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Stationary Distributions and Mean First Passage Times of Perturbed Markov Chains

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Abstract
Stationary distributions of perturbed finite irreducible discrete time Markov chains are intimately connected with the behaviour of associated mean first passage times. This interconnection is explored through the use of generalized matrix inverses. Some interesting qualitative results regarding the nature of the relative and absolute changes to the stationary probabilities are obtained together with some improved bounds.

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1. Introduction

Markov chains subjected to perturbations have received attention in the literature over recent years. The major interest has focussed on the effects of perturbations of the transition probabilities on the stationary distribution of the Markov chain with the derivation of bounds or changes, or relative changes, in the magnitude of the stationary probabilities. Recently sensitivity of the perturbation effects has been considered in terms of the mean first passage times of the original irreducible Markov chain. In this paper we provide further insights into this approach by giving some new derivations and examining some special cases in depth.

2. General perturbations and stationary distributions

Let $P^{(1)} = [p_{ij}^{(1)}]$ be the transition matrix of a finite irreducible, m-state Markov chain. Let $P^{(2)} = [p_{ij}^{(2)}] = P^{(1)} + \boldsymbol{E}$ be the transition matrix of the perturbed Markov chain where $\boldsymbol{E} = [\varepsilon_{ij}]$ is the matrix of perturbations. We assume that the perturbed Markov chain is also irreducible with the same state space $S = \{1, 2,\ldots, m\}$. For i = 1, 2, let $\boldsymbol{\pi}^{(i)\prime} = (\pi_1^{(i)}, \pi_2^{(i)},\ldots, \pi_m^{(i)})$ be the stationary probability vectors for the respective Markov chains.

There are a variety of techniques that we can use to obtain expressions for $\boldsymbol{\pi}^{(1)\prime}$ and $\boldsymbol{\pi}^{(2)\prime}$. In particular, in [9], the author used some generalised matrix inverse techniques to obtain separate expressions for $\boldsymbol{\pi}^{(1)\prime}$ and $\boldsymbol{\pi}^{(2)\prime}$ in the rank one update case when $\boldsymbol{E} = \mathbf{ab}'$ with $\mathbf{b}'\mathbf{e} = 0$.

Since we are interested in the effect that the perturbations $\boldsymbol{E} = [\varepsilon_{ij}]$ have on changes to the stationary probabilities, we use an approach that leads directly to an expression for the difference $\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime}$.

First observe that, since $\boldsymbol{\pi}^{(1)\prime}(I - P^{(1)}) = \mathbf{0}'$ and $\boldsymbol{\pi}^{(2)\prime}(I - P^{(2)}) = \boldsymbol{\pi}^{(2)\prime}(I - P^{(1)} - \boldsymbol{E}) = \mathbf{0}'$,

$$(\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime})(I - P^{(1)}) = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}. \tag{2.1}$$

Equation (2.1) consists of a system of linear equations. Generalized matrix inverses (g-inverses) have an important role in solving such equations. The relevant results (see e.g. [7] or [8]) that we shall make use of are the following:

2.1 A ‘one condition’ g-inverse or an ‘equation solving’ g-inverse of a matrix A is any matrix A^- such that $AA^-A = A$.

2.2 Let $P^{(1)}$ be the transition matrix of a finite irreducible Markov chain with stationary probability vector $\boldsymbol{\pi}^{(1)\prime}$. Let $\mathbf{e}' = (1, 1, \ldots, 1)$ and $\mathbf{t}$ and $\mathbf{u}$ be any vectors.
(a) $I - P^{(1)} + \mathbf{tu}'$ is non-singular if and only if $\boldsymbol{\pi}^{(1)\prime}\mathbf{t} \neq 0$ and $\mathbf{u}'\mathbf{e} \neq 0$.
(b) If $\boldsymbol{\pi}^{(1)\prime}\mathbf{t} \neq 0$ and $\mathbf{u}'\mathbf{e} \neq 0$ then $[I - P^{(1)} + \mathbf{tu}']^{-1}$ is a g-inverse of $I - P^{(1)}$.

2.3 All one condition g-inverses of $I - P^{(1)}$ are of the form $[I - P^{(1)} + \mathbf{tu}']^{-1} + \mathbf{ef}' + \mathbf{g}\boldsymbol{\pi}^{(1)\prime}$ for arbitrary vectors $\mathbf{f}$ and $\mathbf{g}$.

2.4 A necessary and sufficient condition for $\mathbf{x}'A = \mathbf{b}'$ to have a solution is $\mathbf{b}'A^-A = \mathbf{b}'$. If this consistency condition is satisfied the general solution is given by $\mathbf{x}' = \mathbf{b}'A^- + \mathbf{w}'(I - AA^-)$ where $\mathbf{w}'$ is an arbitrary vector.

2.5 The following results are easily established (see Section 3.3, [7])

(a) $\mathbf{u}'[I - P^{(1)} + \mathbf{tu}']^{-1} = \boldsymbol{\pi}^{(1)\prime}/(\boldsymbol{\pi}^{(1)\prime}\mathbf{t})$. (2.2)

(b) $[I - P^{(1)} + \mathbf{tu}']^{-1}\mathbf{t} = \mathbf{e}/(\mathbf{u}'\mathbf{e})$. (2.3)

(c) $[I - P^{(1)} + \mathbf{tu}']^{-1}(I - P^{(1)}) = I - \mathbf{eu}'/(\mathbf{u}'\mathbf{e})$. (2.4)

(d) $(I - P^{(1)})[I - P^{(1)} + \mathbf{tu}']^{-1} = I - \mathbf{t}\boldsymbol{\pi}^{(1)\prime}/(\boldsymbol{\pi}^{(1)\prime}\mathbf{t})$. (2.5)

2.6 Hunter, ([6]), established that Kemeny and Snell’s ‘fundamental matrix’, ([12]), $Z^{(1)} \equiv [I - P^{(1)} + \Pi^{(1)}]^{-1}$, where $\Pi^{(1)} = \mathbf{e}\boldsymbol{\pi}^{(1)\prime}$, is a one condition g-inverse of $I - P^{(1)}$. Meyer, ([14]), showed that the ‘group inverse’ $A^{\#(1)} \equiv Z^{(1)} - \Pi^{(1)}$ is also a g-inverse of $I - P^{(1)}$.

Theorem 2.1: *If* G *is any g-inverse of* $I - P^{(1)}$ *then, for any general perturbation* $\boldsymbol{E}$,

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,G(I - \Pi^{(1)}). \tag{2.6}$$

Proof: By taking G as the general form $[I - P^{(1)} + \mathbf{tu}']^{-1} + \mathbf{ef}' + \mathbf{g}\boldsymbol{\pi}^{(1)\prime}$ and using results from §2.5, the facts that $\Pi^{(1)} = \mathbf{e}\boldsymbol{\pi}^{(1)\prime}$, $\boldsymbol{\pi}^{(1)\prime}\mathbf{e} = 1$ and $P^{(1)}\mathbf{e} = \mathbf{e}$, it follows that

$$(I - P^{(1)})G(I - \Pi^{(1)}) = I - \Pi^{(1)}. \tag{2.7}$$

Thus, from (2.1) and (2.7),

$$(\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime})(I - \Pi^{(1)}) = (\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime})(I - P^{(1)})G(I - \Pi^{(1)}) = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,G(I - \Pi^{(1)}).$$

Further $(\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime})(I - \Pi^{(1)}) = (\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime})(I - \mathbf{e}\boldsymbol{\pi}^{(1)\prime}) = \boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime}$ and (2.6) follows.

An alternative approach to solving (2.1) is to use the results of §2.4 with $\mathbf{x}' = \boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime}$, $A = I - P^{(1)}$, $\mathbf{b}' = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}$ and $A^- = G$, as above. The consistency condition is satisfied and the general solution has the form $\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,G + \mathbf{w}'\{\mathbf{t}/(\boldsymbol{\pi}^{(1)\prime}\mathbf{t}) - (I - P^{(1)})\mathbf{g}\}\boldsymbol{\pi}^{(1)\prime}$ with $\mathbf{w}'$ arbitrary. Since $(\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime})\mathbf{e} = 0$, the arbitrariness of $\mathbf{w}'$ is eliminated with $\mathbf{w}'\{\mathbf{t}/(\boldsymbol{\pi}^{(1)\prime}\mathbf{t}) - (I - P^{(1)})\mathbf{g}\} = -\boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,G\,\mathbf{e}$ and equation (2.6) follows.

❑

Theorem 2.1 is a new result and all known results for the difference $\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime}$ can be obtained from this result. In particular we have the following special cases.

Corollary 2.1.1:

(i) *If* $G = [I - P^{(1)} + \mathbf{tu}']^{-1} + \mathbf{ef}' + \mathbf{g}\boldsymbol{\pi}^{(1)\prime}$ *with* $\boldsymbol{\pi}^{(1)\prime}\mathbf{t} \neq 0$, $\mathbf{u}'\mathbf{e} \neq 0$, $\mathbf{f}'$ *and* $\mathbf{g}$ *arbitrary vectors, then*

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,[I - P^{(1)} + \mathbf{tu}']^{-1}(I - \Pi^{(1)}). \tag{2.8}$$

(ii) *If* $G = [I - P^{(1)} + \mathbf{eu}']^{-1} + \mathbf{ef}' + \mathbf{g}\boldsymbol{\pi}^{(1)'}$ *with* $\boldsymbol{\pi}^{(1)'}\mathbf{t} \neq 0$, $\mathbf{u'e} \neq 0$, $\mathbf{f}'$ *and* $\mathbf{g}$ *arbitrary vectors, then*

$$\boldsymbol{\pi}^{(2)'} - \boldsymbol{\pi}^{(1)'} = \boldsymbol{\pi}^{(2)'}\boldsymbol{E}\,[I - P^{(1)} + \mathbf{eu}']^{-1}. \tag{2.9}$$

(iii) *If* $G = [I - P^{(1)} + \mathbf{eu}']^{-1} + \mathbf{ef}'$ *with* $\mathbf{u'e} \neq 0$, *and* $\mathbf{f}'$ *an arbitrary vector, then*

$$\boldsymbol{\pi}^{(2)'} - \boldsymbol{\pi}^{(1)'} = \boldsymbol{\pi}^{(2)'}\boldsymbol{E}\,G. \tag{2.10}$$

(iv) *If* $Z^{(1)}$ *is the 'fundamental matrix' of* $I - P^{(1)}$, *then*

$$\boldsymbol{\pi}^{(2)'} - \boldsymbol{\pi}^{(1)'} = \boldsymbol{\pi}^{(2)'}\boldsymbol{E}\,Z^{(1)}. \tag{2.11}$$

(v) *If* $A^{\#(1)}$ *is the 'group inverse' of* $I - P^{(1)}$, *then*

$$\boldsymbol{\pi}^{(2)'} - \boldsymbol{\pi}^{(1)'} = \boldsymbol{\pi}^{(2)'}\boldsymbol{E}\,A^{\#(1)}. \tag{2.12}$$

Proof: (i) For (2.8), substitution of G into (2.6) leads to the $\mathbf{ef}'$ term vanishing since $\boldsymbol{E}\mathbf{e} = \mathbf{0}$. Similarly the $\mathbf{g}\boldsymbol{\pi}^{(1)'}$ term cancels since $\boldsymbol{\pi}^{(1)'}(I - \mathbf{e}\boldsymbol{\pi}^{(1)'}) = \mathbf{0}'$.

(ii) Equation (2.9) follows from (2.8) upon substitution of $\mathbf{t} = \mathbf{e}$ since, from (2.3), $[I - P^{(1)} + \mathbf{eu}']^{-1}\mathbf{e} = \mathbf{e}/(\mathbf{u'e})$ and $\boldsymbol{E}\mathbf{e} = \mathbf{0}$.

(iii) Substitution of the form of G into (2.6) leads to the $\boldsymbol{\pi}^{(2)'}\boldsymbol{E}\,G\Pi^{(1)}$ term vanishing since $\boldsymbol{E}\,G\Pi^{(1)} = \boldsymbol{E}([I - P^{(1)} + \mathbf{eu}']^{-1} + \mathbf{ef}')\mathbf{e}\boldsymbol{\pi}^{(1)'} = \boldsymbol{E}\mathbf{e}\{1/(\mathbf{u'e}) + (\mathbf{f'e})\}\boldsymbol{\pi}^{(1)'} = \mathbf{0}$.

(iv) Equation (2.11) follows from (2.9) or (2.10) with $\mathbf{u}' = \boldsymbol{\pi}^{(1)'}$.

(v) Equation (2.12) follows from (2.10) with $\mathbf{u}' = \boldsymbol{\pi}^{(1)'}$ and $\mathbf{f}' = \boldsymbol{\pi}^{(1)'}$.

❑

The general result (2.8) is new. The other results, or special cases of them, appear in the literature but with ad hoc derivations. Result (2.11) was initially derived by Schweitzer ([18]). Result (2.12) is due to Meyer ([15]). A special case of results (2.9) and (2.10), (with $\mathbf{f}' = \mathbf{0}'$ and $\mathbf{g} = \mathbf{0}$), appears in Seneta [19], while result (2.10) appears in Seneta [21].

The results obtained by the author in [9], in the case that $\boldsymbol{E} = \mathbf{ab}'$ where $\mathbf{b'e} = \mathbf{0}$, can also be obtained from Corollary 2.1.1.

Corollary 2.1.2:

(i) *If* $\mathbf{u'e} \neq 0$, $\pi^{(1)'}\mathbf{t} \neq 0$, $\boldsymbol{\alpha}' = \mathbf{u}'[I - P^{(1)} + \mathbf{tu}']^{-1}$ *and* $\boldsymbol{\beta}' = \mathbf{b}'[I - P^{(1)} + \mathbf{tu}']^{-1}$, *then*

$$\boldsymbol{\pi}^{(1)'} = \frac{\boldsymbol{\alpha}'}{\boldsymbol{\alpha}'\mathbf{e}} \quad and \quad \boldsymbol{\pi}^{(2)'} = \frac{(1 - \boldsymbol{\beta}'\mathbf{a})\boldsymbol{\pi}^{(1)'} + (\boldsymbol{\pi}^{(1)}\mathbf{a})\boldsymbol{\beta}'}{(1 - \boldsymbol{\beta}'\mathbf{a}) + (\boldsymbol{\pi}^{(1)'}\mathbf{a})(\boldsymbol{\beta}'\mathbf{e})} = \frac{(1 - \boldsymbol{\beta}'\mathbf{a})\boldsymbol{\alpha}' + (\boldsymbol{\alpha}'\mathbf{a})\boldsymbol{\beta}'}{(1 - \boldsymbol{\beta}'\mathbf{a})(\boldsymbol{\alpha}'\mathbf{e}) + (\boldsymbol{\alpha}'\mathbf{a})(\boldsymbol{\beta}'\mathbf{e})}. \tag{2.13}$$

(ii) *If* $\mathbf{u'e} \neq 0$, $\boldsymbol{\pi}^{(1)'}\mathbf{a} \neq 0$, $\boldsymbol{\alpha}' = \mathbf{u}'[I - P^{(1)} + \mathbf{au}']^{-1}$ *and* $\boldsymbol{\beta}' = \mathbf{b}'[I - P^{(1)} + \mathbf{au}']^{-1}$, *then*

$$\boldsymbol{\pi}^{(1)'} = \frac{\boldsymbol{\alpha}'}{\boldsymbol{\alpha}'\mathbf{e}} \quad and \quad \boldsymbol{\pi}^{(2)'} = \frac{\boldsymbol{\alpha}' + \boldsymbol{\beta}'}{\boldsymbol{\alpha}'\mathbf{e} + \boldsymbol{\beta}'\mathbf{e}}. \tag{2.14}$$

(iii) *If* $\mathbf{u'e} \neq 0$, $\boldsymbol{\alpha}' = \mathbf{u}'[I - P^{(1)} + \mathbf{eu}']^{-1}$ *and* $\boldsymbol{\beta}' = \mathbf{b}'[I - P^{(1)} + \mathbf{eu}']^{-1}$, *then*

$$\boldsymbol{\pi}^{(1)'} = \boldsymbol{\alpha}' \quad and \quad \boldsymbol{\pi}^{(2)'} = \boldsymbol{\pi}^{(1)'} + \left(\frac{\boldsymbol{\alpha}'\mathbf{a}}{1 - \boldsymbol{\beta}'\mathbf{a}}\right)\boldsymbol{\beta}'. \tag{2.15}$$

(iv) If $Z^{(1)} = [I - P^{(1)} + \mathbf{e}\boldsymbol{\pi}^{(1)\prime}]^{-1}$ is the fundamental matrix of $I - P^{(1)}$, *then*

$$\boldsymbol{\pi}^{(2)\prime} = \boldsymbol{\pi}^{(1)\prime} + \left(\frac{\boldsymbol{\pi}^{(1)\prime}\mathbf{a}}{1-\boldsymbol{\beta}'\mathbf{a}}\right)\mathbf{b}'Z^{(1)} = \boldsymbol{\pi}^{(1)\prime} + \left(\boldsymbol{\pi}^{(1)\prime}\mathbf{a}\right)\mathbf{b}'Z^{(1)}. \tag{2.16}$$

Proof: (i) The expression for $\boldsymbol{\pi}^{(1)\prime}$ follows from (2.2) since $\boldsymbol{\alpha}' = \boldsymbol{\pi}^{(1)\prime}/(\boldsymbol{\pi}^{(1)\prime}\mathbf{t})$ and $\boldsymbol{\alpha}'\mathbf{e} = 1/(\boldsymbol{\pi}^{(1)\prime}\mathbf{t})$. Further, from (2.8),

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = (\boldsymbol{\pi}^{(2)\prime}\mathbf{a})\boldsymbol{\beta}'(I - \mathbf{e}\boldsymbol{\pi}^{(1)\prime}) = (\boldsymbol{\pi}^{(2)\prime}\mathbf{a})\boldsymbol{\beta}' - (\boldsymbol{\pi}^{(2)\prime}\mathbf{a})(\boldsymbol{\beta}'\mathbf{e})\boldsymbol{\pi}^{(1)\prime}. \tag{2.17}$$

Post-multiplication of (2.17) by $\mathbf{a}$ and solving for $\boldsymbol{\pi}^{(2)\prime}\mathbf{a}$ yields

$$\boldsymbol{\pi}^{(2)\prime}\mathbf{a} = \frac{\boldsymbol{\pi}^{(1)\prime}\mathbf{a}}{1-(\boldsymbol{\beta}'\mathbf{a}) + (\boldsymbol{\beta}'\mathbf{e})(\boldsymbol{\pi}^{(1)\prime}\mathbf{a})}. \tag{2.18}$$

Substitution for $\boldsymbol{\pi}^{(2)\prime}\mathbf{a}$ into (2.15) and solving for $\boldsymbol{\pi}^{(2)\prime}$ yields the expressions (2.13).

(ii) Follows from (2.13) with $\mathbf{t} = \mathbf{a}$ by noting that $\mathbf{a}'\mathbf{a} = 0$ and $\mathbf{b}'\mathbf{a} = 1$.

(iii) Follows from (2.13) with $\mathbf{t} = \mathbf{e}$ by noting that $\mathbf{a}'\mathbf{e} = 1$ and $\mathbf{b}'\mathbf{e} = 0$.

(iv) Follows from (2.15) with $\mathbf{u}' = \boldsymbol{\pi}^{(1)\prime}$, since $\mathbf{a}' = \boldsymbol{\pi}^{(1)\prime}\mathbf{Z}^{(1)} = \boldsymbol{\pi}^{(1)\prime}$ and $\mathbf{b}' = \mathbf{b}'Z^{(1)}$.

Note also from (2.18) that

$$\boldsymbol{\pi}^{(2)\prime}\mathbf{a} = \frac{\boldsymbol{\pi}^{(1)\prime}\mathbf{a}}{1-\mathbf{b}'Z^{(1)}\mathbf{a}}.$$

❑

For a summary on the current known results concerning absolute norm-wise error bounds on the differences between the two stationary probability vectors of the form

$$\| \boldsymbol{\pi}^{(1)} - \boldsymbol{\pi}^{(2)} \|_p \le \kappa_l \| \boldsymbol{E} \|_q$$

where (p,q) = (∞, ∞) or (1, ∞) depending on *l*, see Cho and Meyer [2]. They summarise and compare results due to Schweitzer [18], Meyer [15], Haviv and Van der Heyden [5], Kirkland et al. [13], Funderlic and Meyer [4], Meyer [16], Seneta [20], Seneta [21], Seneta [22], Ipsen and Meyer [11], Cho and Meyer [3].

Results for component-wise bounds of the form

$$| \pi_j^{(1)} - \pi_j^{(2)} | \le \kappa_l \| \boldsymbol{E} \|_\infty$$

and relative error bounds of the form

$$\left| \frac{\pi_j^{(1)} - \pi_j^{(2)}}{\pi_j^{(1)}} \right| \le \kappa_l \| \boldsymbol{E} \|_\infty$$

are also discussed in Cho and Meyer [2]. For relative error bounds see also O'Cinneide [17] and Xue [23].

3. General perturbations and mean first passage times

In Hunter [7], (see also [8]), a general technique for finding mean first passage times of a finite irreducible discrete time Markov chains, using generalised inverses, was developed. The key result is as follows:

Let $M^{(1)} = [m_{ij}^{(1)}]$ be the mean first passage time matrix of a finite irreducible, Markov chain with transition matrix $P^{(1)}$. If G is any generalised matrix inverse of $I - P^{(1)}$, then

$$M^{(1)} = [G\Pi^{(1)} - E(G\Pi^{(1)})_d + I - G + EG_d]D^{(1)}, \tag{3.1}$$

where $E = \mathbf{ee'} = [1]$ and $D^{(1)} = M_d^{(1)} = (\Pi_d^{(1)})^{-1}$.

The general result given by (2.6) in Theorem 2.1 expressing the difference $\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime}$, under general perturbations $\boldsymbol{E}$, lends itself to re-expression in terms of mean first passage times.

Theorem 3.1: *If* $M^{(1)}$ *is the mean first passage time matrix of the finite irreducible, Markov chain with transition matrix* $P^{(1)}$, *then for any general perturbation* $\boldsymbol{E}$ *of* $P^{(1)}$,

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = -\boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}(M^{(1)} - M_d^{(1)})(M_d^{(1)})^{-1}. \tag{3.2}$$

Proof: From (3.1) observe that if G is any g-inverse of $I - P^{(1)}$,
$(M^{(1)} - M_d^{(1)})(M_d^{(1)})^{-1} = G\Pi^{(1)} - E(G\Pi^{(1)})_d - G + EG_d = EH_d - H$ where $H = G(I - \Pi^{(1)})$.

Thus $$H = EH_d - (M^{(1)} - M_d^{(1)})(M_d^{(1)})^{-1}.$$

Now $\boldsymbol{E}G(I - \Pi^{(1)}) = \boldsymbol{E}EH_d - \boldsymbol{E}(M^{(1)} - M_d^{(1)})(M_d^{(1)})^{-1} = -\boldsymbol{E}(M^{(1)} - M_d^{(1)})(M_d^{(1)})^{-1}$ since $\boldsymbol{E}EH_d = \boldsymbol{E}\mathbf{ee}'H_d = 0$ and (3.2) follows from (2.6).

❑

Result 3.2 is new. However, Theorem 3.1 can be further simplified.

Theorem 3.2: Let $N^{(1)} = [n_{ij}^{(1)}] = [(1 - \delta_{ij})m_{ij}^{(1)}/m_{jj}^{(1)}] = [(1 - \delta_{ij})m_{ij}^{(1)}\pi_j^{(1)}]$ *then, for any general perturbation* $\boldsymbol{E}$,

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = -\boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}N^{(1)}. \tag{3.3}$$

Proof: Equation (3.3) follows directly from (3.2) since $N^{(1)} = (M^{(1)} - M_d^{(1)})(M_d^{(1)})^{-1}$.

❑

Result (3.3) shows that elemental expressions for $\pi_j^{(2)} - \pi_j^{(1)}$ can be expressed in terms of $n_{ij}^{(1)} = m_{ij}^{(1)}/m_{jj}^{(1)} = m_{ij}^{(1)}\pi_j^{(1)}$ $(i \neq j)$ with $n_{jj}^{(1)} = 0$.

Note also the negative signs in each of (3.2) and (3.3). We shall in the future consider the difference $\boldsymbol{\pi}^{(1)\prime} - \boldsymbol{\pi}^{(2)\prime}$ when using these forms.

A bound for $\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime}$, in terms of the mean first passage times, was first derived by Cho and Meyer [3]. Their derivation was based upon the observation that the elements of the group inverse $A^{\#(1)}$ can be expressed in terms of the $m_{ij}^{(1)}$, viz. $a_{ij}^{\#(1)} = a_{jj}^{\#(1)} - \pi_j^{(1)}m_{ij}^{(1)}$, $(i \neq j)$, with $m_{jj}^{(1)} = 1/\pi_j^{(1)}$.

This is also related to the observation that there is a similar connection between the elements of the fundamental matrix $Z^{(1)}$ and the mean first passage times $m_{jj}^{(1)}$, (see viz. [1]). $\pi_j^{(1)}m_{ij}^{(1)} = z_{jj}^{(1)} - z_{ij}^{(1)}$, $(i \neq j)$, with $m_{ij}^{(1)} = 1/\pi_j^{(1)}$.

Further links between stationary distributions and mean first passage times in Markov chains, using generalised inverses, are explored in [10].

The following theorem gives the Cho and Meyer [3] bounds. Their proof uses the results of equation (2.12), the properties of the group inverse $A^{\#(1)}$ as mentioned above, and an inequality that appears in Haviv and Van der Heyden [5]. We can however provide a simpler more direct proof from an elemental expression of (3.3):

Theorem 3.3: (Cho and Meyer, 2000). For a general perturbation $\boldsymbol{E} = [\varepsilon_{ij}]$,

(i)
$$\left|\frac{\pi_j^{(1)} - \pi_j^{(2)}}{\pi_j^{(1)}}\right| \leq \frac{\|\boldsymbol{E}\|_\infty}{2} \max_{i \neq j} \{m_{ij}^{(1)}\}, \tag{3.4}$$

(ii)
$$\left|\pi_j^{(1)} - \pi_j^{(2)}\right| \leq \frac{\|\boldsymbol{E}\|_\infty}{2} \max_{i \neq j} \{m_{ij}^{(1)}\}, \tag{3.5}$$

where $\|\boldsymbol{E}\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{m} |\varepsilon_{ij}|$.

Proof: The proof is based upon the result (see [5]) that for any vectors **c** and **d** such that **c'e** = 0, then

$$|\mathbf{c'd}| \leq \left(\sum_{i=1}^{m} |c_i|\right) \frac{\max_{r,s} |d_r - d_s|}{2}. \tag{3.6}$$

From (3.3),

$$\pi_j^{(1)} - \pi_j^{(1)} = \textstyle\sum_{l=1}^{m} \alpha_l^{(2)} n_{lj}^{(1)} \text{ where } \alpha_l^{(2)} = \sum_{k=1}^{m} \pi_k^{(2)} \varepsilon_{kl}, \text{ and } \sum_{l=1}^{m} \alpha_l^{(2)} = 0, \text{ since } \sum_{l=1}^{m} \varepsilon_{kl} = 0,$$

so that applying (3.6) yields

$$\left|\pi_j^{(1)} - \pi_j^{(1)}\right| \leq \sum_{l=1}^{m} \left|\alpha_l^{(2)}\right| \left(\frac{\max_{r,s}\left|n_{rj}^{(1)} - n_{sj}^{(1)}\right|}{2}\right). \tag{3.7}$$

Now

$$\sum_{l=1}^{m} \left|\alpha_l^{(2)}\right| \leq \sum_{l=1}^{m} \pi_l^{(2)} \sum_{k=1}^{m} |\varepsilon_{kl}| \leq \sum_{l=1}^{m} \pi_l^{(2)} \left\{\max_{1 \le k \le m} \sum_{k=1}^{m} |\varepsilon_{kl}|\right\} = \|\boldsymbol{E}\|_\infty .$$

Equations (3.4) and (3.5) follow by observing that $n_{ij}^{(1)} = (1-\delta_{ij}) m_{ij}^{(1)} \pi_j^{(1)}$ and, because of the positivity of the mean first passage times, that

$$\max_{r,s \neq j} \left|m_{rj}^{(1)} - m_{sj}^{(1)}\right| = \max_{r \neq j} m_{rj}^{(1)} - \min_{s \neq j} m_{sj}^{(1)} \leq \max_{i \neq j} m_{ij}^{(1)}, \tag{3.8}$$

together with the observation that $|\pi_j^{(1)}| \leq 1$.

❑

While Theorem 3.3 reproduces Cho and Meyer's bound, a closer examination of (3.8) shows that we can in fact improve the universal bounds, for fixed j, given by (3.3) and (3.4)

Corollary 3.3.1: For any general perturbation,

(i)
$$\left|\frac{\pi_j^{(1)} - \pi_j^{(2)}}{\pi_j^{(1)}}\right| \leq \frac{\|\boldsymbol{E}\|_\infty}{2} [\max_{i \neq j} \{m_{ij}^{(1)}\} - \min_{i \neq j} \{m_{ij}^{(1)}\}] \leq \frac{\|\boldsymbol{E}\|_\infty}{2} [\max_{i \neq j} \{m_{ij}^{(1)}\} - 1], \tag{3.9}$$

(ii)
$$\left|\pi_j^{(1)} - \pi_j^{(2)}\right| \leq \frac{\|\boldsymbol{E}\|_\infty}{2} [\max_{i \neq j} \{m_{ij}^{(1)}\} - \min_{i \neq j} \{m_{ij}^{(1)}\}] \leq \frac{\|\boldsymbol{E}\|_\infty}{2} [\max_{i \neq j} \{m_{ij}^{(1)}\} - 1]. \tag{3.10}$$

Proof: Inequalities (3.9) and (3.10) follow from (3.8) by observing that $m_{ij}^{(1)} = E(T_{ij} | X_0 = i)$ and, since in an irreducible Markov chain each state can be reached, $T_{ij} \geq 1$ implying that $m_{ij}^{(1)} \geq 1$. Such an inequality obviously holds for the minimum over all $i \neq j$.

❑

The results (3.9) and (3.10) are new improved bounds for a fixed index j. We show later that, for the special cases considered in this paper, we can get improved bounds.

4. Single row perturbations

Let $\mathbf{p}_r^{(i)'} = \mathbf{e}_r P^{(i)}$ so that $\mathbf{p}_r^{(i)'}$ is the r[th] row of the transition matrix $P^{(i)}$. Now let $\boldsymbol{E} = \mathbf{e}_r \mathbf{e}_r'$ where $\mathbf{e}_r' = \mathbf{p}_r^{(2)'} - \mathbf{p}_r^{(1)'}$. This implies that the perturbation of interest results from changing the rth row of the transition matrix $P^{(1)}$ by the rth row of the transition matrix $P^{(2)}$.

Suppose that $\mathbf{e}_r' = (e_1, e_2, \ldots, e_m)$ where $\mathbf{e}_r'e = 0$. Substitution in equation (3.3) yields $\boldsymbol{\pi}^{(1)'} - \boldsymbol{\pi}^{(2)'} = \boldsymbol{\pi}^{(2)'}\mathbf{e}_r\mathbf{e}_r'N^{(1)} = \pi_r^{(2)}\mathbf{e}_r'N^{(1)}$ so that in elemental form, for j = 1, 2, …, m,

$$\pi_j^{(1)} - \pi_j^{(2)} = \pi_r^{(2)} \sum_{i \neq j} \varepsilon_i n_{ij}^{(1)} = \pi_j^{(1)} \pi_r^{(2)} \sum_{i \neq j} \varepsilon_i m_{ij}^{(1)}. \tag{4.1}$$

4.1 Two-element perturbations in a single row

The simplest perturbation arises from decreasing the (r,a)[th] element of $P^{(1)}$ by an amount ε and increasing the (r,b)[th] element of $P^{(1)}$ by the same amount to obtain the new transition matrix $P^{(2)}$. Thus $p_{ra}^{(2)} = p_{ra}^{(1)} - \varepsilon$ and $p_{rb}^{(2)} = p_{rb}^{(1)} + \varepsilon$, ($\varepsilon_a = -\varepsilon$, $\varepsilon_b = \varepsilon$). We assume that the stochastic and irreducible nature of both $P^{(1)}$ and $P^{(2)}$ is preserved. This requires $\varepsilon < p_{ra}^{(1)} \le 1$, and $0 \le p_{rb}^{(1)} < 1 - \varepsilon$. For this special case we obtain the following results.

Theorem 4.1: Suppose that the transition probability $p_{ra}^{(1)}$ *in an irreducible chain is decreased by an amount* ε *while* $p_{rb}^{(1)}$ *is increased by an amount* ε. *If the resulting chain is irreducible then expressions for difference in the stationary probabilities* $\pi_j^{(1)} - \pi_j^{(2)}$ *are given by*

$$\pi_j^{(1)} - \pi_j^{(2)} = \begin{cases} \varepsilon \pi_a^{(1)} \pi_r^{(2)} m_{ba}^{(1)} & j = a, \\ -\varepsilon \pi_b^{(1)} \pi_r^{(2)} m_{ab}^{(1)} & j = b, \\ \varepsilon \pi_j^{(1)} \pi_r^{(2)} (m_{bj}^{(1)} - m_{aj}^{(1)}) & j \neq a,\ b. \end{cases} \tag{4.2}$$

❑

First note that for those states j ≠ a, b, we can make the general observation that $\pi_j^{(2)} \ge \pi_j^{(1)}$ if and only if $m_{aj}^{(1)} \ge m_{bj}^{(1)}$, reflecting the influence of mean first passage times on stationary probabilities. Thus, irrespective of the magnitude of the perturbations at a and b, the stationary probability at j (≠ a, b) will increase if, in the unperturbed chain, the mean passage time from state a to state j is greater than the mean passage time from state b to state j. Thus the "distance" a and b are from particular states will influence the changes in the stationary probabilities at those states.

Cho and Meyer [3] considered this special case. Equations (4.2) correct some minor errors in their results (for the j = a and b cases). While they noted the sensitivities of the mean first passage times on the *relative* changes to stationary probabilities they did not notice any directional influence upon the *absolute* changes to the stationary probabilities at states a and b. We discuss these in more detail shortly.

Prior to considering general relationships between the stationary probabilities, we establish a useful new relationship between the mean first passage times between states in a Markov chain.

Theorem 4.2: Let m_{ij} *be the mean first passage time from state i to state j in a finite irreducible Markov chain. Then, for all i, j, and k,*

$$m_{ij} \le m_{ik} + m_{kj}. \tag{4.3}$$

Proof: First observe that if $\{X_n\}$ is the underlying Markov chain then $m_{ij} = E[T_{ij}]$, where $T_{ij} = \min\{n: X_n = j | X_0 = i\}$, the number of trials for a first passage from state i to state j (i ≠ j) ([8], p113.)

Now it is obvious, from the sample path of a typical chain, that $T_{ij} \leq T_{ik} + T_{kj}$, since the chain will clearly take at least as many transitions (steps) to move from state i to state j via a first passage through state k starting at state i as it will without making such a forced first passage through state k. Equation (4.3) follows upon taking expectations of the respective random variables (which are all well-defined proper random variables since the chain is irreducible.) While applications of the theorem are meaningful when $i \neq j \neq k$ the theorem obviously holds without such restrictions.

❑

A consequence of equation (4.3) is that $m_{aj} \leq m_{ab} + m_{bj}$ and that $m_{bj} \leq m_{ba} + m_{aj}$ so that, for $j \neq a, b$,

$$- m_{ab} \leq m_{bj} - m_{aj} \leq m_{ba} \tag{4.4}$$

A consequence of this result is the following Corollary to Theorem 4.1.

Corollary 4.1.1: Under the conditions of Theorem 4.1, the maximum relative change in the stationary probabilities $\pi_j^{(1)}, \pi_j^{(2)}$*, is given by the following bound. For* $1 \leq j \leq m$*,*

❑

$$\left|\frac{\pi_j^{(1)} - \pi_j^{(2)}}{\pi_j^{(1)}}\right| \leq \varepsilon\, \pi_r^{(2)} \max\{m_{ab}^{(1)},\ m_{ba}^{(1)}\} = \max\left\{\left|\frac{\pi_a^{(1)} - \pi_a^{(2)}}{\pi_a^{(1)}}\right|, \left|\frac{\pi_b^{(1)} - \pi_b^{(2)}}{\pi_b^{(1)}}\right|\right\}. \tag{4.5}$$

Corollary 4.1.1 provides a new bound. The bound (4.5) cannot be improved, as it is achieved at one of the states j = a or b.

The relevant bound derived by Cho and Meyer [3] for this situation, follows from (3.4), upon observing that $\| \boldsymbol{E} \| = 2\varepsilon$, viz.

$$\left|\frac{\pi_j^{(1)} - \pi_j^{(2)}}{\pi_j^{(1)}}\right| \leq \varepsilon \max_{i \neq j} \{m_{ij}^{(1)}\}. \tag{4.6}$$

Thus the bound (4.5) is a significant improvement over (4.6) in this two-element case.

The significance of (4.5) for the *relative changes* in the stationary probabilities at any state j is that the bound depends only the mean first passage times associated with the states where the perturbations take place (a and b) and not upon any other mean first passage times. The general bound (4.6) depends on all of the mean first passage times between different states. The advantage of (4.5) is that if one wishes to make perturbations at two states, say a and b, that will not, for example, unduly affect stationary probabilities at other states, then knowledge of the transitions between these two states will sometimes lead to simple estimates of the mean first passage times $m_{ab}^{(1)}$ and $m_{ba}^{(1)}$. Consequently one can estimate the relative changes between the two stationary probabilities at a and b in advance of any detailed calculation. Two states "close together" with small mean first passage times will achieve tighter bounds than states "far apart".

Corollary 4.1.2: Under the conditions of Theorem 4.1,

$$-\varepsilon\, \pi_r^{(2)} m_{ab}^{(1)} = \frac{\pi_b^{(1)} - \pi_b^{(2)}}{\pi_b^{(1)}} \leq \frac{\pi_j^{(1)} - \pi_j^{(2)}}{\pi_j^{(1)}} \leq \frac{\pi_a^{(1)} - \pi_a^{(2)}}{\pi_a^{(1)}} = \varepsilon\, \pi_r^{(2)} m_{ba}^{(1)},\ 1 \leq j \leq m. \tag{4.7}$$

Further, $1 - \varepsilon\, \pi_r^{(2)} m_{ba}^{(1)} = \dfrac{\pi_a^{(2)}}{\pi_a^{(1)}} \leq \dfrac{\pi_j^{(2)}}{\pi_j^{(1)}} \leq \dfrac{\pi_b^{(2)}}{\pi_b^{(1)}} = 1 + \varepsilon\, \pi_r^{(2)} m_{ab}^{(1)},\ 1 \leq j \leq m.$ (4.8)

Also, $\dfrac{\pi_a^{(2)}}{\pi_a^{(1)}} < 1 < \dfrac{\pi_b^{(2)}}{\pi_b^{(1)}}$*, so that* $\pi_a^{(2)} < \pi_a^{(1)}$ *and* $\pi_b^{(1)} < \pi_b^{(2)}$. (4.9)

❑

We can immediately make some interesting observations from Corollary 4.1.2.

First note, as a consequence of (4.7) and (4.8), that while the stationary probabilities at all other states either increase or decrease following a perturbation, the *relative change* in magnitude at any state never exceeds the relative changes exhibited at the two states a and b. i.e. the *minimal and maximal relative changes* occur at states a and b, respectively.

Secondly, as a consequence of (4.9), that when we decrease the transition probability at a single element, the a-th, in the rth row of the transition matrix of a finite irreducible Markov chain and make the corresponding increase in the b-th element in the same row, then the *absolute* stationary probabilities for the perturbed Markov chain at the a-th and b-th positions, correspondingly decrease and increase. i.e.

$$\text{If } p_{ra}^{(2)} < p_{ra}^{(1)} \text{ and } p_{rb}^{(2)} > p_{rb}^{(1)} \text{ then } \pi_a^{(2)} < \pi_a^{(1)} \text{ and } \pi_b^{(2)} > \pi_b^{(1)}.$$

This observation was also noted by Burnley, [1]. We cannot however make any statement regarding the *absolute* changes in the stationary probabilities at a typical state j. The stationary probabilities at any general state (apart from a and b) may increase or decrease.

Our interest now is to investigate whether the results of Theorem 4.1 or the inequalities given in Corollaries 4.1.1 and 4.1.2 also hold for other more general perturbations within a single row. In other words can we conclude that in a general single row perturbation with minimal and maximal perturbations at states a and b, respectively, do inequalities (4.7) and (4.8) hold for the relative changes and (4.9) for the absolute changes?

We consider first three-element perturbations before considering the more general setting.

4.2 Three-element perturbations in a single row

The following theorem follows from the general equation (4.1).

Theorem 4.3: Suppose that three perturbations are carried out in the rth row of a transition matrix at states a, b and c. Let $\varepsilon_i = p_{ri}^{(2)} - p_{ri}^{(1)}$ *and suppose that the perturbations can be expressed as* $\varepsilon_a = -m$ *(minimum),* $\varepsilon_b = M$ *(maximum) and* $\varepsilon_c = m - M$ *where* $e_c > (<)\ 0$ *if* $m > (<)\ M$. *Then*

❑

$$\pi_j^{(1)} - \pi_j^{(2)} = \begin{cases} \pi_a^{(1)}\pi_r^{(2)}[Mm_{ba}^{(1)} + (m-M)m_{ca}^{(1)}], & j = a, \\ \pi_b^{(1)}\pi_r^{(2)}[-mm_{ab}^{(1)} + (m-M)m_{cb}^{(1)}], & j = b, \\ \pi_c^{(1)}\pi_r^{(2)}[-mm_{ac}^{(1)} + Mm_{bc}^{(1)}], & j = c, \\ \pi_j^{(1)}\pi_r^{(2)}[-mm_{aj}^{(1)} + Mm_{bj}^{(1)} + (m-M)m_{cj}^{(1)}], & j \neq a,\ b,\ c. \end{cases} \tag{4.10}$$

We can obtain some general bounds from these results. In particular, using equation (4.5):

$$-mm_{ac}^{(1)} - Mm_{cb}^{(1)} \le \frac{\pi_j^{(1)} - \pi_j^{(2)}}{\pi_j^{(1)}\pi_r^{(2)}} \le mm_{ca}^{(1)} + Mm_{bc}^{(1)}, \quad j \neq a,\ b,\ c \tag{4.11}$$

with

$$\frac{\pi_c^{(1)} - \pi_c^{(2)}}{\pi_c^{(1)}\pi_r^{(2)}} = -mm_{ac}^{(1)} + Mm_{bc}^{(1)}. \tag{4.13}$$

$$-mm_{ac}^{(1)} - Mm_{cb}^{(1)} \le \frac{\pi_b^{(1)} - \pi_b^{(2)}}{\pi_b^{(1)}\pi_r^{(2)}} \le mm_{ca}^{(1)} - Mm_{cb}^{(1)} \le \frac{\pi_a^{(1)} - \pi_a^{(2)}}{\pi_a^{(1)}\pi_r^{(2)}} \le mm_{ca}^{(1)} + Mm_{bc}^{(1)}, \tag{4.12}$$

From these above results we see that the inequalities (4.11) also hold for j = a, b, and c and hence generally. Consequently, we obtain the following general bounds.

Corollary 4.3.1: Under the conditions of Theorem 4.3, for all j,

$$\left|\frac{\pi_j^{(1)} - \pi_j^{(2)}}{\pi_j^{(1)}}\right| \le \max(m, M)\pi_r^{(2)} \max\{m_{ac}^{(1)} + m_{cb}^{(1)},\ m_{bc}^{(1)} + m_{ca}^{(1)}\}. \tag{4.14}$$

❑

Whereas for the two-element perturbation case the general bound, (4.5), involved the maximum of two individual mean first passage times the three-element case involves the maximum of the sums of two mean first passage times and a total of only four specific mean first passage times involving the states a, b and c.

The comparable bound found by Cho and Meyer [3] in this case is given by (3.4) as

$$\left|\frac{\pi_j^{(1)} - \pi_j^{(2)}}{\pi_j^{(1)}}\right| \le \max(\mathrm{m}, \mathrm{M}) \max_{i \ne j}\{m_{ij}^{(1)}\}, \tag{4.15}$$

since $\|\boldsymbol{E}\|_\infty = \max(\mathrm{m} + \mathrm{M} + |\mathrm{m} - \mathrm{M}|) = 2\max(\mathrm{m}, \mathrm{M})$.

The bound given by (4.14) will be an improvement over that given by (4.15) if

$$\max\{m_{ac}^{(1)} + m_{cb}^{(1)},\ m_{bc}^{(1)} + m_{ca}^{(1)}\} \le \max_{i \ne j}\{m_{ij}^{(1)}\}.$$

This is likely to be the case when the states a, b and c are "closely located" and the chain contains some states that are some "distance apart".

Note also that, from (4.3), $m_{ac}^{(1)} + m_{cb}^{(1)}$ and $m_{bc}^{(1)} + m_{ca}^{(1)}$ are upper bounds, respectively, for $m_{ab}^{(1)}$ and $m_{ba}^{(1)}$ so that by including an additional perturbation the bound given by (4.14) is larger than that given by (4.5) for the two-element case.

In the two-element case, (4.8) in Corollary 4.1.2 provided bounds on the ratios $\pi_j^{(2)}/\pi_j^{(1)}$ for all j. The equivalent results in the three element case can be derived from (4.11), viz. for all j,

$$1 - \pi_r^{(2)}(mm_{ca}^{(1)} + Mm_{bc}^{(1)}) \le \frac{\pi_j^{(2)}}{\pi_j^{(1)}} \le 1 + \pi_r^{(2)}(mm_{ac}^{(1)} + Mm_{cb}^{(1)}). \tag{4.16}$$

The difference however in this three-element case, over the two-element perturbation situation, is that whereas in the two element case the minimal (and maximal) bounds to the *relative probabilities* for all j are achieved at states a (or b), where the extreme perturbation changes take place, this is not in fact the case for the three-element case. From (4.12), it follows that the minimal bound in (4.16) is in fact a lower bound on the minimal value of $\pi_a^{(2)}/\pi_a^{(1)}$ while the maximal bound in (4.16) is an upper bound on $\pi_b^{(2)}/\pi_b^{(1)}$. There is no guarantee that these bounds will be achieved at those values for states a and b.

Let us explore these bounds in more detail.

Corollary 4.3.2: Under the conditions of Theorem 4.3,

$$(i)\quad \frac{\pi_a^{(2)}}{\pi_a^{(1)}} \le \frac{\pi_b^{(2)}}{\pi_b^{(1)}}. \tag{4.17}$$

$$(ii)\ \text{If}\quad \varepsilon_c > 0\ \ (\text{i.e. } \mathrm{m} > \mathrm{M}),\qquad \frac{\pi_a^{(2)}}{\pi_a^{(1)}} \le \min\left\{1, \frac{\pi_b^{(2)}}{\pi_b^{(1)}},\ \frac{\pi_c^{(2)}}{\pi_c^{(1)}}\right\}. \tag{4.18}$$

$$(iii)\ \text{If}\quad \varepsilon_c < 0\ \ (\text{i.e. } m < M),\quad \max\left\{1, \frac{\pi_a^{(2)}}{\pi_a^{(1)}}, \frac{\pi_c^{(2)}}{\pi_c^{(1)}}\right\} \le \frac{\pi_b^{(2)}}{\pi_b^{(1)}}. \tag{4.19}$$

Proof:

(i) Result (4.17) follows directly from equation (4.10) for the cases j = a and b together with appropriate versions of equation (4.4).

(ii) Results (4.18) follow from (4.10) using the results for j = a and j = c and (4.17).

(iii) Results (4.19) follow from (4.10) using the results for j = b and j = c and (4.17).

❑

An important observation comparing result (4.17) of Corollary 4.3.2 with results (4.8) and (4.9) of Corollary 4.1.2 is that, while in each case the r*elative* change in the stationary probability at state a (where there is a negative perturbation) is always smaller than the relative change of the stationary probability at state b (where there is a positive perturbation), only under certain circumstances is there an *absolute* decrease (resp. increase) at state a (resp. b) in the three-element situation, as opposed to this always occurring in the two-element case. The situations described in Corollary 4.3.2 where $\pi_a^{(2)} < \pi_a^{(1)}$ or $\pi_b^{(2)} > \pi_b^{(1)}$ are sensible in that for the $\varepsilon_c > 0$ case the only decrease that occurs is at state a and this is of greater magnitude than the positive increases that occur at states b and c (since $\varepsilon_a + \varepsilon_b + \varepsilon_c = 0$.). Similarly for the situations where the increase occurs at state b when $\varepsilon_c < 0$.

The situation is, however, not entirely as the result of the nature of the size of the perturbation at state c. For example, even if $\varepsilon_c < 0$, it is still possible to have a decrease in the stationary probability at state a as a result of the effects of the relationships between the mean first passage times between the states a, b and c. The following corollary summarises the situation.

Corollary 4.3.3: Under the conditions of Theorem 4.3

$$(i)\ \textit{If either } m_{ba}^{(1)} > m_{ca}^{(1)} \textit{ or } mm_{ca}^{(1)} > Mm_{cb}^{(1)} \textit{ then } \pi_a^{(2)} < \pi_a^{(1)}. \tag{4.20}$$

$$(ii)\ \textit{If either } m_{cb}^{(1)} < m_{ab}^{(1)} \textit{ or } mm_{ca}^{(1)} < Mm_{cb}^{(1)} \textit{ then } \pi_b^{(1)} < \pi_b^{(2)}. \tag{4.21}$$

$$(iii)\ Mm_{bc}^{(1)} < mm_{ac}^{(1)} \textit{ if and only if } \pi_c^{(1)} < \pi_c^{(2)}. \tag{4.22}$$

$$(iv)\ \textit{If } m_{bj}^{(1)} > m_{cj}^{(1)} > m_{aj}^{(1)} \textit{ then } \pi_j^{(1)} > \pi_j^{(2)},\ (j \ne a, b, c) \tag{4.23}$$

$$(v)\ \textit{If } m_{bj}^{(1)} < m_{cj}^{(1)} < m_{aj}^{(1)} \textit{ then } \pi_j^{(1)} < \pi_j^{(2)},\ (j \ne a, b, c) \tag{4.24}$$

Proof:

(i) and (ii): The first condition for (4.20) and (4.21) follow directly from equation (4.10) by considering the coefficients of m and M. The second condition follows, in both cases, directly from (4.10) together with appropriate versions of equation (4.4).

(iii): Results (4.22) follows directly from (4.10) for the case j = c.

(iv) and (v): Results (4.23) and (4.24) follow from equation (4.10) by considering the condition under which both the coefficients of M and m are either positive or negative, respectively.

❑

Thus, for example, if $\varepsilon_c < 0$ (i.e. m > M) so that the condition of (4.18) is not satisfied, but if either $m_{ba}^{(1)} > m_{ca}^{(1)}$ or $m_{ca}^{(1)} \ge m_{cb}^{(1)}$, then respectively either the first or the second condition of (4.20) holds and consequently it is still true that $\pi_a^{(2)} < \pi_a^{(1)}$.

The interrelations between the mean first passage times between those states where the perturbations occur play an important role in establishing the absolute changes in the stationary probabilities.

We have not been able to establish simple general necessary and sufficient conditions under which either $\pi_j^{(2)} < \pi_j^{(1)}$ or $\pi_j^{(2)} > \pi_j^{(1)}$ apart from checking the right hand side of equations (4.10) for conditions of non-negativity and negativity.

The observation that we made in the two-element case that the minimal (resp. maximal) *absolute* changes to the stationary probabilities occur at those states where the perturbations are the smallest (resp. largest) in magnitude need not hold in general in the three-element perturbation situation. This is substantiated by numerical calculations for some specific chains.

4.3 Multiple-element perturbations in a single row

The following theorem follows from the general equation (4.1). Since $\mathbf{e}_r'e = 0$, some of the perturbations ε_j $(1 \leq j \leq m)$ will be negative and some will be positive but the perturbations will all sum to zero.

Theorem 4.4: Suppose that multiple perturbations are carried out in the rth row of a transition matrix. Let $\varepsilon_i = p_{ri}^{(2)} - p_{ri}^{(1)}$. *Let the minimal negative perturbation occur at state a with* $\varepsilon_a = -m = \min\{\varepsilon_j, 1 \leq j \leq m\}$ *and the maximal positive perturbation occur at state b with* $\varepsilon_b = M = \max\{\varepsilon_j, 1 \leq j \leq m\}$. *Let P be set of states with positive perturbations (excluding b),* $P = \{j \mid \varepsilon_j > 0 \text{ with } j \neq b\}$. *Let N be the set of states with negative perturbations (excluding a),* $N = \{j \mid \varepsilon_j < 0 \text{ with } j \neq a\}$.

$$\pi_j^{(1)} - \pi_j^{(2)} = \begin{cases} \pi_a^{(1)}\pi_r^{(2)}[Mm_{ba}^{(1)} + \sum_{k\in P}\varepsilon_k m_{ka}^{(1)} + \sum_{k\in N}\varepsilon_k m_{ka}^{(1)}], & j = a, \\ \pi_b^{(1)}\pi_r^{(2)}[-mm_{ab}^{(1)} + \sum_{k\in P}\varepsilon_k m_{kb}^{(1)} + \sum_{k\in N}\varepsilon_k m_{kb}^{(1)}], & j = b, \\ \pi_j^{(1)}\pi_r^{(2)}[-mm_{aj}^{(1)} + Mm_{bj}^{(1)} + \sum_{k\in P, k\neq j}\varepsilon_k m_{kj}^{(1)} + \sum_{k\in N, k\neq j}\varepsilon_k m_{kj}^{(1)}], & j \neq a, b. \end{cases} \tag{4.25}$$

❑

General results for this situation are difficult to obtain. We can however obtain the following results concerning the relative relationships between the $\pi_a^{(1)}$, $\pi_a^{(2)}$, $\pi_b^{(1)}$ and $\pi_b^{(2)}$.

Corollary 4.4.1: Under the conditions of Theorem 4.4,

$$\frac{\pi_a^{(1)} - \pi_a^{(2)}}{\pi_a^{(1)}} \geq \frac{\pi_b^{(1)} - \pi_b^{(2)}}{\pi_b^{(1)}} \text{ and hence } \frac{\pi_a^{(2)}}{\pi_a^{(1)}} \leq \frac{\pi_b^{(2)}}{\pi_b^{(1)}} \text{ ,when } m - \sum_{k\in P}\varepsilon_k = M + \sum_{k\in N}\varepsilon_k \geq 0. \tag{4.26}$$

Proof: From (4.21) it is easy to see that

$$\frac{\pi_a^{(1)} - \pi_a^{(2)}}{\pi_a^{(1)}\pi_r^{(2)}} - \frac{\pi_b^{(1)} - \pi_b^{(2)}}{\pi_b^{(1)}\pi_r^{(2)}} = Mm_{ba}^{(1)} + \sum_{k\in P}\varepsilon_k\left(m_{ka}^{(1)} - m_{kb}^{(1)}\right) + \sum_{k\in N}(-\varepsilon_k)\left(m_{kb}^{(1)} - m_{ka}^{(1)}\right). \tag{4.27}$$

Further, since $-m_{ab}^{(1)} \leq m_{ka}^{(1)} - m_{kb}^{(1)}$ and $-m_{ba}^{(1)} \leq m_{kb}^{(1)} - m_{ka}^{(1)}$, and from (4.27), since $-m + \sum_{k\in N}\varepsilon_k + \sum_{k\in P}\varepsilon_k + M = 0$, $M + \sum_{k\in N}\varepsilon_k = m - \sum_{k\in P}\varepsilon_k$, it follows that

$$\frac{\pi_a^{(1)} - \pi_a^{(2)}}{\pi_a^{(1)}\pi_r^{(2)}} - \frac{\pi_b^{(1)} - \pi_b^{(2)}}{\pi_b^{(1)}\pi_r^{(2)}} \geq \left(m - \sum_{k\in P}\varepsilon_k\right)m_{ab}^{(1)} + \left(M + \sum_{k\in N}\varepsilon_k\right)m_{ba}^{(1)}, \tag{4.28}$$

leading to the stated results. ❑

The more general result that $\frac{\pi_a^{(1)} - \pi_a^{(2)}}{\pi_a^{(1)}} \geq \frac{\pi_b^{(1)} - \pi_b^{(2)}}{\pi_b^{(1)}}$ and hence that $\frac{\pi_a^{(2)}}{\pi_a^{(1)}} \leq \frac{\pi_b^{(2)}}{\pi_b^{(1)}}$, have already been shown to hold in the two-element and three-element perturbation cases (as exhibited by Corollaries 4.1.2 and 4.3.2). These special cases also follow from Corollary 4.4.1, since it can be easily verified that the conditions of (4.26) hold in these situations.

The more general result also holds, for example, in the four-element case when $\varepsilon_a = -m$, $\varepsilon_b = M$, $\varepsilon_c \neq 0$, $\varepsilon_d \neq 0$. If ε_c and ε_d are both positive, or both negative, then it is easy to verify that the conditions of (4.22) are satisfied. When $\varepsilon_c < 0$ and $\varepsilon_d > 0$ then, since $-m \le \varepsilon_c < 0$, it follows that $M - m \le M + \varepsilon_c = m - \varepsilon_d < M$, implying that the conditions of Corollary 4.4.1 are satisfied when $M - m \ge 0$. Further, since $0 < \varepsilon_d \le M$, it follows that $m - N \le m - \varepsilon_d = M + \varepsilon_c < m$ and thus the conditions of Corollary 4.4.1 are satisfied if $m - M \ge 0$. Consequently, if either $M - m \ge$ or ≤ 0, i.e. generally, the results of (4.22) are satisfied.

However the results of Corollary 4.4.1 do not necessarily hold in all situations. For example, it is easy to construct a multi-element example where the conditions of (4.22) are violated, (e.g. $\varepsilon_1 = -m = -0.20$, $\varepsilon_2 = -0.15$, $\varepsilon_3 = -0.05$, $\varepsilon_4 = 0.12$, $\varepsilon_5 = 0.13$, $\varepsilon_6 = M = 0.15$).

We have been unable to obtain specific generalisations of the more general results concerning bounds on $(\pi_j^{(1)} - \pi_j^{(2)})/\pi_j^{(1)}$ and $\pi_j^{(2)}/\pi_j^{(1)}$ or $\pi_j^{(1)} - \pi_j^{(2)}$ that we obtained in the two-element case (Corollaries 4.1.1 and 4.1.2) and the three-element case (Corollaries 4.3.1, 4.3.2 and 4.3.3). In fact, examples can be constructed to show that some of the generalisations do not hold in more general settings. Further, general conditions under which $\pi_a^{(2)} < \pi_a^{(1)}$ and/or $\pi_b^{(2)} > \pi_b^{(1)}$ hold have not been found in this more general setting. What is clear however is that both relative and absolute changes in the stationary probabilities can occur at states other than a and b (where $\varepsilon_a = -m$, $\varepsilon_b = M$) of magnitude exceeding those at states a and b.

5. Concluding remarks

The results derived for the two-element case are elegant. The changes to the stationary probabilities that occur at any state in this situation can be easily determined from a knowledge of the mean first passage times, as exhibited by equations (4.2). If we can update the mean first passage times following a two-element perturbation then a useful procedure could be to consider a multiple-perturbation as a sequence of two-element perturbations.

In a sequel to this paper we explore further procedures, using generalized matrix inverses, that will enable us to update the matrix of mean first passage times $M^{(1)}$, following a perturbation on the transition matrix $P^{(1)}$, to obtain $M^{(2)}$, the matrix of mean first passage times of the perturbed chain, without the necessity of calculating a further matrix inverse.

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