)}(p,t_m) - Q^{(n-1)}(p,t_m)\right),$$

where $H^{(n-1)}(p,t_m)$ lies between $Q^{(n-1)}(p,t_m)$ and $Y^{(n-1)}(p,t_m)$. Taking into account that

$$\left|Y^{(n-1)}(p,t_m) - Q^{(n-1)}(p,t_m)\right| \le \overline{U}^{(n-1)}(p,t_m) - \underline{U}^{(n-1)}(p,t_m).$$

In the notation (1.78), we estimate $G^{(n-1)}(p,t_m)$ as follows:

$$\left\|G^{(n-1)}(\cdot,t_m)\right\|_{\Lambda^h} \le \xi_m \left\|W^{(n-1)}(\cdot,t_m)\right\|^2_{\overline{\Lambda}^h}.$$

From here, (1.80) and using (1.51), we conclude (1.79). □

1.4 General overview of the thesis

In Chapter 2, the nonlinear difference scheme for approximating the elliptic problems is presented. For solving the nonlinear difference scheme, the point Jacobi and point Gauss-Seidel iterative methods are constructed and their monotone properties are proved. The uniqueness of a solution of the nonlinear difference scheme is given. We prove that the numerical solution converges to the unique solution of the nonlinear elliptic problem and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme and the error between the numerical solution and the exact solution of the elliptic problem. We prove that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method. Initial upper and lower solutions to start the point monotone iterative methods are constructed. Numerical experiments are presented.

In Chapter 3, for solving nonlinear systems of elliptic differential equations with quasi-monotone nondecreasing and nonincreasing reaction functions, we present the nonlinear difference scheme which approximates the nonlinear elliptic systems. We construct the point monotone Jacobi and Gauss-Seidel methods for solving the nonlinear difference scheme and prove their monotone properties. The existence and uniqueness of a solution of the nonlinear difference scheme with quasi-monotone nondecreasing and quasi-monotone nonincreasing reaction functions are proved. We prove that the numerical solution converges to the unique solution of the nonlinear elliptic problem and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme and the error between the numerical solution and the exact solution of the elliptic system. We prove that the point monotone Gauss-Seidel methods converge faster than the point monotone Jacobi methods for the quasi-monotone nondecreasing and quasi-monotone nonincreasing cases. Constructions of initial upper and lower solutions to start the point monotone iterative methods are presented.

In Chapter 4, for solving nonlinear systems of elliptic differential equations, we construct the block monotone Jacobi and Gauss-Seidel methods with quasi-monotone nondecreasing and nonincreasing reaction functions and prove their monotone properties. We prove that the numerical solution converges to the unique solution of the nonlinear elliptic problem and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme and the error between the numerical solution and the exact solution of the elliptic system. For the quasi-monotone nondecreasing and nonincreasing cases, we prove that the block monotone Gauss-Seidel methods converge faster than the block monotone Jacobi methods. Numerical experiments are presented.

In Chapter 5, for solving nonlinear systems of parabolic differential equations, the two classes of coupled parabolic systems with quasi-monotone nondecreasing and nonincreasing reaction functions are considered. We present a nonlinear difference scheme which approximates the parabolic system. For solving the nonlinear difference scheme, we construct the point monotone Jacobi and Gauss-Seidel methods and prove their monotone properties on each time level. The existence and uniqueness of a solution of the nonlinear difference scheme, for the quasi-monotone nondecreasing and nonincreasing cases, are proved. We prove that the numerical solution converges to the unique solution of the nonlinear parabolic problem and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme, and the error between the numerical solution and the exact solution of the parabolic problem. We prove that for the quasi-monotone nondecreasing and nonincreasing cases, the point monotone Gauss-Seidel methods converge faster than the point monotone Jacobi methods. For quasi-monotone nondecreasing and nonincreasing cases, on each time level, we construct initial upper and lower solutions to start the point monotone iterative methods. Numerical experiments are presented.

In Chapter 6, for solving the nonlinear parabolic systems with quasi-monotone nondecreasing and nonincreasing reaction functions, we construct the block monotone Jacobi and Gauss-Seidel iterative methods and prove their monotone properties on each time level. For the quasi-monotone nondecreasing and nonincreasing cases, we prove that the numerical solution converges to the unique solution of the nonlinear parabolic problem and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme and the error between the numerical solution and the exact solution of the parabolic problem. Numerical experiments are presented.

Chapter 2

Jacobi and Gauss-Seidel methods for elliptic boundary value problems

In this chapter, for solving nonlinear elliptic problems, based on the method of upper and lower solutions, we employ point monotone Jacobi and Gauss-Seidel iterative methods. Some properties of solutions to the continuous problem are reviewed. Difference schemes which approximate the nonlinear continuous problem are presented. In the view of the upper and lower solutions method, the point monotone Jacobi and Gauss-Seidel methods are constructed. Convergence analysis of the point monotone iterative methods are introduced. We construct initial upper and lower solutions to start the monotone iterative methods. Numerical experiments illustrate the theoretical results.

By comparing the numerical results in this chapter with [61], we conclude that to attain the required stopping test, the numbers of iterations for the point monotone methods are almost double of the numbers of iterations for the block monotone methods in [61].

The numerical experiments give a motivation to investigate block monotone iterative methods rather than point monotone iterative methods for solving nonlinear differential problems.

2.1 Properties of solutions to the nonlinear elliptic problem

We consider properties of the nonlinear elliptic boundary value problem

$$-\,\mathrm{L}u(x,y)+f(x,y,u)=0,\quad (x,y)\in\omega, \tag{2.1}$$
$$\omega=\{(x,y):0<x<l_1,\quad 0<y<l_2\},\quad u(x,y)=g(x,y),\quad (x,y)\in\partial\omega,$$

where l_1 and l_2 are constants and $\partial\omega$ is the boundary of ω. The differential operator L is defined by

$$\mathrm{L}u(x,y)\equiv D^{(x)}(x,y)u_{xx}+D^{(y)}(x,y)u_{yy}+v^{(x)}(x,y)u_x+v^{(y)}(x,y)u_y, \tag{2.2}$$

where $D^{(x)}(x,y)$ and $D^{(y)}(x,y)$ are positive functions. It is assumed that the functions $f(x,y)$, $g(x,y)$, $D^{(x)}(x,y)$, $D^{(y)}(x,y)$, $v^{(x)}(x,y)$ and $v^{(y)}(x,y)$ are smooth in their respective domains. It is clear that the differential operator $\mathrm{L}u(x,y)$ in (2.2) is uniformly elliptic which is a special case of (1.1) and the coefficient matrix

$$\begin{bmatrix} D^{(x)}(x,y) & 0 \\ 0 & D^{(y)}(x,y) \end{bmatrix},$$

is positive definite and bounded.

Two functions $\widetilde{u}(x,y)$ and $\widehat{u}(x,y)$ are called ordered upper and lower solutions to (2.1), if they satisfy the inequalities

$$\widehat{u}(x,y)\le\widetilde{u}(x,y),\quad (x,y)\in\overline{\omega}, \tag{2.3a}$$

$$-\mathrm{L}\widehat{u}(x,y)+f(x,y,\widehat{u})\le 0\le -\mathrm{L}\widetilde{u}(x,y)+f(x,y,\widetilde{u}),\quad (x,y)\in\omega, \tag{2.3b}$$

$$\widehat{u}(x,y)\le g(x,y)\le\widetilde{u}(x,y),\quad (x,y)\in\partial\omega. \tag{2.3c}$$

For given ordered upper $\widetilde{u}(x,y)$ and lower $\widehat{u}(x,y)$ solutions, a sector $\langle\widehat{u},\widetilde{u}\rangle$ is defined in the form

$$\langle\widehat{u},\widetilde{u}\rangle=\{u(x,y):\quad \widehat{u}(x,y)\le u(x,y)\le\widetilde{u}(x,y),\quad (x,y)\in\overline{\omega}\}\,.$$

To ensure the existence of a solution to (2.1), in the sector $\langle\widehat{u},\widetilde{u}\rangle$, the function $f(x,y,u)$ is assumed to satisfy the constraint

$$f_u(x,y,u)\le c(x,y),\quad u(x,y)\in\langle\widehat{u},\widetilde{u}\rangle,\quad (x,y)\in\overline{\omega},\quad \Big(f_u=\frac{\partial f}{\partial u}\Big), \tag{2.4}$$

where c is a nonnegative bounded function. The following theorem states the existence of a solution to problem (2.1).

Theorem 2.1.1. *Let $\widetilde{u}(x,y)$, $\widehat{u}(x,y)$ be ordered upper and lower solutions of (2.1), and f satisfy (2.4). Then problem (2.1) has a solution $u^*(x,y) \in \langle \widehat{u}, \widetilde{u} \rangle$.*

The proof of the theorem is given in Theorem 3.2.1, [59].

For uniqueness of a solution to (2.1), the function $f(x,y,u)$ is assumed to satisfy the two-sided constraints

$$0 < f_u(x,y,u) \le c(x,y), \quad u(x,y) \in \langle \widehat{u}, \widetilde{u} \rangle, \quad (x,y) \in \overline{\omega}. \tag{2.5}$$

Theorem 2.1.2. *Let $\widetilde{u}$, $\widehat{u}$ be ordered upper and lower solutions of (2.1), and f satisfy (2.5). Then problem (2.1) has a unique solution $u^*(x,y) \in \langle \widehat{u}(x,y), \widetilde{u}(x,y) \rangle$.*

The proof of the theorem is given in Theorem 3.3.1, [59].

2.2 The nonlinear difference scheme

On $\overline{\omega}$, we introduce a rectangular mesh $\overline{\Lambda}^h = \overline{\Lambda}^{hx} \times \overline{\Lambda}^{hy}$:

$$\begin{aligned}
\overline{\Lambda}^{hx} &= \{x_i, \quad i = 0,1,\dots,N_x; \quad x_0 = 0, \quad x_{N_x} = l_1; \quad h_x = x_{i+1} - x_i\}, \\
\overline{\Lambda}^{hy} &= \{y_j, \quad j = 0,1,\dots,N_y; \quad y_0 = 0, \quad y_{N_y} = l_2; \quad h_y = y_{j+1} - y_j\},
\end{aligned}$$

where x_i and y_j are equally spaced. By using the central difference approximations for the first and second derivatives, we introduce the nonlinear difference scheme in the form

$$\mathcal{A}_{ij} U_{ij} + f_{ij}(U_{ij}) = 0, \quad (i,j) \in \Omega^h, \quad U_{ij} = g_{ij}, \quad (i,j) \in \partial\Omega^h, \tag{2.6}$$

where Ω^h is the set of indices of interior mesh points in $\overline{\Lambda}^h$, $\partial\Omega^h$ is the set of indices of boundary mesh points in $\overline{\Lambda}^h$ and the central difference approximations for the first and second derivatives are given by

$$\begin{aligned}
\mathcal{D}_x^2 U_{ij} &= \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h_x^2}, \quad \mathcal{D}_y^2 U_{ij} = \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{h_y^2}, \\
\mathcal{D}_x^1 U_{ij} &= \frac{U_{i+1,j} - U_{i-1,j}}{2h_x}, \quad \mathcal{D}_y^1 U_{ij} = \frac{U_{i,j+1} - U_{i,j-1}}{2h_y}.
\end{aligned} \tag{2.7}$$

When no confusion arises, we write $f(x_i, y_j, U(x_i, y_j)) = f_{ij}(U_{ij})$. The difference operator $\mathcal{A}_{ij}U_{ij}$ in (2.6) is defined by

$$\begin{aligned}
\mathcal{A}_{ij}U_{ij} &= \mathcal{A}^{(x)}_{ij}U_{ij} + \mathcal{A}^{(y)}_{ij}U_{ij}, \\
\mathcal{A}^{(x)}_{ij}U_{ij} &= \frac{1}{h_x^2}\left[-l_{ij}U_{i-1,j} + 2D^{(x)}_{ij}U_{ij} - r_{ij}U_{i+1,j}\right], \\
\mathcal{A}^{(y)}_{ij}U_{ij} &= \frac{1}{h_y^2}\left[-b_{ij}U_{i,j-1} + 2D^{(y)}_{ij}U_{ij} - q_{ij}U_{i,j+1}\right], \\
l_{ij} &= \frac{D^{(x)}_{ij}}{h_x^2} - \frac{v^{(x)}_{ij}}{2h_x}, \quad r_{ij} = \frac{D^{(x)}_{ij}}{h_x^2} + \frac{v^{(x)}_{ij}}{2h_x}, \\
b_{ij} &= \frac{D^{(y)}_{ij}}{h_y^2} - \frac{v^{(y)}_{ij}}{2h_y}, \quad q_{ij} = \frac{D^{(y)}_{ij}}{h_y^2} + \frac{v^{(y)}_{ij}}{2h_y}.
\end{aligned} \tag{2.8}$$

To insure that l_{ij}, r_{ij}, b_{ij} and q_{ij} are positive, we choose the step sizes h_x and h_y, which satisfy the inequalities

$$h_x < \frac{2D^{(x)}_{ij}}{|v^{(x)}_{ij}|}, \quad h_y < \frac{2D^{(y)}_{ij}}{|v^{(y)}_{ij}|}, \quad i = 1, 2, \dots, N_x - 1, \quad j = 1, 2, \dots, N_y - 1.$$

Remark 2.2.1. *If the effect of convection $v(x, y)$ dominates diffusion $D(x, y)$ in (2.2) to the extent that these conditions require prohibitively small h_x and h_y, then an upwind difference scheme can be used to remove any restriction on h_x and h_y, that is,*

$$\mathcal{D}'_x U_{ij} = \begin{cases} \frac{U_{i+1,j} - U_{ij}}{h_x}, & if\ v^{(x)}_{ij} \le 0, \\ \frac{U_{ij} - U_{i-1,j}}{h_x}, & if\ v^{(x)}_{ij} \ge 0, \end{cases} \tag{2.9}$$

$$\mathcal{D}'_y U_{ij} = \begin{cases} \frac{U_{i,j+1} - U_{ij}}{h_y}, & if\ v^{(y)}_{ij} \le 0, \\ \frac{U_{ij} - U_{i,j-1}}{h_y}, & if\ v^{(y)}_{ij} \ge 0. \end{cases} \tag{2.10}$$

We introduce the linear problem

$$\mathcal{A}_{ij}W_{ij} + c^*_{ij}W_{ij} = \Phi_{ij}, \quad (i, j) \in \Omega^h, \quad W_{ij} = g_{ij}, \quad (i, j) \in \partial\Omega^h, \tag{2.11}$$

where c^*_{ij} is a nonnegative mesh function. We formulate the maximum principle for the difference operator $\mathcal{A}_{ij} + c^*_{ij}$, $(i, j) \in \Omega^h$.

Lemma 2.2.2. *If a mesh function W_{ij} satisfies the conditions*

$$\mathcal{A}_{ij}W_{ij} + c^*_{ij}W_{ij} \ge 0\ (\le 0), \quad (i, j) \in \Omega^h, \quad W_{ij} \ge 0\ (\le 0), \quad (i, j) \in \partial\Omega^h,$$

then $W_{ij} \ge 0\ (\le 0)$, $\quad (i, j) \in \overline{\Omega}^h = \Omega^h \cup \partial\Omega^h$.

The proof of the lemma is given in Lemma 1.2.1, Chapter 1.

Two mesh functions $\widetilde{U}_{ij}$ and $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ are called ordered upper and lower solutions of (2.6), if they satisfy the inequalities

$$\widehat{U}_{ij} \leq \widetilde{U}_{ij}, \quad (i,j) \in \overline{\Omega}^h, \tag{2.12a}$$

$$\mathcal{A}_{ij}\widehat{U}_{ij} + f_{ij}(\widehat{U}_{ij}) \leq 0 \leq \mathcal{A}_{ij}\widetilde{U}_{ij} + f_{ij}(\widetilde{U}_{ij}), \quad (i,j) \in \Omega^h, \tag{2.12b}$$

$$\widehat{U}_{ij} \leq g_{ij} \leq \widetilde{U}_{ij}, \quad (i,j) \in \partial\Omega^h. \tag{2.12c}$$

For given ordered upper $\widetilde{U}_{ij}$ and lower $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ solutions, a sector $\langle \widehat{U}, \widetilde{U} \rangle$ is defined as follows

$$\langle \widehat{U}, \widetilde{U} \rangle = \left\{ U_{ij} : \quad \widehat{U}_{ij} \leq U_{ij} \leq \widetilde{U}_{ij}, \quad (i,j) \in \overline{\Omega}^h \right\}.$$

In the sector $\langle \widehat{U}, \widetilde{U} \rangle$, we assume that the function f in (2.1) satisfies the constraint

$$\frac{\partial f_{ij}(U_{ij})}{\partial u} \leq c_{ij}, \quad U_{ij} \in \langle \widehat{U}, \widetilde{U} \rangle, \quad (i,j) \in \overline{\Omega}^h, \tag{2.13}$$

where c_{ij}, $(i,j) \in \overline{\Omega}^h$ is a nonnegative bounded mesh function.

We introduce the notation

$$\Gamma_{ij}(U_{ij}) \equiv c_{ij}U_{ij} - f_{ij}(U_{ij}), \quad (i,j) \in \overline{\Omega}^h, \tag{2.14}$$

where c_{ij} is defined in (2.13), and prove a monotone property of Γ_{ij}.

Lemma 2.2.3. *Suppose that U_{ij} and V_{ij}, $(i,j) \in \overline{\Omega}^h$ are mesh functions in $\langle \widehat{U}, \widetilde{U} \rangle$, which satisfy $U_{ij} \geq V_{ij}$, $(i,j) \in \overline{\Omega}^h$, and (2.13) is satisfied. Then*

$$\Gamma_{ij}(U_{ij}) \geq \Gamma_{ij}(V_{ij}), \quad (i,j) \in \overline{\Omega}^h. \tag{2.15}$$

Proof. From (2.14), we have

$$\Gamma_{ij}(U_{ij}) - \Gamma_{ij}(V_{ij}) = c_{ij}(U_{ij} - V_{ij}) - [f_{ij}(U_{ij}) - f_{ij}(V_{ij})].$$

From here and using the mean-value theorem, we obtain

$$\Gamma_{ij}(U_{ij}) - \Gamma_{ij}(V_{ij}) = \left(c_{ij} - \frac{\partial f_{ij}(Q_{ij})}{\partial u} \right) (U_{ij} - V_{ij}),$$

where, $V_{ij} \leq Q_{ij} \leq U_{ij}$, $(i,j) \in \overline{\Omega}^h$. From here, (2.13) and taking into account that $U_{ij} \geq V_{ij}$, , $(i,j) \in \overline{\Omega}^h$, we conclude (2.15). □

2.3 The point monotone Jacobi and Gauss-Seidel iterative methods

Write down the difference scheme (2.6) at an interior mesh point $(i,j) \in \Omega^h$ in the form

$$d_{ij}U_{ij} - l_{ij}U_{i-1,j} - r_{ij}U_{i+1,j} - b_{ij}U_{i,j-1} - q_{ij}U_{i,j+1} = -f_{ij}(U_{ij}), \quad (i,j) \in \Omega^h, \quad (2.16)$$
$$d_{ij} = l_{ij} + r_{ij} + b_{ij} + q_{ij}, \quad l_{ij}, r_{ij}, b_{ij}, q_{ij} > 0, \quad (2.17)$$

where l_{ij}, r_{ij}, b_{ij} and q_{ij} are defined in (2.8).

We now present the point monotone Jacobi and Gauss–Seidel methods for the nonlinear difference scheme (2.16). The upper $\{\overline{U}_{ij}^{(n)}\}$ and lower $\{\underline{U}_{ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$ sequences of solutions are calculated by the following point Jacobi and Gauss–Seidel methods

$$\begin{aligned}
&\mathcal{L}_{ij}Z_{ij}^{(n)} = -\mathcal{K}_{ij}(U_{ij}^{(n-1)}), \quad (i,j) \in \Omega^h, \ n \geq 1, \qquad (2.18)\\
&Z_{ij}^{(1)} = g_{ij} - U_{ij}^{(0)}, \quad Z_{ij}^{(n)} = 0, \quad n \geq 2, \quad (i,j) \in \partial\Omega^h,\\
&\mathcal{L}_{ij}Z_{ij}^{(n)} \equiv (d_{ij} + c_{ij})Z_{ij}^{(n)} - \eta\left(l_{ij}Z_{i-1,j}^{(n)} + b_{ij}Z_{i,j-1}^{(n)}\right),\\
&Z_{ij}^{(n)} = U_{ij}^{(n)} - U_{ij}^{(n-1)}, \quad (i,j) \in \overline{\Omega}^h,\\
&\mathcal{K}_{ij}(U_{ij}^{(n-1)}) \equiv d_{ij}U_{ij}^{(n-1)} - l_{ij}U_{i-1,j}^{(n-1)} - r_{ij}U_{i+1,j}^{(n-1)} - b_{ij}U_{i,j-1}^{(n-1)}\\
&\qquad - q_{ij}U_{i,j+1}^{(n-1)} + f_{ij}(U_{ij}^{(n-1)}),
\end{aligned}$$

where $\mathcal{K}_{ij}(U_{ij}^{(n-1)})$, $(i,j) \in \Omega^h$ is the residual of the nonlinear difference scheme (2.16) on $U_{ij}^{(n-1)}$, $(i,j) \in \overline{\Omega}^h$, and c_{ij} is defined in (2.13). For $\eta = 0$ and $\eta = 1$, we have, respectively, the point Jacobi and Gauss-Seidel methods.

Theorem 2.3.1. *Let $\widetilde{U}_{ij}$ and $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ be ordered upper and lower solutions (2.12). Suppose that the function f in (2.1) satisfies (2.13). Then the upper $\{\overline{U}_{ij}^{(n)}\}$ and lower $\{\underline{U}_{ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$ sequences generated by (2.18) with, respectively, $\overline{U}_{ij}^{(0)} = \widetilde{U}_{ij}$ and $\underline{U}_{ij}^{(0)} = \widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ converge monotonically from above to a maximal solution $\overline{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, and from below to a minimal solution $\underline{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, that is,*

$$\underline{U}_{ij}^{(n-1)} \leq \underline{U}_{ij}^{(n)} \leq \underline{U}_{ij} \leq \overline{U}_{ij} \leq \overline{U}_{ij}^{(n)} \leq \overline{U}_{ij}^{(n-1)}, \quad (i,j) \in \overline{\Omega}^h. \quad (2.19)$$

Proof. Since $\overline{U}_{ij}^{(0)}$, $(i,j) \in \Omega^h$ is an initial upper solution, from (2.12b), it follows that $\mathcal{K}_{ij}(\overline{U}_{ij}^{(0)}) \geq 0$, $(i,j) \in \Omega^h$, and from (2.18), we have

$$\begin{aligned}
&(d_{ij} + c_{ij})\overline{Z}_{ij}^{(1)} - \eta l_{ij}\overline{Z}_{i-1,j}^{(1)} - \eta b_{ij}\overline{Z}_{i,j-1}^{(1)} \leq 0, \quad (i,j) \in \Omega^h, \qquad (2.20)\\
&\overline{Z}_{ij}^{(1)} \leq 0, \quad (i,j) \in \partial\Omega^h.
\end{aligned}$$

From here, $\eta = 0,1$ and $b_{i,1} > 0$ in (2.17), for $j = 1$ in (2.20), we obtain

$$(d_{i,1}+c_{i,1})\overline{Z}_{i,1}^{(1)} - \eta l_{i,1}\overline{Z}_{i-1,1}^{(1)} \le 0, \quad i = 1,2,\dots,N_x - 1, \quad \overline{Z}_{i,1}^{(1)} \le 0, \quad i = 0, N_x. \tag{2.21}$$

Taking into account that $\eta = 0,1$, $l_{1,1} > 0$ in (2.17) and using the maximum principle in Lemma 2.2.2, for $i = 1$ in (2.21), we have $\overline{Z}_{1,1}^{(1)} \le 0$. From here, $l_{2,1} > 0$ in (2.17) and using the maximum principle in Lemma 2.2.2, for $i = 2$ in (2.21), we obtain $\overline{Z}_{2,1}^{(1)} \le 0$. By induction on i, we can prove that $\overline{Z}_{i,1}^{(1)} \le 0, \quad i = 0,1,\dots,N_x$.

By a similar manner, for $j = 2$ in (2.20), we conclude that $\overline{Z}_{i,2}^{(1)} \le 0, \quad i = 0,1,\dots,N_x$. By induction on j, we can prove that

$$\overline{Z}_{ij}^{(1)} \le 0, \quad (i,j) \in \overline{\Omega}^h. \tag{2.22}$$

Similarly, for an initial lower solution $\underline{U}_{ij}^{(0)}$, $(i,j) \in \overline{\Omega}^h$, we have

$$\underline{Z}_{ij}^{(1)} \ge 0, \quad (i,j) \in \overline{\Omega}^h. \tag{2.23}$$

We now prove that $\overline{U}_{ij}^{(1)}$ and $\underline{U}_{ij}^{(1)}$, $(i,j) \in \overline{\Omega}^h$ are ordered upper and lower solutions (2.12). Letting $W_{ij}^{(n)} = \overline{U}_{ij}^{(n)} - \underline{U}_{ij}^{(n)}$, $(i,j) \in \overline{\Omega}^h$, using notation (2.14), from (2.18), we conclude that

$$(d_{ij}+c_{ij})W_{ij}^{(1)} - \eta l_{ij}W_{i-1,j}^{(1)} - \eta b_{ij}W_{i,j-1}^{(1)} = r_{ij}W_{i+1,j}^{(0)} + q_{ij}W_{i,j+1}^{(0)} + \Gamma_{ij}(\overline{U}_{ij}^{(0)}) - \Gamma_{ij}(\underline{U}_{ij}^{(0)}),$$
$$(i,j) \in \Omega^h, \quad W_{ij}^{(1)} = 0, \quad (i,j) \in \partial\Omega^h.$$

From here, (2.16) and taking into account that $\overline{U}_{ij}^{(0)} \ge \underline{U}_{ij}^{(0)}$, $(i,j) \in \overline{\Omega}^h$, by Lemma 2.2.3, we conclude that

$$(d_{ij}+c_{ij})W_{ij}^{(1)} - \eta l_{ij}W_{i-1,j}^{(1)} - \eta b_{ij}W_{i,j-1}^{(1)} \ge 0, \quad (i,j) \in \Omega^h, \quad W_{ij}^{(1)} = 0, \ (i,j) \in \partial\Omega^h. \tag{2.24}$$

From here, $\eta = 0,1$ and $l_{i,1} > 0$ in (2.17), for $j = 1$ in (2.24), we obtain

$$(d_{i,1}+c_{i,1})W_{i,1}^{(1)} - \eta l_{i,1}W_{i-1,1}^{(1)} \ge 0, \quad i = 1,2,\dots,N_x - 1, \quad W_{i,1}^{(1)} = 0, \quad i = 0, N_x.$$

From here, by Lemma 2.2.2, for $i = 1$, we have $W_{1,1}^{(1)} \ge 0$. From here, $l_{2,1} > 0$ in (2.17) and using Lemma 2.2.2, for $i = 2$, we conclude that $W_{2,1}^{(1)} \ge 0$. By induction on i, we can prove that $W_{i,1}^{(1)} \ge 0, \quad i = 0,1,\dots,N_x$.

By a similar manner, for $j = 2$ in (2.24), we can prove that $W_{i,2}^{(1)} \ge 0, \quad i = 0,1,\dots,N_x$. By induction on j, we can prove that

$$W_{ij}^{(1)} \ge 0, \quad (i,j) \in \overline{\Omega}^h, \tag{2.25}$$

that is, we prove (2.12a). We now prove (2.12b). From (2.18) and using the mean-value theorem, we conclude that

$$\mathcal{K}_{ij}(\overline{U}_{ij}^{(1)}) = -\Big(c_{ij} - \frac{\partial f_{ij}(\overline{Q}_{ij}^{(1)})}{\partial u}\Big)\overline{Z}_{ij}^{(1)} - r_{ij}\overline{Z}_{i+1,j}^{(1)} - q_{ij}\overline{Z}_{i,j+1}^{(1)}, \tag{2.26}$$

where $\overline{U}_{ij}^{(1)} \leq \overline{Q}_{ij}^{(1)} \leq \overline{U}_{ij}^{(0)}$, $(i,j) \in \Omega^h$. From (2.23) and (2.25), it follows that $\partial f_{ij}(\overline{Q}_{ij}^{(1)})/\partial u$ satisfies (2.13), and from (2.13), (2.17), (2.22) and (2.26), we conclude that

$$\mathcal{K}_{ij}(\overline{U}_{ij}^{(1)}) \geq 0, \quad (i,j) \in \Omega^h,$$

which means that $\overline{U}_{ij}^{(1)}$, $(i,j) \in \overline{\Omega}^h$ satisfies (2.12b). By a similar argument, we can prove that

$$\mathcal{K}_{ij}(\underline{U}_{ij}^{(1)}) \leq 0, \quad (i,j) \in \Omega^h,$$

which means that $\underline{U}_{ij}^{(1)}$, $(i,j) \in \overline{\Omega}^h$ satisfies (2.12b). From the boundary condition on $\partial\Omega^h$ in (2.18), it is clear that $\overline{U}_{ij}^{(1)}$ and $\underline{U}_{ij}^{(1)}$ satisfy (2.12c). Thus, we prove that $\overline{U}_{ij}^{(1)}$ and $\underline{U}_{ij}^{(1)}$, $(i,j) \in \overline{\Omega}^h$ are ordered upper and lower solutions (2.12).

Now, by induction on n, we can prove that $\{\overline{U}_{ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$ is a monotone decreasing sequence of upper solutions and $\{\underline{U}_{ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$ is a monotone increasing sequence of lower solutions.

We now prove that the sequence $\{\overline{U}_{ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$ converges monotonically from above to a maximal solution $\overline{U}_{ij}$ and the sequence $\{\underline{U}_{ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$ converges monotonically from below to a minimal solution $\underline{U}_{ij}$. From (2.19), we conclude that $\lim \overline{U}_{ij}^{(n)} = \overline{U}_{ij}$ and $\lim \underline{U}_{ij}^{(n)} = \underline{U}_{ij}$ as $n \to \infty$ exist and

$$\lim_{n\to\infty} \overline{Z}_{ij}^{(n)} = 0, \quad \lim_{n\to\infty} \underline{Z}_{ij}^{(n)} = 0, \quad (i,j) \in \overline{\Omega}^h.$$

Similar to (2.26), we obtain

$$\mathcal{K}_{ij}(\overline{U}_{ij}^{(n)}) = -\Big(c_{ij} - \frac{\partial f_{ij}(\overline{Q}_{ij}^{(n)})}{\partial u}\Big)\overline{Z}_{ij}^{(n)} - r_{ij}\overline{Z}_{i+1,j}^{(n)} - q_{ij}\overline{Z}_{i,j+1}^{(n)}, \quad (i,j) \in \Omega^h,$$

where $\overline{U}_{ij}^{(n)} \leq \overline{Q}_{ij}^{(n)} \leq \overline{U}_{ij}^{(n-1)}$, $(i,j) \in \overline{\Omega}^h$. By taking limit of the both sides, we conclude that

$$\mathcal{K}_{ij}(\overline{U}_{ij}) = 0, \quad (i,j) \in \Omega^h,$$

which means that $\overline{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ is a maximal solution to (2.6). Similarly, we can prove that

$$\mathcal{K}_{ij}(\underline{U}_{ij}) = 0, \quad (i,j) \in \Omega^h,$$

which means that $\underline{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ is a minimal solution to (2.6). Thus, we prove the theorem. □

To prove the uniqueness of a solution to the nonlinear difference scheme (2.6), we assume that the reaction function f in (2.1) satisfies the following two-sided constraint

$$0 < \underline{c}_{ij} \leq \frac{\partial f_{ij}(U_{ij})}{\partial u} \leq c_{ij}, \quad U_{ij} \in \langle \widehat{U}, \widetilde{U} \rangle, \quad (i,j) \in \overline{\Omega}^h, \tag{2.27}$$

where $\underline{c}_{ij}$ and c_{ij} are positive bounded mesh functions.

Theorem 2.3.2. *Let $\widetilde{U}_{ij}$ and $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ be ordered upper and lower solutions (2.12), and f in (2.1) satisfy (2.27). Then the nonlinear difference scheme (2.6) has a unique solution.*

Proof. To prove the uniqueness of a solution to the nonlinear difference scheme (2.6), it suffices to check that $\underline{U}_{ij} = \overline{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, where $\underline{U}_{ij}$ and $\overline{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ are the minimal and maximal solutions from (2.19). Letting $V_{ij} = \overline{U}_{ij} - \underline{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, from (2.6), we have

$$\mathcal{L}_{ij} V_{ij} + f_{ij}(\overline{U}_{ij}) - f_{ij}(\underline{U}_{ij}) = 0, \quad (i,j) \in \Omega^h, \quad V_{ij} = 0, \quad (i,j) \in \partial\Omega^h.$$

From here and using the mean-value theorem, we obtain

$$\left(\mathcal{L}_{ij} + \frac{\partial f_{ij}(Q_{ij})}{\partial u} \right) V_{ij} = 0, \quad (i,j) \in \Omega^h, \quad V_{ij} = 0, \quad (i,j) \in \partial\Omega^h,$$

where $\underline{U}_{ij} \leq Q_{ij} \leq \overline{U}_{ij}$, $(i,j) \in \Omega^h$. From here and the left inequality in (2.27), by using the maximum principle in Lemma (2.2.2), we conclude that

$$V_{ij} = 0, \quad (i,j) \in \overline{\Omega}^h.$$

Thus, we prove the theorem. □

2.4 Convergence analysis of the point monotone iterative methods

A stopping test for the point monotone iterative methods (2.18) is chosen in the form

$$\left\| \mathcal{K}(U^{(n)}) \right\|_{\Omega^h} \leq \delta, \quad \left\| \mathcal{K}(U^{(n)}) \right\|_{\Omega^h} = \max_{(i,j) \in \Omega^h} \left| \mathcal{K}_{ij}(U_{ij}^{(n)}) \right|, \tag{2.28}$$

where δ is a prescribed accuracy and $\mathcal{K}_{ij}(U_{ij}^{(n)})$ is defined in (2.18).

In the following lemma, we give a bound on the magnitude of the solution to the linear problem (2.11).

Lemma 2.4.1. *The following bound on the magnitude of the solution to the linear problem (2.11) with a positive mesh function c_{ij} holds*

$$\|W\|_{\overline{\Omega}^h} \leq \max\left\{\|g\|_{\partial\Omega^h}, \max_{(i,j)\in\Omega^h} \frac{|\Phi(ij)|}{c_{ij}}\right\}, \tag{2.29}$$

where

$$\|g\|_{\partial\Omega^h} = \max_{(i,j)\in\partial\Omega^h} |g_{ij}|.$$

The proof of the lemma is given in Lemma 1.2.1, Chapter 1.

Theorem 2.4.2. *Suppose that the two-sided constraint in (2.27) is satisfied. Then for the sequence of solutions $\{U_{ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$, generated by the point monotone iterative methods (2.18), (2.28), we have the following estimate*

$$\left\|U^{(n_\delta)} - U^*\right\|_{\overline{\Omega}^h} \leq \underline{c}^{-1}\delta,$$

where U_{ij}^*, $(i,j) \in \overline{\Omega}^h$ is the unique solution of the nonlinear difference scheme (2.6), $\underline{c}_{ij}$, $(i,j) \in \Omega^h$ is defined in (2.27), and n_δ is the minimal number of iterations subject to (2.28).

Proof. From (2.18), for $U_{ij}^{(n_\delta)}$ and U_{ij}^*, $(i,j) \in \overline{\Omega}^h$, we have

$$\begin{aligned}
&\mathcal{A}_{ij}U_{ij}^{(n_\delta)} + f_{ij}(U_{ij}^{(n_\delta)}) = \mathcal{K}_{ij}(U_{ij}^{(n_\delta)}), \quad (i,j) \in \Omega^h, \quad U_{ij}^{(n_\delta)} = g_{ij}, \quad (i,j) \in \partial\omega^h,\\
&\mathcal{A}_{ij}U_{ij}^* + f_{ij}(U_{ij}^*) = 0, \quad (i,j) \in \Omega^h, \quad U_{ij}^* = g_{ij}, \quad (i,j) \in \partial\Omega^h.
\end{aligned}$$

Letting $W_{ij}^{(n_\delta)} = U_{ij}^{(n_\delta)} - U_{ij}^*$, $(i,j) \in \overline{\Omega}^h$, from here and using the mean-value theorem, we obtain that

$$\mathcal{A}_{ij}W_{ij}^{(n_\delta)} + \frac{\partial f_{ij}(Q_{ij}^{(n_\delta)})}{\partial u}W_{ij}^{(n_\delta)} = \mathcal{K}_{ij}(U_{ij}^{(n_\delta)}), \quad (i,j) \in \Omega^h, \quad W_{ij}^{(n_\delta)} = 0, \quad (i,j) \in \partial\Omega^h,$$

where $Q_{ij}^{(n_\delta)}$ lies between $U_{ij}^{(n_\delta)}$ and U_{ij}^*. From here and using (2.29), we conclude that

$$\|W^{(n_\delta)}\|_{\overline{\Omega}^h} \leq \underline{c}^{-1}\left\|\mathcal{K}(U^{(n_\delta)})\right\|_{\Omega^h}.$$

From here and (2.28), we prove the theorem. □

Theorem 2.4.3. *Let the assumptions in Theorem 2.4.2 be satisfied. Then for the sequence of solutions $\{U_{ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$, generated by (2.18), (2.28), the following* *estimate holds*

$$\left\|U^{(n_\delta)} - u^*\right\|_{\overline{\Omega}^h} \le \underline{c}^{-1}(\delta + \|E(h)\|_{\overline{\Omega}^h}),$$

where $u^*(x,y)$ is the exact solution to (2.1), E_{ij} is the truncation error of the exact solution $u^*(x,y)$ on the nonlinear difference scheme (2.6), and n_δ is the minimal number of iterations subject to the stopping test (2.28).

Proof. We denote $e_{ij} = U^*_{ij} - u^*_{ij}$, $(i,j) \in \overline{\Omega}^h$, where the mesh function U^*_{ij}, $(i,j) \in \overline{\Omega}^h$, is the unique solution of the nonlinear difference scheme (2.6). From (2.6), by using the mean-value theorem, we obtain that

$$\mathcal{A}_{ij} e_{ij} + \frac{\partial f_{ij}(Y_{ij})}{\partial u_\alpha} e_{ij} = -E_{ij}(h), \quad (i,j) \in \Omega^h, \quad e_{ij} = 0, \quad (i,j) \in \partial\Omega^h,$$

where Y_{ij} lies between u^*_{ij} and U^*_{ij}. From here and (2.27), by Lemma 2.4.1, it follows that

$$\|e\|_{\overline{\Omega}^h} \le \underline{c}^{-1} \|E(h)\|_{\Omega^h}. \tag{2.30}$$

We estimate $\left\|U^{(n_\delta)} - u^*\right\|_{\overline{\Omega}^h}$ as follows

$$\left\|U^{(n_\delta)} - U^* + U^* - u^*\right\|_{\overline{\Omega}^h} \le \left\|U^{(n_\delta)} - U^*\right\|_{\overline{\Omega}^h} + \|U^* - u^*\|_{\overline{\Omega}^h}.$$

From here, (2.30) and using the estimate from Theorem 2.4.2, we prove the theorem. □

2.5 Construction of initial upper and lower solutions

To start the monotone iterative methods (2.18), an initial iteration is needed. In this section, we discuss the construction of initial iterations $\widetilde{U}_{ij}$ and $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$.

2.5.1 Bounded functions

Assume that the functions f and g in (2.1) satisfy the following conditions:

$$f(x,y,0) \le 0, \quad g(x,y) \ge 0, \quad f(x,y,u) \ge -M, \quad u(x,y) \ge 0, \quad (x,y) \in \overline{\omega}, \tag{2.31}$$

where $M = \text{const} > 0$.

We introduce the mesh function

$$\widehat{U}_{ij} = 0, \quad (i,j) \in \overline{\Omega}^h, \tag{2.32}$$

and the linear problem

$$\mathcal{A}_{ij} \widetilde{U}_{ij} = M, \quad (i,j) \in \Omega^h, \quad \widetilde{U}_{ij} = g_{ij}, \quad (i,j) \in \partial\Omega^h. \tag{2.33}$$

Lemma 2.5.1. *Assume that the assumptions in (2.31) are satisfied. Then the mesh functions from (2.32) and (2.33) are ordered lower and upper solutions (2.12).*

Proof. Letting $W_{ij} = \widetilde{U}_{ij} - \widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, from (2.32) and (2.33), we have

$$\mathcal{A}_{ij} W_{ij} = M, \quad (i,j) \in \Omega^h, \quad W_{ij} = g_{ij}, \quad (i,j) \in \partial\Omega^h,$$

where $\mathcal{A}_{ij}$ is defined in (2.6). From here, (2.31) and the maximum principle in Lemma 2.2.2, we conclude that

$$W_{ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h.$$

Thus, we prove (2.12a). Now we prove (2.12b). From (2.33), by the maximum principle in Lemma 2.2.2, we obtain

$$\widetilde{U}_{ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h. \tag{2.34}$$

From (2.31), (2.33) and (2.34), we have

$$\mathcal{A}_{ij} \widetilde{U}_{ij} + f_{ij}(\widetilde{U}_{ij}) \geq 0, \quad (i,j) \in \Omega^h,$$

that is, $\widetilde{U}_{ij}$, $(i,j) \in \Omega^h$ satisfies (2.12b). From (2.33), it is clear that $\widetilde{U}_{ij}$, $(i,j) \in \partial\Omega^h$ satisfies (2.12c). Thus, $\widetilde{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ is an upper solution (2.12). From (2.31) and (2.32), we conclude that

$$\mathcal{L}_{ij} \widehat{U}_{ij} + f(x_i, y_j, \widehat{U}) \leq 0, \quad (i,j) \in \Omega^h, \quad \widehat{U}_{ij} \leq g_{ij}, \quad (i,j) \in \partial\Omega^h,$$

hence, $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ is a lower solution (2.12). Thus, $\widehat{U}_{ij}$ and $\widetilde{U}_{ij}$, $(ij) \in \overline{\Omega}^h$ from (2.32) and (2.33) are ordered lower and upper solutions (2.12) to the nonlinear difference scheme (2.6). □

2.5.2 Constant upper and lower solutions

Assume that the functions f and g in (2.1) satisfy the conditions

$$f(x,y,0) \leq 0, \quad g(x,y) \geq 0, \quad u(x,y) \geq 0, \quad (x,y) \in \overline{\omega}, \tag{2.35}$$

and there exists a positive constant K, such that

$$f(x,y,K) \geq 0, \quad g(x,y) \leq K, \quad (x,y) \in \overline{\omega}. \tag{2.36}$$

Introduce the constant mesh function

$$\widetilde{U}_{ij} = K, \quad (i,j) \in \overline{\Omega}^h. \tag{2.37}$$

The following lemma states that the mesh functions from (2.32) and (2.37) are ordered lower and upper solutions (2.12).

Lemma 2.5.2. *Assume that (2.35) and (2.36) are satisfied. Then the mesh functions from (2.32) and (2.37) are ordered lower and upper solutions (2.12).*

Proof. Letting $W_{ij} = \widetilde{U}_{ij} - \widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, from (2.32) and (2.37), we conclude that

$$\mathcal{A}_{ij} W_{ij} = 0, \quad (i,j) \in \Omega^h, \quad W_{ij} > 0, \quad (i,j) \in \partial\Omega^h.$$

From here and Lemma 2.2.2, we obtain that $W_{ij} \geq 0$, $(i,j) \in \overline{\Omega}^h$. Thus, we prove (2.12a). From (2.36) and (2.37), we have

$$\mathcal{A}_{ij}\widetilde{U}_{ij} + f_{ij}(\widetilde{U}_{ij}) \geq 0, \quad (i,j) \in \Omega^h, \quad \widetilde{U}_{ij} \geq g_{ij}, \quad (i,j) \in \partial\Omega^h.$$

Thus, $\widetilde{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ from (2.37) satisfies (2.12b), (2.12c). From (2.32) and (2.35), we obtain that

$$\mathcal{A}_{ij}\widehat{U}_{ij} + f_{ij}(\widehat{U}_{ij}) \leq 0, \quad (i,j) \in \Omega^h, \quad \widehat{U}_{ij} \leq g_{ij}, \quad (i,j) \in \partial\Omega^h,$$

that is, $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ from (2.32) satisfies (2.12b), (2.12c). Thus, we prove that $\widehat{U}_{ij}$ and $\widetilde{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, from (2.32) and (2.37) are ordered lower and upper solutions (2.12) to the nonlinear difference scheme (2.6). □

2.6 Applications

Here, we construct initial upper and lower solutions for two applied problems.

2.6.1 The enzyme kinetics model [9]

In the enzyme-substrate reaction scheme, if the effect of inhibition is taken into consideration, then the scheme is governed by (2.1) with $\mathrm{L}u(x,y) = \triangle u(x,y)$ and the reaction function f is given by

$$f(u) = \frac{\sigma u}{1 + au + bu^2}, \quad u \geq 0, \tag{2.38}$$

where σ, a and b are positive constants. Problem (2.1) is reduced to

$$-\triangle u + \frac{\sigma u}{1 + au + bu^2} = 0, \quad (x,y) \in \omega, \quad u(x,y) = g(x,y) \geq 0, \quad (x,y) \in \partial\omega.$$

The nonlinear difference scheme (2.6) has the form

$$\mathcal{A}_{ij} U_{ij} + \frac{\sigma U_{ij}}{1 + aU_{ij} + bU_{ij}^2} = 0, \quad (i,j) \in \Omega^h, \quad U_{ij} = g_{ij}, \quad (i,j) \in \partial\Omega^h, \tag{2.39}$$

where the difference operator $\mathcal{A}_{ij}$, $(i,j) \in \Omega^h$ is defined in (2.6) with $D = 1$ and $v_{ij} = 0$, $(i,j) \in \Omega^h$.

We now show that

$$\widetilde{U}_{ij} = K, \quad \widehat{U}_{ij} = 0, \quad (i,j) \in \overline{\Omega}^h, \quad K = \max_{(i,j)\in\partial\Omega^h} g_{ij}, \tag{2.40}$$

are ordered upper and lower solutions to (2.39).

From (2.38) and $g(x,y) \geq 0$, it follows (2.35). From (2.38) and (2.40), we conclude that (2.36) is satisfied. Thus, Lemma 2.5.2 holds for $\widetilde{U}_{ij}$ and $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ from (2.40). From (2.38), we have

$$f_u(x,y,u) = \frac{\sigma(1 - bu^2)}{(1 + au + bu^2)^2}.$$

We assume that $b < 1/K^2$, and hence, in the sector $\langle \widehat{U}, \widetilde{U} \rangle = \langle 0, K \rangle$, we conclude that

$$0 < \frac{\sigma(1 - bK^2)}{(1 + aK + bK^2)^2} < \frac{\partial f_{ij}(U_{ij})}{\partial u} \leq \sigma, \quad (i,j) \in \overline{\Omega}^h, \quad b < \frac{1}{K^2}. \tag{2.41}$$

The assumptions in (2.27) are satisfied with $\underline{c}_{ij} = \sigma(1 - bK^2)/(1 + aK + bK^2)^2$ and $c_{ij} = \sigma$. From here, we conclude that Theorems 2.3.1 and 2.3.2 hold for the enzyme kinetics model (2.39).

2.6.2 The chemical reactor model [42]

In the chemical reactor, when the isothermal reaction is irreversible, the temperature is constant and the mass concentration is described by (2.1) with $\mathrm{L}u(x,y) = \triangle u(x,y)$, and the reaction function f in the form

$$f(u) = \sigma u^p, \quad u \geq 0, \tag{2.42}$$

where σ and p are positive constants with $p \geq 1$. Problem (2.1) is reduced to

$$-\triangle u + \sigma u^p = 0, \quad (x,y) \in \omega, \quad u(x,y) = g(x,y) \geq 0, \quad (x,y) \in \partial\omega.$$

The nonlinear difference scheme (2.6) has the form

$$\mathcal{A}_{ij} U_{ij} + \sigma U_{ij}^p = 0, \quad (i,j) \in \Omega^h, \quad U_{ij} = g_{ij}, \quad (i,j) \in \partial\Omega^h, \tag{2.43}$$

where the difference operator $\mathcal{A}_{ij}$, $(i,j) \in \Omega^h$ is defined in (2.6). We introduce the linear problem

$$\mathcal{A}_{ij}\widetilde{U}_{ij} = 0, \quad (i,j) \in \Omega^h, \quad \widetilde{U}_{ij} = g_{ij}, \quad (i,j) \in \partial\Omega^h. \tag{2.44}$$

Now we show that $\widehat{U}_{ij}$ and $\widetilde{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ from, respectively, (2.32) and (2.44) are ordered lower and upper solutions (2.12). Letting $W_{ij} = \widetilde{U}_{ij} - \widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, from (2.32) and (2.44), we have

$$\mathcal{A}_{ij}W_{ij} = 0, \quad (i,j) \in \Omega^h.$$

From here, by using Lemma 2.2.2, we conclude that

$$W_{ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h.$$

Thus, we prove (2.12a). From (2.44), by using Lemma 2.2.2, we obtain

$$\widetilde{U}_{ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h. \tag{2.45}$$

From (2.32), (2.42) and (2.44), we conclude that

$$\mathcal{A}_{ij}\widetilde{U}_{ij} + f_{ij}(\widetilde{U}_{ij}) = f_{ij}(\widetilde{U}_{ij}) \geq 0, \quad (i,j) \in \Omega^h, \quad \widetilde{U}_{ij} \geq 0, \quad (i,j) \in \partial\Omega^h,$$

that is, $\widetilde{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ satisfies (2.12b) and (2.12c). From (2.32) and (2.42), we have

$$\mathcal{A}_{ij}\widehat{U}_{ij} + f_{ij}(\widehat{U}_{ij}) = 0, \quad (i,j) \in \Omega^h, \quad \widehat{U}_{ij} \leq g_{ij}, \quad (i,j) \in \partial\Omega^h,$$

that is, $\widehat{U}_{ij} = 0$, $(i,j) \in \overline{\Omega}^h$ satisfies (2.12b) and (2.12c). Thus, we prove that $\widehat{U}_{ij}$ and $\widetilde{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ from, respectively, (2.32) and (2.44) are ordered lower and upper solutions (2.12) to (2.43).

From (2.42), in the sector $\langle 0, \widetilde{U} \rangle$, we obtain

$$0 \leq \frac{\partial f_{ij}(U_{ij})}{\partial u} \leq c, \quad (i,j) \in \overline{\Omega}^h,$$

where $c = p\,\sigma \left(\max_{(i,j)\in\overline{\Omega}^h} \widetilde{U}_{ij} \right)^{p-1}$. From here, we conclude that Theorem 2.3.1 holds for the chemical reactor model (2.43).

2.7 Comparison of the point monotone Jacobi and Gauss–Seidel methods

In the following theorem, we show that the point monotone Gauss–Seidel method with $\eta = 1$ in (2.18) converges faster than the point monotone Jacobi method with $\eta = 0$ in (2.18).

Theorem 2.7.1. *Let $\widetilde{U}_{ij}$ and $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, be ordered upper and lower solutions (2.12). Assume that the function f in (2.1) satisfies (2.13). Suppose that the sequences $\{(U_{ij}^{(n)})_J\}$ and $\{(U_{ij}^{(n)})_{GS}\}$, $(i,j) \in \overline{\Omega}^h$, are, respectively, the sequences generated by the point monotone Jacobi method with $\eta = 0$ in (2.18) and the point monotone Gauss–Seidel method with $\eta = 1$ in (2.18), where $(\overline{U}_{ij}^{(0)})_J = (\overline{U}_{ij}^{(0)})_{GS} = \widetilde{U}_{ij}$ and $(\underline{U}_{ij}^{(0)})_J = (\underline{U}_{ij}^{(0)})_{GS} = \widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, then*

$$(\underline{U}_{ij}^{(n)})_J \leq (\underline{U}_{ij}^{(n)})_{GS} \leq (\overline{U}_{ij}^{(n)})_{GS} \leq (\overline{U}_{ij}^{(n)})_J, \quad (i,j) \in \overline{\Omega}^h. \tag{2.46}$$

Proof. Letting $W_{ij}^{(n)} = \left(U_{ij}^{(n)}\right)_{\mathtt{GS}} - \left(U_{ij}^{(n)}\right)_{\mathtt{J}}$, $(i,j) \in \overline{\Omega}^h$, from (2.18), we obtain

$$\begin{aligned} \mathcal{A}_{ij}\overline{W}_{ij}^{(n)} &= c_{ij}\overline{W}_{ij}^{(n-1)} + \eta l_{ij}\left(\left(\overline{U}_{i-1,j}^{(n)}\right)_{\mathtt{GS}} - \left(\overline{U}_{i-1,j}^{(n-1)}\right)_{\mathtt{J}}\right) + r_{ij}\overline{W}_{i+1,j}^{(n-1)} \\ &\quad + \eta b_{ij}\left(\left(\overline{U}_{i,j-1}^{(n)}\right)_{\mathtt{GS}} - \left(\overline{U}_{i,j-1}^{(n-1)}\right)_{\mathtt{J}}\right) + q_{ij}\overline{W}_{i,j+1}^{(n-1)} \\ &\quad - \left[f_{ij}((\overline{U}_{ij}^{(n-1)})_{\mathtt{GS}}) - f_{ij}((\overline{U}_{ij}^{(n-1)})_{\mathtt{J}})\right], \quad (i,j) \in \Omega^h, \\ \overline{W}_{ij}^{(n)} &= 0, \quad (i,j) \in \partial\Omega^h. \end{aligned} \tag{2.47}$$

By using Theorem 2.3.1, we have $\left(\overline{U}_{ij}^{(n)}\right)_{GS} \leq \left(\overline{U}_{ij}^{(n-1)}\right)_{GS}$. From here, $\eta = 0, 1$, (2.17) and (2.47), we obtain

$$\begin{aligned} \mathcal{A}_{ij}\overline{W}_{ij}^{(n)} &\leq c_{ij}\overline{W}_{ij}^{(n-1)} + \eta l_{ij}\overline{W}_{i-1,j}^{(n-1)} + r_{ij}\overline{W}_{i+1,j}^{(n-1)} + \eta b_{ij}\overline{W}_{i,j-1}^{(n-1)} + q_{ij}\overline{W}_{i,j+1}^{(n-1)} \\ &\quad - \left[f_{ij}((\overline{U}_{ij}^{(n-1)})_{\mathtt{GS}}) - f_{ij}((\overline{U}_{ij}^{(n-1)})_{\mathtt{J}})\right], \quad (i,j) \in \Omega^h, \\ \overline{W}_{ij}^{(n)} &= 0, \quad (i,j) \in \partial\Omega^h. \end{aligned}$$

Using notation (2.14), we write the above inequality in the form

$$\begin{aligned} \mathcal{A}_{ij}\overline{W}_{ij}^{(n)} &\leq \eta l_{ij}\overline{W}_{i-1,j}^{(n-1)} + r_{ij}\overline{W}_{i+1,j}^{(n-1)} + \eta b_{ij}\overline{W}_{i,j-1}^{(n-1)} + q_{ij}\overline{W}_{i,j+1}^{(n-1)} \\ &\quad + \Gamma_{ij}((\overline{U}_{ij}^{(n-1)})_{\mathtt{GS}}) - \Gamma_{ij}((\overline{U}_{ij}^{(n-1)})_{\mathtt{J}}), \quad (i,j) \in \Omega^h, \\ \overline{W}_{ij}^{(n)} &= 0, \quad (i,j) \in \partial\Omega^h, \end{aligned} \tag{2.48}$$

where

$$\Gamma_{ij}((\overline{U}_{ij}^{(n-1)})_{\mathtt{J}}) = c_{ij}(\overline{U}_{ij}^{(n-1)})_J - f_{ij}((\overline{U}_{ij}^{(n-1)})_{\mathtt{J}}),$$
$$\Gamma_{ij}((\overline{U}_{ij}^{(n-1)})_{\mathtt{GS}}) = c_{ij}(\overline{U}_{ij}^{(n-1)})_{\mathtt{GS}} - f_{ij}((\overline{U}_{ij}^{(n-1)})_{\mathtt{GS}}).$$

From $\eta = 0, 1$, (2.16) and the fact that $\left(\overline{U}_{ij}^{(0)}\right)_{\mathtt{GS}} = \left(\overline{U}_{ij}^{(0)}\right)_{\mathtt{J}}$, $(i,j) \in \overline{\Omega}^h$, for $n = 1$ in (2.48), we conclude that

$$\mathcal{A}_{ij}\overline{W}_{ij}^{(1)} \le 0, \quad (i,j) \in \Omega^h, \quad \overline{W}_{ij}^{(1)} = 0, \quad (i,j) \in \partial\Omega^h.$$

By using the maximum principle in Lemma 2.2.2, we obtain

$$\overline{W}_{ij}^{(1)} \le 0, \quad (i,j) \in \overline{\Omega}^h.$$

From here, (2.16), using the monotone property (2.15), for $n = 2$ in (2.48), we conclude that

$$\mathcal{A}_{ij}\overline{W}_{ij}^{(2)} \le 0, \quad (i,j) \in \Omega^h, \quad \overline{W}_{ij}^{(2)} = 0, \quad (i,j) \in \partial\Omega^h.$$

By using Lemma 2.2.2, we obtain that

$$\overline{W}_{ij}^{(2)} \le 0, \quad (i,j) \in \overline{\Omega}^h.$$

By induction on n, we can prove that

$$\overline{W}_{ij}^{(n)} \le 0, \quad (i,j) \in \overline{\Omega}^h, \quad n \ge 1.$$

Thus, we prove (2.46) for upper sequences. By a similar argument, we can prove (2.46) for lower sequences. □

2.8 Numerical experiments

Test 1

We consider the test problem

$$-(u_{xx} + u_{yy}) + \sigma u(u-1) = q(x,y), \quad (0 < x < 1, 0 < y < 2), \tag{2.49}$$
$$u(0,y) = \sin(\pi y/2), \quad u(1,y) = 0, \quad u(x,0) = u(x,2) = 0.$$

The function

$$u(x,y) = (1 - x^2)\sin(\pi y/2),$$

is the analytical solution of the model problem (2.49), when $\sigma = \pi^2/4$ and

$$q(x, y) = 2\sin(\pi y/2) + (\pi^2/4)(1 - x^2)^2 \sin^2(\pi y/2).$$

By using Lemma 2.5.2, it follows that for the model problem (2.49), the pair $\widetilde{U}_{ij} = K$ and $\widehat{U}_{ij} = 0$, $(i, j) \in \overline{\Omega}^h$ are ordered upper and lower solutions, such that, (2.35) and (2.36) are satisfied whenever $\frac{\pi^2}{4}K(K-1) - q(x, y) \geq 0$. For $K \geq 2$, the last inequality holds true, and we take $\widetilde{U}_{ij} = 2$ and $\widehat{U}_{ij} = 0$, $(i, j) \in \overline{\Omega}^h$.

Taking into account that $f_u(u) = \sigma(2u - 1)$, we conclude that $f_u \leq 3\pi^2/4$, and, hence, we choose $c_{ij} = 3\pi^2/4$ in (2.13). The space step sizes h_x and h_y are taken as $h_x = h_y = 0.05$. The stopping criterion of the monotone iterative methods (2.18) is chosen as in [61]

$$\|\overline{U}^{(n)} - \underline{U}^{(n)}\| \leq \delta, \tag{2.50}$$

where the notation of the norm from (2.29) is in use, $\overline{U}_{ij}^{(n)}$ and $\underline{U}_{ij}^{(n)}$, $(i, j) \in \overline{\Omega}^h$ are the upper and lower sequences generated by (2.18), and δ is a prescribed accuracy. We set $\delta = 10^{-5}$.

Under the same conditions, the test problem (2.49) was considered in [61] and solved by the block monotone Jacobi and Gauss-Seidel methods.

In Tables 2.1, 2.3 and in Tables 2.2, 2.4, we present upper and lower approximate solutions generated by, respectively, the point monotone methods (2.18) and the block monotone methods from [61]. The exact solution and the required number of iterations n_δ to reach the stopping test (2.50) are given as well.

The numerical results confirm the theoretical estimates (2.19) and (2.46) obtained, respectively, in Theorem 2.3.1 and Theorem 2.7.1.

Comparing our numerical results and the results from [61], we conclude that the numbers of iterations n_δ in the point monotone methods are almost double of the numbers of iterations in the block monotone methods from [61]. That gives us a motivation to investigate the block monotone approach for solving nonlinear differential problems.

Since the exact solution for our test problem is known, we investigate the numerical error $E(N)$ and order of convergence $\gamma(N)$ to the exact solution with respect to $1/N$, $N_x = N_y = N$ as follows

$$E(N) = \left[\max_{(i,j)\in\overline{\omega}^h} \left|U_{ij}^{(n_\delta)} - u_{ij}^*\right|\right], \quad \gamma(N) = \log_2\left(\frac{E(N)}{E(2N)}\right),$$

where $U_{ij}^{(n_\delta)}$, $(i, j) \in \overline{\Omega}^h$, is the numerical solution generated by (2.18), (2.50), u^* is the exact solution to the continuous problem and n_δ is the minimal number of iterations subject to (2.50).

In Table 2.5, for different values of N ($N_x = N_y = N$), we present $E(N)$ and $\gamma(N)$. The data in the table indicate that the numerical solution of the nonlinear difference scheme (2.6) converges to the exact solution with second-order accuracy.

From the numerical experiments, we conclude that the sequence of solutions generated by (2.18) has a linear rate of convergence q, such that, q is defined in the form

$$q = \frac{\|U^{(n)}\|_{\overline{\Omega}^h}}{\|U^{(n-1)}\|_{\overline{\Omega}^h}} < 1, \quad n \geq 2.$$

Table 2.1: Solutions by the point monotone Jacobi method for Test 1.

Solution	y_j/x_i	0	1/4	1/2	3/4	1	n_δ
$\overline{U}_{ij}$		0.382683	0.358796	0.287050	0.167448	0	
$\underline{U}_{ij}$	1/4	0.382683	0.358793	0.287045	0.167445	0	
u_{ij}		0.382683	0.358766	0.287013	0.167424	0	
$\overline{U}_{ij}$		0.707107	0.662967	0.530396	0.309403	0	1598 ($\overline{U}_{ij}$)
$\underline{U}_{ij}$	1/2	0.707107	0.662962	0.530389	0.309398	0	1566 ($\underline{U}_{ij}$)
u_{ij}		0.707107	0.662913	0.530330	0.309400	0	
$\overline{U}_{ij}$		0.923880	0.866206	0.692994	0.404253	0	
$\underline{U}_{ij}$	3/4	0.923880	0.866200	0.692984	0.404246	0	
u_{ij}		0.923880	0.866137	0.692910	0.404197	0	
$\overline{U}_{ij}$		1	0.937574	0.750089	0.437560	0	
$\underline{U}_{ij}$	1	1	0.937568	0.750080	0.437553	0	
u_{ij}		1	0.937500	0.750000	0.437500	0	

Test 2

As the second test problem, we consider the enzyme kinetics model Section 2.6.1 in the form

$$\begin{aligned} &-D(u_{xx} + u_{yy}) + \frac{\sigma u}{1 + au + bu^2} = 0, \quad 0 < x < 1,\ 0 < y < 1, \qquad (2.51)\\ &u(0, y) = 1, \quad u(1, y) = 1, \quad 0 \leq y \leq 1,\\ &u(x, 0) = 1, \quad u(x, 1) = 1, \quad 0 \leq x \leq 1. \end{aligned}$$

We choose $a = 1$, $b = 0.1$ and $\sigma = 10$. The upper solution $\widetilde{U}_{ij} = K$ and the lower solution $\widehat{U}_{ij} = 0$ from (2.40). We choose $K = 1$. It is clear that b and K satisfy the

Table 2.2: Solutions by the block monotone Jacobi method for Test 1.

Solution	y_j/x_i	0	1/4	1/2	3/4	1	n_δ
$\overline{U}_{ij}$		0.3832	0.3592	0.2874	0.1676	0	
$\underline{U}_{ij}$	1/4	0.3822	0.3583	0.2867	0.1672	0	
u_{ij}		0.3827	0.3588	0.2870	0.1674	0	
$\overline{U}_{ij}$		0.7080	0.6638	0.5310	0.3097	0	953 ($\overline{U}_{ij}$)
$\underline{U}_{ij}$	1/2	0.7063	0.6621	0.5297	0.3090	0	922 ($\underline{U}_{ij}$)
u_{ij}		0.7071	0.6629	0.5303	0.3094	0	
$\overline{U}_{ij}$		0.9250	0.8672	0.6937	0.4047	0	
$\underline{U}_{ij}$	3/4	0.9229	0.8652	0.6921	0.4038	0	
u_{ij}		0.9239	0.8661	0.6929	0.4042	0	
$\overline{U}_{ij}$		1.0012	0.9386	0.7509	0.4380	0	
$\underline{U}_{ij}$	0.9989	0.9365	0.7492	0.4370	0.437553	0	
u_{ij}		1.000	0.9375	0.7500	0.4375	0	

Table 2.3: Solutions by the point monotone Gauss-Seidel method for Test 1.

Solution	y_j/x_i	0	1/4	1/2	3/4	1	n_δ
$\overline{U}_{ij}$		0.382683	0.358795	0.287047	0.167447	0	
$\underline{U}_{ij}$	1/4	0.382683	0.358794	0.287046	0.167446	0	
u_{ij}		0.382683	0.358766	0.287013	0.167424	0	
$\overline{U}_{ij}$		0.707107	0.662964	0.530392	0.309340	0	921 ($\overline{U}_{ij}$)
$\underline{U}_{ij}$	1/2	0.707107	0.662964	0.530391	0.309340	0	880 ($\underline{U}_{ij}$)
u_{ij}		0.707107	0.662913	0.530330	0.309359	0	
$\overline{U}_{ij}$		0.923880	0.866203	0.692989	0.404249	0	
$\underline{U}_{ij}$	3/4	0.923880	0.866202	0.692987	0.404242	0	
u_{ij}		0.923880	0.866137	0.692909	0.404197	0	
$\overline{U}_{ij}$		1	0.937571	0.750084	0.437556	0	
$\underline{U}_{ij}$	1	1	0.937569	0,750083	0.437555	0	
u_{ij}		1	0.937500	0.750000	0.437500	0	

Table 2.4: Solutions by the block monotone Gauss-Seidel method from [61] for Test 1.

Solution	y_j/x_i	0	1/4	1/2	3/4	1	n_δ
$\overline{U}_{ij}$		0.3831	0.3591	0.2873	0.1676	0	
$\underline{U}_{ij}$	1/4	0.3825	0.3586	0.2868	0.1673	0	
u_{ij}		0.3872	0.3588	0.2870	0.1674	0	
$\overline{U}_{ij}$		0.7078	0.6636	0.5308	0.3096	0	505 ($\overline{U}_{ij}$)
$\underline{U}_{ij}$	1/2	0.7067	0.6626	0.5300	0.3092	0	508 ($\underline{U}_{ij}$)
u_{ij}		0.7071	0.6629	0.5303	0.3094	0	
$\overline{U}_{ij}$		0.9247	0.8669	0.6935	0.4046	0	
$\underline{U}_{ij}$	3/4	0.9234	0.8657	0.6926	0.4040	0	
u_{ij}		0.9239	0.8661	0.6929	0.4042	0	
$\overline{U}_{ij}$		1.0008	0.9383	0.7506	0.4379	0	
$\underline{U}_{ij}$	0.9996	0.9371	0.7497	0,4373	0.437555	0	
u_{ij}		1.0000	0.9375	0.7500	0.4375	0	

Table 2.5: Order of convergence of the nonlinear scheme (2.6) for Test 1.

N	8	16	32	64	128
E	2.082e-03	5.280e-04	1.327e-04	3.376e-05	9.015e-06
γ	1.98	1.99	1.97	1.91	

inequality $b < 1/K^2$ in (2.41). From (2.41), the bounded $c_{ij} = \sigma$, $(i,j) \in \overline{\Omega}^h$ where c_{ij} is defined in (2.27).

The exact solution for our test problem is unknown, and the numerical solution is compared to a corresponding reference solution. We investigate the numerical error and numerical order of convergence with respect to $1/N$, $N_x = N_y = N$. We define the numerical error $E(N)$ and the order of convergence $\gamma(N)$ of the numerical solution as follows

$$E(N) = \left[\max_{(i,j)\in\overline{\omega}^h} \left| \widetilde{U}_{ij} - \widetilde{U}_{ij}^{ref} \right| \right], \quad \gamma(N) = \log_2 \left(\frac{E(N)}{E(2N)} \right),$$

where $\widetilde{U}_{ij}^{ref}$ is the reference solution. A stopping test for the monotone iterative methods (2.18) is chosen in the form of (2.28). In our tests, we choose the reference solution with $N_{ref} = 512$ and $\delta = 10^{-6}$ in (2.28).

In Table 2.6, for different values of N ($N_x = N_y = N$), we present $E(N)$ and $\gamma(N)$.

The data in the table indicate that the numerical solution of the nonlinear difference scheme (2.6) converges to the reference solution with the second-order accuracy.

In Table 2.7, we present the number of iterations to find the approximate solution for (2.51) by the point monotone point Jacobi method with $\eta = 0$ in (2.18) and the point monotone Gauss-Seidel method with $\eta = 1$ in (2.18), with different values of diffusion coefficient D and number of mesh points N. In Figure 2.1, we show the convergence of numerical solutions, obtained by the point Gauss-Seidel method with $\eta = 1$ in (2.18) and $N = 128$ to the reference solution $N_{ref} = 512$, where the dashed line represents the numerical solution and the solid blue line refers to the reference solution with respect to x and fixed value of y. In the subgraph 2.1a, starting from the initial lower solution $\widehat{U} = 0$, we show the convergence of the numerical lower solutions at $n_\delta = 80$ and $n_\delta = 1000$ to the reference solution. Similarly, starting from the initial upper solution $\widetilde{U} = 1$, the subgraph 2.1b shows the convergence of the numerical upper solutions at $n_\delta = 80$ and $n_\delta = 1000$ to the reference solution.

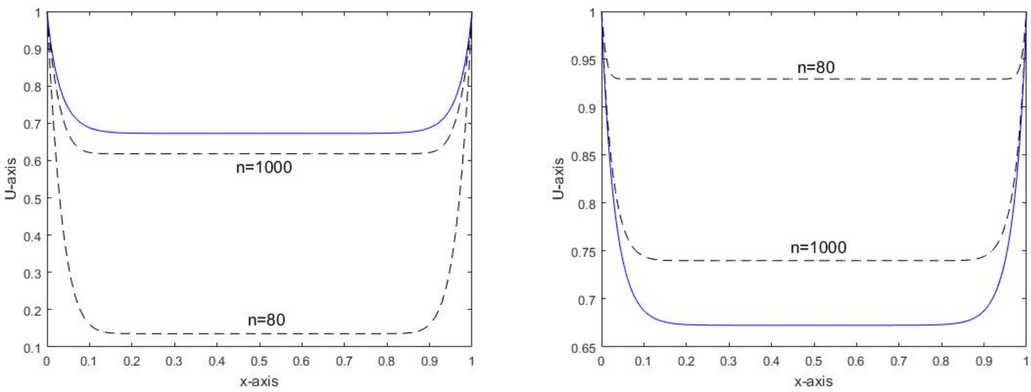
Table 2.6: Order of convergence of the nonlinear scheme (2.6) for Test 2.

N	16	32	64	128	256
E	3.916e-02	1.066e-02	2.866e-03	7.054e-04	1.434e-04
γ	1.88	1.89	2.02	2.30	

Table 2.7: Numbers of iterations for Test 2. Over line and under line iterations refer to, respectively, point monotone Jacobi Gauss-Seidel methods.

$D \setminus N$	16	32	64	128	256
1	$\frac{671}{339}$	$\frac{2677}{1342}$	$\frac{10702}{5355}$	$\frac{42802}{21405}$	$\frac{172305}{85600}$
10^{-1}	$\frac{142}{77}$	$\frac{543}{278}$	$\frac{2146}{1081}$	$\frac{8558}{4287}$	$\frac{34206}{17102}$
10^{-2}	$\frac{20}{15}$	$\frac{58}{34}$	$\frac{209}{110}$	$\frac{811}{412}$	$\frac{3222}{1620}$
10^{-3}	$\frac{6}{6}$	$\frac{12}{10}$	$\frac{28}{19}$	$\frac{88}{49}$	$\frac{329}{170}$
10^{-4}	$\frac{3}{3}$	$\frac{5}{4}$	$\frac{7}{7}$	$\frac{15}{12}$	$\frac{40}{25}$
10^{-5}	$\frac{2}{2}$	$\frac{3}{3}$	$\frac{4}{4}$	$\frac{5}{5}$	$\frac{9}{8}$
10^{-6}	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{4}{4}$
10^{-7}	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{3}{3}$

Figure 2.1: Convergence of lower and upper solutions to the reference solution for Test 2.



(a) Lower solutions. (b) Upper solutions.

2.9 Conclusions to Chapter 2

Theoretical results

Far solving nonlinear elliptic problems, we constructed and investigated monotone properties of point Jacobi and Gauss-Seidel iterative methods. The nonlinear elliptic problem (2.1) is approximated by using the central difference approximations for the first and second derivatives. For solving the nonlinear difference scheme (2.6), the point Jacobi and Gauss-Seidel iterative methods are constructed. We prove that the sequences of upper and lower solutions, generated by the point iterative methods, converge monotonically to the solutions of the nonlinear difference scheme. In Theorem 2.3.2, we prove the uniqueness of a solution under the conditions that the nonlinear reaction function is bounded from below and above. By using the stopping test (2.28), based on the norm of the residual, we prove that the numerical solution converges to the unique solution of the nonlinear elliptic problem (2.6) and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme (2.6) in Theorem 2.4.2 and between the numerical solution and the exact solution of the elliptic problem (2.1) in Theorem 2.4.3. In Theorem 2.7.1, we prove that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method. In Lemmas 2.5.1 and 2.5.2, under assumptions (2.31) and (2.35) on the reaction function, we construct initial upper and lower solutions to start the point monotone iterative methods.

Numerical results

The numerical experiments show that the numerical solution of the nonlinear difference scheme (2.6) converges to the reference solution with the second order accuracy.

The numerical sequences of upper and lower solutions, generated by the point monotone methods (2.18) with stopping (2.28), converge monotonically. The point monotone Gauss-Seidel method with $\eta = 1$ in (2.18) converges faster than the point monotone Jacobi method with $\eta = 0$ in (2.18) which confirms Theorem 2.7.1. The block monotone methods from [61] converge faster than the corresponding point monotone methods (2.18). In Test 2, for fixed diffusion coefficient D, the numbers of iterations increase with increasing N. For fixed values of N and small values of D, the numbers of iterations are independent of D.

Chapter 3

Jacobi and Gauss-Seidel methods for systems of elliptic problems

This chapter deals with numerical methods for solving nonlinear elliptic systems. We derive the point monotone Jacobi and Gauss–Seidel methods for solving difference schemes which approximate the coupled systems of elliptic problems. In the view of the method of upper and lower solutions, two monotone upper and lower sequences of solutions are constructed. Convergence estimates for the point monotone iterative methods are introduced. Constructions of initial upper and lower solutions are presented. The sequences of solutions generated by the point monotone Gauss–Seidel method converge faster than those generated by the Jacobi method.

3.1 Properties of solutions to systems of nonlinear elliptic problems

We consider properties of systems of nonlinear elliptic boundary value problems

$$- \mathrm{L}_\alpha u_\alpha(x,y) + f_\alpha(x,y,u) = 0, \quad (x,y) \in \omega, \tag{3.1}$$
$$\omega = \{(x,y) : 0 < x < l_1, \quad 0 < y < l_2\}, \quad u_\alpha(x,y) = g_\alpha(x,y), \quad (x,y) \in \partial\omega, \quad \alpha = 1,2,$$

where l_1, l_2 are positive constants, $u = (u_1, u_2)$ and $\partial\omega$ is the boundary of ω. The differential operators L_α, $\alpha = 1,2$, are defined by

$$\mathrm{L}_\alpha u_\alpha(x,y) \equiv D_\alpha^{(x)}(x,y)u_{\alpha,xx} + D_\alpha^{(y)}(x,y)u_{\alpha,yy} + v_\alpha^{(x)}(x,y)u_{\alpha,x} + v_\alpha^{(x)}(x,y)u_{\alpha,y},$$

where $D_\alpha^{(x)}(x,y)$, $D_\alpha^{(y)}(x,y)$, $\alpha = 1,2$, are positive functions. It is assumed that the functions $f_\alpha(x,y,u)$, $g_\alpha(x,y)$, $D_\alpha^{(x)}(x,y)$, $D_\alpha^{(y)}(x,y)$, $v_\alpha^{(x)}(x,y)$ and $v_\alpha^{(y)}(x,y)$, $\alpha = 1,2$, are smooth in their respective domains.

3.1.1 Quasi-monotone nondecreasing case

Two vector functions $\widetilde{u}(x,y)=(\widetilde{u}_1,\widetilde{u}_2)$ and $\widehat{u}(x,y)=(\widehat{u}_1,\widehat{u}_2)$, are called ordered upper and lower solutions to (3.1), if they satisfy the inequalities

$$\widehat{u}(x,y)\le\widetilde{u}(x,y),\quad (x,y)\in\overline{\omega}, \tag{3.2a}$$
$$-\mathrm{L}_\alpha\widehat{u}_\alpha(x,y)+f_\alpha(x,y,\widehat{u})\le 0\le -\mathrm{L}_\alpha\widetilde{u}_\alpha(x,y)+f_\alpha(x,y,\widetilde{u}),\quad (x,y)\in\omega, \tag{3.2b}$$
$$\widehat{u}(x,y)\le g(x,y)\le\widetilde{u}(x,y),\quad (x,y)\in\partial\omega. \tag{3.2c}$$

For a given ordered upper $\widetilde{u}$ and lower $\widehat{u}$ solutions, a sector $\langle\widehat{u},\widetilde{u}\rangle$ is defined as follows

$$\langle\widehat{u},\widetilde{u}\rangle=\{u(x,y):\quad \widehat{u}(x,y)\le u(x,y)\le\widetilde{u}(x,y),\quad (x,y)\in\overline{\omega}\}.$$

In the sector $\langle\widehat{u},\widetilde{u}\rangle$, the functions $f_\alpha(x,y,u)$, $\alpha=1,2$, are assumed to satisfy the constraint

$$\frac{\partial f_\alpha(x,y,u)}{\partial u_\alpha}\le c_\alpha(x,y),\quad u\in\langle\widehat{u},\widetilde{u}\rangle,\quad (x,y)\in\overline{\omega},\quad \alpha=1,2, \tag{3.3}$$
$$-\frac{\partial f_\alpha(x,y,u)}{\partial u_{\alpha'}}\ge 0,\quad u\in\langle\widehat{u},\widetilde{u}\rangle,\quad (x,y)\in\overline{\omega},\quad \alpha'\ne\alpha,\quad \alpha,\alpha'=1,2, \tag{3.4}$$

where $c_\alpha(x,y)$, $\alpha=1,2$, are nonnegative bounded functions. The functions $f_\alpha(x,y,u)$, $\alpha=1,2$, are called quasi-monotone nondecreasing in $\langle\widehat{u},\widetilde{u}\rangle$, if they satisfy (3.4).

Theorem 3.1.1. *Let $\widetilde{u}=(\widetilde{u}_1,\widetilde{u}_2)$ and $\widehat{u}=(\widehat{u}_1,\widehat{u}_2)$ be ordered upper and lower solutions (3.2). Assume that the functions $f_\alpha(x,y,u)$, $\alpha=1,2$, in (3.1) satisfy (3.3) and (3.4). Then a solution to the nonlinear problem (3.1) exists.*

The proof of the theorem is given in Theorem 8.4.1, [59].

We assume that the reaction functions f_α, $\alpha=1,2$, in (3.1) satisfy the conditions

$$0<\underline{c}_\alpha(x,y)\le\frac{\partial f_\alpha(x,y,u)}{\partial u_\alpha}\le c_\alpha(x,y),\quad (x,y)\in\overline{\omega},\quad u\in(-\infty,\infty),\quad \alpha=1,2, \tag{3.5}$$
$$0\le-\frac{\partial f_\alpha(x,y,u)}{\partial u_{\alpha'}}\le q_{\alpha\alpha'}(x,y),\quad (x,y)\in\overline{\omega},\quad u\in(-\infty,\infty),\quad \alpha'\ne\alpha,\quad \alpha,\alpha'=1,2, \tag{3.6}$$
$$0<\beta=\max_{\alpha=1,2}\left[\max_{(x,y)\in\overline{\omega}}\left(\frac{q_{\alpha\alpha'}(x,y)}{\underline{c}_\alpha(x,y)}\right)\right]<1,\quad (x,y)\in\overline{\omega},\quad u\in(-\infty,\infty),\quad \alpha'\ne\alpha,$$
$$\alpha,\alpha'=1,2. \tag{3.7}$$

Introduce the linear problem

$$\mathrm{L}_\alpha w_\alpha(x,y)+c^*_\alpha(x,y)w_\alpha(x,y)=\phi_\alpha(x,y),\quad (x,y)\in\omega^h, \tag{3.8}$$
$$w_\alpha(x,y)=g_\alpha(x,y),\quad (x,y)\in\partial\omega^h,\quad \alpha=1,2,$$

where $c_\alpha^*(x,y)$, $\alpha=1,2$, are positive bounded functions. We give a bound on the magnitude of the solution to the linear problem (3.8) in the following lemma.

Lemma 3.1.2. *The following bound on the magnitude of the solution to the linear problem (3.8) holds*

$$\|w_\alpha\|_{\overline{\omega}} \leq \max\left\{\|g_\alpha\|_{\partial\omega}, \left\|\frac{\phi_\alpha}{c_\alpha^*}\right\|_\omega\right\}, \quad \alpha=1,2, \tag{3.9}$$

where

$$\|g_\alpha\|_{\partial\omega} = \max_{(x,y)\in\partial\omega} |g_\alpha(x,y)|, \quad \left\|\frac{\phi_\alpha}{c_\alpha^*}\right\|_\omega = \max_{(x,y)\in\omega}\left|\frac{\phi_\alpha(x,y)}{c_\alpha^*(x,y)}\right|.$$

The proof of the lemma is given in Lemma 1.2.1 from Chapter 1.

Theorem 3.1.3. *Let assumptions (3.5)–(3.7) be satisfied. Then the continuous problem (3.1) has a unique solution.*

Proof. The existence of solutions to the nonlinear problem (3.1) is given in Theorem 3.1.1. Suppose that $u^*(x,y)=(u_1^*(x,y),u_2^*(x,y))$ and $u^{**}(x,y)=(u_1^{**}(x,y),u_2^{**}(x,y))$, $(x,y)\in\overline{\omega}$ are two solutions to (3.1). Letting $z_\alpha(x,y)=u_\alpha^*(x,y)-u_\alpha^{**}(x,y)$, $(x,y)\in\overline{\omega}$, $\alpha=1,2$, from (3.1) for $z_\alpha(x,y)$, we have

$$\begin{gathered}-\mathrm{L}_\alpha z_\alpha(x,y)+f_\alpha(x,y,u_\alpha^*,u_{\alpha'}^*)- \\ f_\alpha(x,y,u_\alpha^{**},u_{\alpha'}^*)+f_\alpha(x,y,u_\alpha^{**},u_{\alpha'}^*)-f_\alpha(x,y,u_\alpha^{**},u_{\alpha'}^{**})=0, \\ (x,y)\in\omega, \quad z_\alpha(x,y)=0, \quad (x,y)\in\partial\omega, \quad \alpha=1,2.\end{gathered}$$

From here and using the mean-value theorem, we obtain

$$\begin{gathered}-\mathrm{L}_\alpha z_\alpha(x,y)+\frac{\partial f_\alpha(x,y,q_\alpha,u_{\alpha'}^*)}{\partial u_\alpha}z_\alpha(x,y)=-\frac{\partial f_\alpha(x,y,u_\alpha^{**},k_{\alpha'})}{\partial u_{\alpha'}}z_{\alpha'}(x,y), \quad (x,y)\in\omega, \\ z_\alpha(x,y)=0, \quad (x,y)\in\partial\omega, \quad \alpha'\neq\alpha, \quad \alpha,\alpha'=1,2,\end{gathered}$$

where the functions $q_\alpha(x,y)$, $k_\alpha(x,y)$ lie between $u_\alpha^*(x,y)$ and $u_\alpha^{**}(x,y)$, $\alpha=1,2$. From here and (3.5), by using estimate (3.9), we conclude that

$$\|z_\alpha\|_{\overline{\omega}} \leq \left\|\frac{(f_\alpha(u_\alpha^{**},k_{\alpha'}))_{u_{\alpha'}}z_{\alpha'}}{\left(f_\alpha(q_\alpha,u_{\alpha'}^*)\right)_{u_\alpha}}\right\|_\omega \leq \left\|\frac{(f_\alpha(u_\alpha^{**},k_{\alpha'}))_{u_{\alpha'}}}{\left(f_\alpha(q_\alpha,u_{\alpha'}^*)\right)_{u_\alpha}}\right\|_\omega \|z_{\alpha'}\|_\omega.$$

Using (3.5)–(3.7), we obtain

$$\|z_\alpha\|_{\overline{\omega}} \leq \beta\|z_{\alpha'}\|_\omega.$$

Letting $z=\max_{\alpha=1,2}\|z_\alpha\|_{\overline{\omega}}$, we have $z(1-\beta)\leq 0$. From here, (3.7) and taking into account that $z\geq 0$, we conclude that $z=0$. Thus, we prove the theorem. □

3.1.2 Quasi-monotone nonincreasing case

Introduce the following notation:

$$\mathcal{F}_\alpha(x,y,u_\alpha,u_{\alpha'}) = \begin{cases} \mathcal{F}_1(x,y,u_1,u_2), & \alpha=1, \\ \mathcal{F}_2(x,y,u_1,u_2), & \alpha=2, \end{cases} \quad \alpha' \neq \alpha. \tag{3.10}$$

Two vector functions $\widetilde{u}(x,y) = (\widetilde{u}_1, \widetilde{u}_2)$ and $\widehat{u}(x,y) = (\widehat{u}_1, \widehat{u}_2)$, are called ordered upper and lower solutions to (3.1) in the case of quasi-monotone nonincreasing reaction functions f_α, $\alpha = 1, 2$, if they satisfy the inequalities

$$\widehat{u}(x,y) \leq \widetilde{u}(x,y), \quad (x,y) \in \overline{\omega}, \tag{3.11a}$$

$$-\,\mathrm{L}_\alpha \widehat{u}_\alpha(x,y) + f_\alpha(x,y,\widehat{u}_\alpha,\widetilde{u}_{\alpha'}) \leq 0 \leq -\mathrm{L}_\alpha \widetilde{u}_\alpha(x,y) + f_\alpha(x,y,\widetilde{u}_\alpha,\widehat{u}_{\alpha'}), \quad (x,y) \in \omega, \tag{3.11b}$$

$$\widehat{u}_\alpha(x,y) \leq g_\alpha(x,y) \leq \widetilde{u}_\alpha(x,y), \quad (x,y) \in \partial\omega, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2. \tag{3.11c}$$

For a given ordered upper $\widetilde{u}$ and lower $\widehat{u}$ solutions, a sector $\langle \widehat{u}, \widetilde{u} \rangle$ is defined as follows

$$\langle \widehat{u}, \widetilde{u} \rangle = \{ u(x,y); \quad \widehat{u}(x,y) \leq u(x,y) \leq \widetilde{u}(x,y), \quad (x,y) \in \overline{\omega} \}.$$

In the sector $\langle \widehat{u}, \widetilde{u} \rangle$, the vector function $f(x,y,u)$ is assumed to satisfy the constraint

$$\frac{\partial f_\alpha(x,y,u)}{\partial u_\alpha} \leq c_\alpha(x,y), \quad u \in \langle \widehat{u}, \widetilde{u} \rangle, \quad (x,y) \in \overline{\omega}, \quad \alpha = 1,2, \tag{3.12}$$

$$-\frac{\partial f_\alpha(x,y,u)}{\partial u_{\alpha'}} \leq 0, \quad u \in \langle \widehat{u}, \widetilde{u} \rangle, \quad (x,y) \in \overline{\omega}, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2, \tag{3.13}$$

where $c_\alpha(x,y)$, $\alpha = 1, 2$, are nonnegative bounded functions. The vector function $f(x,y,u)$ is called quasi-monotone nonincreasing in $\langle \widehat{u}, \widetilde{u} \rangle$, if it satisfies (3.13).

Theorem 3.1.4. *Let $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2)$ and $\widehat{u} = (\widehat{u}_1, \widehat{u}_2)$ be ordered upper and lower solutions (3.11). Assume that the functions $f_\alpha(x,y,u)$, $\alpha = 1, 2$, in (3.1) satisfy (3.12) and (3.13). Then a solution to the nonlinear problem (3.1) exists.*

The proof of the theorem is given in Theorem 8.4.2, [59].

We assume that the reaction functions f_α, $\alpha = 1, 2$, in (3.1) satisfy the conditions (3.5), (3.7) and

$$q_{\alpha\alpha'}(x,y) \leq -\frac{\partial f_\alpha(x,y,u)}{\partial u_{\alpha'}} \leq 0, \quad (x,y) \in \overline{\omega}, \quad u \in (-\infty,\infty), \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2, \tag{3.14}$$

Theorem 3.1.5. *Let assumptions (3.5), (3.7) and (3.14) be satisfied. Then the continuous problem (3.1) has a unique solution.*

Proof. The existence of a solution to the nonlinear problem (3.1) is given in Theorem 3.1.4. The proof of uniqueness of a solution repeats the proof of Theorem 3.1.3. □

3.2 The nonlinear difference scheme

On $\overline{\omega}$, we introduce a rectangular mesh $\overline{\Lambda}^h = \overline{\Lambda}^{hx} \times \overline{\Lambda}^{hy}$:

$$\begin{aligned} \overline{\Lambda}^{hx} &= \{x_i, \quad i = 0, 1, \dots, N_x; \quad x_0 = 0, \quad x_{N_x} = l_1; \quad h_x = x_{i+1} - x_i\}, \\ \overline{\Lambda}^{hy} &= \{y_j, \quad j = 0, 1, \dots, N_y; \quad y_0 = 0, \quad y_{N_y} = l_2; \quad h_y = y_{j+1} - y_j\}. \end{aligned} \tag{3.15}$$

We denote by Ω^h and $\partial\Omega^h$ the sets of indices which correspond to interior and boundary mesh points, such that

$$\begin{aligned} \Omega^h &= \{(i, j) : \quad i = 1, 2, \dots, N_x - 1, \quad j = 1, 2, \dots, N_y - 1\}, \\ \partial\Omega^h &= \{(i, j) : \quad i = 0, N_x, \quad j = 0, 1, \dots, N_y; \quad i = 0, 1, \dots, N_x, \quad j = 0, N_y\}. \end{aligned}$$

For $(i, j) \in \overline{\Omega}^h = \Omega^h \cup \partial\Omega^h$, we introduce the notation

$$\mathcal{T}_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}) = \begin{cases} \mathcal{T}_{1,ij}(U_{1,ij}, U_{2,ij}), & \alpha = 1, \\ \mathcal{T}_{2,ij}(U_{1,ij}, U_{2,ij}), & \alpha = 2, \end{cases} \quad \alpha' \neq \alpha. \tag{3.16}$$

By using the central difference approximations for the first and second derivatives on the 5-point stencil, we introduce the nonlinear difference scheme

$$\begin{aligned} &\mathcal{A}_{\alpha,ij} U_{\alpha,ij} + f_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}) = 0, \quad (i, j) \in \Omega^h, \\ &U_{\alpha,ij} = g_{\alpha,ij}, \quad (i, j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2, \end{aligned} \tag{3.17}$$

where $f_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij})$ is defined by (3.16), Ω^h is the set of indices of interior mesh points in $\overline{\Lambda}^h$, $\partial\Omega^h$ is the set of indices of the boundary mesh points in $\overline{\Lambda}^h$ and the central difference approximations for the first and second derivatives are given by

$$\begin{aligned} &\mathcal{D}_x^2 U_{\alpha,ij} = \frac{U_{\alpha,i-1,j} - 2U_{\alpha,ij} + U_{\alpha,i+1,j}}{h_x^2}, \quad \mathcal{D}_y^2 U_{\alpha,ij} = \frac{U_{\alpha,i,j-1} - 2U_{\alpha,ij} + U_{\alpha,i,j+1}}{h_y^2}, \\ &\mathcal{D}_x^1 U_{\alpha,ij} = \frac{U_{\alpha,i+1,j} - U_{\alpha,i-1,j}}{2h_x}, \quad \mathcal{D}_y^1 U_{\alpha,ij} = \frac{U_{\alpha,i,j+1} - U_{\alpha,i,j-1}}{2h_y}, \quad \alpha = 1, 2. \end{aligned} \tag{3.18}$$

The difference operators $\mathcal{A}_{\alpha,ij}U_{\alpha,ij}$, $\alpha = 1, 2$, in (3.17) are defined by

$$\begin{aligned}
\mathcal{A}_{\alpha,ij}U_{\alpha,ij} &= \mathcal{A}^{(x)}_{\alpha,ij}U_{\alpha,ij} + \mathcal{A}^{(y)}_{\alpha,ij}U_{\alpha,ij}, \\
\mathcal{A}^{(x)}_{\alpha,ij}U_{\alpha,ij} &= \frac{1}{h_x^2}\left[-l_{\alpha,ij}U_{\alpha,i-1,j} + 2D^{(x)}_{\alpha,ij}U_{\alpha,ij} - r_{\alpha,ij}U_{\alpha,i+1,j}\right], \\
\mathcal{A}^{(y)}_{\alpha,ij}U_{\alpha,ij} &= \frac{1}{h_y^2}\left[-b_{\alpha,ij}U_{\alpha,i,j-1} + 2D^{(y)}_{\alpha,ij}U_{\alpha,ij} - q_{\alpha,ij}U_{\alpha,i,j+1}\right], \\
l_{\alpha,ij} &= \frac{D^{(x)}_{\alpha,ij}}{h_x^2} - \frac{v^{(x)}_{\alpha,ij}}{2h_x}, \quad r_{\alpha,ij} = \frac{D^{(x)}_{\alpha,ij}}{h_x^2} + \frac{v^{(x)}_{\alpha,ij}}{2h_x}, \\
b_{\alpha,ij} &= \frac{D^{(y)}_{\alpha,ij}}{h_y^2} - \frac{v^{(y)}_{\alpha,ij}}{2h_y}, \quad q_{\alpha,ij} = \frac{D^{(y)}_{\alpha,ij}}{h_y^2} + \frac{v^{(y)}_{\alpha,ij}}{2h_y}, \quad \alpha = 1, 2.
\end{aligned}$$

To insure that $l_{\alpha,ij}$, $r_{\alpha,ij}$, $b_{\alpha,ij}$ and $q_{\alpha,ij}$, $\alpha = 1, 2$, are positive, we choose

$$h_x < \frac{2D^{(x)}_{\alpha,ij}}{|v^{(x)}_{\alpha,ij}|}, \quad h_y < \frac{2D^{(y)}_{\alpha,ij}}{|v^{(y)}_{\alpha,ij}|}.$$

Remark 3.2.1. *If the effect of convection $v(x, y)$ dominates diffusion $D(x, y)$ to the extent that these conditions require prohibitively small h_x and h_y, then an upwind difference scheme for the first derivatives can be used to remove any restriction on h_x and h_y, that is, for $\alpha = 1, 2$,*

$$\mathcal{D}'_x U_{\alpha,ij} = \begin{cases} \frac{U_{\alpha,i+1,j} - U_{\alpha,ij}}{h_x}, & if\ v^{(x)}_{\alpha,ij} \leq 0, \\ \frac{U_{\alpha,ij} - U_{\alpha,i-1,j}}{h_x}, & if\ v^{(x)}_{\alpha,ij} \geq 0, \end{cases}$$

$$\mathcal{D}'_y U_{\alpha,ij} = \begin{cases} \frac{U_{\alpha,i,j+1} - U_{\alpha,ij}}{h_y}, & if\ v^{(y)}_{\alpha,ij} \leq 0, \\ \frac{U_{\alpha,ij} - U_{\alpha,i,j-1}}{h_y}, & if\ v^{(y)}_{\alpha,ij} \geq 0, \end{cases}$$

We introduce the linear version of problem (3.17) in the form

$$\begin{aligned}
&\mathcal{A}_{\alpha,ij}W_{\alpha,ij} + c^*_{\alpha,ij}W_{\alpha,ij} = \Phi_{\alpha,ij}, \quad (i,j) \in \Omega^h, \\
&W_{\alpha,ij} = g_{\alpha,ij}, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1, 2,
\end{aligned} \tag{3.19}$$

where $c^*_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, are nonnegative bounded functions. We formulate the maximum principle for the difference operators $\mathcal{A}_{\alpha,ij} + c^*_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1, 2$.

Lemma 3.2.2. *If $W_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, satisfy the conditions*

$$\begin{aligned}
&\mathcal{A}_{\alpha,ij}W_{\alpha,ij} + c^*_{\alpha,ij}W_{\alpha,ij} \geq 0 \ (\leq 0), \quad (i,j) \in \Omega^h, \\
&W_{\alpha,ij} \geq 0 \ (\leq 0), \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1, 2,
\end{aligned}$$

then $W_{\alpha,ij} \geq 0 \ (\leq 0), \quad (i,j) \in \overline{\Omega}^h, \ \alpha = 1,2.$

The proof of the lemma is given in Lemma 1.2.1 from Chapter 1.

Remark 3.2.3. *In this remark, we discuss the mean-value theorem for vector-valued functions. Assume that* $\mathcal{F}_\alpha(x,y,u_\alpha,u_{\alpha'})$, $\alpha' \neq \alpha$, $\alpha = 1,2$, *are smooth functions, then we have*

$$\mathcal{F}_\alpha(x,y,u_\alpha,u_{\alpha'}) - \mathcal{F}_\alpha(x,y,w_\alpha,u_{\alpha'}) = \frac{\partial \mathcal{F}_\alpha(h_\alpha,u_{\alpha'})}{\partial u_\alpha}[u_\alpha - w_\alpha], \tag{3.20}$$

$$\mathcal{F}_\alpha(x,y,u_\alpha,u_{\alpha'}) - \mathcal{F}_\alpha(x,y,u_\alpha,w_{\alpha'}) = \frac{\partial \mathcal{F}_\alpha(u_\alpha,p_{\alpha'})}{\partial u_{\alpha'}}[u_{\alpha'} - w_{\alpha'}],$$

where $h_\alpha(x,y)$, $p_\alpha(x,y)$ *lie between* $u_\alpha(x,y)$ *and* $w_\alpha(x,y)$, $\alpha = 1,2$, *and notation (3.10) is in use.*

3.2.1 Quasi-monotone nondecreasing case

Two vector mesh functions $\widetilde{U}_{ij} = (\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij})$ and $\widehat{U}_{ij} = (\widehat{U}_{1,ij}, \widehat{U}_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, are called ordered upper and lower solutions of (3.17), if they satisfy the inequalities

$$\widehat{U}_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \tag{3.21a}$$

$$\mathcal{A}_{\alpha,ij}\widehat{U}_{\alpha,ij} + f_{\alpha,ij}(\widehat{U}_{ij}) \leq 0 \leq \mathcal{A}_{\alpha,ij}\widetilde{U}_{\alpha,ij} + f_{\alpha,ij}(\widetilde{U}_{ij}), \quad (i,j) \in \Omega^h, \tag{3.21b}$$

$$\widehat{U}_{\alpha,ij} \leq g_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij}, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1,2. \tag{3.21c}$$

For a given pair of ordered upper and lower solutions $\widetilde{U}_{ij}$ and $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, we define the sector

$$\langle \widehat{U}, \widetilde{U} \rangle = \left\{ U_{ij} : \widehat{U}_{ij} \leq U_{ij} \leq \widetilde{U}_{ij}, \quad (i,j) \in \overline{\Omega}^h \right\}.$$

In the sector $\langle \widehat{U}, \widetilde{U} \rangle$, we assume that the functions $f_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, in (3.17), satisfy the constraints

$$\frac{\partial f_{\alpha,ij}(U_{ij})}{\partial u_\alpha} \leq c_{\alpha,ij}, \quad U \in \langle \widehat{U}, \widetilde{U} \rangle, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \tag{3.22}$$

$$-\frac{\partial f_{\alpha,ij}(U_{ij})}{\partial u_{\alpha'}} \geq 0, \quad U \in \langle \widehat{U}, \widetilde{U} \rangle, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2, \tag{3.23}$$

where $c_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are nonnegative bounded functions in $\overline{\Omega}^h$. We say that the functions $f_{\alpha,ij}(U_{ij})$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are quasi-monotone nondecreasing in $\langle \widehat{U}, \widetilde{U} \rangle$ if they satisfy (3.23).

We introduce the notation

$$\Gamma_{\alpha,ij}(U_{\alpha,ij}, V_{\alpha',ij}) = c_{\alpha,ij}U_{\alpha,ij} - f_{\alpha,ij}(U_{\alpha,ij}, V_{\alpha',ij}), \quad (i,j) \in \overline{\Omega}^h, \tag{3.24}$$
$$\alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2,$$

where $c_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, are nonnegative bounded functions, and notation (3.16) is in use. We give a monotone property of $\Gamma_{\alpha,ij}(U_{\alpha,ij}, V_{\alpha',ij})$, $(i,j) \in \overline{\Omega}^h$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1, 2$ from (3.24).

Lemma 3.2.4. *Suppose that $U_{ij} = (U_{1,ij}, U_{2,ij})$ and $V_{ij} = (V_{1,ij}, V_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, are vector functions in $\langle \widehat{U}, \widetilde{U} \rangle$, such that $U_{ij} \geq V_{ij}$, $(i,j) \in \overline{\Omega}^h$, and assume that (3.22) and (3.23) are satisfied. Then*

$$\Gamma_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}) \geq \Gamma_{\alpha,ij}(V_{\alpha,ij}, V_{\alpha',ij}), \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2. \tag{3.25}$$

Proof. From (3.24), we have

$$\begin{aligned} \Gamma_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}) - \Gamma_{\alpha,ij}(V_{\alpha,ij}, V_{\alpha',ij}) &= c_{\alpha,ij}(U_{\alpha,ij} - V_{\alpha,ij}) \\ &\quad - \left[f_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}) - f_{\alpha,ij}(V_{\alpha,ij}, U_{\alpha',ij})\right] \\ &\quad - \left[f_{\alpha,ij}(V_{\alpha,ij}, U_{\alpha',ij}) - f_{\alpha,ij}(V_{\alpha,ij}, V_{\alpha',ij})\right], \end{aligned} \tag{3.26}$$
$$(i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.$$

Using the mean-value theorem (3.20), we obtain that

$$\Gamma_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}) - \Gamma_{\alpha,ij}(V_{\alpha,ij}, V_{\alpha',ij}) =$$
$$\Big(c_{\alpha,ij} - \big(f_{\alpha,ij}(Q_{\alpha,ij}, U_{\alpha',ij})\big)_{u_\alpha}\Big)(U_{\alpha,ij} - V_{\alpha,ij}) - \big(f_{\alpha,ij}(V_{\alpha,ij}, Y_{\alpha',ij})\big)_{u_{\alpha'}}(U_{\alpha',ij} - V_{\alpha',ij}),$$
$$V_{\alpha,ij} \leq Q_{\alpha,ij}, Y_{\alpha,ij} \leq U_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.$$

Taking into account that $U_{\alpha,ij} \geq V_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, from (3.22) and (3.23), we conclude (3.25). □

3.2.1.1 Applied problems

The gas-liquid interaction model

Consider the gas-liquid interaction model where a dissolved gas A and a dissolved reactant B interact in a bounded diffusion medium ω (more details are given in [34]). The chemical reaction scheme is given by $A + k_1 B \to k_2 P$ and is called the second order reaction, where k_1 and k_2 are the rate constants and P is the product. Denote by $z_1(x,y)$ and $z_2(x,y)$ the concentrations of the dissolved gas A and the reactant B. Then the above reaction scheme is governed by (3.1) with $\mathrm{L}_\alpha z_\alpha = D_\alpha \triangle z_\alpha$, $f_\alpha = \sigma_\alpha z_1 z_2$,

$\alpha = 1, 2$, where σ_1 is the rate constant, $\sigma_2 = k_1\sigma_1$. By choosing a suitable positive constant $\rho_1 > 0$ and letting $u_1 = \rho_1 - z_1 \geq 0$, $u_2 = z_2$, we have

$$f_1(u_1, u_2) = -\sigma_1(\rho_1 - u_1)u_2, \quad f_2(u_1, u_2) = \sigma_2(\rho_1 - u_1)u_2, \tag{3.27}$$

and system (3.1) is reduced to

$$\begin{aligned} &- D_\alpha \triangle u_\alpha + f_\alpha(u_1, u_2) = 0, \quad (x, y) \in \omega, \quad \alpha = 1, 2, \\ &u_1(x, y) = g_1^*(x, y) \geq 0, \quad u_2(x, y) = g_2(x, y) \geq 0, \quad (x, y) \in \partial\omega, \end{aligned}$$

where $g_1^* = \rho_1 - g_1 \geq 0$ and $g_1 \geq 0$ on $\partial\omega$. The nonlinear difference scheme (3.17) for the model is presented in the form

$$\begin{aligned} &\mathcal{A}_{\alpha,ij}U_{\alpha,ij} + f_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}) = 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2, \\ &U_{1,ij} = g_{1,ij}^*, \quad U_{2,ij} = g_{2,ij}, \quad (i,j) \in \partial\Omega^h, \end{aligned} \tag{3.28}$$

where f_α, $\alpha = 1, 2$, are defined in (3.27), and

$$\mathcal{A}_{\alpha,ij}U_{\alpha,ij} = -D_\alpha(\mathcal{D}_x^2 + \mathcal{D}^2{}_y)U_{\alpha,ij}, \quad (i,j) \in \Omega^h, \quad \alpha = 1, 2,$$

where $\mathcal{D}_x^2$, $\mathcal{D}_y^2$ are defined in (3.18). We introduce the linear problems

$$\mathcal{A}_{\alpha,ij}V_{\alpha,ij} = 0, \quad (i,j) \in \Omega^h, \quad \alpha = 1, 2, \tag{3.29}$$

$$V_{1,ij} = g_{1,ij}^*, \quad V_{2,ij} = g_{2,ij}, \quad (i,j) \in \partial\Omega^h.$$

We now show that

$$(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = (\rho_1, V_{2,ij}), \quad (\widehat{U}_{1,ij}, \widehat{U}_{2,ij}) = (V_{1,ij}, 0), \quad (i,j) \in \overline{\Omega}^h, \tag{3.30}$$

are ordered upper and lower solutions (3.21) to (3.28). Letting $W_{\alpha,ij} = \widetilde{U}_{\alpha,ij} - \widehat{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, from (3.27) and (3.29), we have

$$\mathcal{A}_{\alpha,ij}W_{\alpha,ij} = 0, \quad (i,j) \in \Omega^h, \quad \alpha = 1, 2.$$

From here and using Lemma 3.2.2, we conclude that $W_{\alpha,ij} \geq 0$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$. Thus, we prove (3.21a). From (3.27), (3.29) and (3.30), we obtain

$$\begin{aligned} &\mathcal{A}_{\alpha,ij}\widetilde{U}_{\alpha,ij} + f_{\alpha,ij}(\widetilde{U}_{\alpha,ij}, \widetilde{U}_{\alpha',ij}) = 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq, \alpha, \quad \alpha, \alpha' = 1, 2, \\ &\mathcal{A}_{\alpha,ij}\widehat{U}_{\alpha,ij} + f_{\alpha,ij}(\widehat{U}_{\alpha,ij}, \widehat{U}_{\alpha',ij}) = 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq, \alpha, \quad \alpha, \alpha' = 1, 2. \end{aligned}$$

Hence, we conclude (3.21b). From (3.30), it follows (3.21c). Thus, we prove that $\widetilde{U}_{\alpha,ij}$

and $\widehat{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from (3.30) are ordered upper and lower solutions (3.21). From (3.27), in the sector $\langle \widehat{U}, \widetilde{U} \rangle$, we have

$$\begin{aligned}
&\frac{\partial f_{1,ij}}{\partial u_1}(U_{1,ij}, U_{2,ij}) = \sigma_1 U_{2,ij} \leq \sigma_1 V_{2,ij}, \quad (i,j) \in \overline{\Omega}^h, \\
&\frac{\partial f_{2,ij}}{\partial u_2}(U_{1,ij}, U_{2,ij}) = \sigma_2(\rho_1 - U_{1,ij}) \leq \sigma_2 \rho_1, \quad (i,j) \in \overline{\Omega}^h, \\
&- \frac{\partial f_{1,ij}}{\partial u_2}(U_{1,ij}, U_{2,ij}) = \sigma_1(\rho_1 - U_{1,ij}) \geq 0, \quad (i,j) \in \overline{\Omega}^h, \\
&- \frac{\partial f_{2,ij}}{\partial u_1}(U_{1,ij}, U_{2,ij}) = \sigma_2 U_{2,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h.
\end{aligned}$$

Thus, the assumptions in (3.22) are satisfied with

$$c_{1,ij} = \sigma_1 V_{2,ij}, \quad c_{2,ij} = \sigma_2 \rho_1, \quad (i,j) \in \overline{\Omega}^h.$$

From here, we conclude that f_α, $\alpha = 1,2$, from (3.27) satisfy (3.22) and quasi-monotone nondecreasing property (3.23).

Enzyme-substrate reaction diffusion model

In the enzyme–substrate reaction problem, the chemical reaction scheme is expressed by

$$E + S \rightleftharpoons ES \rightarrow E + P,$$

where E, S and P represent, respectively, enzyme, substrate and product. The usual enzyme concentration law is given by

$$E + C = E_0, \tag{3.31}$$

where $C = ES$ is the enzyme substrate complex, and E_0 is the total enzyme (more details are given in [41]). Let $z_1(x,y)$ and $z_2(x,y)$ be, respectively, the concentrations of the enzyme and the substrate. Then the above reactant scheme is governed by (3.1) with $\mathrm{L}_\alpha z_\alpha = D_\alpha \triangle z_\alpha$, $\alpha = 1,2$, $f_1(z_1,z_2) = a_1 z_1 z_2 - b_1(E_0 - z_2)$, $f_2(z_1,z_2) = a_2 z_1 z_2 - b_2(E_0 - z_2)$, where a_α, b_α, $\alpha = 1,2$, are positive constants. Letting $u_1 = z_1$, $u_2 = E_0 - z_2 \geq 0$, we have

$$f_1(u_1,u_2) = a_1 u_1(E_0 - u_2) - b_1 u_2, \quad f_2(u_1,u_2) = -a_2 u_1(E_0 - u_2) + b_2 u_2. \tag{3.32}$$

System (3.1) is reduced to

$$\begin{aligned}
&- D_\alpha \triangle u_\alpha + f_\alpha(u_\alpha, u_{\alpha'}) = 0, \quad (x,y) \in \omega, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \\
&u_1(x,y) = g_1(x,y) \geq 0, \quad u_2(x,y) = g_2^*(x,y), \quad (x,y) \in \partial\omega,
\end{aligned}$$

where $g_1 \geq 0$ on $\partial\omega$ and $g_2^* = E_0 - g_2 \geq 0$. The nonlinear difference scheme (3.17) for the model is presented in the form

$$\mathcal{A}_{\alpha,ij}U_{\alpha,ij} + f_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}) = 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2, \tag{3.33}$$
$$U_{1,ij} = g_{1,ij}, \quad U_{2,ij} = g_{2,ij}^*, \quad (i,j) \in \partial\Omega^h,$$

where f_α, $\alpha = 1, 2$, are defined in (3.32), and

$$\mathcal{A}_{\alpha,ij}U_{\alpha,ij} = -D_\alpha(\mathcal{D}_x^2 + \mathcal{D}_y^2)U_{\alpha,ij}, \quad (i,j) \in \Omega^h, \quad \alpha = 1, 2,$$

where $\mathcal{D}_x^2$, $\mathcal{D}_y^2$ are defined in (3.18).

Introduce the linear problem

$$\mathcal{A}_{1,ij}V_{ij} = \Phi_{ij}, \quad (i,j) \in \Omega^h, \quad V_{ij} = g_{1,ij}, \quad (i,j) \in \partial\Omega^h, \tag{3.34}$$

where Φ_{ij}, $(i,j) \in \overline{\Omega}^h$, is any positive mesh function, such that $\Phi_{ij} \geq b_1 E_0$, $(i,j) \in \overline{\Omega}^h$. We now show that

$$(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = (V_{ij}, E_0), \quad (\widehat{U}_{1,ij}, \widehat{U}_{2,ij}) = (0, 0), \quad (i,j) \in \overline{\Omega}^h, \tag{3.35}$$

are ordered upper and lower solutions (3.21) to (3.33). Letting $W_{\alpha,ij} = \widetilde{U}_{\alpha,ij} - \widehat{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$. From (3.34) and (3.35), we conclude that

$$\mathcal{A}_{1,ij}W_{1,ij} = \Phi_{ij}, \quad (i,j) \in \Omega^h, \quad W_{1,ij} \geq 0, \quad (i,j) \in \partial\Omega^h,$$
$$\mathcal{A}_{2,ij}W_{2,ij} = 0, \quad (i,j) \in \Omega^h, \quad W_{2,ij} > 0, \quad (i,j) \in \partial\Omega^h.$$

From here, by Lemma 3.2.2, we obtain that

$$W_{\alpha,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1, 2. \tag{3.36}$$

Thus, we prove (3.21a). From (3.32), (3.34) and (3.35), we have

$$\mathcal{A}_{1,ij}\widetilde{U}_{1,ij} + f_{1,ij}(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = \Phi_{ij} - b_1 E_0 \geq 0, \quad (i,j) \in \Omega^h,$$
$$\mathcal{A}_{2,ij}\widetilde{U}_{2,ij} + f_{1,ij}(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = b_2 E_0 > 0, \quad (i,j) \in \Omega^h,$$

that is, $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1, 2$, from (3.35) satisfy (3.21b). From (3.32) and (3.35), we have

$$\mathcal{A}_{\alpha,ij}\widehat{U}_{\alpha,ij} + f_{\alpha,ij}(\widehat{U}_{\alpha,ij}, \widehat{U}_{\alpha',ij}) = 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2,$$

that is, $\widehat{U}_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1, 2$, satisfy (3.21b). From (3.36), it follows (3.21c) is satisfied. Thus, we prove that $(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij})$ and $(\widehat{U}_{1,ij}, \widehat{U}_{2,ij})$, $(i,j) \in \overline{\Omega}^h$ from (3.35) are ordered upper and lower solutions (3.21) to the nonlinear difference scheme (3.17). From (3.32) and (3.35), in the sector $\langle \widehat{U}, \widetilde{U} \rangle$, we have

$$
\begin{aligned}
&\frac{\partial f_{1,ij}}{\partial u_1}(U_{1,ij}, U_{2,ij}) = a_1(E_0 - U_{2,ij}) \leq a_1 E_0, \quad (i,j) \in \overline{\Omega}^h, \\
&\frac{\partial f_{2,ij}}{\partial u_2}(U_{1,ij}, U_{2,ij}) = a_2 U_{1,ij} + b_2 \leq a_2 V_{ij} + b_2, \quad (i,j) \in \overline{\Omega}^h, \\
&-\frac{\partial f_{1,ij}}{\partial u_2}(U_{1,ij}, U_{2,ij}) = a_1 U_{1,ij} + b_1 \geq 0, \quad (i,j) \in \overline{\Omega}^h, \\
&-\frac{\partial f_{2,ij}}{\partial u_1}(U_{1,ij}, U_{2,ij}) = a_2(E_0 - U_{2,ij}) \geq 0, \quad (i,j) \in \overline{\Omega}^h.
\end{aligned}
$$

Thus, the assumptions in (3.22) are satisfied with

$$
c_{1,ij} = a_1 E_0, \quad c_{2,ij} = a_2 V_{ij} + b_2, \quad (i,j) \in \overline{\Omega}^h.
$$

From here, we conclude that f_α, $\alpha = 1,2$, from (3.32) satisfy (3.22) and quasi-monotone nondecreasing property (3.23).

3.2.2 Quasi-monotone nonincreasing case

Two vector mesh functions $\widetilde{U}_{ij} = (\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij})$, $\widehat{U}_{ij} = (\widehat{U}_{1,ij}, \widehat{U}_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, are called ordered upper and lower solutions of (3.17), if they satisfy the inequalities

$$
\widehat{U}_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \tag{3.37a}
$$

$$
\mathcal{A}_{\alpha,ij} \widehat{U}_{\alpha,ij} + f_{\alpha,ij}(\widehat{U}_{\alpha,ij}, \widetilde{U}_{\alpha',ij}) \leq 0 \leq \mathcal{A}_{\alpha,ij} \widetilde{U}_{\alpha,ij} + f_{\alpha,ij}(\widetilde{U}_{\alpha,ij}, \widehat{U}_{\alpha',ij}), \quad (i,j) \in \Omega^h, \tag{3.37b}
$$

$$
\widehat{U}_{\alpha,ij} \leq g_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij}, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \tag{3.37c}
$$

where notation (3.16) is in use.

For a given pair of ordered upper and lower solutions $\widetilde{U}_{ij}$ and $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$, we define the sector

$$
\langle \widehat{U}, \widetilde{U} \rangle = \left\{ U_{ij} : \widehat{U}_{ij} \leq U_{ij} \leq \widetilde{U}_{ij}, \quad (i,j) \in \overline{\Omega}^h \right\}.
$$

In the sector $\langle \widehat{U}, \widetilde{U} \rangle$, we assume that the functions $f_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, in (3.17), satisfy the constraints

$$\frac{\partial f_{\alpha,ij}(U_{ij})}{\partial u_\alpha} \le c_{\alpha,ij}, \quad U \in \langle \widehat{U}, \widetilde{U} \rangle, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \tag{3.38}$$

$$-\frac{\partial f_{\alpha,ij}(U_{ij})}{\partial u_{\alpha'}} \le 0, \quad U \in \langle \widehat{U}, \widetilde{U} \rangle, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2, \tag{3.39}$$

where $c_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are nonnegative bounded functions in $\overline{\Omega}^h$. We say that the functions $f_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are quasi-monotone nonincreasing in $\langle \widehat{U}, \widetilde{U} \rangle$ if they satisfy (3.39).

We give a monotone property of $\Gamma_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij})$, $(i,j) \in \overline{\Omega}^h$, $\alpha' \ne \alpha$, $\alpha, \alpha' = 1,2$, from (3.24) in the quasi-monotone nonincreasing case (3.39).

Lemma 3.2.5. *Suppose that* $U_{ij} = (U_{1,ij}, U_{2,ij})$ *and* $V_{ij} = (V_{1,ij}, V_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, *are vector functions in* $\langle \widehat{U}, \widetilde{U} \rangle$, *such that* $U_{ij} \ge V_{ij}$, $(i,j) \in \overline{\Omega}^h$. *Assume that (3.38) and (3.39) are satisfied. Then*

$$\Gamma_{\alpha,ij}(U_{\alpha,ij}, V_{\alpha',ij}) \ge \Gamma_{\alpha,ij}(V_{\alpha,ij}, U_{\alpha',ij}), \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2. \tag{3.40}$$

Proof. From (3.24), we have

$$\begin{aligned}\Gamma_{\alpha,ij}(U_{\alpha,ij}, V_{\alpha',ij}) - \Gamma_\alpha(V_{\alpha,ij}, U_{\alpha',ij}) = {} & c_{\alpha,ij}(U_{\alpha,ij} - V_{\alpha,ij}) \\ & - \left[f_{\alpha,ij}(U_{\alpha,ij}, V_{\alpha',ij}) - f_{\alpha,ij}(V_{\alpha,ij}, V_{\alpha',ij})\right] \\ & + \left[f_{\alpha,ij}(V_{\alpha,ij}, U_{\alpha',ij}) - f_{\alpha,ij}(V_{\alpha,ij}, V_{\alpha',ij})\right],\end{aligned}$$

$$(i,j) \in \overline{\Omega}^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2.$$

Using the mean-value theorem (3.20), we obtain that

$$\Gamma_{\alpha,ij}(U_{\alpha,ij}, V_{\alpha',ij}) - \Gamma_\alpha(V_{\alpha,ij}, U_{\alpha',ij}) =$$
$$\Big(c_{\alpha,ij} - \big(f_{\alpha,ij}(Q_{\alpha,ij}, V_{\alpha',ij})\big)_{u_\alpha}\Big)(U_{\alpha,ij} - V_{\alpha,ij}) + \big(f_{\alpha,ij}(V_{\alpha,ij}, Y_{\alpha',ij})\big)_{u_{\alpha'}}(U_{\alpha',ij} - V_{\alpha',ij}),$$
$$V_{\alpha,ij} \le Q_{\alpha,ij}, Y_{\alpha,ij} \le U_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2.$$

Taking into account that $U_{\alpha,ij} \ge V_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from (3.38) and (3.39), we conclude (3.40). □

3.2.2.1 Applied problems

The gas-liquid interaction model

We now consider the gas-liquid model from Section 3.2.1.1 with the reaction functions given in the original form

$$f_\alpha(u_1, u_2) = \sigma_\alpha u_1 u_2, \quad \alpha = 1, 2. \tag{3.41}$$

System (3.1) is reduced to

$$\begin{aligned} &-D_\alpha \triangle u_\alpha + f_\alpha(u_1, u_2) = 0, \quad (x, y) \in \omega, \\ &u_\alpha(x, y) = g_\alpha(x, y) \geq 0, \quad (x, y) \in \partial\omega, \quad \alpha = 1, 2. \end{aligned}$$

The nonlinear difference scheme (3.17) for the model is presented in the form

$$\begin{aligned} &\mathcal{A}_{\alpha,ij} U_{\alpha,ij} + f_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}) = 0, \quad (i, j) \in \Omega^h, \\ &U_{\alpha,ij} = g_{\alpha,ij}, \quad (i, j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2, \end{aligned} \tag{3.42}$$

where f_α, $\alpha = 1, 2$, are defined in (3.41), and

$$\mathcal{A}_{\alpha,ij} U_{\alpha,ij} = -D_\alpha(\mathcal{D}_x^2 + \mathcal{D}_y^2) U_{\alpha,ij}, \quad (i, j) \in \Omega^h, \quad \alpha = 1, 2,$$

where $\mathcal{D}_x^2$, $\mathcal{D}_y^2$ are defined in (3.18).

We introduce the linear problems

$$\mathcal{A}_{\alpha,ij} V_{\alpha,ij} = 0, \quad (i, j) \in \Omega^h, \tag{3.43}$$

$$V_{\alpha,ij} = g_{\alpha,ij}, \quad (i, j) \in \partial\Omega^h, \quad \alpha = 1, 2.$$

We show that

$$(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = (V_{1,ij}, V_{2,ij}), \quad (\widehat{U}_{1,ij}, \widehat{U}_{2,ij}) = (0, 0), \quad (i, j) \in \overline{\Omega}^h, \tag{3.44}$$

are ordered upper and lower solutions (3.37) to (3.42). Letting $W_{\alpha,ij} = \widetilde{U}_{\alpha,ij} - \widehat{U}_{\alpha,ij}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$. From (3.42) and (3.43), we have

$$\mathcal{A}_{\alpha,ij} W_{\alpha,ij} = 0, \quad (i, j) \in \Omega^h, \quad \alpha = 1, 2.$$

From here and using Lemma 3.2.2, we conclude that $W_{\alpha,ij} \geq 0$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$.

Thus, we prove (3.37a). From (3.42)–(3.44), we obtain

$$\mathcal{A}_{\alpha,ij}\widetilde{U}_{\alpha,ij} + f_{\alpha,ij}(\widetilde{U}_{\alpha,ij},\widehat{U}_{\alpha',ij}) = 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq, \alpha, \quad \alpha,\alpha' = 1,2,$$
$$\mathcal{A}_{\alpha,ij}\widehat{U}_{\alpha,ij} + f_{\alpha,ij}(\widehat{U}_{\alpha,ij},\widetilde{U}_{\alpha',ij}) = 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq, \alpha, \quad \alpha,\alpha' = 1,2.$$

Hence, we conclude (3.37b). From (3.44), it follows (3.37c). Thus, we prove that the mesh functions $\widetilde{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from (3.44) are ordered upper and lower solutions (3.37). From (3.41), in the sector $\langle \widehat{U}, \widetilde{U} \rangle$, we have

$$\frac{\partial f_{1,ij}}{\partial u_1}(U_{1,ij},U_{2,ij}) = \sigma_1 U_{2,ij} \le \sigma_1 V_{2,ij}, \quad (i,j) \in \overline{\Omega}^h,$$
$$\frac{\partial f_{2,ij}}{\partial u_2}(U_{1,ij},U_{2,ij}) = \sigma_2 U_{1,ij} \le \sigma_2 V_{1,ij}, \quad (i,j) \in \overline{\Omega}^h,$$
$$-\frac{\partial f_{1,ij}}{\partial u_2} = -\sigma_1 U_{1,ij} \le 0, \quad (i,j) \in \overline{\Omega}^h,$$
$$-\frac{\partial f_{2,ij}}{\partial u_1} = -\sigma_2 U_{2,ij} \le 0, \quad (i,j) \in \overline{\Omega}^h.$$

Thus, the assumptions in (3.38) are satisfied with

$$c_{1,ij} = \sigma_1 V_{2,ij}, \quad c_{2,ij} = \sigma_2 V_{1,ij}, \quad (i,j) \in \overline{\Omega}^h.$$

From here, we conclude that f_α, $\alpha = 1,2$, from (3.41) satisfy (3.38) and quasi-monotone nonincreasing property (3.39).

The Volterra–Lotka competition model in ecology

The coexistence of the competing species in ecology is closely related to the existence of a positive steady-state solution and the asymptotic behavior of the time-dependent solution in relation to the steady-state solution. The Volterra–Lotka competition model is governed by (3.1) with $\mathrm{L}_\alpha u_\alpha = \triangle u_\alpha$, and

$$f_\alpha(u_1,u_2) = -u_\alpha(a_\alpha - b_\alpha u_1 - d_\alpha u_2), \quad \alpha = 1,2, \tag{3.45}$$

where a_α, b_α and d_α, $\alpha = 1,2$, are positive constants. System (3.1) is reduced to

$$-D_\alpha \triangle u_\alpha + f_\alpha(u_1,u_2) = 0, \quad (x,y) \in \omega, \quad u_\alpha(x,y) = 0, \quad (x,y) \in \partial\omega, \quad \alpha = 1,2.$$

The nonlinear difference scheme (3.17) for the model is presented in the form

$$\mathcal{A}_{\alpha,ij}U_{\alpha,ij} + f_{\alpha,ij}(U_{\alpha,ij},U_{\alpha',ij}) = 0, \quad (i,j) \in \Omega^h, \tag{3.46}$$
$$U_{\alpha,ij} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2,$$

where f_α, $\alpha = 1, 2$, are defined in (3.45), and

$$\mathcal{A}_{\alpha,ij} U_{\alpha,ij} = -D_\alpha (\mathcal{D}_x^2 + \mathcal{D}_y^2) U_{\alpha,ij}, \quad (i,j) \in \Omega^h, \quad \alpha = 1, 2,$$

where $\mathcal{D}_x^2$, $\mathcal{D}_y^2$ are defined in (2.8). We now show that

$$(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = \left(\frac{a_1}{b_1}, \frac{a_2}{d_2}\right), \quad (\widehat{U}_{1,ij}, \widehat{U}_{2,ij}) = (0,0), \quad (i,j) \in \overline{\Omega}^h, \tag{3.47}$$

are ordered upper and lower solutions (3.37) to (3.46). From (3.47), it follows (3.37a). From (3.45) and (3.47), we obtain

$$\begin{aligned} &\mathcal{A}_{1,ij} \widetilde{U}_{1,ij} + f_{1,ij}(\widetilde{U}_{1,ij}, \widehat{U}_{2,ij}) = 0, \quad (i,j) \in \Omega^h, \\ &\mathcal{A}_{2,ij} \widetilde{U}_{2,ij} + f_{2,ij}(\widehat{U}_{1,ij}, \widetilde{U}_{2,ij}) = 0, \quad (i,j) \in \Omega^h. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} &\mathcal{A}_{1,ij} \widehat{U}_{1,ij} + f_{1,ij}(\widehat{U}_{1,ij}, \widetilde{U}_{2,ij}) = 0, \quad (i,j) \in \Omega^h, \\ &\mathcal{A}_{2,ij} \widetilde{U}_{2,ij} + f_{2,ij}(\widetilde{U}_{1,ij}, \widehat{U}_{2,ij}) = 0, \quad (i,j) \in \Omega^h. \end{aligned}$$

Hence, we conclude (3.37b). From (3.47), it follows (3.37c). Thus, the mesh functions $\widetilde{U}_{\alpha,ij}$ and $\widehat{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, from (3.47) are ordered upper and lower solutions (3.37). From (3.45), in the sector $\langle \widehat{U}, \widetilde{U} \rangle$, we have

$$\begin{aligned} &\frac{\partial f_{1,ij}}{\partial u_1}(U_{1,ij}, U_{2,ij}) = -a_1 + 2b_1 U_{1,ij} + d_1 U_{2,ij} \le 2a_1 + \frac{d_1 a_2}{d_2}, \quad (i,j) \in \overline{\Omega}^h, \\ &\frac{\partial f_{2,ij}}{\partial u_2}(U_{1,ij}, U_{2,ij}) = -a_2 + b_2 U_{1,ij} + 2d_2 U_{2,ij} \le a_2 + \frac{a_1 b_2}{b_1}, \quad (i,j) \in \overline{\Omega}^h, \\ &-\frac{\partial f_{1,ij}}{\partial u_2} = -d_1 U_{1,ij} \le 0, \quad -\frac{\partial f_{2,ij}}{\partial u_1} = -b_2 U_{2,ij} \le 0, \quad (i,j) \in \overline{\Omega}^h. \end{aligned}$$

Thus, assumptions (3.38) are satisfied with

$$c_{1,ij} = 2a_1 + \frac{d_1 a_2}{d_2}, \quad c_{2,ij} = a_2 + \frac{a_1 b_2}{b_1}, \quad (i,j) \in \overline{\Omega}^h.$$

From here, we conclude that f_α, $\alpha = 1, 2$, from (3.45) satisfy (3.38) and quasi-monotone nonincreasing property (3.39).

3.3 The monotone Jacobi and Gauss-Seidel methods

At interior mesh points (x_i, y_j), $(i,j) \in \Omega^h$, the difference scheme (3.17) can be written in the following form

$$d_{\alpha,ij}U_{\alpha,ij} - l_{\alpha,ij}U_{\alpha,i-1,j} - r_{\alpha,ij}U_{\alpha,i+1,j} - b_{\alpha,ij}U_{\alpha,i,j-1} - q_{\alpha,ij}U_{\alpha,i,j+1} = \quad (3.48)$$
$$-f_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}), \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2,$$
$$d_{\alpha,ij} = l_{\alpha,ij} + r_{\alpha,ij} + b_{\alpha,ij} + q_{\alpha,ij}, \quad l_{\alpha,ij}, r_{\alpha,ij}, b_{\alpha,ij}, q_{\alpha,ij} > 0, \quad (3.49)$$

where $l_{\alpha,ij}$, $r_{\alpha,ij}$, $b_{\alpha,ij}$ and $q_{\alpha,ij}$, $\alpha = 1,2$, are defined in (3.17).

3.3.1 Quasi-monotone nondecreasing case

The definition of the ordered upper $\widetilde{U}_{ij}$ and lower $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ solutions (3.21) can be written in the form

$$\widehat{U}_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \quad (3.50a)$$
$$\mathcal{K}_{\alpha,ij}(\widehat{U}_{\alpha,ij}, \widehat{U}_{\alpha',ij}) \leq 0 \leq \mathcal{K}_{\alpha,ij}(\widetilde{U}_{\alpha,ij}, \widetilde{U}_{\alpha',ij}), \quad (i,j) \in \Omega^h, \quad (3.50b)$$
$$\widehat{U}_{\alpha,ij} \leq g_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij}, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2, \quad (3.50c)$$

where $\mathcal{K}_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij})$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, are the residuals of the nonlinear difference scheme (3.48) on $U_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, and notation (3.16) is in use.

We now present the point monotone Jacobi and Gauss-Seidel methods for the difference scheme (3.48). Upper $\{\overline{U}^{(n)}_{\alpha,ij}\}$ and lower $\{\underline{U}^{(n)}_{\alpha,ij}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, sequences of solutions are calculated by the following point Jacobi and Gauss-Seidel iterative methods:

$$\mathcal{L}_{\alpha,ij}Z^{(n)}_{\alpha,ij} = -\mathcal{K}_{\alpha,ij}(U^{(n-1)}_{\alpha,ij}, U^{(n-1)}_{\alpha',ij}), \quad (i,j) \in \Omega^h, \quad n \geq 1, \quad (3.51)$$
$$Z^{(1)}_{\alpha,ij} = g_{\alpha,ij} - U^{(0)}_{\alpha,ij}, \quad Z^{(n)}_{\alpha,ij} = 0, \quad n \geq 2, \quad (i,j) \in \partial\Omega^h,$$
$$\mathcal{L}_{\alpha,ij}Z^{(n)}_{\alpha,ij} = (d_{\alpha,ij} + c_{\alpha,ij})Z^{(n)}_{\alpha,ij} - \eta\left(l_{\alpha,ij}Z^{(n)}_{\alpha,i-1,j} + b_{\alpha,ij}Z^{(n)}_{\alpha,i,j-1}\right),$$
$$Z^{(n)}_{\alpha,ij} = U^{(n)}_{\alpha,ij} - U^{(n-1)}_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h,$$
$$\mathcal{K}_{\alpha,ij}(U^{(n-1)}_{\alpha,ij}, U^{(n-1)}_{\alpha',ij}) = d_{\alpha,ij}U^{(n-1)}_{\alpha,ij} - l_{\alpha,ij}U^{(n-1)}_{\alpha,i-1,j} - r_{\alpha,ij}U^{(n-1)}_{\alpha,i+1,j}$$
$$-b_{\alpha,ij}U^{(n-1)}_{\alpha,i,j-1} - q_{\alpha,ij}U^{(n-1)}_{\alpha,i,j+1} + f_{\alpha,ij}(U^{(n-1)}_{\alpha,ij}, U^{(n-1)}_{\alpha',ij}),$$
$$\alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2,$$

where $\mathcal{K}_{\alpha,ij}(U^{(n-1)}_{\alpha,ij}, U^{(n-1)}_{\alpha',ij})$, $(i,j) \in \Omega^h$, $\alpha' \neq \alpha$, $\alpha,\alpha' = 1,2$, are the residuals of the nonlinear difference scheme (3.48) on $U^{(n-1)}_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, and notation (3.16) is in use. For $\eta = 0$ and $\eta = 1$, we have, respectively, the point Jacobi and Gauss-Seidel methods.

Remark 3.3.1. *For quasi-monotone nondecreasing functions (3.38), upper and lower solutions are independent, hence, by using (3.51), we calculate either the sequence* $\{\overline{U}^{(n)}_{1,ij}, \overline{U}^{(n)}_{2,ij}\}$*,* $(i,j) \in \overline{\Omega}^h$ *or the sequence* $\{\underline{U}^{(n)}_{1,ij}, \underline{U}^{(n)}_{2,ij}\}$*,* $(i,j) \in \overline{\Omega}^h$*.*

Theorem 3.3.2. *Let* $\widetilde{U}_{ij} = (\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij})$ *and* $\widehat{U}_{ij} = (\widehat{U}_{1,ij}, \widehat{U}_{2,ij})$*,* $(i,j) \in \overline{\Omega}^h$*, be ordered upper and lower solutions (3.50). Suppose that the functions* f_α*,* $\alpha = 1,2$*, in (3.1) satisfy (3.22) and (3.23). Then upper* $\{\overline{U}^{(n)}_{\alpha,ij}\}$ *and lower* $\{\underline{U}^{(n)}_{\alpha,ij}\}$*,* $(i,j) \in \overline{\Omega}^h$*,* $\alpha = 1,2$*, sequences generated by (3.51) with, respectively,* $\overline{U}^{(0)}_{ij} = \widetilde{U}_{ij}$ *and* $\underline{U}^{(0)}_{ij} = \widehat{U}_{ij}$*,* $(i,j) \in \overline{\Omega}^h$*, converge monotonically from above to a maximal solution* $\overline{U}_{ij}$*,* $(i,j) \in \overline{\Omega}^h$*, and from below to a minimal solution* $\underline{U}_{ij}$*,* $(i,j) \in \overline{\Omega}^h$*,*

$$\underline{U}^{(n-1)}_{\alpha,ij} \le \underline{U}^{(n)}_{\alpha,ij} \le \underline{U}_{\alpha,ij} \le \overline{U}_{\alpha,ij} \le \overline{U}^{(n)}_{\alpha,ij} \le \overline{U}^{(n-1)}_{\alpha,ij} \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \tag{3.52}$$

If $S_{ij} = (S_{1,ij}, S_{2,ij})$ *is any other solution in* $\langle \widehat{U}, \widetilde{U} \rangle$*, then*

$$\underline{U}_{ij} \le S_{ij} \le \overline{U}_{ij}, \quad (i,j) \in \overline{\Omega}^h. \tag{3.53}$$

Proof. Since $\overline{U}^{(0)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are initial upper solutions (3.21), it follows that $\mathcal{K}_{\alpha,ij}(\overline{U}^{(0)}_{\alpha,ij}, \overline{U}^{(0)}_{\alpha',ij}) \ge 0$, $(i,j) \in \Omega^h$, $\alpha' \ne \alpha$, $\alpha, \alpha' = 1,2$. From here and (3.51), we have

$$(d_{\alpha,ij} + c_{\alpha,ij})\overline{Z}^{(1)}_{\alpha,ij} - \eta l_{\alpha,ij}\overline{Z}^{(1)}_{\alpha,i-1,j} - \eta b_{\alpha,ij}\overline{Z}^{(1)}_{\alpha,i,j-1} \le 0, \quad (i,j) \in \Omega^h, \tag{3.54}$$
$$\overline{Z}^{(1)}_{\alpha,ij} \le 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1,2.$$

From here, $\eta = 0,1$, $b_{\alpha,i,1} \ge 0$ in (3.48) and $\overline{Z}^{(1)}_{\alpha,i,0} \le 0$, for $j = 1$ in (3.54), we obtain

$$\begin{aligned} &(d_{\alpha,i,1} + c_{\alpha,i,1})\overline{Z}^{(1)}_{\alpha,i,1} - \eta l_{\alpha,i,1}\overline{Z}^{(1)}_{\alpha,i-1,1} \le 0, \quad i = 1,2,\ldots,N_x - 1, \\ &\overline{Z}^{(1)}_{\alpha,i,1} \le 0, \quad i = 0, N_x, \quad \alpha = 1,2. \end{aligned} \tag{3.55}$$

Taking into account that $\eta = 0,1$, $l_{\alpha,1,1} > 0$ in (3.48), $\overline{Z}^{(1)}_{\alpha,0,1} \le 0$, by using the maximum principle in Lemma 3.2.2, for $i = 1$ in (3.55), we have $\overline{Z}^{(1)}_{\alpha,1,1} \le 0$, $\alpha = 1,2$. From here, for $i = 2$ in (3.55), by Lemma 3.2.2, we have $\overline{Z}^{(1)}_{\alpha,2,1} \le 0, \quad \alpha = 1,2$. By induction on i, we can prove that $\overline{Z}^{(1)}_{\alpha,i,1} \le 0$, $i = 0,1,\ldots,N_x, \quad \alpha = 1,2$.

By induction on $j \ge 1$, we can prove that

$$\overline{Z}^{(1)}_{\alpha,ij} \le 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \tag{3.56}$$

Similarly, for initial lower solutions $\underline{U}^{(0)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, we can prove that

$$\underline{Z}^{(1)}_{\alpha,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \tag{3.57}$$

We now prove that $\overline{U}^{(1)}_{\alpha,ij}$ and $\underline{U}^{(1)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are ordered upper and lower solutions (3.50). Letting $W^{(n)}_{\alpha,ij} = \overline{U}^{(n)}_{\alpha,ij} - \underline{U}^{(n)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $n \geq 0$, using notation (3.24), from (3.51), we conclude that

$$\begin{aligned}
&\mathcal{L}_{\alpha,ij} W^{(1)}_{\alpha,ij} = r_{\alpha,ij} W^{(0)}_{\alpha,i+1,j} + q_{\alpha,ij} W^{(0)}_{\alpha,i,j+1} + \Gamma_{\alpha,ij}(\overline{U}^{(0)}_{\alpha,ij}, \overline{U}^{(0)}_{\alpha',ij}) - \Gamma_{\alpha,ij}(\underline{U}^{(0)}_{\alpha,ij}, \underline{U}^{(0)}_{\alpha',ij}),\\
&(i,j) \in \Omega^h, \quad W^{(1)}_{\alpha,ij} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2.
\end{aligned}$$

From here, (3.48), (3.51) and taking into account that $\overline{U}^{(0)}_{\alpha,ij} \geq \underline{U}^{(0)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, by Lemma 3.2.4, we obtain

$$\begin{aligned}
&(d_{\alpha,ij} + c_{\alpha,ij}) W^{(1)}_{\alpha,ij} - \eta l_{\alpha,ij} W^{(1)}_{\alpha,i-1,j} - \eta b_{\alpha,ij} W^{(1)}_{\alpha,i,j-1} \geq 0, \quad (i,j) \in \Omega^h, \\
&W^{(1)}_{\alpha,ij} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1,2.
\end{aligned} \tag{3.58}$$

From here and taking into account that $W^{(1)}_{\alpha,i,0} = 0$, $\alpha = 1,2$, for $j = 1$ in (3.58), we conclude that

$$\begin{aligned}
&(d_{\alpha,i,1} + c_{\alpha,i,1}) W^{(1)}_{\alpha,i,1} - \eta l_{\alpha,i,1} W^{(1)}_{\alpha,i-1,1} \geq 0, \quad i = 1,2,\ldots,N_x - 1,\\
&W^{(1)}_{\alpha,i,1} = 0, \quad i = 0, N_x, \quad \alpha = 1,2.
\end{aligned}$$

Taking into account that $W^{(1)}_{\alpha,0,1} = 0$, $\alpha = 1,2$, by Lemma 3.2.2, for $i = 1$ in (5.50), we have $W^{(1)}_{\alpha,1,1} \geq 0$, $\alpha = 1,2$. From here, $\eta = 0,1$, $l_{\alpha,2,1} > 0$, $\alpha = 1,2$, in (3.48) and using Lemma 3.2.2, for $i = 2$, we obtain that $W^{(1)}_{\alpha,2,1} \geq 0$, $\alpha = 1,2$. By induction on i, we can prove that

$$W^{(1)}_{\alpha,i,1} \geq 0, \quad i = 0,1,\ldots,N_x, \quad \alpha = 1,2.$$

By induction on $j \geq 1$, we can prove that

$$W^{(1)}_{\alpha,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \tag{3.59}$$

Thus, we prove (3.50a).

From (3.51) and using the mean-value theorem, we conclude that

$$\begin{aligned}
\mathcal{K}_{\alpha,ij}(\overline{U}^{(1)}_{\alpha,ij}, \overline{U}^{(1)}_{\alpha',ij}) \;=\; & -\Big(c_{\alpha,ij} - \Big(f_{\alpha,ij}(\overline{Q}^{(1)}_{\alpha,ij}, \overline{U}^{(0)}_{\alpha',ij})\Big)_{u_\alpha}\Big)\overline{Z}^{(1)}_{\alpha,ij} \\
& + \Big(f_{\alpha,ij}(\overline{U}^{(0)}_{\alpha,ij}, \overline{Y}^{(1)}_{\alpha',ij})\Big)_{u_{\alpha'}} \overline{Z}^{(1)}_{\alpha',ij} - \eta l_{\alpha,ij} \overline{Z}^{(1)}_{\alpha,i-1,j} - r_{\alpha,ij} \overline{Z}^{(1)}_{\alpha,i+1,j} \\
& - \eta b_{\alpha,ij} \overline{Z}^{(1)}_{\alpha,i,j-1} - q_{\alpha,ij} \overline{Z}^{(1)}_{\alpha,i,j+1}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,
\end{aligned} \tag{3.60}$$

where $\overline{U}^{(1)}_{\alpha,ij} \le \overline{Q}^{(1)}_{\alpha,ij}, \overline{Y}^{(1)}_{\alpha,ij} \le \overline{U}^{(0)}_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$. From (3.57) and (3.59), we conclude that $\Big(f_{\alpha,ij}(\overline{Q}^{(1)}_{\alpha,ij}, \overline{U}^{(0)}_{\alpha',ij})\Big)_{u_\alpha}$ and $\Big(f_{\alpha,ij}(\overline{U}^{(0)}_{\alpha,ij}, \overline{Y}^{(1)}_{\alpha',ij})\Big)_{u_{\alpha'}}$ satisfy (3.22) and (3.23). From (3.22), (3.23), (3.48), (3.56) and (3.60), it follows that

$$\mathcal{K}_{\alpha,ij}(\overline{U}^{(1)}_{\alpha,ij}, \overline{U}^{(1)}_{\alpha',ij}) \ge 0, \quad (i,j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2.$$

Thus, $\overline{U}^{(1)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, satisfy (3.50b). By a similar manner, we can prove that

$$\mathcal{K}_{\alpha,ij}(\underline{U}^{(1)}_{\alpha,ij}, \underline{U}^{(1)}_{\alpha',ij}) \le 0, \quad (i,j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2,$$

that is, $\underline{U}^{(1)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, satisfy (3.50b). From the boundary conditions on $\partial\Omega^h$ in (3.51), it follows that $\overline{U}^{(1)}_{\alpha,ij}$ and $\underline{U}^{(1)}_{\alpha,ij}$, $(i,j) \in \partial\Omega^h$, $\alpha = 1,2$ satisfy (3.50c).

Thus, we prove that $\overline{U}^{(1)}_{\alpha,ij}$ and $\underline{U}^{(1)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are ordered upper and lower solutions (3.50).

By induction on n, we can prove that $\{\overline{U}^{(n)}_{\alpha,ij}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are monotone decreasing sequences of upper solutions and $\{\underline{U}^{(n)}_{\alpha,ij}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are monotone increasing sequences of lower solutions which satisfy (3.52). From (3.52), it follows that $\lim \overline{U}^{(n)}_{\alpha,ij} = \overline{U}_{\alpha,ij}$ and $\lim \underline{U}^{(n)}_{\alpha,ij} = \underline{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, as $n \to \infty$ exist and

$$\lim_{n\to\infty} \overline{Z}^{(n)}_{\alpha,ij} = 0, \quad \lim_{n\to\infty} \underline{Z}^{(n)}_{\alpha,ij} = 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \tag{3.61}$$

Similar to (3.60), we have

$$\begin{aligned} \mathcal{K}_{\alpha,ij}(\overline{U}^{(n)}_{\alpha,ij}, \overline{U}^{(n)}_{\alpha',ij}) = &-\Big(c_{\alpha,ij} - \Big(f_{\alpha,ij}(\overline{Q}^{(n)}_{\alpha,ij}, \overline{U}^{(n-1)}_{\alpha',ij})\Big)_{u_\alpha}\Big)\overline{Z}^{(n)}_{\alpha,ij} \\ &+ \Big(f_{\alpha,ij}(\overline{U}^{(n-1)}_{\alpha,ij}, \overline{Y}^{(n)}_{\alpha',ij})\Big)_{u_{\alpha'}} \overline{Z}^{(n)}_{\alpha',ij} - \eta l_{\alpha,ij}\overline{Z}^{(n)}_{\alpha,i-1,j} \\ &- r_{\alpha,ij}\overline{Z}^{(n)}_{\alpha,i+1,j} - \eta b_{\alpha,ij}\overline{Z}^{(n)}_{\alpha,i,j-1} - q_{\alpha,ij}\overline{Z}^{(n)}_{\alpha,i,j+1}, \\ (i,j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2,& \end{aligned} \tag{3.62}$$

where

$$\overline{U}^{(n)}_{\alpha,ij} \le \overline{Q}^{(n)}_{\alpha,ij}, \overline{Y}^{(n)}_{\alpha,ij} \le \overline{U}^{(n-1)}_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2.$$

By taking limit of both sides, we conclude that

$$\mathcal{K}_{\alpha,ij}(\overline{U}_{\alpha,ij}, \overline{U}_{\alpha',ij}) = 0, \quad (i,j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2.$$

Thus, $\overline{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are maximal solutions to the nonlinear difference scheme (3.17). By a similar argument, we can prove that

$$\mathcal{K}_{\alpha,ij}(\underline{U}_{\alpha,ij}, \underline{U}_{\alpha',ij}) = 0, \quad (i,j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2,$$

that is, $\underline{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are minimal solutions to the nonlinear difference scheme (3.17).

Now, we prove (3.53). We assume that $S_{ij} = (S_{1,ij}, S_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, is another solution in $\langle \widehat{U}, \widetilde{U} \rangle$. We consider the sector $\langle S, \widetilde{U} \rangle$, which means that we treat S_{ij}, $(i,j) \in \overline{\Omega}^h$, as a lower solution. Since $\{\underline{S}^{(n)}_{\alpha,ij}\} = \{S_{\alpha,ij}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, is a constant sequence for all n, then from (3.52), we conclude that $S_{\alpha,ij} \leq \overline{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$.

Now, we consider the sector $\langle \widehat{U}, S \rangle$, which means that we treat S_{ij}, $(i,j) \in \overline{\Omega}^h$, as an upper solution. Similarly, since $\{\overline{S}^{(n)}_{\alpha,ij}\} = \{S_{\alpha,ij}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, is a constant sequence for all n, then from (3.52), we conclude that $\underline{U}_{\alpha,ij} \leq S_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. Thus, we prove (3.53). □

3.3.2 Quasi-monotone nonincreasing case

The definition of the ordered upper $\widetilde{U}_{ij}$ and lower $\widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$ solutions (3.37) can be written in the form

$$\widehat{U}_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \tag{3.63a}$$

$$\mathcal{K}_{\alpha,ij}(\widehat{U}_{\alpha,ij}, \widetilde{U}_{\alpha',ij}) \leq 0 \leq \mathcal{K}_{\alpha,ij}(\widetilde{U}_{\alpha,ij}, \widehat{U}_{\alpha',ij}), \quad (i,j) \in \Omega^h, \tag{3.63b}$$

$$\widehat{U}_{\alpha,ij} \leq g_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij}, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \tag{3.63c}$$

where $\mathcal{K}_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij})$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, are the residuals of the nonlinear difference scheme (3.48) on $U_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, and notation (3.16) is in use.

In the case of quasi-monotone nonincreasing reaction functions, for solving the nonlinear difference scheme (3.48), we introduce the point Jacobi and Gauss-Seidel iterative methods in the forms

$$\begin{aligned}
&\mathcal{L}_{\alpha,ij}\overline{Z}^{(n)}_{\alpha,ij} = -\mathcal{K}_{\alpha,ij}(\overline{U}^{(n-1)}_{\alpha,ij}, \underline{U}^{(n-1)}_{\alpha',ij}), \quad (i,j) \in \Omega^h, \\
&\mathcal{L}_{\alpha,ij}\underline{Z}^{(n)}_{\alpha,ij} = -\mathcal{K}_{\alpha,ij}(\underline{U}^{(n-1)}_{\alpha,ij}, \overline{U}^{(n-1)}_{\alpha',ij}), \quad (i,j) \in \Omega^h, \\
&\mathcal{L}_{\alpha,ij}Z^{(n)}_{\alpha,ij} = (d_{\alpha,ij} + c_{\alpha,ij})Z^{(n)}_{\alpha,ij} - \eta\left(l_{\alpha,ij}Z^{(n)}_{\alpha,i-1,j} + b_{\alpha,ij}Z^{(n)}_{\alpha,i,j-1}\right), \\
&Z^{(n)}_{\alpha,ij} = U^{(n)}_{\alpha,ij} - U^{(n-1)}_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \\
&Z^{(n)}_{\alpha,ij} = \begin{cases} g_{\alpha,ij} - U^{(0)}_{\alpha,ij}, & n = 1, \\ 0, & n \geq 2, \end{cases} \qquad (i,j) \in \partial\Omega^h, \\
&\mathcal{K}_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}) = d_{\alpha,ij}U_{\alpha,ij} - l_{\alpha,ij}U_{\alpha,i-1,j} - r_{\alpha,ij}U_{\alpha,i+1,j} - b_{\alpha,ij}U_{\alpha,i,j-1} \\
&\qquad\qquad - q_{\alpha,ij}U_{\alpha,i,j+1} + f_{\alpha,ij}(U_{\alpha,ij}, U_{\alpha',ij}), \\
&\alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,
\end{aligned} \tag{3.64}$$

where $\mathcal{K}_{\alpha,ij}(U^{(n-1)}_{\alpha,ij}, U^{(n-1)}_{\alpha',ij})$, $(i,j) \in \Omega^h$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1, 2$, are the residuals of the difference equations (3.48) on $U^{(n-1)}_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1, 2$, and notation (3.16) is in use. For $\eta = 0$ and $\eta = 1$, we have, respectively, the point Jacobi and point Gauss-Seidel methods.

Remark 3.3.3. *For quasi-monotone nonincreasing functions, upper and lower solutions are coupled, hence, by using (3.64), we calculate either the sequence* $\{\overline{U}^{(n)}_{1,ij}, \underline{U}^{(n)}_{2,ij}\}$, $(i,j) \in \overline{\Omega}^h$ *or the sequence* $\{\underline{U}^{(n)}_{1,ij}, \overline{U}^{(n)}_{2,ij}\}$, $(i,j) \in \overline{\Omega}^h$.

Theorem 3.3.4. *Let the pair* $\widetilde{U}_{ij} = (\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij})$ *and* $\widehat{U}_{ij} = (\widehat{U}_{1,ij}, \widehat{U}_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, *be ordered upper and lower solutions (3.63). Assume that the functions* f_α, $\alpha = 1, 2$, *in (3.1) satisfy equations (3.38) and (3.39). Then the sequences* $\{\overline{U}^{(n)}_{1,ij}, \underline{U}^{(n)}_{2,ij}\}$, $\{\underline{U}^{(n)}_{1,ij}, \overline{U}^{(n)}_{2,ij}\}$, $(i,j) \in \overline{\Omega}^h$, *generated by (3.64) with* $\{\overline{U}^{(0)}_{1,ij}, \underline{U}^{(0)}_{2,ij}\} = \{\widetilde{U}_{1,ij}, \widehat{U}_{2,ij}\}$ *and* $\{\underline{U}^{(0)}_{1,ij}, \overline{U}^{(0)}_{2,ij}\} = \{\widehat{U}_{1,ij}, \widetilde{U}_{2,ij}\}$, $(i,j) \in \overline{\Omega}^h$, *converge monotonically to their respective solutions* $(\overline{U}_{1,ij}, \underline{U}_{2,ij})$ *and* $(\underline{U}_{1,ij}, \overline{U}_{2,ij})$, *such that*

$$\underline{U}^{(n-1)}_{\alpha,ij} \leq \underline{U}^{(n)}_{\alpha,ij} \leq \underline{U}_{\alpha,i} \leq \overline{U}_{\alpha,ij} \leq \overline{U}^{(n)}_{\alpha,ij} \leq \overline{U}^{(n-1)}_{\alpha,ij}, \ (i,j) \in \overline{\Omega}^h, \quad \alpha = 1, 2, \tag{3.65}$$

If $S_{ij} = (S_{1,ij}, S_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, *is any other solution in* $\langle \widehat{U}, \widetilde{U} \rangle$, *then*

$$\underline{U}_{ij} \leq S_{ij} \leq \overline{U}_{ij}, \quad (i,j) \in \overline{\Omega}^h. \tag{3.66}$$

Proof. In the case of the sequence $\{\overline{U}^{(n)}_{1,ij}, \underline{U}^{(n)}_{2,ij}\}$, $(i,j) \in \overline{\Omega}^h$, $(\overline{U}^{(0)}_{1,ij}, \underline{U}^{(0)}_{2,ij}) = (\widetilde{U}_{1,ij}, \widehat{U}_{2,ij})$, $(i,j) \in \overline{\Omega}^h$ are initial upper and lower solutions (3.63). Hence, it follows that the residuals $\mathcal{K}_{1,ij}(\overline{U}^{(n-1)}_{1,ij}, \underline{U}^{(n-1)}_{2,ij}) \geq 0$, $\mathcal{K}_{2,ij}(\overline{U}^{(n-1)}_{1,ij}, \underline{U}^{(n-1)}_{2,ij}) \leq 0$, $(i,j) \in \Omega^h$, from (3.64), we have

$$\begin{aligned}
&(d_{1,ij} + c_{1,ij})\overline{Z}^{(1)}_{1,ij} - \eta l_{1,ij}\overline{Z}^{(1)}_{1,i-1,j} - \eta b_{1,ij}\overline{Z}^{(1)}_{1,i,j-1} \leq 0, \quad (i,j) \in \Omega^h, \\
&(d_{2,ij} + c_{2,ij})\underline{Z}^{(1)}_{2,ij} - \eta l_{2,ij}\underline{Z}^{(1)}_{2,i-1,j} - \eta b_{2,ij}\underline{Z}^{(1)}_{2,i,j-1} \geq 0, \quad (i,j) \in \Omega^h, \\
&\overline{Z}^{(1)}_{1,ij} \leq 0, \quad \underline{Z}^{(1)}_{2,ij} \geq 0, \quad (i,j) \in \partial\Omega^h.
\end{aligned} \tag{3.67}$$

For here, $\eta = 0, 1$, $b_{\alpha,i,1} > 0$ in (3.49) and $\overline{Z}^{(1)}_{1,i,0} \leq 0$, $\underline{Z}^{(1)}_{2,i,0} \geq 0$, $i = 0, N_x$, for $j = 1$ in (3.67), we obtain

$$\begin{aligned}
&(d_{1,ij} + c_{1,ij})\overline{Z}^{(1)}_{1,i,1} - \eta l_{1,i,1}\overline{Z}^{(1)}_{1,i-1,1} \leq 0, \quad (i,j) \in \Omega^h, \\
&(d_{2,ij} + c_{2,ij})\underline{Z}^{(1)}_{2,i,1} - \eta l_{2,i,1}\underline{Z}^{(1)}_{2,i-1,1} \geq 0, \quad (i,j) \in \Omega^h, \\
&i = 1, 2, \ldots, N_x - 1, \quad \overline{Z}^{(1)}_{1,i,1} \leq 0, \quad \underline{Z}^{(1)}_{2,i,1} \geq 0, \quad i = 0, N_x.
\end{aligned} \tag{3.68}$$

Taking into account that $\eta = 0, 1$, $l_{\alpha,i,1} > 0$ in (3.49), $\overline{Z}^{(1)}_{1,0,1} \leq 0$, $\underline{Z}^{(1)}_{2,0,1} \geq 0$, and using the maximum principle in Lemma 3.2.2, for $i = 1$ in (3.68), we have $\overline{Z}^{(1)}_{1,1,1} \leq 0$, $\underline{Z}^{(1)}_{2,1,1} \geq 0$. From here, by using Lemma 3.2.2, for $i = 2$ in (3.68), we have $\overline{Z}^{(1)}_{1,2,1} \leq 0$, $\underline{Z}^{(1)}_{2,2,1} \geq 0$. By induction on i and j, we can prove that

$$\overline{Z}^{(1)}_{1,ij} \leq 0, \quad \underline{Z}^{(1)}_{2,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h. \tag{3.69}$$

Similarly, for the sequence $\{\underline{U}^{(1)}_{1,ij}, \overline{U}^{(1)}_{2,ij}\}$, $(i,j) \in \overline{\Omega}^h$, from (3.64), we conclude that

$$\underline{Z}^{(1)}_{1,ij} \geq 0, \quad \overline{Z}^{(1)}_{2,ij} \leq 0, \quad (i,j) \in \overline{\Omega}^h. \tag{3.70}$$

We now prove that $\overline{U}^{(1)}_{\alpha,ij}$ and $\underline{U}^{(1)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are ordered upper and lower solutions (3.63). Letting $W^{(1)}_{\alpha,ij} = \overline{U}^{(1)}_{\alpha,ij} - \underline{U}^{(1)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, using notation (3.24), from (3.64), we conclude that

$$\begin{aligned} &\mathcal{L}_{\alpha,ij} W^{(1)}_{\alpha,ij} = r_{\alpha,ij} W^{(0)}_{\alpha,i+1,j} + q_{\alpha,ij} W^{(0)}_{\alpha,i,j+1} + \Gamma_{\alpha,ij}(\overline{U}^{(0)}_{\alpha,ij}, \underline{U}^{(0)}_{\alpha',ij}) - \Gamma_{\alpha,ij}(\underline{U}^{(0)}_{\alpha,ij}, \overline{U}^{(0)}_{\alpha',ij}), \\ &(i,j) \in \Omega^h, \quad W^{(1)}_{\alpha,ij} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2. \end{aligned}$$

From here, (3.49) and taking into account that $\overline{U}^{(0)}_{\alpha,ij} \geq \underline{U}^{(0)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, by using Lemma 3.2.5, we obtain

$$\begin{aligned} &(d_{\alpha,ij} + c_{\alpha,ij}) W^{(1)}_{\alpha,ij} - \eta l_{\alpha,ij} W^{(1)}_{\alpha,i-1,j} - \eta b_{\alpha,ij} W^{(1)}_{\alpha,i,j-1} \geq 0, \quad (i,j) \in \Omega^h, \\ &W^{(1)}_{\alpha,ij} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1,2. \end{aligned} \tag{3.71}$$

Since $W^{(1)}_{\alpha,i,0} = 0$, $\alpha = 1,2$, for $j = 1$ in (3.71), it follows that

$$\begin{aligned} &(d_{\alpha,i,1} + c_{\alpha,i,1}) W^{(1)}_{\alpha,i,1} - \eta l_{\alpha,i,1} W^{(1)}_{\alpha,i-1,1} \geq 0, \quad i = 1,2,\ldots,N_x - 1, \\ &W^{(1)}_{\alpha,i,1} = 0, \quad i = 0, N_x, \quad \alpha = 1,2. \end{aligned} \tag{3.72}$$

From here, $W^{(1)}_{\alpha,0,1} = 0$, $\alpha = 1,2$, by using Lemma 3.2.2, for $i = 1$ in (3.72), we have $W^{(1)}_{\alpha,1,1} \geq 0$, $\alpha = 1,2$. From here, $\eta = 0,1$, $l_{\alpha,2,1} > 0$, $\alpha = 1,2$, in (3.49) and using Lemma 3.2.2, for $i = 2$ in (3.2.2), we obtain $W^{(1)}_{\alpha,2,1} \geq 0$, $\alpha = 1,2$. By induction on i and j, we can prove

$$W^{(1)}_{\alpha,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \tag{3.73}$$

Thus, we prove (3.63a).

From (3.64) and using the mean-value theorem (3.24), we conclude that

$$\begin{aligned}\mathcal{K}_{1,ij}(\overline{U}^{(1)}_{1,ij},\underline{U}^{(1)}_{2,ij}) \quad &= \quad -\left(c_{1,ij}-\left(f_{1,ij}(\overline{Q}^{(1)}_{1,ij},\underline{U}^{(1)}_{2,ij})\right)_{u_1}\right)\overline{Z}^{(1)}_{1,ij} \\ &\quad +\left(f_{1,ij}(\overline{U}^{(0)}_{1,ij},\underline{Q}^{(1)}_{2,ij})\right)_{u_2}\underline{Z}^{(1)}_{2,ij}-\eta l_{1,ij}\overline{Z}^{(1)}_{1,i-1,j}-r_{1,ij}\overline{Z}^{(1)}_{1,i+1,j} \\ &\quad -\eta b_{1,ij}\overline{Z}^{(1)}_{1,i,j-1}-q_{1,ij}\overline{Z}^{(1)}_{1,i,j+1}, \quad (i,j)\in\Omega^h,\end{aligned} \tag{3.74}$$

where

$$\overline{U}^{(1)}_{1,ij}\le\overline{Q}^{(1)}_{1,ij}\le\overline{U}^{(0)}_{1,ij}, \quad \underline{U}^{(0)}_{2,ij}\le\underline{Q}^{(1)}_{2,ij}\le\underline{U}^{(1)}_{2,ij}, \quad (i,j)\in\overline{\Omega}^h.$$

From (3.69), (3.70) and (3.73), it follows that the partial derivatives $\left(f_1(\overline{Q}^{(1)}_{1,ij},\underline{U}^{(1)}_{2,ij})\right)_{u_1}$ and $\left(f_1(\overline{U}^{(0)}_{1,ij},\underline{Q}^{(1)}_{2,ij})\right)_{u_2}$ satisfy (3.38) and (3.39). From here, (3.49), (3.69), (3.70) and (3.74), we obtain that

$$\mathcal{K}_{1,ij}(\overline{U}^{(1)}_{1,ij},\underline{U}^{(1)}_{2,ij})\ge 0, \quad (i,j)\in\Omega^h. \tag{3.75}$$

Similarly, we can prove that

$$\mathcal{K}_{2,ij}(\overline{U}^{(1)}_{1,ij},\underline{U}^{(1)}_{2,ij})\le 0, \quad (i,j)\in\Omega^h. \tag{3.76}$$

By a similar manner, for the sequence $\{\underline{U}^{(n)}_{1,ij},\overline{U}^{(n)}_{2,ij}\}$, $(i,j)\in\overline{\Omega}^h$, from (3.64), we can prove that

$$\mathcal{K}_{1,ij}(\underline{U}^{(1)}_{1,ij},\overline{U}^{(1)}_{2,ij})\le 0, \quad \mathcal{K}_{2,ij}(\underline{U}^{(1)}_{1,ij},\overline{U}^{(1)}_{2,ij})\ge 0, \quad (i,j)\in\Omega^h. \tag{3.77}$$

From the boundary conditions on $\partial\Omega^h$ in (3.64), it follows that $\overline{U}^{(1)}_{\alpha,ij}$, $\underline{U}^{(1)}_{\alpha,ij}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, satisfy (3.63c). Thus, from here, (3.73), (3.75)–(3.77), we conclude that $\overline{U}^{(1)}_{\alpha,ij}$ and $\underline{U}^{(1)}_{\alpha,ij}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, are ordered upper and lower solutions (3.63).

By induction on n, we can prove that $\{\overline{U}^{(n)}_{\alpha,ij}\}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, are monotone decreasing sequence of upper solutions and $\{\underline{U}^{(n)}_{\alpha,ij}\}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, are monotone increasing sequence of lower solutions which satisfy (3.65). From (3.65), it follows that $\lim\overline{U}^{(n)}_{\alpha,ij}=\overline{U}_{\alpha,ij}$ and $\lim\underline{U}^{(n)}_{\alpha,ij}=\underline{U}_{\alpha,ij}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, as $n\to\infty$ exist and

$$\lim_{n\to\infty}\overline{Z}^{(n)}_{\alpha,ij}=0, \quad \lim_{n\to\infty}\underline{Z}^{(n)}_{\alpha,ij}=0, \quad (i,j)\in\overline{\Omega}^h, \quad \alpha=1,2. \tag{3.78}$$

Similar to (3.74), for any $n \geq 1$, we have

$$\begin{aligned}\mathcal{K}_{1,ij}(\overline{U}^{(n)}_{1,ij}, \underline{U}^{(n)}_{2,ij}) = &-\left(c_{1,ij} - f_{1,ij}(\overline{Q}^{(n)}_{1,ij}, \underline{U}^{(n)}_{2,ij})_{u_1}\right) \overline{Z}^{(n)}_{1,ij} \\ &+ f_{1,ij}(\overline{U}^{(n-1)}_{1,ij}, \underline{Q}^{(n)}_{2,ij})_{u_2} \underline{Z}^{(n)}_{2,ij} - \eta l_{1,ij} \overline{Z}^{(n)}_{1,i-1,j} \\ &- r_{1,ij} \overline{Z}^{(n)}_{1,i+1,j} - \eta b_{1,ij} \overline{Z}^{(n)}_{1,i,j-1} - q_{1,ij} \overline{Z}^{(n)}_{1,i,j+1}, \quad (i,j) \in \Omega^h,\end{aligned} \tag{3.79}$$

where

$$\overline{U}^{(n)}_{1,ij} \leq \overline{Q}^{(n)}_{1,ij} \leq \overline{U}^{(n-1)}_{1,ij}, \quad \underline{U}^{(n-1)}_{2,ij} \leq \underline{Q}^{(n)}_{2,ij} \leq \underline{U}^{(n)}_{2,ij}, \quad (i,j) \in \overline{\Omega}^h.$$

By taking the limit of both sides and using (3.78), we obtain that

$$\mathcal{K}_{1,ij}(\overline{U}_{1,ij}, \underline{U}_{2,ij}) = 0, \quad (i,j) \in \Omega^h. \tag{3.80}$$

Similarly, we have

$$\mathcal{K}_{2,ij}(\overline{U}_{1,ij}, \underline{U}_{2,ij}) = 0, \quad (i,j) \in \Omega^h. \tag{3.81}$$

In a similar manner, we can prove that

$$\mathcal{K}_{1,ij}(\underline{U}_{1,ij}, \overline{U}_{2,ij}) = 0, \quad \mathcal{K}_{2,ij}(\underline{U}_{1,ij}, \overline{U}_{2,ij}) = 0, \quad (i,j) \in \Omega^h. \tag{3.82}$$

Thus, from (3.80)–(3.82), we conclude that $\overline{U}_{\alpha,ij}$, $\underline{U}_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1, 2$, are, respectively, maximal and minimal solutions to the nonlinear difference scheme (3.17).

Now, we prove (3.66). We assume that $S_{ij} = (S_{1,ij}, S_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, is another solution in $\langle \widehat{U}, \widetilde{U} \rangle$. We consider the sector $\langle S, \widetilde{U} \rangle$, which means that we treat S_{ij}, $(i,j) \in \overline{\Omega}^h$, as a lower solution. Since $\{\underline{S}^{(n)}_{\alpha,ij}\} = \{S_{\alpha,ij}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, is a constant sequence for all n, then from (3.65), we conclude that $S_{\alpha,ij} \leq \overline{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$. Now, we consider the sector $\langle \widehat{U}, S \rangle$, which means that we treat S_{ij}, $(i,j) \in \overline{\Omega}^h$, as an upper solution. Similarly, since $\{\overline{S}^{(n)}_{\alpha,ij}\} = \{S_{\alpha,ij}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, is a constant sequence for all n, then from (3.65), we conclude that $\underline{U}_{\alpha,ij} \leq S_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$. Thus, we prove (3.66). □

3.4 Existence and uniqueness of solutions to the nonlinear difference problem (3.17)

We give a bound on the magnitude of the solution to the linear problem (3.19).

Lemma 3.4.1. *The following bound on the magnitude of the solution to the linear problem (3.19) with positive functions $c_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, holds*

$$\|W_\alpha\|_{\overline{\Omega}^h} \leq \max\left\{ \|g_\alpha\|_{\partial\Omega^h}, \left\| \frac{\Phi_\alpha}{c_\alpha} \right\|_{\Omega^h} \right\}, \quad \alpha = 1, 2, \tag{3.83}$$

where

$$\|g_\alpha\|_{\partial\Omega^h} = \max_{(i,j)\in\partial\Omega^h} |g_{\alpha,ij}|, \qquad \left\|\frac{\Phi_\alpha}{c_\alpha}\right\|_{\Omega^h} = \max_{(i,j)\in\Omega^h} \left|\frac{\Phi_{\alpha,ij}}{c_{\alpha,ij}}\right|.$$

The proof of the lemma is given in Lemma 1.2.1 from Chapter 1.

3.4.1 Quasi-monotone nondecreasing case

Theorem 3.4.2. *Let $\widetilde{U}_{ij} = (\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij})$ and $\widehat{U}_{ij} = (\widehat{U}_{1,ij}, \widehat{U}_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, be ordered upper and lower solutions (3.21). Suppose that the functions f_α, $\alpha = 1, 2$, in (3.1) satisfy (3.22) and (3.23). Then a solution to the nonlinear difference problem (3.17) exists.*

Proof. From (3.52), it follows that $\underline{U}_{\alpha,ij}$ and $\overline{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, are solutions to (3.17). Thus, we prove the theorem. □

Theorem 3.4.3. *Let assumptions (3.5)–(3.7) be satisfied. Then the nonlinear difference scheme (3.17) has a unique solution.*

Proof. Suppose that $U^*_{ij} = (U^*_{1,ij}, U^*_{2,ij})$ and $U^{**}_{ij} = (U^{**}_{1,ij}, U^{**}_{2,ij})$, $(i,j) \in \overline{\Omega}^h$ are two solutions to (3.17). Letting $V_{\alpha,ij} = U^*_{\alpha,ij} - U^{**}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, from (3.17), we have

$$\begin{aligned}
&\mathcal{A}_{\alpha,ij} V_{\alpha,ij} + f_{\alpha,ij}(U^*_{\alpha,ij}, U^*_{\alpha',ij}) - f_{\alpha,ij}(U^{**}_{\alpha,ij}, U^*_{\alpha',ij}) \\
&\qquad\qquad\qquad + f_{\alpha,ij}(U^{**}_{\alpha,ij}, U^*_{\alpha',ij}) - f_{\alpha,ij}(U^{**}_{\alpha,ij}, U^{**}_{\alpha',ij}) = 0, \\
&(i,j) \in \Omega^h, \quad V_{\alpha,ij} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1, 2.
\end{aligned}$$

From here and using the mean-value theorem, we obtain

$$\begin{aligned}
&\mathcal{A}_{\alpha,ij} V_{\alpha,ij} + \frac{\partial f_{\alpha,ij}(Q_{\alpha,ij}, U^*_{\alpha',ij})}{\partial u_\alpha} V_{\alpha,ij} = -\frac{\partial f_{\alpha,ij}(U^{**}_{\alpha,ij}, Y_{\alpha',ij})}{\partial u_{\alpha'}} V_{\alpha',ij}, \quad (i,j) \in \Omega, \\
&V_{\alpha,ij} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2,
\end{aligned}$$

where $Q_{\alpha,ij}$, $Y_{\alpha,ij}$ lie between $U^*_{\alpha,ij}$ and $U^{**}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$. From here, by using estimate (3.83), we conclude that

$$\|V_\alpha\|_{\overline{\Omega}^h} \le \left\| \frac{(f_\alpha(U^{**}_\alpha, Y_{\alpha'}))_{u_{\alpha'}} V_{\alpha'}}{\big(f_\alpha(Q_\alpha, U^*_{\alpha'})\big)_{u_\alpha}} \right\|_{\overline{\Omega}^h} \le \left\| \frac{(f_\alpha(U^{**}_\alpha, Y_{\alpha'}))_{u_{\alpha'}}}{\big(f_\alpha(Q_\alpha, U^*_{\alpha'})\big)_{u_\alpha}} \right\|_{\overline{\Omega}^h} \|V_{\alpha'}\|_{\overline{\Omega}^h}.$$

Then from here and (3.5)–(3.7), we obtain

$$\|V_\alpha\|_{\overline{\Omega}^h} \le \beta \|V_{\alpha'}\|_{\overline{\Omega}^h}.$$

Letting $v = \max_{\alpha=1,2} \|V_\alpha\|_{\overline{\Omega}^h}$, we have $v(1-\beta) \leq 0$. From here, (3.7) and taking into account that $v \geq 0$, we conclude that $v = 0$. Thus, we prove the theorem. □

3.4.2 Quasi-monotone nonincreasing case

Theorem 3.4.4. *Let $\widetilde{U}_{ij} = (\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij})$ and $\widehat{U}_{ij} = (\widehat{U}_{1,ij}, \widehat{U}_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, be ordered upper and lower solutions (3.37). Suppose that the functions f_α, $\alpha = 1,2$, in (3.1) satisfy (3.38) and (3.39). Then a solution to the nonlinear difference problem (3.17) exists.*

Proof. From (3.65), it follows that $\{\overline{U}_{1,ij}, \underline{U}_{2,ij}\}$ and $\{\underline{U}_{1,ij}, \overline{U}_{2,ij}\}$, $(i,j) \in \overline{\Omega}^h$ are solutions to (3.17). Thus, we prove the theorem. □

Theorem 3.4.5. *Let assumptions (3.5), (3.7) and (3.14) be satisfied. Then the nonlinear difference scheme (3.17) has a unique solution.*

The proof of the theorem repeats the proof of Theorem 3.4.3.

3.5 Convergence analysis

3.5.1 Quasi-monotone nondecreasing case

A stopping test for the point monotone iterative methods (3.51) is chosen in the form

$$\max_{\alpha=1,2} \left\| \mathcal{K}_\alpha(U_\alpha^{(n)}, U_{\alpha'}^{(n)}) \right\|_{\Omega^h} \leq \delta, \tag{3.84}$$
$$\left\| \mathcal{K}_\alpha(U_\alpha^{(n)}, U_{\alpha'}^{(n)}) \right\|_{\Omega^h} = \max_{(i,j)\in\Omega^h} \left| \mathcal{K}_{\alpha,ij}(U_{\alpha,ij}^{(n)}, U_{\alpha',ij}^{(n)}) \right|, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,$$

where $\mathcal{K}_{\alpha,ij}(U_{\alpha,ij}^{(n)}, U_{\alpha',ij}^{(n)})$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, are defined in (3.51) and δ is a prescribed accuracy.

Theorem 3.5.1. *Assume that the assumptions in Theorem 3.4.3 are satisfied. Then for the sequences $\{U_{\alpha,ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, generated by the point monotone iterative methods (3.51), (3.84), we have the estimate*

$$\max_{\alpha=1,2} \left\| U_\alpha^{(n_\delta)} - U_\alpha^* \right\|_{\overline{\Omega}^h} \leq \frac{1}{(1-\beta)\varrho}\delta, \quad \varrho = \min_{\alpha=1,2} \left\{ \min_{(x,y)\in\overline{\omega}} \underline{c}_\alpha(x,y) \right\} > 0, \tag{3.85}$$

where $U_{\alpha,ij}^*$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, is the unique solution of the nonlinear difference scheme (3.17), and n_δ is a minimal number of iterations subject to the stopping test (3.84).

Proof. From (3.17), for $\overline{U}^{(n_\delta)}_{\alpha,ij}$ and $U^*_{\alpha,ij}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, we have

$$\begin{aligned}&\mathcal{A}_{\alpha,ij}\overline{U}^{(n_\delta)}_{\alpha,ij}+f_{\alpha,ij}(\overline{U}^{(n_\delta)}_{\alpha,ij},\overline{U}^{(n_\delta)}_{\alpha',ij})=\mathcal{K}_{\alpha,ij}\left(\overline{U}^{(n_\delta)}_{\alpha,ij},\overline{U}^{(n_\delta)}_{\alpha',ij}\right),\quad (i,j)\in\Omega^h,\\ &\overline{U}^{(n_\delta)}_{\alpha,ij}=g_{\alpha,ij},\quad (i,j)\in\partial\Omega^h,\quad \alpha=1,2,\end{aligned}$$

$$\begin{aligned}&\mathcal{A}_{\alpha,ij}U^*_{\alpha,ij}+f_{\alpha,ij}(U^*_{\alpha,ij},U^*_{\alpha',ij})=0,\quad (i,j)\in\Omega^h,\\ &U^*_{\alpha,ij}=g_{\alpha,ij},\quad (i,j)\in\partial\Omega^h,\quad \alpha=1,2.\end{aligned}$$

Letting $W^{(n_\delta)}_{\alpha,ij}=\overline{U}^{(n_\delta)}_{\alpha,ij}-U^*_{\alpha,ij}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, we have

$$\begin{aligned}\mathcal{A}_{\alpha,ij}W^{(n_\delta)}_{\alpha,ij}&+f_{\alpha,ij}(\overline{U}^{(n_\delta)}_{\alpha,ij},\overline{U}^{(n_\delta)}_{\alpha',ij})-f_{\alpha,ij}(U^*_{\alpha,ij},\overline{U}^{(n_\delta)}_{\alpha',ij})+f_{\alpha,ij}(U^*_{\alpha,ij},\overline{U}^{(n_\delta)}_{\alpha',ij})\\ &-f_{\alpha,ij}(U^*_{\alpha,ij},U^*_{\alpha',ij})=\mathcal{K}_{\alpha,ij}\left(\overline{U}^{(n_\delta)}_{\alpha,ij},\overline{U}^{(n_\delta)}_{\alpha',ij}\right),\\ (i,j)\in\Omega^h,&\quad W^{(n_\delta)}_{\alpha,ij}=0,\quad (i,j)\in\partial\Omega^h,\quad \alpha=1,2.\end{aligned}$$

From here, by the mean-value theorem, we obtain

$$\begin{aligned}\mathcal{A}_{\alpha,ij}W^{(n_\delta)}_{\alpha,ij}+\left(f_{\alpha,ij}(Q^{(n_\delta)}_{\alpha,ij},\overline{U}^{(n_\delta)}_{\alpha',ij})\right)_{u_\alpha}W^{(n_\delta)}_{\alpha,ij}&=-\left(f_\alpha(U^*_{\alpha,ij},Y^{(n_\delta)}_{\alpha',ij})\right)_{u_{\alpha'}}W^{(n_\delta)}_{\alpha',ij}\\ &+\mathcal{K}_{\alpha,ij}\left(\overline{U}^{(n_\delta)}_{\alpha,ij},\overline{U}^{(n_\delta)}_{\alpha',ij}\right),\quad (i,j)\in\Omega^h,\\ W^{(n_\delta)}_{\alpha,ij}=0,\quad (i,j)\in\partial\Omega^h,\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2,&\end{aligned}$$

where

$$U^*_{\alpha,ij}\le Q^{(n_\delta)}_{\alpha,ij},Y^{(n_\delta)}_{\alpha,ij}\le\overline{U}^{(n_\delta)}_{\alpha,ij},\quad (i,j)\in\overline{\Omega}^h,\quad \alpha=1,2.$$

From here, (3.5), (3.6) and using (3.83), we conclude that

$$\|W^{(n_\delta)}_\alpha\|_{\overline{\Omega}^h}\le\left\|\frac{\mathcal{K}_\alpha\left(\overline{U}^{(n_\delta)}_\alpha,\overline{U}^{(n_\delta)}_{\alpha'}\right)}{\left(f_\alpha(Q^{(n_\delta)}_\alpha,\overline{U}^{(n_\delta)}_{\alpha'})\right)_{u_\alpha}}\right\|_{\Omega^h}+\left\|\frac{\left(f_\alpha(U^*_\alpha,Y^{(n_\delta)}_{\alpha'})\right)_{u_{\alpha'}}}{\left(f_\alpha(Q^{(n_\delta)}_\alpha,\overline{U}^{(n_\delta)}_{\alpha'})\right)_{u_\alpha}}\right\|_{\Omega^h}\left\|W^{(n_\delta)}_{\alpha'}\right\|_{\Omega^h}.$$

Letting $w^{(n_\delta)}=\max_{\alpha=1,2}\left\|W^{(n_\delta)}_\alpha\right\|_{\overline{\Omega}^h}$. From here, (3.5)–(3.7), we obtain

$$w^{(n_\delta)}\le\frac{1}{\varrho}\max_{\alpha=1,2}\left\|\mathcal{K}_\alpha\left(\overline{U}^{(n_\delta)}_\alpha,\overline{U}^{(n_\delta)}_{\alpha'}\right)\right\|+\beta w^{(n_\delta)},$$

where ϱ is defined in (3.85). From here and (3.84), we prove (3.85). □

Theorem 3.5.2. *Let the assumptions in Theorem 3.4.3 be satisfied. Then for the sequences* $\{U^{(n)}_{\alpha,ij}\}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, *generated by the point monotone iterative* *methods (3.51), (3.84), the following estimate holds*

$$\max_{\alpha=1,2} \left\| U_\alpha^{(n_\delta)} - u_\alpha^* \right\|_{\overline{\Omega}^h} \leq \frac{1}{(1-\beta)\varrho} \left[\delta + \max_{\alpha=1,2} \|E_\alpha\|_{\Omega^h} \right], \tag{3.86}$$

$$\|E_\alpha\|_{\overline{\Omega}^h} = \max_{(i,j)\in\overline{\Omega}^h} |E_{\alpha,ij}|, \tag{3.87}$$

where $u_\alpha^*(x,y)$, $\alpha = 1,2$, are the exact solutions to (3.1), $E_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, are the truncation errors of the exact solutions $u_\alpha^*(x,y)$, $\alpha = 1,2$, on the nonlinear difference scheme (3.17), and n_δ is the minimal number of iterations subject to the stopping test (3.84).

Proof. We denote $E_{\alpha,ij} = u_{\alpha,ij}^* - U_{\alpha,ij}^*$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, where the mesh functions $U_{\alpha,ij}^*$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are the unique solutions of the nonlinear difference scheme (3.17). From (3.17), we obtain that

$$\begin{aligned} \mathcal{A}_{\alpha,ij}E_{\alpha,ij} + f_{\alpha,ij}(u_{\alpha,ij}^*, u_{\alpha',ij}^*) - f_{\alpha,ij}(U_{\alpha,ij}^*, u_{\alpha',ij}^*) + f_{\alpha,ij}(U_{\alpha,ij}^*, u_{\alpha',ij}^*) \\ - f_{\alpha,ij}(U_{\alpha,ij}^*, U_{\alpha',ij}^*) = E_{\alpha,ij}, \\ (i,j) \in \Omega^h, \quad E_{\alpha,ij} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1,2. \end{aligned}$$

By the mean-value theorem, we have

$$\left(\mathcal{A}_{\alpha,ij} + \frac{\partial f_{\alpha,ij}(Q_{\alpha,ij}, u_{\alpha',ij}^*)}{\partial u_\alpha} \right) E_{\alpha,ij} = -\frac{\partial f_{\alpha,ij}(U_{\alpha,ij}^*, Y_{\alpha',ij})}{\partial u_{\alpha'}} E_{\alpha',ij} + E_{\alpha,ij},$$

$$(i,j) \in \Omega^h, \quad E_{\alpha,ij} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2,$$

where $Q_{\alpha,ij}$, $Y_{\alpha,ij}$ lie between $u_{\alpha,ij}^*$ and $U_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. From (3.5) and (3.6), by using (3.83), it follows that

$$\|E_\alpha\|_{\overline{\Omega}^h} \leq \left\| \frac{q_{\alpha\alpha'}}{\underline{c}_\alpha} \right\|_{\overline{\Omega}^h} \|E_{\alpha'}\|_{\overline{\Omega}^h} + \left\| \frac{E_\alpha}{\underline{c}_\alpha} \right\|_{\overline{\Omega}^h}, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2.$$

Letting $e = \max_{\alpha=1,2} \|E_\alpha\|_{\overline{\Omega}^h}$, from (3.7), we have

$$e \leq \beta e + \max_{\alpha=1,2} \left\| \frac{E_\alpha}{\underline{c}_\alpha} \right\|_{\overline{\Omega}^h}, \quad \alpha = 1,2.$$

From here, we conclude that

$$e \leq \frac{1}{1-\beta} \max_{\alpha=1,2} \left\| \frac{E_\alpha}{\underline{c}_\alpha} \right\|_{\overline{\Omega}^h}, \quad \alpha = 1,2. \tag{3.88}$$

We estimate $\max_{\alpha=1,2}\left\|U_\alpha^{(n_\delta)}-u_\alpha^*\right\|_{\overline{\Omega}^h}$ as follows

$$\max_{\alpha=1,2}\|U_\alpha^{(n_\delta)}-U_\alpha^*+U_\alpha^*-u_\alpha^*\|_{\overline{\Omega}^h}\le\max_{\alpha=1,2}\|U_\alpha^{(n_\delta)}-U_\alpha^*\|_{\overline{\Omega}^h}+\max_{\alpha=1,2}\|U_\alpha^*-u_\alpha^*\|_{\overline{\Omega}^h}.$$

From here, (3.85) and (3.88), we prove the theorem. □

3.5.2 Quasi-monotone nonincreasing case

For the sequences $\{\overline{U}_{1,ij}^{(n)},\underline{U}_{2,ij}^{(n)}\}$ and $\{\underline{U}_{1,ij}^{(n)},\overline{U}_{2,ij}^{(n)}\}$, $(i,j)\in\overline{\Omega}^h$, generated by (3.64), we introduce the notation

$$K=\max\left\{\left\|\mathcal{K}_1\left(\overline{U}_1^{(n)},\underline{U}_2^{(n)}\right)\right\|_{(i,j)\in\Omega^h};\ \left\|\mathcal{K}_2\left(\overline{U}_1^{(n)},\underline{U}_2^{(n)}\right)\right\|_{(i,j)\in\Omega^h}\right\},\tag{3.89a}$$

for the sequence $\{\overline{U}_{1,ij}^{(n)},\underline{U}_{2,ij}^{(n)}\}$, $(i,j)\in\Omega^h$, and

$$K=\max\left\{\left\|\mathcal{K}_1\left(\underline{U}_1^{(n)},\overline{U}_2^{(n)}\right)\right\|_{(i,j)\in\Omega^h};\ \left\|\mathcal{K}_2\left(\underline{U}_1^{(n)},\overline{U}_2^{(n)}\right)\right\|_{(i,j)\in\Omega^h}\right\},\tag{3.89b}$$

for the sequence $\{\underline{U}_{1,ij}^{(n)},\overline{U}_{2,ij}^{(n)}\}$, $(i,j)\in\Omega^h$, where the residuals $\mathcal{K}_{\alpha,ij}\left(U_{\alpha,ij}^{(n)},U_{\alpha',ij}^{(n)}\right)$, $(i,j)\in\Omega^h$, $\alpha'\neq\alpha$, $\alpha,\alpha'=1,2$, are defined in (3.64), and the notation of the norm (3.83) is in use. A stopping test for the point monotone iterative methods (3.64) is chosen in the form

$$K\le\delta,\tag{3.90}$$

where $\mathcal{K}_\alpha$, $\alpha=1,2$, are defined in (3.89) and δ is a prescribed accuracy.

Theorem 3.5.3. *Assume that the assumptions in Theorem 3.4.5 are satisfied. Then for the sequences $\{\overline{U}_{1,ij}^{(n)},\underline{U}_{2,ij}^{(n)}\}$, $\{\underline{U}_{1,ij}^{(n)},\overline{U}_{2,ij}^{(n)}\}$, $(i,j)\in\overline{\Omega}^h$, generated by the point monotone iterative methods (3.64), (3.90), we have the estimates*

$$\max\left\{\|\overline{U}_1^{(n_\delta)}-U_1^*\|_{\overline{\Omega}^h};\|\underline{U}_2^{(n_\delta)}-U_2^*\|_{\overline{\Omega}^h}\right\}\le\frac{1}{(1-\beta)\varrho}\delta,\tag{3.91}$$
$$\max\left\{\|\underline{U}_1^{(n_\delta)}-U_1^*\|_{\overline{\Omega}^h};\|\overline{U}_2^{(n_\delta)}-U_2^*\|_{\overline{\Omega}^h}\right\}\le\frac{1}{(1-\beta)\varrho}\delta,$$
$$\varrho=\min_{\alpha=1,2}\left\{\min_{(x,y)\in\overline{\omega}}\underline{c}_\alpha(x,y)\right\}>0,$$

where $U_{\alpha,ij}^*$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, are the unique solutions of the nonlinear difference scheme (3.17), and n_δ is a minimal number of iterations subject to (3.90).

Proof. We consider the case of the sequence $\{\overline{U}_{1,ij}^{(n)},\underline{U}_{2,ij}^{(n)}\}$, $(i,j)\in\overline{\Omega}^h$. From (3.17), for

$\overline{U}^{(n_\delta)}_{1,ij}$, $\underline{U}^{(n_\delta)}_{2,ij}$ and $U^*_{\alpha,ij}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, we have

$$\begin{aligned}
&\mathcal{A}_{1,ij}\overline{U}^{(n_\delta)}_{1,ij}+f_{1,ij}(\overline{U}^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij})=\mathcal{K}_{1,ij}\left(\overline{U}^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij}\right),\quad (i,j)\in\Omega^h,\\
&\mathcal{A}_{2,ij}\underline{U}^{(n_\delta)}_{2,ij}+f_{2,ij}(\overline{U}^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij})=\mathcal{K}_{2,ij}\left(\overline{U}^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij}\right),\quad (i,j)\in\Omega^h,\\
&\overline{U}^{(n_\delta)}_{1,ij}=g_{1,ij},\quad \underline{U}^{(n_\delta)}_{2,ij}=g_{2,ij},\quad (i,j)\in\partial\Omega^h,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{A}_{\alpha,ij}U^*_{\alpha,ij}+f_{\alpha,ij}(U^*_{\alpha,ij},U^*_{\alpha',ij})=0,\quad (i,j)\in\Omega^h,\\
&U^*_{\alpha,ij}=g_{\alpha,ij},\quad (i,j)\in\partial\Omega^h,\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2,
\end{aligned}$$

Letting $\overline{W}^{(n_\delta)}_{1,ij}=\overline{U}^{(n_\delta)}_{1,ij}-U^*_{1,ij}$ and $\underline{W}^{(n_\delta)}_{2,ij}=U^*_{2,ij}-\underline{U}^{(n_\delta)}_{2,ij}$, $(i,j)\in\overline{\Omega}^h$, we obtain

$$\begin{aligned}
\mathcal{A}_{1,ij}\overline{W}^{(n_\delta)}_{1,ij}&+f_{1,ij}(\overline{U}^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij})-f_{1,ij}(U^*_{1,ij},\underline{U}^{(n_\delta)}_{2,ij})+f_{1,ij}(U^*_{1,ij},\underline{U}^{(n_\delta)}_{2,ij})\\
&\qquad-f_{1,ij}(U^*_{1,ij},U^*_{2,ij})=\mathcal{K}_{1,ij}\left(\overline{U}^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij}\right),\\
\mathcal{A}_{2,ij}\underline{W}^{(n_\delta)}_{2,ij}&+f_{2,ij}(U^*_{1,ij},U^*_{2,ij})-f_{2,ij}(U^*_{1,ij},\underline{U}^{(n_\delta)}_{2,ij})+f_{2,ij}(U^*_{1,ij},\underline{U}^{(n_\delta)}_{2,ij})\\
&\qquad-f_{2,ij}(\overline{U}^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij})=-\mathcal{K}_{2,ij}(\overline{U}^{(n_\delta)}_{1,ij}\underline{U}^{(n_\delta)}_{2,ij}),\\
(i,j)\in\Omega^h,&\quad \overline{U}^{(n_\delta)}_{1,ij}=g_{1,ij},\quad \underline{U}^{(n_\delta)}_{2,ij}=g_{2,ij},\quad (i,j)\in\partial\Omega^h.
\end{aligned}$$

From here, by the mean-value theorem, we obtain

$$\begin{aligned}
\mathcal{A}_{1,ij}\overline{W}^{(n_\delta)}_{1,ij}&+\left(f_{1,ij}(Q^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij})\right)_{u_1}\overline{W}^{(n_\delta)}_{1,ij}=\left(f_1(U^*_{1,ij},Y^{(n_\delta)}_{2,ij})\right)_{u_2}\underline{W}^{(n_\delta)}_{2,ij}\\
&\qquad+\mathcal{K}_{1,ij}\left(\overline{U}^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij}\right),\\
\mathcal{A}_{2,ij}\underline{W}^{(n_\delta)}_{2,ij}&+\left(f_{2,ij}(U^*_{1,ij},Y^{(n_\delta)}_{2,ij})\right)_{u_2}\underline{W}^{(n_\delta)}_{2,ij}=\left(f_2(Q^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij})\right)_{u_1}\overline{W}^{(n_\delta)}_{1,ij}\\
&\qquad-\mathcal{K}_{2,ij}\left(\overline{U}^{(n_\delta)}_{1,ij},\underline{U}^{(n_\delta)}_{2,ij}\right),\\
(i,j)\in\Omega^h,&\quad \overline{W}^{(n_\delta)}_{1,ij}=0,\quad \underline{W}^{(n_\delta)}_{2,ij}=0,\quad (i,j)\in\partial\Omega^h,
\end{aligned}$$

where

$$U^*_{1,ij}\le Q_{1,ij}\le\overline{U}^{(n_\delta)}_{1,ij},\quad \underline{U}^{(n_\delta)}_{2,ij}\le Y_{2,ij}\le U^*_{2,ij},\quad (i,j)\in\overline{\Omega}^h.$$

From here, (3.5), (3.14) and using (3.83), we obtain

$$\|\overline{W}^{(n_\delta)}_1\|_{\overline{\Omega}^h}\le\left\|\frac{\mathcal{K}_1\left(\overline{U}^{(n_\delta)}_1,\underline{U}^{(n_\delta)}_2\right)}{\left(f_1(Q^{(n_\delta)}_1,\underline{U}^{(n_\delta)}_2)\right)_{u_1}}\right\|_{\Omega^h}+\left\|\frac{\left(f_1(U^*_1,Y^{(n_\delta)}_2)\right)_{u_2}}{\left(f_1(Q^{(n_\delta)}_1,\underline{U}^{(n_\delta)}_2)\right)_{u_1}}\right\|_{\Omega^h}\left\|\underline{W}^{(n_\delta)}_2\right\|_{\Omega^h},$$

$$\|\underline{W}_2^{(n_\delta)}\|_{\overline{\Omega}^h} \leq \left\| \frac{\mathcal{K}_2\left(\overline{U}_1^{(n_\delta)}, \underline{U}_2^{(n_\delta)}\right)}{\left(f_2(U_1^*, Y_2^{(n_\delta)})\right)_{u_2}} \right\|_{\Omega^h} + \left\| \frac{\left(f_2(Q_1^{(n_\delta)}, U_2^{(n_\delta)})\right)_{u_1}}{\left(f_2(U_1^*, Y_2^{(n_\delta)})\right)_{u_2}} \right\|_{\Omega^h} \left\|\overline{W}_1^{(n_\delta)}\right\|_{\Omega^h}.$$

Letting $w^{(n_\delta)} = \max\left\{\|\overline{W}_1^{(n_\delta)}\|_{\overline{\Omega}^h}, \|\underline{W}_2^{(n_\delta)}\|_{\overline{\Omega}^h}\right\}$. From here, (3.5), (3.7) and (3.14), we obtain

$$w^{(n_\delta)} \leq \frac{1}{\varrho} \max_{\alpha=1,2} \left\|\mathcal{K}_\alpha\left(\overline{U}_1^{(n_\delta)}, \underline{U}_2^{(n_\delta)}\right)\right\| + \beta w^{(n_\delta)},$$

where ϱ is defined in (3.91). From here and (3.90), we prove (3.91).

By a similar argument, we can prove (3.91) for the sequence $\{\underline{U}_{1,ij}^{(n)}, \overline{U}_{2,ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$. □

Theorem 3.5.4. *Let the assumptions in Theorem 3.5.3 be satisfied. Then for the sequences* $\{\overline{U}_{1,ij}^{(n)}, \underline{U}_{2,ij}^{(n)}\}$ *and* $\{\underline{U}_{1,ij}^{(n)}, \overline{U}_{2,ij}^{(n)}\}$, $(i,j) \in \overline{\Omega}^h$ *generated by the point monotone iterative methods (3.64), (3.90), the following estimate holds*

$$\max\left\{\|\overline{U}_1^{(n_\delta)} - u_1^*\|_{\overline{\Omega}^h}; \|\underline{U}_2^{(n_\delta)} - u_2^*\|_{\overline{\Omega}^h}\right\} \leq \frac{1}{(1-\beta)\varrho}\left[\delta + \max_{\alpha=1,2} \|E_\alpha\|_{\Omega^h}\right], \tag{3.92}$$

$$\max\left\{\|\underline{U}_1^{(n_\delta)} - u_1^*\|_{\overline{\Omega}^h}; \|\overline{U}_2^{(n_\delta)} - u_2^*\|_{\overline{\Omega}^h}\right\} \leq \frac{1}{(1-\beta)\varrho}\left[\delta + \max_{\alpha=1,2} \|E_\alpha\|_{\Omega^h}\right],$$

$$\|E_\alpha\|_{\overline{\Omega}^h} = \max_{(i,j)\in\overline{\Omega}^h} |E_{\alpha,ij}|, \quad \alpha = 1,2,$$

where $u_\alpha^*(x,y)$, $\alpha = 1,2$, *are the exact solutions to (3.1),* $E_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, *are the truncation errors of the exact solutions* $u_\alpha^*(x,y)$, $\alpha = 1,2$, *on the nonlinear difference scheme (3.17), and* n_δ *is the minimal number of iterations subject to the stopping test (3.90).*

The proof of the theorem repeats the proof of Theorem 3.5.2.

3.6 Constructions of initial upper and lower solutions

We discuss constructions of upper and lower solutions which are used as initial iterations in the monotone iterative methods (3.51) and (3.64).

3.6.1 Quasi-monotone nondecreasing case

3.6.1.1 Bounded functions

Assume that the functions $f_\alpha(x, y, u)$ and $g_\alpha(x, y)$, $\alpha = 1, 2$, in (3.1) satisfy the following conditions

$$\begin{aligned} &-M_\alpha \leq f_\alpha(x, y, \mathbf{0}) \leq 0, \quad u_\alpha(x, y) \geq 0, \quad (x, y) \in \overline{\omega}, \\ &g_\alpha(x, y) \geq 0, \quad (x, y) \in \partial\omega, \quad \alpha = 1, 2, \end{aligned} \tag{3.93}$$

where $M_\alpha = \text{const} > 0$, $\alpha = 1, 2$, and $\mathbf{0}$ is the zero vector (0,0).

We introduce the mesh functions

$$\widehat{U}_{\alpha,ij} = 0, \quad (i, j) \in \overline{\Omega}^h, \quad \alpha = 1, 2, \tag{3.94}$$

and the mesh functions $\widetilde{U}_{\alpha,ij}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, which are solutions of the following linear problems:

$$\mathcal{A}_{\alpha,ij}\widetilde{U}_{\alpha,ij} = M_\alpha, \quad (i, j) \in \Omega^h, \quad \widetilde{U}_{\alpha,ij} = g_{\alpha,ij}, \quad (i, j) \in \partial\Omega^h, \quad \alpha = 1, 2, \tag{3.95}$$

where $\mathcal{A}_{\alpha,ij}$, $(i, j) \in \Omega^h$, $\alpha = 1, 2$, are defined in (3.17).

Lemma 3.6.1. *Assume that the assumptions in (3.93) are satisfied. Then the mesh functions from (3.94) and (3.95) are ordered lower and upper solutions (3.21).*

Proof. Letting $W_{\alpha,ij} = \widetilde{U}_{\alpha,ij} - \widehat{U}_{\alpha,ij}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, from (3.94) and (3.95), we have

$$\mathcal{A}_{\alpha,ij}W_{\alpha,ij} = M_\alpha, \quad (i, j) \in \Omega^h, \quad W_{\alpha,ij} = 0, \quad (i, j) \in \partial\Omega^h, \quad \alpha = 1, 2.$$

From here, $M_\alpha > 0$, $\alpha = 1, 2$, and using the maximum principle in Lemma 3.2.2, we conclude that

$$\widetilde{U}_{\alpha,ij} - \widehat{U}_{\alpha,ij} \geq 0, \quad (i, j) \in \overline{\Omega}^h, \quad \alpha = 1, 2.$$

Thus, we prove (3.21a). From (3.93) and (3.95), we have

$$\mathcal{A}_{\alpha,ij}\widetilde{U}_{\alpha,ij} + f_{\alpha,ij}(\widetilde{U}_{\alpha,ij}, \widetilde{U}_{\alpha',ij}) \geq 0, \quad (i, j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.$$

Thus, $\widetilde{U}_{\alpha,ij}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, satisfy (3.21b). From (3.93) and (3.94), we obtain

$$\mathcal{A}_{\alpha,ij}\widehat{U}_{\alpha,ij} + f_{\alpha,ij}(\widehat{U}_{\alpha,ij}, \widehat{U}_{\alpha',ij}) \leq 0, \quad (i, j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2,$$

that is, $\widehat{U}_{\alpha,ij}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, satisfy (3.21b). From (3.94) and (3.95), it follows that $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, satisfy (3.21c). Thus, $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$,

$(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from (3.94) and (3.95) are ordered lower and upper solutions (3.21) to the nonlinear difference scheme (3.17). □

The gas-liquid interaction model

Consider the gas-liquid interaction model which is presented in Section 3.2.1.1. Since the reaction functions $f_1(u_1,u_2) = -\sigma_1(\rho_1 - u_1)u_2$, $f_2(u_1,u_2) = \sigma_2(\rho_1 - u_1)u_2$, satisfy the assumptions in (3.93), with any positive constants M_α, $\alpha = 1,2$. Hence, by using Lemma 3.6.1, it follows that the mesh functions $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from, respectively, (3.94) and (3.95) are ordered lower and upper solutions to (3.28).

3.6.2 Constant upper and lower solutions

Assume that the functions $f_\alpha(x,y,u)$ and $g_\alpha(x,y)$, $\alpha = 1,2$, in (3.1) satisfy the conditions

$$f_\alpha(x,y,\mathbf{0}) \le 0, \quad u_\alpha(x,y) \ge 0, \quad (x,y) \in \overline{\omega}, \quad \alpha = 1,2, \tag{3.96}$$

and there exist positive constants M_1, M_2 such that

$$f_\alpha(x,y,M) \ge 0, \quad (x,y) \in \overline{\omega}, \quad 0 \le g_\alpha(x,y) \le M_\alpha, \quad (i,j) \in \partial\omega, \quad \alpha = 1,2, \tag{3.97}$$

where $M = (M_1, M_2)$. Introduce the constant mesh functions

$$\widetilde{U}_{\alpha,ij} = M_\alpha, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \tag{3.98}$$

Lemma 3.6.2. *Assume that (3.96) and (3.97) are satisfied. Then the mesh functions from (3.94) and (3.98) are ordered lower and upper solutions (3.21).*

Proof. From (3.94) and (3.98), we obtain (3.21a). From (3.97) and (3.98), we have

$$\mathcal{A}_{\alpha,ij}\widetilde{U}_{\alpha,ij} + f_{\alpha,ij}(\widetilde{U}_{\alpha,ij}, \widetilde{U}_{\alpha',ij}) \ge 0, \quad (i,j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha,\alpha' = 1,2.$$

Thus, $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, satisfy (3.21b). From (3.94) and (3.96), we obtain

$$\mathcal{A}_{\alpha,ij}\widehat{U}_{\alpha,ij} + f_{\alpha,ij}(\widehat{U}_{\alpha,ij}, \widehat{U}_{\alpha',ij}) \le 0, \quad (i,j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha,\alpha' = 1,2.$$

Hence, $\widehat{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, satisfy (3.21b). From (3.94) and (3.98), it follows that $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, satisfy (3.21c). Thus, we prove that $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from (3.94) and (3.98) are ordered lower and upper solutions (3.21) to the nonlinear difference scheme (3.17). □

The gas-liquid interaction model

Consider the gas-liquid interaction model which is presented in Section 3.2.1.1. Since the reaction functions $f_1(u_1,u_2) = -\sigma_1(\rho_1 - u_1)u_2$, $f_2(u_1,u_2) = \sigma_2(\rho_1 - u_1)u_2$, satisfy the assumptions in (3.96) and (3.97), with M_α, $\alpha = 1,2$ are given by

$$M_\alpha = \varrho_\alpha, \quad \alpha = 1,2, \quad \varrho_1 \geq \max_{(i,j)\in\partial\Omega^h} g^*_{1,ij}, \quad \varrho_2 \geq \max_{(i,j)\in\partial\Omega^h} g_{2,ij}. \tag{3.99}$$

By using Lemma 3.6.3, it follows that the mesh functions $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from, respectively, (3.94) and (3.99) are ordered lower and upper solutions to (3.28).

3.6.3 Quasi-monotone nonincreasing case

From the definition of upper and lower solutions (3.37) for quasi-monotone nonincreasing functions, it follows that lower and upper solutions are coupled. Thus, we give sufficient conditions for the existence of coupled lower and upper solutions.

3.6.3.1 Bounded functions

Assume that the functions $f_\alpha(x,y,u)$ and $g_\alpha(x,y)$, $\alpha = 1,2$, in (3.1) satisfy the following conditions

$$\begin{aligned} & -M_\alpha \leq f_\alpha(x,y,u_\alpha,0_{\alpha'}) \leq 0, \quad f_\alpha(x,y,0_\alpha,u_{\alpha'}) \leq 0, \quad u_\alpha(x,y) \geq 0, \quad (x,y) \in \overline{\omega}, \\ & g_\alpha(x,y) \geq 0, \quad (x,y) \in \partial\omega, \quad \alpha \neq \alpha', \quad \alpha = 1,2, \end{aligned} \tag{3.100}$$

where $M_\alpha = \text{const} > 0$, $\alpha = 1,2$, and 0_α means $u_\alpha(x,y) = 0$, $(x,y) \in \overline{\omega}$, $\alpha = 1,2$.

Let $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, be solutions of the linear problems (3.95) and the mesh functions $\widehat{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from (3.94).

We show that $\widetilde{U}_{\alpha,ij}$ and $\widehat{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from (3.94) and (3.95) are coupled pairs of ordered upper and lower solutions.

Lemma 3.6.3. *Assume that (3.100) is satisfied. Then the mesh functions from (3.94) and (3.95) are ordered lower and upper solutions (3.37).*

Proof. From (3.95), by using the maximum principle in Lemma 3.2.2, we conclude that $\widetilde{U}_{\alpha,ij} \geq 0$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. From here and (3.94), it follows that $\widetilde{U}_{\alpha,ij} \geq \widehat{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. Hence, we prove (3.37a). From (3.94), (3.95) and (3.100), we have

$$\mathcal{A}_{\alpha,ij}\widetilde{U}_{\alpha,ij} + f_{\alpha,ij}(\widetilde{U}_{\alpha,ij},\widehat{U}_{\alpha',ij}) \geq 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2.$$

From (3.94) and (3.100), we obtain

$$\mathcal{A}_{\alpha,ij}\widehat{U}_{\alpha,ij} + f_{\alpha,ij}(\widehat{U}_{\alpha,ij}, \widetilde{U}_{\alpha',ij}) \leq 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.$$

From here, we conclude that $\widetilde{U}_{\alpha,ij}$ and $\widehat{U}_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1, 2$, satisfy (3.37b). From (3.94) and (3.95), it follows that $\widetilde{U}_{\alpha,ij}$ and $\widehat{U}_{\alpha,ij}$, $(i,j) \in \Omega^h$, $\alpha = 1, 2$, satisfy (3.37c). Thus, $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, from (3.94) and (3.95) are ordered lower and upper solutions (3.37) to the nonlinear difference scheme (3.17). □

The gas-liquid interaction model

Consider the gas-liquid interaction model which is presented in Section 3.2.2.1. Since the reaction functions $f_\alpha(u_1, u_2) = \sigma_\alpha u_1 u_2$, $\alpha = 1, 2$, satisfy the assumptions in (3.100), with any positive constants M_α, $\alpha = 1, 2$. Hence, by using Lemma 3.6.3, it follows that the mesh functions $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, from, respectively, (3.94) and (3.95) are ordered lower and upper solutions to (3.21).

3.6.4 Constant upper and lower solutions

Assume that the functions $f_\alpha(x, y, u)$ and $g_\alpha(x, y)$, $\alpha = 1, 2$, in (3.1) satisfy the conditions

$$\begin{aligned} &f_\alpha(x, y, M_\alpha, 0_{\alpha'}) \geq 0, \quad f_\alpha(x, y, 0_\alpha, M_{\alpha'}) \leq 0, \quad u_\alpha(x, y) \geq 0, \quad (x, y) \in \overline{\omega}, \\ &0 \leq g_\alpha(x, y) \leq M_\alpha, \quad (x, y) \in \partial\omega, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2, \end{aligned} \tag{3.101}$$

where M_α, $\alpha = 1, 2$, are positive constants, 0_α means that $u_\alpha(x, y) = 0$, $(x, y) \in \overline{\omega}$, $\alpha = 1, 2$.

Lemma 3.6.4. *Assume that the assumptions in (3.101) are satisfied. Then the mesh functions from (3.94) and (3.98) are ordered lower and upper solutions (3.37).*

Proof. From (3.94) and (3.98), we obtain (3.37a). From (3.94), (3.98) and (3.101) , we have

$$\begin{aligned} &\mathcal{A}_{\alpha,ij}\widetilde{U}_{\alpha,ij} + f_{\alpha,ij}(\widetilde{U}_{\alpha,ij}, \widehat{U}_{\alpha',ij}) \geq 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2. \\ &\mathcal{A}_{\alpha,ij}\widehat{U}_{\alpha,ij} + f_{\alpha,ij}(\widehat{U}_{\alpha,ij}, \widehat{U}_{\alpha',ij}) \leq 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2. \end{aligned}$$

Hence, $\widetilde{U}_{\alpha,ij}$ and $\widehat{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, satisfy (3.37b). From (3.94) and (3.98), it follows that $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, satisfy (3.37c). Thus, we prove that $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, from (3.94) and (3.98) are ordered lower and upper solutions (3.37) to the nonlinear difference scheme (3.17). □

The Volterra-Lotka competition model

Consider the Volterra–Lotka competition model which is presented in Section 3.2.2.1. The reaction functions $f_\alpha(u_1,u_2) = -u_\alpha(a_\alpha - b_\alpha u_1 - d_\alpha u_2)$, $\alpha = 1,2$, satisfy the assumptions in (3.101), with positive constants M_α, $\alpha = 1,2$, such that

$$M_\alpha \geq \max_{(i,j)\in\partial\omega} g_{\alpha,ij}, \quad \alpha = 1,2.$$

Hence, by using Lemma 3.6.4, it follows that the mesh functions $\widehat{U}_{\alpha,ij}$ and $\widetilde{U}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from, respectively, (3.94) and (3.98) are ordered lower and upper solutions to (3.46).

3.7 Comparison of convergence rates of the point monotone Jacobi and Gauss–Seidel methods

3.7.1 Quasi-monotone nondecreasing case

The following theorem shows that the point monotone Gauss–Seidel method with $\eta = 1$ in (3.51) converges faster than the point monotone Jacobi method with $\eta = 0$ in (3.51).

Theorem 3.7.1. *Let $\widetilde{U}_{ij} = (\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij})$ and $\widehat{U}_{ij} = (\widehat{U}_{1,ij}, \widehat{U}_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, be ordered upper and lower solutions (3.17), the functions f_α, $\alpha = 1,2$, in (3.1) satisfy (3.22) and (3.23). Suppose that the sequences $\{(U^{(n)}_{\alpha,ij})_J\}$ and $\{(U^{(n)}_{\alpha,ij})_{GS}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are generated by the point monotone Jacobi method with $\eta = 0$ in (3.51) and by the point monotone Gauss–Seidel method with $\eta = 1$ in (3.51), where $(\overline{U}^{(0)}_{ij})_J = (\overline{U}^{(0)}_{ij})_{GS} = \widetilde{U}_{ij}$ and $(\underline{U}^{(0)}_{ij})_J = (\underline{U}^{(0)}_{ij})_{GS} = \widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$. Then*

$$(\underline{U}^{(n)}_{\alpha,ij})_J \leq (\underline{U}^{(n)}_{\alpha,ij})_{GS} \leq (\overline{U}^{(n)}_{\alpha,ij})_{GS} \leq (\overline{U}^{(n)}_{\alpha,ij})_J, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \tag{3.102}$$

Proof. Letting $\underline{W}^{(n)}_{\alpha,ij} = \left(\underline{U}^{(n)}_{\alpha,ij}\right)_{\mathtt{GS}} - \left(\underline{U}^{(n)}_{\alpha,ij}\right)_{\mathtt{J}}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from (3.51), we have

$$\begin{aligned}
(d_{\alpha,ij} + c_{\alpha,ij})(\underline{W}^{(n)}_{\alpha,ij}) &= c_{\alpha,ij}\underline{W}^{(n-1)}_{\alpha,ij} + \eta l_{\alpha,ij}\left((\underline{U}^{(n)}_{\alpha,i-1,j})_{\mathtt{GS}} - (\underline{U}^{(n-1)}_{\alpha,i-1,j})_{\mathtt{J}}\right) \\
&\quad + r_{\alpha,ij}\underline{W}^{(n-1)}_{\alpha,i+1,j} + \eta b_{\alpha,ij}\left((\underline{U}^{(n)}_{\alpha,i,j-1})_{\mathtt{GS}} - (\underline{U}^{(n-1)}_{\alpha,i,j-1})_{\mathtt{J}}\right) \\
&\quad + q_{\alpha,ij}\underline{W}^{(n-1)}_{\alpha,i,j+1} - f_{\alpha,ij}\left((\underline{U}^{(n-1)}_{\alpha,ij})_{\mathtt{GS}}, (\underline{U}^{(n-1)}_{\alpha',ij})_{\mathtt{GS}}\right) \\
&\quad + f_{\alpha,ij}\left((\underline{U}^{(n-1)}_{\alpha,ij})_{\mathtt{J}}, (\underline{U}^{(n-1)}_{\alpha',ij})_{\mathtt{J}}\right), \quad (i,j) \in \Omega^h, \\
\underline{W}^{(n)}_{\alpha,ij} &= 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1,2.
\end{aligned}$$

From here, $\eta = 0,1$, (3.49) and taking into account (3.24) for $\left(\underline{U}^{(n-1)}_{\alpha,ij}\right)_{\mathtt{GS}} \leq \left(\underline{U}^{(n)}_{\alpha,ij}\right)_{\mathtt{GS}}$,

$(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, we obtain

$$
\begin{aligned}
(d_{\alpha,ij} + c_{\alpha,ij})\underline{W}^{(n)}_{\alpha,ij} \geq{} & c_{\alpha,ij}\underline{W}^{(n-1)}_{\alpha,ij} + \eta l_{\alpha,ij}\underline{W}^{(n-1)}_{\alpha,i-1,j} + r_{\alpha,ij}\underline{W}^{(n-1)}_{\alpha,i+1,j} \\
& + \eta b_{\alpha,ij}\underline{W}^{(n-1)}_{\alpha,i,j-1} + q_{\alpha,ij}\underline{W}^{(n-1)}_{\alpha,i,j+1} \\
& - f_{\alpha,ij}\left((\underline{U}^{(n-1)}_{\alpha,ij})_{\mathtt{GS}}, (\underline{U}^{(n-1)}_{\alpha',ij})_{\mathtt{GS}}\right) \\
& + f_{\alpha,ij}\left((\underline{U}^{(n-1)}_{\alpha,ij})_{\mathtt{J}}, (\underline{U}^{(n-1)}_{\alpha',ij})_{\mathtt{J}}\right), \quad (i,j) \in \Omega^h, \\
\underline{W}^{(n)}_{\alpha,ij} = 0, \quad & (i,j) \in \partial\Omega^h, \quad \alpha = 1,2.
\end{aligned}
\tag{3.103}
$$

For $n = 1$ in (3.103), in view of $(\underline{U}^{(0)}_{\alpha,ij})_{\mathtt{GS}} = (\underline{U}^{(0)}_{\alpha,ij})_{\mathtt{J}}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, and using the maximum principle in Lemma 3.2.2, we conclude that

$$
\underline{W}^{(1)}_{\alpha,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \tag{3.104}
$$

Using notation (3.24), for $n = 2$ in (3.103), we have

$$
\begin{aligned}
(d_{\alpha,ij} + c_{\alpha,ij})\underline{W}^{(2)}_{\alpha,ij} \geq{} & \eta l_{\alpha,ij}\underline{W}^{(1)}_{\alpha,i-1,j} + r_{\alpha,ij}\underline{W}^{(1)}_{\alpha,i+1,j} + \eta b_{\alpha,ij}\underline{W}^{(1)}_{\alpha,i,j-1} \\
& + q_{\alpha,ij}\underline{W}^{(1)}_{\alpha,i,j+1} + \Gamma_{\alpha,ij}((\underline{U}^{(1)}_{\alpha,ij})_{\mathtt{GS}}, (\underline{U}^{(1)}_{\alpha',ij})_{\mathtt{GS}}) \\
& - \Gamma_{\alpha,ij}((\underline{U}^{(1)}_{\alpha,ij})_{\mathtt{J}}, (\underline{U}^{(1)}_{\alpha',ij})_{\mathtt{J}}), \quad (i,j) \in \Omega^h, \\
\underline{W}^{(2)}_{\alpha,ij} = 0, \quad & (i,j) \in \partial\Omega^h, \quad \alpha = 1,2,
\end{aligned}
$$

From here, $\eta = 0,1$, (3.49) and (3.104), by using Lemma 3.2.2, we obtain that

$$
\underline{W}^{(2)}_{\alpha,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2.
$$

By induction on n, we can prove that

$$
\underline{W}^{(n)}_{\alpha,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad n \geq 1.
$$

Thus, we prove (3.102) for the case of lower solutions. By the same manner, we can prove (3.102) for the case of upper solutions. □

3.7.2 Quasi-monotone nonincreasing case

In the case of quasi-monotone nonincreasing reaction functions, the following theorem shows that the point monotone Gauss–Seidel method with $\eta = 1$ in (3.64) converges faster than the block monotone Jacobi method with $\eta = 0$ in (3.64).

Theorem 3.7.2. *Let $\widetilde{U}_{ij} = (\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij})$ and $\widehat{U}_{ij} = (\widehat{U}_{1,ij}, \widehat{U}_{2,ij})$, $(i,j) \in \overline{\Omega}^h$, be ordered upper and lower solutions (3.63). Assume that functions f_α, $\alpha = 1,2$, satisfy (3.38) and*

(3.39). Suppose that the sequences $\{(\overline{U}^{(n)}_{1,ij})_P, (\underline{U}^{(n)}_{2,i})_P\}$ *and* $\{(\underline{U}^{(n)}_{1,ij})_P, (\overline{U}^{(n)}_{2,ij})_P\}$, $(i,j) \in \overline{\Omega}^h$, $P = J$ *or* $P = GS$, *are the sequences generated by the point monotone Jacobi method with* $\eta = 0$ *in (3.64) or the point monotone Gauss–Seidel method with* $\eta = 1$ *in (3.64), where* $(\overline{U}^{(0)}_{ij})_J = (\overline{U}^{(0)}_{ij})_{GS} = \widetilde{U}_{ij}$ *and* $(\underline{U}^{(0)}_{ij})_J = (\underline{U}^{(0)}_{ij})_{GS} = \widehat{U}_{ij}$, $(i,j) \in \overline{\Omega}^h$. *Then*

$$(\underline{U}^{(n)}_{\alpha,ij})_J \leq (\underline{U}^{(n)}_{\alpha,ij})_{GS} \leq (\overline{U}^{(n)}_{\alpha,ij})_{GS} \leq (\overline{U}^{(n)}_{\alpha,ij})_J, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2.$$

Proof. The proof of the theorem repeats the proof of Theorem 3.7.1, where $\Gamma_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are written in the form

$$\Gamma_{\alpha,ij}(\overline{U}^{(n)}_{\alpha,ij}, \underline{U}^{(n)}_{\alpha',ij}) = c_{\alpha,ij}\overline{U}^{(n)}_{\alpha,ij} - f_{\alpha,ij}(\overline{U}^{(n)}_{\alpha,ij}, \underline{U}^{(n)}_{\alpha',ij}),$$
$$\Gamma_{\alpha,ij}(\underline{U}^{(n)}_{\alpha,ij}, \overline{U}^{(n)}_{\alpha',ij}) = c_{\alpha,ij}\underline{U}^{(n)}_{\alpha,ij} - f_{\alpha,ij}(\underline{U}^{(n)}_{\alpha,ij}, \overline{U}^{(n)}_{\alpha',ij}),$$

and the monotone property (3.25) for $\Gamma_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, is in use. □

3.8 Numerical experiments

We present numerical experiments for numerical solutions of test problems with quasi-monotone nondecreasing or nonincreasing reaction functions f_α, $\alpha = 1,2$, in (3.1). Exact solutions for our test problems are unknown, and numerical solutions are compared to corresponding reference solutions. We investigate the numerical error and numerical order of convergence with respect to $1/N$, $N_x = N_y = N$. We define the numerical error $E(N)$ and the order of convergence $\gamma(N)$ of the numerical solution similar to the definition in ([37], p.79), in the following forms:

$$E(N) = \max_{\alpha=1,2}\left[\max_{(i,j)\in\overline{\Omega}^h}\left|U^{(n_\delta)}_{\alpha,ij} - U^{ref}_{\alpha,ij}\right|\right], \quad \gamma(N) = \log_2\left(\frac{E(N)}{E(2N)}\right), \tag{3.105}$$

where $U^{(n_\delta)}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are the approximate solutions generated by either the monotone iterative methods (3.51), (3.84) or (3.64), (3.90), and $U^{ref}_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are the reference solutions. In our tests, we choose the reference solutions with $N = 256$ and $\delta = 10^{-5}$ in (3.84) and (3.90).

3.8.1 Quasi-monotone nondecreasing case

Test 1

As the first test problem with quasi-monotone nondecreasing reaction functions (3.23), we consider the gas-liquid interaction model in 3.2.1.1, where $\mathrm{L}_\alpha u_\alpha = D_\alpha(u_{\alpha,xx}+$

$u_{\alpha,yy}$), $\alpha = 1, 2$, in (3.1). The reaction functions are given by

$$f_1(u_1, u_2) = -\sigma_1(1 - u_1)u_2, \quad f_2(u_1, u_2) = \sigma_2(1 - u_1)u_2, \tag{3.106}$$

where $u_\alpha \geq 0$, $\alpha = 1, 2$, are concentrations of, respectively, the gas and liquid, and $\sigma_\alpha = \text{const} > 0$, $\alpha = 1, 2$, are reaction rates. We choose the boundary conditions $g_1(x, y) = 0$, $g_2(x, y) = 1$ in (3.1). The pairs $(\widetilde{U}_1, \widetilde{U}_2) = (1, 1)$ and $(\widehat{U}_1, \widehat{U}_2) = (0, 0)$ are ordered upper and lower solutions. Indeed, all the assumptions in (3.93) and (3.96) with $M_\alpha = 1$, $\alpha = 1, 2$, are satisfied. From here, on $\langle \widehat{U}, \widetilde{U} \rangle$, we conclude the inequalities

$$\begin{aligned} &\frac{\partial f_{1,ij}}{\partial u_1} = \sigma_1 U_{2,ij} \leq 1, \quad -\frac{\partial f_{1,ij}}{\partial u_2} = \sigma_1(1 - U_{1,ij}) \geq 0, \quad (i, j) \in \overline{\Omega}^h, \\ &\frac{\partial f_{2,ij}}{\partial u_2} = \sigma_2(1 - U_{1,ij}) \leq 1, \quad -\frac{\partial f_{2,ij}}{\partial u_1} = \sigma_2 U_{2,ij} \geq 0, \quad (i, j) \in \overline{\Omega}^h. \end{aligned}$$

Thus, f_α, $\alpha = 1, 2$, satisfy (3.22) and (3.23) with $c_\alpha = 1$, $\alpha = 1, 2$. We calculate sequences of upper solutions generated by (3.51), (3.84) with the initial iteration $(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = (1, 1)$, $(i, j) \in \overline{\Omega}^h$. We take $D_1 = 1$, $D_2 = 0.1$, in (3.1) and $\sigma_\alpha = 1$, $\alpha = 1, 2$, in (3.106).

In Table 3.1, for different values of N ($N_x = N_y = N$), we present $E(N)$ and $\gamma(N)$ from (3.105). The data in the table indicate that the numerical solution of the nonlinear difference scheme (3.17) converges to the reference solution with second-order accuracy which confirms the theoretical error estimate for the central difference scheme. Numbers of iterations n_δ and execution times (`CPU`) are given in Table 3.2. The computer used to run our codes has Windows 10 Enterprise operating system, Intel(R) Core(TM) i5-6500 processor and 8GB installed memory (RAM). From these results, we conclude that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method, which confirms Theorem 3.7.1; the point monotone Gauss–Seidel method is approximately twice as fast as the point monotone Jacobi method. In Figure 3.1, we show the convergence of numerical solutions, obtained by the point Gauss-Seidel method with $\eta = 1$ in (3.51) and $N = 128$ to the reference solution $N_{ref} = 256$, where the dashed line represents the numerical solution and the solid blue line refers to the reference solution with respect to x and fixed value of $y = 0.5$. In the subgraph 3.1a, staring from the initial lower solution $\widehat{U} = 0$, we show the convergence of the numerical lower solution U_2 at $n_\delta = 100$ and $n_\delta = 2000$ to the reference solution. Similarly, starting from the initial upper solution $\widetilde{U} = 1$, the subgraph 3.1b shows the convergence of the numerical upper solution U_1 at $n_\delta = 300$ and $n_\delta = 6000$ to the reference solution.

Test 2

As the second test problem with quasi-monotone nondecreasing reaction functions

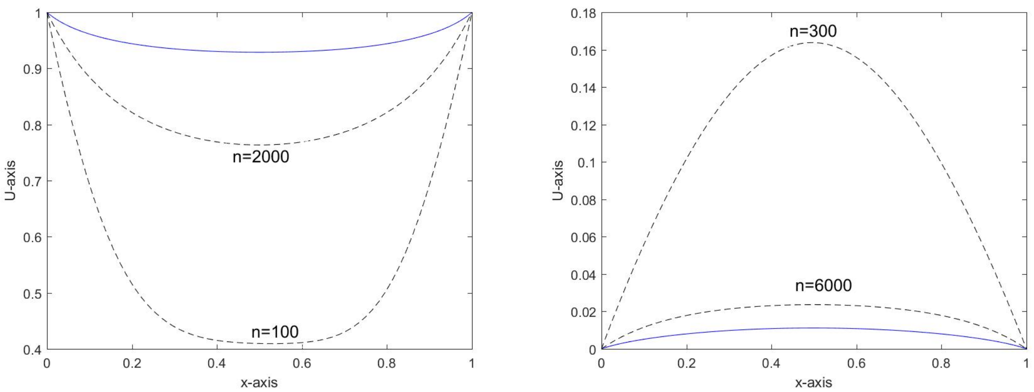
Table 3.1: Order of convergence of the nonlinear scheme (3.17) for Test 1 by using the point monotone Gauss-Seidel method.

N	8	16	32	64	128
E	7.060e-03	1.798e-03	4.466e-04	1.065e-04	2.130e-05
γ	1.97	2.01	2.07	2.32	

Table 3.2: Numbers of iterations n_δ and CPU times for Test 1.

N	8	16	32	64	128
the point Jacobi method					
n_δ	190	771	3092	12378	49520
CPU(s)	0.01	0.07	1.09	16.15	261.28
the point Gauss-Seidel method					
n_δ	97	388	1548	6191	24762
CPU(s)	0.005	0.04	0.53	8.58	141.37

Figure 3.1: Convergence of lower and upper solutions calculated by the point monotone Gauss-Seidel method ($N = 128$) to the reference solution for test 1.



(a) Lower solutions. (b) Upper solutions.

(3.23), we consider system (3.1) with $\mathrm{L}_\alpha u_\alpha(x, y) = D_\alpha(u_{\alpha,xx} + u_{\alpha,yy})$, $\alpha = 1, 2$, and the reaction functions in the forms

$$f_1(u_1, u_2) = \sigma_1 u_1(1 + e^{-u_2}), \quad f_2(u_1, u_2) = \sigma_2 \left(1 + \frac{1}{1 + u_1}\right) u_2, \tag{3.107}$$

where σ_α, $\alpha = 1, 2$, are positive constants. We choose the boundary conditions $g_\alpha(x, y) = 1$, $\alpha = 1, 2$, in (3.1). The pairs $(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = (1, 1)$ and $(\widehat{U}_{1,ij}, \widehat{U}_{2,ij}) = (0, 0)$, $(i, j) \in \overline{\Omega}^h$, are ordered upper and lower solutions. Indeed, all the assumptions in (3.96) and (3.97) with $M_\alpha = 1$, $\alpha = 1, 2$, are satisfied. From here, on the sector $\langle \widehat{U}, \widetilde{U} \rangle$, we conclude the inequalities

$$
\begin{aligned}
&\sigma_1(1+e^{-1}) \le \frac{\partial f_1}{\partial u_1} = \sigma_1(1+e^{-u_2}) \le 2\sigma_1, \quad 0 \le -\frac{\partial f_1}{\partial u_2} = \sigma_1 u_1 e^{-u_2} \le \sigma_1,\\
&\frac{3}{2}\sigma_2 \le \frac{\partial f_2}{\partial u_2} = \sigma_2(1+\frac{1}{1+u_1}) \le 2\sigma_2, \quad 0 \le -\frac{\partial f_2}{\partial u_1} = \frac{\sigma_2 u_2}{(1+u_1)^2} \le \sigma_2.
\end{aligned}
$$

Thus, f_α, $\alpha = 1, 2$, satisfy (3.5)–(3.7) with $\underline{c}_1 = \sigma_1(1+e^{-1})$, $\underline{c}_2 = 3\sigma_2/2$, $c_1 = 2\sigma_1$, $c_2 = 2\sigma_2$, $q_{12} = \sigma_1$ and $q_{21} = \sigma_2$. We calculate sequences of upper solutions generated by (3.51), (3.84) with the initial iteration $(\widetilde{U}_1, \widetilde{U}_2) = (1, 1)$. We take $D_\alpha = 0.1$, $\alpha = 1, 2$, in (3.1) and $\sigma_\alpha = 1$, $\alpha = 1, 2$, in (3.107).

In Table 3.3, for different values of N, we present $E(N)$ and $\gamma(N)$ from (3.105). The data in the table indicate that the numerical solution of the nonlinear difference scheme (3.17) converges to the reference solution with second-order accuracy which confirms the theoretical error estimate for the central difference scheme.

Numbers of iterations n_δ and execution (`CPU`) times are given in Table 3.4. From these results, we conclude that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method, which confirms Theorem 3.7.1. The numerical data indicate that the point monotone Gauss–Seidel method is approximately twice as fast as the point monotone Jacobi method.

Table 3.3: Order of convergence of the nonlinear scheme (3.17) for Test 2.

N	8	16	32	64	128
E	1.413e-02	3.800e-03	9.567e-04	2.294e-04	4.595e-05
γ	1.89	1.99	2.06	2.32	

Table 3.4: Numbers of iterations n_δ and `CPU` times for Test 2.

N	8	16	32	64	128
the point Jacobi method					
n_δ	89	353	1409	5632	22525
`CPU`(s)	0.02	0.05	0.70	10.90	174.46
the point Gauss-Seidel method					
n_δ	46	178	706	2818	11264
`CPU`(s)	0.01	0.02	0.37	5.78	92.29

3.8.2 Quasi-monotone nonincreasing case

Test 3

As the first test problem with quasi-monotone nonincreasing reaction functions (3.39), we consider the Volterra-Lotka competition model which is presented in Section 3.2.2.1, where $\mathrm{L}_\alpha u_\alpha = D_\alpha(u_{\alpha,xx} + u_{\alpha,yy})$, $\alpha = 1, 2$, in (3.1) and the reaction functions are given by

$$f_\alpha(u_1, u_2) = -u_\alpha(a_\alpha - b_\alpha u_1 - d_\alpha u_2), \quad \alpha = 1, 2. \tag{3.108}$$

We choose the boundary conditions $g_\alpha(x, y) = 1$, $\alpha = 1, 2$, in (3.1). The pairs $(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = (a_1/b_1, a_2/d_2)$ and $(\widehat{U}_{1,ij}, \widehat{U}_{2,ij}) = (0, 0)$, $(i, j) \in \overline{\Omega}^h$ are ordered upper and lower solutions. Indeed, all the assumptions in (3.101) are satisfied. From here, on $\langle \widehat{U}, \widetilde{U} \rangle$, we conclude the inequalities

$$\begin{aligned}
&\frac{\partial f_{1,ij}}{\partial u_1}(U_{1,ij}, U_{2,ij}) = -a_1 + 2b_1 U_{1,ij} + d_1 U_{2,ij} \le 2a_1 + \frac{d_1 a_2}{d_2}, \quad (i, j) \in \overline{\Omega}^h, \\
&\frac{\partial f_{2,ij}}{\partial u_2}(U_{1,ij}, U_{2,ij}) = -a_2 + b_2 U_{1,ij} + 2d_2 U_{2,ij} \le a_2 + \frac{a_1 b_2}{b_1}, \quad (i, j) \in \overline{\Omega}^h, \\
&-\frac{\partial f_{1,ij}}{\partial u_2} = -d_1 U_{1,ij} \le 0, \quad -\frac{\partial f_{2,ij}}{\partial u_1} = -b_2 U_{2,ij} \le 0, \quad (i, j) \in \overline{\Omega}^h.
\end{aligned}$$

Thus, f_α, $\alpha = 1, 2$, satisfy (3.38), (3.39) with $c_{1,ij} = 2a_1 + d_1 a_2/d_2$ and $c_{2,ij} = a_2 + a_1 b_2/b_1$, $(i, j) \in \overline{\Omega}^h$. We calculate the sequence $\{\overline{U}^{(n)}_{1,ij}, \underline{U}^{(n)}_{2,ij}\}$, $(i, j) \in \overline{\Omega}^h$, generated by (3.64), (3.90) with the initial iteration $(\widetilde{U}_{1,ij}, \widehat{U}_{2,ij}) = (a_1/b_1, 0)$, $(i, j) \in \overline{\Omega}^h$. We take $D_1 = 1$, $D_2 = 0.1$ in (3.1) and $a_\alpha = 1$, $b_\alpha = 1$ and $d_\alpha = 1$, $\alpha = 1, 2$, in (3.108).

In Table 3.5, for different values of N, we present $E(N)$ and $\gamma(N)$ from (3.105). The data in the table indicate that the numerical solution of the nonlinear difference scheme (3.17) converges to the reference solution with the second-order accuracy which confirms the theoretical error estimate for the central difference scheme. Numbers of iterations n_δ and execution (`CPU`) times are given in Table 3.6. From these results, we conclude that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method, which confirms Theorem 3.7.2. The numerical data indicate that the point monotone Gauss–Seidel method is approximately twice as fast as the point monotone Jacobi method.

Table 3.5: Order of convergence of the nonlinear scheme (3.17) for Test 3.

N	8	16	32	64	128
E	6.193e-3	1.590e-3	3.960e-04	9.448e-05	1.890e-05
γ	1.96	2.01	2.07	2.32	

Table 3.6: Numbers of iterations n_δ and CPU times for Test 3.

N	8	16	32	64	128
the point Jacobi method					
n_δ	157	626	2501	10002	40007
CPU(s)	0.02	0.08	1.11	17.31	287.70
the point Gauss–Seidel method					
n_δ	77	311	1249	5000	20002
CPU(s)	0.01	0.05	0.59	9.26	152.73

Test 4

As the second test problem with quasi-monotone nonincreasing reaction functions (3.39), we consider the Belousov-Zhabotinskii reaction diffusion model ([59], some background to the model is also given in [65]), which includes the metal-ion-catalyzed oxidation by bromate ion of organic materials. the chemical reaction scheme is given by

$$A_1 + Y \to X, \quad X + Y \to P_1, \quad A_2 + X \to 2X + Z, \quad 2X \to P_2, \quad Z \to \lambda Y,$$

where A_1 and A_2 are constants which represent reactants, P_1 and P_2 are products, λ is the stoichiometric factor, and X, Y and Z are, respectively, the concentrations of the intermediates $HBrO_2$ (bromous acid), Br^- (bromide ion) and Ce(IV)(cerium). A simplified system of two equations [39] of the above reactant scheme is governed by (3.1) with $\mathrm{L}_\alpha u_\alpha = D_\alpha \triangle u_\alpha$, $\alpha = 1, 2$, where u_1 and u_2 represent, respectively, the concentrations X and Y. The reaction functions are given by

$$f_1 = -u_1(a - bu_1 - \sigma_1 u_2), \quad f_2 = \sigma_2 u_1 u_2, \tag{3.109}$$

where a, b, σ_α, $\alpha = 1, 2$, are positive constants.

We choose the boundary conditions $g_\alpha(x, y) = 1$, $\alpha = 1, 2$, in (3.1). The pairs $(\widetilde{U}_1, \widetilde{U}_2) = (M_1, M_2)$ and $(\widehat{U}_1, \widehat{U}_2) = (0, 0)$ are ordered upper and lower solutions. Indeed, all the assumptions in (3.101) are satisfied, where M_α, $\alpha = 1, 2$, are chosen in the following form:

$$M_1 \geq \max\left(\frac{a}{b}, \max_{(x,y)\in\partial\omega} g_1(x, y)\right), \quad M_2 \geq \max_{(x,y)\in\partial\omega} g_2(x, y).$$

From here, on $\langle \widehat{U}, \widetilde{U} \rangle$, we conclude the inequalities

$$
\begin{aligned}
&\frac{\partial f_{1,ij}}{\partial u_1}(U_{1,ij}, U_{2,ij}) = 2bU_{1,ij} + \sigma_1 U_{2,ij} - a \le 2bM_1 + \sigma_1 M_2, \quad (i,j) \in \overline{\Omega}^h, \\
&\frac{\partial f_{2,ij}}{\partial u_2}(U_{1,ij}, U_{2,ij}) = \sigma_2 U_{1,ij} \le \sigma_2 M_1, \quad (i,j) \in \overline{\Omega}^h, \\
&-\frac{\partial f_{1,ij}}{\partial u_2}(U_{1,ij}, U_{2,ij}) = -\sigma_1 U_{1,ij} \le 0, \quad (i,j) \in \overline{\Omega}^h, \\
&-\frac{\partial f_{2,ij}}{\partial u_1}(U_{1,ij}, U_{2,ij}) = -\sigma_2 U_{2,ij} \le 0, \quad (i,j) \in \overline{\Omega}^h.
\end{aligned}
$$

Thus, f_α, $\alpha = 1,2$, satisfy (3.38) and (3.39) with $c_{1,ij} = 2bM_1 + \sigma_1 M_2$ and $c_{2,ij} = \sigma_2 M_1$, $(i,j) \in \overline{\Omega}^h$. We calculate the sequence $\{\overline{U}^{(n)}_{1,ij}, \underline{U}^{(n)}_{2,ij}\}$, $(i,j) \in \overline{\Omega}^h$, generated by (3.64), (3.90) with the initial iteration $(\widetilde{U}_{1,ij}, \widehat{U}_{2,ij}) = (1,0)$, $(i,j) \in \overline{\Omega}^h$. We take $D_1 = 1$, $D_2 = 0.1$ in (3.1), and $a = 1$, $b = 1$ and $\sigma_\alpha = 1$, $\alpha = 1,2$, in (3.109).

In Table 3.7, for different values of N, we present $E(N)$ and $\gamma(N)$ from (3.105). The data in the table indicate that the numerical solution of the nonlinear difference scheme (3.17) converges to the reference solution with the second-order accuracy which confirms the theoretical error estimate for the central difference scheme. Numbers of iterations n_δ and execution (`CPU`) times are given in Table 3.8. From these results, we conclude that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method, which confirms Theorem 3.7.2. Numerical data indicate that the point monotone Gauss–Seidel method is approximately twice as fast as the point monotone Jacobi method.

Table 3.7: Order of convergence of the nonlinear scheme (3.17) for Test 4.

N	8	16	32	64	128
E	6.208e-3	1.587e-3	3.948e-04	9.416e-05	1.884e-05
γ	1.97	2.01	2.07	2.32	

Table 3.8: Numbers of iterations n_δ and `CPU` times for Test 4.

N	8	16	32	64	128
the point Jacobi method					
n_δ	145	566	2248	8980	35906
`CPU`(s)	0.08	0.06	0.74	11.58	200.21
the point Gauss–Seidel method					
n_δ	78	288	1129	4495	17958
`CPU`(s)	0.05	0.03	0.41	6.28	102.21

3.9 Conclusions to Chapter 3

Theoretical results

For solving nonlinear elliptic systems with quasi-monotone nondecreasing and nonincreasing reaction functions, we constructed and investigated monotone properties of point Jacobi and Gauss-Seidel iterative methods. The coupled system of nonlinear elliptic problems (3.1) is approximated by using the central difference approximations for the first and second derivatives. For solving the nonlinear difference scheme (3.17) with quasi-monotone nondecreasing (3.23) and quasi-monotone nonincreasing (3.39) reaction functions, the point Jacobi and point Gauss-Seidel iterative methods for the coupled system are constructed. In Theorems 3.3.2 and 3.3.4, we prove that the sequences of upper and lower solutions, generated by the point iterative methods for problems with quasi-monotone nondecreasing (3.23) and quasi-monotone nonincreasing (3.39) reaction functions, converge monotonically to the solutions of the nonlinear difference scheme. In Theorems 3.4.2, 3.4.3 and 3.4.4, 3.4.5, for, respectively, quasi-monotone nondecreasing and nonincreasing cases, we prove the existence and uniqueness of a solution under the conditions that the nonlinear reaction functions are bounded from below and above. By using the stopping tests (3.84) and (3.90), based on the norms of residuals, for quasi-monotone nondecreasing and nonincreasing cases, we prove that the numerical solution converges to the unique solution of the nonlinear elliptic problem (3.1) and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme (3.17) in Theorems 3.5.1 and 3.5.3 and the error between the numerical solution and the exact solution of the elliptic system (3.1) in Theorems 3.5.2 and 3.5.4. We prove that the point monotone Gauss-Seidel methods converge faster than the point monotone Jacobi methods for the quasi-monotone nondecreasing and quasi-monotone nonincreasing cases, respectively, in Theorems 3.7.1 and 3.7.2. In Lemmas 3.6.1, 3.6.2 and 3.6.3, 3.6.4, respectively, for the quasi-monotone nondecreasing and quasi-monotone nonincreasing reaction functions, under assumptions (3.93), (3.96) and (3.100), (3.101), we construct initial upper and lower solutions to start the point monotone iterative methods.

Numerical results

The numerical experiments show that the numerical solution of the nonlinear difference scheme (3.17) converges to the reference solution with second-order accuracy. The numerical sequences of upper and lower solutions generated by the point monotone methods (3.23) with stopping (3.84) and the point monotone methods (3.39) with stoping (3.90) converge monotonically. The point monotone Gauss-Seidel method with $\eta = 1$ in (3.23) and $\eta = 1$ in (3.64) converges faster than the point monotone Jacobi method with $\eta = 0$ in (3.23) and $\eta = 0$ in (3.64) which confirm, respectively, Theorems

3.7.1 and 3.7.2. The point monotone Gauss-Seidel method is approximately twice as fast as the point monotone Jacobi method.

Chapter 4

Block Jacobi and Gauss-Seidel methods for systems of elliptic problems

This chapter deals with numerical methods for solving nonlinear elliptic systems by block iterative methods based on the Jacobi and Gauss–Seidel methods. The idea of these methods is the decomposition technique which reduces a domain into a series of nonoverlapping one dimensional intervals by slicing the domain into a finite number of thin strips, and then solving a two-point boundary-value problem for each strip by a standard computational scheme such as the Thomas algorithm [51]. In the view of the method of upper and lower solutions, two monotone upper and lower sequences of solutions are constructed. Convergence rates for the block monotone iterative methods are estimated in similar way as in Section 3.5. Constructions of initial upper and lower solutions are similar to Section 3.6. We show that the sequences of solutions generated by the block monotone Gauss–Seidel method converges faster than by the block monotone Jacobi method.

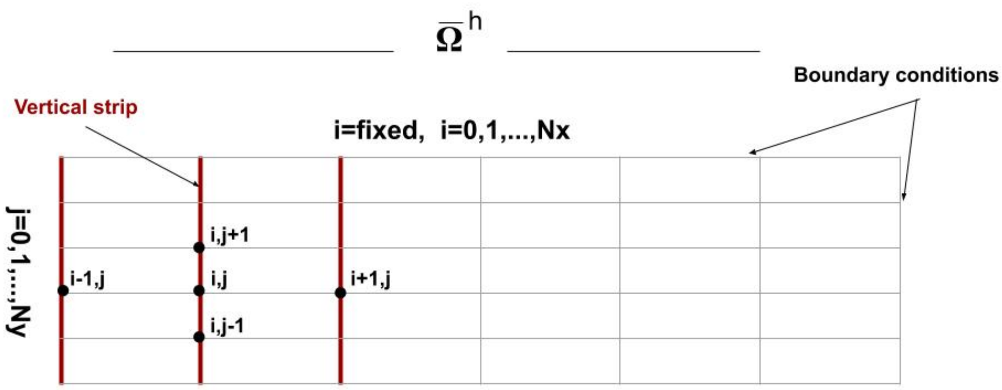
4.1 The block monotone Jacobi and Gauss-Seidel methods

We decompose the mesh $\overline{\Lambda}^h = \overline{\Lambda}^{hx} \times \overline{\Lambda}^{hy}$, which is defined in (3.15), into strips. For $x_i =$ fixed, $i = 0, 1, \ldots, N_x$, we introduce vertical strips $\overline{\Lambda}_i^h$, in the form

$$\overline{\Lambda}_i^h = \{(x_i, y_j), \quad j = 0, 1, \ldots, N_y\}, \quad i = 0, 1, \ldots, N_x. \tag{4.1}$$

Figure 4.1 illustrates the decomposition of the domain $\overline{\Omega}^h$.

Figure 4.1: Fragment of the domain decomposition



For the value of i, we consider the following notation:

$$\overline{\mathcal{I}} \equiv \mathcal{I} \cup \partial\mathcal{I}, \quad \mathcal{I} = \{1, 2, \dots, N_x - 1\}, \quad \partial\mathcal{I} = \{0, N_x\}. \tag{4.2}$$

For the nonlinear difference scheme (3.48), (3.49), we define vectors and diagonal matrices by

$$\begin{aligned}
&U_{\alpha,i} = (U_{\alpha,i,1}, \dots, U_{\alpha,i,N_y-1})^T, \quad i \in \overline{\mathcal{I}}, \\
&F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) = \left(f_{\alpha,i,1}(U_{\alpha,i,1}, U_{\alpha',i,1}), \dots, f_{\alpha,i,N_y-1}(U_{\alpha,i,N_y-1}, U_{\alpha',i,N_y-1})\right)^T, \\
&L_{\alpha,i} = \mathrm{diag}(l_{\alpha,i,1}, \dots, l_{\alpha,i,N_y-1}), \quad R_{\alpha,i} = \mathrm{diag}(r_{\alpha,i,1}, \dots, r_{\alpha,i,N_y-1}), \\
&B_{\alpha,i} = \mathrm{diag}(b_{\alpha,i,1}, \dots, b_{\alpha,i,N_y-1}), \quad Q_{\alpha,i} = \mathrm{diag}(q_{\alpha,i,1}, \dots, q_{\alpha,i,N_y-1}), \\
&L_{\alpha,i} > O, \quad R_{\alpha,i} > O, \quad B_{\alpha,i} > O, \quad Q_{\alpha,i} > O, \\
&i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2,
\end{aligned} \tag{4.3}$$

where the following notation is in use

$$F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) = \begin{cases} F_{1,i}(U_{1,i}, U_{2,i}), & \alpha = 1, \\ F_{2,i}(U_{1,i}, U_{2,i}), & \alpha = 2, \end{cases} \quad \alpha' \neq \alpha, \quad i \in \overline{\mathcal{I}}, \tag{4.4}$$

with symmetry $F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) = F_{\alpha,i}(U_{\alpha',i}, U_{\alpha,i})$. The terms $L_{\alpha,1}U_{\alpha,0}$ and $R_{\alpha,N_x-1}U_{\alpha,N_x}$, $\alpha = 1, 2$, are included in the boundaries.

Then the difference scheme (3.48), (3.49) can be presented in the form

$$A_{\alpha,i}U_{\alpha,i} - L_{\alpha,i}U_{\alpha,i-1} - R_{\alpha,i}U_{\alpha,i+1} = -F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}), \quad i \in \mathcal{I}, \tag{4.5}$$
$$U_{\alpha,i,0} = g_{\alpha,i,0}, \quad U_{\alpha,i,N_y} = g_{\alpha,i,N_y}, \quad i \in \mathcal{I}, \quad \alpha = 1, 2,$$

with the tridiagonal matrices $A_{\alpha,i}$, $i \in \mathcal{I}$, $\alpha = 1, 2$,

$$A_{\alpha,i} = \begin{bmatrix} d_{\alpha,i,1} & -q_{\alpha,i,1} & & 0 \\ -b_{\alpha,i,2} & d_{\alpha,i,2} & -q_{\alpha,i,2} & \\ \ddots & \ddots & \ddots & \\ & -b_{\alpha,i,N_y-2} & d_{\alpha,i,N_y-2} & -q_{\alpha,i,N_y-2} \\ 0 & & -b_{\alpha,i,N_y-1} & d_{\alpha,i,N_y-1} \end{bmatrix}.$$

The elements of the matrices $L_{\alpha,i}$ and $R_{\alpha,i}$, $i \in \mathcal{I}$, $\alpha = 1, 2$, contain the coupling coefficients of a mesh point (i, j) to, respectively, mesh points $(i-1, j)$ and $(i+1, j)$, $j = 1, 2, \ldots, N_y - 1$.

Remark 4.1.1. *let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two real $n \times r$ matrices. Then, $A \geq B (> B)$ if $a_{ij} \geq b_{ij} (> b_{ij})$ for all $1 \leq i \leq n$, $1 \leq j \leq r$. if O is the null matrix and $A \geq O (> O)$, we say that A is a nonnegative(positive) matrix.*

Lemma 4.1.2. *If $H = [h_{ij}]$ is a real, irreducibly diagonally dominant $N \times N$ matrix with $h_{ij} \leq 0$ for all $i \neq j$, and $h_{ii} > 0$ for all $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$, then*

$$H^{-1} > O, \tag{4.6}$$

where O is the $N \times N$ null matrix.

The proof of the lemma is given in Corollary 3.20, [71].

4.1.1 Quasi-monotone nondecreasing case

In the case of the quasi-monotone nondecreasing functions f_α, $\alpha = 1, 2$, in (3.1), we say that mesh functions

$$(\widetilde{U}_{1,i}, \widetilde{U}_{2,i}), \quad (\widehat{U}_{1,i}, \widehat{U}_{2,i}), \quad i \in \overline{\mathcal{I}},$$

are called ordered upper and lower solutions of (4.5), if they satisfy the inequalities

$$\widehat{U}_{\alpha,i} \leq \widetilde{U}_{\alpha,i}, \quad i \in \overline{\mathcal{I}}, \tag{4.7a}$$

$$\mathcal{K}_{\alpha,i}(\widehat{U}_{\alpha,i}, \widehat{U}_{\alpha',i}) \leq \mathbf{0} \leq \mathcal{K}_{\alpha,i}(\widetilde{U}_{\alpha,i}, \widetilde{U}_{\alpha',i}), \quad i \in \mathcal{I}, \tag{4.7b}$$

$$\mathcal{K}_{\alpha,i}(U_{\alpha,i}, V_{\alpha',i}) \equiv A_{\alpha,i}U_{\alpha,i} - L_{\alpha,i}U_{\alpha,i-1} - R_{\alpha,i}U_{\alpha,i+1} + F_{\alpha,i}(U_{\alpha,i}, V_{\alpha',i}),$$

$$\widehat{U}_{\alpha,i} \leq g_{\alpha,i} \leq \widetilde{U}_{\alpha,i}, \quad i \in \partial\mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2, \tag{4.7c}$$

where notation (4.4) is in use.

For a given pair of ordered upper and lower solutions $(\widetilde{U}_{1,i}, \widetilde{U}_{2,i})$, $(\widehat{U}_{1,i}, \widehat{U}_{2,i})$, $i \in \overline{\mathcal{I}}$, we define the sector

$$\langle \widehat{U}, \widetilde{U} \rangle = \left\{ U_{\alpha,i} : \quad \widehat{U}_{\alpha,i} \leq U_{\alpha,i} \leq \widetilde{U}_{\alpha,i}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2 \right\}. \tag{4.8}$$

Remark 4.1.3. *Similar to Remark 3.2.3, we state the mean-value theorem for mesh vector-functions. Assume that $F_\alpha(x, y, u_\alpha, u_{\alpha'})$, $(x, y, t) \in \overline{Q}_T$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1, 2$, are smooth functions, then we have*

$$F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) - F_{\alpha,i}(V_{\alpha,i}, U_{\alpha',i}) = \left(F_{\alpha,i}(Q_{\alpha,i}, U_{\alpha',i})\right)_{u_\alpha} [U_{\alpha,i} - V_{\alpha,i}], \tag{4.9}$$

$$F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) - F_{\alpha,i}(U_{\alpha,i}, V_{\alpha',i}) = \left(F_{\alpha,i}(U_{\alpha,i}, Y_{\alpha',i})\right)_{u_{\alpha'}} [U_{\alpha',i} - V_{\alpha',i}],$$

where $Q_{\alpha,i}$ and $Y_{\alpha,i}$ lie between $U_{\alpha,i}$ and $V_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, and notation (4.4) is in use. The partial derivatives $(F_{\alpha,i})_{u_\alpha}$ and $(F_{\alpha,i})_{u_{\alpha'}}$ are the diagonal matrices

$$(F_{\alpha,i})_{u_\alpha} = \operatorname{diag}\left(\left(f_{\alpha,i,1}(Q_{\alpha,i,1}, U_{\alpha',i,1})\right)_{u_\alpha}, \ldots, \left(f_{\alpha,i,N_y-1}(Q_{\alpha,i,N_y-1}, U_{\alpha',i,N_y-1})\right)_{u_\alpha} \right), \tag{4.10}$$

$$(F_{\alpha,i})_{u_{\alpha'}} = \operatorname{diag}\left(\left(f_{\alpha,i,1}(U_{\alpha,i,1}, Y_{\alpha',i,1})\right)_{u_{\alpha'}}, \ldots, \left(f_{\alpha,i,N_y-1}(U_{\alpha,i,N_y-1}, Y_{\alpha',i,N_y-1})\right)_{u_{\alpha'}} \right).$$

We rewrite notation (3.16) in vector form

$$\Gamma_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) = C_{\alpha,i}U_{\alpha,i} - F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}), \tag{4.11}$$

$$C_{\alpha,i} = \operatorname{diag}(c_{\alpha,i,1}, \ldots, c_{\alpha,i,N_y-1}), \quad i \in \overline{\mathcal{I}}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha',$$

where $c_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, are nonnegative bounded functions, and notation (4.4) is in use. We give a monotone property of $\Gamma_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i})$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$.

Lemma 4.1.4. *Let (3.22) and (3.23) hold, and $U_{\alpha,i}$, $V_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, be any mesh functions in $\langle \widehat{U}, \widetilde{U} \rangle$ such that $U_{\alpha,i} \geq V_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$. Then*

$$\Gamma_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) \geq \Gamma_{\alpha,i}(V_{\alpha,i}, V_{\alpha',i}), \quad i \in \overline{\mathcal{I}}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2. \tag{4.12}$$

Proof. From (4.11), we have

$$
\begin{aligned}
\Gamma_{\alpha,i}(U_{\alpha,i},U_{\alpha',i})-\Gamma_{\alpha,i}(V_{\alpha,i},V_{\alpha',i}) &= C_{\alpha,i}[U_{\alpha,i}-V_{\alpha,i}] \\
&\quad -[F_{\alpha,i}(U_{\alpha,i},U_{\alpha',i})-F_{\alpha,i}(V_{\alpha,i},U_{\alpha',i})] \\
&\quad -[F_{\alpha,i}(V_{\alpha,i},U_{\alpha',i})-F_{\alpha,i}(V_{\alpha,i},V_{\alpha',i})],
\end{aligned}
$$

$$
i\in\overline{\mathcal{I}},\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2.
$$

Using the mean-value theorem (4.9), we obtain that

$$
\begin{aligned}
&\Gamma_{\alpha,i}(U_{\alpha,i},U_{\alpha',i})-\Gamma_{\alpha,i}(V_{\alpha,i},V_{\alpha',i}) = \\
&(C_{\alpha,i}-(F_{\alpha,i}(Q_{\alpha,i},U_{\alpha,i}))_{u_\alpha})(U_{\alpha,i}-V_{\alpha,i})-(F_{\alpha,i}(V_{\alpha,i},Y_{\alpha',i}))_{u_{\alpha'}}\left(U_{\alpha',i}-V_{\alpha',i}\right), \\
&V_{\alpha,i}\le Q_{\alpha,i}\le Y_{\alpha,i}\le U_{\alpha,i},\quad i\in\overline{\mathcal{I}},\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2,
\end{aligned}
$$

where the partial derivatives are defined in (4.10). Taking into account that $U_{\alpha,i}\ge V_{\alpha,i}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, from (3.22) and (3.23), we conclude (4.12). □

We now construct block iterative methods for solving (4.5). Upper $\{\overline{U}^{(n)}_{\alpha,i}\}$ and lower $\{\underline{U}^{(n)}_{\alpha,i}\}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, sequences of solutions are calculated by the following block Jacobi and Gauss-Seidel iterative methods:

$$
\begin{aligned}
&A_{\alpha,i}Z^{(n)}_{\alpha,i}-\eta L_{\alpha,i}Z^{(n)}_{\alpha,i-1}+C_{\alpha,i}Z^{(n)}_{\alpha,i}=-\mathcal{K}_{\alpha,i}(U^{(n-1)}_{\alpha,i},U^{(n-1)}_{\alpha',i}),\quad i\in\mathcal{I}, \qquad (4.13)\\
&Z^{(n)}_{\alpha,i}=\begin{cases} g_{\alpha,i}-U^{(0)}_{\alpha,i}, & n=1,\\ \mathbf{0}, & n\ge 2,\end{cases}\qquad i\in\partial\mathcal{I},\\
&Z^{(n)}_{\alpha,i}=U^{(n)}_{\alpha,i}-U^{(n-1)}_{\alpha,i},\quad i\in\overline{\mathcal{I}},\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2,
\end{aligned}
$$

where $\mathcal{K}_{\alpha,i}(U^{(n-1)}_{\alpha,i},U^{(n-1)}_{\alpha',i})$, $i\in\mathcal{I}$, $\alpha'\neq\alpha$, $\alpha,\alpha'=1,2$, are the residuals of the difference scheme (4.5) on $U^{(n-1)}_{\alpha,i}$, $i\in\mathcal{I}$, $\alpha=1,2$, which are defined in (4.7). If $\eta=0$ and $\eta=1$, we have, respectively, the block Jacobi and block Gauss–Seidel iterative methods.

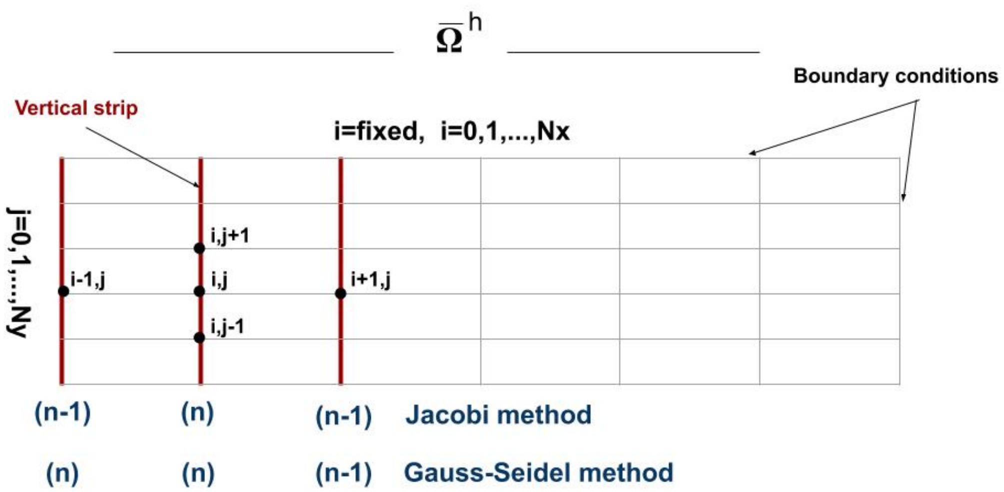
Remark 4.1.5. *For quasi-monotone nondecreasing functions (3.23), upper and lower solutions are independent, hence, by using (4.13), we calculate either the sequence* $\{\overline{U}^{(n)}_{1,i},\overline{U}^{(n)}_{2,i}\}$, $i\in\overline{\mathcal{I}}$ *or the sequence* $\{\underline{U}^{(n)}_{1,i},\underline{U}^{(n)}_{2,i}\}$, $i\in\overline{\mathcal{I}}$.

Remark 4.1.6. *The basic advantage of the block Jacobi iterative method (4.13) with* $\eta=0$ *is that the Thomas algorithm can be used for each subsystem* (α,i), $i\in\mathcal{I}$, $\alpha=1,2$, *and all the subsystems can be computed in parallel.*

The advantage of the block Gauss–Seidel method (4.13) with $\eta=1$ *is that the Thomas algorithm for solving tridiagonal systems can be used for each subsystem* (α,i), $\mathcal{I}$, $\alpha=1,2$. *Since* $U^{(n)}_{\alpha,0}$, $\alpha=1,2$, *are given, and from (4.6),* $(A_{\alpha,i}+C_{\alpha,i})^{-1}>0$, $i\in\mathcal{I}$, $\alpha=1,2$, *then the tridiagonal systems (4.13) for* $i=1$ *are well-defined and can be solved* *for* $U_{\alpha,1}^{(n)}$*,* $\alpha = 1,2$*, by the Thomas algorithm. Now, the tridiagonal systems (4.13) for* $i = 2$ *are well-defined and can be solved for* $U_{\alpha,2}^{(n)}$*,* $\alpha = 1,2$*, by the Thomas algorithm. Thus, starting from* $i = 1$ *and finishing off with* $i = N_x - 1$*, we solve only the tridiagonal systems for* $U_{\alpha,i}^{(n)}$*,* $i \in \mathcal{I}$*,* $\alpha = 1,2$*.*

Figure 4.2 illustrates the implementation of block Jacobi and Gauss-Seidel methods.

Figure 4.2: Implementation of the block Jacobi and Gauss-Seidel methods



Theorem 4.1.7. *Let* $(\widetilde{U}_{1,i}, \widetilde{U}_{2,i})$ *and* $(\widehat{U}_{1,i}, \widehat{U}_{1,i})$*,* $i \in \overline{\mathcal{I}}$*, be ordered upper and lower solutions (4.7) of (4.5). Suppose that the functions* f_α*,* $\alpha = 1,2$*, in (3.1) satisfy (3.22) and (3.23). Then upper* $\{\overline{U}_{\alpha,i}^{(n)}\}$ *and lower* $\{\underline{U}_{\alpha,i}^{(n)}\}$*,* $i \in \overline{\mathcal{I}}$*,* $\alpha = 1,2$*, sequences generated by (4.13) with, respectively,* $(\overline{U}_{1,i}^{(0)}, \overline{U}_{2,i}^{(0)}) = (\widetilde{U}_{1,i}, \widetilde{U}_{2,i})$ *and* $(\underline{U}_{1,i}^{(0)}, \underline{U}_{2,i}^{(0)}) = (\widehat{U}_{1,i}, \widehat{U}_{2,i})$*,* $i \in \overline{\mathcal{I}}$*, converge monotonically from above to a maximal solution* $(\overline{U}_{1,i}, \overline{U}_{2,i})$*,* $i \in \overline{\mathcal{I}}$*, and from below to a minimal solution* $(\underline{U}_{1,i}, \underline{U}_{2,i})$*,* $i \in \overline{\mathcal{I}}$*,*

$$\underline{U}_{\alpha,i}^{(n-1)} \le \underline{U}_{\alpha,i}^{(n)} \le \underline{U}_{\alpha,i} \le \overline{U}_{\alpha,i} \le \overline{U}_{\alpha,i}^{(n)} \le \overline{U}_{\alpha,i}^{(n-1)} \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1,2. \tag{4.14}$$

If $S_{\alpha,i}$*,* $i \in \overline{\mathcal{I}}$*,* $\alpha = 1,2$*, are any other solutions in* $\langle \widehat{U}, \widetilde{U} \rangle$*, then*

$$\underline{U}_{\alpha,i} \le S_{\alpha,i} \le \overline{U}_{\alpha,i}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1,2. \tag{4.15}$$

Proof. Letting $W^{(n)}_{\alpha,i} = \overline{U}^{(n)}_{\alpha,i} - \underline{U}^{(n)}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, in notation (4.11), from (4.13), we have

$$(A_{\alpha,i} + C_{\alpha,i})W^{(1)}_{\alpha,i} - \eta L_{\alpha,i}W^{(1)}_{\alpha,i-1} = R_{\alpha,i}W^{(0)}_{\alpha,i+1} + \Gamma_{\alpha,i}(\overline{U}^{(0)}_{\alpha,i}, \overline{U}^{(0)}_{\alpha',i}) - \Gamma_{\alpha,i}(\underline{U}^{(0)}_{\alpha,i}, \underline{U}^{(0)}_{\alpha',i}),$$
$$i \in \mathcal{I}, \quad W^{(1)}_{\alpha,i} = \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.$$

Taking into account that $\underline{U}^{(0)}_{\alpha,i} = \widehat{U}_{\alpha,i} \leq \overline{U}^{(0)}_{\alpha,i} = \widetilde{U}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $R_{\alpha,i} > O$, $i \in \mathcal{I}$, $\alpha = 1, 2$, from (4.3), using (4.12), it follows that

$$(A_{\alpha,i} + C_{\alpha,i})W^{(1)}_{\alpha,i} - \eta L_{\alpha,i}W^{(1)}_{\alpha,i-1} \geq \mathbf{0}, \quad i \in \mathcal{I}, \quad W^{(1)}_{\alpha,i} = \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad \alpha = 1, 2. \tag{4.16}$$

Since $W^{(1)}_{\alpha,0} = \mathbf{0}$ and from (4.6), $(A_{\alpha,1} + C_{\alpha,1})^{-1} > O$, $\alpha = 1, 2$, for $i = 1$ in (4.16), by using (4.12), we conclude that $W^{(1)}_{\alpha,1} \geq \mathbf{0}$, $\alpha = 1, 2$. From here, $\eta = 0, 1$, $L_{\alpha,2} > O$, $\alpha = 1, 2$, from (4.3), and using (4.12), for $i = 2$, we obtain that $W^{(1)}_{\alpha,2} \geq \mathbf{0}$, $\alpha = 1, 2$. By induction on i, we can prove that

$$W^{(1)}_{\alpha,i} \geq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2.$$

Thus, we prove (4.7a). Since $\widetilde{U}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are upper solutions (4.7), it follows that $\mathcal{K}_{\alpha,i}(\widetilde{U}_{\alpha,i}, \widetilde{U}_{\alpha',i}) \geq 0$, $i \in \mathcal{I}$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1, 2$. From here and (4.13), we have

$$(A_{\alpha,i} + C_{\alpha,i})\overline{Z}^{(1)}_{\alpha,i} - \eta L_{\alpha,i}\overline{Z}^{(1)}_{\alpha,i-1} \leq \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha = 1, 2. \tag{4.17}$$

Taking into account that $\eta = 0, 1$, $L_{\alpha,i} \geq O$ from (4.3), $(A_{\alpha,i} + C_{\alpha,i})^{-1} > O$ from (4.6), $\overline{Z}^{(1)}_{\alpha,0} \leq \mathbf{0}$, $i \in \mathcal{I}$, $\alpha = 1, 2$, for $i = 1$ in (4.17), we conclude that $\overline{Z}^{(1)}_{\alpha,1} \leq \mathbf{0}$, $\alpha = 1, 2$. By induction on i, we can prove that

$$\overline{Z}^{(1)}_{\alpha,i} \leq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2. \tag{4.18}$$

Similarly, for initial lower solutions $\underline{U}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, we can prove that

$$\underline{Z}^{(1)}_{\alpha,i} \leq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2. \tag{4.19}$$

From (4.13) and using notation (4.11), we have

$$\mathcal{K}_{\alpha,i}(\overline{U}^{(1)}_{\alpha,i}, \overline{U}^{(1)}_{\alpha',i}) = -R_{\alpha,i}\overline{Z}^{(1)}_{\alpha,i+1} + \Gamma_{\alpha,i}(\overline{U}^{(0)}_{\alpha,i}, \overline{U}^{(0)}_{\alpha',i}) - \Gamma_{\alpha,i}(\overline{U}^{(1)}_{\alpha,i}, \overline{U}^{(1)}_{\alpha',i}), \tag{4.20}$$
$$i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.$$

Taking into account that $R_{\alpha,i} \geq O$, $\alpha = 1,2$, (4.18), by using 4.12, we conclude that

$$\mathcal{K}_{\alpha,i}(\overline{U}^{(1)}_{\alpha,i}, \overline{U}^{(1)}_{\alpha',i}) \geq \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2.$$

Thus, $\overline{U}^{(1)}_{\alpha,i}$, $i \in \mathcal{I}$, $\alpha = 1,2$, satisfy (4.7b). By a similar manner, we can prove

$$\mathcal{K}_{\alpha,i}(\underline{U}^{(1)}_{\alpha,i}, \underline{U}^{(1)}_{\alpha',i}) \geq \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,$$

that is $\underline{U}^{(1)}_{\alpha,i}$, $i \in \mathcal{I}$, $\alpha = 1,2$, satisfy (4.7b). From the boundary conditions on $i \in \partial\mathcal{I}$ in (4.11), it follows that $\overline{U}^{(1)}_{\alpha,i}$ and $\underline{U}^{(1)}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, satisfy (4.7c).

Thus, we prove that $\overline{U}^{(1)}_{\alpha,i}$ and $\underline{U}^{(1)}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are ordered upper and lower solutions (4.7).

By induction on n, we can prove that $\{\overline{U}^{(n)}_{\alpha,i}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are a monotone decreasing sequence of upper solutions and $\{\underline{U}^{(n)}_{\alpha,i}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are a monotone increasing sequence of lower solutions which satisfy (4.14).

From (4.14), we conclude that $\lim \overline{U}^{(n)}_{\alpha,i} = \overline{U}_{\alpha,i}$ and $\lim \underline{U}^{(n)}_{\alpha,i} = \underline{U}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, as $n \to \infty$ exist, and

$$\lim_{n\to\infty} \overline{Z}^{(n)}_{\alpha,i} = \mathbf{0}, \quad \lim_{n\to\infty} \underline{Z}^{(n)}_{\alpha,i} = \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1,2.$$

Similar to (4.20), for $n \geq 1$, we conclude that

$$\begin{aligned}&\mathcal{K}_{\alpha,i}(\overline{U}^{(n)}_{\alpha,i}, \overline{U}^{(n)}_{\alpha',i}) = -R_{\alpha,i}\overline{Z}^{(n)}_{\alpha,i+1} + \Gamma_{\alpha,i}(\overline{U}^{(n-1)}_{\alpha,i}, \overline{U}^{(n-1)}_{\alpha',i}) - \Gamma_{\alpha,i}(\overline{U}^{(n)}_{\alpha,i}, \overline{U}^{(n)}_{\alpha',i}),\\ &i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2.\end{aligned}$$

By taking the limit of both sides, (4.14) and using (4.11), we conclude that

$$\mathcal{K}_{\alpha,i}(\overline{U}_{\alpha,i}, \overline{U}_{\alpha',i}) = \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2.$$

Thus, $\overline{U}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are maximal solutions to the nonlinear difference scheme (4.5). By a similar argument, we can prove

$$\mathcal{K}_{\alpha,i}(\underline{U}_{\alpha,i}, \underline{U}_{\alpha',i}) = \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,$$

that is, $\underline{U}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are minimal solutions to the nonlinear difference scheme (4.5).

Now, we prove (4.15). We assume that $S_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are other solutions in $\langle \widehat{U}, \widetilde{U} \rangle$. We consider the sector $\langle S, \widetilde{U} \rangle$, which means that we treat $S_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, as lower solutions. Since $\{\underline{S}^{(n)}_{\alpha,i}\} = \{S_{\alpha,i}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, is a constant sequence for all n, then from (4.14), we conclude that $S_{\alpha,i} \leq \overline{U}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$.

Now, we consider the sector $\langle \widehat{U}, S\rangle$, which means that we treat $S_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, as upper solutions. Similarly, since $\{\overline{S}^{(n)}_{\alpha,i}\} = \{S_{\alpha,i}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, is a constant sequence for all n, then from (4.14), we conclude that $\underline{U}_{\alpha,i} \leq S_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$. Thus, we prove (4.15). □

4.1.2 Quasi-monotone nonincreasing case

In the case of the quasi-monotone nonincreasing functions f_α, $\alpha = 1,2$, (3.39), we say that mesh functions

$$(\widetilde{U}_{1,i}, \widetilde{U}_{2,i}), \quad (\widehat{U}_{1,i}, \widehat{U}_{2,i}), \quad i \in \overline{\mathcal{I}},$$

are called ordered upper and lower solutions of (4.5), if they satisfy the inequalities

$$\widehat{U}_{\alpha,i} \leq \widetilde{U}_{\alpha,i}, \quad i \in \overline{\mathcal{I}}, \tag{4.21a}$$

$$\mathcal{K}_{\alpha,i}(\widehat{U}_{\alpha,i}, \widetilde{U}_{\alpha',i}) \leq \mathbf{0} \leq \mathcal{K}_{\alpha,i}(\widetilde{U}_{\alpha,i}, \widehat{U}_{\alpha',i}), \quad i \in \mathcal{I}, \tag{4.21b}$$

$$\widehat{U}_{\alpha,i} \leq g_{\alpha,i} \leq \widetilde{U}_{\alpha,i}, \quad i \in \partial\mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \tag{4.21c}$$

where $\mathcal{K}_{\alpha,i}(\widehat{U}_{\alpha,i}, \widetilde{U}_{\alpha',i})$, $i \in \mathcal{I}$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1,2$, are defined in (4.7).

Lemma 4.1.8. *Let (3.38) and (3.39) hold, and* $(U_{1,i}, U_{2,i})$, $(V_{1,i}, V_{2,i})$, $i \in \overline{\mathcal{I}}$ *be any functions in the sector* $\langle \widehat{U}, \widetilde{U}\rangle$ *(4.8) such that* $U_{\alpha,i} \geq V_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$. *Then*

$$\Gamma_{\alpha,i}(U_{\alpha,i}, V_{\alpha',i}) \geq \Gamma_{\alpha,i}(V_{\alpha,i}, U_{\alpha',i}), \quad i \in \overline{\mathcal{I}}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2. \tag{4.22}$$

Proof. From (4.11), we have

$$\begin{aligned}\Gamma_{\alpha,i}(U_{\alpha,i}, V_{\alpha',i}) - \Gamma_\alpha(V_{\alpha,i}, U_{\alpha',i}) = {} & C_{\alpha,i}(U_{\alpha,i} - V_{\alpha,i}) \\ & - \left[F_{\alpha,i}(U_{\alpha,i}, V_{\alpha',i}) - F_{\alpha,i}(V_{\alpha,i}, V_{\alpha',i})\right] \\ & + \left[F_{\alpha,i}(V_{\alpha,i}, U_{\alpha',i}) - F_{\alpha,i}(V_{\alpha,i}, V_{\alpha',i})\right], \\ i \in \overline{\mathcal{I}}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, &\end{aligned}$$

where notation (4.4) is in use. Using the mean-value theorem (4.9), we obtain that

$$\begin{aligned}&\Gamma_{\alpha,i}(U_{\alpha,i}, V_{\alpha',i}) - \Gamma_\alpha(V_{\alpha,i}, U_{\alpha',i}) = \\ &\Big(C_{\alpha,i} - \left(F_{\alpha,i}(Q_{\alpha,i}, V_{\alpha',i})\right)_{u_\alpha}\Big)(U_{\alpha,i} - V_{\alpha,i}) + \left(F_{\alpha,i}(V_{\alpha,i}, Y_{\alpha',i})\right)_{u_{\alpha'}}(U_{\alpha',i} - V_{\alpha',i}), \\ &V_{\alpha,i} \leq Q_{\alpha,i}, Y_{\alpha,i} \leq U_{\alpha,i}, \quad i \in \overline{\mathcal{I}}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,\end{aligned}$$

where the partial derivatives are defined in (4.10). Taking into account that $U_{\alpha,i} \geq V_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, from (3.38) and (3.39), we conclude (4.22). □

In the case of quasi-monotone nonincreasing reaction functions (3.39), for solving the nonlinear difference scheme (4.5), we introduce the block Jacobi and Gauss-Seidel iterative methods in the forms

$$\begin{aligned}
&A_{\alpha,i}\overline{Z}^{(n)}_{\alpha,i} - \eta L_{\alpha,i}\overline{Z}^{(n)}_{\alpha,i-1} + C_{\alpha,i}\overline{Z}^{(n)}_{\alpha,i} = -\mathcal{K}_{\alpha,i}(\overline{U}^{(n-1)}_{\alpha,i}, \underline{U}^{(n-1)}_{\alpha',i}), \quad i \in \mathcal{I}, \qquad (4.23)\\
&A_{\alpha,i}\underline{Z}^{(n)}_{\alpha,i} - \eta L_{\alpha,i}\underline{Z}^{(n)}_{\alpha,i-1} + C_{\alpha,i}\underline{Z}^{(n)}_{\alpha,i} = -\mathcal{K}_{\alpha,i}(\underline{U}^{(n-1)}_{\alpha,i}, \overline{U}^{(n-1)}_{\alpha',i}), \quad i \in \mathcal{I},\\
&Z^{(n)}_{\alpha,i} = U^{(n)}_{\alpha,i} - U^{(n-1)}_{\alpha,i}, \quad i \in \overline{\mathcal{I}},\\
&Z^{(n)}_{\alpha,i} = \begin{cases} g_{\alpha,i} - U^{(0)}_{\alpha,i}, & n = 1,\\ \mathbf{0}, & n \geq 2, \end{cases} \quad i \in \partial\mathcal{I},
\end{aligned}$$

where $\mathcal{K}_{\alpha,i}(U^{(n-1)}_{\alpha,i}, U^{(n-1)}_{\alpha',i})$, $i \in \mathcal{I}$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1, 2$, are the residuals of the difference scheme (4.5) on $U^{(n-1)}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, which are defined in (4.7). For $\eta = 0$ and $\eta = 1$ in (4.23), we have, respectively, the block Jacobi and block Gauss–Seidel methods.

Remark 4.1.9. *For quasi-monotone nonincreasing functions f_α, $\alpha = 1, 2$, (3.39), upper and lower solutions are coupled, hence, by using (4.23), we calculate either the sequence $\{\overline{U}^{(n)}_{1,i}, \underline{U}^{(n)}_{2,i}\}$, $i \in \overline{\mathcal{I}}$ or the sequence $\{\underline{U}^{(n)}_{1,i}, \overline{U}^{(n)}_{2,i}\}$, $i \in \overline{\mathcal{I}}$.*

Remark 4.1.10. *The basic advantages of the block Jacobi iterative method with $\eta = 0$ in (4.13) and the block Gauss–Seidel method with $\eta = 1$ in (4.13) are the Thomas algorithm can be used for each subsystem (α, i), $i \in \mathcal{I}$, $\alpha = 1, 2$, as in the case of quasi-monotone nondecreasing reaction functions, which are indicated in Remark 4.1.6.*

Theorem 4.1.11. *Let $(\widetilde{U}_{1,i}, \widetilde{U}_{2,i})$ and $(\widehat{U}_{1,i}, \widehat{U}_{2,i})$, $i \in \overline{\mathcal{I}}$ be ordered upper and lower solutions (4.7). Assume that the functions f_α, $\alpha = 1, 2$, in (3.1) satisfy equations (3.38) and (3.39). Then the sequences $\{\overline{U}^{(n)}_{1,i}, \underline{U}^{(n)}_{2,i}\}$, $\{\underline{U}^{(n)}_{1,i}, \overline{U}^{(n)}_{2,i}\}$, $i \in \overline{\mathcal{I}}$, generated by the monotone methods (4.23) with $\{\overline{U}^{(0)}_{1,i}, \underline{U}^{(0)}_{2,i}\} = \{\widetilde{U}_{1,i}, \widehat{U}_{2,i}\}$ and $\{\underline{U}^{(0)}_{1,i}, \overline{U}^{(0)}_{2,i}\} = \{\widehat{U}_{1,i}, \widetilde{U}_{2,i}\}$, $i \in \overline{\mathcal{I}}$, converge monotonically to their respective solutions $(\overline{U}_{1,i}, \underline{U}_{2,i})$ and $(\underline{U}_{1,i}, \overline{U}_{2,i})$, $i \in \overline{\mathcal{I}}$, such that (4.14) holds. If $S_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are any other solution in $\langle \widehat{U}, \widetilde{U} \rangle$, then (4.15) holds.*

Proof. In the case of the sequence $\{\overline{U}^{(n)}_{1,i}, \underline{U}^{(n)}_{2,i}\}$, $i \in \overline{\mathcal{I}}$, $(\overline{U}^{(0)}_{1,i}, \underline{U}^{(0)}_{2,i}) = (\widetilde{U}_{1,i}, \widehat{U}_{2,i})$, $i \in \overline{\mathcal{I}}$, are initial upper and lower solutions (4.21). Hence, it follows that $\mathcal{K}_{1,i}(\overline{U}^{(0)}_{1,i}, \underline{U}^{(0)}_{2,i}) \geq \mathbf{0}$, $\mathcal{K}_{2,i}(\overline{U}^{(0)}_{1,i}, \underline{U}^{(0)}_{2,i}) \leq \mathbf{0}$, $i \in \mathcal{I}$, from (4.23), we conclude that

$$\begin{aligned}
&(A_{1,i} + C_{1,i})\overline{Z}^{(1)}_{1,i} - \eta L_{1,i}\overline{Z}^{(1)}_{1,i-1} \leq \mathbf{0}, \quad i \in \mathcal{I},\\
&(A_{2,i} + C_{2,i})\underline{Z}^{(1)}_{2,i} - \eta L_{2,i}\underline{Z}^{(1)}_{2,i-1} \geq \mathbf{0}, \quad i \in \mathcal{I},\\
&\overline{Z}^{(1)}_{1,i} \leq \mathbf{0}, \quad \underline{Z}^{(1)}_{2,i} \geq \mathbf{0}, \quad i \in \partial\mathcal{I}.
\end{aligned}$$

Taking into account that $(A_{\alpha,i} + C_{\alpha,i})^{-1} > O$ from (4.6), $\eta = 0, 1$, $L_{\alpha,i} \geq O$, $\alpha = 1, 2$ from (4.3), for $i = 1$, $\overline{Z}^{(1)}_{1,0} \leq \mathbf{0}$, $\underline{Z}^{(1)}_{2,0} \geq \mathbf{0}$, we conclude that $\overline{Z}^{(1)}_{1,1} \leq \mathbf{0}$, $\underline{Z}^{(1)}_{2,1} \geq \mathbf{0}$. By induction on i, we can prove that

$$\overline{Z}_{1,i}^{(1)} \leq \mathbf{0}, \quad \underline{Z}_{2,i}^{(1)} \geq \mathbf{0}, \quad i \in \overline{\mathcal{I}}. \tag{4.24}$$

Similarly, for the sequence $\{\underline{U}_{1,i}^{(n)}, \overline{U}_{2,i}^{(n)}\}$, $i \in \overline{\mathcal{I}}$, from (4.23), we conclude that

$$\begin{aligned}
&(A_{1,i} + C_{1,i})\underline{Z}_{1,i}^{(1)} - \eta L_{1,i}\underline{Z}_{1,i-1}^{(1)} \geq \mathbf{0}, \quad i \in \mathcal{I},\\
&(A_{2,i} + C_{2,i})\overline{Z}_{2,i}^{(1)} - \eta L_{2,i}\overline{Z}_{2,i-1}^{(1)} \leq \mathbf{0}, \quad i \in \mathcal{I},\\
&\underline{Z}_{1,i}^{(1)} \geq \mathbf{0}, \quad \overline{Z}_{2,i}^{(1)} \leq \mathbf{0}, \quad i \in \partial\mathcal{I}.
\end{aligned}$$

Taking into account that $(A_{\alpha,i} + C_{\alpha,i})^{-1} > O$ from (4.6), $\eta = 0, 1$, $L_{\alpha,i} \geq O$, $\alpha = 1, 2$ from (4.3), for $i = 1$, $\underline{Z}_{1,0}^{(1)} \geq \mathbf{0}$, $\overline{Z}_{2,0}^{(1)} \leq \mathbf{0}$, we conclude that $\underline{Z}_{1,1}^{(1)} \geq \mathbf{0}$, $\overline{Z}_{2,1}^{(1)} \leq \mathbf{0}$. By induction on i, we can prove that

$$\underline{Z}_{1,i}^{(1)} \geq \mathbf{0}, \quad \overline{Z}_{2,i}^{(1)} \leq \mathbf{0}, \quad i \in \overline{\mathcal{I}}. \tag{4.25}$$

We now prove that $\overline{U}_{\alpha,i}^{(1)}$ and $\underline{U}_{\alpha,i}^{(1)}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are ordered upper and lower solutions (4.21). Letting $W_{\alpha,i}^{(1)} = \overline{U}_{\alpha,i}^{(1)} - \underline{U}_{\alpha,i}^{(1)}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, in notation (4.11), from (4.23), we have

$$\begin{aligned}
&(A_{\alpha,i} + C_{\alpha,i})W_{\alpha,i}^{(1)} = \eta L_{\alpha,i}W_{\alpha,i-1}^{(1)} + R_{\alpha,i}W_{\alpha,i+1}^{(0)} + \Gamma_{\alpha,i}(\overline{U}_{\alpha,i}^{(0)}, \underline{U}_{\alpha',i}^{(0)}) - \Gamma_{\alpha,i}(\underline{U}_{\alpha,i}^{(0)}, \overline{U}_{\alpha',i}^{(0)}),\\
&i \in \mathcal{I}, \quad W_{\alpha,i}^{(1)} = \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.
\end{aligned}$$

From here with $i = 1$, taking into account that $L_{\alpha,i} > O$, $R_{\alpha,i} > O$ from (4.3), $\eta = 0, 1$, $\underline{U}_{\alpha,i}^{(0)} = \widehat{U}_{\alpha,i}$, $\overline{U}_{\alpha,i}^{(0)} = \widetilde{U}_{\alpha,i}$, $\widehat{U}_{\alpha,i} \leq \widetilde{U}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $W_{\alpha,0}^{(1)} = \mathbf{0}$ and $(A_{\alpha,1} + C_{\alpha,1})^{-1} > O$ from (4.6), we conclude that $W_{\alpha,1}^{(1)} \geq \mathbf{0}$, $\alpha = 1, 2$. For $i = 2$, taking into account that $W_{\alpha,1}^{(1)} \geq 0$, $\alpha = 1, 2$ and using similar arguments as for $i = 1$, we prove that $W_{\alpha,2}^{(1)} \geq \mathbf{0}$, $\alpha = 1, 2$. By induction on i, we can prove that

$$W_{\alpha,i}^{(1)} \geq 0, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2. \tag{4.26}$$

Thus, we prove (4.21a).

From (4.23), we have

$$\begin{aligned}
&\mathcal{K}_{\alpha,i}(\overline{U}_{\alpha,i}^{(1)}, \underline{U}_{\alpha',i}^{(1)}) = -C_{\alpha,i}\overline{Z}_{\alpha,i}^{(1)} - R_{\alpha,i}\overline{Z}_{\alpha,i+1}^{(1)} + F_{\alpha,i}(\overline{U}_{\alpha,i}^{(1)}, \underline{U}_{\alpha',i}^{(1)}) - F_{\alpha,i}(\overline{U}_{\alpha,i}^{(0)}, \underline{U}_{\alpha',i}^{(0)}),\\
&i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.
\end{aligned}$$

From here, in notation (4.11), we obtain that

$$\mathcal{K}_{\alpha,i}(\overline{U}^{(1)}_{\alpha,i},\underline{U}^{(1)}_{\alpha',i}) = -R_{\alpha,i}\overline{Z}^{(1)}_{\alpha,i+1} + \Gamma_{\alpha,i}(\overline{U}^{(0)}_{\alpha,i},\underline{U}^{(0)}_{\alpha',i}) - \Gamma_{\alpha,i}(\overline{U}^{(1)}_{\alpha,i},\underline{U}^{(1)}_{\alpha',i}),$$
$$i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2.$$

Taking into account that $R_{\alpha,i} \geq O$, by using (4.22), we conclude that

$$\mathcal{K}_{\alpha,i}(\overline{U}^{(1)}_{\alpha,i},\underline{U}^{(1)}_{\alpha',i}) \geq \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2. \tag{4.27}$$

Similarly, we can prove

$$\mathcal{K}_{\alpha,i}(\underline{U}^{(1)}_{\alpha,i},\overline{U}^{(1)}_{\alpha',i}) \leq \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2. \tag{4.28}$$

From the boundary conditions with $i \in \partial\mathcal{I}$ in (4.23), it follows that $\overline{U}^{(1)}_{\alpha,i}$, $\underline{U}^{(1)}_{\alpha,i}$, $i \in \partial\mathcal{I}$, $\alpha = 1,2$, satisfy (4.7c). Thus, from here, (4.26)–(4.28), we conclude that $\overline{U}^{(1)}_{\alpha,i}$ and $\underline{U}^{(1)}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are ordered upper and lower solutions (4.21).

By induction on n, we can prove that $\{\overline{U}^{(n)}_{\alpha,i}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are monotone decreasing sequence of upper solutions and $\{\underline{U}^{(n)}_{\alpha,i}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are monotone increasing sequence of lower solutions which satisfy (4.21).

From (4.14), we conclude that $\lim \overline{U}^{(n)}_{\alpha,i} = \overline{U}_{\alpha,i}$ and $\lim \underline{U}^{(n)}_{\alpha,i} = \underline{U}_{\alpha,i}$ $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, as $n \to \infty$ exist, and

$$\lim_{n\to\infty} \overline{Z}^{(n)}_{\alpha,i} = \mathbf{0}, \quad \lim_{n\to\infty} \underline{Z}^{(n)}_{\alpha,i} = \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1,2.$$

From here and (4.23), we conclude that

$$\mathcal{K}_{\alpha,i}(\overline{U}_{\alpha,i},\underline{U}_{\alpha',i}) = \mathbf{0}, \quad \mathcal{K}_{\alpha,i}(\underline{U}_{\alpha,i},\overline{U}_{\alpha,i}) = \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2,$$

which means that $(\overline{U}_{1,i},\underline{U}_{2,i})$ and $(\underline{U}_{1,i},\overline{U}_{2,i})$, $i \in \overline{\mathcal{I}}$, are solutions to the nonlinear difference scheme (4.5).

The proof of (4.15) repeats the proof in Theorem 3.3.2, Chapter 3. □

4.2 Convergence analysis and constructions of initial iterates

4.2.1 The quasi-monotone nondecreasing case

A stopping test for the block monotone iterative methods (4.13) is chosen in the form

$$\begin{aligned} &\left\|\mathcal{K}_\alpha(U_\alpha^{(n)}, U_{\alpha'}^{(n)})\right\|_{\Omega^h} \le \delta, \\ &\left\|\mathcal{K}_\alpha(U_\alpha^{(n)}, U_{\alpha'}^{(n)})\right\|_{\Omega^h} = \max_{i\in\mathcal{I}} \left|\mathcal{K}_{\alpha,i}(U_{\alpha,i}^{(n)}, U_{\alpha',i}^{(n)})\right|, \quad \alpha' \ne \alpha, \quad \alpha,\alpha' = 1,2, \end{aligned} \tag{4.29}$$

where $\mathcal{K}_{\alpha,i}(U_{\alpha,i}^{(n)}, U_{\alpha',i}^{(n)})$, $i \in \mathcal{I}$, $\alpha = 1,2$, are defined in (4.7) and δ is a prescribed accuracy.

Theorem 4.2.1. *Assume that the assumptions in Theorem 3.4.3 are satisfied. Then for the sequences $\{U_{\alpha,i}^{(n)}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, generated by the block monotone iterative methods (4.13), (4.29), we have the estimate (3.85) from Theorem 3.5.1 in Chapter 3.*

Proof. The proof of the theorem repeats the proof of Theorem 3.5.1 with $U_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, rather than $U_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. □

Theorem 4.2.2. *Let the assumptions in Theorem 3.4.3 be satisfied. Then for the sequences $\{U_{\alpha,i}^{(n)}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, generated by the block monotone iterative methods (4.13), (4.29), the estimate (3.86) from Chapter 3, holds.*

Proof. The proof of the theorem repeats the proof of Theorem 3.5.2 with $U_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, rather than $U_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. □

4.2.2 The quasi-monotone nonincreasing case

For the sequences $\{\overline{U}_{1,i}^{(n)}, \underline{U}_{2,i}^{(n)}\}$ and $\{\underline{U}_{1,i}^{(n)}, \overline{U}_{2,i}^{(n)}\}$, $i \in \overline{\mathcal{I}}$, generated by (4.23), we introduce the notation

$$K = \begin{cases} \max\left\{\left\|\mathcal{K}_1\left(\overline{U}_1^{(n)}, \underline{U}_2^{(n)}\right)\right\|_{\mathcal{I}};\ \left\|\mathcal{K}_2\left(\overline{U}_1^{(n)}, \underline{U}_2^{(n)}\right)\right\|_{\mathcal{I}}\right\} \text{ for } \{\overline{U}_{1,i}^{(n)}, \underline{U}_{2,i}^{(n)}\}, \\ \max\left\{\left\|\mathcal{K}_1\left(\underline{U}_1^{(n)}, \overline{U}_2^{(n)}\right)\right\|_{\mathcal{I}};\ \left\|\mathcal{K}_2\left(\underline{U}_1^{(n)}, \overline{U}_2^{(n)}\right)\right\|_{\mathcal{I}}\right\} \text{ for } \{\underline{U}_{1,i}^{(n)}, \overline{U}_{2,i}^{(n)}\}, \end{cases} \tag{4.30}$$

where the residuals $\mathcal{K}_{\alpha,i}\left(U_{\alpha,i}^{(n)}, U_{\alpha',i}^{(n)}\right)$, $i \in \mathcal{I}$, $\alpha' \ne \alpha$, $\alpha,\alpha' = 1,2$, are defined in (4.7). A stopping test for the block monotone iterative methods (4.23) is chosen in the form

$$K \le \delta, \tag{4.31}$$

where K is defined in (4.30).

Theorem 4.2.3. *Assume that the assumptions in Theorem 3.4.5 are satisfied. Then for the sequences* $\{\overline{U}_{1,i}^{(n)}, \underline{U}_{2,i}^{(n)}\}$, $\{\underline{U}_{1,i}^{(n)}, \overline{U}_{2,i}^{(n)}\}$, $i \in \overline{\mathcal{I}}$, *generated by the block monotone iterative methods (4.23), (4.31), we have the estimate (3.91) from Chapter 3 holds.*

Proof. The proof of the theorem repeats the proof of Theorem 3.5.3 with $U_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, rather than $U_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$. $\square$

Theorem 4.2.4. *Let the assumptions in Theorem 3.5.3 be satisfied. Then for the sequences* $\{\overline{U}_{1,i}^{(n)}, \underline{U}_{2,i}^{(n)}\}$ *and* $\{\underline{U}_{1,i}^{(n)}, \overline{U}_{2,i}^{(n)}\}$, $i \in \overline{\mathcal{I}}$, *generated by the block monotone iterative methods (4.23), (4.31), we have the estimate (3.92) from Chapter 3 holds.*

Proof. The proof of the theorem repeats the proof of Theorem 3.5.4 from Chapter 3 with $U_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, rather than $U_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$. $\square$

4.2.3 Constructions of initial upper and lower iterates

In Section 3.6, for quasi-monotone nondecreasing and quasi-monotone nonincreasing reaction functions, we consider the constructions of initial upper and lower solutions in the cases of bounded reaction functions and constant initial iterates.

Constructions of initial iterates only depend on properties of corresponding reaction functions f_α, $\alpha = 1, 2$. Hence, the constructed initial iterates from Section 3.6 can be used as starting iterates for the block monotone iterative methods (4.13) and (4.23).

4.3 Comparison of convergence rates of the block monotone Jacobi and Gauss–Seidel methods

4.3.1 The quasi-monotone nondecreasing case

In the case of quasi-monotone nondecreasing reaction functions (3.23), the following theorem shows that the block monotone Gauss–Seidel method with $\eta = 1$ in (4.13) converges faster than the block monotone Jacobi method with $\eta = 0$ in (4.13).

Theorem 4.3.1. *Let* $(\widetilde{U}_{1,i}, \widetilde{U}_{2,i})$ *and* $(\widehat{U}_{1,i}, \widehat{U}_{2,i})$, $i \in \overline{\mathcal{I}}$, *be ordered upper and lower solutions (4.7), and the functions* f_α, $\alpha = 1, 2$, *in (3.1) satisfy (3.22) and (3.23). Suppose that the sequences* $\{(U_{\alpha,i}^{(n)})_J\}$ *and* $\{(U_{\alpha,i}^{(n)})_{GS}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, *are generated by the block monotone Jacobi method with* $\eta = 0$ *in (4.13) and by the block monotone Gauss–Seidel method with* $\eta = 1$ *in (4.13), where* $(\overline{U}_{\alpha,i}^{(0)})_J = (\overline{U}_{\alpha,i}^{(0)})_{GS} = \widetilde{U}_{\alpha,i}$ *and* $(\underline{U}_{\alpha,i}^{(0)})_J = (\underline{U}_{\alpha,i}^{(0)})_{GS} = \widehat{U}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$. *Then*

$$(\underline{U}_{\alpha,i}^{(n)})_J \le (\underline{U}_{\alpha,i}^{(n)})_{GS} \le (\overline{U}_{\alpha,i}^{(n)})_{GS} \le (\overline{U}_{\alpha,i}^{(n)})_J, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2. \tag{4.32}$$

Proof. Letting $\overline{W}^{(n)}_{\alpha,i} = (\overline{U}^{(n)}_{\alpha,i})_{\mathtt{J}} - (\overline{U}^{(n)}_{\alpha,i})_{\mathtt{GS}}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, from (4.13), we have

$$\begin{aligned} A_{\alpha,i}\overline{W}^{(n)}_{\alpha,i} + C_{\alpha,i}\overline{W}^{(n)}_{\alpha,i} &= \eta L_{\alpha,i}\left((\overline{U}^{(n)}_{\alpha,i-1})_{\mathtt{J}} - (\overline{U}^{(n-1)}_{\alpha,i-1})_{\mathtt{GS}}\right) + R_{\alpha,i}\overline{W}^{(n-1)}_{\alpha,i+1} \\ &\quad + \Gamma_{\alpha,i}\left((U^{(n-1)}_{\alpha,i})_{\mathtt{J}}, (U^{(n-1)}_{\alpha',i})_{\mathtt{J}}\right) \\ &\quad - \Gamma_{\alpha,i}\left((U^{(n-1)}_{\alpha,i})_{\mathtt{GS}}, (U^{(n-1)}_{\alpha',i})_{\mathtt{GS}}\right), \\ i \in \mathcal{I}, \quad \overline{W}^{(n)}_{\alpha,i} &= \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2. \end{aligned}$$

From here and taking into account that $\left(\overline{U}^{(n-1)}_{\alpha,i}\right)_{\mathtt{GS}} \le \left(\overline{U}^{(n)}_{\alpha,i}\right)_{\mathtt{GS}}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, it follows that

$$\begin{aligned} A_{\alpha,i}\overline{W}^{(n)}_{\alpha,i} + C_{\alpha,i}\overline{W}^{(n)}_{\alpha,i} &= \eta L_{\alpha,i}\overline{W}^{(n)}_{\alpha,i-1} + R_{\alpha,i}\overline{W}^{(n-1)}_{\alpha,i+1} + \Gamma_{\alpha,i}\left((U^{(n-1)}_{\alpha,i})_{\mathtt{J}}, (U^{(n-1)}_{\alpha',i})_{\mathtt{J}}\right) \\ &\quad - \Gamma_{\alpha,i}\left((U^{(n-1)}_{\alpha,i})_{\mathtt{GS}}, (U^{(n-1)}_{\alpha',i})_{\mathtt{GS}}\right), \qquad (4.33) \\ i \in \mathcal{I}, \quad \overline{W}^{(n)}_{\alpha,i} &= \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2. \end{aligned}$$

Taking into account that $(A_{\alpha,i} + C_{\alpha,i})^{-1} > O$ from (4.6), $L_{\alpha,i} \ge O$, $R_{\alpha,i} \ge O$, $i \in \mathcal{I}$, $\alpha = 1, 2$, from (4.3), $\eta = 0, 1$, $(\underline{U}^{(0)}_{\alpha,i})_{\mathtt{GS}} = (\underline{U}^{(0)}_{\alpha,i})_{\mathtt{J}}$, $\overline{W}^{(0)}_{\alpha,i} = \mathbf{0}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, and using the monotone property (4.12), we conclude for $n = 1$ in (4.33) that

$$\overline{W}^{(1)}_{\alpha,i} \ge \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2.$$

Similarly, from here and (4.33) with $n = 2$, we obtain that $\overline{W}^{(2)}_{\alpha,i} \ge \mathbf{0}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$. By induction on n, we can prove that $\overline{W}^{(n)}_{\alpha,i} \ge \mathbf{0}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$. Thus, we prove (4.32) for upper solutions. By following the same manner, we can prove (4.32) for lower solutions. □

4.3.2 The quasi-monotone nonincreasing case

In the case of quasi-monotone nonincreasing reaction functions (3.39), the following theorem shows that the block monotone Gauss–Seidel method with $\eta = 1$ in (4.13) converges faster than the block monotone Jacobi method with $\eta = 0$ in (4.13).

Theorem 4.3.2. *Let $(\widetilde{U}_{1,i}, \widehat{U}_{2,i})$ and $(\widehat{U}_{1,i}, \widetilde{U}_{2,i})$, $i \in \overline{\mathcal{I}}$ be ordered upper and lower solutions (4.21), and the functions f_α, $\alpha = 1, 2$, in (3.1) satisfy (3.38) and (3.39). Suppose that the sequences $\{(U^{(n)}_{\alpha,i})_J\}$ and $\{(U^{(n)}_{\alpha,i})_{GS}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are generated by the block monotone Jacobi method with $\eta = 0$ in (4.23) and by the block monotone Gauss–Seidel method with $\eta = 1$ in (4.23), where $(\overline{U}^{(0)}_{\alpha,i})_J = (\overline{U}^{(0)}_{\alpha,i})_{GS} = \widetilde{U}_{\alpha,i}$ and $(\underline{U}^{(0)}_{\alpha,i})_J = (\underline{U}^{(0)}_{\alpha,i})_{GS} = \widehat{U}_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$. Then (4.32) holds.*

Proof. The proof of the theorem repeats the proof of Theorem 4.3.1, where $\Gamma_{\alpha,i}$, $i \in \overline{\mathcal{I}}$,

$\alpha = 1, 2$, are written in the form

$$\Gamma_{\alpha,i}(\overline{U}^{(n)}_{\alpha,i}, \underline{U}^{(n)}_{\alpha',i}) = C_{\alpha,i}\overline{U}^{(n)}_{\alpha,i} - F_{\alpha,i}(\overline{U}^{(n)}_{\alpha,i}, \underline{U}^{(n)}_{\alpha',i}),$$
$$\Gamma_{\alpha,i}(\underline{U}^{(n)}_{\alpha,i}, \overline{U}^{(n)}_{\alpha',i}) = C_{\alpha,i}\underline{U}^{(n)}_{\alpha,i} - F_{\alpha,i}(\underline{U}^{(n)}_{\alpha,i}, \overline{U}^{(n)}_{\alpha',i}),$$

and the monotone property (4.22) for $\Gamma_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, is in use.

4.4 Numerical experiments

We present numerical experiments for test problems with quasi-monotone nondecreasing (4.7) and quasi-monotone nonincreasing (4.21) reaction functions f_α, $\alpha = 1, 2$, in (3.1). Exact solutions for our test problems are unknown, and numerical solutions are compared to corresponding reference solutions. The approximate solutions $U^{(n_m)}_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, are generated by either the block monotone methods (4.13), (4.29) or the block monotone methods (4.23), (4.31). In our tests, we choose the reference solutions with $N = 256$ and $\delta = 10^{-5}$ in (4.29) and (4.31). The reference solutions are calculated by the corresponding block method. □

4.4.1 Quasi-monotone nondecreasing case

Test 1

As the first test problem with quasi-monotone nondecreasing reaction functions (3.23), we consider Test 1 from Section 3.8.1 with the same data sets.

We calculate sequences of upper solutions generated by the block monotone iterative method (4.13), (4.29) and the initial iteration $(\widetilde{U}_{1,i}, \widetilde{U}_{2,i}) = (1, 1)$, $i \in \overline{\mathcal{I}}$.

In Table 4.1, we give number of iterations n_δ and execution (`CPU`) times for the block iterative methods and for the point monotone iterative methods from Table 3.2. From these results, we conclude that the block monotone Gauss-Seidel method converges faster than the block monotone Jacobi method, which confirms Theorem 4.3.1. Numerical data indicate that the block monotone Gauss–Seidel method is approximately twice as fast as the block monotone Jacobi method. The data in Table 4.1 show that the block monotone methods converge faster than the corresponding point monotone methods.

Table 4.1: Number of iterations n_δ and CPU times for Test 1.

N	8	16	32	64	128
			the block Jacobi method		
n_δ	101	397	1577	6299	25189
CPU(s)	0.02	0.11	0.91	14.17	225.99
			the block Gauss–Seidel method		
n_δ	51	180	762	3084	12370
CPU(s)	0.01	0.06	0.47	7.34	117.62
			the point Jacobi method		
n_δ	190	771	3092	12378	49520
CPU(s)	0.01	0.07	1.09	16.15	261.28
			the point Gauss–Seidel method		
n_δ	97	388	1548	6191	24762
CPU(s)	0.005	0.04	0.53	8.58	141.37

Test 2

As the second test problem with quasi-monotone nondecreasing reaction functions (3.23), we consider Test 2 from Section 3.8.1 with the same data sets.

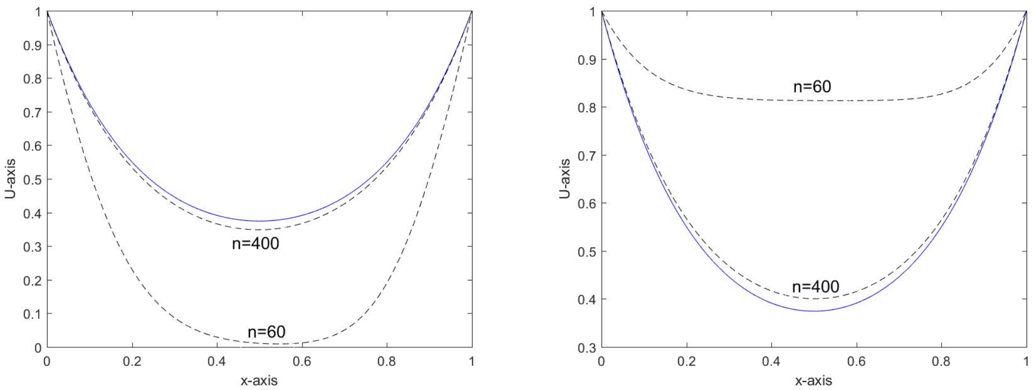
We calculate sequences of upper solutions generated by the block monotone iterative method (4.13), (4.29) and the initial iteration $(\widetilde{U}_{1,i}, \widetilde{U}_{2,i}) = (1, 1)$, $i \in \overline{\mathcal{I}}$.

In Table 4.2, we give numbers of iterations n_δ and execution (CPU) times for the block iterative methods and for the point monotone iterative methods from Table 3.4. From these results, we conclude that the block monotone Gauss-Seidel method converges faster than the block monotone Jacobi method, which confirms Theorem 4.3.1. Numerical data indicate that the block monotone Gauss–Seidel method is approximately twice as fast as the block monotone Jacobi method. The data in Table 4.2 show that the block monotone methods converge faster than the corresponding point monotone methods. In Figure 4.3, we show the convergence of numerical solutions, obtained by the block Gauss-Seidel method with $\eta = 1$ in (4.13) and $N = 64$ to the reference solution $N_{ref} = 256$, where the dashed line represents the numerical solution and the solid blue line refers to the reference solution with respect to x and fixed value of $y = 0.5$. In the subgraph 4.3a, staring from the initial lower solution $\widehat{U} = 0$, we show the convergence of the numerical lower solutions at $n_\delta = 60$ and $n_\delta = 400$ to the reference solution. Similarly, starting from the initial upper solution $\widetilde{U} = 1$, the subgraph 4.3b shows the convergence of the numerical upper solutions at $n_\delta = 60$ and $n_\delta = 400$ to the reference solution.

Table 4.2: Number of iterations n_δ and CPU times for Test 2.

N	8	16	32	64	128
the block Jacobi method					
n_δ	48	181	709	2820	11266
CPU(s)	0.01	0.04	0.4	5.74	88.26
the block Gauss–Seidel method					
n_δ	41	86	403	1645	6612
CPU(s)	0.05	0.06	0.28	3.41	55.39
the point Jacobi method					
n_δ	89	353	1409	5632	22525
CPU(s)	0.02	0.05	0.70	10.90	174.46
the point Gauss-Seidel method					
n_δ	46	178	706	2818	11264
CPU(s)	0.01	0.02	0.37	5.78	92.29

Figure 4.3: Convergence of lower and upper solutions to the reference solution for Test 2.



(a) Lower solutions. (b) Upper solutions.

4.4.2 Quasi-monotone nonincreasing case

Test 3

As the first test problem with quasi-monotone nonincreasing reaction functions (3.39), we consider the Volterra–Lotka competition model from Section 3.8.1 with the same data sets.

We calculate sequences of upper solutions generated by the block monotone iterative method (4.23), (4.31) and the initial iteration $(\widetilde{U}_{1,i}, \widehat{U}_{2,i}) = (1, 0)$, $i \in \overline{\mathcal{I}}$.

In Table 4.3, we give numbers of iterations n_δ and execution (CPU) times for the block monotone iterative methods and for the point monotone iterative methods from

Table 3.6. From these results, we conclude that the block monotone Gauss-Seidel method converges faster than the block monotone Jacobi method, which confirms Theorem 4.3.2. Numerical data indicate that the block monotone Gauss–Seidel method is approximately twice as fast as the block monotone Jacobi method. The data in Tables 4.3 show that the block monotone methods converge faster than the corresponding point monotone methods.

Table 4.3: Number of iterations n_δ and `CPU` times for Test 3.

N	8	16	32	64	128
the block Jacobi method					
n_δ	84	327	1301	5196	20776
`CPU`(s)	0.02	0.05	0.58	8.80	142.48
the block Gauss–Seidel method					
n_δ	48	147	617	2493	9994
`CPU`(s)	0.01	0.02	0.28	4.39	71.55
the point Jacobi method					
n_δ	155	623	2498	9999	40003
`CPU`(s)	0.03	0.17	1.29	18.61	281
the point Gauss–Seidel method					
n_δ	80	314	1251	5002	20004
`CPU`(s)	0.02	0.08	0.68	10.12	148.51

Test 4

As the second test problem with quasi-monotone nonincreasing reaction functions (3.39), we consider the Belousov-Zhabotinskii reaction diffusion model from Section 3.8.2 with the same data sets.

We calculate sequences of upper solutions generated by the block monotone iterative methods (4.23), (4.31) and the initial iteration $(\widetilde{U}_{1,i}, \widehat{U}_{2,i}) = (1, 0)$, $i \in \overline{\mathcal{I}}$.

In Table 4.4, we give numbers of iterations n_δ and execution (`CPU`) for the block monotone iterative methods and for the point monotone iterative methods from table 3.8. From these results, we conclude that the block monotone Gauss-Seidel method converges faster than the block monotone Jacobi method, which confirms Theorem 4.3.2. Numerical data indicate that the block monotone Gauss–Seidel method is approximately twice as fast as the block monotone Jacobi method. The data in Table 4.4 show that the block monotone methods converge faster than the corresponding point monotone methods.

Table 4.4: Number of iterations n_δ and `CPU` times for Test 4.

N	8	16	32	64	128
the block Jacobi method					
n_δ	80	289	1131	4500	17973
`CPU`(s)	0.006	0.06	0.63	9.72	153.84
the block Gauss–Seidel method					
n_δ	40	138	559	2242	8974
`CPU`(s)	0.004	0.03	0.26	3.95	63.86
the point Jacobi method					
n_δ	157	626	2501	10002	40007
`CPU`(s)	0.02	0.08	1.11	17.31	287.70
the point Gauss–Seidel method					
n_δ	77	311	1249	5000	20002
`CPU`(s)	0.01	0.05	0.59	9.26	152.73

4.5 Numerical experiments with convective terms

In the case when the elliptic problem (3.1) contains the convective terms, the implementation of the block monotone Gauss-Seidel method depends on approximations of the partial derivatives $u_{\alpha,x}$ and on the signs of the coefficients $v_\alpha^{(x)}$, $\alpha = 1, 2$, in convective terms.

If the central difference approximations (2.7) are in use, then the implementation of the block Gauss-Seidel method can be started from either $i = 0$ or $i = N_x$, that is, it can be started from either vertical sides of the computational domain.

When the one-sided difference approximations (2.9) are in use, we consider the following cases:

(i) If $v_\alpha^{(x)} \geq 0$, $\alpha = 1, 2$, then the backward difference approximations from (2.9) are in use, and the implementation of the block Gauss-Seidel method (4.13) is started from the left vertical side.

(ii) If $v_\alpha^{(x)} \leq 0$, $\alpha = 1, 2$, then the forward difference approximations from (2.9) are in use, and the implementation of the block Gauss-Seidel method is started from the right vertical side

$$A_{\alpha,i}Z_{\alpha,i}^{(n)} - R_{\alpha,i}Z_{\alpha,i-1}^{(n)} + C_{\alpha,i}Z_{\alpha,i}^{(n)} = -\mathcal{K}_{\alpha,i}(U_{\alpha,i}^{(n-1)}, U_{\alpha',i}^{(n-1)}), \quad i = \mathcal{I}, \mathcal{I}-1, \dots, 1, \tag{4.34}$$

$$Z_{\alpha,i}^{(n)} = \begin{cases} g_{\alpha,i} - U_{\alpha,i}^{(0)}, & n = 1, \\ \mathbf{0}, & n \geq 2, \end{cases} \quad i \in \partial\mathcal{I},$$

$$Z_{\alpha,i}^{(n)} = U_{\alpha,i}^{(n)} - U_{\alpha,i}^{(n-1)}, \quad i \in \overline{\mathcal{I}}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.$$

(iii) If, for example, $v_1^{(x)} \geq 0$ and $v_2^{(x)} \leq 0$, then the backward difference approximation from (2.9) for $u_{1,x}$ and the forward difference approximation from (2.9) for $u_{2,x}$ are in use. The implementation of the block Gauss-Seidel method is started from the left vertical side for $Z_{1,i}^{(n)}$, $i = 1, 2, \dots, \mathcal{I}$, (4.13) and from the right vertical side for $Z_{2,i}^{(n)}$, $i = \mathcal{I}, \mathcal{I}-1, \dots, 1$.

As a test problem with convective terms, we consider Test 2 from Section 3.8.1 with the constant coefficients $v_1^{(x)} = v$, $v_2^{(x)} = v$ and $v_\alpha^{(y)} = 0$, $\alpha = 1, 2$, in the elliptic problem (3.1). We choose the constant diffusion coefficients $D_1 = D$, $D_2 = D$, the initial iteration $(\widetilde{U}_{1,i}; \widetilde{U}_{2,i}) = (\mathbf{1}; \mathbf{1})$, $i \in \overline{\mathcal{I}}$ and calculate sequences of upper solutions generated by the block monotone Gauss-Seidel method with $\eta = 1$ in (4.13) and the stopping test (4.29).

In Table 4.5, for $v = 1, 10, 100$, different values of N and $D = 1,\ 10^{-1},\ 10^{-2},\ 10^{-3}$, by using the central difference approximations for $u_{\alpha,x}$, $\alpha = 1, 2$, we present numbers of iterations n_δ to satisfy the stopping test (4.29). From the numerical data in Table 4.5, we conclude that for fixed value of D, numbers of iterations are independent of the coefficient v in the convective terms, and for $N =$ fixed, numbers of iterations decrease when D decreases.

In Table 4.6, for $v = 1, 10, 100$, different values of N and $D = 1,\ 10^{-1},\ 10^{-2}$, 10^{-3}, by using the backward difference approximations for $u_{\alpha,x}$, $\alpha = 1, 2$, numbers of iterations n_δ are given. From the numerical data in Table 4.6, we conclude that for fixed values of D and N, numbers of iterations for $D = \mathrm{O}(1)$ decrease very fast when the coefficient v in the convective terms increases; number of iterations for sufficiently small values of D is almost independent of v; for fixed values of N and v, numbers of iterations decrease when D decreases. From the numerical data in Tables 4.5 and 4.6 , we can conclude that for $D = 1$ and $v = 1$, numbers of iterations are almost the same for both the central and backward difference approximations of $u_{\alpha,x}$, $\alpha = 1, 2$. For $D \leq 10^{-1}$ and $v = 1, 10, 100$, numbers of iterations for the backward difference approximations are less than for the central difference approximations. Thus, when the convective terms dominate the diffusion terms, the block monotone Gauss-Seidel method with the one-sided difference approximations of the first partial derivatives are more efficient than the block monotone Gauss-Seidel method with the central difference approximations.

Table 4.5: Number of iterations by using the central difference approximations.

D/N	16	32	64	128	256
			$v = 1, 10, 100$		
1	141	598	2422	9715	38886
10^{-1}	74	343	1400	5623	22516
10^{-2}	26	54	240	976	3919
10^{-3}	13	20	34	97	406

Table 4.6: Number of iterations by using the backward difference approximations.

D/N	16	32	64	128	256
			$v = 1$		
1	141	595	2403	9627	38511
10^{-1}	55	251	992	3897	15399
10^{-2}	31	54	114	347	1159
10^{-3}	14	15	18	24	42
			$v = 10$		
1	80	338	1314	5132	20229
10^{-1}	51	63	137	349	1282
10^{-2}	16	18	22	31	51
10^{-3}	16	16	17	19	21
			$v = 100$		
1	51	101	268	349	1307
10^{-1}	19	22	28	37	60
10^{-2}	18	19	21	22	26
10^{-3}	18	19	20	21	22

4.6 Conclusions to Chapter 4

Theoretical results

For solving nonlinear elliptic systems with quasi-monotone nondecreasing and nonincreasing reaction functions, we construct and investigate monotone properties of block Jacobi and block Gauss-Seidel iterative methods. For solving the nonlinear difference scheme (3.17) with quasi-monotone nondecreasing (3.23) and quasi-monotone nonincreasing (3.39) reaction functions, the block Jacobi and block Gauss-Seidel iterative methods are constructed. In Theorems 4.1.7 and 4.1.11, we prove that the sequences of upper and lower solutions, generated by the block monotone iterative methods for problems with quasi-monotone nondecreasing (3.23) and quasi-monotone nonincreasing (3.39) reaction functions, converge monotonically to the solutions of the nonlinear difference scheme. By using the stopping test (4.29) and (4.31), based on the norms of residuals, respectively, for the quasi-monotone nondecreasing and nonincreasing cases, we prove that the numerical solution converges to the unique solution of the nonlinear elliptic problem (3.1) and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme (3.17) in Theorems 4.2.1 and 3.5.2, and the error between the numerical solution and the exact solution of the elliptic system (3.1) in Theorems 4.2.3 and 4.2.4. We prove that the block monotone Gauss-Seidel methods converge faster than the block monotone Jacobi methods in Theorems 4.3.1 and 4.3.2, respectively, for the quasi-monotone nondecreasing and nonincreasing reaction cases. The construction methods of initial iterates from Section 3.6 depend only on properties of corresponding reaction functions and can be used as starting iterates for the block iterative methods (4.13) and (4.23).

Numerical results

The numerical sequences of solutions generated by block monotone methods (4.13) with stopping (4.29) and the block monotone methods (4.23) with stopping (4.31) converge monotonically. The block monotone Gauss-Seidel methods with $\eta = 1$ in (4.13) and (4.23) converge faster than the block monotone Jacobi methods with $\eta = 0$ in (4.13) and (4.23) which confirm, respectively, Theorems 4.3.1 and 4.3.2. The block monotone Gauss-Seidel methods are approximately twice as fast as the block Jacobi methods. For fixed diffusion coefficient D, the numbers of iterations n_δ increase with increasing N. The block monotone methods converge faster than the corresponding point monotone methods. The number of iterations n_δ and CPU times for the block Jacobi methods are very close to the data for the point Gauss-Seidel methods. When the convective terms dominate the diffusion terms, the block monotone Gauss-Seidel method with the one-sided difference approximations of the first derivatives are more efficient than the block monotone Gauss-Seidel method with the central difference approximations.

Chapter 5

Jacobi and Gauss-Seidel methods for systems of parabolic problems

This chapter deals with investigating numerical methods for solving coupled system of nonlinear parabolic problems by point iterative methods based on Jacobi and Gauss–Seidel methods. In the view of the method of upper and lower solutions, two monotone upper and lower sequences of solutions are constructed. Convergence rates for the point monotone iterative methods are estimated. We show that the sequences of solutions generated by the point monotone Gauss–Seidel method converge faster than by the point monotone Jacobi method. Constructions of initial upper and lower solutions are presented.

5.1 Properties of solutions to systems of nonlinear parabolic problems

We consider the system of nonlinear parabolic problems in the form

$$\begin{aligned} & u_{\alpha,t} - \mathrm{L}_\alpha u_\alpha(x,y,t) + f_\alpha(x,y,t,u) = 0, \quad (x,y,t) \in Q_T = \omega \times (0,T], \\ & \omega = \{(x,y) : 0 < x < l_1, \quad 0 < y < l_2\}, \\ & u_\alpha(x,y,t) = g_\alpha(x,y,t), \quad (x,y,t) \in \partial Q_T = \partial\omega \times (0,T], \\ & u_\alpha(x,y,0) = \psi_\alpha(x,y), \quad (x,y) \in \overline{\omega}, \quad \alpha = 1,2, \end{aligned} \tag{5.1}$$

where $u = (u_1, u_2)$, $\partial\omega$ is the boundary of ω and l_1, l_2 are positive constants. The differential operators L_α, $\alpha = 1,2$, are defined by

$$\mathrm{L}_\alpha u_\alpha(x,y,t) \equiv D_\alpha^{(x)}(x,y,t)u_{\alpha,xx} + D_\alpha^{(y)}(x,y,t)u_{\alpha,yy} + v_\alpha^{(x)}(x,y,t)u_{\alpha,x} + v_\alpha^{(y)}(x,y,t)u_{\alpha,y},$$

where $D_\alpha^{(x)}(x,y,t)$, $D_\alpha^{(y)}(x,y,t)$, $\alpha=1,2$, are positive functions. It is assumed that the functions $f_\alpha(x,y,t,u)$, $g_\alpha(x,y,t)$, $D_\alpha^{(x)}(x,y,t)$, $D_\alpha^{(y)}(x,y,t)$, $v_\alpha^{(x)}(x,y,t)$ and $v_\alpha^{(y)}(x,y,t)$, $\alpha=1,2$, are smooth in their respective domains.

5.1.1 Quasi-monotone nondecreasing case

Two vector functions $\widetilde{u}=(\widetilde{u}_1,\widetilde{u}_2)$ and $\widehat{u}=(\widehat{u}_1,\widehat{u}_2)$, are called ordered upper and lower solutions to (5.1), if they satisfy the inequalities

$$\widehat{u}(x,y,t)\le\widetilde{u}(x,y,t),\quad (x,y,t)\in\overline{Q}_T, \tag{5.2a}$$

$$\widehat{u}_{\alpha,t}-\mathrm{L}_\alpha\widehat{u}_\alpha+f_\alpha(x,y,t,\widehat{u})\le 0\le\widetilde{u}_{\alpha,t}-\mathrm{L}_\alpha\widetilde{u}_\alpha+f_\alpha(x,y,t,\widetilde{u}),\quad (x,y,t)\in Q_T, \tag{5.2b}$$

$$\widehat{u}(x,y,t)\le g(x,y,t)\le\widetilde{u}(x,y,t),\quad (x,y,t)\in\partial Q_T, \tag{5.2c}$$

$$\widehat{u}(x,y,0)\le\psi(x,y)\le\widetilde{u}(x,y,0),\quad (x,y)\in\overline{\omega},\quad \alpha=1,2,$$

For given ordered upper $\widetilde{u}$ and lower $\widehat{u}$ solutions, a sector $\langle\widehat{u},\widetilde{u}\rangle$ is defined as follows

$$\langle\widehat{u},\widetilde{u}\rangle=\left\{u(x,y,t):\quad \widehat{u}(x,y,t)\le u(x,y,t)\le\widetilde{u}(x,y,t),\quad (x,y,t)\in\overline{Q}_T\right\}. \tag{5.3}$$

In the sector $\langle\widehat{u},\widetilde{u}\rangle$, the functions $f_\alpha(x,y,t,u)$, $\alpha=1,2$, are assumed to satisfy the constraints

$$0\le\frac{\partial f_\alpha(x,y,t,u)}{\partial u_\alpha}\le c_\alpha(x,y,t),\quad u\in\langle\widehat{u},\widetilde{u}\rangle,\quad (x,y,t)\in\overline{Q}_T,\quad \alpha=1,2, \tag{5.4}$$

$$-\frac{\partial f_\alpha(x,y,t,u)}{\partial u_{\alpha'}}\ge 0,\quad u\in\langle\widehat{u},\widetilde{u}\rangle,\quad (x,y,t)\in\overline{Q}_T,\quad \alpha'\ne\alpha,\quad \alpha,\alpha'=1,2, \tag{5.5}$$

where $c_\alpha(x,y,t)$, $\alpha=1,2$, are nonnegative bounded functions. The reaction functions $f_\alpha(x,y,t,u)$, $\alpha=1,2$, are called quasi-monotone nondecreasing in $\langle\widehat{u},\widetilde{u}\rangle$, if they satisfy (5.5).

Theorem 5.1.1. *Let $\widetilde{u}=(\widetilde{u}_1,\widetilde{u}_2)$ and $\widehat{u}=(\widehat{u}_1,\widehat{u}_2)$ be ordered upper and lower solutions (5.2). Assume that the functions $f_\alpha(x,y,t,u)$, $\alpha=1,2$, in (5.1) satisfy (5.5). Then problem (5.1) has a unique solution in the sector $\langle\widehat{u},\widetilde{u}\rangle$.*

The proof of the theorem is given in Theorem 8.3.1, [59].

5.2 Quasi-monotone nonincreasing case

Introduce the following notation:

$$\mathcal{F}_\alpha(x,y,t,u_\alpha,u_{\alpha'})=\begin{cases}\mathcal{F}_1(x,y,t,u_1,u_2), & \alpha=1,\\ \mathcal{F}_2(x,y,t,u_1,u_2), & \alpha=2.\end{cases} \tag{5.6}$$

Two mesh functions $\widetilde{u}_\alpha(x,y,t)$ and $\widehat{u}_\alpha(x,y,t)$, $\alpha=1,2$, are called ordered upper and lower solutions to (5.1) in the case of quasi-monotone nonincreasing reaction functions f_α, $\alpha=1,2$, if they satisfy the inequalities

$$\widehat{u}_\alpha(x,y,t) \le \widetilde{u}_\alpha(x,y,t), \quad (x,y,t)\in\overline{Q}_T, \tag{5.7a}$$

$$\begin{aligned} &\widehat{u}_{\alpha,t} - \mathrm{L}_\alpha \widehat{u}_\alpha + f_\alpha(x,y,t,\widehat{u}_\alpha,\widetilde{u}_{\alpha'}) \le 0, \quad (x,y,t)\in Q_T, \\ &\widetilde{u}_{\alpha,t} - \mathrm{L}_\alpha \widetilde{u}_\alpha + f_\alpha(x,y,t,\widetilde{u}_\alpha,\widehat{u}_{\alpha'}) \ge 0, \quad (x,y,t)\in Q_T, \end{aligned} \tag{5.7b}$$

$$\begin{aligned} &\widehat{u}_\alpha(x,y,t) \le g_\alpha(x,y,t) \le \widetilde{u}_\alpha(x,y,t), \quad (x,y,t)\in\partial Q_T, \\ &\widehat{u}_\alpha(x,y,0) \le \psi_\alpha(x,y) \le \widetilde{u}_\alpha(x,y,0), \quad (x,y)\in\overline{\omega}, \quad \alpha'\ne\alpha, \quad \alpha,\alpha'=1,2, \end{aligned} \tag{5.7c}$$

where notation (5.6) is in use.

In the sector $\langle\widehat{u},\widetilde{u}\rangle$ from (5.3), the functions $f_\alpha(x,y,t,u)$, $\alpha=1,2$, are assumed to satisfy (5.4) and the constraint

$$-\frac{\partial f_\alpha(x,y,t,u)}{\partial u_{\alpha'}} \le 0, \quad u\in\langle\widehat{u},\widetilde{u}\rangle, \quad (x,y,t)\in\overline{Q}_T, \quad \alpha'\ne\alpha, \quad \alpha,\alpha'=1,2. \tag{5.8}$$

The reaction functions $f_\alpha(x,y,t,u)$, $\alpha=1,2$, are called quasi-monotone nonincreasing in $\langle\widehat{u},\widetilde{u}\rangle$, if they satisfy (5.8).

Theorem 5.2.1. *Let $\widetilde{u}=(\widetilde{u}_1,\widetilde{u}_2)$ and $\widehat{u}=(\widehat{u}_1,\widehat{u}_2)$ be ordered upper and lower solutions (5.7). Assume that the functions $f_\alpha(x,y,t,u)$, $\alpha=1,2$, in (5.1) satisfy (5.4) and (5.8). Then problem (5.1) has a unique solution in the sector $\langle\widehat{u},\widetilde{u}\rangle$.*

The proof of the theorem is given in Theorem 8.3.2, [59].

5.3 The nonlinear difference scheme

On $\overline{\omega}$ and $[0,T]$, we introduce a rectangular mesh $\overline{\mathcal{A}}^h=\overline{\Lambda}^{h_x}\times\overline{\Lambda}^{h_y}$ and $\overline{\Lambda}^\tau$, such that

$$\begin{aligned} &\overline{\Lambda}^{h_x} = \{x_i, \quad i=0,1,\ldots,N_x; \quad x_0=0, \quad x_{N_x}=l_1; \;\; h_x=x_{i+1}-x_i\}, \\ &\overline{\Lambda}^{h_y} = \{y_j, \quad j=0,1,\ldots,N_y; \quad y_0=0, \quad y_{N_y}=l_2; \;\; h_y=y_{j+1}-y_j\}, \\ &\overline{\Lambda}^{\tau} = \{t_m, \quad m=0,1,\ldots,N_\tau; \quad t_0=0, \quad t_{N_\tau}=T; \;\; \tau=t_m-t_{m-1}\}. \end{aligned} \tag{5.9}$$

We denote by Ω^h, $\partial\Omega^h$ and Ω^τ the sets of indices which correspond to interior space mesh points, boundary space mesh points and time mesh points, such that

$$\begin{aligned} &\Omega^h = \{(i,j): \quad i=1,2,\ldots,N_x-1, \quad j=1,2,\ldots,N_y-1\}, \\ &\partial\Omega^h = \{(i,j): \quad i=0,N_x, \quad j=0,1,\ldots,N_y; \quad i=0,1,\ldots,N_x, \quad j=0,N_y\}, \\ &\overline{\Omega}^\tau = \{m: \quad m=0,1,\ldots,N_\tau\}. \end{aligned}$$

For $(i,j,m) \in \overline{\Omega}^h \times \overline{\Omega}^\tau = (\Omega^h \cup \partial\Omega^h) \times \overline{\Omega}^\tau$, we introduce the notation

$$\mathcal{T}_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) = \begin{cases} \mathcal{T}_{1,ij,m}(U_{1,ij,m}, U_{2,ij,m}), & \alpha = 1, \\ \mathcal{T}_{2,ij,m}(U_{1,ij,m}, U_{2,ij,m}), & \alpha = 2. \end{cases} \tag{5.10}$$

By using the central difference approximations for the first and second derivatives on the 5-point stencil, we introduce the nonlinear two-time levels difference scheme

$$\begin{aligned} &\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\right) U_{\alpha,ij,m} + f_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) - \tau^{-1} U_{\alpha,ij,m-1} = 0, \\ &(i,j,m) \in \Omega^{h\tau} = \Omega^h \times \Omega^\tau, \quad U_{\alpha,ij,m} = g_{\alpha,ij,m}, \quad (i,j,m) \in \partial\Omega^{h\tau} = \partial\Omega^h \times \Omega^\tau, \\ &U_{\alpha,ij,0} = \psi_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \end{aligned} \tag{5.11}$$

where $f_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m})$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1,2$, are defined by (5.10) and the central difference approximations for the first and second derivatives are given by

$$\begin{aligned} \mathcal{D}_x^2 U_{\alpha,ij,m} &= \frac{U_{\alpha,i-1,j,m} - 2U_{\alpha,ij,m} + U_{\alpha,i+1,j,m}}{h_x^2}, \\ \mathcal{D}_y^2 U_{\alpha,ij,m} &= \frac{U_{\alpha,i,j-1,m} - 2U_{\alpha,ij,m} + U_{\alpha,i,j+1,m}}{h_y^2}, \\ \mathcal{D}_x^1 U_{\alpha,ij,m} &= \frac{U_{\alpha,i+1,j,m} - U_{\alpha,i-1,j,m}}{2h_x}, \\ \mathcal{D}_y^1 U_{\alpha,ij,m} &= \frac{U_{\alpha,i,j+1,m} - U_{\alpha,i,j-1,m}}{2h_y}, \quad \alpha = 1,2, \quad m \geq 1. \end{aligned} \tag{5.12}$$

The difference operators $\mathcal{A}_{\alpha,ij,m} U_{\alpha,ij,m}$, $\alpha = 1,2$, in (5.11), are defined by

$$\begin{aligned} &\mathcal{A}_{\alpha,ij,m} U_{\alpha,ij,m} = \mathcal{A}^{(x)}_{\alpha,ij,m} U_{\alpha,ij,m} + \mathcal{A}^{(y)}_{\alpha,ij,m} U_{\alpha,ij,m}, \\ &\mathcal{A}^{(x)}_{\alpha,ij,m} U_{\alpha,ij,m} = -l_{\alpha,ij,m} U_{\alpha,i-1,j,m} + \frac{2D^{(x)}_{\alpha,ij,m} U_{\alpha,ij,m}}{h_x^2} - r_{\alpha,ij,m} U_{\alpha,i+1,j,m}, \\ &\mathcal{A}^{(y)}_{\alpha,ij,m} U_{\alpha,ij,m} = -b_{\alpha,ij,m} U_{\alpha,i,j-1,m} + \frac{2D^{(y)}_{\alpha,ij,m} U_{\alpha,ij,m}}{h_y^2} - q_{\alpha,ij,m} U_{\alpha,i,j+1,m}, \\ &l_{\alpha,ij,m} = \frac{D^{(x)}_{\alpha,ij,m}}{h_x^2} - \frac{v^{(x)}_{\alpha,ij,m}}{2h_x}, \quad r_{\alpha,ij,m} = \frac{D^{(x)}_{\alpha,ij,m}}{h_x^2} + \frac{v^{(x)}_{\alpha,ij,m}}{2h_x}, \\ &b_{\alpha,ij,m} = \frac{D^{(y)}_{\alpha,ij,m}}{h_y^2} - \frac{v^{(y)}_{\alpha,ij,m}}{2h_y}, \quad q_{\alpha,ij,m} = \frac{D^{(y)}_{\alpha,ij,m}}{h_y^2} + \frac{v^{(y)}_{\alpha,ij,m}}{2h_y}, \quad \alpha = 1,2, \quad m \geq 1. \end{aligned} \tag{5.13}$$

To ensure that $l_{\alpha,ij,m}$, $r_{\alpha,ij,m}$, $b_{\alpha,ij,m}$ and $q_{\alpha,ij}$, $\alpha = 1,2$, are positive, we choose space step sizes h_x and h_y such that

$$h_x < \frac{2D^{(x)}_{\alpha,ij,m}}{|v^{(x)}_{\alpha,ij}|}, \quad h_y < \frac{2D^{(y)}_{\alpha,ij,m}}{|v^{(y)}_{\alpha,ij,m}|}. \tag{5.14}$$

Remark 5.3.1. *If the effect of convection dominates diffusion to the extent that these conditions require prohibitively small h_x and h_y, then an upwind difference scheme for the first derivatives can be used to remove any restrictions on h_x and h_y, that is, for $\alpha = 1,2$,*

$$\mathcal{D}'_x U_{\alpha,ij,m} = \begin{cases} \frac{U_{\alpha,i+1,j,m}-U_{\alpha,ij,m}}{h_x}, & if\ v^{(x)}_{\alpha,ij,m} \le 0, \\ \frac{U_{\alpha,ij,m}-U_{\alpha,i-1,j,m}}{h_x}, & if\ v^{(x)}_{\alpha,ij,m} \ge 0, \end{cases}$$

$$\mathcal{D}'_y U_{\alpha,ij,m} = \begin{cases} \frac{U_{\alpha,i,j+1,m}-U_{\alpha,ij,m}}{h_y}, & if\ v^{(y)}_{\alpha,ij,m} \le 0, \\ \frac{U_{\alpha,ij,m}-U_{\alpha,i,j-1,m}}{h_y}, & if\ v^{(y)}_{\alpha,ij,m} \ge 0. \end{cases}$$

On each time level t_m, $m \ge 1$, we introduce the linear version of problem (5.11)

$$\begin{aligned} &\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1} + c^*_{\alpha,ij,m}\right) W_{\alpha,ij,m} = \varphi_{\alpha,ij,m}, \quad (i,j) \in \Omega^h, \\ &U_{\alpha,ij,m} = g_{\alpha,ij,m}, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1,2, \end{aligned} \tag{5.15}$$

where $c^*_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \ge 1$, are nonnegative bounded mesh functions. In the following lemma, we formulate the maximum principle for the difference operators $\mathcal{A}_{\alpha,ij,m} + \tau^{-1} + c^*_{\alpha,ij,m}$, $\alpha = 1,2$.

Lemma 5.3.2. *If $W_{\alpha,ij,m}$, $\alpha = 1,2$, satisfy the conditions*

$$\begin{aligned} &\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1} + c^*_{\alpha,ij,m}\right) W_{\alpha,ij,m} \ge 0 \ (\le 0), \quad (i,j) \in \Omega^h, \\ &W_{\alpha,ij,m} \ge 0 \ (\le 0), \quad (i,j) \in \partial\Omega^h, \end{aligned}$$

then $W_{\alpha,ij,m} \ge 0 \ (\le 0), \quad (i,j) \in \overline{\Omega}^h$.

The proof is given in Lemma 1.2.1 from Chapter 1.

Remark 5.3.3. *In this remark, we state the mean-value theorem for vector-valued functions. Assume that $f_\alpha(x,y,t,u_\alpha,u_{\alpha'})$, $\alpha' \ne \alpha$, $\alpha, \alpha' = 1,2$, are smooth functions, then we have*

$$f_\alpha(x,y,t,u_\alpha,u_{\alpha'}) - f_\alpha(x,y,t,w_\alpha,u_{\alpha'}) = \left(f_\alpha(x,y,t,q_\alpha,u_{\alpha'})\right)_{u_\alpha}[u_\alpha - w_\alpha], \tag{5.16}$$

$$f_\alpha(x,y,t,u_\alpha,u_{\alpha'}) - f_\alpha(x,y,t,u_\alpha,w_{\alpha'}) = \left(f_\alpha(x,y,t,u_\alpha,h_{\alpha'}\right)_{u_{\alpha'}}[u_{\alpha'} - w_{\alpha'}],$$

where $q_\alpha(x,y,t)$ and $h_\alpha(x,y,t)$ lie between $u_\alpha(x,y,t)$ and $w_\alpha(x,y,t)$, $(x,y,t) \in \overline{Q}_T$, $\alpha = 1,2$, and notation (5.10) is in use.

5.3.1 Quasi-monotone nondecreasing case

On each time level $t_m \in \Omega^\tau$, $m \geq 1$, two vector mesh functions

$$\widetilde{U}_{ij,m} = (\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m}), \quad \widehat{U}_{ij,m} = (\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m}), \quad (i,j) \in \overline{\Omega}^h,$$

are called ordered upper and lower solutions of (5.10), if they satisfy the inequalities

$$\widehat{U}_{\alpha,ij,m} \leq \widetilde{U}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \tag{5.17a}$$
$$\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\right) \widehat{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widehat{U}_{ij,m}) - \tau^{-1}\widehat{U}_{\alpha,ij,m-1} \leq 0, \quad (i,j) \in \Omega^h, \tag{5.17b}$$
$$\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\right) \widetilde{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widetilde{U}_{ij,m}) - \tau^{-1}\widetilde{U}_{\alpha,ij,m-1} \geq 0, \quad (i,j) \in \Omega^h,$$
$$\widehat{U}_{\alpha,ij,m} \leq g_{\alpha,ij,m} \leq \widetilde{U}_{\alpha,ij,m}, \quad (i,j) \in \partial\Omega^h, \tag{5.17c}$$
$$\widehat{U}_{\alpha,ij,0} \leq \psi_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij,0}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \geq 1.$$

For a given pair of ordered upper and lower solutions $\widetilde{U}_{ij,m}$ and $\widehat{U}_{ij,m}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, we define the sector

$$\langle \widehat{U}_m, \widetilde{U}_m \rangle\rangle = \left\{ U_{ij,m} : \widehat{U}_{ij,m} \leq U_{ij,m} \leq \widetilde{U}_{ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1 \right\}. \tag{5.18}$$

In the sector $\langle \widehat{U}_m, \widetilde{U}_m \rangle$ from (5.18), we assume that the functions $f_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, satisfy the constraints

$$\frac{\partial f_{\alpha,ij,m}(U_{ij,m})}{\partial u_\alpha} \leq c_{\alpha,ij,m}, \quad U \in \langle \widehat{U}_m, \widetilde{U}_m \rangle, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \tag{5.19}$$
$$-\frac{\partial f_{\alpha,ij,m}}{\partial u_{\alpha'}} \geq 0, \quad U \in \langle \widehat{U}_m, \widetilde{U}_m \rangle, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \tag{5.20}$$

where $c_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are nonnegative bounded functions. We say that the functions $f_{\alpha,ij,m}(U_{ij,m})$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are quasi-monotone nondecreasing in the sector $\langle \widehat{U}_m, \widetilde{U}_m \rangle$ from (5.18) if they satisfy (5.20).

Remark 5.3.4. *For quasi-monotone nondecreasing functions (5.20), upper and lower solutions (5.17) are independent.*

We introduce the notation

$$\Gamma_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) = c_{\alpha,ij,m} U_{\alpha,ij,m} - f_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}), \tag{5.21}$$
$$(i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \quad m \geq 1.$$

where $c_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are defined in (5.19) and notation (5.10) is in use. We give a monotone property of $\Gamma_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m})$, $(i,j) \in \overline{\Omega}^h$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1,2$, $m \geq 1$.

Lemma 5.3.5. *Suppose that* $U_{ij,m} = (U_{1,ij,m}, U_{2,ij,m})$ *and* $V_{ij,m} = (V_{1,ij,m}, V_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, *are two vector functions in the sector* $\langle \widehat{U}_m, \widetilde{U}_m \rangle$ *from (5.18), such that* $U_{ij,m} \geq V_{ij,m}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, *and (5.19), (5.20) are satisfied. Then*

$$\Gamma_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) \geq \Gamma_{\alpha,ij,m}(V_{\alpha,ij,m}, V_{\alpha',ij,m}), \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \quad m \geq 1. \tag{5.22}$$

Proof. From (5.21), we have

$$\Gamma_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) - \Gamma_{\alpha,ij,m}(V_{\alpha,ij,m}, V_{\alpha',ij,m}) =$$
$$c_{\alpha,ij,m}(U_{\alpha,ij,m} - V_{\alpha,ij,m}) - \left[f_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) - f_{\alpha,ij,m}(V_{\alpha,ij,m}, U_{\alpha',ij,m})\right] -$$
$$\left[f_{\alpha,ij,m}(V_{\alpha,ij,m}, U_{\alpha',ij,m}) - f_{\alpha,ij,m}(V_{\alpha,ij,m}, V_{\alpha',ij,m})\right], \quad (i,j) \in \overline{\Omega}^h,$$
$$\alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \quad m \geq 1.$$

Using the mean-value theorem (5.16), we have

$$\Gamma_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) - \Gamma_{\alpha,ij,m}(V_{\alpha,ij,m}, V_{\alpha',ij,m}) =$$
$$\left(c_{\alpha,ij,m} - \left(f_{\alpha,ij,m}(Q_{\alpha,ij,m}, U_{\alpha',ij,m})\right)_{u_\alpha}\right)(U_{\alpha,ij,m} - V_{\alpha,ij,m}) -$$
$$\left(f_{\alpha,ij,m}(V_{\alpha,ij,m}, Y_{\alpha',ij,m})\right)_{u_{\alpha'}}(U_{\alpha',ij,m} - V_{\alpha',ij,m}),$$
$$V_{\alpha,ij,m} \leq Q_{\alpha,ij,m}, Y_{\alpha,ij,m} \leq U_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \quad m \geq 1.$$

Taking into account that $U_{\alpha,ij,m} \geq V_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, from (5.19) and (5.20), we conclude (5.22). □

5.3.1.1 Applied problems

The gas-liquid interaction model

In section 3.2.1.1, we consider the steady-state gas-liquid interaction model. Here, we consider the time-dependent gas-liquid interaction model in the form

$$u_{\alpha,t} - D_\alpha \triangle u_\alpha + f_\alpha(u_1, u_2) = 0, \quad (x,y,t) \in Q_T,$$
$$u_1(x,y,t) = g_1^*(x,y,t) \geq 0, \quad u_2(x,y,t) = g_2(x,y,t) \geq 0, \quad (x,y,t) \in \partial Q_T,$$
$$u_\alpha(x,y,0) = \psi_\alpha(x,y), \quad (x,y) \in \overline{\omega}, \quad \alpha = 1,2,$$

where the reaction functions f_α, $\alpha = 1,2$, are defined in (3.27), $g_1^* = \rho_1 - g_1 \geq 0$, $g_2 \geq 0$ on $\partial\omega$ and $\psi_\alpha \geq 0$, $\alpha = 1,2$, in $\overline{\omega}^h$. The nonlinear difference scheme (5.11) for the model is presented in the form

$$
\begin{aligned}
&(\mathcal{A}_{\alpha,ij,m}+\tau^{-1})U_{\alpha,ij,m}+f_{\alpha,ij,m}(U_{\alpha,ij,m},U_{\alpha',ij,m})-\tau^{-1}U_{\alpha,ij,m-1}=0, \quad (i,j)\in\Omega^h,\\
&U_{1,ij,m}=g^*_{1,ij,m}, \quad U_{2,ij,m}=g_{2,ij,m}, \quad (i,j)\in\partial\Omega^h, \quad m\geq 1, \qquad (5.23)\\
&U_{\alpha,ij,0}=\psi_{\alpha,ij}, \quad (i,j)\in\overline{\Omega}^h, \quad \alpha'\neq\alpha, \quad \alpha,\alpha'=1,2,
\end{aligned}
$$

where f_α, $\alpha=1,2$, are defined in (3.27), and

$$
\mathcal{A}_{\alpha,ij,m}U_{\alpha,ij,m}=-D_\alpha(\mathcal{D}_x^2+\mathcal{D}_y^2)U_{\alpha,ij,m}, \quad (i,j)\in\Omega^h, \quad \alpha=1,2, \quad m\geq 1,
$$

where $\mathcal{D}_x^2$, $\mathcal{D}_y^2$ are defined in (5.12). Introduce the following mesh functions

$$
\begin{aligned}
&\widetilde{U}_{\alpha,ij,m}=\begin{cases}\psi_{\alpha,ij}, & m=0,\\ K_\alpha, & m\geq 1,\end{cases} \quad (i,j)\in\overline{\Omega}^h, \quad \alpha=1,2, \qquad (5.24)\\
&\widehat{U}_{\alpha,ij,m}=\begin{cases}\psi_{\alpha,ij}, & m=0,\\ 0, & m\geq 1,\end{cases} \quad (i,j)\in\overline{\Omega}^h, \quad \alpha=1,2,
\end{aligned}
$$

where K_α, $\alpha=1,2$, satisfy the conditions

$$
\begin{aligned}
&K_1\geq\max\left(\max_{(x,y,t)\in\partial Q_T}g_1^*(x,y,t),\max(\rho_1,\max_{(x,y)\in\overline{\omega}}\psi_1(x,y))\right),\\
&K_2\geq\max\left(\max_{(x,y,t)\in\partial Q_T}g_2(x,y,t),\max_{(x,y)\in\overline{\omega}}\psi_2(x,y)\right).
\end{aligned}
$$

We now show that these mesh functions are ordered upper and lower solutions (5.17) to (5.23).

From (5.24), we conclude (5.17a). From (3.27) and (5.24), for $m=1$, we have

$$
\begin{aligned}
&(\mathcal{A}_{1,ij,1}+\tau^{-1})\widetilde{U}_{1,ij,1}+f_{1,ij,1}(\widetilde{U}_{1,ij,1},\widetilde{U}_{2,ij,1})-\tau^{-1}\widetilde{U}_{1,ij,0}=\\
&(\mathcal{A}_{1,ij,1}+\tau^{-1})K_1-\sigma_1(\rho_1-K_1)K_2-\tau^{-1}\psi_{1,ij}=\\
&\tau^{-1}(K_1-\psi_{1,ij})-\sigma_1(\rho_1-K_1)K_2\geq 0.
\end{aligned}
$$

From (3.27) and (5.24), for $m=2$, we have

$$
\begin{aligned}
&(\mathcal{A}_{1,ij,1}+\tau^{-1})\widetilde{U}_{1,ij,2}+f_{1,ij,2}(\widetilde{U}_{1,ij,2},\widetilde{U}_{2,ij,2})-\tau^{-1}\widetilde{U}_{1,ij,1}=\\
&(\mathcal{A}_{1,ij,1}+\tau^{-1})K_1-\sigma_1(\rho_1-K_1)K_2-\tau^{-1}K_1=\\
&-\sigma_1(\rho_1-K_1)K_2\geq 0.
\end{aligned}
$$

By induction on $m \geq 1$, we can prove that

$$\big(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\big)\, \widetilde{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widetilde{U}_{\alpha,ij,m}, \widetilde{U}_{\alpha',ij,m}) - \tau^{-1}\widetilde{U}_{\alpha,ij,m-1} \geq 0, \quad (i,j) \in \Omega^h,$$
$$\alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \quad m \geq 1.$$

Similarly, we can prove that

$$\big(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\big)\, \widehat{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widehat{U}_{\alpha,ij,m}, \widehat{U}_{\alpha',ij,m}) - \tau^{-1}\widehat{U}_{\alpha,ij,m-1} \leq 0, \quad (i,j) \in \Omega^h,$$
$$\alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \quad m \geq 1.$$

Hence, we conclude (5.17b). From (5.24), it follows (5.17c). Thus, we prove that $\widetilde{U}_{\alpha,ij,m}$ and $\widetilde{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, from (5.24) are ordered upper and lower solutions (5.17). From (3.27), in the sector $\langle \widehat{U}_m, \widetilde{U}_m \rangle$, for $m \geq 1$, we have

$$\frac{\partial f_{1,ij,m}}{\partial u_1}(U_{1,ij,m}, U_{2,ij,m}) = \sigma_1 U_{2,ij,m} \leq \sigma_1 K_2, \quad (i,j) \in \overline{\Omega}^h,$$
$$\frac{\partial f_{2,ij,m}}{\partial u_2}(U_{1,ij,m}, U_{2,ij,m}) = \sigma_2(\varrho_1 - U_{1,ij,m}) \leq \sigma_2 \rho_1, \quad (i,j) \in \overline{\Omega}^h,$$
$$-\frac{\partial f_{1,ij,m}}{\partial u_2} = \sigma_1(\varrho_1 - U_{1,ij,m}) \geq 0, \quad -\frac{\partial f_{2,ij,m}}{\partial u_1} = \sigma_2 U_{2,ij,m} \geq 0, \quad (i,j) \in \overline{\Omega}^h.$$

Thus, the assumptions in (5.19) are satisfied with

$$c_{1,ij,m} = \sigma_1 K_2, \quad c_{2,ij,m} = \sigma_2 \rho_1, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1.$$

From here, we conclude that f_α, $\alpha = 1,2$, from (3.27) satisfy (5.19) and possess quasi-monotone nondecreasing property (5.20).

The Volterra-Lotka cooperation model

Consider the Volterra-Lotka cooperation model in an ecological system (more details are given in [59]). The model is governed by (5.1) with $\mathrm{L}_\alpha u_\alpha = \triangle u_\alpha$, $\alpha = 1,2$, and

$$f_1 = -u_1(1 - u_1 + a_1 u_2), \quad f_2 = -u_2(1 + a_2 u_1 - u_2), \tag{5.25}$$

where u_1 and u_2 are the populations of two cooperating species, the parameters a_α, $\alpha = 1,2$, are positive constants which describe the interaction of the two species, which satisfy the inequality

$$a_1 < \frac{1}{a_2}. \tag{5.26}$$

System (5.1) is reduced to

$$
\begin{aligned}
&u_{\alpha,t} - D_\alpha \triangle u_\alpha + f_\alpha(u_1,u_2) = 0, \quad (x,y,t) \in Q_T,\\
&u_\alpha(x,y,t) = 0, \quad (x,y,t) \in \partial Q_T, \quad u_\alpha(x,y,0) = \psi_\alpha(x,y), \quad (x,y) \in \overline{\omega}, \quad \alpha = 1,2.
\end{aligned}
$$

The nonlinear difference scheme (5.11) for the model is presented in the form

$$
\begin{aligned}
&\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\right) U_{\alpha,ij,m} + f_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) - \tau^{-1} U_{\alpha,ij,m-1} = 0, \quad (i,j) \in \Omega^h,\\
&U_{\alpha,ij,m} = 0, \quad (i,j) \in \partial\Omega^h, \quad m \geq 1, \quad U_{\alpha,ij,0} = \psi_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h,\\
&\alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,
\end{aligned} \tag{5.27}
$$

where f_α, $\alpha = 1,2$, are defined in (5.25), and

$$
\mathcal{A}_{\alpha,ij,m} U_{\alpha,ij,m} = -D_\alpha(\mathcal{D}_x^2 + \mathcal{D}_y^2) U_{\alpha,ij,m}, \quad (i,j) \in \Omega^h, \quad \alpha = 1,2, \quad m \geq 1,
$$

where $\mathcal{D}_x^2$, $\mathcal{D}_y^2$ are defined in (5.12). Introduce the following mesh functions

$$
\begin{aligned}
&(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m}) = (M_1, M_2), \quad (\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m}) = (0,0), \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1, \\
&M_1 = a_1 M_2 + 1,\\
&M_2 \geq \max\left\{\frac{a_2+1}{1-a_1a_2}, \max_{(i,j)\in\overline{\Omega}^h} \psi_{2,ij}, \max_{(i,j)\in\overline{\Omega}^h} g_{2,ij},\right.\\
&\qquad\qquad \left.\frac{1}{a_1}\Big(\max_{(i,j)\in\overline{\Omega}^h} \psi_{1,ij} - 1\Big), \frac{1}{a_1}\Big(\max_{(i,j)\in\overline{\Omega}^h} g_{1,ij} - 1\Big)\right\}.
\end{aligned} \tag{5.28}
$$

We now show that these mesh functions are ordered upper and lower solutions (5.17) to (5.27). From (5.28), it follows (5.17a). From (5.25), (5.26) and (5.28), we have

$$
\begin{aligned}
&\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\right) \widetilde{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widetilde{U}_{\alpha,ij,m}, \widetilde{U}_{\alpha',ij,m}) - \tau^{-1} \widetilde{U}_{\alpha,ij,m-1} \geq 0,\\
&\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\right) \widehat{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widehat{U}_{\alpha,ij,m}, \widehat{U}_{\alpha',ij,m}) - \tau^{-1} \widehat{U}_{\alpha,ij,m-1} \leq 0,\\
&(i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2.
\end{aligned}
$$

Hence, we conclude (5.17b). From (5.28), it follows (5.17c). Thus, $\widetilde{U}_{\alpha,ij,m}$ and $\widehat{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, from (5.28) are ordered upper and lower solutions (5.17).

From (5.25), in the sector $\langle \widehat{U}_m, \widetilde{U}_m \rangle$, for $m \geq 1$, we have

$$\frac{\partial f_{1,ij,m}}{\partial u_1}(U_{1,ij,m}, U_{2,ij,m}) = 2U_{1,ij,m} - a_1 U_{2,ij,m} - 1 \leq 2M_1, \quad (i,j) \in \overline{\Omega}^h,$$
$$\frac{\partial f_{2,ij,m}}{\partial u_2}(U_{1,ij,m}, U_{2,ij,m}) = 2U_{2,ij,m} - a_2 U_{1,ij,m} - 1 \leq 2M_2, \quad (i,j) \in \overline{\Omega}^h,$$
$$-\frac{\partial f_{1,ij,m}}{\partial u_2} = a_1 U_{1,ij,m} \geq 0, \quad -\frac{\partial f_{2,ij,m}}{\partial u_1} = a_2 U_{2,ij,m} \geq 0, \quad (i,j) \in \overline{\Omega}^h.$$

Thus, the assumptions in (5.19) are satisfied with

$$c_{1,ij,m} = 2M_1, \quad c_{2,ij,m} = 2M_2, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1.$$

From here, we conclude that f_α, $\alpha = 1,2$, from (5.25) satisfy (5.19) and possess quasi-monotone nondecreasing property (5.20).

5.3.2 Quasi-monotone nonincreasing case

On each time level $t_m \in \Omega^\tau$, $m \geq 1$, two vector mesh functions

$$\widetilde{U}_{ij,m} = (\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m}), \quad \widehat{U}_{ij,m} = (\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m}), \quad (i,j) \in \overline{\Omega}^h,$$

are called ordered upper and lower solutions of (5.10), if they satisfy the inequalities

$$\widehat{U}_{\alpha,ij,m} \leq \widetilde{U}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \tag{5.29a}$$
$$\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\right) \widetilde{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widetilde{U}_{\alpha,ij,m}, \widehat{U}_{\alpha',ij,m}) - \tau^{-1}\widetilde{U}_{\alpha,ij,m-1} \geq 0, \quad (i,j) \in \Omega^h, \tag{5.29b}$$
$$\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\right) \widehat{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widehat{U}_{\alpha,ij,m}, \widetilde{U}_{\alpha',ij,m}) - \tau^{-1}\widehat{U}_{\alpha,ij,m-1} \leq 0, \quad (i,j) \in \Omega^h,$$
$$\widehat{U}_{\alpha,ij,m} \leq g_{\alpha,ij,m} \leq \widetilde{U}_{\alpha,ij,m}, \quad (i,j) \in \partial\Omega^h, \tag{5.29c}$$
$$\widehat{U}_{\alpha,ij,0} \leq \psi_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij,0}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \quad m \geq 1,$$

where notation (5.10) is in use.

In the sector $\langle \widehat{U}_m, \widetilde{U}_m \rangle$ from (5.18), we assume that the functions $f_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, in (5.11), satisfy (5.19) and the constraint

$$-\frac{\partial f_{\alpha,ij,m}(U_{\alpha,ij,m})}{\partial u_{\alpha'}} \leq 0, \quad U \in \langle \widehat{U}_m, \widetilde{U}_m \rangle, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2. \tag{5.30}$$

We say that the functions $f_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are quasi-monotone nonincreasing in the sector $\langle \widehat{U}_m, \widetilde{U}_m \rangle$ if they satisfy (5.30).

Remark 5.3.6. *For quasi-monotone nonincreasing functions f_α, $\alpha = 1,2$, (5.30), upper and lower solutions (5.29) are coupled.*

We give a monotone property of $\Gamma_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, from (5.21) in the quasi-monotone nonincreasing case.

Lemma 5.3.7. *Suppose that $U_{ij,m} = (U_{1,ij,m}, U_{2,ij,m})$ and $V_{ij,m} = (V_{1,ij,m}, V_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are two vector functions in $\langle \widehat{U}_m, \widetilde{U}_m \rangle$, such that $U_{ij,m} \geq V_{ij,m}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, and (5.19) and (5.30) are satisfied. Then*

$$\Gamma_\alpha(U_{1,ij,m}, V_{2,ij,m}) \geq \Gamma_\alpha(V_{1,ij,m}, U_{2,ij,m}), \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \geq 1. \tag{5.31}$$

Proof. From (5.21), we have

$$\begin{aligned} &\Gamma_{\alpha,ij,m}(U_{1,ij,m}, V_{2,ij,m}) - \Gamma_{\alpha,ij,m}(V_{1,ij,m}, U_{2,ij,m}) = \\ &c_{\alpha,ij,m}(U_{\alpha,ij,m} - V_{\alpha,ij,m}) - [f_{\alpha,ij,m}(U_{1,ij,m}, V_{2,ij,m}) - f_{\alpha,ij,m}(V_{1,ij,m}, V_{2,ij,m})] \\ &+ [f_{\alpha,ij,m}(V_{1,ij,m}, U_{2,ij,m}) - f_{\alpha,ij,m}(V_{1,ij,m}, V_{2,ij,m})]. \end{aligned} \tag{5.32}$$

Using the mean-value theorem (5.16), we obtain

$$\begin{aligned} &\Gamma_{\alpha,ij,m}(U_{\alpha,ij,m}, V_{\alpha',ij,m}) - \Gamma_{\alpha,ij,m}(V_{\alpha,ij,m}, U_{\alpha',ij,m}) = \\ &\left(c_{\alpha,ij,m} - \left(f_{\alpha,ij,m}(Q_{\alpha,ij,m}, V_{\alpha',ij,m})\right)_{u_\alpha}\right)(U_{\alpha,ij,m} - V_{\alpha,ij,m}) + \\ &\left(f_{\alpha,ij,m}(V_{\alpha,ij,m}, Y_{\alpha',ij,m})\right)_{u_{\alpha'}}(U_{\alpha',ij,m} - V_{\alpha',ij,m}), \\ &V_{\alpha,ij,m} \leq Q_{\alpha,ij,m}, Y_{\alpha,ij,m} \leq U_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \quad m \geq 1. \end{aligned}$$

Taking into account that $U_{\alpha,ij,m} \geq V_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, from (5.19) and (5.30), we conclude (5.31). □

5.3.2.1 Applied problems

The Belousov-Zhabotinskii reaction diffusion system

The Belousov-Zhabotinskii reaction diffusion model includes the metal-ion-catalyzed oxidation by bromate ion of organic materials ([59], some background to the model is also given in [65]). The chemical reaction scheme is given by

$$A_1 + Y \to X, \quad X + Y \to P_1, \quad A_2 + X \to 2X + Z, \quad 2X \to P_2, \quad Z \to \lambda Y,$$

where A_1 and A_2 are constants which represent reactants, P_1 and P_2 are products, λ is the stoichiometric factor, and X, Y and Z are, respectively, the concentrations of the intermediates HBrO2 (bromous acid), Br^- (bromide ion) and Ce(IV)(cerium). A simplified system of two equations of the above reactant scheme is governed by (5.1) with $L_\alpha u_\alpha = D_\alpha \triangle u_\alpha$, $\alpha = 1,2$, where u_1 and u_2 represent, respectively, the concentrations of X and Y [39]. The reaction functions are given by

$$f_1 = -u_1(a - bu_1 - \sigma_1 u_2), \quad f_2 = \sigma_2 u_1 u_2, \tag{5.33}$$

where a, b, σ_α, $\alpha = 1, 2$, are positive constants. System (5.1) is reduced to

$$\begin{aligned}
&u_{\alpha,t} - D_\alpha \triangle u_\alpha + f_\alpha(u_1, u_2) = 0, \quad (x, y, t) \in Q_T, \\
&u_\alpha(x, y, t) = g_\alpha(x, y, t) \geq 0, \quad (x, y, t) \in \partial Q_T, \\
&u_\alpha(x, y, 0) = \psi_\alpha(x, y) \geq 0, \quad (x, y) \in \overline{\omega}, \quad \alpha = 1, 2.
\end{aligned}$$

The nonlinear difference scheme (5.11) for the model is presented in the form

$$\begin{aligned}
&(\mathcal{A}_{\alpha,ij,m} + \tau^{-1})U_{\alpha,ij,m} + f_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) - \tau^{-1}U_{\alpha,ij,m-1} = 0, \quad (i, j) \in \Omega^h, \\
&U_{\alpha,ij,m} = g_{\alpha,ij,m}, \quad (i, j) \in \partial\Omega^h, \quad U_{\alpha,ij,0} = \psi_{\alpha,ij}, \quad (i, j) \in \overline{\Omega}^h, \\
&\alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2, \quad m \geq 1.
\end{aligned} \tag{5.34}$$

where f_α, $\alpha = 1, 2$, are defined in (5.33), and

$$\mathcal{A}_{\alpha,ij,m}U_{\alpha,ij,m} = -D_\alpha(\mathcal{D}_x^2 + \mathcal{D}^2{}_y)U_{\alpha,ij,m}, \quad (i, j) \in \Omega^h, \quad \alpha = 1, 2, \quad m \geq 1,$$

where $\mathcal{D}_x^2$, $\mathcal{D}_y^2$ are defined in (5.12). We introduce the mesh functions

$$\widetilde{U}_{\alpha,ij,m} = \begin{cases} \psi_{\alpha,ij}, & m = 0, \\ K_\alpha, & m \geq 1, \end{cases} \quad (i, j) \in \overline{\Omega}^h, \quad \alpha = 1, 2, \tag{5.35}$$

$$\widehat{U}_{\alpha,ij,m} = \begin{cases} \psi_{\alpha,ij}, & m = 0, \\ 0, & m \geq 1, \end{cases} \quad (i, j) \in \overline{\Omega}^h, \quad \alpha = 1, 2,$$

where K_α, $\alpha = 1, 2$, satisfy the conditions

$$\begin{aligned}
K_1 &\geq \max\left(a/b, \max_{(x,y,t)\in\partial Q_T} g_1(x, y, t), \max_{(x,y)\in\overline{\omega}} \psi_1(x, y)\right), \\
K_2 &\geq \max\left(\max_{(x,y,t)\in\partial Q_T} g_2(x, y, t), \max_{(x,y)\in\overline{\omega}} \psi_2(x, y)\right).
\end{aligned}$$

We now show that these mesh functions are ordered upper and lower solutions (5.29) to (5.34). From (5.35), it follows (5.29a). From (5.33) and (5.35), for $m = 1$, we have

$$\begin{aligned}
&\left(\mathcal{A}_{1,ij,1} + \tau^{-1}\right) \widetilde{U}_{1,ij,1} + f_{1,ij,1}(\widetilde{U}_{1,ij,1}, \widehat{U}_{2,ij,1}) - \tau^{-1}\widetilde{U}_{1,ij,0} = \\
&\tau^{-1}K_1 - K_1(a - bK_1) - \tau^{-1}\psi_{1,ij} \geq 0, \\
&\left(\mathcal{A}_{2,ij,1} + \tau^{-1}\right) \widetilde{U}_{2,ij,1} + f_{2,ij,1}(\widehat{U}_{1,ij,1}, \widetilde{U}_{2,ij,1}) - \tau^{-1}\widetilde{U}_{2,ij,0} = \\
&\tau^{-1}K_2 - \tau^{-1}\psi_{2,ij} \geq 0, \quad (i,j) \in \Omega^h.
\end{aligned}$$

From (5.33) and (5.35), for $m = 2$, we obtain

$$\begin{aligned}
&\left(\mathcal{A}_{\alpha,ij,2} + \tau^{-1}\right) \widetilde{U}_{\alpha,ij,2} + f_{\alpha,ij,2}(\widetilde{U}_{\alpha,ij,2}, \widehat{U}_{\alpha',ij,2}) - \tau^{-1}\widetilde{U}_{1,ij,1} = 0, \quad (i,j) \in \Omega^h, \\
&\alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.
\end{aligned}$$

By induction on m, $m \geq 1$, we can prove that

$$\begin{aligned}
&\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\right) \widetilde{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widetilde{U}_{\alpha,ij,m}, \widehat{U}_{\alpha',ij,m}) - \tau^{-1}\widetilde{U}_{\alpha,ij,m-1} \geq 0, \quad (i,j) \in \Omega^h, \\
&\alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.
\end{aligned}$$

From (5.33) and (5.35), for $m = 1$, we have

$$\begin{aligned}
&\left(\mathcal{A}_{\alpha,ij,1} + \tau^{-1}\right) \widehat{U}_{\alpha,ij,1} + f_{\alpha,ij,1}(\widehat{U}_{\alpha,ij,1}, \widetilde{U}_{\alpha',ij,1}) - \tau^{-1}\widehat{U}_{\alpha,ij,0} = -\tau^{-1}\psi_{\alpha,ij} \leq 0, \\
&(i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.
\end{aligned}$$

From (5.33) and (5.35), for $m = 2$, we obtain

$$\begin{aligned}
&\left(\mathcal{A}_{\alpha,ij,2} + \tau^{-1}\right) \widehat{U}_{\alpha,ij,2} + f_{\alpha,ij,2}(\widehat{U}_{\alpha,ij,2}, \widetilde{U}_{\alpha',ij,2}) - \tau^{-1}\widehat{U}_{\alpha,ij,1} = -\tau^{-1}K_\alpha \leq 0, \\
&(i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.
\end{aligned}$$

By induction on m, $m \geq 1$, we can prove that

$$\begin{aligned}
&\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1}\right) \widehat{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widehat{U}_{\alpha,ij,m}, \widetilde{U}_{\alpha',ij,m}) - \tau^{-1}\widehat{U}_{\alpha,ij,m-1} \leq 0, \quad (i,j) \in \Omega^h, \\
&(i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.
\end{aligned}$$

Hence, we conclude (5.29b). From (5.35), it follows (5.29c). Thus, $\widetilde{U}_{\alpha,ij,m}$ and $\widetilde{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \geq 1$, from (5.35) are ordered upper and lower solutions (5.29).

From (5.33), in the sector $\langle \widehat{U}_m, \widetilde{U}_m \rangle$, for $m \geq 1$, we have

$$\frac{\partial f_{1,ij,m}}{\partial u_1}(U_{1,ij,m}, U_{2,ij,m}) = 2bU_{1,ij,m} + \sigma_1 U_{2,ij,m} - a \leq 2bK_1 + \sigma_1 K_2, \quad (i,j) \in \overline{\Omega}^h,$$
$$\frac{\partial f_{2,ij,m}}{\partial u_2}(U_{1,ij,m}, U_{2,ij,m}) = \sigma_2 U_{1,ij,m} \leq \sigma_2 K_1, \quad (i,j) \in \overline{\Omega}^h,$$
$$-\frac{\partial f_{1,ij,m}}{\partial u_2}(U_{1,ij,m}, U_{2,ij,m}) = -\sigma_1 U_{1,ij,m} \leq 0, \quad (i,j) \in \overline{\Omega}^h,$$
$$-\frac{\partial f_{2,ij,m}}{\partial u_1}(U_{1,ij,m}, U_{2,ij,m}) = -\sigma_2 U_{2,ij,m} \leq 0, \quad (i,j) \in \overline{\Omega}^h.$$

Thus, the assumptions in (5.19) are satisfied with

$$c_{1,ij,m} = 2bK_1 + \sigma_1 K_2, \quad c_{2,ij,m} = \sigma_2 K_1, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1.$$

From here, we conclude that f_α, $\alpha = 1, 2$, from (5.33) satisfy (5.19) and possess quasi-monotone nonincreasing property (5.30).

Enzyme-substrate reaction diffusion model

In section 3.2.1.1, we consider the steady-state enzyme substrate reaction diffusion model. Here, we consider the time-dependent enzyme substrate reaction diffusion model with the reaction functions given in the original form [59]

$$f_1 = a_1 u_1 u_2 - b_1(E_0 - u_2), \quad f_2 = a_2 u_1 u_2 - b_2(E_0 - u_2). \tag{5.36}$$

System (5.1) is reduced to

$$\begin{aligned} &u_{\alpha,t} - D_\alpha \triangle u_\alpha + f_\alpha(u_1, u_2) = 0, \quad (x,y,t) \in Q_T, \\ &u_\alpha(x,y,t) = g_\alpha(x,y,t) \geq 0, \quad (x,y,t) \in \partial Q_T, \\ &u_\alpha(x,y,0) = \psi_\alpha(x,y), \quad (x,y) \in \overline{\omega}, \quad \alpha = 1, 2, \\ &E_0 \geq \max_{(x,y) \in \overline{\omega}} \psi_2(x,y), \end{aligned} \tag{5.37}$$

where the reaction functions f_α, $\alpha = 1, 2$, are defined in (5.36) and E_0 is defined in (3.31). The nonlinear difference scheme (5.11) for the model is presented in the form

$$\begin{aligned} &(\mathcal{A}_{\alpha,ij,m} + \tau^{-1})U_{\alpha,ij,m} + f_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) - \tau^{-1}U_{\alpha,ij,m-1} = 0, \quad (i,j) \in \Omega^h, \\ &U_{\alpha,ij,m} = g_{\alpha,ij,m} \geq 0, \quad (i,j) \in \partial\Omega^h, \quad U_{\alpha,ij,0} = \psi_{\alpha,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \\ &\alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2, \quad m \geq 1. \end{aligned} \tag{5.38}$$

where f_α, $\alpha = 1, 2$, are defined in (5.36), and

$$\mathcal{A}_{\alpha,ij,m} U_{\alpha,ij,m} = -D_\alpha (\mathcal{D}_x^2 + \mathcal{D}^2{}_y) U_{\alpha,ij,m}, \quad (i,j) \in \Omega^h, \quad \alpha = 1, 2, \quad m \geq 1,$$

where $\mathcal{D}_x^2$, $\mathcal{D}_y^2$ are defined in (5.12). Denote by $V_{ij,m}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, solutions to the linear problems

$$\begin{aligned} &(\mathcal{A}_{1,ij,m} + \tau^{-1}) V_{ij,m} = \tau^{-1} V_{ij,m-1} + M_0, \quad (i,j) \in \Omega^h, \\ &V_{ij,m} = g_{1,ij,m}, \quad (i,j) \in \partial\Omega^h, \quad V_{ij,0} = \psi_{1,ij}, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1, \\ &M_0 = \text{const} > b_1 E_0. \end{aligned} \tag{5.39}$$

We show that the functions

$$(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m}) = (V_{ij,m}, E_0), \quad (\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m}) = (0,0), \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1, \tag{5.40}$$

are ordered upper and lower solutions (5.29) to (5.38). Firstly, we prove that $V_{ij,m} \geq 0$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$. From (5.39), for $m = 1$, we obtain that

$$\begin{aligned} &(\mathcal{A}_{1,ij,1} + \tau^{-1}) V_{ij,1} = \tau^{-1} \psi_{1,ij} + M_0, \quad (i,j) \in \Omega^h, \\ &V_{ij,1} = g_{1,ij,1}, \quad (i,j) \in \partial\Omega^h, \quad V_{ij,0} = \psi_{1,ij}, \quad i,j \in \overline{\Omega}^h. \end{aligned}$$

Taking into account that $\psi_{1,ij} \geq 0$, $(i,j) \in \overline{\Omega}^h$, we have

$$\begin{aligned} &(\mathcal{A}_{1,ij,1} + \tau^{-1}) V_{ij,m} \geq 0, \quad (i,j) \in \Omega^h, \\ &V_{ij,1} = g_{1,ij,1}, \quad (i,j) \in \partial\Omega^h, \quad V_{ij,0} = \psi_{1,ij}, \quad (i,j) \in \overline{\Omega}^h. \end{aligned}$$

Using the maximum principle in Lemma 5.4.1, we obtain

$$V_{ij,1} \geq 0, \quad (i,j) \in \overline{\Omega}^h.$$

From here and (5.39), for $m = 2$, by using the maximum principle in Lemma 5.4.1, we have

$$V_{ij,2} \geq 0, \quad (i,j) \in \overline{\Omega}^h.$$

By induction on m, $m \geq 1$, we can prove that

$$V_{ij,m} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1.$$

From here, taking into account that the total enzyme $E_0 > 0$ and (5.39), it follows that the upper and lower solutions from (5.40) satisfy (5.29a). From (5.36), (5.39) and

(5.40), for $\alpha = 1$, we have

$$\left(\mathcal{A}_{1,ij,m} + \tau^{-1}\right) \widetilde{U}_{1,ij,m} + f_{1,ij,m}(\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m}) - \tau^{-1}\widetilde{U}_{1,ij,m-1} = M_0 - b_1 E_0 \geq 0,$$
$$(i,j) \in \Omega^h, \quad m \geq 1.$$

From (5.36), (5.37) and (5.40), for $\alpha = 2$ and $m = 1$, we obtain

$$\left(\mathcal{A}_{2,ij,1} + \tau^{-1}\right) \widetilde{U}_{2,ij,1} + f_{2,ij,1}(\widehat{U}_{1,ij,1}, \widetilde{U}_{2,ij,1}) - \tau^{-1}\widetilde{U}_{2,ij,0} = \tau^{-1}(E_0 - \psi_{2,ij}) \geq 0,$$
$$(i,j) \in \Omega^h.$$

For $m = 2$, it follows that

$$\left(\mathcal{A}_{2,ij,2} + \tau^{-1}\right) \widetilde{U}_{2,ij,2} + f_{2,ij,2}(\widehat{U}_{1,ij,2}, \widetilde{U}_{2,ij,2}) - \tau^{-1}\widetilde{U}_{2,ij,1} = 0, \quad (i,j) \in \Omega^h.$$

By induction on m, $m \geq 1$, we can prove that

$$\left(\mathcal{A}_{2,ij,m} + \tau^{-1}\right) \widetilde{U}_{2,ij,m} + f_{2,ij,2}(\widehat{U}_{1,ij,m}, \widetilde{U}_{2,ij,m}) - \tau^{-1}\widetilde{U}_{2,ij,m-1} \geq 0, \quad (i,j) \in \Omega^h, \quad m \geq 1.$$

From (5.36) and (5.40), for $\alpha = 1$ and $m = 1$, we have

$$\left(\mathcal{A}_{1,ij,1} + \tau^{-1}\right) \widehat{U}_{1,ij,1} + f_{1,ij,1}(\widehat{U}_{1,ij,1}, \widetilde{U}_{2,ij,1}) - \tau^{-1}\widehat{U}_{1,ij,0} = -\tau^{-1}\psi_{1,ij} \leq 0, \quad (i,j) \in \Omega^h.$$

For $m = 2$, we have

$$\left(\mathcal{A}_{1,ij,2} + \tau^{-1}\right) \widehat{U}_{1,ij,2} + f_{1,ij,2}(\widehat{U}_{1,ij,2}, \widetilde{U}_{2,ij,2}) - \tau^{-1}\widehat{U}_{1,ij,1} = 0, \quad (i,j) \in \Omega^h.$$

By induction on m, $m \geq 1$, we can prove that

$$\left(\mathcal{A}_{1,ij,m} + \tau^{-1}\right) \widehat{U}_{1,ij,m} + f_{1,ij,m}(\widehat{U}_{1,ij,m}, \widetilde{U}_{2,ij,m}) - \tau^{-1}\widehat{U}_{1,ij,m-1} \leq 0, \quad (i,j) \in \Omega^h,$$
$$m \geq 1.$$

From (5.36) and (5.40), for $\alpha = 2$ and $m = 1$, we have

$$\left(\mathcal{A}_{2,ij,1} + \tau^{-1}\right) \widehat{U}_{2,ij,1} + f_{2,ij,1}(\widetilde{U}_{1,ij,1}, \widehat{U}_{2,ij,1}) - \tau^{-1}\widehat{U}_{2,ij,0} = -(b_2 E_0 + \tau^{-1}\psi_{2,ij}) \leq 0,$$
$$(i,j) \in \Omega^h.$$

For $m = 2$, it follows that

$$\left(\mathcal{A}_{2,ij,2} + \tau^{-1}\right) \widehat{U}_{2,ij,2} + f_{2,ij,2}(\widetilde{U}_{1,ij,2}, \widehat{U}_{2,ij,2}) - \tau^{-1}\widehat{U}_{2,ij,1} = -b_2 E_0 \leq 0, \quad (i,j) \in \Omega^h.$$

By induction on m, $m \geq 1$, we can prove that

$$\left(\mathcal{A}_{2,ij,m} + \tau^{-1}\right) \widehat{U}_{2,ij,m} + f_{2,ij,m}(\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m}) - \tau^{-1}\widehat{U}_{2,ij,m-1} \leq 0, \quad (i,j) \in \Omega^h,$$
$$m \geq 1.$$

Hence, we conclude (5.29b). From (5.40), it follows (5.29c). Thus, $\widetilde{U}_{\alpha,ij,m}$ and $\widetilde{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, from (5.40) are ordered upper and lower solutions (5.29). From (5.36), in the sector $\langle \widehat{U}_m, \widetilde{U}_m \rangle$, for $m \geq 1$, we have

$$\begin{aligned}
&\frac{\partial f_{1,ij,m}}{\partial u_1}(U_{1,ij,m}, U_{2,ij,m}) = a_1 U_{2,ij,m} \leq a_1 E_0, \quad (i,j) \in \overline{\Omega}^h, \\
&\frac{\partial f_{2,ij,m}}{\partial u_2}(U_{1,ij,m}, U_{2,ij,m}) = a_2 U_{1,ij,m} + b_2 \leq a_2 V_{ij,m} + b_2, \quad (i,j) \in \overline{\Omega}^h, \\
&-\frac{\partial f_{1,ij,m}}{\partial u_2}(U_{1,ij,m}, U_{2,ij,m}) = -(a_1 U_{1,ij,m} + b_1) \leq 0, \quad (i,j) \in \overline{\Omega}^h, \\
&-\frac{\partial f_{2,ij,m}}{\partial u_1}(U_{1,ij,m}, U_{2,ij,m}) = -a_2 U_{2,ij,m} \leq 0, \quad (i,j) \in \overline{\Omega}^h.
\end{aligned}$$

Thus, the assumptions in (5.19) are satisfied with

$$c_{1,ij,m} = a_1 E_0, \quad c_{2,ij,m} = a_2 V_{ij,m} + b_2, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1.$$

From here, we conclude that f_α, $\alpha = 1,2$, from (5.36) satisfy (5.19) and possess quasi-monotone nonincreasing property (5.30).

5.4 The point monotone Jacobi and Gauss-Seidel methods

On each time level $m \geq 1$, at interior mesh points $(i,j) \in \Omega^h$, the difference scheme (5.11), (5.13) can be written in the following form

$$\begin{aligned}
&d_{\alpha,ij,m} U_{\alpha,ij,m} - l_{\alpha,ij,m} U_{\alpha,i-1,j,m} - r_{\alpha,ij,m} U_{\alpha,i+1,j,m} - b_{\alpha,ij,m} U_{\alpha,i,j-1,m} \qquad (5.41a) \\
&- q_{\alpha,ij,m} U_{\alpha,i,j+1,m} + \tau^{-1} U_{\alpha,ij,m} = -f_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) + \tau^{-1} U_{\alpha,ij,m-1}, \\
&(i,j) \in \Omega^h, \\
&U_{\alpha,ij,m} = g_{\alpha,ij,m}, \quad (i,j) \in \partial\Omega^h, \quad m \geq 1, \quad U_{\alpha,ij,0} = \psi_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \\
&\alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \\
&d_{\alpha,ij,m} = l_{\alpha,ij,m} + r_{\alpha,ij,m} + b_{\alpha,ij,m} + q_{\alpha,ij,m}, \quad l_{\alpha,ij,m}, r_{\alpha,ij,m}, b_{\alpha,ij,m}, q_{\alpha,ij,m} > 0, \\
&\qquad (5.41b)
\end{aligned}$$

where $l_{\alpha,ij,m}$, $r_{\alpha,ij,m}$, $b_{\alpha,ij,m}$ and $q_{\alpha,ij,m}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, $m \geq 1$, are defined in (5.13).

In the following lemma, we formulate the maximum principle for the difference operators $d_{\alpha,ij,m} + \tau^{-1} + c^*_{\alpha,ij,m}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, $m \geq 1$.

Lemma 5.4.1. *If $W_{\alpha,ij,m}$, $\alpha = 1,2$, satisfy the conditions*

$$\left(d_{\alpha,ij,m} + \tau^{-1} + c^*_{\alpha,ij,m}\right) W_{\alpha,ij,m} \geq 0 \ (\leq 0), \quad (i,j) \in \Omega^h,$$
$$W_{\alpha,ij,m} \geq 0 \ (\leq 0), \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1,2 \quad m \geq 1,$$

then $W_{\alpha,ij,m} \geq 0$ (≤ 0), $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$.

The proof is given in Lemma 1.2.1 from Chapter 1.

5.4.1 Quasi-monotone nondecreasing case

The definition of the ordered upper $\widetilde{U}_{\alpha,ij,m}$ and lower $\widehat{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, solutions (5.17) can be written in the form

$$\widehat{U}_{\alpha,ij,m} \leq \widetilde{U}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \tag{5.42a}$$
$$\mathcal{K}_{\alpha,ij,m}(\widehat{U}_{\alpha,ij,m}, \widehat{U}_{\alpha,ij,m-1}, \widehat{U}_{\alpha',ij,m}) \leq 0 \leq \mathcal{K}_{\alpha,ij,m}(\widetilde{U}_{\alpha,ij,m}, \widetilde{U}_{\alpha,ij,m-1}, \widetilde{U}_{\alpha',ij,m}), \tag{5.42b}$$
$$\mathcal{K}_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha,ij,m-1}, U_{\alpha',ij,m}) \equiv$$
$$(d_{\alpha,ij,m} + \tau^{-1})U_{\alpha,ij,m} - l_{\alpha,ij,m}U_{\alpha,i-1,j,m} - r_{\alpha,ij,m}U_{\alpha,i+1,j,m} - b_{\alpha,ij,m}U_{\alpha,i,j-1,m}$$
$$- q_{\alpha,ij,m}U_{\alpha,i,j+1,m} + f_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha',ij,m}) - \tau^{-1}U_{\alpha,ij,m-1},$$
$$(i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,$$
$$\widehat{U}_{\alpha,ij,m} \leq g_{\alpha,ij,m} \leq \widetilde{U}_{\alpha,ij,m}, \quad (i,j) \in \partial\Omega^h, \quad m \geq 1, \tag{5.42c}$$
$$\widehat{U}_{\alpha,ij,0} \leq \psi_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij,0}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2,$$

where $\mathcal{K}_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha,ij,m-1}, U_{\alpha',ij,m})$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, $m \geq 1$, are the residuals of the nonlinear difference scheme (5.41) on $U_{\alpha,ij,m}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, $m \geq 1$, and notation (5.10) is in use.

On each time level $m \geq 1$, we present the point Jacobi and Gauss-Seidel methods for the difference scheme (5.41). Upper $\{\overline{U}^{(n)}_{\alpha,ij,m}\}$ and lower $\{\underline{U}^{(n)}_{\alpha,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, sequences of solutions are calculated by the following point Jacobi and Gauss-Seidel iterative methods:

$$\mathcal{L}_{\alpha,ij,m}Z^{(n)}_{\alpha,ij,m} = -\mathcal{K}_{\alpha,ij,m}(U^{(n-1)}_{\alpha,ij,m}, U_{\alpha,ij,m-1}, U^{(n-1)}_{\alpha',ij,m}), \quad (i,j) \in \Omega^h, \quad n \geq 1, \tag{5.43}$$
$$Z^{(1)}_{\alpha,ij,m} = g_{\alpha,ij,m} - U^{(0)}_{\alpha,ij,m}, \quad Z^{(n)}_{\alpha,ij,m} = 0, \quad n \geq 2, \quad (i,j) \in \partial\Omega^h,$$
$$U_{\alpha,ij,0} = \psi_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \quad U_{\alpha,ij,m} = U^{(n_m)}_{\alpha,ij,m},$$
$$\mathcal{L}_{\alpha,ij,m}Z^{(n)}_{\alpha,ij,m} \equiv (d_{\alpha,ij,m} + c_{\alpha,ij,m})Z^{(n)}_{\alpha,ij,m}$$
$$- \eta\left(l_{\alpha,ij,m}Z^{(n)}_{\alpha,i-1,j,m} + b_{\alpha,ij,m}Z^{(n)}_{\alpha,i,j-1,m}\right) + \tau^{-1}Z^{(n)}_{\alpha,ij,m},$$
$$Z^{(n)}_{\alpha,ij,m} = U^{(n)}_{\alpha,ij,m} - U^{(n-1)}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2, \quad m \geq 1,$$

where $\mathcal{K}_{\alpha,ij,m}(U^{(n-1)}_{\alpha,ij,m}, U_{\alpha,ij,m-1}, U^{(n-1)}_{\alpha',ij,m})$, $(i,j) \in \Omega^h$, $\alpha' \neq \alpha$, $\alpha,\alpha' = 1,2$, $m \geq 1$, are the residuals of the nonlinear difference scheme (5.41) on $U^{(n-1)}_{\alpha,ij,m}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, $m \geq 1$, which are defined in (5.42) and notation (5.10) is in use. The mesh functions $U_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are the approximate solutions on time level $m \geq 1$, where n_m is a number of iterations on time level $m \geq 1$. For $\eta = 0$ and $\eta = 1$, we have, respectively, the point Jacobi and Gauss-Seidel methods.

Remark 5.4.2. *For quasi-monotone nondecreasing functions (5.19), upper and lower solutions are independent, hence, by using (5.43), we calculate either the sequence* $\{\overline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, *or the sequence* $\{\underline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$.

Theorem 5.4.3. *Let* $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ *and* $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, *be ordered upper and lower solutions (5.42). Suppose that the functions* f_α, $\alpha = 1,2$, *in (5.1) satisfy (5.19) and (5.20). Then upper* $\{\overline{U}^{(n)}_{\alpha,ij,m}\}$ *and lower* $\{\underline{U}^{(n)}_{\alpha,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, *sequences generated by (5.43) with, respectively,* $\overline{U}^{(0)}_{\alpha,ij,m} = \widetilde{U}_{\alpha,ij,m}$ *and* $\underline{U}^{(0)}_{\alpha,ij,m} = \widehat{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, *converge monotonically, such that,*

$$\underline{U}^{(n-1)}_{\alpha,ij,m} \leq \underline{U}^{(n)}_{\alpha,ij,m} \leq \overline{U}^{(n)}_{\alpha,ij,m} \leq \overline{U}^{(n-1)}_{\alpha,ij,m} \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \geq 1. \tag{5.44}$$

Proof. Since $\overline{U}^{(0)}_{\alpha,ij,1}$, $(i,i) \in \overline{\Omega}^h$, $\alpha = 1,2$, are upper solutions (5.42) with respect to $U_{\alpha,ij,0} = \psi_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, it follows that $\mathcal{K}_{\alpha,ij,1}(\overline{U}^{(0)}_{\alpha,ij,1}, \psi_{\alpha,ij}, \overline{U}^{(0)}_{\alpha',ij,1}) \geq 0$, $(i,j) \in \Omega^h$, $\alpha' \neq \alpha$, $\alpha,\alpha' = 1,2$. From here and (5.43), we have

$$(d_{\alpha,ij,1} + \tau^{-1} + c_{\alpha,ij,1})\overline{Z}^{(1)}_{\alpha,ij,1} - \eta l_{\alpha,ij,1}\overline{Z}^{(1)}_{\alpha,i-1,j,1} - \eta b_{\alpha,ij,1}\overline{Z}^{(1)}_{\alpha,i,j-1,1} \leq 0, \tag{5.45}$$
$$(i,j) \in \Omega^h, \quad \overline{Z}^{(1)}_{\alpha,ij,1} \leq 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1,2.$$

From here, $\eta = 0,1$, $b_{\alpha,i,1,1} > 0$, $\alpha = 1,2$, in (5.41b) and $\overline{Z}^{(1)}_{\alpha,i,0,1} \leq 0$, $i = 1,2\ldots,N_x-1$,

$\alpha = 1, 2$, for $j = 1$ in (5.45), we obtain

$$(d_{\alpha,i,1,1} + (\tau^{-1} + c_{\alpha,i,1,1})I)\overline{Z}^{(1)}_{\alpha,i,1,1} - \eta l_{\alpha,i,1,1}\overline{Z}^{(1)}_{\alpha,i-1,1,1} \leq 0, \quad i = 1, 2, \ldots, N_x - 1,$$
$$\overline{Z}^{(1)}_{\alpha,i,1,1} \leq 0, \quad i = 0, N_x, \quad \alpha = 1, 2. \tag{5.46}$$

Taking into account that $\eta = 0, 1$, $l_{\alpha,1,1,1} > 0$, $\alpha = 1, 2$, from (5.41b), $\overline{Z}^{(1)}_{\alpha,0,1,1} \leq 0$, $\alpha = 1, 2$, for $i = 1$ in (5.46), by using the maximum principle in Lemma 5.4.1, we have $\overline{Z}^{(1)}_{\alpha,1,1,1} \leq 0$, $\alpha = 1, 2$. From here, for $i = 2$ in (5.46), by Lemma 5.4.1, we have $\overline{Z}^{(1)}_{\alpha,2,1,1} \leq 0$, $\alpha = 1, 2$. By induction on i, we can prove that $\overline{Z}^{(1)}_{\alpha,i,1,1} \leq 0$, $i = 0, 1, \ldots, N_x$, $\alpha = 1, 2$.

By induction on $j \geq 1$, we can prove that

$$\overline{Z}^{(1)}_{\alpha,ij,1} \leq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1, 2. \tag{5.47}$$

Similarly, for initial lower solutions $\underline{U}^{(0)}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, we can prove that

$$\underline{Z}^{(1)}_{\alpha,ij,1} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1, 2. \tag{5.48}$$

We now prove that $\overline{U}^{(1)}_{\alpha,ij,1}$ and $\underline{U}^{(1)}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, are ordered upper and lower solutions (5.42). Letting $W^{(1)}_{\alpha,ij,1} = \overline{U}^{(1)}_{\alpha,ij,1} - \underline{U}^{(1)}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, using notation (5.21) and taking into account that $W^{(1)}_{\alpha,ij,0} = 0$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, from (5.43), we conclude that

$$\mathcal{L}_{\alpha,ij,1}W^{(1)}_{\alpha,ij,1} = r_{\alpha,ij,1}W^{(0)}_{\alpha,i+1,j,1} + q_{\alpha,ij,1}W^{(0)}_{\alpha,i,j+1,1} + \Gamma_{\alpha,ij,1}(\overline{U}^{(0)}_{\alpha,ij,1}, \overline{U}^{(0)}_{\alpha',ij,1})$$
$$- \Gamma_{\alpha,ij,1}(\underline{U}^{(0)}_{\alpha,ij,1}, \underline{U}^{(0)}_{\alpha',ij,1}), \quad (i,j) \in \Omega^h,$$
$$W^{(1)}_{\alpha,ij,1} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.$$

From here, (5.41b), (5.43) and taking into account that $\overline{U}^{(0)}_{\alpha,ij,1} \geq \underline{U}^{(0)}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, by using (5.22), we obtain

$$(d_{\alpha,ij,1} + \tau^{-1} + c_{\alpha,ij,1})W^{(1)}_{\alpha,ij,1} - \eta l_{\alpha,ij,1}W^{(1)}_{\alpha,i-1,j,1} - \eta b_{\alpha,ij,1}W^{(1)}_{\alpha,i,j-1,1} \geq 0, \quad (i,j) \in \Omega^h,$$
$$W^{(1)}_{\alpha,ij,1} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1, 2. \tag{5.49}$$

From here and taking into account that $W^{(1)}_{\alpha,i,0,1} = 0$, $i = 1, 2, \ldots, N_x - 1$, $\alpha = 1, 2$, for $j = 1$ in (5.49), it follows that

$$(d_{\alpha,i,1,1} + \tau^{-1} + c_{\alpha,i,1,1})W^{(1)}_{\alpha,i,1,1} - \eta l_{\alpha,i,1,1}W^{(1)}_{\alpha,i-1,1,1} \geq 0, \quad i = 1, 2, \ldots, N_x - 1, \tag{5.50}$$
$$W^{(1)}_{\alpha,i,1,1} = 0, \quad i = 0, N_x, \quad \alpha = 1, 2.$$

Taking into account that $W^{(1)}_{\alpha,0,1,1} = 0$, $\alpha = 1,2$, by Lemma 5.4.1, for $i = 1$ in (5.50), we have $W^{(1)}_{\alpha,1,1,1} \geq 0, \quad \alpha = 1,2$. From here, $\eta = 0,1$, $l_{\alpha,2,1,1} > 0$, $\alpha = 1,2$, in (5.41b) and using Lemma 5.4.1, for $i = 2$, we obtain that $W^{(1)}_{\alpha,2,1,1} \geq 0$, $\alpha = 1,2$. By induction on i, we can prove that

$$W^{(1)}_{\alpha,i,1,1} \geq 0, \quad i = 0,1,\dots,N_x, \quad \alpha = 1,2.$$

By induction on $j \geq 1$, we can prove that

$$W^{(1)}_{\alpha,ij,1} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \tag{5.51}$$

Thus, we prove (5.42a).

From (5.43) and using notation (5.21), we conclude that

$$\begin{aligned}\mathcal{K}_{\alpha,ij,1}(\overline{U}^{(1)}_{\alpha,ij,1}, \psi_{\alpha,ij}, \overline{U}^{(1)}_{\alpha',ij,1}) = & -\eta l_{\alpha,ij,1}\overline{Z}^{(1)}_{\alpha,i-1,j,1} - r_{\alpha,ij,1}\overline{Z}^{(1)}_{\alpha,i+1,j,1} \\ & - \eta b_{\alpha,ij,1}\overline{Z}^{(1)}_{\alpha,i,j-1,1} - q_{\alpha,ij,1}\overline{Z}^{(1)}_{\alpha,i,j+1,1} \\ & + \Gamma_{\alpha,ij,1}(\overline{U}^{(0)}_{\alpha,ij,1}, \overline{U}^{(0)}_{\alpha',ij,1}) - \Gamma_{\alpha,ij,1}(\overline{U}^{(1)}_{\alpha,ij,1}, \overline{U}^{(1)}_{\alpha',ij,1})\end{aligned} \tag{5.52}$$

$$(i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2.$$

From $\eta = 0,1$, (5.41b) and (5.47), by using (5.22), we obtain that

$$\mathcal{K}_{\alpha,ij,1}(\overline{U}^{(1)}_{\alpha,ij,1}, \psi_{\alpha,ij}, \overline{U}^{(1)}_{\alpha',ij,1}) \geq 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2.$$

Thus, $\overline{U}^{(1)}_{\alpha,ij,1}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, satisfy (5.42b). By a similar manner, we can prove that

$$\mathcal{K}_{\alpha,ij,1}(\underline{U}^{(1)}_{\alpha,ij,1}, \psi_{\alpha,ij}, \underline{U}^{(1)}_{\alpha',ij,1}) \leq 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2.$$

Hence, $\underline{U}^{(1)}_{\alpha,ij,1}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, satisfy (5.42b). From the boundary and initial conditions in (5.43), it follows that $\overline{U}^{(1)}_{\alpha,ij,1}$ and $\underline{U}^{(1)}_{\alpha,ij,1}$ satisfy (5.42c).

Thus, we prove that $\overline{U}^{(1)}_{\alpha,ij,1}$ and $\underline{U}^{(1)}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are ordered upper and lower solutions (5.42). By induction on $n \geq 1$, we can prove (5.44) on the first time level $m = 1$.

On the second time level $m = 2$, taking into account that $\overline{U}^{(0)}_{\alpha,ij,2} = \widetilde{U}_{\alpha,ij,2}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from (5.42), we obtain

$$\begin{aligned}&\mathcal{K}_{\alpha,ij,2}(\widetilde{U}_{\alpha,ij,2}, \overline{U}_{\alpha,ij,1}, \widetilde{U}_{\alpha',ij,2}) = \\ &(d_{\alpha,ij,2} + \tau^{-1})\widetilde{U}_{\alpha,ij,2} - l_{\alpha,ij,2}\widetilde{U}_{\alpha,i-1,j,2} - r_{\alpha,ij,2}\widetilde{U}_{\alpha,i+1,j,2} - b_{\alpha,ij,2}\widetilde{U}_{\alpha,i,j-1,2} \\ &- q_{\alpha,ij,2}\widetilde{U}_{\alpha,i,j+1,2} + f_{\alpha,ij,2}(\widetilde{U}_{\alpha,ij,2}, \widetilde{U}_{\alpha',ij,2}) - \tau^{-1}\widetilde{U}_{\alpha,ij,1},\end{aligned}$$

where $\overline{U}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are the approximate solutions on the first time level $m = 1$, which are defined in (5.43). From here and taking into account that from (5.44), $\overline{U}_{\alpha,ij,1} \leq \widetilde{U}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, it follows that

$$\begin{aligned}&\mathcal{K}_{\alpha,ij,2}(\widetilde{U}_{\alpha,ij,2}, \overline{U}_{\alpha,ij,1}, \widetilde{U}_{\alpha',ij,2}) \geq \mathcal{K}_{\alpha,ij,2}(\widetilde{U}_{\alpha,ij,2}, \widetilde{U}_{\alpha,ij,1}, \widetilde{U}_{\alpha',ij,2}) \geq 0, \\ &(i,j) \in \Omega^h, \quad \alpha = 1,2, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,\end{aligned} \tag{5.53}$$

which means that $\overline{U}^{(0)}_{\alpha,ij,2} = \widetilde{U}_{\alpha,ij,2}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are upper solutions with respect to $\overline{U}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. Similarly, we can obtain that

$$\mathcal{K}_{\alpha,ij,2}\left(\widehat{U}_{\alpha,ij,2}, \underline{U}_{\alpha,ij,1}, \widehat{U}_{\alpha',ij,2}\right) \leq 0, \quad (i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,$$

which means that $\underline{U}^{(0)}_{\alpha,ij,2} = \widehat{U}_{\alpha,ij,2}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are lower solutions with respect to $\underline{U}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. From here, (5.43) and (5.53), on the second time level $m = 2$, we have

$$\begin{aligned}&(d_{\alpha,ij,2} + \tau^{-1} + c_{\alpha,ij,2})\overline{Z}^{(1)}_{\alpha,ij,2} - \eta l_{\alpha,ij,2}\overline{Z}^{(1)}_{\alpha,i-1,j,2} - \eta b_{\alpha,ij,2}\overline{Z}^{(1)}_{\alpha,i,j-1,2} \leq 0, \\ &(i,j) \in \Omega^h, \quad \overline{Z}^{(1)}_{\alpha,ij,1} \leq 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha = 1,2.\end{aligned} \tag{5.54}$$

From here, $\eta = 0,1$, $b_{\alpha,i,1,2} > 0$, $\alpha = 1,2$, in (5.41b) and $\overline{Z}^{(1)}_{\alpha,i,0,2} \leq 0$, $i = 1,2\ldots,N_x-1$, $\alpha = 1,2$, for $j = 1$ in (5.54), we obtain

$$\begin{aligned}&(d_{\alpha,i,1,2} + \tau^{-1} + c_{\alpha,i,1,2})\overline{Z}^{(1)}_{\alpha,i,1,2} - \eta l_{\alpha,i,1,2}\overline{Z}^{(1)}_{\alpha,i-1,1,2} \leq 0, \quad i = 1,2,\ldots,N_x-1, \\ &\overline{Z}^{(1)}_{\alpha,i,1,2} \leq 0, \quad i = 0, N_x, \quad \alpha = 1,2.\end{aligned} \tag{5.55}$$

Taking into account that $\eta = 0,1$, $l_{\alpha,1,1,2} > 0$, $\alpha = 1,2$, in (5.41b), $\overline{Z}^{(1)}_{\alpha,0,1,2} \leq 0$, $\alpha = 1,2$, for $i = 1$ in (5.55), by using the maximum principle in Lemma 5.4.1, we have $\overline{Z}^{(1)}_{\alpha,1,1,2} \leq 0$, $\alpha = 1,2$. From here, for $i = 2$ in (5.55), by Lemma 5.4.1, we have $\overline{Z}^{(1)}_{\alpha,2,1,2} \leq 0$, $\alpha = 1,2$. By induction on i, we can prove that $\overline{Z}^{(1)}_{\alpha,i,1,2} \leq 0$, $i = 0,1,\ldots,N_x$, $\alpha = 1,2$.

By induction on $j \geq 1$, we can prove that

$$\overline{Z}^{(1)}_{\alpha,ij,2} \leq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2.$$

Similarly, for initial lower solutions $\underline{U}^{(0)}_{\alpha,ij,2}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, we can prove that

$$\underline{Z}^{(1)}_{\alpha,ij,2} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2.$$

The proof that $\overline{U}^{(1)}_{\alpha,ij,2}$ and $\underline{U}^{(1)}_{\alpha,ij,2}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, are ordered upper and lower solutions (5.42) repeats the proof on the first time level $m=1$. By induction on $m \geq 1$, we can prove (5.44) for $m \geq 1$. $\square$

5.4.2 Quasi-monotone nonincreasing case

In the case of quasi-monotone nonincreasing functions (5.30), on each time level $m \geq 1$, we say that mesh functions

$$(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m}), \quad (\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m}), \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1,$$

are ordered upper and lower solutions of (5.41), if they satisfy the inequalities

$$\widehat{U}_{\alpha,ij,m} \leq \widetilde{U}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \tag{5.56a}$$
$$\mathcal{K}_{\alpha,ij,m}(\widehat{U}_{\alpha,ij,m}, \widehat{U}_{\alpha,ij,m-1}, \widetilde{U}_{\alpha',ij,m}) \leq 0 \leq \mathcal{K}_{\alpha,ij,m}(\widetilde{U}_{\alpha,ij,m}, \widetilde{U}_{\alpha,ij,m-1}, \widehat{U}_{\alpha',ij,m}), \tag{5.56b}$$
$$(i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,$$
$$\widehat{U}_{\alpha,ij,m} \leq g_{\alpha,ij,m} \leq \widetilde{U}_{\alpha,ij,m}, \quad (i,j) \in \partial\Omega^h, \quad m \geq 1, \tag{5.56c}$$
$$\widehat{U}_{\alpha,ij,0} \leq \psi_{\alpha,ij} \leq \widetilde{U}_{\alpha,ij,0}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2,$$

where $\mathcal{K}_{\alpha,ij,m}(U_{\alpha,ij,m}, U_{\alpha,ij,m-1}, U_{\alpha',ij,m})$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, $m \geq 1$, are the residuals of the nonlinear difference scheme (5.41) on $U_{\alpha,ij,m}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, $m \geq 1$, which are defined in (5.42) and notation (5.10) is in use.

We now present the point Jacobi and block Gauss–Seidel methods for the nonlinear difference scheme (5.41) when the reaction functions f_α, $\alpha = 1,2$, are quasi-monotone nonincreasing (5.30).

For solving the nonlinear difference scheme (5.41), on each time level t_m, $m \geq 1$, we construct the point iterative Jacobi and point iterative Gauss-Seidel methods in the forms

$$\mathcal{L}_{\alpha,ij,m}\overline{Z}^{(n)}_{\alpha,ij,m} = -\mathcal{K}_{\alpha,ij,m}(\overline{U}^{(n-1)}_{\alpha,ij,m}, \overline{U}_{\alpha,ij,m-1}, \underline{U}^{(n-1)}_{\alpha',ij,m}), \tag{5.57}$$
$$\mathcal{L}_{\alpha,ij,m}\underline{Z}^{(n)}_{\alpha,ij,m} = -\mathcal{K}_{\alpha,ij,m}(\underline{U}^{(n-1)}_{\alpha,ij,m}, \underline{U}_{\alpha,ij,m-1}, \overline{U}^{(n-1)}_{\alpha',ij,m}), \quad (i,j) \in \Omega^h, \quad n \geq 1,$$
$$Z^{(1)}_{\alpha,ij,m} = g_{\alpha,ij,m} - U^{(0)}_{\alpha,ij,m}, \quad Z^{(n)}_{\alpha,ij,m} = 0, \quad n \geq 2, \quad (i,j) \in \partial\Omega^h,$$
$$U_{\alpha,ij,0} = \psi_{\alpha,ij}, \quad (i,j) \in \overline{\Omega}^h, \quad U_{\alpha,ij,m} = U^{(n_m)}_{\alpha,ij,m},$$
$$Z^{(n)}_{\alpha,ij,m} = U^{(n)}_{\alpha,ij,m} - U^{(n-1)}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \quad m \geq 1,$$

where the difference operators $\mathcal{L}_{\alpha,ij,m}Z^{(n)}_{\alpha,ij,m}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, $m \geq 1$, are defined in (5.43), $\mathcal{K}_{\alpha,ij,m}(U^{(n-1)}_{\alpha,ij,m}, U_{\alpha,ij,m-1}, U^{(n-1)}_{\alpha',ij,m})$, $(i,j) \in \Omega^h$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1,2$, $m \geq 1$, are the residuals of the nonlinear difference scheme (5.41) on $U^{(n-1)}_{\alpha,ij,m}$, $(i,j) \in \Omega^h$, $\alpha = 1,2$, $m \geq 1$, which are defined in (5.42) and notation (5.10) is in use. The mesh functions

$U_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are the approximate solutions on time level $m \geq 1$, where n_m is a number of iterations on time level $m \geq 1$. For $\eta = 0$ and $\eta = 1$, we have, respectively, the point Jacobi and Gauss-Seidel methods.

Remark 5.4.4. *For quasi-monotone nonincreasing functions f_α, $\alpha = 1,2$, (5.30), upper and lower solutions are coupled, hence, by using (5.57), we calculate either the sequence $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, or the sequence $\{\underline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$.*

In the following theorem, we prove the monotone property of the point iterative methods (5.57).

Theorem 5.4.5. *Suppose that $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ and $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, are ordered upper and lower solutions (5.56) to (5.41). Suppose that f_α, $\alpha = 1,2$, in (5.1) satisfy (5.19) and (5.30). Then the sequences $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$ and $\{\underline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n-1)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, generated by (5.57), with $(\overline{U}^{(0)}_{1,ij,m}, \underline{U}^{(0)}_{2,ij,m}) = (\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$ and $(\underline{U}^{(0)}_{1,ij,m}, \overline{U}^{(0)}_{2,ij,m}) = (\widehat{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, converge monotonically, such that,*

$$\underline{U}^{(n-1)}_{\alpha,ij,m} \leq \underline{U}^{(n)}_{\alpha,i,m} \leq \overline{U}^{(n)}_{\alpha,ij,m} \leq \overline{U}^{(n-1)}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \geq 1. \tag{5.58}$$

Proof. On the first time level $m = 1$, in the case of the sequence $\{\overline{U}^{(n)}_{1,ij,1}, \underline{U}^{(n)}_{2,ij,1}\}$, $(\overline{U}^{(0)}_{1,ij,1}, \underline{U}^{(0)}_{2,ij,1}) = (\widetilde{U}_{1,ij,1}, \widehat{U}_{2,ij,1})$, $(i,j) \in \overline{\Omega}^h$, are initial upper and lower solutions (5.56) with respect to $U_{\alpha,ij,0} = \psi_{\alpha,ij}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. Hence, it follows that $\mathcal{K}_{1,ij,1}\left(\widetilde{U}_{1,ij,1}, \psi_{1,ij}, \widehat{U}_{2,ij,1}\right) \geq 0$, $(i,j) \in \Omega^h$, and $\mathcal{K}_{2,ij,1}\left(\widetilde{U}_{1,ij,1}, \psi_{2,ij}, \widehat{U}_{2,ij,1}\right) \leq 0$, $(i,j) \in \Omega^h$. From here and (5.57), we have

$$\begin{aligned}
&(d_{1,ij,1} + \tau^{-1} + c_{1,ij,1})\overline{Z}^{(1)}_{1,ij,1} - \eta l_{1,ij,1}\overline{Z}^{(1)}_{1,i-1,j,1} - \eta b_{1,ij,1}\overline{Z}^{(1)}_{1,i,j-1,1} \leq 0, \\
&(d_{2,ij,1} + (\tau^{-1} + c_{2,ij,1}))\underline{Z}^{(1)}_{2,ij,1} - \eta l_{2,ij}\underline{Z}^{(1)}_{2,i-1,j,1} - \eta b_{2,ij,1}\underline{Z}^{(1)}_{2,i,j-1,1} \geq 0, \quad (i,j) \in \Omega^h, \\
&\overline{Z}^{(1)}_{1,ij,1} \leq 0, \quad \underline{Z}^{(1)}_{2,ij,1} \geq 0, \quad (i,j) \in \partial\Omega^h.
\end{aligned} \tag{5.59}$$

For here, $\eta = 0,1$, $b_{\alpha,i,1,1} > 0$ in (5.41b) and $\overline{Z}^{(1)}_{1,i,0,1} \leq 0$, $\underline{Z}^{(1)}_{2,i,0,1} \geq 0$, $i = 0, N_x$, for $j = 1$ in (5.59), we obtain

$$\begin{aligned}
&(d_{1,ij,1} + \tau^{-1} + c_{1,ij,1})\overline{Z}^{(1)}_{1,i,1,1} - \eta l_{1,i,1,1}\overline{Z}^{(1)}_{1,i-1,1,1} \leq 0, \quad (i,j) \in \Omega^h, \\
&(d_{2,ij,1} + (\tau^{-1} + c_{2,ij,1}))\underline{Z}^{(1)}_{2,i,1,1} - \eta l_{2,i,1,1}\underline{Z}^{(1)}_{2,i-1,1,1} \geq 0, \quad (i,j) \in \Omega^h, \\
&i = 1,2,\ldots,N_x - 1, \quad \overline{Z}^{(1)}_{1,i,1,1} \leq 0, \quad \underline{Z}^{(1)}_{2,i,1,1} \geq 0, \quad i = 0, N_x.
\end{aligned} \tag{5.60}$$

Taking into account that $\eta = 0,1$, $l_{\alpha,i,1,1} > 0$ in (5.41b), $\overline{Z}^{(1)}_{1,0,1,1} \leq 0$, $\underline{Z}^{(1)}_{2,0,1,1} \geq 0$, and using the maximum principle in Lemma 5.4.1, for $i = 1$ in (5.60), we have $\overline{Z}^{(1)}_{1,1,1,1} \leq 0$,

$\underline{Z}^{(1)}_{2,1,1,1} \geq 0$. From here, by using Lemma 5.4.1, for $i=2$ in (5.60), we have $\overline{Z}^{(1)}_{1,2,1,1} \leq 0$, $\underline{Z}^{(1)}_{2,2,1,1} \geq 0$. By induction on i and j, we can prove that

$$\overline{Z}^{(1)}_{1,ij,1} \leq 0, \quad \underline{Z}^{(1)}_{2,ij,1} \geq 0, \quad (i,j) \in \overline{\Omega}^h. \tag{5.61}$$

Similarly, for $(\underline{U}^{(1)}_{1,ij,1}, \overline{U}^{(1)}_{2,ij,1})$, $(i,j) \in \overline{\Omega}^h$, from (5.57), we can prove that

$$\underline{Z}^{(1)}_{1,ij,1} \geq 0, \quad \overline{Z}^{(1)}_{2,ij,1} \leq 0, \quad (i,j) \in \overline{\Omega}^h. \tag{5.62}$$

We now prove that $\overline{U}^{(1)}_{\alpha,ij,1}$ and $\underline{U}^{(1)}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha=1,2$, are ordered upper and lower solutions (5.56). Letting $W^{(1)}_{\alpha,ij,1} = \overline{U}^{(1)}_{\alpha,ij,1} - \underline{U}^{(1)}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha=1,2$, using notation (5.21), from (5.57), we conclude that

$$\begin{aligned}
\mathcal{L}_{\alpha,ij,1} W^{(1)}_{\alpha,ij} &= r_{\alpha,ij,1} W^{(0)}_{\alpha,i+1,j,1} + q_{\alpha,ij,1} W^{(0)}_{\alpha,i,j+1,1} + \Gamma_{\alpha,ij,1}(\overline{U}^{(0)}_{\alpha,ij,1}, \underline{U}^{(0)}_{\alpha',ij,1}) \\
&\quad - \Gamma_{\alpha,ij,1}(\underline{U}^{(0)}_{\alpha,ij,1}, \overline{U}^{(0)}_{\alpha',ij,1}), \quad (i,j) \in \Omega^h, \\
W^{(1)}_{\alpha,ij,1} &= 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha'=1,2.
\end{aligned}$$

From here, (5.41b) and taking into account that $\overline{U}^{(0)}_{\alpha,ij,1} \geq \underline{U}^{(0)}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha=1,2$, by using (5.31), we obtain

$$\begin{aligned}
&(d_{\alpha,ij,1} + \tau^{-1} + c_{\alpha,ij,1}) W^{(1)}_{\alpha,ij,1} - \eta l_{\alpha,ij,1} W^{(1)}_{\alpha,i-1,j,1} - \eta b_{\alpha,ij,1} W^{(1)}_{\alpha,i,j-1,1} \geq 0, \quad (i,j) \in \Omega^h, \qquad (5.63) \\
&W^{(1)}_{\alpha,ij,1} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha=1,2.
\end{aligned}$$

Since $W^{(1)}_{\alpha,i,0,1} = 0$, $\alpha=1,2$, for $j=1$ in (5.63), it follows that

$$\begin{aligned}
&(d_{\alpha,i,1,1} + c_{\alpha,i,1,1}) W^{(1)}_{\alpha,i,1,1} - \eta l_{\alpha,i,1,1} W^{(1)}_{\alpha,i-1,1,1} \geq 0, \quad i=1,2,\ldots,N_x-1, \qquad (5.64) \\
&W^{(1)}_{\alpha,i,1,1} = 0, \quad i=0,N_x, \quad \alpha=1,2.
\end{aligned}$$

From here, $W^{(1)}_{\alpha,0,1,1} = 0$, $\alpha=1,2$, for $i=1$ in (5.64), by using Lemma 5.4.1, we have $W^{(1)}_{\alpha,1,1,1} \geq 0$, $\alpha=1,2$. From here, $\eta=0,1$, $l_{\alpha,2,1,1} > 0$, $\alpha=1,2$, in (5.41b), for $i=2$ in (5.64), by using Lemma 5.4.1, we obtain $W^{(1)}_{\alpha,2,1,1} \geq 0$, $\alpha=1,2$. By induction on i and j, we can prove that

$$W^{(1)}_{\alpha,ij} \geq 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha=1,2. \tag{5.65}$$

Thus, we prove (5.56a) on the first time level $m=1$.

From (5.57) and using notation (5.21), we conclude that

$$\mathcal{K}_{\alpha,ij,1}(\overline{U}^{(1)}_{\alpha,ij,1},\psi_{\alpha,ij},\underline{U}^{(1)}_{\alpha',ij,1}) = -r_{\alpha,ij,1}\overline{Z}^{(1)}_{\alpha,i+1,j,1} - q_{\alpha,ij,1}\overline{Z}^{(1)}_{\alpha,i+1,j,1} + \Gamma_{\alpha,ij,1}(\overline{U}^{(0)}_{\alpha,ij,1},\underline{U}^{(0)}_{\alpha',ij,1}) - \Gamma_{\alpha,ij,1}(\overline{U}^{(1)}_{\alpha,ij,1},\underline{U}^{(1)}_{\alpha',ij,1}), \quad (5.66)$$

$(i,j)\in\Omega^h,\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2.$

From here, (5.41b) and (5.61), by using (5.31), we obtain that

$$\mathcal{K}_{\alpha,ij,1}(\overline{U}^{(1)}_{\alpha,ij,1},\psi_{\alpha,ij},\underline{U}^{(1)}_{\alpha',ij,1}) \geq 0,\quad (i,j)\in\Omega^h,\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2. \quad (5.67)$$

Similarly, we can prove that

$$\mathcal{K}_{\alpha,ij,1}(\underline{U}^{(1)}_{\alpha,ij,1},\psi_{\alpha,ij},\overline{U}^{(1)}_{\alpha',ij,1}) \leq 0,\quad (i,j)\in\Omega^h,\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2. \quad (5.68)$$

From the boundary and initial conditions in (5.57), it follows that $\overline{U}^{(1)}_{\alpha,ij,1}$, $\underline{U}^{(1)}_{\alpha,ij,1}$, satisfy (5.56c). Thus, from here, (5.65), (5.67) and (5.68), we conclude that $\overline{U}^{(1)}_{\alpha,ij,1}$ and $\underline{U}^{(1)}_{\alpha,ij,1}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, are ordered upper and lower solutions (5.56).

By induction on n, we can prove that $\{\overline{U}^{(n)}_{\alpha,ij,1}\}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, are monotone decreasing sequences of upper solutions and $\{\underline{U}^{(n)}_{\alpha,ij,1}\}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, are monotone increasing sequences of lower solutions which satisfy (5.58) on the first time level $m=1$.

On the second time level $m=2$, for the sequence $\{\overline{U}^{(n)}_{1,ij,2},\underline{U}^{(n)}_{2,ij,2}\}$, $(i,j)\in\overline{\Omega}^h$, we have $\overline{U}^{(0)}_{1,ij,2}=\widetilde{U}_{1,ij,2}$ and $\underline{U}^{(0)}_{2,ij,2}=\widehat{U}_{2,ij,2}$, $(i,j)\in\overline{\Omega}^h$. From (5.42), we obtain that

$$\begin{aligned}&\mathcal{K}_{1,ij,2}(\widetilde{U}_{1,ij,2},\overline{U}_{1,ij,1},\widehat{U}_{2,ij,2}) =\\ &(d_{1,ij,2}+\tau^{-1})\widetilde{U}_{1,ij,2} - l_{1,ij,2}\widetilde{U}_{1,i-1,j,2} - r_{1,ij,2}\widetilde{U}_{1,i+1,j,2} - b_{1,ij,2}\widetilde{U}_{1,i,j-1,2}\\ &- q_{1,ij,2}\widetilde{U}_{1,i,j+1,2} + f_{1,ij,2}(\widetilde{U}_{1,ij,2},\widehat{U}_{2,ij,2}) - \tau^{-1}\overline{U}_{1,ij,1},\end{aligned}$$

$$\begin{aligned}&\mathcal{K}_{2,ij,2}(\widetilde{U}_{1,ij,2},\underline{U}_{2,ij,1},\widehat{U}_{2,ij,2}) =\\ &(d_{2,ij,2}+\tau^{-1})\widehat{U}_{2,ij,2} - l_{2,ij,2}\widehat{U}_{2,i-1,j,2} - r_{2,ij,2}\widehat{U}_{2,i+1,j,2} - b_{2,ij,2}\widehat{U}_{2,i,j-1,2}\\ &- q_{2,ij,2}\widehat{U}_{2,i,j+1,2} + f_{2,ij,2}(\widetilde{U}_{1,ij,2},\widehat{U}_{2,ij,2}) - \tau^{-1}\underline{U}_{2,ij,1},\end{aligned}$$

where $\overline{U}_{1,ij,1}$ and $\underline{U}_{2,ij,1}$, $(i,j)\in\overline{\Omega}^h$, are the approximate solutions on the first time level $m=1$, which are defined in (5.57). From here and taking into account that from

(5.58), $\overline{U}_{1,ij,1} \le \widetilde{U}_{1,ij,1}$ and $\widehat{U}_{2,ij,1} \le \underline{U}_{2,ij,1}$, $(i,j) \in \overline{\Omega}^h$, it follows that

$$\mathcal{K}_{1,ij,2}(\widetilde{U}_{1,ij,2}, \overline{U}_{1,ij,1}, \widehat{U}_{2,ij,2}) \ge \mathcal{K}_{1,ij,2}(\widetilde{U}_{1,ij,2}, \widetilde{U}_{1,ij,1}, \widehat{U}_{2,ij,2}) \ge 0, \quad (i,j) \in \Omega^h, \tag{5.69}$$
$$\mathcal{K}_{2,ij,2}(\widetilde{U}_{1,ij,2}, \underline{U}_{2,ij,1}, \widehat{U}_{2,ij,2}) \le \mathcal{K}_{2,ij,2}(\widetilde{U}_{1,ij,2}, \widehat{U}_{2,ij,1}, \widehat{U}_{2,ij,2}) \le 0, \quad (i,j) \in \Omega^h,$$

which means that $\overline{U}^{(0)}_{1,ij,2} = \widetilde{U}_{1,ij,2}$ and $\underline{U}^{(0)}_{2,ij,2} = \widehat{U}_{2,ij,2}$, $(i,j) \in \overline{\Omega}^h$ are upper and lower solutions with respect to $\overline{U}_{1,ij,1}$ and $\underline{U}_{2,ij,1}$, $(i,j) \in \overline{\Omega}^h$.

Similarly, we can obtain that

$$\mathcal{K}_{1,ij,2}\left(\widehat{U}_{1,ij,2}, \underline{U}_{1,ij,1}, \widetilde{U}_{2,ij,2}\right) \le 0, \quad \mathcal{K}_{2,ij,2}\left(\widehat{U}_{1,ij,2}, \overline{U}_{2,ij,1}, \widetilde{U}_{2,ij,2}\right) \ge 0, \quad (i,j) \in \Omega^h,$$

which means that $\widetilde{U}_{2,ij,2}$ and $\widehat{U}_{1,ij,2}$, $(i,j) \in \overline{\Omega}^h$, are upper and lower solutions with respect to $\overline{U}_{2,ij,1}$ and $\underline{U}_{1,ij,1}$, $(i,j) \in \overline{\Omega}^h$. From here, (5.57) and (5.69), on the second time level $m = 2$, we have

$$(d_{1,ij,2} + \tau^{-1} + c_{1,ij,2})\overline{Z}^{(1)}_{1,ij,2} - \eta l_{1,ij,2}\overline{Z}^{(1)}_{1,i-1,j,2} - \eta b_{1,ij,2}\overline{Z}^{(1)}_{1,i,j-1,2} \le 0, \tag{5.70}$$
$$(d_{2,ij,2} + \tau^{-1} + c_{2,ij,2})\underline{Z}^{(1)}_{2,ij,2} - \eta l_{2,ij}\underline{Z}^{(1)}_{2,i-1,j,2} - \eta b_{2,ij,2}\underline{Z}^{(1)}_{2,i,j-1,2} \ge 0, \quad (i,j) \in \Omega^h,$$
$$\overline{Z}^{(1)}_{1,ij,2} \le 0, \quad \underline{Z}^{(1)}_{2,ij,2} \ge 0, \quad (i,j) \in \partial\Omega^h.$$

For here, $\eta = 0,1$, $b_{\alpha,i,1,2} > 0$ in (5.41b) and $\overline{Z}^{(1)}_{1,i,0,2} \le 0$, $\underline{Z}^{(1)}_{2,i,0,2} \ge 0$, $i = 0, N_x$, for $j = 1$ in (5.70), we obtain

$$(d_{1,i,1,2} + (\tau^{-1} + c_{1,i,1,2}))\overline{Z}^{(1)}_{1,i,1,2} - \eta l_{1,i,1,2}\overline{Z}^{(1)}_{1,i-1,1,2} \le 0, \quad (i,j) \in \Omega^h, \tag{5.71}$$
$$(d_{2,i,1,2} + (\tau^{-1} + c_{2,i,1,2}))\underline{Z}^{(1)}_{2,i,1,2} - \eta l_{2,i,1,2}\underline{Z}^{(1)}_{2,i-1,1,2} \ge 0, \quad (i,j) \in \Omega^h,$$
$$i = 1, 2, \ldots, N_x - 1, \quad \overline{Z}^{(1)}_{1,i,1,2} \le 0, \quad \underline{Z}^{(1)}_{2,i,1,2} \ge 0, \quad i = 0, N_x.$$

Taking into account that $\eta = 0,1$, $l_{\alpha,i,1,2} > 0$ in (5.41b), $\overline{Z}^{(1)}_{1,0,1,2} \le 0$, $\underline{Z}^{(1)}_{2,0,1,2} \ge 0$, and using the maximum principle in Lemma 5.4.1, for $i = 1$ in (5.71), we have $\overline{Z}^{(1)}_{1,1,1,2} \le 0$, $\underline{Z}^{(1)}_{2,1,1,2} \ge 0$. From here, by using Lemma 5.4.1, for $i = 2$ in (5.71), we have $\overline{Z}^{(1)}_{1,2,1,2} \le 0$, $\underline{Z}^{(1)}_{2,2,1,2} \ge 0$. By induction on i and j, we can prove that

$$\overline{Z}^{(1)}_{1,ij,2} \le 0, \quad \underline{Z}^{(1)}_{2,ij,2} \ge 0, \quad (i,j) \in \overline{\Omega}^h.$$

Similarly, for the sequence $\{\underline{U}^{(1)}_{1,ij,2}, \overline{U}^{(1)}_{2,ij,2}\}$, $(i,j) \in \overline{\Omega}^h$, from (5.57), we can prove that

$$\underline{Z}^{(1)}_{1,ij,2} \ge 0, \quad \overline{Z}^{(1)}_{2,ij,2} \le 0, \quad (i,j) \in \overline{\Omega}^h.$$

The proof, that $\overline{U}^{(1)}_{\alpha,ij,2}$ and $\underline{U}^{(1)}_{\alpha,ij,2}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, are ordered upper and lower solutions (5.56), repeats the proof on the first time level $m = 1$. By induction on m, we can prove (5.58) for $m \geq 1$. In a similar manner, we can prove the theorem for the sequence $\{\underline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$. $\square$

5.5 Existence and uniqueness of a solution to the nonlinear difference scheme (5.41)

We give estimates of the solution to the linear problem (5.15).

Lemma 5.5.1. *The following estimates of the solution to the linear problem (5.15) hold*

$$\|W_{\alpha,m}\|_{\overline{\Omega}^h} \leq \max\left\{\|g_{\alpha,m}\|_{\partial\Omega^h}, \max_{(i,j)\in\Omega^h} \frac{|\varphi_{\alpha,ij,m}|}{c^*_{\alpha,ij,m} + \tau^{-1}}\right\}, \quad \alpha = 1,2, \tag{5.72}$$

where

$$\|W_{\alpha,m}\|_{\overline{\Omega}^h} = \max_{(i,j)\in\overline{\Omega}^h} |W_{\alpha,ij,m}|, \quad \|g_{\alpha,m}\|_{\partial\Omega^h} = \max_{(i,j)\in\partial\Omega^h} |g_{\alpha,ij,m}|.$$

The proof of the lemma is given in Lemma 1.2.1, Chapter 1.

5.5.1 Quasi-monotone nondecreasing case

In the following theorem, we prove the existence of a solution to the nonlinear difference scheme (5.41) based on Theorem 5.4.3.

Theorem 5.5.2. *Let $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ and $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, be ordered upper and lower solutions (5.42) to (5.41). Suppose that f_α, $\alpha = 1,2$, in (5.1) satisfy (5.19) and (5.20). Then a solution of the nonlinear difference scheme (5.41) exists in the sector $\langle \widehat{U}_m, \widetilde{U}_m\rangle$, $m \geq 1$, from (5.18).*

Proof. We consider the case of upper solutions based on the point Gauss–Seidel method with $\eta = 1$ in (5.43). On the first time level $m = 1$, from (5.44), we conclude that $\lim_{n\to\infty} \overline{U}^{(n)}_{\alpha,ij,1} = \overline{V}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, exist and

$$\overline{V}_{\alpha,ij,1} \leq \overline{U}^{(n)}_{\alpha,ij,1} \leq \overline{U}^{(n-1)}_{\alpha,ij,1} \leq \widetilde{U}_{\alpha,ij,1}, \quad \lim_{n\to\infty} \overline{Z}^{(n)}_{\alpha,ij,1} = 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \tag{5.73}$$

where $\overline{U}^{(0)}_{\alpha,ij,1} = \widetilde{U}_{\alpha,ij,1}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. From (5.42) and (5.43), we have

$$\mathcal{K}_{\alpha,ij,1}(\overline{U}^{(n)}_{\alpha,ij,1}, \psi_{\alpha,ij}, \overline{U}^{(n)}_{\alpha',ij,1}) = -(c_{\alpha,ij,1}\overline{Z}^{(n)}_{\alpha,ij,1} + r_{\alpha,ij,1}\overline{Z}^{(n)}_{\alpha,i+1,j,1} + q_{\alpha,ij,1}\overline{Z}^{(n)}_{\alpha,i,j+1,1})$$
$$+ f_{\alpha,ij,1}(\overline{U}^{(n)}_{\alpha,ij,1}, \overline{U}^{(n)}_{\alpha',ij,1}) - f_{\alpha,ij,1}(\overline{U}^{(n-1)}_{\alpha,ij,1}, \overline{U}^{(n-1)}_{\alpha',ij,1}),$$
$$(i,j) \in \Omega^h, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2.$$

By the mean-value theorem (5.16), we have

$$\begin{aligned}
\mathcal{K}_{\alpha,ij,1}(\overline{U}^{(n)}_{\alpha,ij,1},\psi_{\alpha,ij},\overline{U}^{(n)}_{\alpha',ij,1}) &= -[c_{\alpha,ij,1}-(f_{\alpha,ij,1})_{u_\alpha}]\overline{Z}^{(n)}_{\alpha,ij,1}\\
&\quad -(f_{\alpha,ij,1})_{u_{\alpha'}}\overline{Z}^{(n)}_{\alpha',ij,1}-r_{\alpha,ij,1}\overline{Z}^{(n)}_{\alpha,i+1,j,1}\\
&\quad -q_{\alpha,ij,1}\overline{Z}^{(n)}_{\alpha,i,j+1,1},
\end{aligned}$$

$$(i,j)\in\Omega^h,\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2,$$

$$(f_{\alpha,ij,1})_{u_\alpha}=(f_{\alpha,ij,1}(Q^{(n)}_{\alpha,ij,1},\overline{U}^{(n)}_{\alpha',ij,1}))_{u_\alpha},\quad (f_{\alpha,ij,1})_{u_{\alpha'}}=(f_{\alpha,ij,1}(\overline{U}^{(n)}_{\alpha,ij,1},Y^{(n)}_{\alpha',ij,1}))_{u_{\alpha'}},$$

$$\overline{U}^{(n)}_{\alpha,ij,1}\le Q^{(n)}_{\alpha,ij,1}\le U^{(n-1)}_{\alpha',ij,1},\quad \overline{U}^{(n)}_{\alpha,ij,1}\le Y^{(n)}_{\alpha,ij,1}\le \overline{U}^{(n-1)}_{\alpha',ij,1}.$$

By taking limits of the both sides and using (5.73), we conclude that

$$\mathcal{K}_{\alpha,ij,1}(\overline{V}_{\alpha,ij,1},\psi_{\alpha,ij},\overline{V}_{\alpha',ij,1})=0,\quad (i,j)\in\Omega^h,\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2.$$

Thus, $\overline{V}_{\alpha,ij,1}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, solve (5.41) on the first time level $m=1$.

By the assumption of the theorem that $\widetilde{U}_{\alpha,ij,2}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, are upper solutions and from (5.73), it follows that $\widetilde{U}_{\alpha,ij,2}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, are upper solutions with respect to $\overline{V}_{\alpha,ij,1}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$. Indeed, from (5.73), it follows that $\overline{V}_{\alpha,ij,1}\le\widetilde{U}_{\alpha,ij,1}$, $(i,j)\in\overline{\Omega}$, $\alpha=1,2$, and we have

$$\begin{aligned}
\mathcal{K}_{\alpha,ij,2}(\widetilde{U}_{\alpha,ij,2},\overline{U}_{\alpha,ij,1},\widetilde{U}_{\alpha',ij,2}) &= (d_{\alpha,ij,2}+\tau^{-1})\widetilde{U}_{\alpha,ij,2}-l_{\alpha,ij,2}\widetilde{U}_{\alpha,i-1,j,2}-r_{\alpha,ij,2}\widetilde{U}_{\alpha,i+1,j,2}\\
&\quad -b_{\alpha,ij,2}\widetilde{U}_{\alpha,i,j-1,2}-q_{\alpha,ij,2}\widetilde{U}_{\alpha,i,j+1,2}\\
&\quad +f_{\alpha,ij,2}(\widetilde{U}_{\alpha,ij,2},\widetilde{U}_{\alpha',ij,2})-\tau^{-1}\overline{U}_{\alpha,ij,1},\\
&\ge \mathcal{K}_{\alpha,ij,2}(\widetilde{U}_{\alpha,ij,2},\widetilde{U}_{\alpha,ij,1},\widetilde{U}_{\alpha',ij,2})\ge 0,
\end{aligned}$$

$$(i,j)\in\Omega^h,\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2.$$

Using a similar argument as in (5.73), we can prove that the limits

$$\lim_{n\to\infty}\overline{U}^{(n)}_{\alpha,ij,2}=\overline{V}_{\alpha,ij,2},\quad (i,j)\in\overline{\Omega}^h,\quad \alpha=1,2,$$

exist and solve (5.41) on the second time level $m=2$.

By induction on $m\ge1$, we can prove that

$$\lim_{n\to\infty}\overline{U}^{(n)}_{\alpha,ij,m}=\overline{V}_{\alpha,ij,m},\quad (i,j)\in\overline{\Omega}^h,\quad \alpha=1,2,\quad m\ge1,$$

are solutions of the nonlinear difference scheme (5.41). Similarly, we can prove that $\underline{V}_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\ge1$, defined by

$$\lim_{n\to\infty}\underline{U}^{(n)}_{\alpha,ij,m}=\underline{V}_{\alpha,ij,m},\quad (i,j)\in\overline{\Omega}^h,\quad \alpha=1,2,\quad m\ge1,$$

are solutions to the nonlinear difference scheme (5.41). □

We now assume that in the sector $\langle \widehat{U}_m, \widetilde{U}_m \rangle$ from (5.18), the reaction functions f_α, $\alpha = 1, 2$, satisfy the two-sided constrains

$$\underline{c}_{\alpha,ij,m} \le \frac{\partial f_{\alpha,ij,m}(U_{1,ij,m}, U_{2,ij,m})}{\partial u_\alpha} \le c_{\alpha,ij,m}, \quad U_{\alpha,ij,m} \in \langle \widehat{U}_m, \widetilde{U}_m \rangle, \tag{5.74}$$

$$0 \le -\frac{\partial f_{\alpha,ij,m}(U_{1,ij,m}, U_{2,ij,m})}{\partial u_{\alpha'}} \le q_{\alpha,ij,m}, \quad U_{\alpha,ij,m} \in \langle \widehat{U}_m, \widetilde{U}_m \rangle, \tag{5.75}$$

$$(i,j) \in \overline{\Omega}^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1, 2,$$

where $c_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \ge 1$, are defined in (5.19), $q_{\alpha,ij,m}$ and $\underline{c}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \ge 1$, are, respectively, nonnegative bounded and bounded functions. It is assumed that the time step τ satisfies the assumption

$$\tau < \max_{m \ge 1} \frac{1}{\beta_m}, \tag{5.76}$$

$$\beta_m = \max(0, q_m - \underline{c}_m) = \begin{cases} 0, & \text{if } q_m - \underline{c}_m \le 0, \\ q_m - \underline{c}_m, & \text{if } q_m - \underline{c}_m > 0, \end{cases}$$

$$\underline{c}_m = \min_{\alpha=1,2} \left[\min_{(i,j) \in \overline{\Omega}^h} \underline{c}_{\alpha,ij,m} \right], \quad q_m = \max_{\alpha=1,2} \|q_{\alpha,m}\|_{\overline{\Omega}^h},$$

where the notation of the discrete norm from (5.72) is in use. When $\beta_m = 0$, $m \ge 1$, then there is no restriction on τ.

Theorem 5.5.3. *Let $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ and $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \ge 1$, be ordered upper and lower solutions (5.42) to (5.41). Suppose that functions f_α, $\alpha = 1, 2$, in (5.1) satisfy (5.74), (5.75), and assumption (5.76) on the time step τ is satisfied. Then the nonlinear difference scheme (5.41) has a unique solution.*

Proof. Firstly, we show that if $V^*_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \ge 1$, are any other solutions in $\langle \widehat{U}_m, \widetilde{U}_m \rangle$, then

$$\underline{V}_{\alpha,ij,m} \le V^*_{\alpha,ijj,m} \le \overline{V}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1, 2, \quad m \ge 1, \tag{5.77}$$

where $\overline{V}_{\alpha,ij,m}$ and $\underline{V}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \ge 1$, are the solutions to the nonlinear difference scheme (5.41), which are defined in Theorem 5.5.2. Using $(V^*_{1,ij,m}, V^*_{2,ij,m})$ and $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \ge 1$, as initial upper and lower iterations, the sequence $\{\underline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \ge 1$, remains unchanged and converges to the solution $(\underline{V}_{1,ij,m}, \underline{V}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \ge 1$. Taking into account that the sequence

$\{\overline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, with

$$(\overline{U}^{(0)}_{1,ij,m}, \overline{U}^{(0)}_{2,ij,m}) = (V^*_{1,ij,m}, V^*_{2,ij,m}), \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1,$$

consists of the single element $(V^*_{1,ij,m}, V^*_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, from (5.44), it follows that

$$V^*_{\alpha,ij,m} \geq \underline{V}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \geq 1. \tag{5.78}$$

Similarly, by using $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{1,ij,m})$ and $(V^*_{1,ij,m}, V^*_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, as initial upper and lower iterations, the sequence $\{\overline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, remains unchanged and converges to the solution $(\overline{V}_{1,ij,m}, \overline{V}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$. Taking into account that the sequence $\{\underline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, with

$$(\underline{U}^{(0)}_{1,ij,m}, \underline{U}^{(0)}_{2,ij,m}) = (V^*_{1,ij,m}, V^*_{2,ij,m}), \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1,$$

consists of the single element $(V^*_{1,ij,m}, V^*_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, from (5.44), it follows that

$$V^*_{\alpha,ij,m} \leq \overline{V}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \geq 1.$$

From here and (5.78), we conclude (5.77).

Taking into account (5.77), for the uniqueness of a solution to the nonlinear difference scheme (5.41), it suffices to prove that

$$\overline{V}_{\alpha,ij,m} = \underline{V}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad ,m \geq 1.$$

From (5.44) and Theorem 5.5.2, we have

$$\underline{U}^{(n)}_{\alpha,ij,m} \leq \underline{V}_{\alpha,ij,m} \leq \overline{V}_{\alpha,ij,m} \leq \overline{U}^{(n)}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \ \alpha = 1,2, \quad m \geq 1. \tag{5.79}$$

Letting $W_{\alpha,ij,m} = \overline{V}_{\alpha,ij,m} - \underline{V}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, from (5.11), by using the mean-value theorem (5.16), we obtain for $m \geq 1$

$$\begin{aligned}
&\big(\mathcal{A}_{\alpha,ij,m} + (\tau^{-1} + (f_{\alpha,ij,m}(Q_{\alpha,ij,m}, \overline{V}_{\alpha',ij,m}))_{u_\alpha})\big) W_{\alpha,ij,m} = \\
&- (f_{\alpha,ij,m}(\underline{V}_{\alpha,ij,m}, Y_{\alpha',ij,m}))_{u_{\alpha'}} W_{\alpha',ij,m} + \frac{1}{\tau} W_{\alpha,ij,m-1}, \quad (i,j) \in \Omega^h, \\
&W_{\alpha,ij,m} = 0, \quad (i,j) \in \partial\omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \\
&\underline{V}_{\alpha,ij,m} \leq Q_{\alpha,ij,m}, Y_{\alpha,ij,m} \leq \overline{V}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2.
\end{aligned} \tag{5.80}$$

From here and (5.79), it follows that the partial derivatives satisfy (5.74) and (5.75). From here for $\alpha = 1$, (5.74), (5.75), taking into account that $W_{\alpha,ij,0} = 0$, $(i,j) \in \overline{\Omega}^h$,

$\alpha = 1, 2$ and using (5.72), we conclude that

$$w_1 \leq \frac{\tau q_1}{1 + \tau \underline{c}_1} w_1, \quad w_m = \max_{\alpha=1,2} \|W_{\alpha,m}\|_{\overline{\Omega}^h}, \quad m \geq 1.$$

From the above inequality, by the assumption on τ in (5.76) and $w_1 \geq 0$, we conclude that $w_1 = 0$. On the second time level $m = 2$, taking into account that $w_1 = 0$, by the similar manner, we conclude that $w_2 = 0$. Now, by induction on m, $m \geq 1$, we can prove that $w_m = 0$, $m \geq 1$. Thus, we prove the theorem. □

5.5.2 Quasi-monotone nonincreasing case

In the following theorem, we prove the existence of a solution to (5.41) based on Theorem 5.4.5.

Theorem 5.5.4. *Let* $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ *and* $(\widehat{U}_{1j,i,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \geq 1$, *be ordered upper and lower solutions (5.56) to (5.41). Suppose that* f_α, $\alpha = 1, 2$, *in (5.1) satisfy (5.19) and (5.30). Then a solution of the nonlinear difference scheme (5.41) exists in the sector* $\langle \widehat{U}_m, \widetilde{U}_m \rangle$, $m \geq 1$, *from (5.18).*

Proof. We consider a sequence $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, generated by the point monotone Gauss-Seidel method with $\eta = 1$ in (5.57).

On the first time level $m = 1$, from (5.58), we conclude that $\lim_{n\to\infty} \overline{U}^{(n)}_{1,ij,1} = \overline{V}_{1,ij,1}$ and $\lim_{n\to\infty} \underline{U}^{(n)}_{2,ij,1} = \underline{V}_{2,ij,1}$, $(i,j) \in \overline{\Omega}^h$, exist and

$$\begin{aligned} &\overline{V}_{1,ij,1} \leq \overline{U}^{(n)}_{1,ij,1} \leq \overline{U}^{(n-1)}_{1,ij,1} \leq \widetilde{U}_{1,ij,1}, \quad \lim_{n\to\infty} \overline{Z}^{(n)}_{1,ij,1} = 0, \quad (i,j) \in \overline{\Omega}^h, \\ &\widehat{U}_{2,ij,1} \leq \underline{U}^{(n-1)}_{2,ij,1} \leq \underline{U}^{(n)}_{2,ij,1} \leq \underline{V}_{2,ij,1}, \quad \lim_{n\to\infty} \underline{Z}^{(n)}_{2,ij,1} = 0, \quad (i,j) \in \overline{\Omega}^h, \end{aligned} \tag{5.81}$$

where $\overline{U}^{(0)}_{1,ij,1} = \widetilde{U}_{1,ij,1}$ and $\underline{U}^{(0)}_{2,ij,1} = \widehat{U}_{2,ij,1}$, $(i,j) \in \overline{\Omega}^h$. From (5.57), for $\alpha = 1$, we have

$$\begin{aligned} \mathcal{K}_{1,ij,1}(\overline{U}^{(n)}_{1,ij,1}, \psi_{1,ij}, \underline{U}^{(n)}_{2,ij,1}) = &-(c_{1,ij,1}\overline{Z}^{(n)}_{1,ij,1} + r_{1,ij,1}\overline{Z}^{(n)}_{1,i+1,j,1}) + q_{1,ij,1}\overline{Z}^{(n)}_{1,i,j+1,1}) \\ &+ f_{1,ij,1}(\overline{U}^{(n)}_{1,ij,1}, \underline{U}^{(n)}_{2,ij,1}) - f_{1,ij,1}(\overline{U}^{(n-1)}_{1,ij,1}, \underline{U}^{(n-1)}_{2,ij,1}), \end{aligned}$$

$(i,j) \in \Omega^h$.

By the mean-value theorem (5.16), we have

$$\mathcal{K}_{1,ij,1}(\overline{U}^{(n)}_{1,ij,1},\psi_{1,ij},\underline{U}^{(n)}_{2,ij,1}) = [c_{1,ij,1}-(f_{1,ij,1})_{u_1})]\overline{Z}^{(n)}_{1,ij,1}$$
$$-(f_{1,ij,1})_{u_2}\underline{Z}^{(n)}_{2,ij,1}-r_{1,ij,1}\overline{Z}^{(n)}_{1,i+1,j,1}-q_{1,ij,1}\overline{Z}^{(n)}_{1,i,j+1,1},$$
$$(i,j)\in\Omega,$$
$$(f_{1,ij,1})_{u_1}=(f_{1,ij,1}(Q^{(n)}_{1,ij,1},\underline{U}^{(n)}_{2,ij,1}))_{u_1},\quad (f_{1,ij,1})_{u_2}=(f_{1,i,1}(\overline{U}^{(n)}_{1,ij,1},Y^{(n)}_{2,ij,1}))_{u_2},$$
$$\overline{U}^{(n)}_{1,ij,1}\le Q^{(n)}_{1,ij,1}\le\overline{U}^{(n-1)}_{1,ij,1},\quad \underline{U}^{(n-1)}_{2,ij,1}\le Y^{(n)}_{2,ij,1}\le\underline{U}^{(n)}_{2,ij,1}.$$

From (5.81), by taking limit of the both sides, we conclude that

$$\mathcal{K}_{1,ij,1}(\overline{V}_{1,ij,1},\psi_{1,ij},\underline{V}_{2,ij,1})=0,\quad (i,j)\in\Omega^h.$$

Similarly, we can prove that

$$\mathcal{K}_{2,ij,1}(\overline{V}_{1,ij,1},\psi_{2,ij},\underline{V}_{2,ij,1})=0,\quad (i,j)\in\Omega^h.$$

Thus, $(\overline{V}_{1,ij,1},\underline{V}_{2,ij,1})$, $(i,j)\in\overline{\Omega}^h$, solve (5.41) on the first time level $m=1$.

By the assumptions of the theorem that $\widetilde{U}_{1,ij,2}$, and $\widehat{U}_{2,ij,2}$, $(i,j)\in\overline{\Omega}^h$, are upper and lower solutions and from (5.56), it follows that $\widetilde{U}_{1,ij,2}$, and $\widehat{U}_{2,ij,2}$, $(i,j)\in\overline{\Omega}^h$, are upper and lower solutions solutions with respect to $\overline{V}_{1,ij,1}$, and $\underline{V}_{2,ij,1}$, $(i,j)\in\overline{\Omega}^h$. Indeed, from (5.81), it follows that $\overline{V}_{1,ij,1}\le\widetilde{U}_{1,ij,1}$, $\underline{V}_{2,ij,1}\ge\widehat{U}_{2,ij,1}$, $(i,j)\in\overline{\Omega}^h$. From here and (5.41), we have

$$\mathcal{K}_{1,ij,2}(\widetilde{U}_{1,ij,2},\overline{V}_{1,ij,1},\widehat{U}_{2,ij,2})=(d_{1,ij,2}+\tau^{-1})\widetilde{U}_{1,ij,2}-l_{1,ij,2}\widetilde{U}_{1,i-1,j,2}-r_{1,ij,2}\widetilde{U}_{1,i+1,j,2}$$
$$-b_{1,ij,2}\widetilde{U}_{1,i,j-1,2}-q_{1,ij,2}\widetilde{U}_{1,i,j+1,2}$$
$$+f_{1,ij,2}(\widetilde{U}_{1,ij,2},\widehat{U}_{2,ij,2})+\tau^{-1}\overline{V}_{1,ij,1}$$
$$\ge\mathcal{K}_{1,ij,2}(\widetilde{U}_{1,ij,2},\widetilde{U}_{1,ij,1},\widehat{U}_{2,ij,2})\ge 0,\quad (i,j)\in\Omega^h,$$

$$\mathcal{K}_{2,ij,2}(\widetilde{U}_{1,ij,2},\underline{V}_{2,ij,1},\widehat{U}_{2,ij,2})=(d_{2,ij,2}+\tau^{-1})\widehat{U}_{2,ij,2}-l_{2,ij,2}\widehat{U}_{2,i-1,j,2}-r_{2,ij,2}\widehat{U}_{2,i+1,j,2}$$
$$-b_{2,ij,2}\widehat{U}_{2,i,j-1,2}-q_{2,ij,2}\widehat{U}_{2,i,j+1,2}$$
$$+f_{2,ij,2}(\widetilde{U}_{1,ij,2},\widehat{U}_{2,ij,2})+\tau^{-1}\underline{V}_{2,ij,1}$$
$$\le\mathcal{K}_{2,ij,2}(\widetilde{U}_{1,ij,2},\widehat{U}_{2,ij,1},\widehat{U}_{2,ij,2})\le 0,\quad (i,j)\in\Omega^h,$$

which means that $\widetilde{U}_{1,ij,2}$ and $\widehat{U}_{2,ij,2}$, $(i,j)\in\overline{\Omega}^h$, are upper and lower solutions with respect to $\overline{V}_{1,ij,1}$ and $\underline{V}_{2,ij,1}$, $(i,j)\in\overline{\Omega}^h$.

Using a similar argument as in (5.81), we can prove that the limits

$$\lim_{n\to\infty} \overline{U}^{(n)}_{1,ij,2} = \overline{V}_{1,ij,2}, \quad \lim_{n\to\infty} \underline{U}^{(n)}_{2,ij,2} = \underline{V}_{2,ij,2} \quad (i,j) \in \overline{\Omega}^h,$$

exist and $(\overline{V}_{1,ij,2}, \underline{V}_{2,ij,2})$, $(i,j) \in \overline{\Omega}^h$, solves (5.41) on the second time level $m = 2$.

By induction on $m \geq 1$, we can prove that

$$\lim_{n\to\infty} \overline{U}^{(n)}_{1,ij,m} = \overline{V}_{1,ij,m}, \quad \lim_{n\to\infty} \underline{U}^{(n)}_{2,ij,m} = \underline{V}_{2,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1.$$

Thus, $(\overline{V}_{1,ij,m}, \underline{V}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, are solutions of the nonlinear difference scheme (5.41).

Similarly, for a sequence $\{\underline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, we can prove that

$$\lim_{n\to\infty} \underline{U}^{(n)}_{1,ij,m} = \underline{V}_{1,ij,m}, \quad \lim_{n\to\infty} \overline{U}^{(n)}_{2,ij,m} = \overline{V}_{2,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1,$$

and $(\underline{V}_{1,ij,m}, \overline{V}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, are solutions of the nonlinear difference scheme (5.41). □

We now assume that in the sector $\langle \widehat{U}_m, \widetilde{U}_m \rangle$, $m \geq 1$, the reaction functions f_α, $\alpha = 1, 2$, satisfy (5.74) and the two-sided constrains

$$q_{\alpha,ij,m} \leq -\frac{\partial f_{\alpha,ij,m}(U_{1,ij,m}, U_{2,ij,m})}{\partial u_{\alpha'}} \leq 0, \quad U_{\alpha,ij,m} \in \langle \widehat{U}_m, \widetilde{U}_m \rangle, \tag{5.82}$$
$$(i,j) \in \overline{\Omega}^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2,$$

where $c_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \geq 1$, are defined in (5.19), $q_{\alpha,ij,m}$ and $\underline{c}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \geq 1$, are, respectively, nonpositive bounded and bounded functions.

Theorem 5.5.5. *Let $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ and $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, be ordered upper and lower solutions (5.56) to (5.41). Suppose that functions f_α, $\alpha = 1, 2$, in (5.1) satisfy (5.74), (5.82), and assumption (5.76) on the time step τ is satisfied. Then the nonlinear difference scheme (5.41) has a unique solution.*

Proof. Firstly, we show that if $V^*_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \geq 1$, are any other solutions in $\langle \widehat{U}_m, \widetilde{U}_m \rangle$, $m \geq 1$, then

$$\underline{V}_{\alpha,ij,m} \leq V^*_{\alpha,ij,m} \leq \overline{V}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1, 2, \quad m \geq 1, \tag{5.83}$$

where $(\overline{V}_{1,ij,m},\ \underline{V}_{2,ij,m})$ and $(\underline{V}_{1,ij,m},\ \overline{V}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, are the solutions to the nonlinear difference scheme (5.41), which are defined in Theorem 5.5.4. Using

$(\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$ and $(V^*_{1,ij,m}, V^*_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, as initial iterations, the sequence $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, remains unchanged and converges to the solution $(\overline{V}_{1,ij,m}, \underline{V}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$. Taking into account that the sequence $\{\underline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, with

$$(\underline{U}^{(0)}_{1,ij,m}, \overline{U}^{(0)}_{2,ij,m}) = (V^*_{1,ij,m}, V^*_{2,ij,m}), \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1,$$

consists of the single element $(V^*_{1,ij,m}, V^*_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, from (5.58), it follows that

$$V^*_{1,ij,m} \leq \overline{V}_{1,ij,m}, \quad V^*_{2,ij,m} \geq \underline{V}_{2,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1. \tag{5.84}$$

Similarly, by using $(\widehat{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ and $(V^*_{1,ij,m}, V^*_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, as initial iterations, the sequence $\{\underline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, remains unchanged and converges to the solution $(\underline{V}_{1,ij,m}, \overline{V}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$. Taking into account that the sequence $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, with

$$(\overline{U}^{(0)}_{1,ij,m}, \underline{U}^{(0)}_{2,ij,m}) = (V^*_{1,ij,m}, V^*_{2,ij,m}), \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1,$$

consists of the single element $(V^*_{1,ij,m}, V^*_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, from (5.58), it follows that

$$V^*_{1,ij,m} \geq \underline{V}_{1,ij,m}, \quad V^*_{2,ij,m} \leq \overline{V}_{2,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1.$$

From here and (5.84), we conclude (5.83).

Taking into account (5.83), for the uniqueness of a solution to the nonlinear difference scheme (5.41), it suffices to prove that

$$\overline{V}_{\alpha,ij,m} = \underline{V}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \geq 1.$$

From (5.58) and Theorem 5.5.4, we have

$$\underline{U}^{(n)}_{\alpha,ij,m} \leq \underline{V}_{\alpha,ij,m} \leq \overline{V}_{\alpha,ij,m} \leq \overline{U}^{(n)}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \geq 1. \tag{5.85}$$

Letting $W_{\alpha,ij,m} = \overline{V}_{\alpha,ij,m} - \underline{V}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, from (5.11), by using the mean-value theorem (5.16), we obtain for $m \geq 1$

$$\begin{aligned} &\big(\mathcal{A}_{\alpha,ij,m} + (\tau^{-1} + (f_{\alpha,ij,m}(Q_{\alpha,ij,m}, \overline{V}_{\alpha',ij,m}))_{u_\alpha})\big) W_{\alpha,ij,m} = \\ &- (f_{\alpha,ij,m}(\underline{V}_{\alpha,ij,m}, Y_{\alpha',ij,m}))_{u_{\alpha'}} W_{\alpha',ij,m} + \frac{1}{\tau} W_{\alpha,ij,m-1}, \quad (i,j) \in \Omega^h, \\ &W_{\alpha,ij,m} = 0, \quad (i,j) \in \partial\Omega^h, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \\ &\underline{V}_{\alpha,ij,m} \leq Q_{\alpha,ij,m}, Y_{\alpha,ij,m} \leq \overline{V}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2. \end{aligned} \tag{5.86}$$

From here and (5.85), it follows that the partial derivatives satisfy (5.74), (5.82). From here for $m = 1$, (5.74), (5.82), taking into account that $W_{\alpha,ij,0} = 0$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, and using (5.72), we conclude that

$$w_1 \leq \frac{\tau q_1}{1 + \tau \underline{c}_1} w_1, \quad w_m = \max_{\alpha=1,2} \|W_{\alpha,m}\|_{\overline{\Omega}^h}.$$

From here, by the assumption on τ in (5.76) and $w_1 \geq 0$, we conclude that $w_1 = 0$. On the second time level $m = 2$, taking into account that $w_1 = 0$, by the similar manner, we conclude that $w_2 = 0$. Now, by induction on $m \geq 1$, we can prove that $w_m = 0$, $m \geq 1$. Thus, we prove the theorem. □

5.6 Comparison of convergence of the point monotone Jacobi and Gauss–Seidel methods

5.6.1 Quasi-monotone nondecreasing case

The following theorem shows that the point monotone Gauss–Seidel method with $\eta = 1$ in (5.43) converges faster than the point monotone Jacobi method with $\eta = 0$ in (5.43).

Theorem 5.6.1. *Let $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ and $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, be ordered upper and lower solutions (5.42) to (5.41), the functions f_α, $\alpha = 1,2$, in (5.1) satisfy (5.19) and (5.20). Suppose that the sequences $\{(U^{(n)}_{\alpha,ij,m})_J\}$ and $\{(U^{(n)}_{\alpha,ij,m})_{GS}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are generated by the point monotone Jacobi method with $\eta = 0$ in (5.43) and by the point monotone Gauss–Seidel method with $\eta = 1$ in (5.43), where $(\overline{U}^{(0)}_{ij,m})_J = (\overline{U}^{(0)}_{ij,m})_{GS} = \widetilde{U}_{ij}$ and $(\underline{U}^{(0)}_{ij,m})_J = (\underline{U}^{(0)}_{ij,m})_{GS} = \widehat{U}_{ij,m}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$. Then*

$$(\underline{U}^{(n)}_{\alpha,ij,m})_J \leq (\underline{U}^{(n)}_{\alpha,ij,m})_{GS} \leq (\overline{U}^{(n)}_{\alpha,ij,m})_{GS} \leq (\overline{U}^{(n)}_{\alpha,ij,m})_J, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \geq 1. \tag{5.87}$$

Proof. Letting $\underline{W}^{(n)}_{\alpha,ij,m} = \left(\underline{U}^{(n)}_{\alpha,ij,m}\right)_{\mathtt{GS}} - \left(\underline{U}^{(n)}_{\alpha,ij,m}\right)_{\mathtt{J}}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, $\alpha = 1,2$, from (5.43) and using notation (5.21), we have

$$\begin{aligned}
&(d_{\alpha,ij,m} + \tau^{-1} + c_{\alpha,ij,m})(\underline{W}^{(n)}_{\alpha,ij,m}) = \\
&\eta l_{\alpha,ij,m}\left((\underline{U}^{(n)}_{\alpha,i-1,j,m})_{\mathtt{GS}} - (\underline{U}^{(n-1)}_{\alpha,i-1,j,m})_{\mathtt{J}}\right) + r_{\alpha,ij,m}\underline{W}^{(n-1)}_{\alpha,i+1,j,m} \\
&+ \eta b_{\alpha,ij,m}\left((\underline{U}^{(n)}_{\alpha,i,j-1,m})_{\mathtt{GS}} - (\underline{U}^{(n-1)}_{\alpha,i,j-1,m})_{\mathtt{J}}\right) + q_{\alpha,ij,m}\underline{W}^{(n-1)}_{\alpha,i,j+1,m} \\
&+ \Gamma_{\alpha,ij,m}\left((\underline{U}^{(n-1)}_{\alpha,ij,m}), \underline{U}^{(n-1)}_{\alpha',ij,m})\right)_{\mathtt{GS}} - \Gamma_{\alpha,ij,m}\left((\underline{U}^{(n-1)}_{\alpha,ij,m}), \underline{U}^{(n-1)}_{\alpha',ij,m})\right)_{\mathtt{J}} \\
&+ \tau^{-1}\left((\underline{U}_{\alpha,ij,m-1})_{\mathtt{GS}} - (\underline{U}_{\alpha,ij,m-1})_{\mathtt{J}}\right), \quad (i,j) \in \Omega^h, \\
&\underline{W}^{(n)}_{\alpha,ij,m} = 0, \quad (i,j) \in \partial\Omega^h, \quad m \geq 1, \quad \underline{W}_{\alpha,ij,0} = 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2.
\end{aligned}$$

From here and taking into account that $\left(\underline{U}^{(n-1)}_{\alpha,ij,m}\right)_{\mathtt{GS}} \le \left(\underline{U}^{(n)}_{\alpha,ij,m}\right)_{\mathtt{GS}}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \ge 1$, we obtain

$$\begin{aligned}(d_{\alpha,ij,m} + \tau^{-1} + c_{\alpha,ij,m})\underline{W}^{(n)}_{\alpha,ij,m} \ge{}& \eta l_{\alpha,ij,m}\underline{W}^{(n-1)}_{\alpha,i-1,j,m} + r_{\alpha,ij,m}\underline{W}^{(n-1)}_{\alpha,i+1,j,m} \\ &+ \eta b_{\alpha,ij,m}\underline{W}^{(n-1)}_{\alpha,i,j-1,m} + q_{\alpha,ij,m}\underline{W}^{(n-1)}_{\alpha,i,j+1,m} \\ &+ \Gamma_{\alpha,ij,m}\left((\underline{U}^{(n-1)}_{\alpha,ij,m}), \underline{U}^{(n-1)}_{\alpha',ij,m})\right)_{\mathtt{GS}} \\ &- \Gamma_{\alpha,ij,m}\left((\underline{U}^{(n-1)}_{\alpha,ij,m}), \underline{U}^{(n-1)}_{\alpha',ij,m})\right)_{\mathtt{J}} \\ &+ \tau^{-1}\left((\underline{U}_{\alpha,ij,m-1})_{\mathtt{GS}} - (\underline{U}_{\alpha,ij,m-1})_{\mathtt{J}}\right), \quad (i,j) \in \Omega^h,\end{aligned} \tag{5.88}$$

$$\underline{W}^{(n)}_{\alpha,ij,m} = 0, \quad (i,j) \in \partial\Omega^h, \quad m \ge 1, \quad \underline{W}_{\alpha,ij,0} = 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2.$$

Taking into account that $\eta = 0,1$, (5.41b) and $(\underline{U}^{(0)}_{\alpha,ij,m})_{\mathtt{GS}} = (\underline{U}^{(0)}_{\alpha,ij,m})_{\mathtt{J}}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \ge 1$, for $n = 1$ in (5.88), on the first time level $m = 1$, by using the maximum principle in Lemma 5.4.1, we conclude that

$$\underline{W}^{(1)}_{\alpha,ij,1} \ge 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2.$$

Similarly, from here, $\eta = 0,1$, (5.41b) and (5.88) with $n = 2$, by using (5.22) and Lemma 5.4.1, we obtain that $\underline{W}^{(2)}_{\alpha,ij,1} \ge 0$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. By induction on n, $n \ge 1$, we can prove that $\underline{W}^{(n)}_{\alpha,ij,1} \ge 0$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$.

On the second time level $m = 2$, taking into account that $\underline{W}^{(0)}_{\alpha,ij,2} = 0$ and $\underline{W}_{\alpha,ij,1} \ge 0$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, from (5.41b) and (5.22), by using Lemma (5.4.1), we have

$$\overline{W}^{(1)}_{\alpha,ij,2} \ge 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2.$$

Similarly, from here and (5.88) with $n = 2$, by using (5.31), on the second time level $m = 2$, we obtain that $\underline{W}^{(2)}_{\alpha,ij,2} \ge 0$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$. By induction on n, we can prove that $\underline{W}^{(n)}_{\alpha,ij,2} \ge 0$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$.

By induction on $m \ge 1$, we can prove that

$$\underline{W}^{(n)}_{\alpha,ij,m} \ge 0, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \ge 1.$$

Thus, we prove (5.87) for the case of lower solutions. By the same manner, we can prove (5.87) for the case of upper solutions. □

5.6.2 Quasi-monotone nonincreasing case

Theorem 5.6.2. *Let $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ and $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \ge 1$, be ordered upper and lower solutions (5.56) to (5.41). Suppose that the functions f_α,*

$\alpha = 1,2$, in (5.1) satisfy (5.19) and (5.30). The sequences $\left\{(\overline{U}^{(n)}_{1,ij,m})_J, (\underline{U}^{(n)}_{2,ij,m})_J\right\}$, $\left\{(\underline{U}^{(n)}_{1,ij,m})_J, (\overline{U}^{(n)}_{2,ij,m})_J\right\}$ and $\left\{(\overline{U}^{(n)}_{1,ij,m})_{GS}, (\underline{U}^{(n)}_{2,ij,m})_{GS}\right\}$, $\left\{(\underline{U}^{(n)}_{1,ij,m})_{GS}, (\overline{U}^{(n)}_{2,ij,m})_{GS}\right\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are, respectively, the sequences generated by the point monotone Jacobi method with $\eta = 0$ in (5.57), and the point monotone Gauss–Seidel method with $\eta = 1$ in (5.57), where $(\overline{U}^{(0)}_{\alpha,ij,m})_J = (\overline{U}^{(0)}_{\alpha,ij,m})_{GS} = \widetilde{U}_{\alpha,ij,m}$ and $(\underline{U}^{(0)}_{\alpha,ij,m})_J = (\underline{U}^{(0)}_{\alpha,i,m})_{GS} = \widehat{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$. Then the inequalities in (5.87) hold true.

Proof. The proof of the theorem repeats the proof of Theorem 5.6.1, where $\Gamma_{\alpha,ij,m}$, $i \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are now written in the form

$$\Gamma_{\alpha,ij,m}(\overline{U}^{(n)}_{\alpha,ij,m}, \underline{U}^{(n)}_{\alpha',ij,m}) = c_{\alpha,ij,m}\overline{U}^{(n)}_{\alpha,ij,m} - f_{\alpha,ij,m}(\overline{U}^{(n)}_{\alpha,ij,m}, \underline{U}^{(n)}_{\alpha',ij,m}),$$
$$\Gamma_{\alpha,ij,m}(\underline{U}^{(n)}_{\alpha,ij,m}, \overline{U}^{(n)}_{\alpha',ij,m}) = c_{\alpha,ij,m}\underline{U}^{(n)}_{\alpha,ij,m} - f_{\alpha,ij,m}(\underline{U}^{(n)}_{\alpha,ij,m}, \overline{U}^{(n)}_{\alpha',ij,m}),$$

and the monotone properties (5.31) for $\Gamma_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are in use. □

5.7 Convergence analysis of the point monotone iterative methods

5.7.1 Quasi-monotone nondecreasing case

Instead of (5.74), we now assume that for $m \geq 1$,

$$q_m \leq \frac{\partial f_{\alpha,ij,m}(U_{1,ij,m}, U_{2,ij,m})}{\partial u_\alpha} \leq c_{\alpha,ij,m}, \quad U_{\alpha,ij,m} \in \langle \widehat{U}_m, \widetilde{U}_m \rangle, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \tag{5.89}$$

where q_m is defined in (5.76).

Remark 5.7.1. *The assumption $\partial f_\alpha / \partial u_\alpha \geq q_m \geq 0$ in (5.89) can always be obtained by a change of variables. Indeed, we introduce the functions $z_\alpha(x,y,t) = e^{-\lambda t} u_\alpha(x,y,t)$, $\alpha = 1,2$, where λ is a constant. Now, $z = (z_1, z_2)$ satisfies (5.1) with*

$$\widetilde{f}_\alpha = \lambda z_\alpha + e^{-\lambda t} f_\alpha(x,y,t,e^{\lambda t} z_\alpha), \quad \alpha = 1,2,$$

instead of f_α, $\alpha = 1,2$, and we have

$$\frac{\partial \widetilde{f}_\alpha}{\partial z_\alpha} = \lambda + \frac{\partial f_\alpha}{\partial u_\alpha}, \quad \frac{\partial \widetilde{f}_\alpha}{\partial z_{\alpha'}} = \frac{\partial f_\alpha}{\partial u_{\alpha'}}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2.$$

Thus, if $\lambda \geq \max_{m \geq 1}(q_m, |\underline{c}_m|)$, where q_m and $\underline{c}_m$ are defined in (5.76), then from here and (5.74), we conclude that $\partial \widetilde{f}_\alpha / \partial z_\alpha$, $\alpha = 1,2$, satisfy (5.89).

A stopping test for the point monotone iterative methods (5.43) is chosen in the form

$$\max_{\alpha=1,2}\left[\max_{(i,j)\in\Omega^h}\left|\mathcal{K}_{\alpha,ij,m}(U^{(n)}_{\alpha,ij,m},U_{\alpha,ij,m-1},U^{(n)}_{\alpha',ij,m})\right|\right]\le\delta, \tag{5.90}$$

where $\mathcal{K}_{\alpha,ij,m}(U^{(n)}_{\alpha,ij,m},U_{\alpha,ij,m-1},U^{(n)}_{\alpha',ij,m})$, $(i,j)\in\Omega^h$, $\alpha'\ne\alpha$, $\alpha,\alpha'=1,2$, $m\ge1$, are residuals of the nonlinear difference scheme (5.11), $U^{(n)}_{\alpha,ij,m}$, $(i,j)\in\Omega^h$, $\alpha=1,2$, $m\ge1$, are generated by (5.43), and δ is a prescribed accuracy. On each time level $m\ge1$, we set up $U_{\alpha,ij,m}=U^{(n_m)}_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\ge1$, such that n_m is the minimal number of iterations subject to (5.90).

Theorem 5.7.2. *Suppose that the assumptions in Theorem 5.5.3 are satisfied. Then for the sequences* $\{U^{(n)}_{\alpha,ij,m}\}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\ge1$, *generated by the point monotone iterative methods (5.43), (5.90), we have the estimate*

$$\max_{m\ge1}\max_{\alpha=1,2}\|U_{\alpha,m}-U^*_{\alpha,m}\|_{\overline{\Omega}^h}\le T\delta, \tag{5.91}$$

where $U_{\alpha,ij,m}=U^{(n_m)}_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\ge1$, *and* $U^*_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\ge1$, *are the unique solutions to the nonlinear difference scheme (5.11) and* T *is the final time.*

Proof. Letting $W_{\alpha,ij,m}=U_{\alpha,ij,m}-U^*_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\ge1$, from (5.11), by using the mean-value theorem (5.16), we obtain

$$\begin{aligned}&\left(\mathcal{A}_{\alpha,ij,m}+\tau^{-1}+(f_{\alpha,ij,m}(Q_{\alpha,ij,m},U_{\alpha',ij,m}))_{u_\alpha}\right)W_{\alpha,ij,m}=\\&-(f_{\alpha,ij,m}(U^*_{\alpha,ij,m},Y_{\alpha',ij,m}))_{u_{\alpha'}}W_{\alpha',ij,m}+\mathcal{K}_{\alpha,ij,m}(\overline{U}_{\alpha,ij,m},\overline{U}_{\alpha,ij,m-1},\overline{U}_{\alpha',ij,m})\\&+\tau^{-1}W_{\alpha,ij,m-1},\\&(i,j)\in\Omega^h,\quad \alpha'\ne\alpha,\quad \alpha,\alpha'=1,2,\\&W_{\alpha,ij,m}=0,\quad (i,j)\in\partial\Omega,\quad m\ge1,\quad W_{\alpha,ij,0}=0,\quad (i,j)\in\overline{\Omega}^h,\quad \alpha=1,2,\end{aligned}$$

where $Q_{\alpha,ij,m}$ and $Y_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\ge1$, lie between $U^*_{\alpha,ij,m}$ and $\overline{U}_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\ge1$, and the partial derivatives satisfy (5.75) and (5.89). From here, (5.75) and (5.89), by using (5.72), we obtain that

$$w_m\le\frac{1}{\tau^{-1}+q_m}\left(q_mw_m+\delta+\tau^{-1}w_{m-1}\right),\quad w_m=\max_{\alpha=1,2}\|W_{\alpha,m}\|_{\overline{\Omega}^h},\quad m\ge1.$$

Solving this inequality for w_m, we have

$$w_m\le\tau\delta+w_{m-1},\quad m\ge1. \tag{5.92}$$

Since $w_0=0$, for $m=1$ in (5.92), we have $w_1\le\tau\delta$. For $m=2$, it follows that

$w_2 \leq \delta(\tau + \tau)$, and by induction on $m \geq 1$, we can prove that

$$w_m \leq \delta \sum_{l=1}^{m} \tau.$$

Since $\sum_{l=1}^{m} \tau \leq T$, we prove (5.91). □

Theorem 5.7.3. *Let the assumptions in Theorem 5.7.2 be satisfied. Then for the sequence of solutions* $\{U^{(n)}_{\alpha,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, *generated by (5.43), (5.90), the following estimate holds*

$$\max_{m\geq 1} \max_{\alpha=1,2} \|U_{\alpha,m} - u^*_{\alpha,m}\|_{\overline{\Omega}^h} \leq T(\delta + \max_{m\geq 1} E_m), \tag{5.93}$$

$$E_m = \max_{\alpha=1,2} \|E_{\alpha,m}\|_{\overline{\Omega}^h}, \quad m \geq 1,$$

where $U_{\alpha,ij,m} = U^{(n_m)}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, $u^*_\alpha(x,y,t)$, $\alpha = 1,2$, *are the exact solutions to (5.1), and* $E_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, *are the truncation errors of the exact solutions on the nonlinear difference scheme (5.11).*

Proof. We denote $V_{\alpha,ij,m} = U^*_{\alpha,ij,m} - u^*_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, where $U^*_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, are the unique solutions of the nonlinear difference scheme (5.11). From (5.11), by using the mean-value theorem (5.16), we obtain that

$$\begin{aligned}
&\left(\mathcal{A}_{\alpha,ij,m} + \tau^{-1} + (f_{\alpha,ij,m}(Q_{\alpha,ij,m}, U^*_{\alpha',ij,m}))_{u_\alpha}\right) V_{\alpha,ij,m} = \\
&- (f_{\alpha,ij,m}(u^*_{\alpha,ij,m}, Y_{\alpha',ij,m}))_{u_{\alpha'}} V_{\alpha',ij,m} + \frac{1}{\tau} V_{\alpha',ij,m-1} - E_{\alpha,ij,m}, \quad (i,j) \in \Omega^h, \\
&V_{\alpha,ij,m} = 0, \quad (i,j) \in \partial\Omega^h, \quad V_{\alpha,ij,0} = 0, \quad (i,j) \in \overline{\Omega}^h, \\
&\alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2,
\end{aligned}$$

where $Q_{\alpha,ij,m}$ and $Y_{\alpha,ij,m}$ lie between $U^*_{\alpha,ij,m}$ and $u^*_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$. From here, (5.75), (5.89) and using (5.72), it follows that

$$v_m \leq \frac{1}{\tau^{-1} + q_m}(q_m v_m + \tau^{-1} v_{m-1} + E_m), \quad v_m = \max_{\alpha=1,2} \|V_{\alpha,m}\|_{\overline{\Omega}^h}, \quad m \geq 1.$$

Solving for v_m, we obtain

$$v_m \leq v_{m-1} + \tau E_m.$$

From here, taking into account that $v_0 = 0$, by induction on $m \geq 1$, we obtain that

$$v_m \leq \sum_{l=1}^{m} \tau E_l.$$

Since $\sum_{l=1}^{m} \tau \leq T$, where T is the final time, we have

$$v_m \leq T \max_{m \geq 1} E_m, \quad m \geq 1. \tag{5.94}$$

We now estimate the left hand side in (5.93) as follows

$$\|U_{\alpha,m} \pm U^*_{\alpha,m} - u^*_{\alpha,m}\|_{\overline{\Omega}^h} \leq \|U_{\alpha,m} - U^*_{\alpha,m}\|_{\overline{\Omega}^h} + \|U^*_{\alpha,m} - u^*_{\alpha,m}\|_{\overline{\Omega}^h}.$$

From here, (5.91) and (5.94), we prove (5.93). □

5.7.2 Quasi-monotone nonincreasing case

Stopping tests for the sequences $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$ and $\{\underline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, generated by the point monotone iterative methods (5.57), are chosen in the forms

$$\max\Big\{ \max_{(i,j)\in\Omega^h} \mathcal{K}_{1,ij,m}(\overline{U}^{(n)}_{1,ij,m}, \overline{U}_{1,ij,m-1}, \underline{U}^{(n)}_{2,ij,m}); \\ \max_{(i,j)\in\Omega^h} \mathcal{K}_{2,ij,m}(\overline{U}^{(n)}_{1,ij,m}, \underline{U}_{2,ij,m-1}, \underline{U}^{(n)}_{2,ij,m})\Big\} \leq \delta, \tag{5.95a}$$

$$\max\Big\{ \max_{(i,j)\in\Omega^h} \mathcal{K}_{1,ij,m}(\underline{U}^{(n)}_{1,ij,m}, \underline{U}_{1,ij,m-1}, \overline{U}^{(n)}_{2,ij,m}); \\ \max_{(i,j)\in\Omega^h} \mathcal{K}_{2,ij,m}(\underline{U}^{(n)}_{1,ij,m}, \overline{U}_{2,ij,m-1}, \overline{U}^{(n)}_{2,ij,m})\Big\} \leq \delta, \tag{5.95b}$$

where $\mathcal{K}_{\alpha,ij,m}(U^{(n)}_{\alpha,ij,m}, U_{\alpha,ij,m-1}, U^{(n)}_{\alpha',ij,m})$, $(i,j) \in \Omega^h$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1,2$, $m \geq 1$, are residuals of the nonlinear difference scheme (5.41), which are defined in (5.42), and δ is a prescribed accuracy. On each time level $m \geq 1$, we set up

$$(\overline{U}_{1,ij,m}, \underline{U}_{2,ij,m}) = (\overline{U}^{(n_m)}_{1,ij,m}, \underline{U}^{(n_m)}_{1,ij,m}), \quad (\underline{U}_{1,ij,m}, \overline{U}_{2,ij,m}) = (\underline{U}^{(n_m)}_{1,ij,m}, \overline{U}^{(n_m)}_{1,ij,m}), \\ (i,j) \in \overline{\Omega}^h, \quad m \geq 1,$$

such that n_m is the minimal number of iterations subject to (5.95).

Theorem 5.7.4. *Let $\widetilde{U}_{\alpha,ij,m}$ and $\widehat{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, be ordered upper and lower solutions (5.56) to (5.41). Suppose that the functions f_α, $\alpha = 1,2$, satisfy (5.82) and (5.89), and assumption (5.76) on the time step τ holds. Then for sequences $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$ and $\{\underline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, generated by (5.57), (5.95)* *with*

$$(\overline{U}^{(0)}_{1,ij,m},\underline{U}^{(0)}_{2,ij,m})=(\widetilde{U}_{1,ij,m},\widehat{U}_{2,ij,m}),\quad (\underline{U}^{(0)}_{1,ij,m},\overline{U}^{(0)}_{2,ij,m})=(\widehat{U}_{1,ij,m},\widetilde{U}_{2,ij,m}),$$
$$(i,j)\in\overline{\Omega}^h,\quad m\geq 1,$$

the following estimates hold

$$\max_{m\geq 1}\left\{\max\left[\|\overline{U}_{1,m}-U^*_{1,m}\|_{\overline{\Omega}^h};\|\underline{U}_{2,m}-U^*_{2,m}\|_{\overline{\Omega}^h}\right]\right\}\leq T\delta, \tag{5.96}$$

$$\max_{m\geq 1}\left\{\max\left[\|\underline{U}_{1,m}-U^*_{1,m}\|_{\overline{\Omega}^h};\|\overline{U}_{2,m}-U^*_{2,m}\|_{\overline{\Omega}^h}\right]\right\}\leq T\delta,$$

where $U_{\alpha,ij,m}=U^{(n_m)}_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\geq 1$, *and* $U^*_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\geq 1$, *are the unique solutions to the nonlinear difference scheme (5.41).*

Proof. We consider the case of the sequence $\{\overline{U}^{(n)}_{1,ij,m},\underline{U}^{(n)}_{2,ij,m}\}$, $(i,j)\in\overline{\Omega}^h$, $m\geq 1$, that is, the point monotone iterative methods (5.57), (5.95) generate the numerical solutions $(\overline{U}_{1,ij,m},\underline{U}_{2,ij,m})$, $(i,j)\in\overline{\Omega}^h$, $m\geq 1$. Letting $W_{1,ij,m}=\overline{U}_{1,ij,m}-U^*_{1,ij,m}$, $W_{2,ij,m}=\underline{U}_{2,ij,m}-U^*_{2,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $m\geq 1$, from (5.11), by using the mean-value theorem (5.16), we obtain

$$\begin{aligned}
&\left(\mathcal{A}_{1,ij,m}+\tau^{-1}+(f_{1,ij,m}(Q_{1,ij,m},\underline{U}_{2,ij,m}))_{u_1}\right)W_{1,ij,m}=\\
&-(f_{1,ij,m}(U^*_{1,ij,m},Y_{2,ij,m}))_{u_2}W_{2,ij,m}+\mathcal{K}_{1,ij,,m}(\overline{U}_{1,ij,m},\overline{U}_{1,ij,m-1},\underline{U}_{2,ij,m})\\
&+\tau^{-1}W_{1,ij,m-1},\\
&\left(\mathcal{A}_{2,ij,m}+\tau^{-1}+(f_{2,ij,m}(\overline{U}_{1,ij,m},Q_{2,ij,m}))_{u_2}\right)W_{2,ij,m}=\\
&-(f_{2,ij,m}(Y_{1,ij,m},U^*_{2,ij,m}))_{u_1}W_{1,ij,m}+\mathcal{K}_{2,ij,,m}(\overline{U}_{1,ij,m},\underline{U}_{2,ij,m-1},\underline{U}_{2,ij,m})\\
&+\tau^{-1}W_{2,ij,m-1},\quad (i,j)\in\Omega^h,\quad m\geq 1,\\
&U^*_{1,ij,m}\leq Q_{1,ij,m},Y_{1,ij,m}\leq\overline{U}_{1,ij,m},\quad \underline{U}_{2,ij,m}\leq Q_{2,ij,m},Y_{2,ij,m}\leq U^*_{2,ij,m},
\end{aligned}$$

where the partial derivatives satisfy (5.82) and (5.89). From here, (5.82), (5.89) and using (5.72), we obtain

$$w_m\leq\frac{1}{\tau^{-1}+q_m}(q_mw_m+\delta+\tau^{-1}w_{m-1}),\quad w_m=\max_{\alpha=1,2}\|W_{\alpha,m}\|_{\overline{\Omega}^h},\quad m\geq 1.$$

Solving this inequality for w_m, we have

$$w_m\leq\tau\delta+w_{m-1},\quad m\geq 1.$$

From here, taking into account that $w_0 = 0$, by induction on $m \geq 1$, we obtain that

$$w_m \leq \delta \sum_{l=1}^{m} \tau.$$

Since $\sum_{l=1}^{m} \tau \leq T$, we prove the theorem for the sequence $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$. The case of the sequence $\{\underline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, is proved in a similar manner. □

Theorem 5.7.5. *Let the assumptions in Theorem 5.7.4 be satisfied. Then for sequences* $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$ *and* $\{\underline{U}^{(n)}_{1,ij,m}, \overline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, *generated by (5.57), (5.95) with*

$$(\overline{U}^{(0)}_{1,ij,m}, \underline{U}^{(0)}_{2,ij,m}) = (\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m}), \quad (\underline{U}^{(0)}_{1,ij,m}, \overline{U}^{(0)}_{2,ij,m}) = (\widehat{U}_{1,ij,m}, \widetilde{U}_{2,ij,m}),$$
$$(i,j) \in \overline{\Omega}^h, \quad m \geq 1,$$

the following estimates hold

$$\max_{m \geq 1} \max \left[\|\overline{U}_{1,m} - u^*_{1,m}\|_{\overline{\Omega}^h},\ \|\underline{U}_{2,m} - u^*_{2,m}\|_{\overline{\Omega}^h} \right] \leq T(\delta + \max_{m \geq 1} E_m),$$
$$\max_{m \geq 1} \max \left[\|\underline{U}_{1,m} - u^*_{1,m}\|_{\overline{\Omega}^h},\ \|\overline{U}_{2,m} - u^*_{2,m}\|_{\overline{\Omega}^h} \right] \leq T(\delta + \max_{m \geq 1} E_m),$$
$$E_m = \max_{\alpha=1,2} \|E_{\alpha,m}\|_{\overline{\Omega}^h}, \quad m \geq 1,$$

where $u^*_\alpha(x,y,t)$, $\alpha = 1,2$, *are the exact solutions to (5.1), and* $E_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, *are the truncation errors of the exact solutions on the nonlinear difference scheme (5.41).*

Proof. The proof of this theorem repeats the proof of Theorem 5.7.3. □

5.8 Construction of initial upper and lower solutions

We discuss constructions of upper and lower solutions which are used as initial iterations in the monotone iterative methods (5.43) and (5.57).

5.8.1 Quasi-monotone nondecreasing case

5.8.1.1 Bounded reaction functions

We assume that functions f_α, g_α and ψ_α, $\alpha = 1, 2$, in (5.1) satisfy the conditions

$$\begin{aligned} & f_\alpha(x, y, t, \mathbf{0}) \le 0, \quad -K_\alpha \le f_\alpha(x, y, t, u), \quad u_\alpha(x, y, t) \ge 0, \quad (x, y, t) \in \overline{Q}_T, \\ & g_\alpha(x, y, t) \ge 0, \quad (x, y, t) \in \partial Q_T, \quad \psi_\alpha(x, y) \ge 0, \quad (x, y) \in \overline{\omega}, \quad \alpha = 1, 2, \end{aligned} \tag{5.97}$$

where $K_\alpha = \text{const} > 0$, $\alpha = 1, 2$, and $\mathbf{0}$ is the zero vector $(0, 0)$. We introduce the mesh functions

$$\widehat{U}_{\alpha,ij,m} = 0, \quad (i, j) \in \overline{\Omega}^h, \quad \alpha = 1, 2, \quad m \ge 1, \tag{5.98}$$

and the mesh functions $\widetilde{U}_{\alpha,ij,m}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \ge 1$, which are solutions of the linear problems

$$\begin{aligned} & \mathcal{A}_{\alpha,ij,m}\widetilde{U}_{\alpha,ij,m} = \tau^{-1}\widetilde{U}_{\alpha,ij,m-1} + K_\alpha, \quad (i, j) \in \Omega^h, \\ & \widetilde{U}_{\alpha,ij,m} = g_{\alpha,ij,m}, \quad (i, j) \in \partial\Omega^h, \quad \widetilde{U}_{\alpha,ij,0} = \psi_{\alpha,ij}, \quad (i, j) \in \overline{\Omega}^h, \quad \alpha = 1, 2, \quad m \ge 1, \end{aligned} \tag{5.99}$$

where the difference operators $\mathcal{A}_{\alpha,ij,m}$, $(i, j) \in \Omega^h$, $\alpha = 1, 2$, $m \ge 1$, are defined in (5.13). We show that under assumptions (5.97), $\widetilde{U}_{\alpha,ij,m}$ and $\widehat{U}_{\alpha,ij,m}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \ge 1$, are ordered upper and lower solutions (5.42) to (5.11). From (5.97)–(5.99), by using Lemma 5.4.1, we conclude that $\widetilde{U}_{\alpha,ij,1} \ge 0$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, and

$$0 = \widehat{U}_{\alpha,ij,1} \le \widetilde{U}_{\alpha,ij,1}, \quad (i, j) \in \overline{\Omega}^h, \quad \alpha = 1, 2.$$

By induction on m, we can prove that

$$0 = \widehat{U}_{\alpha,ij,m} \le \widetilde{U}_{\alpha,ij,m}, \quad (i, j) \in \overline{\Omega}^h, \quad \alpha = 1, 2, \quad m \ge 1.$$

By using (5.99), the residuals of the difference equations (5.11) on $\widetilde{U}_{\alpha,ij,m}$, $(i, j) \in \Omega^h$, $\alpha = 1, 2$, can be presented in the form

$$\begin{aligned} & \mathcal{K}_{\alpha,ij,m}(\widetilde{U}_{\alpha,ij,m}, \widetilde{U}_{\alpha,ij,m-1}, \widetilde{U}_{\alpha',ij,m}) = K_\alpha + f_{\alpha,ij,m}(\widetilde{U}_{\alpha,ij,m}, \widetilde{U}_{\alpha',ij,m}), \\ & (i, j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1, 2, \quad m \ge 1. \end{aligned}$$

Using (5.97), we obtain that

$$\mathcal{K}_{\alpha,ij,m}(\widetilde{U}_{\alpha,ij,m}, \widetilde{U}_{\alpha,ij,m-1}, \widetilde{U}_{\alpha',ij,m}) \ge 0, \quad (i, j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1, 2, \quad m \ge 1.$$

From here and taking into account that

$$\mathcal{K}_{\alpha,ij,m}(\widehat{U}_{\alpha,ij,m}, \widehat{U}_{\alpha,ij,m-1}, \widehat{U}_{\alpha',ij,m}) \le 0, \quad (i,j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2, \quad m \ge 1,$$

where $\widehat{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \ge 1$, from (5.98), we conclude that $\widehat{U}_{\alpha,ij,m}$ from (5.98) and $\widetilde{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \ge 1$, from (5.99) are ordered lower and upper solutions (5.42) to (5.11).

5.8.1.2 Constant upper and lower solutions

We now assume that functions f_α, g_α and ψ_α, $\alpha = 1,2$, in (5.1) satisfy the conditions

$$\begin{aligned} &f_\alpha(x,y,t,\mathbf{0}) \le 0, \quad f_\alpha(x,y,t,K) \ge 0, \quad u_\alpha(x,y,t) \ge 0, \quad (x,y,t) \in \overline{Q}_T, \\ &0 \le g_\alpha(x,y,t) \le K_\alpha, \quad (x,y,t) \in \partial Q_T, \quad 0 \le \psi_\alpha(x,y) \le K_\alpha, \quad (x,y) \in \overline{\omega}, \end{aligned} \tag{5.100}$$

where $K = (K_1, K_2)$ and K_α, $\alpha = 1,2$, are positive constants. On each time level $m \ge 1$, we introduce the constant mesh functions

$$\widetilde{U}_{\alpha,ij,m} = K_\alpha, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1,2, \quad m \ge 1. \tag{5.101}$$

From (5.98) and (5.101), on each time level $m \ge 1$, we have

$$\begin{aligned} &\mathcal{K}_{\alpha,ij,m}(\widehat{U}_{\alpha,ij,m}, \widehat{U}_{\alpha,ij,m-1}, \widehat{U}_{\alpha',ij,m}) = 0, \quad \mathcal{K}_{\alpha,ij,m}(\widetilde{U}_{\alpha,ij,m}, \widetilde{U}_{\alpha,ij,m-1}, \widetilde{U}_{\alpha',ij,m}) \ge 0, \\ &(i,j) \in \Omega^h, \quad \alpha' \ne \alpha, \quad \alpha, \alpha' = 1,2, \quad m \ge 1. \end{aligned}$$

Thus, under assumptions (5.100), $\widehat{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \ge 1$, from (5.98) and $\widetilde{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \ge 1$, from (5.101) are ordered lower and upper solutions (5.42) to (5.11).

5.8.2 Quasi-monotone nonincreasing case

5.8.2.1 Bounded reaction functions

We assume that functions f_α, g_α and ψ_α, $\alpha = 1,2$, in (5.1) satisfy the conditions

$$\begin{aligned} &f_\alpha(x,y,t,0_\alpha,u_{\alpha'}) \le 0, \quad -K_\alpha \le f_\alpha(x,y,t,u_\alpha,0_{\alpha'}), \quad u_\alpha(x,y,t) \ge 0, \\ &(x,y,t) \in \overline{Q}_T, \quad g_\alpha(x,y,t) \ge 0, \ (x,y,t) \in \partial Q_T, \quad \psi_\alpha(x,y) \ge 0, \ (x,y) \in \overline{\omega}, \end{aligned} \tag{5.102}$$

where $K_\alpha = \text{const} > 0$, $\alpha = 1,2$, the notation 0_α, $\alpha = 1,2$, means that $u_\alpha = 0$, $\alpha = 1,2$, and notation (5.10) is in use.

We show that under assumptions (5.102), $\widehat{U}_{\alpha,ij,m}$ and $\widetilde{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \ge 1$, from, respectively, (5.98) and (5.99) are ordered lower and upper solutions

(5.56) to (5.11). From (5.98), (5.99) and (5.102), by using Lemma 5.4.1, we conclude that $\widetilde{U}_{\alpha,ij,1} \geq 0$, $\alpha = 1, 2$, and

$$0 = \widehat{U}_{\alpha,ij,1} \leq \widetilde{U}_{\alpha,ij,1}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1, 2.$$

By induction on $m \geq 1$, we can prove that

$$0 = \widehat{U}_{\alpha,ij,m} \leq \widetilde{U}_{\alpha,ij,m}, \quad (i,j) \in \overline{\Omega}^h, \quad \alpha = 1, 2, \quad m \geq 1. \tag{5.103}$$

Consider the case of the sequence $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, where

$$(\overline{U}^{(0)}_{1,ij,m}, \underline{U}^{(0)}_{2,ij,m}) = (\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m}), \quad (i,j) \in \overline{\Omega}^h, \quad m \geq 1.$$

By using (5.99), the residual of the first difference equation in (5.11) on $(\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, can be presented in the form

$$\mathcal{K}_{1,ij,m}(\widetilde{U}_{1,ij,m}, \widetilde{U}_{1,ij,m-1}, \widehat{U}_{2,ij,m}) = K_1 + f_{1,ij,m}(\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m}), \quad (i,j) \in \Omega^h, \quad m \geq 1.$$

From here, (5.98) and (5.102), we conclude that

$$\mathcal{K}_{1,ij,m}(\widetilde{U}_{1,ij,m}, \widetilde{U}_{1,ij,m-1}, \widehat{U}_{2,ij,m}) \geq 0, \quad (i,j) \in \Omega^h, \quad m \geq 1.$$

From (5.98), (5.102) and (5.103), for the residual of the second difference equation in (5.11) on $(\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, it follows the inequalities

$$\mathcal{K}_{2,ij,m}(\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m-1}, \widehat{U}_{2,ij,m}) \leq 0, \quad (i,j) \in \Omega^h, \quad m \geq 1.$$

Similarly, for the case $(\widehat{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, we have

$$\mathcal{K}_{1,ij,m}(\widehat{U}_{1,ij,m}, \widehat{U}_{1,ij,m-1}, \widetilde{U}_{2,ij,m}) \leq 0, \quad \mathcal{K}_{2,ij,m}(\widehat{U}_{1,ij,m}, \widetilde{U}_{2,ij,m-1}, \widetilde{U}_{2,ij,m}) \geq 0,$$
$$(i,j) \in \Omega^h, \quad m \geq 1.$$

Thus, $\widehat{U}_{\alpha,ij,m}$ and $\widehat{U}_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \geq 1$, from, respectively, (5.98) and (5.99) are ordered lower and upper solutions (5.56) to (5.11).

5.8.2.2 Constant upper and lower solutions

We now assume that functions f_α, g_α and ψ_α, $\alpha = 1, 2$, in (5.1) satisfy the conditions

$$f_\alpha(x, y, t, 0_\alpha, u_{\alpha'}) \le 0, \quad f_\alpha(x, y, t, K_\alpha, 0_{\alpha'}) \ge 0, \quad u_\alpha(x, y, t) \ge 0, \tag{5.104}$$
$$(x, y, t) \in \overline{Q}_T, \quad 0 \le g_\alpha(x, y, t) \le K_\alpha, \quad (x, y, t) \in \partial Q_T,$$
$$0 \le \psi_\alpha(x, y) \le K_\alpha, \quad (x, y) \in \overline{\omega},$$

where $K_\alpha = \text{const} > 0$, $\alpha = 1, 2$, and notation (5.10) is in use.

We show that under assumptions (5.104), $\widehat{U}_{\alpha,ij,m}$ and $\widetilde{U}_{\alpha,ij,m}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \ge 1$, from, respectively, (5.98) and (5.101) are ordered lower and upper solutions (5.56) to (5.11). From (5.98), (5.101) and (5.104), for the case of $(\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \ge 1$, we have

$$\mathcal{K}_{1,ij,m}(\widetilde{U}_{1,ij,m}, \widetilde{U}_{1,ij,m-1}, \widehat{U}_{2,ij,m}) \ge 0, \quad \mathcal{K}_{2,ij,m}(\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m-1}, \widehat{U}_{2,ij,m}) \le 0,$$
$$(i, j) \in \Omega^h, \quad m \ge 1.$$

Similarly, for the case $(\widehat{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \ge 1$, we have

$$\mathcal{K}_{1,ij,m}(\widehat{U}_{1,ij,m}, \widehat{U}_{1,ij,m-1}, \widetilde{U}_{2,ij,m}) \le 0, \quad \mathcal{K}_{2,ij,m}(\widehat{U}_{1,ij,m}, \widetilde{U}_{2,ij,m-1}, \widetilde{U}_{2,ij,m}) \ge 0,$$
$$(i, j) \in \Omega^h, \quad m \ge 1.$$

Thus, under assumptions (5.104), $\widehat{U}_{\alpha,ij,m}$ and $\widehat{U}_{\alpha,ij,m}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \ge 1$, from, respectively, (5.98) and (5.101) are ordered lower and upper solutions (5.56) to (5.11).

5.9 Numerical experiments

We present numerical experiments, implemented by the point monotone Jacobi and Gauss-Seidel methods, for test problems with quasi-monotone nondecreasing (5.20) and quasi-monotone nonincreasing (5.30) reaction functions f_α, $\alpha = 1, 2$, in (5.1). Exact solutions of our test problems are unknown, and numerical solutions are compared to corresponding reference solutions. In our tests, we choose the reference solutions with $N = 256$ and $\delta = 10^{-5}$ in the stopping tests (5.90) and (5.95). The reference solutions are calculated by the corresponding block method.

5.9.1 Quasi-monotone nondecreasing case

Test 1

As the first test problem with quasi-monotone nondecreasing reaction functions

(5.20), we consider the Volterra-Lotka cooperating model from Section 5.3.1.1, where $L_\alpha u_\alpha = D_\alpha(u_{\alpha,xx} + u_{\alpha,yy})$, $\alpha = 1, 2$, in (5.1). The reaction functions are given by

$$f_1(u_1, u_2) = -u_1(1 - u_1 + a_1 u_2), \quad f_2(u_1, u_2) = -u_2(1 + a_2 u_1 - u_2), \tag{5.105}$$

where $u_\alpha \geq 0$, $\alpha = 1, 2$, are the populations of two species with a symbiotic relationship and a_α, $\alpha = 1, 2$, are positive constants which describe the interaction of the two species.

As ordered upper and lower solutions, we choose the pairs $(\widetilde{U}_1, \widetilde{U}_2) = (M_1, M_2)$ and $(\widehat{U}_1, \widehat{U}_2) = (0, 0)$. Then all the assumptions in (5.100) with $M_1 = 3$ and $M_2 = 2$, are satisfied. From here, in the sector $\langle \mathbf{0}, M \rangle$, $M = (M_1, M_2)$, we conclude the inequalities

$$\frac{\partial f_1}{\partial u_1} = 2u_1 - a_1 u_2 - 1 \leq 2M_1 = 6, \quad -\frac{\partial f_1}{\partial u_2} = a_1 u_1 \geq 0,$$
$$\frac{\partial f_2}{\partial u_2} = 2u_2 - a_2 u_1 - 1 \leq 2M_2 = 4, \quad -\frac{\partial f_2}{\partial u_1} = a_2 u_2 \geq 0.$$

Thus, f_α, $\alpha = 1, 2$, satisfy (5.19) and (5.20) with $c_1 = 6$ and $c_2 = 4$. We choose the initial iteration $(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = (3, 2)$, $(i, j) \in \overline{\Omega}^h$ and calculate sequences of upper solutions generated by (5.43), (5.90). We take $D_1 = 0.7$, $D_2 = 1$, $a_1 = 0.5$, $a_2 = 1$, $g_\alpha(x, y, t) = 0$, $(x, y, t) \in \partial Q_T$, $\alpha = 1, 2$, and $\psi_\alpha(x, y) = 1$, $(x, y) \in \bar{\omega}$, $\alpha = 1, 2$, in (5.1).

In Table 5.1, for different values of N, $T = 2$ and $\tau = 0.01$, we present average numbers of iterations n_δ per a time step and corresponding CPU times for the point monotone methods (5.43). From these results, we conclude that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi methods, which confirms Theorem 5.6.1; the point monotone Gauss–Seidel method is approximately twice as fast as the point monotone Jacobi method.

Table 5.1: Average numbers of iterations n_δ and CPU times for Test 1.

N	8	16	32	64	128
the point Jacobi method					
n_δ	11.98	35.88	135.27	533.09	2958.82
CPU(s)	0.13	0.91	13.42	212.16	1287.19
the point Gauss-Seidel method					
n_δ	6.99	19.50	69.27	268.10	1680.77
CPU(s)	0.12	0.56	7.34	115.24	733.43

Test 2

As the second test problem with quasi-monotone nondecreasing reaction functions (5.20), we consider the time dependent case of Test 2 from Section 3.8.1 with the same data sets and initial functions $\psi_\alpha(x, y) = \sin(\pi x) \sin(\pi y)$, $(x, y) \in \overline{\omega}$, $\alpha = 1, 2$.

We choose the initial iteration $(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = (1,1)$, $(i,j) \in \overline{\Omega}^h$ and calculate sequences of upper solutions generated by the point monotone iterative methods (5.43), (5.90).

In Table 5.2, for different values of N, $\tau = 0.5$ and $\tau = 0.01$, we give average numbers of iterations n_δ and execution (CPU) times for the point iterative methods (5.43). From these results, we conclude that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method, which confirms Theorem 5.6.1; the point monotone Gauss–Seidel method is approximately twice as fast as the point monotone Jacobi method.

Table 5.2: Average numbers of iterations n_δ and CPU times for Test 2.

N	8	16	32	64	128
the point Jacobi method					
n_δ	7.62	17.36	52.80	193.92	752
CPU(s)	0.07	0.13	1.44	20.31	325.65
the point Gauss–Seidel method					
n_δ	5.86	11.24	29.46	99.78	379.78
CPU(s)	0.06	0.09	0.82	11.26	173.81

5.9.2 Quasi-monotone nonincreasing case

Test 3

As the first test problem with quasi-monotone nonincreasing reaction functions (5.30), we consider the Belousov-Zhabotinskii reaction diffusion model which is presented in Section 5.3.2.1, where $\mathrm{L}_\alpha u_\alpha = D_\alpha(u_{\alpha,xx} + u_{\alpha,yy})$, $\alpha = 1,2$, in (5.1) and the reaction functions are given by

$$f_1 = -u_1(a - bu_1 - \sigma_1 u_2), \quad f_2 = \sigma_2 u_1 u_2. \tag{5.106}$$

where σ_α, $\alpha = 1,2$, a and b are positive constants. We choose the following boundary and initial conditions $g_\alpha(x,y) = 1$, $(x,y) \in \partial\omega^h$, $\psi_\alpha(x,y) = 0$, $(x,y) \in \overline{\omega}$, $\alpha = 1,2$, in (5.1).

The pairs $(\widetilde{U}_{1,ij}, \widetilde{U}_{2,ij}) = (K_1, K_2)$ and $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m}) = (0,0)$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, are ordered upper and lower solutions. Indeed, all the assumptions in (5.104) are satisfied. From here, on $\langle \widehat{U}, \widetilde{U} \rangle$, we conclude the inequalities

$$\begin{aligned}
&\frac{\partial f_{1,ij,m}}{\partial u_1}(U_{1,ij,m}, U_{2,ij,m}) = 2bU_{1,ij,m} + \sigma_1 U_{2,ij,m} - a \leq 2bK_1 + \sigma_1 K_2, \quad (i,j) \in \overline{\Omega}^h, \\
&\frac{\partial f_{2,ij,m}}{\partial u_2}(U_{1,ij,m}, U_{2,ij,m}) = \sigma_2 U_{1,ij,m} \leq \sigma_2 K_1, \quad (i,j) \in \overline{\Omega}^h, \\
&-\frac{\partial f_{1,ij,m}}{\partial u_2}(U_{1,ij,m}, U_{2,ij,m}) = -\sigma_1 U_{1,ij,m} \leq 0, \quad (i,j) \in \overline{\Omega}^h, \\
&-\frac{\partial f_{2,ij,m}}{\partial u_1}(U_{1,ij,m}, U_{2,ij,m}) = -\sigma_2 U_{2,ij,m} \leq 0, \quad (i,j) \in \overline{\Omega}^h.
\end{aligned}$$

Thus, f_α, $\alpha = 1, 2$, satisfy (5.19) and (5.30) with $c_{1,ij,m} = 2bK_1 + \sigma_1 K_2$ and $c_{2,ij,m} = \sigma_2 K_1$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$. We choose the initial iteration $(\widetilde{U}_{1,ij,m}, \widehat{U}_{2,ij,m}) = (K_1, 0)$, $(i,j) \in \overline{\Omega}^h$ and calculate the sequence $\{\overline{U}^{(n)}_{1,ij,m}, \underline{U}^{(n)}_{2,ij,m}\}$, $(i,j) \in \overline{\Omega}^h$, $m \geq 1$, generated by (5.57), (5.95). We take $D_\alpha = 1$, $\alpha = 1, 2$, in (5.1), $a = 1$, $b = 1$ and $\sigma_\alpha = 1$, $\alpha = 1, 2$, in (5.106).

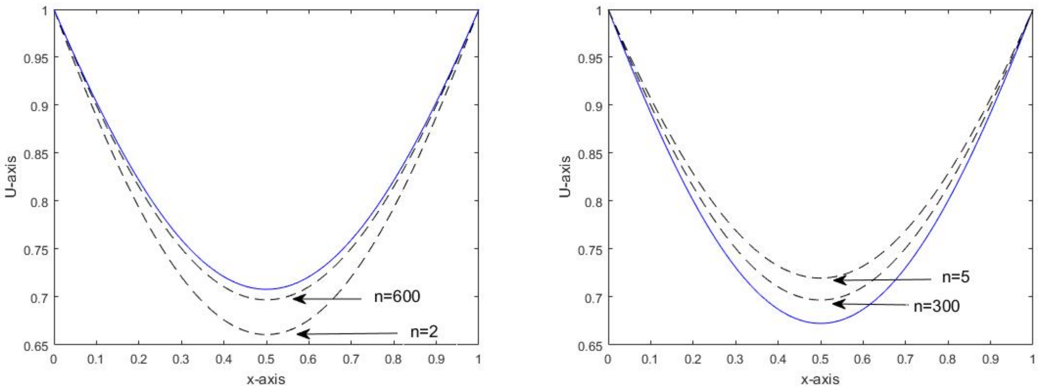
In Table 5.3, for different values of N, $T = 1$ and $\tau = 0.01$, we give average numbers of iterations n_δ and execution (CPU) times for the point monotone iterative methods (5.57). From these results, we conclude that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method, which confirms Theorem 5.6.2; the point monotone Gauss–Seidel method is approximately twice as fast as the point monotone Jacobi method.

In Figure 5.1, we show the convergence of numerical solutions, obtained by the point Gauss-Seidel method with $\eta = 1$ in (5.57) and $N = 64$ to the reference solution $N_{ref} = 256$, where the dashed line represents the numerical solution and the solid blue line refers to the reference solution with respect to x and fixed value of $y = 0.5$. In subgraph 5.1a, staring from the initial lower solution $\widehat{U}_{2,10} = 0$, on the time level $t_{10} = 0.1$, we show the convergence of the numerical lower solutions $\underline{U}^{(n)}_{2,10}$ at $n = 2$ and $n = 600$ to the reference solution. Similarly, starting from the initial upper solution $\widetilde{U}_{1,10} = 1$, on the time level $t_{10} = 0.1$, subgraph 5.1b shows the convergence of the numerical upper solutions $\overline{U}^{(n)}_{1,10}$ at $n = 5$ and $n = 300$ to the reference solution.

Table 5.3: Average numbers of iterations n_δ and CPU times for Test 3.

N	8	16	32	64	128
the point Jacobi method					
n_δ	15.34	50.83	196.43	779.99	3115.91
CPU(s)	0.15	0.66	9.64	155.46	1612.87
the point Gauss–Seidel method					
n_δ	9.21	27.16	100.04	391.93	1624.43
CPU(s)	0.08	0.37	5.19	80.32	741.89

Figure 5.1: Convergence of lower and upper solutions to the reference solution for Test 3.



(a) Lower solutions. (b) Upper solutions.

Test 4

As the second test problem with quasi-monotone nonincreasing reaction functions (5.30), we consider the time dependent case of Test 3 from Section 3.8.2 with the same data sets and initial functions $\psi_\alpha(x,y) = \sin(\pi x)\sin(\pi y)$, $(x,y) \in \overline{\omega}^h$, $\alpha = 1,2$.

We choose the initial iteration $(\widetilde{U}_{1,ij}, \widehat{U}_{2,ij}) = (1,0)$, $(i,j) \in \overline{\Omega}^h$ and calculate sequences of upper solutions generated by the point monotone iterative method (5.57), (5.95).

In Table 5.4, for different values of N, $T = 0.5$ and $\tau = 0.01$, we give average numbers of iterations n_δ and execution (CPU) times for the point iterative method (5.57). From these results, we conclude that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method, which confirms Theorem 5.6.2; the point monotone Gauss–Seidel method is approximately twice as fast as the point monotone Jacobi method.

Table 5.4: Average numbers of iterations n_δ and CPU times for Test 4.

N	8	16	32	64	128
the point Jacobi method					
n_δ	21.14	74.58	287.66	1139.54	4547.02
CPU(s)	0.09	0.49	7.14	112.98	1889.27
the point Gauss–Seidel method					
n_δ	12.70	39.66	146.32	572.46	2276.22
CPU(s)	0.07	0.27	3.77	57.85	942.17

5.10 Conclusions to Chapter 5

Theoretical results

For solving nonlinear parabolic systems with quasi-monotone nondecreasing and nonincreasing reaction functions, we construct and investigate monotone properties of point Jacobi and point Gauss-Seidel iterative methods. The coupled system of nonlinear parabolic problems (5.1) is approximated by the nonlinear implicit difference scheme, where for the spatial derivatives, the central difference approximations are in use. For solving the nonlinear difference scheme (5.11) with quasi-monotone nondecreasing (5.20) and quasi-monotone nonincreasing (5.30) reaction functions, the point Jacobi and point Gauss-Seidel iterative methods are constructed. In Theorems 5.4.3 and 5.4.5, on each time level, we prove that the sequences of upper and lower solutions, generated by the point monotone iterative methods for problems with quasi-monotone nondecreasing (5.20) and quasi-monotone nonincreasing (5.30) reaction functions, converge monotonically. In Theorems 5.5.2 and 5.5.3, respectively, for quasi-monotone nondecreasing and nonincreasing cases, we prove the existence and uniqueness of a solution of the nonlinear difference scheme (5.11). Taking into account the fact that on each time level, in general, the nonlinear discrete problems can be solved only inexactly, we introduce the stopping tests on each time level. By using the stopping test (5.90) and (5.95), based on the norms of residuals, respectively, for the quasi-monotone nondecreasing and nonincreasing cases, we prove that the numerical solution converges to the unique solution of the nonlinear difference scheme and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme (5.11) in Theorems 5.7.2 and 5.7.3, and the error between the numerical solution and the exact solution of the parabolic problem (5.1) in Theorems 5.7.4 and 5.7.5. We prove that the point monotone Gauss-Seidel methods converge faster than the point monotone Jacobi methods for the quasi-monotone nondecreasing and nonincreasing, respectively, in Theorems 5.6.1 and 5.6.2. For quasi-monotone nondecreasing and nonincreasing cases, on each time level, we construct initial upper and lower solutions to start the point monotone iterative methods.

Numerical results

The numerical sequences of upper and lower solutions, generated by the point monotone iterative methods (5.43) and (5.57) with stopping tests (5.90) and (5.95), respectively, for the quasi-monotone nondecreasing and nonincreasing cases, converge monotonically. The point monotone Gauss-Seidel methods with $\eta = 1$ in (5.43) and $\eta = 1$ in (5.57) converge faster than the point monotone Jacobi methods with $\eta = 0$ in (5.43) and $\eta = 0$ in (5.57) which confirm, respectively, Theorems 5.6.1 and 5.6.2. The point monotone Gauss-Seidel methods are approximately twice as fast as the point monotone Jacobi methods.

Chapter 6

Block Jacobi and Gauss-Seidel methods for systems of parabolic problems

This chapter deals with numerical methods for solving nonlinear parabolic systems by block iterative methods based on the Jacobi and Gauss Seidel methods. The idea of these methods is the decomposition technique which on each time level reduces a domain into a series of nonoverlapping one dimensional intervals by slicing the domain into a finite number of thin strips, and then solving a two-point boundary-value problem for each strip by a standard computational scheme such as the Thomas algorithm [48]. In the view of the method of upper and lower solutions, on each time level, two monotone upper and lower sequences of solutions are constructed. Convergence rates for the block monotone iterative methods are estimated in similar way as in Section 5.7. Constructions of initial upper and lower solutions are similar to Section 5.8. We show that the sequences of solutions generated by the block monotone Gauss-Seidel method converges faster than by the block monotone Jacobi method.

6.1 The block monotone Jacobi and Gauss-Seidel methods

On each time level $m \geq 1$, we decompose the mesh $\overline{\Lambda}^h = \overline{\Lambda}^{hx} \times \overline{\Lambda}^{hy}$, from (5.9), into vertical strips similar to (4.1).

For the nonlinear difference scheme (5.11), on each time level $m \geq 1$, we define vectors and diagonal matrices by

$$
\begin{aligned}
&U_{\alpha,i,m} = (U_{\alpha,i,1,m}, \ldots, U_{\alpha,i,N_y-1,m})^T, \quad i \in \overline{\mathcal{I}} = \{0, 1, \ldots, N_x\}, \qquad (6.1)\\
&F_{\alpha,i,m}(U_{1,i,m}, U_{2,i,m}) =\\
&(f_{\alpha,i,1,m}(U_{1,i,1,m}, U_{2,i,1,m}), \ldots, f_{\alpha,i,N_y-1,m}(U_{1,i,N_y-1,m}, U_{2,i,N_y-1,m}))^T,\\
&L_{\alpha,i,m} = \operatorname{diag}(l_{\alpha,i,1,m}, \ldots, l_{\alpha,i,N_y-1,m}), \quad R_{\alpha,i,m} = \operatorname{diag}(r_{\alpha,i,1,m}, \ldots, r_{\alpha,i,N_y-1,m}),\\
&i \in \mathcal{I} = \{1, 2, \ldots, N_x - 1\}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2,\\
&\psi_{\alpha,i} = (\psi_{\alpha,i,0}, \ldots, \psi_{\alpha,i,N_y})^T, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2.
\end{aligned}
$$

where the following notation is in use

$$
F_{\alpha,i,m}(U_{\alpha,i,m}, U_{\alpha',i,m}) = \begin{cases} F_{1,i,m}(U_{1,i,m}, U_{2,i,m}), & \alpha = 1, \\ F_{2,i,m}(U_{1,i,m}, U_{2,i,m}), & \alpha = 2, \end{cases} \quad i \in \overline{\mathcal{I}}, \quad m \geq 1, \qquad (6.2)
$$

with symmetry $F_{\alpha,i,m}(U_{\alpha,i,m}, U_{\alpha',i,m}) = F_{\alpha,i,m}(U_{\alpha',m}, U_{\alpha,i,m})$. The terms $L_{\alpha,1,m}U_{\alpha,0,m}$ and $R_{\alpha,N_x-1}U_{\alpha,N_x,m}$ are included in the boundaries. Thus, the difference scheme (5.11), (5.13) can be presented in the form

$$
\begin{aligned}
&A^{\tau}_{\alpha,i,m}U_{\alpha,i,m} - L_{\alpha,i,m}U_{\alpha,i-1,m} - R_{\alpha,i,m}U_{\alpha,i+1,m} = \qquad (6.3)\\
&- F_{\alpha,i,m}(U_{\alpha,i,m}, U_{\alpha',i,m}) + \tau^{-1}U_{\alpha,i,m-1}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2,\\
&A^{\tau}_{\alpha,i,m}U_{\alpha,i,m} = (A_{\alpha,i,m} + \tau^{-1}I)U_{\alpha,i,m},\\
&U_{\alpha,i,m} = g_{\alpha,i,m}, \quad i \in \partial\mathcal{I}, \quad m \geq 1, \quad U_{\alpha,i,0} = \psi_{\alpha,i}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2,
\end{aligned}
$$

where I is the identity matrix, and the tridiagonal matrices $A_{\alpha,i,m}$, $i \in \mathcal{I}$, $\alpha = 1, 2$, $m \geq 1$, are defined by

$$
A_{\alpha,i,m} = \begin{bmatrix}
d_{\alpha,i,1,m} & -q_{\alpha,i,1,m} & & & 0 \\
-b_{\alpha,i,2,m} & d_{\alpha,i,2,m} & -q_{\alpha,i,2,m} & & \\
& \ddots & \ddots & \ddots & \\
& & -b_{\alpha,i,N_y-2,m} & d_{\alpha,i,N_y-2,m} & -q_{\alpha,i,N_y-2,m} \\
0 & & & -b_{\alpha,i,N_y-1,m} & d_{\alpha,i,N_y-1,m}
\end{bmatrix}.
$$

The elements of the matrices $L_{\alpha,i,m}$ and $R_{\alpha,i,m}$, $i \in \mathcal{I}$, $\alpha = 1, 2$, $m \geq 1$, contain the coupling coefficients of a mesh point (i, j, m) to, respectively, mesh points $(i - 1, j, m)$ and $(i + 1, j, m)$, $j = 1, 2, \ldots, N_y - 1$.

6.1.1 Quasi-monotone nondecreasing case

In the case of the quasi-monotone nondecreasing functions f_α, $\alpha = 1, 2$, (5.20), we say that mesh functions

$$(\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m}), \quad (\widehat{U}_{1,i,m}, \widehat{U}_{2,i,m}), \quad i \in \overline{\mathcal{I}}, \quad m \geq 1,$$

are ordered upper and lower solutions of (6.3), if they satisfy the inequalities

$$\widehat{U}_{\alpha,i,m} \leq \widetilde{U}_{\alpha,i,m}, \quad i \in \overline{\mathcal{I}}, \tag{6.4a}$$

$$\mathcal{K}_{\alpha,i,m}(\widehat{U}_{\alpha,i,m}, \widehat{U}_{\alpha,i,m-1}, \widehat{U}_{\alpha',i,m}) \leq \mathbf{0} \leq \mathcal{K}_{\alpha,i,m}(\widetilde{U}_{\alpha,i,m}, \widetilde{U}_{\alpha,i,m-1}, \widetilde{U}_{\alpha',i,m}), \quad i \in \mathcal{I}, \tag{6.4b}$$

$$\begin{aligned} \mathcal{K}_{\alpha,i,m}(U_{\alpha,i,m}, U_{\alpha,i,m-1}, U_{\alpha',i,m}) \equiv{}& A^\tau_{\alpha,i,m} U_{\alpha,i,m} - L_{\alpha,i,m} U_{\alpha,i-1,m} - R_{\alpha,i,m} U_{\alpha,i+1,m} \\ &+ F_{\alpha,i,m}(U_{\alpha,i,m}, U_{\alpha',i,m}) - \tau^{-1} U_{\alpha,i,m-1}, \end{aligned}$$

$$\alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2,$$

$$\widehat{U}_{\alpha,i,m} \leq g_{\alpha,i,m} \leq \widetilde{U}_{\alpha,i,m}, \quad i \in \partial\mathcal{I}, \quad m \geq 1, \; \widehat{U}_{\alpha,i,0} \leq \psi_{\alpha,i} \leq \widetilde{U}_{\alpha,i,0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2, \tag{6.4c}$$

where notation (6.2) is in use. On each time level $m \geq 1$, for a given pair of ordered upper and lower solutions $(\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m})$, $(\widehat{U}_{1,i,m}, \widehat{U}_{2,i,m})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, we define the sectors

$$\langle \widehat{U}_m, \widetilde{U}_m \rangle = \left\{ U_{\alpha,i,m} : \quad \widehat{U}_{\alpha,i,m} \leq U_{\alpha,i,m} \leq \widetilde{U}_{\alpha,i,m}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2, \quad m \geq 1 \right\}. \tag{6.5}$$

Remark 6.1.1. *Similar to Remark 5.3.3 from Chapter 5, we state the mean-value theorem for vector-valued mesh functions. Assume that $f_\alpha(x, y, t, u_\alpha, u_{\alpha'})$, $(x, y, t) \in \overline{Q}_T$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1, 2$, are smooth functions, then we have*

$$\begin{aligned} & F_{\alpha,i,m}(U_{\alpha,i,m}, U_{\alpha',i,m}) - F_{\alpha,i,m}(V_{\alpha,i,m}, U_{\alpha',i,m}) = \\ & \big(F_{\alpha,i,m}(Q_{\alpha,i,m}, U_{\alpha',i,m})\big)_{u_\alpha} [U_{\alpha,i,m} - V_{\alpha,i,m}], \\ & F_{\alpha,i,m}(U_{\alpha,i,m}, U_{\alpha',i,m}) - F_{\alpha,i,m}(U_{\alpha,i,m}, V_{\alpha',i,m}) = \\ & \big(F_{\alpha,i,m}(U_{\alpha,i,m}, Y_{\alpha',i,m})\big)_{u_{\alpha'}} [U_{\alpha',i,m} - V_{\alpha',i,m}], \end{aligned} \tag{6.6}$$

where $Q_{\alpha,i,m}$ and $Y_{\alpha,i,m}$ lie between $U_{\alpha,i,m}$ and $V_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, and notation (6.2) is in use.

The notation $(F_{\alpha,i,m})_{u_\alpha}$ and $(F_{\alpha,i,m})_{u_{\alpha'}}$ stands for the diagonal matrices

$$\begin{aligned}
&\left(F_{\alpha,i,m}(Q_{\alpha,i,m},U_{\alpha',i,m})\right)_{u_\alpha} = \qquad (6.7)\\
&\operatorname{diag}\left(\left(f_{\alpha,i,1,m}(Q_{\alpha,i,1,m},U_{\alpha',i,m})\right)_{u_\alpha},\ldots,\left(f_{\alpha,i,N_y-1,m}(Q_{\alpha,i,N_y-1,m},U_{\alpha',i,N_y-1,m})\right)_{u_\alpha}\right),\\
&\left(F_{\alpha,i,m}(U_{\alpha,i,m},Y_{\alpha',i,m})\right)_{u_{\alpha'}} =\\
&\operatorname{diag}\left(\left(f_{\alpha,i,1,m}(U_{\alpha,i,1,m},Y_{\alpha',i,m})\right)_{u_{\alpha'}},\ldots,\left(f_{\alpha,i,N_y-1,m}(U_{\alpha,i,N_y-1,m},Y_{\alpha',i,N_y-1,m})\right)_{u_{\alpha'}}\right).
\end{aligned}$$

We rewrite (5.21) in the vector form

$$\begin{aligned}
&\Gamma_{\alpha,i,m}(U_{\alpha,i,m},U_{\alpha',i,m}) = C_{\alpha,i,m}U_{\alpha,i,m} - F_{\alpha,i,m}(U_{\alpha,i,m},U_{\alpha',i,m}), \qquad (6.8)\\
&C_{\alpha,i,m} = \operatorname{diag}(c_{\alpha,i,1,m},\ldots,c_{\alpha,i,N_y-1,m}), \quad i\in\overline{\mathcal{I}}, \quad \alpha'\neq\alpha, \quad \alpha,\alpha'=1,2, \quad m\geq 1,
\end{aligned}$$

where $c_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\geq 1$, are nonnegative bounded functions, and notation (6.2) is in use. We give a monotone property of $\Gamma_{\alpha,i,m}(U_{\alpha,i,m},U_{\alpha',i,m})$, $i\in\overline{\mathcal{I}}$, $\alpha'\neq\alpha$, $\alpha,\alpha'=1,2$, $m\geq 1$.

Lemma 6.1.2. *Suppose that* $(U_{1,i,m},U_{2,ij,m})$ *and* $(V_{1,i,m},V_{2,i,m})$, $(i,j)\in\overline{\Omega}^h$, $m\geq 1$, *are two vector functions in the sector* $\langle\widehat{U}_m,\widetilde{U}_m\rangle$ *from (5.18), such that* $U_{\alpha,i,m}\geq V_{\alpha,i,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\geq 1$, *and (5.19), (5.20) are satisfied. Then*

$$\Gamma_{\alpha,i,m}(U_{\alpha,i,m},U_{\alpha',i,m}) \geq \Gamma_{\alpha,i,m}(V_{\alpha,i,m},V_{\alpha',i,m}), \quad i\in\overline{\mathcal{I}}, \quad \alpha'\neq\alpha, \quad \alpha,\alpha'=1,2, \quad m\geq 1. \qquad (6.9)$$

Proof. From (6.8),

$$\begin{aligned}
&\Gamma_{\alpha,i,m}(U_{\alpha,i,m},U_{\alpha',i,m}) - \Gamma_{\alpha,i,m}(V_{\alpha,i,m},V_{\alpha',i,m}) = C_{\alpha,i,m}(U_{\alpha,i,m}-V_{\alpha,i,m})\\
&-[F_{\alpha,i,m}(U_{\alpha,i,m},U_{\alpha',i,m}) - F_{\alpha,i,m}(V_{\alpha,i,m},U_{\alpha',i,m})]\\
&-[F_{\alpha,i,m}(V_{\alpha,i,m},U_{\alpha',i,m}) - F_{\alpha,i,m}(V_{\alpha,i,m},V_{\alpha',i,m})].
\end{aligned}$$

Using the mean-value theorem (6.6), we have

$$\begin{aligned}
&\Gamma_{\alpha,i,m}(U_{\alpha,i,m},U_{\alpha',i,m}) - \Gamma_{\alpha,i,m}(V_{\alpha,i,m},V_{\alpha',i,m}) =\\
&(C_{\alpha,i,m}-(F_{\alpha,i,m})_{u_\alpha})(U_{\alpha,i,m}-V_{\alpha,i,m}) - (F_{\alpha,i,m})_{u_{\alpha'}}(U_{\alpha',i,m}-V_{\alpha',i,m}),
\end{aligned}$$

where $(F_{\alpha,i,m})_{u_\alpha}$ and $(F_{\alpha,i,m})_{u_{\alpha'}}$ are defined in (6.7). From here, (5.19), (5.20) and the assumptions of the lemma that $U_{\alpha,i,m}\geq V_{\alpha,i,m}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, $m\geq 1$, we conclude (6.9). □

Based on the method of upper and lower solutions, we now present the block Jacobi and block Gauss–Seidel methods for the nonlinear difference scheme (6.3) when the reaction functions f_α, $\alpha = 1,2$, are quasi-monotone nondecreasing (5.20). On each time level t_m, $m \geq 1$, the upper $\{\overline{U}^{(n)}_{\alpha,i,m}\}$ and lower $\{\underline{U}^{(n)}_{\alpha,i,m}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, $m \geq 1$, solutions are calculated by the following block Jacobi and block Gauss-Seidel iterative methods:

$$(A^\tau_{\alpha,i,m} + C_{\alpha,i,m})Z^{(n)}_{\alpha,i,m} - \eta L_{\alpha,i,m} Z^{(n)}_{\alpha,i-1,m} = -\mathcal{K}_{\alpha,i,m}\left(U^{(n-1)}_{\alpha,i,m}, U_{\alpha,i,m-1}, U^{(n-1)}_{\alpha',i,m}\right),$$
$$i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha,\alpha' = 1,2, \tag{6.10}$$
$$Z^{(n)}_{\alpha,i,m} = \begin{cases} g_{\alpha,i,m} - U^{(0)}_{\alpha,i,m}, & n = 1, \\ \mathbf{0}, & n \geq 2, \end{cases} \quad i \in \partial\mathcal{I}, \quad m \geq 1,$$
$$U_{\alpha,i,0} = \psi_{\alpha,i}, \quad i \in \overline{\mathcal{I}}, \quad U_{\alpha,i,m} = U^{(n_m)}_{\alpha,i,m}, \quad \alpha = 1,2,$$

where $\mathcal{K}_{\alpha,i,m}\left(U^{(n-1)}_{\alpha,i,m}, U_{\alpha,i,m-1}, U^{(n-1)}_{\alpha',i,m}\right)$, $\alpha' \neq \alpha$, $\alpha,\alpha' = 1,2$, $m \geq 1$, are defined in (6.4), $\mathbf{0}$ is a zero column vector with $N_x - 1$ components and $U_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, $m \geq 1$, are the approximate solutions on time level $m \geq 1$, where n_m is a number of iterations on time level $m \geq 1$. For $\eta = 0$ and $\eta = 1$, we have, respectively, the block Jacobi and block Gauss–Seidel methods.

Remark 6.1.3. *For quasi-monotone nondecreasing functions (5.20), upper and lower solutions are independent, hence, by using (6.10), we calculate either the sequence* $\{\overline{U}^{(n)}_{1,i}, \overline{U}^{(n)}_{2,i}\}$, $i \in \overline{\mathcal{I}}$ *or the sequence* $\{\underline{U}^{(n)}_{1,i}, \underline{U}^{(n)}_{2,i}\}$, $i \in \overline{\mathcal{I}}$.

Remark 6.1.4. *Basic advantages of the block Jacobi iterative method with* $\eta = 0$ *in (6.10) and the block Gauss–Seidel method with* $\eta = 1$ *in (6.10), are that on each time level* $m \geq 1$*, the Thomas algorithm can be used for solving each subsystem* (α, i), $i \in \mathcal{I}$, $\alpha = 1,2$*, as in the case of elliptic systems with quasi-monotone nondecreasing reaction functions, which are indicated in Remark 4.1.6.*

Theorem 6.1.5. *Let* $(\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m})$ *and* $(\widehat{U}_{1,i,m}, \widehat{U}_{2,i,m})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, *be ordered upper and lower solutions (6.4) to (6.3). Suppose that the functions* f_α, $\alpha = 1,2$, *in (5.1) satisfy (5.19) and (5.20). Then the upper* $\{\overline{U}^{(n)}_{\alpha,i,m}\}$ *and lower* $\{\underline{U}^{(n)}_{\alpha,i,m}\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, $m \geq 1$, *sequences generated by (6.10) with, respectively,* $(\overline{U}^{(0)}_{1,i,m}, \overline{U}^{(0)}_{2,i,m}) = (\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m})$ *and* $(\underline{U}^{(0)}_{1,i,m}, \underline{U}^{(0)}_{2,i,m}) = (\widehat{U}_{1,i,m}, \widehat{U}_{2,i,m})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, *converge monotonically, such that,*

$$\underline{U}^{(n-1)}_{\alpha,i,m} \leq \underline{U}^{(n)}_{\alpha,i,m} \leq \overline{U}^{(n)}_{\alpha,i,m} \leq \overline{U}^{(n-1)}_{\alpha,i,m}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1,2, \quad m \geq 1. \tag{6.11}$$

Proof. Since $\overline{U}^{(0)}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are upper solutions (6.4) with respect to $U_{\alpha,i,0} = \psi_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, it follows that $\mathcal{K}_{\alpha,i,1}(\overline{U}^{(0)}_{\alpha,i,1}, \psi_{\alpha,i}, \overline{U}^{(0)}_{\alpha',i,1}) \geq \mathbf{0}$, $i \in \mathcal{I}$, $\alpha' \neq \alpha$,

$\alpha, \alpha' = 1, 2$. From here and (6.10), we have

$$\left(A^{\tau}_{\alpha,i,1} + C_{\alpha,i,1}\right) \overline{Z}^{(1)}_{\alpha,i,1} \leq \eta L_{\alpha,i,1} \overline{Z}^{(1)}_{\alpha,i-1,1}, \quad i \in \mathcal{I}, \quad \alpha = 1, 2. \tag{6.12}$$

Taking into account that $\left(A^{\tau}_{\alpha,i,1} + C_{\alpha,i,1}\right)^{-1} > O$ from (4.6), $\eta = 0, 1$, $L_{\alpha,i,1} > O$ from (5.41b) and $\overline{Z}^{(1)}_{\alpha,0,1} \leq \mathbf{0}$, $i \in \mathcal{I}$, $\alpha = 1, 2$, for $i = 1$ in (6.12), we conclude that $\overline{Z}^{(1)}_{\alpha,1,1} \leq \mathbf{0}$, $\alpha = 1, 2$. By induction on i, we can prove that

$$\overline{Z}^{(1)}_{\alpha,i,1} \leq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2. \tag{6.13}$$

Similarly, for the lower solutions $\underline{U}^{(0)}_{\alpha,i,1} = \widehat{U}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, we can prove that

$$\underline{Z}^{(1)}_{\alpha,i,1} \geq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2. \tag{6.14}$$

We now prove that $\overline{U}^{(1)}_{\alpha,i,1}$ and $\underline{U}^{(1)}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are ordered upper and lower solutions (5.17) with respect to the vectors $U_{\alpha,i,0} = \psi_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$. Letting $W^{(1)}_{\alpha,i,1} = \overline{U}^{(1)}_{\alpha,i,1} - \underline{U}^{(1)}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, in notation (5.21), from (6.10), we have

$$\begin{aligned}\left(A^{\tau}_{\alpha,i,1} + C_{\alpha,i,1}\right) W^{(1)}_{\alpha,i,1} - \eta L_{\alpha,i,1} W^{(1)}_{\alpha,i-1,1} &= R_{\alpha,i,1} W^{(0)}_{\alpha,i+1,1} + \Gamma_{\alpha,i,1}(\overline{U}^{(0)}_{\alpha,i,1}, \overline{U}^{(0)}_{\alpha',i,1}) \\ &\quad - \Gamma_{\alpha,i,1}(\underline{U}^{(0)}_{\alpha,i,1}, \overline{U}^{(0)}_{\alpha',i,1}), \\ i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2, \quad W^{(1)}_{\alpha,i,1} = \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad \alpha = 1, 2.\end{aligned}$$

Taking into account that $\underline{U}^{(0)}_{\alpha,i,1} = \widehat{U}_{\alpha,i,1} \leq \overline{U}^{(0)}_{\alpha,i,1} = \widetilde{U}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $R_{\alpha,i,1} > O$, $i \in \mathcal{I}$, $\alpha = 1, 2$, from (5.41b), and using (6.9), it follows that

$$(A^{\tau}_{\alpha,i,1} + C_{\alpha,i,1}) W^{(1)}_{\alpha,i,1} - \eta L_{\alpha,i,1} W^{(1)}_{\alpha,i-1,1} \geq 0, \quad i \in \mathcal{I}, \quad W^{(1)}_{\alpha,i,1} = \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad \alpha = 1, 2. \tag{6.15}$$

Since $W^{(1)}_{\alpha,0,1} = \mathbf{0}$ and $(A^{\tau}_{\alpha,1,1} + C_{\alpha,1,1})^{-1} > O$, $\alpha = 1, 2$, from (4.6), for $i = 1$ in (6.15), we conclude that $W^{(1)}_{\alpha,1,1} \geq \mathbf{0}$, $\alpha = 1, 2$. From here, $(A^{\tau}_{\alpha,2,1} + C_{\alpha,2,1})^{-1} > O$, $\eta = 0, 1$, $L_{\alpha,2,1} > O$, $\alpha = 1, 2$ in (5.41b), for $i = 2$, we obtain that $W^{(1)}_{\alpha,2,1} \geq \mathbf{0}$, $\alpha = 1, 2$. By induction on i, we can prove that

$$W^{(1)}_{\alpha,i,1} \geq 0, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2.$$

Thus, we prove (6.4a). From (6.10) and using notation (6.8), we conclude that

$$\mathcal{K}_{\alpha,i,1}(\overline{U}^{(1)}_{\alpha,i,1},\psi_{\alpha,i},\overline{U}^{(1)}_{\alpha',i,1}) = -R_{\alpha,i,1}\overline{Z}^{(1)}_{\alpha,i+1,1} + \Gamma_{\alpha,i,1}(\overline{U}^{(0)}_{\alpha,i,1},\overline{U}^{(0)}_{\alpha',i,1}) - \Gamma_{\alpha,i,1}(\overline{U}^{(1)}_{\alpha,i,1},\overline{U}^{(1)}_{\alpha',i,1}), \quad i\in\mathcal{I},\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2. \tag{6.16}$$

From here, (6.13), $R_{\alpha,i,1} > O$, $i\in\mathcal{I}$, $\alpha=1,2$, in (5.41b), by using (6.9), we obtain

$$\mathcal{K}_{\alpha,i,1}(\overline{U}^{(1)}_{\alpha,i,1},\psi_{\alpha,i},\overline{U}^{(1)}_{\alpha',i,1}) \geq \mathbf{0},\quad i\in\mathcal{I},\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2.$$

Thus, $\overline{U}^{(1)}_{\alpha,i,1}$, $i\in\mathcal{I}$, $\alpha=1,2$, satisfy (6.4b). By a similar manner, we can prove that

$$\mathcal{K}_{\alpha,i,1}(\underline{U}^{(1)}_{\alpha,i,1},\psi_{\alpha,i},\underline{U}^{(1)}_{\alpha',i,1}) \leq \mathbf{0},\quad i\in\mathcal{I},\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2.$$

Hence, $\underline{U}^{(1)}_{\alpha,i,1}$, $i\in\mathcal{I}$, $\alpha=1,2$, satisfy (6.4b). From the boundary conditions in (6.10), it follows that $\overline{U}^{(1)}_{\alpha,i,1}$ and $\underline{U}^{(1)}_{\alpha,i,1}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, satisfy (6.4c).

Thus, we prove that $\overline{U}^{(1)}_{\alpha,i,1}$ and $\underline{U}^{(1)}_{\alpha,i,1}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, are ordered upper and lower solutions (6.4). By induction on $n\geq 1$, we can prove (6.11) on the first time level $m=1$.

On the second time level $m=2$, taking into account that $\overline{U}^{(0)}_{\alpha,i,2} = \widetilde{U}_{\alpha,i,2}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, from (6.3), we obtain

$$\mathcal{K}_{\alpha,i,2}\left(\widetilde{U}_{\alpha,i,2},\overline{U}_{\alpha,i,1},\widetilde{U}_{\alpha',i,2}\right) = A^{\tau}_{\alpha,i,2}\widetilde{U}_{\alpha,i,2} - L_{\alpha,i,2}\widetilde{U}_{\alpha,i-1,2} - R_{\alpha,i,2}\widetilde{U}_{\alpha,i+1,2} + F_{\alpha,i,2}(\widetilde{U}_{\alpha,i,2},\widetilde{U}_{\alpha',i,2}) - \tau^{-1}\overline{U}_{\alpha,i,1}, \quad i\in\mathcal{I},\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2,$$

where $\overline{U}_{\alpha,i,1}$, $i\in\mathcal{I}$, $\alpha=1,2$, are the approximate solutions on the first time level $m=1$, which defined in (6.10). From here and taking into account that from (6.11), $\overline{U}_{\alpha,i,1}\leq\widetilde{U}_{\alpha,i,1}$, $i\in\mathcal{I}$, $\alpha=1,2$, it follows that

$$\mathcal{K}_{\alpha,i,2}\left(\widetilde{U}_{\alpha,i,2},\overline{U}_{\alpha,i,1},\widetilde{U}_{\alpha',i,2}\right) \geq \mathcal{K}_{\alpha,i,2}\left(\widetilde{U}_{\alpha,i,2},\widetilde{U}_{\alpha,i,1},\widetilde{U}_{\alpha',i,2}\right) \geq \mathbf{0}, \quad i\in\mathcal{I},\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2, \tag{6.17}$$

which means that $\overline{U}^{(0)}_{\alpha,i,2} = \widetilde{U}_{\alpha,i,2}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, are upper solutions with respect to $\overline{U}_{\alpha,i,1}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$. Similarly, we can obtain that

$$\mathcal{K}_{\alpha,i,2}\left(\widehat{U}_{\alpha,i,2},\underline{U}_{\alpha,i,1},\widehat{U}_{\alpha',i,2}\right) \leq \mathbf{0},\quad i\in\mathcal{I},\quad \alpha'\neq\alpha,\quad \alpha,\alpha'=1,2,$$

which means that $\underline{U}^{(0)}_{\alpha,i,2} = \widehat{U}_{\alpha,i,2}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, are lower solutions with respect to

$\underline{U}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$. From here, (6.10) and (6.17), on the second time level $m = 2$, we have

$$\left(A^{\tau}_{\alpha,i,2} + C_{\alpha,i,2}\right) \overline{Z}^{(1)}_{\alpha,i,2} \leq \eta L_{\alpha,i,2} \overline{Z}^{(1)}_{\alpha,i-1,2}, \quad i \in \mathcal{I}, \quad \alpha = 1, 2. \tag{6.18}$$

Taking into account that $\eta = 0, 1$, $L_{\alpha,i,2} > O$ from (5.41b), $(A^{\tau}_{\alpha,i,2} + C_{\alpha,i,2})^{-1} > O$, $i \in \mathcal{I}$, $\alpha = 1, 2$, and $\overline{Z}^{(1)}_{\alpha,0,2} \leq \mathbf{0}$, for $i = 1$ in (6.18), it follows that $\overline{Z}^{(1)}_{\alpha,1,2} \leq \mathbf{0}$, $\alpha = 1, 2$. From here and (6.18) with $i = 2$, we conclude that $\overline{Z}^{(1)}_{\alpha,2,2} \leq \mathbf{0}$, $\alpha = 1, 2$. By induction on i, we can prove that

$$\overline{Z}^{(1)}_{\alpha,i,2} \leq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2. \tag{6.19}$$

Similarly, for initial lower solutions $\underline{U}_{\alpha,i,2}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, we can prove that

$$\underline{Z}^{(1)}_{\alpha,i,2} \geq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2. \tag{6.20}$$

The proof that $\overline{U}^{(1)}_{\alpha,i,2}$ and $\underline{U}^{(1)}_{\alpha,i,2}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are ordered upper and lower solutions (6.4) repeats the proof on the first time level $m = 1$. By induction on m, we can prove (6.11) for $m \geq 1$. $\square$

6.1.2 Quasi-monotone nonincreasing case

In the case of the quasi-monotone nonincreasing functions (5.30) , on each time level $m \geq 1$, we say that mesh functions

$$(\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m}), \quad (\widehat{U}_{1,i,m}, \widehat{U}_{2,i,m}), \quad i \in \overline{\mathcal{I}}, \quad m \geq 1,$$

are ordered upper and lower solutions to (6.3), if they satisfy the inequalities

$$\widehat{U}_{\alpha,i,m} \leq \widetilde{U}_{\alpha,i,m}, \quad i \in \overline{\mathcal{I}}, \tag{6.21a}$$

$$\mathcal{K}_{\alpha,i,m}(\widehat{U}_{\alpha,i,m}, \widehat{U}_{\alpha,i,m-1}, \widetilde{U}_{\alpha',i,m}) \leq \mathbf{0} \leq \mathcal{K}_{\alpha,i,m}(\widetilde{U}_{\alpha,i,m}, \widetilde{U}_{\alpha,i,m-1}, \widehat{U}_{\alpha',i,m}), \quad i \in \mathcal{I}, \tag{6.21b}$$

$$\widehat{U}_{\alpha,i,m} \leq g_{\alpha,i,m} \leq \widetilde{U}_{\alpha,i,m}, \quad i \in \partial\mathcal{I}, \quad m \geq 1, \ \widehat{U}_{\alpha,i,0} \leq \psi_{\alpha,i} \leq \widetilde{U}_{\alpha,i,0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2, \tag{6.21c}$$

where $\mathcal{K}_{\alpha,i,m}(U_{\alpha,i,m}, U_{\alpha,i,m-1}, U_{\alpha',i,m})$, $i \in \mathcal{I}$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1, 2$, $m \geq 1$, are defined in (6.4).

Lemma 6.1.6. *Let (5.19) and (5.30) hold, and $U_{\alpha,i,m}$, $V_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, be two mesh functions in $\langle \widehat{U}_m, \widetilde{U}_m \rangle$ such that $U_{\alpha,i,m} \geq V_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$. Then*

$$\Gamma_{\alpha,i,m}(U_{\alpha,i,m}, V_{\alpha',i,m}) \geq \Gamma_{\alpha,i,m}(V_{\alpha,i,m}, U_{\alpha',i,m}), \quad i \in \overline{\mathcal{I}}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2, \quad m \geq 1. \tag{6.22}$$

Proof. From (6.8), we have

$$\Gamma_{\alpha,i,m}(U_{\alpha,i,m},V_{\alpha',i,m})-\Gamma_{\alpha,i,m}(V_{\alpha,i,m},U_{\alpha',i,m})=C_{\alpha,i,m}(U_{\alpha,i,m}-V_{\alpha,i,m})$$
$$-\left[F_{\alpha,i,m}(U_{\alpha,i,m},V_{\alpha',i,m})-F_{\alpha,i,m}(V_{\alpha,i,m},V_{\alpha',i,m})\right]$$
$$+\left[F_{\alpha,i,m}(V_{\alpha,i,m},U_{\alpha',i,m})-F_{\alpha,i,m}(V_{\alpha,i,m},V_{\alpha',i,m})\right],$$

where notation (6.2) is in use. Using the mean-value theorem (5.16), we have

$$\Gamma_{\alpha,i,m}(U_{\alpha,i,m},V_{\alpha',i,m})-\Gamma_{\alpha,i,m}(V_{\alpha,i,m},U_{\alpha',i,m})=$$
$$\Big(C_{\alpha,i,m}-\big(F_{\alpha,i,m}(Q_{\alpha,i,m},V_{\alpha',i,m})\big)_{u_\alpha}\Big)(U_{\alpha,i,m}-V_{\alpha,i,m})$$
$$+\big(F_{\alpha,i,m}(V_{\alpha,i,m},Y_{\alpha',i,m})\big)_{u_{\alpha'}}(U_{\alpha',i,m}-V_{\alpha',i,m}),$$
$$V_{\alpha,i,m}\le Q_{\alpha,i,m},Y_{\alpha,i,m}\le U_{\alpha,i,m},\quad i\in\overline{\mathcal{I}},\quad \alpha'\ne\alpha,\quad \alpha,\alpha'=1,2,\quad m\ge 1,$$

where $(F_{\alpha,i,m})_{u_\alpha}$ and $(F_{\alpha,i,m})_{u_{\alpha'}}$ are defined in (6.6). From here, (5.19), (5.30) and the assumptions of the lemma, we conclude (6.22). □

We now present the block Jacobi and block Gauss–Seidel methods for the nonlinear difference scheme (6.3) when the reaction functions f_α, $\alpha=1,2$, are quasi-monotone nonincreasing (5.30).

For solving the nonlinear difference scheme (6.3), on each time level t_m, $m\ge 1$, we construct the block iterative Jacobi and block iterative Gauss-Seidel methods in the forms

$$(A^\tau_{\alpha,i,m}+C_{\alpha,i,m})\overline{Z}^{(n)}_{\alpha,i,m}-\eta L_{\alpha,i,m}\overline{Z}^{(n)}_{\alpha,i-1,m}=-\mathcal{K}_{\alpha,i,m}\left(\overline{U}^{(n-1)}_{\alpha,i,m},\overline{U}_{\alpha,i,m-1},\underline{U}^{(n-1)}_{\alpha',i,m}\right),$$
$$(A^\tau_{\alpha,i,m}+C_{\alpha,i,m})\underline{Z}^{(n)}_{\alpha,i,m}-\eta L_{\alpha,i,m}\underline{Z}^{(n)}_{\alpha,i-1,m}=-\mathcal{K}_{\alpha,i,m}\left(\underline{U}^{(n-1)}_{\alpha,i,m},\underline{U}_{\alpha,i,m-1},\overline{U}^{(n-1)}_{\alpha',i,m}\right),$$
$$i\in\mathcal{I}, \tag{6.23}$$
$$Z^{(n)}_{\alpha,i,m}=\begin{cases} g_{\alpha,i,m}-U^{(0)}_{\alpha,i,m}, & n=1,\\ \mathbf{0}, & n\ge 2,\end{cases}\quad i=\partial\mathcal{I},$$
$$U_{\alpha,i,0}=\psi_{\alpha,i},\quad i\in\overline{\mathcal{I}},\quad Z^{(n)}_{\alpha,i,m}=U^{(n)}_{\alpha,i,m}-U^{(n-1)}_{\alpha,i,m},\quad U_{\alpha,i,m}=U^{(n_m)}_{\alpha,i,m},$$
$$\alpha'\ne\alpha,\quad \alpha,\alpha'=1,2,\quad m\ge 1,$$

where the residuals $\mathcal{K}_{\alpha,i,m}\left(U^{(n-1)}_{\alpha,i,m},U_{\alpha,i,m-1},U^{(n-1)}_{\alpha',i,m}\right)$, $i\in\mathcal{I}$, $\alpha'\ne\alpha$, $\alpha,\alpha'=1,2$, $m\ge 1$, are defined in (6.4), $\mathbf{0}$ is zero vector with N_x-1 components. The vectors $U_{\alpha,i,m}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, $m\ge 1$, are the approximate solutions on time level $m\ge 1$, where n_m is a number of iterations on time level $m\ge 1$. For $\eta=0$ and $\eta=1$, we have, respectively, the block Jacobi and block Gauss–Seidel methods.

Remark 6.1.7. *For quasi-monotone nonincreasing functions f_α, $\alpha=1,2$, (5.30),* *upper and lower solutions are coupled, hence, by using (6.23), we calculate either the sequence* $\{\overline{U}^{(n)}_{1,i,m}, \underline{U}^{(n)}_{2,i,m}\}$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, *or the sequence* $\{\underline{U}^{(n)}_{1,i,m}, \overline{U}^{(n)}_{2,i,m}\}$, $i \in \overline{\mathcal{I}}$, $m \geq 1$.

Remark 6.1.8. *Basic advantages of the block Jacobi method with* $\eta = 0$ *in (6.23) and the block Gauss–Seidel method with* $\eta = 1$ *in (6.23) are that on each time level* $m \geq 1$, *the Thomas algorithm can be used for solving each subsystem* (α, i), $i \in \mathcal{I}$, $\alpha = 1, 2$, *as in the case of elliptic systems with quasi-monotone nondecreasing reaction functions, which are indicated in Remark 4.1.6.*

In the following theorem, we prove the monotone property of the block iterative methods (6.23).

Theorem 6.1.9. *Let* $(\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m})$ *and* $(\widehat{U}_{1,i,m}, \widehat{U}_{2,i,m})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, *be ordered upper and lower solutions (6.21) to (6.3). Suppose that* f_α, $\alpha = 1, 2$, *in (5.1) satisfy (5.19) and (5.30). Then the sequences* $\{\overline{U}^{(n)}_{1,i,m}, \underline{U}^{(n)}_{2,i,m}\}$ *and* $\{\underline{U}^{(n)}_{1,i,m}, \overline{U}^{(n)}_{2,i,m}\}$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, *generated by (6.23), with* $(\overline{U}^{(0)}_{1,i,m}, \underline{U}^{(0)}_{2,i,m}) = (\widetilde{U}_{1,i,m}, \widehat{U}_{2,i,m})$ *and* $(\underline{U}^{(0)}_{1,i,m}, \overline{U}^{(0)}_{2,i,m}) = (\widehat{U}_{1,i,m}, \widetilde{U}_{2,i,m})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, *are ordered upper and lower solutions and converge monotonically, such that,*

$$\underline{U}^{(n-1)}_{\alpha,i,m} \leq \underline{U}^{(n)}_{\alpha,i,m} \leq \overline{U}^{(n)}_{\alpha,i,m} \leq \overline{U}^{(n-1)}_{\alpha,i,m}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2, \quad m \geq 1. \tag{6.24}$$

Proof. On first time level $m = 1$, in the case of the sequence $\{\overline{U}^{(n)}_{1,i,1}, \underline{U}^{(n)}_{2,i,1}\}$, $(\overline{U}^{(0)}_{1,i,1}, \underline{U}^{(0)}_{2,i,1}) = (\widetilde{U}_{1,i,1}, \widehat{U}_{2,i,1})$, $i \in \overline{\mathcal{I}}$, are initial upper and lower solution (6.21) with respect to $U_{\alpha,i,0} = \psi_{\alpha,i}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$. Hence, it follows that $\mathcal{K}_{1,i,1}\left(\widetilde{U}_{1,i,1}, \psi_{1,i}, \widehat{U}_{2,i,1}\right) \geq \mathbf{0}$, $i \in \mathcal{I}$, and $\mathcal{K}_{2,i,1}\left(\widetilde{U}_{1,i,1}, \psi_{2,i}, \widehat{U}_{2,i,1}\right) \leq \mathbf{0}$, $i \in \mathcal{I}$. From here and (6.23), we have

$$\begin{aligned} &(A^\tau_{1,i,1} + C_{1,i,1})\overline{Z}^{(1)}_{1,i,1} \leq \eta L_{1,i,1}\overline{Z}^{(1)}_{1,i-1,1}, \quad i \in \mathcal{I}, \\ &(A^\tau_{2,i,1} + C_{2,i,1}I)\underline{Z}^{(1)}_{2,i,1} \geq \eta L_{2,i,1}\underline{Z}^{(1)}_{2,i-1,1}, \quad i \in \mathcal{I}, \\ &\overline{Z}^{(1)}_{1,i,1} \leq \mathbf{0}, \quad \underline{Z}^{(1)}_{2,i,1} \geq \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad \overline{Z}^{(1)}_{1,i,0} = \mathbf{0}. \end{aligned} \tag{6.25}$$

Taking into account that $\eta = 0, 1$, $L_{1,1,1} > O$ from (5.41b) and $\overline{Z}^{(1)}_{1,0,1} \leq \mathbf{0}$, $\underline{Z}^{(1)}_{2,0,1} \geq \mathbf{0}$, for $i = 1$ in (6.25), we have $\left(A^\tau_{1,1,1} + C_{1,1,1}\right)\overline{Z}^{(1)}_{1,1,1} \leq \mathbf{0}$, $\left(A^\tau_{2,1,1} + C_{2,1,1}\right)\underline{Z}^{(1)}_{2,1,1} \geq \mathbf{0}$. From here and taking into account that $(A^\tau_{\alpha,1,1} + C_{\alpha,1,1})^{-1} > O$, $\alpha = 1, 2$, where O is the $(N_y - 1) \times (N_y - 1)$ null matrix, it follows that $\overline{Z}^{(1)}_{1,1,1} \leq \mathbf{0}$, $\underline{Z}^{(1)}_{2,1,1} \geq \mathbf{0}$. By induction on i, we can prove that

$$\overline{Z}^{(1)}_{1,i,1} \leq \mathbf{0}, \quad \underline{Z}^{(1)}_{2,i,1} \geq \mathbf{0}, \quad i \in \overline{\mathcal{I}}. \tag{6.26}$$

Similarly, for the sequence $\{\underline{U}^{(n)}_{1,i,1}, \overline{U}^{(n)}_{2,i,1}\}$, from (6.23), we can prove that

$$\underline{Z}^{(1)}_{1,i,1} \geq \mathbf{0}, \quad \overline{Z}^{(1)}_{2,i,1} \leq \mathbf{0}, \quad i \in \overline{\mathcal{I}}. \tag{6.27}$$

We now prove that $\overline{U}^{(1)}_{\alpha,i,1}$ and $\underline{U}^{(1)}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, are ordered upper and lower solutions (6.21). Let $W^{(1)}_{\alpha,i,1} = \overline{U}^{(1)}_{\alpha,i,1} - \underline{U}^{(1)}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$. Using notation (6.8), from (6.23), we have

$$\begin{aligned}
&(A^{\tau}_{\alpha,i,1} + C_{\alpha,i,1})W^{(1)}_{\alpha,i,1} = \eta L_{\alpha,i,1} W^{(1)}_{\alpha,i-1,1} + R_{\alpha,i,1} W^{(0)}_{\alpha,i+1,1} + \Gamma_{\alpha,i,1}(\overline{U}^{(0)}_{\alpha,i,1}, \underline{U}^{(0)}_{\alpha',i,1}) \\
&\qquad\qquad - \Gamma_{\alpha,i,1}(\underline{U}^{(0)}_{\alpha,i,1}, \overline{U}^{(0)}_{\alpha',i,1}), \qquad i \in \mathcal{I} \\
&W^{(1)}_{\alpha,i,1} = \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad W_{\alpha,i,0} = \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2.
\end{aligned}$$

From (6.22), taking into account that $\eta = 0,1$, $R_{\alpha,i,1} > O$, $i \in \mathcal{I}$ from (5.41b) and $W^{(0)}_{\alpha,i,1} \geq \mathbf{0}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, we conclude that

$$\begin{aligned}
&(A^{\tau}_{\alpha,i,1} + C_{\alpha,i,1})W^{(1)}_{\alpha,i,1} \geq \eta L_{\alpha,i,1} W^{(1)}_{\alpha,i-1,1}, \quad i \in \mathcal{I}, \\
&W^{(1)}_{\alpha,i,1} = \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad W^{(1)}_{\alpha,i,0} = \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1,2.
\end{aligned} \tag{6.28}$$

Taking into account that $W^{(1)}_{\alpha,0,1} = \mathbf{0}$ and $(A^{\tau}_{\alpha,i,1} + C_{\alpha,i,1})^{-1} > O$, $i \in \mathcal{I}$, $\alpha = 1,2$, for $i = 1$ in (6.28), we have $W^{(1)}_{\alpha,1,1} \geq \mathbf{0}$, $\alpha = 1,2$. From here, for $i = 2$ in (6.28), by a similar manner, we obtain $W^{(1)}_{\alpha,2,1} \geq \mathbf{0}$, $\alpha = 1,2$. By induction on i, we can prove that

$$W^{(1)}_{\alpha,i,1} \geq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1,2. \tag{6.29}$$

Thus, we prove (6.21a) on the first time level $m = 1$.

From (6.23) and using (6.22), we obtain

$$\begin{aligned}
&\mathcal{K}_{\alpha,i,1}(\overline{U}^{(1)}_{\alpha,i,1}, \psi_{\alpha,i}, \underline{U}^{(1)}_{\alpha',i,1}) = -R_{\alpha,i,1}\overline{Z}^{(1)}_{\alpha,i+1,1} + \Gamma_{\alpha,i,1}(\overline{U}^{(0)}_{\alpha,i,1}, \underline{U}^{(0)}_{\alpha',i,1}) \\
&\qquad\qquad - \Gamma_{\alpha,i,1}(\overline{U}^{(1)}_{\alpha,i,1}, \underline{U}^{(1)}_{\alpha',i,1}), \\
&i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2.
\end{aligned} \tag{6.30}$$

Taking into account that $R_{1,i,1} > O$, $i \in \mathcal{I}$ in (5.41b), from (6.26), (6.27) and (6.30), by using (6.22), we conclude that

$$\mathcal{K}_{\alpha,i,1}\left(\overline{U}^{(1)}_{\alpha,i,1}, \psi_{\alpha,i}, \underline{U}^{(1)}_{\alpha',i,1}\right) \geq \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2. \tag{6.31}$$

Similarly, we can prove that

$$\mathcal{K}_{\alpha,i,1}\left(\underline{U}^{(1)}_{\alpha,i,1}, \psi_{\alpha,i}, \overline{U}^{(1)}_{\alpha',i,1}\right) \leq \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2. \tag{6.32}$$

Thus, (6.31) and (6.32) satisfy (6.21b). From the boundary and initial conditions in (6.23), it follows that $\underline{U}^{(1)}_{\alpha,i,1}$ and $\overline{U}^{(1)}_{\alpha,i,1}$, satisfy (6.21c). Thus, from here, (6.29), (6.31) and (6.32), we conclude that $\overline{U}^{(1)}_{\alpha,i,1}$ and $\underline{U}^{(1)}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are ordered upper and lower solutions (6.21).

By induction on n, we can prove that $\overline{U}^{(n)}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are monotone decreasing sequences of upper solutions and $\underline{U}^{(n)}_{\alpha,i,1}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are monotone increasing sequences of lower solutions which satisfy (6.24).

On the second time level $m = 2$, for the sequence $\{\overline{U}^{(n)}_{1,i,2}, \underline{U}^{(n)}_{2,i,2}\}$, $i \in \overline{\mathcal{I}}$, we have $\overline{U}^{(0)}_{1,i,2} = \widetilde{U}_{1,i,2}$ and $\underline{U}^{(0)}_{2,i,2} = \widehat{U}_{2,i,2}$, $i \in \overline{\mathcal{I}}$. From (6.3), we obtain that

$$\begin{aligned}\mathcal{K}_{1,i,2}(\widetilde{U}_{1,i,2}, \overline{U}_{1,i,1}, \widehat{U}_{2,i,2}) = {} & A^{\tau}_{1,i,2}\widetilde{U}_{1,i,2} - L_{1,i,2}\widetilde{U}_{1,i-1,2} - R_{1,i,2}\widetilde{U}_{1,i+1,2} \\ & + F_{1,i,2}(\widetilde{U}_{1,i,2}, \widehat{U}_{2,i,2}) - \tau^{-1}\overline{U}_{1,i,1},\end{aligned}$$

$$\begin{aligned}\mathcal{K}_{2,i,2}(\widetilde{U}_{1,i,2}, \underline{U}_{2,i,1}, \widehat{U}_{2,i,2}) = {} & A^{\tau}_{2,i,2}\widehat{U}_{2,i,2} - L_{2,i,2}\widehat{U}_{2,i-1,2} - R_{2,i,2}\widehat{U}_{2,i+1,2} \\ & + F_{2,i,2}(\widetilde{U}_{1,i,2}, \widehat{U}_{2,i,2}) - \tau^{-1}\underline{U}_{2,i,1},\end{aligned}$$

where $\overline{U}_{1,i,1}$ and $\underline{U}_{2,i,1}$, $i \in \overline{\mathcal{I}}$, are the approximate solutions on the first time level $m = 1$, which are defined in (6.23). From here and taking into account that from (6.24), $\overline{U}_{1,i,1} \le \widetilde{U}_{1,i,1}$ and $\widehat{U}_{2,i,1} \le \underline{U}_{2,i,1}$, $i \in \overline{\mathcal{I}}$, it follows that

$$\begin{aligned}&\mathcal{K}_{1,i,2}\left(\widetilde{U}_{1,i,2}, \overline{U}_{1,i,1}, \widehat{U}_{2,i,2}\right) \ge \mathcal{K}_{1,i,2}\left(\widetilde{U}_{1,i,2}, \widetilde{U}_{1,i,1}, \widehat{U}_{2,i,2}\right) \ge \mathbf{0}, \\ &\mathcal{K}_{2,i,2}\left(\widetilde{U}_{1,i,2}, \underline{U}_{2,i,1}, \widehat{U}_{2,i,2}\right) \le \mathcal{K}_{2,i,2}\left(\widetilde{U}_{1,i,2}, \widehat{U}_{2,i,1}, \widehat{U}_{1,i,2}\right) \le \mathbf{0}, \quad i \in \mathcal{I},\end{aligned} \tag{6.33}$$

which means that $\overline{U}^{(0)}_{1,i,2} = \widetilde{U}_{1,i,2}$ and $\underline{U}^{(0)}_{2,i,2} = \widehat{U}_{2,i,2}$, $i \in \overline{\mathcal{I}}$ are upper and lower solutions with respect to $\overline{U}_{1,i,1}$ and $\underline{U}_{2,i,1}$, $i \in \overline{\mathcal{I}}$.

Similarly, we can prove that

$$\mathcal{K}_{1,i,2}\left(\widehat{U}_{1,i,2}, \underline{U}_{1,i,1}, \widetilde{U}_{2,i,2}\right) \le \mathbf{0}, \quad \mathcal{K}_{2,i,2}\left(\widehat{U}_{1,i,2}, \overline{U}_{2,i,1}, \widetilde{U}_{2,i,2}\right) \ge \mathbf{0}, \quad i \in \mathcal{I},$$

which means that $\widetilde{U}_{2,i,2}$ and $\widehat{U}_{1,i,2}$, $i \in \overline{\mathcal{I}}$, are upper and lower solutions with respect to $\overline{U}_{2,i,1}$ and $\underline{U}_{1,i,1}$, $i \in \overline{\mathcal{I}}$. From here, (6.23) and (6.33), on the second time level $m = 2$, we have

$$\begin{aligned}&(A^{\tau}_{1,i,2} + C_{1,i,2})\overline{Z}^{(1)}_{1,i,2} \le \eta L_{1,i,2}\overline{Z}^{(1)}_{1,i-1,2}, \quad i \in \mathcal{I}, \\ &(A^{\tau}_{2,i,2} + C_{2,i,2})\underline{Z}^{(1)}_{2,i,2} \ge \eta L_{2,i,2}\underline{Z}^{(1)}_{2,i-1,2}, \quad i \in \mathcal{I}, \\ &\overline{Z}^{(1)}_{1,i,2} \le \mathbf{0}, \quad \underline{Z}^{(1)}_{2,i,2} \ge \mathbf{0}, \quad i \in \partial\mathcal{I}.\end{aligned} \tag{6.34}$$

Taking into account that $\eta = 0, 1$, $L_{\alpha,i,2} > O$ from (5.41b), $\left(A^{\tau}_{\alpha,i,2} + C_{\alpha,i,2}\right)^{-1} > O$, $i \in \mathcal{I}$, $\alpha = 1, 2$, and $\overline{Z}^{(1)}_{1,0,2} \leq \mathbf{0}$, $\underline{Z}^{(1)}_{2,0,2} \geq \mathbf{0}$, for $i = 1$ in (6.34), we conclude that $\overline{Z}^{(1)}_{1,1,2} \leq \mathbf{0}$, $\underline{Z}^{(1)}_{2,1,2} \geq \mathbf{0}$. From here, in a similar manner, for $i = 2$ in (6.34), we conclude that $\overline{Z}^{(1)}_{1,2,2} \leq \mathbf{0}$, $\underline{Z}^{(1)}_{2,2,2} \geq \mathbf{0}$. By induction on i, we can prove that

$$\overline{Z}^{(1)}_{1,i,2} \leq \mathbf{0}, \quad \underline{Z}^{(1)}_{2,i,2} \geq \mathbf{0}, \quad i \in \overline{\mathcal{I}}.$$

The proof, that $\overline{U}^{(1)}_{\alpha,i,2}$ and $\underline{U}^{(1)}_{\alpha,i,2}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are ordered upper and lower solutions (6.21), repeats the proof on the first time level $m = 1$. By induction on n, we can prove that $\overline{U}^{(n)}_{\alpha,i,2}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are monotone decreasing sequence of upper solutions and $\underline{U}^{(n)}_{\alpha,i,2}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, are monotone increasing sequence of lower solutions which satisfy (6.24). By induction on m, we can prove (6.24) for $m \geq 1$. In a similar manner, we can prove the theorem for the sequence $\{\underline{U}^{(n)}_{1,i,m}, \overline{U}^{(n)}_{2,i,m}\}$, $i \in \overline{\mathcal{I}}$, $m \geq 1$. □

6.1.3 Existence and uniqueness of a solution to the nonlinear difference scheme (6.3)

In Section 5.5, for quasi-monotone nondecreasing reaction functions f_α, $\alpha = 1, 2$, (5.20), we prove the existence and uniqueness of a solution to the nonlinear difference scheme (5.41) in, respectively, Theorems 5.5.2 and 5.5.3. The proofs of these results are based on the monotone properties of the point iterative sequences (5.44) in Theorem 5.4.3 and the maximum principle in Lemma 5.4.1. In a similar manner, we prove the existence and uniqueness of a solution by using the monotone properties of the block iterative sequences (6.11) in Theorem 6.1.5 and property (4.6) of irreducibly diagonally dominant matrices in Lemma 4.1.2, Chapter 4.

In the case of quasi-monotone nonincreasing reaction functions (5.30), we prove the existence and uniqueness of a solution to (5.11) in, respectively, Theorems 5.5.4 and 5.5.5 in Chapter 5. As in the quasi-monotone nondecreasing case, the proofs are based on the monotone properties of the point iterative sequences (5.58) in Theorem 5.4.5 and Lemma 5.4.1 in Chapter 4.

In a similar manner, these results can be proved by using the monotone properties of the block iterative sequences (6.24) in Theorem 6.1.9 and Lemma (4.1.2).

6.2 Comparison of convergence of the block monotone Jacobi and block monotone Gauss–Seidel methods

We compare the convergence rates of the block monotone Jacobi and block monotone Gauss–Seidel methods.

6.2.1 Quasi-monotone nondecreasing case

In the case of quasi-monotone nondecreasing reaction functions (5.20), the following theorem shows that the block monotone Gauss–Seidel method with $\eta = 1$ in (6.10), converges faster than the block monotone Jacobi method with $\eta = 0$ in (6.10).

Theorem 6.2.1. *Let $(\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m})$ and $(\widehat{U}_{1,i,m}, \widehat{U}_{2,i,m})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, be ordered upper and lower solutions (6.4) of the nonlinear difference scheme (6.3). Suppose that f_α, $\alpha = 1, 2$, in (5.1) satisfy (5.19) and (5.20). The sequences $\left\{(U^{(n)}_{\alpha,i,m})_J\right\}$ and $\left\{(U^{(n)}_{\alpha,i,m})_{GS}\right\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, are, respectively, the sequences generated by the block monotone Jacobi method with $\eta = 0$ in (6.10), and the block monotone Gauss–Seidel method with $\eta = 1$ in (6.10), where $(\overline{U}^{(0)}_{\alpha,i,m})_J = (\overline{U}^{(0)}_{\alpha,i,m})_{GS} = \widetilde{U}_{\alpha,i,m}$ and $(\underline{U}^{(0)}_{\alpha,i,m})_J = (\underline{U}^{(0)}_{\alpha,i,m})_{GS} = \widehat{U}_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$. Then the following inequalities hold*

$$(\underline{U}^{(n)}_{\alpha,i,m})_J \leq (\underline{U}^{(n)}_{\alpha,i,m})_{GS} \leq (\overline{U}^{(n)}_{\alpha,i,m})_{GS} \leq (\overline{U}^{(n)}_{\alpha,i,m})_J, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2, \quad m \geq 1. \tag{6.35}$$

Proof. Letting $\overline{W}^{(n)}_{\alpha,i,m} = \left(\overline{U}^{(n)}_{\alpha,i,m}\right)_{\mathtt{J}} - \left(\overline{U}^{(n)}_{\alpha,i,m}\right)_{\mathtt{GS}}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, from (6.10) and using notation (6.8), we have

$$\begin{aligned} A^\tau_{\alpha,i,m}\overline{W}^{(n)}_{\alpha,i,m} + C_{\alpha,i,m}\overline{W}^{(n)}_{\alpha,i,m} &= \eta L_{\alpha,i,m}\left((\overline{U}^{(n-1)}_{\alpha,i-1,m})_{\mathtt{J}} - (\overline{U}^{(n)}_{\alpha,i-1,m})_{\mathtt{GS}}\right) + R_{\alpha,i,m}\overline{W}^{(n-1)}_{\alpha,i+1,m} \\ &\quad + \Gamma_{\alpha,i,m}\left(\overline{U}^{(n-1)}_{\alpha,i,m}, \overline{U}^{(n-1)}_{\alpha',i,m}\right)_{\mathtt{J}} - \Gamma_{\alpha,i,m}\left(\overline{U}^{(n-1)}_{\alpha,i,m}, \overline{U}^{(n-1)}_{\alpha',i,m}\right)_{\mathtt{GS}} \\ &\quad + \tau^{-1}\left((\overline{U}_{\alpha,i,m-1})_{\mathtt{J}} - (\overline{U}_{\alpha,i,m-1})_{\mathtt{GS}}\right), \quad i \in \mathcal{I}, \end{aligned}$$

$$\overline{W}^{(n)}_{\alpha,i,m} = \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad m \geq 1, \quad \overline{W}_{\alpha,i,0} = \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2.$$

From here and taking into account that $\left(\overline{U}^{(n)}_{\alpha,i,m}\right)_{\mathtt{GS}} \leq \left(\overline{U}^{(n-1)}_{\alpha,i,m}\right)_{\mathtt{GS}}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, in (6.11), it follows that

$$\begin{aligned} A^\tau_{\alpha,i,m}\overline{W}^{(n)}_{\alpha,i,m} + C_{\alpha,i,m}\overline{W}^{(n)}_{\alpha,i,m} &\geq \eta L_{\alpha,i,m}\overline{W}^{(n-1)}_{\alpha,i,m} + R_{\alpha,i,m}\overline{W}^{(n-1)}_{\alpha,i+1,m} \\ &\quad + \Gamma_{\alpha,i,m}\left(\overline{U}^{(n-1)}_{\alpha,i,m}, \overline{U}^{(n-1)}_{\alpha',i,m}\right)_{\mathtt{J}} \\ &\quad - \Gamma_{\alpha,i,m}\left(\overline{U}^{(n-1)}_{\alpha,i,m}, \overline{U}^{(n-1)}_{\alpha',i,m}\right)_{\mathtt{GS}} \\ &\quad + \tau^{-1}\left((\overline{U}_{\alpha,i,m-1})_{\mathtt{J}} - (\overline{U}_{\alpha,i,m-1})_{\mathtt{GS}}\right), \quad i \in \mathcal{I}, \\ \overline{W}^{(n)}_{\alpha,i,m} = \mathbf{0}, \quad i \in \partial\mathcal{I}, \quad m \geq 1, &\quad \overline{W}_{\alpha,i,0} = \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2. \end{aligned} \tag{6.36}$$

Taking into account that $\eta = 0, 1$, $(A^\tau_{\alpha,i,1} + C_{\alpha,i,1})^{-1} > O$ from (4.6), $L_{\alpha,i,1} > O$, $R_{\alpha,i,1} > O$, $i \in \mathcal{I}$ from (5.41b), $(\overline{U}^{(0)}_{\alpha,i,1})_{\mathtt{GS}} = (\overline{U}^{(0)}_{\alpha,i,1})_{\mathtt{J}}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, for $n = 1$ in (6.36), on the first time level $m = 1$, we conclude that

$$\overline{W}^{(1)}_{\alpha,i,1} \geq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2.$$

Similarly, from here and (6.36) with $n = 2$, by using (6.9), we obtain that $\overline{W}^{(2)}_{\alpha,i,1} \geq \mathbf{0}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$. By induction on n, we can prove that $\overline{W}^{(n)}_{\alpha,i,1} \geq \mathbf{0}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$.

On the second time level $m = 2$, taking into account that $(A^{\tau}_{\alpha,i,2} + C_{\alpha,i,2})^{-1} > O$ from (4.6), $L_{\alpha,i,2} > O$, $R_{\alpha,i,2} > O$, $i \in \mathcal{I}$ from (5.41b), $\overline{W}^{(0)}_{\alpha,i,2} = \mathbf{0}$ and $\overline{W}_{\alpha,i,1} \geq \mathbf{0}$, $i \in \overline{\Omega}^h$, $\alpha = 1, 2$, from (6.36) and using (6.9), we have

$$\overline{W}^{(1)}_{\alpha,i,2} \geq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2.$$

Similarly, from here and (6.36) with $n = 2$, by using (6.9), on the second time level $m = 2$, we obtain that $\overline{W}^{(2)}_{\alpha,i,2} \geq \mathbf{0}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$. By induction on n, we can prove that $\overline{W}^{(n)}_{\alpha,i,2} \geq \mathbf{0}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$.

By induction on $m \geq 1$, we can prove that

$$\overline{W}^{(n)}_{\alpha,i,m} \geq \mathbf{0}, \quad i \in \overline{\mathcal{I}}, \quad \alpha = 1, 2, \quad m \geq 1.$$

Thus, we prove (6.35) for upper solutions. By the same manner, we can prove (6.35) for lower solutions. □

6.2.2 Quasi-monotone nonincreasing case

Theorem 6.2.2. *Let* $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ *and* $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,i,m})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, *be ordered upper and lower solutions (6.21) of the nonlinear difference scheme (6.3). Suppose that* f_α, $\alpha = 1, 2$, *in (5.1) satisfy (5.19) and (5.30). The sequences* $\left\{(\overline{U}^{(n)}_{1,i,m})_J, (\underline{U}^{(n)}_{2,i,m})_J\right\}$, $\left\{(\underline{U}^{(n)}_{1,i,m})_J, (\overline{U}^{(n)}_{2,i,m})_J\right\}$ *and* $\left\{(\overline{U}^{(n)}_{1,i,m})_{GS}, (\underline{U}^{(n)}_{2,i,m})_{GS}\right\}$, $\left\{(\underline{U}^{(n)}_{1,i,m})_{GS}, (\overline{U}^{(n)}_{2,i,m})_{GS}\right\}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, *are, respectively, the sequences generated by the block monotone Jacobi method with* $\eta = 0$ *in (6.23), and the block monotone Gauss–Seidel method with* $\eta = 1$ *in (6.23), where* $(\overline{U}^{(0)}_{\alpha,i,m})_J = (\overline{U}^{(0)}_{\alpha,i,m})_{GS} = \widetilde{U}_{\alpha,i,m}$ *and* $(\underline{U}^{(0)}_{\alpha,i,m})_J = (\underline{U}^{(0)}_{\alpha,i,m})_{GS} = \widehat{U}_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$. *Then the inequalities in (6.35) hold true.*

Proof. The proof of the theorem repeats the proof of Theorem 6.2.1, where $\Gamma_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, are now written in the form

$$\Gamma_{\alpha,i,m}(\overline{U}^{(n)}_{\alpha,i,m}, \underline{U}^{(n)}_{\alpha',i,m}) = C_{\alpha,i,m}\overline{U}^{(n)}_{\alpha,i,m} - F_{\alpha,i,m}(\overline{U}^{(n)}_{\alpha,i,m}, \underline{U}^{(n)}_{\alpha',i,m}),$$
$$\Gamma_{\alpha,i,m}(\underline{U}^{(n)}_{\alpha,i,m}, \overline{U}^{(n)}_{\alpha',i,m}) = C_{\alpha,i,m}\underline{U}^{(n)}_{\alpha,i,m} - F_{\alpha,i,m}(\underline{U}^{(n)}_{\alpha,i,m}, \overline{U}^{(n)}_{\alpha',i,m}),$$

and the monotone property (6.22) for $\Gamma_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, is in use. □

6.3 Convergence analysis of the block monotone iterative methods

6.3.1 Quasi-monotone nondecreasing reaction functions

A stopping test for the block monotone iterative methods (6.10) is chosen in the form

$$\max_{\alpha=1,2}\left[\max_{i\in\mathcal{I}}\left|\mathcal{K}_{\alpha,i,m}(U^{(n)}_{\alpha,i,m},U_{\alpha,i,m-1},U^{(n)}_{\alpha',i,m})\right|\right]\leq\delta, \tag{6.37}$$

where $\mathcal{K}_{\alpha,i,m}(U^{(n)}_{\alpha,i,m},U_{\alpha,i,m-1},U^{(n)}_{\alpha',i,m})$, $i\in\mathcal{I}$, $\alpha'\neq\alpha$, $\alpha,\alpha'=1,2$, $m\geq1$, are residuals of the nonlinear difference scheme (6.3), $U^{(n)}_{\alpha,i,m}$, $i\in\mathcal{I}$, $\alpha=1,2$, $m\geq1$, are generated by (6.10), and δ is a prescribed accuracy. On each time level $m\geq1$, we set up $U_{\alpha,i,m}=U^{(n_m)}_{\alpha,i,m}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, $m\geq1$, such that n_m is the minimal number of iterations subject to (6.37).

Theorem 6.3.1. *Let $\widetilde{U}_{\alpha,i,m}$ and $\widehat{U}_{\alpha,i,m}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, $m\geq1$, be ordered upper and lower solutions (6.4) of (6.3). Suppose that the functions f_α, $\alpha=1,2$, satisfy (5.82) and (5.89). Assume that assumption (5.76) on the time step τ holds, where $q_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\geq1$, from (5.82). Then for sequences $\{\overline{U}^{(n)}_{1,i,m},\overline{U}^{(n)}_{2,i,m}\}$ and $\{\underline{U}^{(n)}_{1,i,m},\underline{U}^{(n)}_{2,i,m}\}$, $i\in\overline{\mathcal{I}}$, $m\geq1$, generated by (6.10), (6.37) with*

$$(\overline{U}^{(0)}_{1,i,m},\overline{U}^{(0)}_{2,i,m})=(\widetilde{U}_{1,i,m},\widetilde{U}_{2,i,m}),\quad(\underline{U}^{(0)}_{1,i,m},\underline{U}^{(0)}_{2,i,m})=(\widehat{U}_{1,i,m},\widehat{U}_{2,i,m}),\quad i\in\overline{\mathcal{I}},\quad m\geq1,$$

the following estimates hold

$$\max_{m\geq1}\max_{\alpha=1,2}\|U_{\alpha,m}-U^*_{\alpha,m}\|_{\overline{\Omega}^h}\leq T\delta, \tag{6.38}$$

where $U_{\alpha,i,m}=U^{(n_m)}_{\alpha,i,m}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, $m\geq1$, and $U^*_{\alpha,i,m}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, $m\geq1$, are the unique solutions to the nonlinear difference scheme (6.3).

Proof. The proof of the theorem repeats the proof of Theorem 5.7.2 from Chapter 5 with $U_{\alpha,i,m}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, $m\geq1$, rather than $U_{\alpha,ij,m}$, $(i,j)\in\overline{\Omega}^h$, $\alpha=1,2$, $m\geq1$. □

Theorem 6.3.2. *Let the assumptions in Theorem 6.3.1 be satisfied. Then for the sequence of solutions $\{U^{(n)}_{\alpha,i,m}\}$, $i\in\overline{\mathcal{I}}$, $\alpha=1,2$, $m\geq1$, generated by (6.10), (6.37), the following estimate holds*

$$\max_{m\geq1}\max_{\alpha=1,2}\|U_{\alpha,m}-u^*_{\alpha,m}\|_{\overline{\omega}^h}\leq T(\delta+\max_{m\geq1}E_m), \tag{6.39}$$

$$E_m=\max_{\alpha=1,2}\|E_{\alpha,m}\|_{\overline{\omega}^h},\quad m\geq1,$$

where $U_{\alpha,i,m} = U^{(n_m)}_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, $m \geq 1$, $u^*_\alpha(x,y,t)$, $\alpha = 1,2$, *are the exact solutions to (5.1), and* $E_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, $m \geq 1$, *are the truncation errors of the exact solutions on the nonlinear difference scheme (5.11).*

Proof. The proof of the theorem repeats the proof of Theorem 5.7.3 from Chapter 5 with $U_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, $m \geq 1$, rather than $U_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$. □

6.3.2 Quasi-monotone nonincreasing case

Stopping tests for the sequences $\{\overline{U}^{(n)}_{1,i,m}, \underline{U}^{(n)}_{2,i,m}\}$ and $\{\underline{U}^{(n)}_{1,i,m}, \overline{U}^{(n)}_{2,i,m}\}$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, generated by the block monotone iterative methods (6.23), are chosen in the forms

$$\max\left\{\max_{i\in\mathcal{I}} \mathcal{K}_{1,i,m}(\overline{U}^{(n)}_{1,i,m}, \overline{U}_{1,i,m-1}, \underline{U}^{(n)}_{2,i,m}); \max_{i\in\mathcal{I}} \mathcal{K}_{2,i,m}(\overline{U}^{(n)}_{1,i,m}, \underline{U}_{2,i,m-1}, \underline{U}^{(n)}_{2,i,m})\right\} \leq \delta, \tag{6.40}$$
$$\max\left\{\max_{i\in\mathcal{I}} \mathcal{K}_{1,i,m}(\underline{U}^{(n)}_{1,i,m}, \underline{U}_{1,i,m-1}, \overline{U}^{(n)}_{2,i,m}); \max_{i\in\mathcal{I}} \mathcal{K}_{2,i,m}(\underline{U}^{(n)}_{1,i,m}, \overline{U}_{2,i,m-1}, \overline{U}^{(n)}_{2,i,m})\right\} \leq \delta,$$

where $\mathcal{K}_{\alpha,i,m}(U^{(n)}_{\alpha,i,m}, U_{\alpha,i,m-1}, U^{(n)}_{\alpha',i,m})$, $i \in \mathcal{I}$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1,2$, are residuals of the nonlinear difference scheme (6.3), which are defined in (6.4), and δ is a prescribed accuracy. On each time level $m \geq 1$, we set up

$$(\overline{U}_{1,i,m}, \underline{U}_{2,i,m}) = (\overline{U}^{(n_m)}_{1,i,m}, \underline{U}^{(n_m)}_{1,i,m}), \quad (\underline{U}_{1,i,m}, \overline{U}_{2,i,m}) = (\underline{U}^{(n_m)}_{1,i,m}, \overline{U}^{(n_m)}_{1,i,m}), \quad i \in \overline{\mathcal{I}}, \quad m \geq 1,$$

such that n_m is the minimal number of iterations subject to (6.40).

Theorem 6.3.3. *Let* $\widetilde{U}_{\alpha,i,m}$ *and* $\widehat{U}_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, $m \geq 1$, *be ordered upper and lower solutions (6.21) of (6.3). Suppose that the functions* f_α, $\alpha = 1,2$, *satisfy (5.82) and (5.89). Assume that assumption (5.76) on the time step* τ *holds, where* $q_{\alpha,ij,m}$, $(i,j) \in \overline{\Omega}^h$, $\alpha = 1,2$, $m \geq 1$, *from (5.82). Then for sequences* $\{\overline{U}^{(n)}_{1,i,m}, \underline{U}^{(n)}_{2,i,m}\}$ *and* $\{\underline{U}^{(n)}_{1,i,m}, \overline{U}^{(n)}_{2,i,m}\}$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, *generated by (6.23), (6.40) with*

$$(\overline{U}^{(0)}_{1,i,m}, \underline{U}^{(0)}_{2,i,m}) = (\widetilde{U}_{1,i,m}, \widehat{U}_{2,i,m}), \quad (\underline{U}^{(0)}_{1,i,m}, \overline{U}^{(0)}_{2,i,m}) = (\widehat{U}_{1,i,m}, \widetilde{U}_{2,i,m}), \quad i \in \overline{\mathcal{I}}, \quad m \geq 1,$$

the following estimates hold

$$\max_{m\geq 1}\left\{\max\left[\|\overline{U}_{1,m} - U^*_{1,m}\|_{\overline{\Omega}^h}; \|\underline{U}_{2,m} - U^*_{2,m}\|_{\overline{\Omega}^h}\right]\right\} \leq T\delta, \tag{6.41}$$
$$\max_{m\geq 1}\left\{\max\left[\|\underline{U}_{1,m} - U^*_{1,m}\|_{\overline{\Omega}^h}; \|\overline{U}_{2,m} - U^*_{2,m}\|_{\overline{\Omega}^h}\right]\right\} \leq T\delta,$$

where $U_{\alpha,i,m} = U^{(n_m)}_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, $m \geq 1$, *and* $U^*_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1,2$, $m \geq 1$, *are the unique solutions to the nonlinear difference scheme (5.11).*

Proof. The proof of the theorem repeats the proof of Theorem 5.7.4 from Chapter 5 with $U_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, rather than $U_{\alpha,ij,m}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \geq 1$. □

Theorem 6.3.4. *Let the assumptions in Theorem 6.3.3 be satisfied. Then for sequences* $\{\overline{U}^{(n)}_{1,i,m}, \underline{U}^{(n)}_{2,i,m}\}$ *and* $\{\underline{U}^{(n)}_{1,i,m}, \overline{U}^{(n)}_{2,i,m}\}$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, *generated by (6.23), (6.40) with*

$$(\overline{U}^{(0)}_{1,i,m}, \underline{U}^{(0)}_{2,i,m}) = (\widetilde{U}_{1,i,m}, \widehat{U}_{2,i,m}), \quad (\underline{U}^{(0)}_{1,i,m}, \overline{U}^{(0)}_{2,i,m}) = (\widehat{U}_{1,i,m}, \widetilde{U}_{2,i,m}), \quad i \in \overline{\mathcal{I}}, \quad m \geq 1,$$

the following estimates hold

$$\max_{m\geq 1} \max \left[\|\overline{U}_{1,m} - u^*_{1,m}\|_{\overline{\Omega}^h}, \ \|\underline{U}_{2,m} - u^*_{2,m}\|_{\overline{\Omega}^h} \right] \leq T(\delta + \max_{m\geq 1} E_m),$$
$$\max_{m\geq 1} \max \left[\|\underline{U}_{1,m} - u^*_{1,m}\|_{\overline{\Omega}^h}, \ \|\overline{U}_{2,m} - u^*_{2,m}\|_{\overline{\Omega}^h} \right] \leq T(\delta + \max_{m\geq 1} E_m),$$
$$E_m = \max_{\alpha=1,2} \|E_{\alpha,m}\|_{\overline{\Omega}^h}, \quad m \geq 1,$$

where $u^*_\alpha(x, y, t)$, $\alpha = 1, 2$, *are the exact solutions to (5.1), and* $E_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, *are the truncation errors of the exact solutions on the nonlinear difference scheme (5.11).*

Proof. The proof of the theorem repeats the proof of Theorem 5.7.5 from Chapter 5 with $U_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, rather than $U_{\alpha,ij,m}$, $(i, j) \in \overline{\Omega}^h$, $\alpha = 1, 2$, $m \geq 1$. □

6.4 Construction of initial upper and lower solutions

In Section 5.8, for quasi-monotone nondecreasing and quasi-monotone nonincreasing reaction functions, we develop the methods of construction of initial upper and lower solutions in the cases of bounded reaction functions and constant initial iterates.

Since these methods depend on only properties of corresponding reaction functions f_α, $\alpha = 1, 2$, hence, the constructed initial iterates from Section 5.8 can be used as starting iterates for the block monotone iterative methods (6.10) and (6.23).

6.5 Numerical experiments

We present numerical experiments, implemented by the block monotone Jacobi and Gauss-Seidel methods, for test problems with quasi-monotone nondecreasing (5.20) and quasi-monotone nonincreasing (5.30) reaction functions f_α, $\alpha = 1, 2$, in (5.1). Exact solutions for our test problems are unknown, and numerical solutions are compared to corresponding reference solutions. The approximate solutions $U^{(n_m)}_{\alpha,i,m}$, $i \in \overline{\mathcal{I}}$, $\alpha = 1, 2$, $m \geq 1$, are generated by either the block monotone methods (6.10), (6.37) or the block monotone methods (6.23), (6.40). In our tests, we choose the reference solutions with

$N = 256$ and $\delta = 10^{-5}$ in (4.29) and (4.31). The reference solutions are calculated by the corresponding block method.

6.5.1 Quasi-monotone nondecreasing case

Test 1

As the first test problem with quasi-monotone nondecreasing reaction functions (5.20), we consider the Volterra-Lotka competition model from Section 5.3.1.1 with the same data sets. We choose the initial iteration $(\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m}) = (\{\mathbf{1}\}, \{\mathbf{1}\})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, where $\{\mathbf{1}\}$ is the vector with $\mathcal{I}$ components of ones and calculate sequences of upper solutions generated by the block monotone iterative method (6.10), (6.37).

In Table 6.1, for different values of N, $T = 2$ and $\tau = 0.01$, we give numbers of iterations n_δ and execution (CPU) times for the block monotone iterative methods and for the point monotone iterative methods from Table 5.1. From these results, we conclude that the block monotone Gauss-Seidel method converges faster than the block monotone Jacobi method, which confirms Theorem 6.2.1; the block monotone Gauss–Seidel method is approximately twice as fast as the block monotone Jacobi method and the block monotone methods converge faster than the corresponding point monotone methods.

Table 6.1: Average numbers of iterations n_δ and CPU times for Test 1.

N	8	16	32	64	128
the block Jacobi method					
n_δ	9.12	23.18	82.89	321.67	1797.83
CPU(s)	0.15	0.80	10.92	181.16	824.01
the block Gauss–Seidel method					
n_δ	6.22	14.14	38.20	158.08	894.23
CPU(s)	0.15	0.49	5.07	84.83	437.52
the point Jacobi method					
n_δ	11.98	35.88	135.27	533.09	2958.82
CPU(s)	0.13	0.91	13.42	212.16	1287.19
the point Gauss-Seidel method					
n_δ	6.99	19.50	69.27	268.10	1680.77
CPU(s)	0.12	0.56	7.34	115.24	733.43

Test 2

As the second test problem with quasi-monotone nondecreasing reaction functions (5.20), we consider Test 2 from Section 5.9.1 with the same data sets. We choose the initial iteration $(\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m}) = (\{\mathbf{1}\}, \{\mathbf{1}\})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$ and calculate sequences of upper solutions generated by the block monotone iterative method (6.10), (6.37).

In Table 6.2, for different values of N, $T = 0.5$, $\tau = 0.01$, we give numbers of iterations n_δ and execution (CPU) times for the block monotone iterative methods and for the point monotone iterative methods from Table 5.2. From these results, we conclude that the block monotone Gauss-Seidel method converges faster than the block monotone Jacobi method, which confirms Theorem 6.2.1; the block monotone Gauss–Seidel method is approximately twice as fast as the block monotone Jacobi method when the number of mesh points N is higher and the block monotone methods converge faster than the corresponding point monotone methods. Also from Table 6.2, it can be noticed that the data for the block Jacobi method are very close to the data of the point Gauss-Seidel method.

In Figure 6.1, we show the convergence of numerical solutions, obtained by the block Gauss-Seidel method with $\eta = 1$ in (6.10) and $N = 64$ to the reference solution $N_{ref} = 256$, where the dashed line represents the numerical solution and the solid blue line refers to the reference solution with respect to x and fixed value of $y = 0.5$. In subgraph 6.1a, starting from the initial lower solution $\widehat{U}_{1,i,5} = \{\mathbf{0}\}$, $i \in \mathcal{I}$, on the time level $t_5 = 0.05$, we show the convergence of the numerical lower solutions $\underline{U}^{(n)}_{1,i,5}$, $i \in \mathcal{I}$, at $n = 200$ to the reference solution. Similarly, starting from the initial upper solution $\widetilde{U}_{1,i,5} = \{\mathbf{1}\}$, $i \in \mathcal{I}$, on the time level $t_5 = 0.05$, subgraph 6.1b shows the convergence of the numerical upper solutions $\overline{U}^{(n)}_{1,i,5}$, $i \in \mathcal{I}$, at $n = 100$ to the reference solution.

Table 6.2: Average numbers of iterations n_δ and CPU times for Test 2.

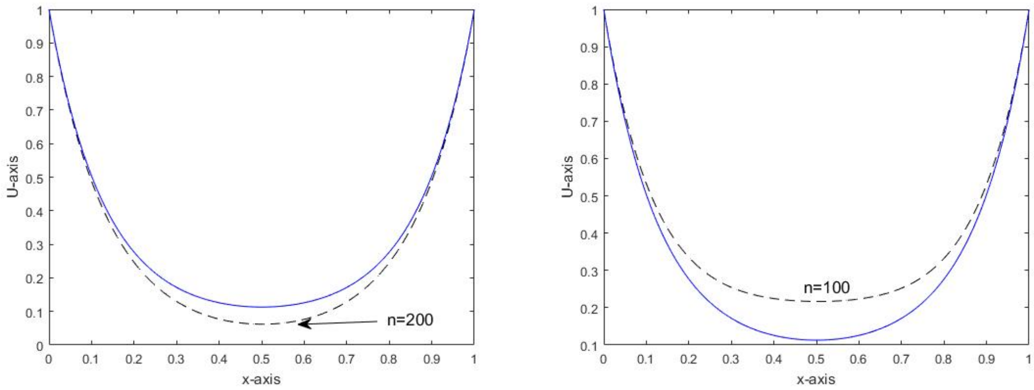
N	8	16	32	64	128
		the block Jacobi method			
n_δ	5.76	12.20	29.36	99.66	379.66
CPU(s)	0.07	0.12	0.88	12.31	185.05
		the block Gauss–Seidel method			
n_δ	4.62	9.54	21.16	40.28	181.54
CPU(s)	0.04	0.09	0.72	5.01	92.23
		the point Jacobi method			
n_δ	7.62	17.36	52.80	193.92	752
CPU(s)	0.07	0.13	1.44	20.31	325.65
		the point Gauss–Seidel method			
n_δ	5.86	11.24	29.46	99.78	379.78
CPU(s)	0.06	0.09	0.82	11.26	173.81

6.5.2 Quasi-monotone nonincreasing case

Test 3

As the first test problem with quasi-monotone nonincreasing reaction functions

Figure 6.1: Convergence of lower and upper solutions to the reference solution for Test 2.



(a) Lower solutions. (b) Upper solutions.

(5.30), we consider Test 3 from Section 5.9.2 with the same data sets. We choose the initial iteration $(\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m}) = (\{\mathbf{1}\}, \{\mathbf{0}\})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$ and calculate the sequence $\{\overline{U}^{(n)}_{1,i,m}, \underline{U}^{(n)}_{2,i,m}\}$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, generated by the block monotone iterative method (6.23), (6.40).

In Table 6.3, for different values of N, $T = 1$ and $\tau = 0.01$, we give average numbers of iterations n_δ and execution (CPU) times for the block monotone iterative methods (6.23) and for the point monotone iterative methods from Table 5.3. From these results, we conclude that the block monotone Gauss-Seidel method converges faster than the block monotone Jacobi method, which confirms Theorem 6.2.2; the block monotone Gauss–Seidel method is approximately twice as fast as the block monotone Jacobi method when we have higher number of mesh points N and the block monotone methods converge faster than the corresponding point monotone methods. Also from Table 6.3, it can be noticed that the data for the block Jacobi method are very close to the data of the point Gauss-Seidel method.

Test 4

As the second test problem with quasi-monotone nonincreasing reaction functions (5.30), we consider Test 4 from Section 5.9.2 with the same data sets. We choose the initial iteration $(\widetilde{U}_{1,i,m}, \widetilde{U}_{2,i,m}) = (\{\mathbf{1}\}, \{\mathbf{0}\})$, $i \in \overline{\mathcal{I}}$, $m \geq 1$ and calculate the sequence $\{\overline{U}^{(n)}_{1,i,m}, \underline{U}^{(n)}_{2,i,m}\}$, $i \in \overline{\mathcal{I}}$, $m \geq 1$, generated by the block monotone iterative method (6.23), (6.40).

In Table 6.4, for different values of N, $T = 0.5$, $\tau = 0.01$, we give numbers of iterations n_δ and execution (CPU) times for the block monotone iterative methods and for the point monotone iterative methods from Table 5.4. From these results, we conclude that the block monotone Gauss-Seidel method converges faster than the block

Table 6.3: Average numbers of iterations n_δ and CPU times for Test 3.

N	8	16	32	64	128
the block Jacobi method					
n_δ	9.33	27.17	100.04	391.75	1559.75
CPU(s)	0.10	0.39	5.52	86.52	737.94
the block Gauss–Seidel method					
n_δ	8.53	16.58	46.09	192.55	776.80
CPU(s)	0.07	0.24	2.66	44.55	397.53
the point Jacobi method					
n_δ	15.34	50.83	196.43	779.99	3115.91
CPU(s)	0.15	0.66	9.64	155.46	1612.87
the point Gauss–Seidel method					
n_δ	9.21	27.16	100.06	391.93	1624.43
CPU(s)	0.08	0.37	5.19	80.32	741.89

monotone Jacobi method, which confirms Theorem 6.2.2; the block monotone Gauss–Seidel method is approximately twice as fast as the block monotone Jacobi method when we have higher number of mesh points N and the block monotone methods converge faster than the corresponding point monotone methods. Also from Table 6.4, it can be noticed that the data for the block Jacobi method are very close to the data of the point Gauss-Seidel method.

Table 6.4: Average numbers of iterations n_δ and CPU times for Test 4.

N	8	16	32	64	128
the block Jacobi method					
n_δ	12.74	39.68	146.32	572.32	2276.06
CPU(s)	0.09	0.29	4.23	65.75	1047.561
the block Gauss–Seidel method					
n_δ	10.10	22.71	67.42	281.08	1133.24
CPU(s)	0.07	0.18	1.92	32.40	530.07
the point Jacobi method					
n_δ	21.14	74.58	287.66	1139.54	4547.02
CPU(s)	0.09	0.49	7.14	112.98	1889.27
the point Gauss–Seidel method					
n_δ	12.70	39.66	146.32	572.46	2276.22
CPU(s)	0.07	0.27	3.77	57.85	942.17

6.6 Conclusions to Chapter 6

Theoretical results

For solving nonlinear parabolic systems with quasi-monotone nondecreasing and nonincreasing reaction functions, we construct and investigate monotone properties of block Jacobi and block Gauss-Seidel iterative methods. For solving the nonlinear difference scheme (6.3) with quasi-monotone nondecreasing and nonincreasing reaction functions, the block Jacobi and block Gauss-Seidel iterative methods are constructed. In Theorems 6.1.5 and 6.1.9, on each time level, we prove that the sequences of upper and lower solutions, generated by the block monotone iterative methods for problems with quasi-monotone nondecreasing (5.20) and quasi-monotone nonincreasing (5.30) reaction functions, converge monotonically. The existence and uniqueness of a solution of the nonlinear difference scheme (6.3) are proved in Chapter 5. Taking into account the fact that on each time level, in general, the nonlinear discrete problems can be solved only inexactly, we introduce the stopping tests on each time level. By using the stopping tests (6.37) and (6.40), respectively, for the quasi-monotone nondecreasing and nonincreasing cases, we prove that the numerical solution converges to the unique solution of the nonlinear parabolic problem (5.1) and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme (6.3) in Theorems 6.3.1 and 6.3.3, and the error between the numerical solution and the exact solution of the parabolic problem (5.1) in Theorems 6.3.2 and 6.3.4. The construction methods of initial iterates from Section 5.8.2 depend only on properties of corresponding reaction functions and can be used as starting iterates for the block iterative methods (6.10) and (6.23).

Numerical results

The numerical sequences of upper and lower solutions, generated by the block monotone iterative methods (6.10) and (6.23) with the stopping tests (6.37) and (6.40), respectively, for the quasi-monotone nondecreasing and nonincreasing cases, converge monotonically. The block monotone Gauss-Seidel methods with $\eta = 1$ in (6.10) and $\eta = 1$ in (4.13) converge faster than the block monotone Jacobi methods with $\eta = 0$ in (6.10) and $\eta = 0$ in (6.23) which confirm, respectively, Theorems 6.2.1 and 6.2.2. The block Gauss-Seidel methods are approximately twice as fast as the block Jacobi methods. The block monotone methods converge faster than the corresponding point monotone methods. The number of iteration n_δ and execution `CPU` time for the block Jacobi methods are very close to the data for the point Gauss-Seidel method.

Chapter 7

Conclusion

In Chapter 1, we review nonlinear elliptic and parabolic problems. Nonlinear difference schemes which approximate elliptic and parabolic problems are presented. For the linear versions of the difference problems, we prove the maximum principle and error estimation. For elliptic and parabolic problems, the iterative methods for solving the nonlinear difference schemes, are constructed. The monotone property of the sequences of solutions, generated by the monotone iterative methods, are proved. Existence and uniqueness of solutions of the nonlinear elliptic and parabolic difference schemes are given. The error between the numerical and exact solutions of the nonlinear difference schemes, for elliptic and parabolic cases, are estimated. Linear and quadratic rates of convergence of the iterative sequences of upper and lower solutions, are discussed.

In Chapter 2, the nonlinear difference scheme for approximating the elliptic problems is presented. For solving the nonlinear difference scheme, the point Jacobi and point Gauss-Seidel iterative methods are constructed. The monotone properties of the sequences of upper and lower solutions, generated by the point iterative methods, are proved. The uniqueness of a solution of the nonlinear difference scheme is given. By using the stopping test, we prove that the numerical solution converges to the unique solution of the nonlinear elliptic problem and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme and the error between the numerical solution and the exact solution of the elliptic problem. We prove that the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method. Initial upper and lower solutions to start the point monotone iterative methods are constructed.

From the numerical experiments, we conclude i) the numerical solution converges to the reference solution with second order accuracy; ii) the numerical sequences of upper and lower solutions, generated by the point monotone methods, converge monotonically; iii) the point monotone Gauss-Seidel method converges faster than the point monotone Jacobi method; iv) the block monotone methods from [61] converge faster than the corresponding point monotone methods.

In Chapter 3, we construct and investigate the point monotone Jacobi and Gauss-Seidel methods for solving nonlinear systems of elliptic differential equations. The two classes of coupled elliptic systems with quasi-monotone nondecreasing and quasi-monotone nonincreasing reaction functions are considered. We present the nonlinear difference scheme which approximates the nonlinear elliptic systems. We prove the monotone properties of the sequences of upper and lower solutions, generated by the point iterative methods for the quasi-monotone nondecreasing and nonincreasing cases. The existence and uniqueness of a solution of the nonlinear difference scheme with quasi-monotone nondecreasing and quasi-monotone nonincreasing reaction functions are proved. By using the stopping tests, based on the norms of the residuals of the nonlinear difference scheme, we prove that the numerical solution converges to the unique solution of the nonlinear elliptic problem and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme and the error between the numerical solution and the exact solution of the elliptic system. We prove that the point monotone Gauss-Seidel methods converge faster than the point monotone Jacobi methods for the quasi-monotone nondecreasing and quasi-monotone nonincreasing cases. Constructions of initial upper and lower solutions to start the point monotone iterative methods are presented.

From the numerical experiments, we conclude i) the numerical solution of the nonlinear difference scheme converges to the reference solution with second-order accuracy; ii) the numerical sequences of upper and lower solutions, generated by the point monotone methods, converge monotonically; iii) the point monotone Gauss-Seidel methods converge faster than the point monotone Jacobi methods; iv) the point monotone Gauss-Seidel methods are approximately twice as fast as the point monotone Jacobi methods.

In Chapter 4, we construct and investigate the block monotone Jacobi and Gauss-Seidel methods for solving nonlinear systems of elliptic differential equations. The two classes of coupled elliptic systems with quasi-monotone nondecreasing and nonincreasing reaction functions are considered. The block monotone iterative methods are based on the decomposition technique which reduces a domain into a series of nonoverlapping one dimensional intervals by slicing the domain into a finite number of thin strips, and then solving a two-point boundary-value problem for each strip by a standard computational scheme such as the Thomas algorithm. The monotone properties of the sequences of upper and lower solutions, generated by the block monotone iterative methods are proved. By using the stopping tests, based on the norms of residuals, for the quasi-monotone nondecreasing and nonincreasing cases, we prove that the numerical solution converges to the unique solution of the nonlinear elliptic problem and estimate the L_∞

discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme and the error between the numerical solution and the exact solution of the elliptic system. For the quasi-monotone nondecreasing and nonincreasing cases, we prove that the block monotone Gauss-Seidel methods converge faster than the block monotone Jacobi methods. These theoretical results were published in [1].

From the numerical experiments, we conclude that i) the numerical sequences of solutions, generated by block monotone methods with the stopping tests, converge monotonically; ii) the block monotone Gauss-Seidel methods converge faster than the block monotone Jacobi methods; iii) the block monotone Gauss-Seidel methods are approximately twice as fast as the block Jacobi methods; iv) the block monotone methods converge faster than the corresponding point monotone methods; v) the numbers of iterations n_δ and CPU times for the block Jacobi methods are very close to the data for the point Gauss-Seidel methods; vi) when the convective terms dominate the diffusion terms, the block monotone Gauss-Seidel method with the one-sided difference approximations of the first partial derivatives are more efficient than the block monotone Gauss-Seidel method with the central difference approximations. The materials on Chapter 4 in the quasi-monotone nondecreasing case were published in [5] and for quasi-monotone nondecreasing and nonincreasing cases, the results are submitted for publication in [3].

In Chapter 5, for solving nonlinear systems of parabolic differential equations, we construct and investigate the point monotone Jacobi and Gauss-Seidel methods. The two classes of coupled parabolic systems with quasi-monotone nondecreasing and nonincreasing reaction functions are considered. We prove that, on each time level, the sequences of upper and lower solutions, generated by the point iterative methods, converge monotonically. The existence and uniqueness of a solution of the nonlinear difference scheme, for the quasi-monotone nondecreasing and nonincreasing cases, are proved. By using the stopping tests, based on the norms of residuals, for the quasi-monotone nondecreasing and nonincreasing reaction functions, we prove that the numerical solution converges to the unique solution of the nonlinear parabolic problem and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme, and the error between the numerical solution and the the exact solution of the parabolic problem. We prove that for the quasi-monotone nondecreasing and nonincreasing cases, the point monotone Gauss-Seidel methods converge faster than the point monotone Jacobi methods. For quasi-monotone nondecreasing and nonincreasing cases, on each time level, we construct initial upper and lower solutions to start the point monotone iterative methods.

From the numerical experiments, we conclude that i) the numerical sequences of upper and lower solutions, generated by the point monotone iterative methods, for the quasi-monotone nondecreasing and nonincreasing cases, on each time level, converge monotonically; ii) the point monotone Gauss-Seidel methods converge faster than the point monotone Jacobi methods; iii) the point monotone Gauss-Seidel methods are approximately twice as fast as the point monotone Jacobi methods.

In Chapter 6, we construct and investigate the block monotone Jacobi and Gauss-Seidel iterative methods for solving the nonlinear parabolic systems with quasi-monotone nondecreasing and nonincreasing reaction functions. We prove that on each time level, the sequences of upper and lower solutions, generated by the block monotone iterative methods, converge monotonically. By using the stopping tests, based on the norms of residuals, for the quasi-monotone nondecreasing and nonincreasing cases, we prove that the numerical solution converges to the unique solution of the nonlinear parabolic problem and estimate the L_∞ discrete-norm of the error between the numerical and exact solutions of the nonlinear difference scheme and the error between the numerical solution and the exact solution of the parabolic problem. These theoretical results were published in [2].

From the numerical experiments, we conclude that i) the numerical sequences of upper and lower solutions, generated by the block monotone iterative methods, for the quasi-monotone nondecreasing and nonincreasing cases, on each time level, converge monotonically; ii) the block monotone Gauss-Seidel methods converge faster than the block monotone Jacobi methods; iii) the block Gauss-Seidel methods are approximately twice as fast as the block Jacobi methods; iv) the block monotone methods converge faster than the corresponding point monotone methods; v) the average numbers of iterations and execution times for the block Jacobi methods are very close to the data for the point Gauss-Seidel method. The materials of this chapter for the quasi-monotone nondecreasing case has been accepted for publication in [6] and for both the quasi-monotone nondecreasing and nonincreasing cases, the results are submitted for publication in [4].

The main goal of the thesis is to develop numerical methods, based on the monotone point and block Jacobi and Gauss-Seidel iterative methods, for solving elliptic and parabolic equations and systems of equations.

The brief conclusions from our theoretical results, obtained in the thesis, are the following:

1. We prove that the iterative sequences of numerical solutions, generated by the point and block monotone iterative methods, converge monotonically.

2. We estimate the L_∞ discrete-norm between the numerical and exact solutions of the nonlinear difference schemes and the error between the numerical solution and the exact solution of the corresponding continuous problem.

3. The existence and uniqueness of a solution of the nonlinear difference schemes are proved.

4. We prove that the point and block monotone Gauss-Seidel methods converge faster than the corresponding point and block monotone Jacobi methods.

The brief findings from our numerical experiments, obtained in the thesis, are the following:

1. The numerical sequences of solutions, generated by the point and block monotone methods, converge monotonically.

2. The point and block monotone Gauss-Seidel methods converge faster than the point and block monotone Jacobi methods.

3. The point and block monotone Gauss-Seidel methods, respectively, are approximately twice as fast as the corresponding point and block monotone Jacobi methods.

4. The block monotone methods converge faster than the corresponding point monotone methods.

5. The numbers of iterations and execution times for the block Jacobi methods are very close to the data for the point Gauss-Seidel methods.

6. When the convective terms dominate the diffusion terms, the block monotone Gauss-Seidel method with the one-sided difference approximations of the first partial derivatives are more efficient than the block monotone Gauss-Seidel method with the central difference approximations.

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