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The density and drag of the accretion wake of a massive body moving through a uniform stellar distribution

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We calculate the change in density within a uniform distribution of field stars (point masses) caused by a single massive body passing through with a constant velocity. Starting with the simplest case in which the field stars are initially stationary this leads to an infinite density wake behind the body. Introducing a small thermalisation within the field stars removes this infinity whilst leading to similar results off the path of the massive body. Results are in good agreement with those previously derived. An approximation can be made for the density in the thermalised case and this can be used to deduce the force exerted on the massive body due to the drag caused by the accretion wake.

1 Introduction

We consider a massive body passing at velocity though a background field of stars. The gravitational attraction of the massive body perturbs the field stars leading to a change in the density of these stars about the body. In particular the field stars tend to concentrate towards the axis behind the massive body leading to a wake of higher density behind the body. The gravitational interaction effectively leads to a drag on the massive body slowing it down. This drag may also be modelled by calculating the force backwards due to the enhanced density in the wake.

We shall consider simplified models of the process. We assume that prior to the influence of the massive body, the field stars are distributed uniformly across space. We assume that the single massive body is moving at a constant velocity and that the wake has already been established so that the density of field stars about and relative to this body is unchanging. For this to be realistic we require the velocity of the massive body to be large in comparison with the relative motion of the field stars to one another. By making use of this assumption we can compute approximate expressions for the density wake and drag.

We calculate the increased density directly in the cases when the field stars are initially stationary and when they have a small initial thermalisation. For the latter case, we also compute the drag exerted by the wake on the massive body.

2 Wake density for a background of stationary field stars

We consider a single massive body mass M moving with speed V in a positive direction along the x-axis through a background field of stars which have unperturbed density ρ_0. There is symmetry about the x-axis. Field stars are perturbed creating an accretion wake. We ignore gravitational interaction between field stars. We calculate the density of the wake ρ that will occur if the massive body continues moving with the same velocity (we neglect any slow down due to drag).

In this section we consider the case where the field stars are initially stationary. We consider the case where they have a non-zero velocity dispersion in the next section.

The problem is equivalent to that of a stationary body bombarded by a cloud of stars. We adopt the rest frame of the massive body (which we locate at the origin) for our ensuing calculations.

2.1 The trajectories of the field stars

In the rest frame of the massive body the field stars have an initial mean velocity of magnitude V towards the negative x-axis. Under gravitational attraction each field star follows a simple hyperbolic orbit about the massive body (at the origin).

Consider a field star of impact parameter p at $x = \infty$. Using its polar coordinates (r, θ), with θ measured relative to the positive x-axis,

$$a(e^2 - 1) = r(1 + e\cos(\theta - \theta_0)) = r + e(x\cos\theta_0 - y\sin\theta_0),$$

where e is eccentricity, a semi-major axis, and θ_0 is the angle at which pericentre occurs.

As $r \to \infty$ we have $x \to \infty$, $y \to p$ and $\theta \to 0$ so that we deduce that

$$(1 + e\cos(\theta - \theta_0)) = 0$$

and

$$ep\sin\theta_0 = a(e^2 - 1).$$

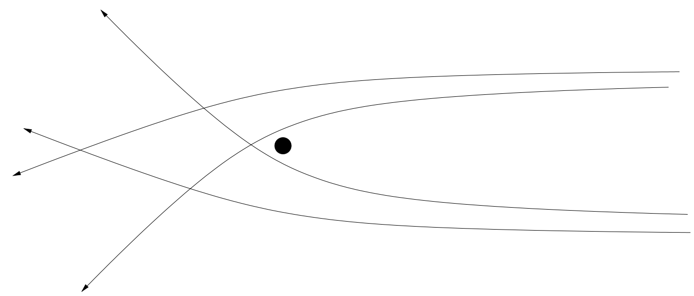

Figure 1: Creation of a wake behind a fast-moving body (after Heggie & Hut(2003) (6)).

Eliminating e and θ_0 we obtain

$$r = \frac{p^2}{a(1-\cos\theta) + p\sin\theta} \tag{1}$$

so that

$$p = \frac{1}{2} r \sin\theta \pm \sqrt{\frac{1}{4} r^2 \sin^2\theta + ra(1-\cos\theta)}.$$

Note that two orbits with differing p pass through each off axis point (Figure 1). We must add together contributions from both when computing the perturbed density.

2.2 Balancing the mass flux

The mass flux of field stars at $x = \infty$ in the annulus of impact parameters from p to $p + dp$ is $2\pi\rho_0 V p dp$. Taking ρ_1 to be the density of this specific mass stream, the corresponding mass flux at (r, θ) is

$$2\pi\rho_1 vr \sin\theta dn = 2\pi\rho_1 vr \sin\theta \cos\phi dr$$

where v is the field star's speed, dn is the width of the flux along the normal direction to its flow, and ϕ is the angle which the normal makes with the radial vector $\mathbf{r}$. The mass flux is conserved and hence we deduce

$$\frac{\rho_1}{\rho_0} = \frac{V \mid p \mid}{vr \sin\theta \cos\phi \mid \frac{\partial r}{\partial p} \mid}.$$

Now $\tan\phi = \frac{1}{r}\frac{dr}{d\theta}$ and $v = \sqrt{V^2 + 2\frac{GM}{r}} = V\sqrt{1 + \frac{2a}{r}}$ (as the energy of the field star is $E = \frac{GM}{2a} = \frac{1}{2}V^2$). So substituting for $\cos\phi$ and v

$$\frac{\rho_1}{\rho_0} = \frac{\mid p \mid \sqrt{1 + (\frac{1}{r}\frac{\partial r}{\partial \theta})^2}}{r \sin\theta \sqrt{1 + \frac{2a}{r}} \mid \frac{\partial r}{\partial p} \mid}.$$

From our expression (1) for $\mathbf{r}$ we derive

$$\frac{1}{r}\frac{\partial r}{\partial \theta} = -\frac{a\sin\theta + p\cos\theta}{a(1-\cos\theta)+p\sin\theta}$$
$$\frac{\partial r}{\partial p} = \frac{2p - r\sin\theta}{a(1-\cos\theta)+p\sin\theta}$$

and with some further work this leads to

$$\begin{aligned}\frac{\rho_1}{\rho_0} &= \frac{p^2}{r\mid\sin\theta\mid\mid 2p - r\sin\theta\mid} \\ &= \frac{p^2}{r\mid\sin\theta\mid\sqrt{r^2\sin^2\theta + 4ra(1-\cos\theta)}}.\end{aligned} \quad (2)$$

The first of these expressions is given in Bisnovatyi-Kogan *et al*(1979) (3) (equation (1.1)), however, these authors do not sum together the terms from both of the different streams, with different values of p, that reach the point (r,θ). We do this now to obtain the overall density enhancement

$$\begin{aligned}\frac{\rho}{\rho_0} &= \frac{r^2\sin^2\theta + 2ra(1-\cos\theta)}{r\sin\theta\sqrt{r^2\sin^2\theta+4ra(1-\cos\theta)}} \\ &= \frac{y^2+2a(r-x)}{\mid y\mid\sqrt{y^2+4a(r-x)}}.\end{aligned} \quad (3)$$

An equivalent expression is derived by a different approach in Danby & Camm (1957) (4) (equation 3).

Substituting $a = \frac{GM}{V^2}$ and using $\frac{\sin^2\theta}{1-\cos\theta} = 1+\cos\theta$

$$\begin{aligned}\frac{\rho}{\rho_0} &= \frac{\left(1+\frac{V^2r(1+\cos\theta)}{2GM}\right)}{\sqrt{\left(1+\frac{V^2r(1+\cos\theta)}{2GM}\right)^2-1}} \\ &= \frac{1}{\sqrt{1-\left(1+\frac{V^2r(1+\cos\theta)}{2GM}\right)^{-2}}}.\end{aligned} \quad (4)$$

We observe that $\frac{\rho}{\rho_0}$ is infinite on the negative x-axis ($\theta = \pi$) but is finite elsewhere. For $\theta \neq \pi$, we have

$$\frac{\rho}{\rho_0} \to 1 \quad \text{as} \quad r \to \infty,$$

so there is no effect at infinite distance as is to be expected. Also for $\theta \neq \pi$, we have

$$\frac{\rho}{\rho_0} \sim \frac{1}{V}\sqrt{\frac{GM}{r(1+\cos\theta)}} = \frac{1}{V\cos(\frac{\theta}{2})}\sqrt{\frac{GM}{2r}} \quad \text{as} \quad r \to 0.$$

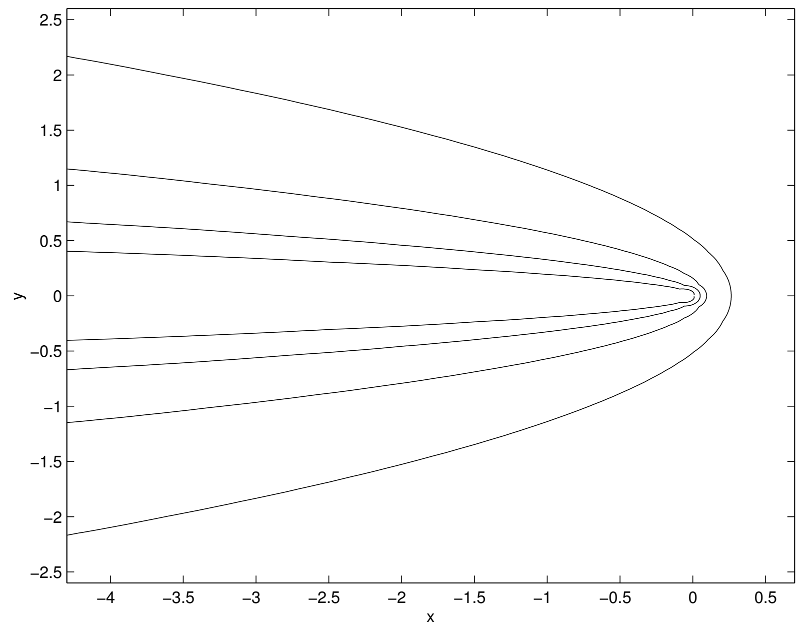

Figure 2: Density contours in the wake behind a fast-moving body (after Heggie & Hut(2003) (6)).

2.3 The contours of constant density

By rearranging (3) we obtain

$$\left(\frac{\rho}{\rho_0}\right)^2 = 1 + \frac{4a^2(r-x)^2}{y^2(y^2+4a(r-x))}.$$

We may solve this expression as a quadratic in $\frac{2a(r-x)}{y^2}$ and then eliminating r yields:

$$\kappa_\rho^2 y^2 + 2\kappa_\rho x = 1$$

where

$$\kappa_\rho = \frac{1}{2a}\left[\left(\frac{\rho}{\rho_0}\right)^2 - 1 + \left(\frac{\rho}{\rho_0}\right)\sqrt{\left(\frac{\rho}{\rho_0}\right)^2 - 1}\right].$$

On the curves of constant density κ_ρ is constant too. Therefore the contours of constant density are parabolae (Figure 2).

3 Wake density for thermalised field stars

We now consider a more realistic situation with thermalised field stars with a Maxwellian distribution of velocities

$$f = \left(2\pi\sigma^2\right)^{-3/2} e^{-(u^2+v^2+w^2)/(2\sigma^2)}.$$

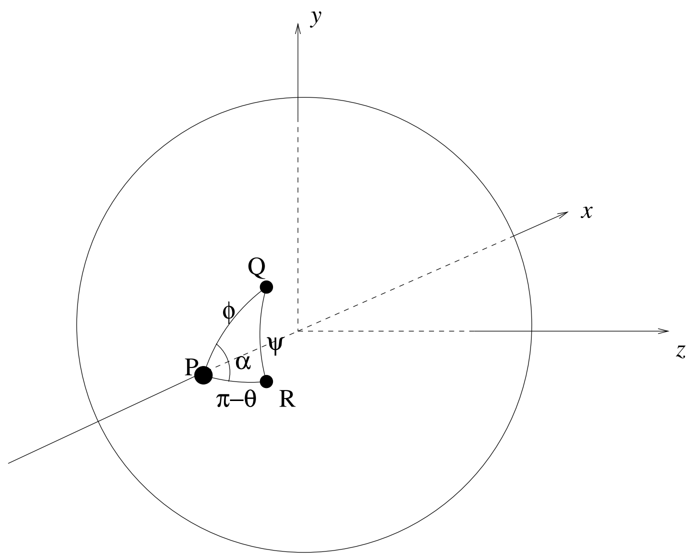

Figure 3: The spherical geometry of the wake.

Figure 3 illustrates the view from downstream with a sphere of radius r overlaid. As found in the previous section, without thermalisation an infinite density wake occurs on the radius through P. For a cohort of field stars with (thermalised) velocity $-\mathbf{W} = (u - V, v, w)$ this wake moves to the radius through Q. We shall compute the density at R where without loss of generality PR lies in the xz-plane. The density enhancement is (cf. non-thermalised result (4))

$$\frac{\rho}{\rho_0} = \int \frac{f(u,v,w)dudvdw}{\sqrt{1 - \left(1 + \frac{W^2 r(1-\cos\psi)}{2GM}\right)^{-2}}}. \tag{5}$$

Now $\cos\phi = \frac{V-u}{W}$, $\tan\alpha = \frac{v}{w}$ and furthermore ψ, ϕ, $\pi - \theta$ and α are related by

$$\cos\psi = -\cos\theta\cos\phi + \sin\theta\sin\phi\cos\alpha$$

and so

$$W^2\left(1 - \cos\psi\right) = W^2\left(1 + \frac{(V-u)}{W}\cos\theta - \frac{w}{W}\sin\theta\right). \tag{6}$$

Henceforth, we will suppose that the massive body has a large velocity relative to that of the field stars ($V >> \sigma$). As f is small for u, v or $w >> \sigma$, we treat the integrand's denominator in (5) for $u, v, w << V$.

3.1 Well away from the negative x-axis

Well away from the negative x-axis ($\theta = \pi$) the dominant term in (6) is constant

$$W^2\left(1 - \cos\psi\right) \approx V^2\left(1 + \cos\theta\right)\ .$$

and (5) reduces to the non-thermalised case

$$\begin{aligned}\frac{\rho}{\rho_0} &= \int \frac{f(u,v,w)du\,dv\,dw}{\sqrt{1-\left(1+\frac{1}{2}\frac{rV^2}{GM}\left(1+\cos\theta\right)\right)^{-2}}}\\ &= \frac{1}{\sqrt{1-\left(1+\frac{V^2 r(1+\cos\theta)}{2GM}\right)^{-2}}}.\end{aligned}$$

3.2 On the negative x-axis

On the negative x-axis expression (6) is dominated by

$$W^2\left(1-\cos\psi\right) \approx \frac{1}{2}\left(v^2+w^2\right)$$

and so (5) becomes

$$\frac{\rho}{\rho_0} = \frac{1}{\left(2\pi\,\sigma^2\right)^{\frac{3}{2}}}\int \frac{e^{-\frac{1}{2}\frac{u^2+v^2+w^2}{\sigma^2}}du\,dv\,dw}{\sqrt{1-\left(1+\frac{1}{4}\frac{r}{GM}\left(v^2+w^2\right)\right)^{-2}}}.$$

Integrating with respect to u and introducing polar coordinates

$$\frac{\rho}{\rho_0} = \frac{1}{2\pi\,\sigma^2}\int_0^\infty\int_0^{2\pi}\frac{e^{-\frac{n^2}{2\sigma^2}}n d\alpha dn}{\sqrt{1-\left(1+\frac{1}{4}\frac{r}{GM}n^2\right)^{-2}}}$$

($n=\sqrt{v^2+w^2}, \alpha = \arctan\frac{v}{w}$). A further integration with respect to α and by parts leads to

$$\frac{\rho}{\rho_0} = \frac{16G^2M^2}{r^2\,\sigma^4}\int_0^\infty e^{-\frac{4GMq}{r\sigma^2}}\sqrt{q(1+q)}dq = ze^z K_1(z)$$

where $q=\frac{1}{8}\frac{rn^2}{GM}$, $z=\frac{2GM}{r\sigma^2}$ and K_1 is the modified Bessel function. (We have made use of standard integrals and expressions (cf. Abramovitz & Stegun(1965) (1).) This expression agrees with one obtained by Danby & Camm(1957) (4). In contrast to the non-thermalised problem the density enhancement is finite except at $r=0$. As we go to an infinite distance away from the massive body $r\to\infty$ ($z\to 0$), $K_1(z)\sim\frac{1}{2}\Gamma(1)\left(\frac{z}{2}\right)^{-1}=z^{-1}$ and so

$$\frac{\rho}{\rho_0}\to 1 \quad \text{as} \quad r\to\infty$$

which is the same as off the axis. As $r\to 0$ ($z\to\infty$), $K_1(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z}$ and so

$$\frac{\rho}{\rho_0}\sim\frac{1}{\sigma}\sqrt{\frac{\pi GM}{2r}} \quad \text{as} \quad r\to 0$$

which is only a minor singularity at $r=0$ and similar in order to those for $\theta\neq\pi$.

3.3 The general case

Close to but not on the negative x-axis $\frac{w}{W}\sin\theta$ is of a similar magnitude to the sum of the two previously dominant terms of (6) considered in the last two subsections. A hybrid approximation valid everywhere is

$$W^2\,(1-\cos\psi) \;\approx\; V^2\left(\frac{1}{2}\left(\frac{w}{V}-\sqrt{2(1+\cos\theta)}\right)^2+\frac{v^2}{2V^2}\right).$$

Substitution into the expression (5) for $\frac{\rho}{\rho_0}$ gives

$$\frac{1}{(2\pi\,\sigma^2)^{\frac{3}{2}}}\int\frac{e^{-\frac{1}{2}\frac{u^2+v^2+w^2}{\sigma^2}}du\,dv\,dw}{\sqrt{1-\left(1+\frac{1}{2}\frac{rV^2}{GM}\left(\frac{1}{2}\left(-\sqrt{2+2\,\cos\theta}+\frac{w}{V}\right)^2+\frac{1}{2}\frac{v^2}{V^2}\right)\right)^{-2}}} \tag{7}$$

The integral with respect to u can be done immediately, and we may use scaled polar coordinates ($n=\sqrt{(v^2+w^2)/2V^2}$, $\alpha=\arctan\frac{v}{w}$) to obtain

$$\frac{V^2}{\pi\sigma^2}\int_0^\infty\int_0^\pi\frac{\left(1+\frac{1}{2}\frac{rV^2}{GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)\right)e^{-\frac{V^2n^2}{\sigma^2}}n d\alpha dn}{\sqrt{\left(1+\frac{1}{2}\frac{rV^2}{GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)\right)^2-1}}. \tag{8}$$

For $\theta\neq\pi$ set $n^2=(1+\cos\theta)p$, $\kappa=\frac{V^2(1+\cos\theta)}{\sigma^2}$ and $k=\frac{1}{2}\frac{rV^2}{GM}(1+\cos\theta)$ to obtain

$$\frac{V^2(1+\cos\theta)}{2\pi\,\sigma^2}\int_0^\infty\int_0^\pi\frac{\left(1+k\left(1-2\sqrt{p}\cos\alpha+p\right)\right)e^{-\kappa p}d\alpha dp}{\sqrt{\left(1+k\left(1-2\sqrt{p}\cos\alpha+p\right)\right)^2-1}}$$

An alternative approach is to set $w=\omega+V\sqrt{2+2\,\cos\theta}$ in (7), use another set of polar coordinates (n',α') and integrate with respect to α' to obtain

$$\frac{\rho}{\rho_0}=\frac{e^{-\frac{V^2(1+\cos\theta)}{\sigma^2}}}{2\pi\,\sigma^2}\int_0^\infty\frac{e^{-\frac{1}{2}\frac{n'^2}{\sigma^2}}I_0(n'V\sqrt{1+\cos\theta}/(\sqrt{2}\sigma^2))\,n'\,dn'}{\sqrt{1-\left(1+\frac{1}{4}\frac{rn'^2}{GM}\right)^{-2}}}.$$

where I_0 is the modified Bessel function of zeroth order. Other general integral expressions for this density have been given by Danby & Bray(1967) (5).

4 Computing the drag due to the accretion wake

As in the previous section we consider the case where the field star velocity distribution is Maxwellian but the dispersion (σ) is much smaller than the relative velocity of the massive body (V). Clearly the drag is infinite as well as the wake density for the non-thermalised case.

To derive the drag we take the general expression for $\frac{\rho}{\rho_0}$ (8), multiply by $GM\rho_0\cos\theta/r^2$ and integrate over space $\mathbf{r}(r,\theta,\phi)$. The spatial integral can be trivially integrated over ϕ and we obtain

$$\frac{2GM\rho_0V^2}{\sigma^2}\int_0^\infty\int_0^\pi ne^{-\frac{V^2n^2}{\sigma^2}}\int_0^\infty\int_0^\pi\left[\frac{\left(1+\frac{rV^2}{2GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)\right)\cos\theta\sin\theta d\theta dr}{\sqrt{\left(1+\frac{rV^2}{2GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)\right)^2-1}}\right]d\alpha dn.$$

Consider the inner part of this expression

$$\int_0^\infty\int_0^\pi\frac{\left(1+\frac{1}{2}\frac{rV^2}{GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)\right)\cos\theta\sin\theta d\theta dr}{\sqrt{\left(1+\frac{1}{2}\frac{rV^2}{GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)\right)^2-1}}.$$

We may begin with the integration with respect to r, however, care is required as there are terms which collectively cancel but individually blow up. (This is to be expected as in any direction towards an infinite radius the density is uniform and non-zero.) The inner expression becomes

$$\left[\frac{2GM}{V^2}\int_0^\pi\frac{\sqrt{\left(1+\frac{rV^2}{2GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)\right)^2-1}}{\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)}\cos\theta\sin\theta d\theta\right]_{r=0}^{r=\infty}.$$

Observe that as $r\to\infty$

$$\begin{aligned}&\sqrt{\left(1+\frac{rV^2}{2GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)\right)^2-1}\\ \sim&\sqrt{\frac{V^2}{2GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)}r+1+O(r^{-1})\end{aligned}$$

and as $r\to 0$

$$\begin{aligned}&\sqrt{\left(1+\frac{1}{2}\frac{rV^2}{GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)\right)^2-1}\\ \sim&\sqrt{\frac{V^2}{GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)}\sqrt{r}+O(r^{3/2})\\ \to&\ 0\end{aligned}$$

So making use of this in our inner integral we have

$$\left[\frac{2GM}{V^2}\int_0^\pi\frac{\frac{1}{2}\sqrt{\frac{V^2}{GM}\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)}\;r+1}{\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)}\cos\theta\sin\theta d\theta\right]^{r=\infty}.$$

Within the integral with respect to θ the term of order r vanishes leaving

$$\begin{aligned}
&\frac{2GM}{V^2}\int_0^{\pi}\frac{\cos\theta\sin\theta d\theta}{\left(1+\cos\theta-2\sqrt{1+\cos\theta}\;n\cos\alpha+n^2\right)}\\
=\;&\frac{4GM}{V^2}\int_0^{\sqrt{2}}\frac{(s^2-1)sds}{(s^2-2sn\cos\alpha+n^2)}
\end{aligned}$$

where we have substituted $s=\sqrt{1+\cos\theta}$ (note that $s>0$).

Reversing the integration order of α and θ in our overall expression for drag and for brevity letting $k=\frac{V^2}{\sigma^2}$, we obtain

$$\begin{aligned}
&\frac{8G^2M^2\rho_0}{\sigma^2}\int_0^{\infty}ne^{-kn^2}\int_0^{\sqrt{2}}\int_0^{\pi}\frac{d\alpha}{(s^2-2sn\cos\alpha+n^2)}(s^2-1)sdsdn\\
=\;&\frac{8G^2M^2\pi\rho_0}{\sigma^2}\int_0^{\infty}ne^{-kn^2}\int_0^{\sqrt{2}}\frac{\pi(s^2-1)s}{|s^2-n^2|}dsdn\\
=\;&\frac{4G^2M^2\pi\rho_0}{\sigma^2}\int_0^{\infty}ne^{-kn^2}\Big[\\
&\text{signum}(\sqrt{2}-n)(2+(n^2-1)\ln|2-n^2|)+(n^2-1)\ln n^2\;\Big]dn.
\end{aligned}$$

For further simplification we subsitute $n^2=p$ and obtain

$$\begin{aligned}
&\frac{2G^2M^2\pi\rho_0}{\sigma^2}\int_0^{\infty}e^{-kp}\Big[\text{signum}(2-p)(2+(p-1)\ln|2-p|)+(p-1)\ln p\Big]dp\\
=\;&\frac{2G^2M^2\pi\rho_0}{\sigma^2}\Big[\int_0^{2}e^{-kp}(2+(p-1)\ln|2-p|)dp\\
&-\int_2^{\infty}e^{-kp}(2+(p-1)\ln|2-p|)dp+\int_0^{\infty}e^{-kp}(p-1)\ln pdp\Big].
\end{aligned}$$

The three integrals on the right-hand side may be calculated in terms of the standard exponential integrals

$$Ei(x)=PV\int_{-\infty}^{x}\frac{e^t}{t}dt\quad\text{and}\quad E_1(x)=PV\int_x^{\infty}\frac{e^{-t}}{t}dt$$

and Euler's constant $\gamma\simeq 0.577215665$. In the first integral by setting $q=2-p$ we find

$$\begin{aligned}
&\int_0^{2}e^{-kp}(2+(p-1)\ln|2-p|)dp\\
=\;&e^{-2k}\int_0^{2}e^{kq}(2+(1-q)\ln q)dq\\
=\;&\frac{e^{-2k}}{k^2}\Big[(1+(1-q)k)e^{kq}\ln q+(1+2k)e^{kq}-(1+k)Ei(kq)\Big]_0^2\\
=\;&\frac{1}{k^2}\Big[(1-k)\ln 2+(1+2k)+e^{-2k}((1+k)(\gamma+\ln k-Ei(2k))-(1+2k))\Big].
\end{aligned}$$

In the second integral set $q = p - 2$:

$$
\begin{aligned}
& - \int_2^\infty e^{-kp}(2 + (p-1)\ln|2-p|)dp \\
= & - e^{-2k} \int_0^\infty e^{-kq}(2 + (1+q)\ln q)dq \\
= & \frac{e^{-2k}}{k^2} \left[(1 + (1+q)k)e^{-kq}\ln q + (1+2k)e^{-kq} + (1+k)E_1(kq)\right]_0^\infty \\
= & - \frac{e^{-2k}}{k^2} \left[(1+2k) + (1+k)(-\gamma - \ln(k))\right].
\end{aligned}
$$

The third integral can be calculated immediately,

$$
\begin{aligned}
& \int_0^\infty e^{-kp}(p-1)\ln p dp \\
= & \frac{1}{k^2} \left[(-1 + (1-q)k)e^{-kq}\ln q - e^{-kq} - (1-k)E_1(kq)\right]_0^\infty \\
= & \frac{1}{k^2} \left[1 + (1-k)(-\gamma - \ln(k))\right].
\end{aligned}
$$

Adding these terms the expression for drag becomes

$$
\begin{aligned}
& \frac{2\pi G^2 M^2 \rho_0}{\sigma^2} \frac{1}{k^2} \Big[(1-k)(\ln 2 - \gamma - \ln(k)) + 2(1+k) \\
+ & e^{-2k}((2(\gamma + \ln(k)) - Ei(2k))(1+k) - 2(1+2k))\Big] \\
= & \frac{2\pi G^2 M^2 \rho_0}{V^2} \Big[(-\ln(2) + \ln(k) + \gamma + 2) \\
+ & \frac{\frac{3}{2} + \ln(2) - \ln(k) - \gamma}{k} - \frac{3}{4} k^{-2} - \frac{1}{2} k^{-3} + O(k^{-4})\Big].
\end{aligned}
$$

We recall that $k = \frac{V^2}{\sigma^2}$ and by assumption k is large (higher order terms were neglected in our earlier approximations for the density). Hence it is as well that with comparatively small values of k, the first term in the expression for drag is dominant. For example with the massive body's velocity ten times the velocity dispersion of the field stars ($V = 10\sigma$), the higher order terms make a difference of under half a percent.

Removing the higher order terms our final expression for the drag force exerted by the density enhancement is

$$
\frac{2\pi G^2 M^2 \rho_0}{V^2}[-\ln(2) + \ln(k) + \gamma + 2] = \frac{4\pi G^2 M^2 \rho_0}{V^2} \ln\left(\frac{V}{\sigma}\frac{e^{1+\gamma/2}}{\sqrt{2}}\right)
$$

The dependency of the drag upon the field star velocity dispersion σ is slight as would be expected. The direct proportionality to initial field density ρ_0 is also not surprising. The drag scales approximately with the square of the ratio of the massive body's mass divided by its velocity, the exponential term providing a slight correction. The factor $\frac{e^{1+\gamma/2}}{\sqrt{2}} \simeq 2.565$ is negligable for large ratios $\frac{V}{\sigma}$.

This result is of the same order as previous expressions derived for the drag by alternative methods such as the Chandrasekhar dynamical friction formula (equation 7-17 in Binney & Tremaine(1987) (2)) and the similar expression in Binney & Tremaine's problem 7-11 although their expression's dependency upon velocity dispersion differs slightly from ours.

References

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