Two Generator Discrete Groups of Isometries and Their Representation

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Abstract

Let M_ϕ and M_ψ be elements of $PSL(2,\mathbb{C})$ representing orientation preserving isometries on the upper half-space model of hyperbolic 3-space ϕ and ψ respectively. The parameters

$$\beta = tr^2(M_\phi) - 4,\ \beta' = tr^2(M_\psi) - 4,\ \gamma = tr[M_\phi, M_\psi] - 2,$$

determine the discrete group $\langle \phi, \psi \rangle$ uniquely up to conjugacy whenever $\gamma \neq 0$. This thesis is concerned with explicitly lifting this parameterisation of $\langle \phi, \psi \rangle$ to $PSO(1,3)$ realised as a discrete 2 generator subgroup of orientation preserving isometries on the hyperboloid model of hyperbolic 3-space. We particularly focus on the case where both ϕ and ψ are elliptic.

Contents

Introduction

The study of hyperbolic 3-manifolds (and orbifolds) is an important part of 3-manifold theory, especially with the recent results of Perelman and Thurston, showing that most 3-manifolds (in a well defined sense) are locally modeled on hyperbolic 3-space.

Discrete subgroups of the matrix group $PSL(2, \mathbb{C})$ are intrinsically linked to this study of hyperbolic 3-manifolds and increased understanding of the classifications of these discrete groups extends to valuable computational tools. These tools inevitably require refinement and extension to find use, as computationally more difficult problems are continually attacked. The work in this thesis will assist in our extension of such computational tools in an effort to find new results including:

- A new proof of the $(\log 3)/2$ theorem of Gabai-Meyerhoff and Thurston [2];
- The search for the optimal Margulis constant for lattices [5]; and
- The classification of 2 generator arithmetic lattices [3].

All of which involve the study of 2 generator discrete groups.

In Chapter 1 we briefly describe three well known models of hyperbolic geometry: $\mathbb{U}^3$, $\mathbb{H}^3$ and $\mathbb{B}^3$. Showing these spaces to be isometric to one another, we focus on the representations of the isometry groups $Isom^+(\mathbb{U}^3)$ and $Isom^+(\mathbb{H}^3)$, which are $PSL(2, \mathbb{C})$ and $PSO(1, 3)$ respectively. Each representation has obvious computational virtues over the other, such as:

- Matrices in $PSL(2, \mathbb{C})$ have a simpler form than those in $PSO(1, 3)$; and
- Computations using $PSO(1, 3)$ involve real linear algebra as opposed to the complex non-linear algebra involved in using $PSL(2, \mathbb{C})$.

We also construct an explicit mapping between representative matrices in $PSL(2, \mathbb{C})$ and $PSO(1, 3)$ in Chapter 2; and in Chapter 3 make note of the known result that

$$\begin{aligned} \beta &= tr^2(M_\phi) - 4, \\ \beta' &= tr^2(M_\psi) - 4, \\ \gamma &= tr[M_\phi, M_\psi] - 2 \end{aligned}$$

parameterise the discrete subgroups $\langle M_\phi, M_\psi \rangle$ of $PSL(2, \mathbb{C})$, representing discrete subgroups $\langle \phi, \psi \rangle$ of $Isom^+(\mathbb{U}^3)$, uniquely up to conjugacy whenever $\gamma \neq 0$.

The major aim of this thesis is to then lift the above parameterisation of 2 generator subgroups of $PSL(2,\mathbb{C})$ to an explicit map

$$\chi : \mathbb{C}^3 \to PSO(1,3) \times PSO(1,3)$$

such that if (β, β', γ) are the complex parameters of a 2 generator discrete group, then

$$\chi(\beta, \beta', \gamma) = (A, B) \in PSO(1,3) \times PSO(1,3)$$

where $\langle A, B \rangle$ and the discrete group in $PSL(2,\mathbb{C})$ associated with these parameters are isomorphic.

This mapping is constructed via the process

$$\mathbb{C}^3 \to PSL(2,\mathbb{C}) \times PSL(2,\mathbb{C}) \to PSO(1,3) \times PSO(1,3),$$

The first part is the aforementioned parameterisation which we then combine with the isomorphism between $PSL(2,\mathbb{C})$ and $PSO(1,3)$ to give us the desired explicit construction.

Particular interest is taken in the case where both ϕ and ψ are elliptic transformations, in which case

$$(\beta, \beta', \gamma) = \left(-\sin^2(\pi/p), -\sin^2(\pi/q), z\right)$$

where p, q are natural numbers greater than or equal to 2 and z is any complex number.

Let $\tilde{\Phi}_{M_\phi}$ and $\tilde{\Phi}_{M_\psi}$ represent the elements of $PSO(1,3)$ such that $\chi(\beta, \beta', \gamma) = (\tilde{\Phi}_{M_\phi}, \tilde{\Phi}_{M_\psi})$. We will describe $\tilde{\Phi}_{M_\phi}$ and $\tilde{\Phi}_{M_\psi}$ explicitly in terms of p, q and z.

This work will allow us, in future research, to make use of the linear algebra from $PSO(1,3)$ in the computation of Dirichlet domains and other important constructs in the computational study of hyperbolic 3-orbifolds.

Chapter 1

Models of Hyperbolic 3-Space

We begin with a basic description of Möbius transformations on $\mathbb{R}^n$, building up a knowledge of orientation preserving Möbius transformations acting on $\hat{\mathbb{R}}^2$ and their actions when extended into $\hat{\mathbb{R}}^3$. We then describe the upper half-space model $\mathbb{U}^3$ of hyperbolic space, and show that these Möbius transformations can be viewed as its orientation preserving isometries.

Moving on we develop the Lorentzian 4-space $\mathbb{R}^{1,3}$ and find the hyperboloid model $\mathbb{H}^3$ of hyperbolic space embedded in this space; showing the relationship between Lorentzian transformations and the isometries of $\mathbb{H}^3$. Lastly, we project $\mathbb{H}^3$ and $\mathbb{U}^3$ into the unit disc, giving us the Conformal disc model of hyperbolic geometry $\mathbb{B}^3$; and giving us a link between the upper half-space and hyperboloid models of hyperbolic 3-space. Much of this is drawn from [1] and [9].

1.1 Möbius Transformations on $\hat{\mathbb{R}}^n$

We start with the definitions of reflections in spheres and hyperplanes in $\mathbb{R}^n$ before proceeding to describe their compositions as Möbius transformations.

1.1.1 Reflections in Spheres

Definition 1.1.1. *(Sphere)*
A sphere of radius r and with center a, in $\mathbb{R}^n$ is defined to be the set

$$S(a, r) = \{x \in \mathbb{R}^n : \|x - a\|_E = r\},$$

where $\|\cdot\|_E$ is the Euclidean norm, which is given by

$$\|x\|_E = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Note that a sphere in $\mathbb{R}^n$ is an $(n-1)$-sphere and that the unit sphere $S(0,1)$ is the sphere commonly denoted S^{n-1}.

Definition 1.1.2. *(Reflection in a Sphere)*
Let S be the $(n-1)$-sphere $S(a,r)$, then the reflection of $\mathbb{R}^n \backslash \{a\}$ in S is the mapping

$$\begin{aligned} \phi_S &: \mathbb{R}^n \backslash \{a\} \longrightarrow \mathbb{R}^n \backslash \{a\}, \\ \phi_S &: x \mapsto a + r^2 \frac{(x-a)}{\|x-a\|_E^2}. \end{aligned}$$

Notice that ϕ_S is a homeomorphism of $\mathbb{R}^n \backslash \{a\}$, $\phi_S^{-1} = \phi_S$ and the fixed points of ϕ_S are the points of S itself.

We can extend the action of ϕ_S continuously so as to act on the extended real space

$$\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\},$$

by defining

$$\phi_S(a) = \infty$$

and

$$\phi_S(\infty) = a.$$

Definition 1.1.3. *(Hyperplane)*
A hyperplane in $\mathbb{R}^n$, with parameters $a \in \mathbb{R}^n$ and $t \in \mathbb{R}$, is defined to be the set

$$P(a,t) = \{x \in \mathbb{R}^n : a \cdot x = t\}.$$

Note that a hyperplane can be viewed as a coset of an $(n-1)$-dimensional vector subspace of $\mathbb{R}^n$. We shall refer to a hyperplane, as defined above, as an $(n-1)$-plane.

Definition 1.1.4. *(Reflection in a Plane)*
Let P be the $(n-1)$-plane $P(a,t)$, then the reflection of $\mathbb{R}^n$ in P is the mapping

$$\begin{aligned} \phi_P &: \mathbb{R}^n \longrightarrow \mathbb{R}^n, \\ \phi_P &: x \mapsto x - 2a \frac{[(x \cdot a) - t]}{\|a\|_E^2}. \end{aligned}$$

Notice that, like ϕ_S, ϕ_P is a homeomorphism of $\mathbb{R}^n$, $\phi_P^{-1} = \phi_P$ and the fixed points of ϕ_P are the points of P itself.

We can extend an $(n-1)$-plane $P(a,t)$ into the extended real space by

$$\hat{P}(a,t) = P(a,t) \cup \{\infty\};$$

and continuously extend its action so that ϕ_P acts on $\hat{\mathbb{R}}^n$ by defining

$$\phi_{\hat{P}}(\infty) = \infty.$$

By extension we shall consider reflections in $(n-1)$-spheres and $(n-1)$-planes to always act on $\hat{\mathbb{R}}^n$. Let ϕ be a reflection in an $(n-1)$-sphere or an $(n-1)$-plane, then ϕ is a conformal, orientation-reversing transformation on $\hat{\mathbb{R}}^n$ and $\phi^2 = I$, the identity transformation[1].

The extension $\mathbb{R}^n \to \hat{\mathbb{R}}^n$ can be considered a stereographic projection, in which case $\hat{\mathbb{R}}^n$ is projected onto S^n; $\mathbb{R}^n$ being projected onto $S^n \backslash \{e_n\}$. In this situation both spheres and planes in $\mathbb{R}^n$ are mapped onto spheres in S^n. Subsequently we shall refer to both planes and spheres, as defined above, collectively as $(n-1)$-spheres. When we wish to distinguish between the two different types of spheres we shall use the notation $S(a,r)$ or $P(a,t)$. When we refer to $\hat{\mathbb{R}}^n$ being reflected in (a composition of) spheres it should be clear that we are referring to reflections in $(n-1)$-spheres.

Given any n-sphere S, the reflection ϕ_S leaves invariant any m-sphere $(m < n)$ orthogonal to S. Additionally let ϕ be a reflection in a $(n-1)$-sphere, or the composition of reflections in such spheres, then the action of ϕ can be determined uniquely from the action of ϕ upon $n+1$ non-collinear points.

1.1.2 Möbius Transformations

Definition 1.1.5. *(Möbius Transformations)*
A Möbius transformation acting on $\hat{\mathbb{R}}^n$ is the finite composition of reflections of $\hat{\mathbb{R}}^n$ in spheres.

Notice that if a Möbius transformation ϕ is a composition of m reflections in spheres ϕ_i

$$\phi = \phi_1 ... \phi_m,$$

then ϕ is orientation preserving if m is even, otherwise ϕ is orientation reversing.

We will use $GM(\hat{\mathbb{R}}^n)$ to denote the set of all Möbius transformations acting on $\hat{\mathbb{R}}^n$. Similarly the set of all orientation preserving Möbius transformations acting on $\hat{\mathbb{R}}^n$ is denoted $M(\hat{\mathbb{R}}^n)$. Let X be any subset of $\hat{\mathbb{R}}^n$, then we use $GM_{\hat{\mathbb{R}}^n}(X)$ and $M_{\hat{\mathbb{R}}^n}(X)$ to respectively denote the subsets of $GM(\hat{\mathbb{R}}^n)$ and $M(\hat{\mathbb{R}}^n)$ leaving X invariant. Notice that $GM_{\hat{\mathbb{R}}^n}(\hat{\mathbb{R}}^n) = GM(\hat{\mathbb{R}}^n)$ and $M_{\hat{\mathbb{R}}^n}(\hat{\mathbb{R}}^n) = M(\hat{\mathbb{R}}^n)$. It is clear that $M_{\hat{\mathbb{R}}^n}(X)$ is a subset of $GM_{\hat{\mathbb{R}}^n}(X)$ and we have the following result.

[1] We shall use I to denote identity matrices, transformations and group elements. Use should be apparent from the context.

Theorem 1.1.6. *Let X be a subset of $\hat{\mathbb{R}}^n$, then the sets $GM_{\hat{\mathbb{R}}^n}(X)$ and $M_{\hat{\mathbb{R}}^n}(X)$ form groups under composition. Further, $M_{\hat{\mathbb{R}}^n}(X)$ is a subgroup of $GM_{\hat{\mathbb{R}}^n}(X)$.*

$GM_{\hat{\mathbb{R}}^n}(X)$ is known as the *general Möbius group* on X and $M_{\hat{\mathbb{R}}^n}(X)$ is known as the *Möbius group* on X.

Our interest will lie in the group $M(\hat{\mathbb{R}}^2)$ of orientation preserving Möbius transformations acting on $\hat{\mathbb{R}}^2$. We begin by associating the extended real plane with the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, allowing us to subsequently consider $M(\hat{\mathbb{R}}^2)$ to be $M(\hat{\mathbb{C}})$, the *complex Möbius group*. We denote the complex Möbius group by $\mathcal{M}$ and will consider it a reference to both $M(\hat{\mathbb{C}})$ and $M(\hat{\mathbb{R}}^2)$, which group we are referring to should be clear from the context.

Theorem 1.1.7. *ϕ is an element of $\mathcal{M}$ if and only if ϕ is a mapping of the form*

$$\begin{aligned} \phi &: \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}, \\ \phi &: z \mapsto \frac{az+b}{cz+d}, \end{aligned} \tag{1.1}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Proof:
A reflection of $\hat{\mathbb{C}}$ in a sphere is a transformation of the form

$$z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$$

and the composition of an even number of such transformations has form

$$\phi : z \mapsto \frac{az+b}{cz+d}.$$

Thus any element of $\mathcal{M}$ is of the form 1.1.

Now

$$\phi = \frac{az+b}{cz+d} = \phi_1\phi_2\phi_3\phi_4(z),$$

where

$$\begin{aligned} \phi_1 &= z + a/c, \\ \phi_2 &= (-(ad-bc)/c^2)z, \\ \phi_3 &= 1/z, \\ \phi_4 &= z + d/c. \end{aligned}$$

Each of the ϕ_i are a composition of an even number of reflections in spheres, thus ϕ is also a composition of an even number of reflections in sphere. Hence we have ϕ is an element of $\mathcal{M}$. □

Notice that any element ϕ in $\mathcal{M}$ is determined by the four complex numbers a, b, c and d, subject to the condition $ad - bc \neq 0$. This makes it apparent that we can use matrices to represent Möbius transformations.

Definition 1.1.8. *A matrix* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc \neq 0$, *is said to induce (or represent) the Möbius transformation*

$$\phi(z) = \frac{az+b}{cz+d} \in \mathcal{M}.$$

The set of such matrices is known as the *general linear group* of 2×2 complex matrices; and is denoted $GL(2, \mathbb{C})$. $GL(2, \mathbb{C})$ is a group under matrix multiplication, and we have the following useful result.

Theorem 1.1.9. *Let* A, B *be matrices inducing Möbius transformations* ϕ_A, ϕ_B *respectively and let* ϕ_{AB} *be the Möbius transformation induced by* AB*, then*

$$\phi_A \phi_B = \phi_{AB}.$$

Proof:
Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

and note that

$$AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix},$$

then

$$\begin{aligned} \phi_A \phi_B &= (a\tfrac{ez+f}{gz+h} + b)/(c\tfrac{ez+f}{gz+h} + d) \\ &= \tfrac{(ae+bg)z+af+hb}{(ce+dg)z+cf+dh} \\ &= \phi_{AB}. \end{aligned}$$

□

This result shows that there is a homomorphism from $GL(2, \mathbb{C})$ to $\mathcal{M}$. There are two groups related to $GL(2, \mathbb{C})$ that are important to our work: the *special linear group* $SL(2, \mathbb{C})$, the subgroup of $GL(2, \mathbb{C})$ containing elements with a determinant of 1; and the *projective special linear group* $PSL(2, \mathbb{C})$, which is the matrix group $SL(2, \mathbb{C}) \backslash \{\pm I\}$.

Theorem 1.1.10. $\mathcal{M}$ *is isomorphic to* $PSL(2, \mathbb{C})$.

Proof:
Let F denote the homomorphism of $GL(2, \mathbb{C})$ onto the induced transformations in $\mathcal{M}$

$$F : GL(2, \mathbb{C}) \to \mathcal{M},$$

then $ker(F) = \{cI : c \in \mathbb{C}\}$.
By the first isomorphism theorem we have

$$F : \frac{GL(2,\mathbb{C})}{\{cI : c \in \mathbb{C}\}} \cong PSL(2,\mathbb{C}) \to \mathcal{M}$$

is an isomorphism. □

We have now shown that the group of orientation preserving Möbius transformations acting on $\hat{\mathbb{R}}^2$ is isomorphic to the matrix group $PSL(2,\mathbb{C})$. This allows us to make use of matrices in $PSL(2,\mathbb{C})$ to uniquely represent Möbius transformations acting on $\hat{\mathbb{C}}$ and hence, elements of $M(\hat{\mathbb{R}}^2)$.

In general, when we discuss the representative matrices of elements in $\mathcal{M}$ we will use elements of $SL(2,\mathbb{C})$ and understand what these matrices represent in both $PSL(2,\mathbb{C})$ and $\mathcal{M}$. Given a Möbius transformation ϕ we will use M_ϕ to denote it's representative matrix (in $SL(2,\mathbb{C})$).

1.1.3 The Poincaré Extension

We now consider $\hat{\mathbb{R}}^n$ to be embedded in $\hat{\mathbb{R}}^{n+1}$, and look at the result of extending the actions of the elements in $GM(\hat{\mathbb{R}}^n)$ into $\hat{\mathbb{R}}^{n+1}$. This will give us a means of linking the elements of $M(\hat{\mathbb{R}}^2)$ with elements of $M(\hat{\mathbb{R}}^3)$.

Identify $\hat{\mathbb{R}}^n$ with the hyperplane $\hat{\mathbb{R}}^n \times \{0\}$ in $\hat{\mathbb{R}}^{n+1}$ by the embedding

$$x \mapsto \tilde{x} = (x, 0).$$

Let ϕ be a reflection of $\mathbb{R}^n$ in any sphere, then we extended ϕ to the Möbius transformation $\tilde{\phi}$ of $\hat{\mathbb{R}}^{n+1}$ thus:

- If ϕ is the reflection of $\hat{\mathbb{R}}^n$ in the $(n-1)$-sphere $P(a,t)$, then $\tilde{\phi}$ is the reflection of $\hat{\mathbb{R}}^{n+1}$ in the n-sphere $P(\tilde{a},t)$; and
- If ϕ is the reflection of $\hat{\mathbb{R}}^n$ in the $(n-1)$-sphere $S(a,r)$, then $\tilde{\phi}$ is the reflection of $\hat{\mathbb{R}}^{n+1}$ in the n-sphere $S(\tilde{a},r)$.

Let ψ be any element of $GM(\hat{\mathbb{R}}^n)$, then there exists some reflections of $\hat{\mathbb{R}}^n$ in spheres ψ_i such that $\psi = \psi_1\psi_2...\psi_n$. This gives us a natural extension of ψ into $GM(\hat{\mathbb{R}}^{n+1})$

$$\psi = \psi_1\psi_2...\psi_n \mapsto \tilde{\psi}_1\tilde{\psi}_2...\tilde{\psi}_n = \tilde{\psi}. \tag{1.2}$$

The above extension is not affected by the choice of ψ_i.

Definition 1.1.11. *(Poincaré Extension)*
The Möbius transformation $\tilde{\psi}$ in $GM(\hat{\mathbb{R}}^{n+1})$ (as described above) is known as the Poincaré extension of the Möbius transformation ψ in $GM(\hat{\mathbb{R}}^n)$.

Definition 1.1.12. *(Upper Half-Space)*
The upper half-space of $\mathbb{R}^{n+1}$ (or $\hat{\mathbb{R}}^{n+1}$) is the set

$$U^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}.$$

Theorem 1.1.13. *Let ϕ be an element of $GM(\hat{\mathbb{R}}^n)$, then the extension $\tilde{\phi}$ leaves the hyperplane $\hat{\mathbb{R}}^n \times \{0\}$ invariant. Furthermore $\tilde{\phi}$ also leaves the half-space U^{n+1} invariant.*

Proof:
As the elements of $GM(\hat{\mathbb{R}}^n)$ leave $\hat{\mathbb{R}}^n$ invariant, it follows immediately that their extensions are elements of $GM_{\hat{\mathbb{R}}^n}(\hat{\mathbb{R}}^n \times \{0\})$ and hence leave $\hat{\mathbb{R}}^n \times \{0\}$ invariant. Given this fact, and as $\tilde{\psi}$ is a continuous mapping, ψ must either also preserve U^{n+1} or switch it with the lower half-space; but in equation 1.2 each $\tilde{\psi}_i$ composing $\tilde{\psi}$ preserves U^{n+1}, so therefore does $\tilde{\psi}$.

Suppose $\tilde{\phi}$ interchanges the two half-spaces, then it follows that $\tilde{\phi}$ must be composed of a reflection in the hyperplane $\hat{\mathbb{R}}^n \times \{0\}$; however there is no ϕ in $GM(\hat{\mathbb{R}}^n)$ extending to such a transformation. Hence ϕ must leave U^{n+1} invariant. $\square$

Theorem 1.1.14. *A Möbius transformation on $\hat{\mathbb{R}}^{n+1}$ leaves U^{n+1} invariant if and only if it is the Poincaré extension of a Möbius transformation on $\hat{\mathbb{R}}^n$.*

Trivially we have the following corollary.

Theorem 1.1.15. *$GM(\hat{\mathbb{R}}^n)$ is isomorphic to $GM_{\hat{\mathbb{R}}^n}(U^{n+1})$ and $M(\hat{\mathbb{R}}^n)$ is isomorphic to $M_{\hat{\mathbb{R}}^n}(U^{n+1})$.*

Corollary 1.1.16. *Every Möbius transformation of U^n is the finite composition of reflections of $\hat{\mathbb{R}}^n$ in spheres orthogonal to $\hat{\mathbb{R}}^{n-1}$.*

When $n = 2$ Theorem 1.1.15 implies that the groups $PSL(2, \mathbb{C})$ and $M_{\hat{\mathbb{R}}^3}(U^3)$ are isomorphic. Thus our representation of the reflections in spheres acting on $\hat{\mathbb{C}}$, serves equally well as a representation of the reflections in spheres acting on $\hat{\mathbb{R}}^3$, and preserving U^3.

1.1.4 Conjugacy Classes and Invariants

Having described the groups $M(\hat{\mathbb{R}}^n)$ and $M_{\hat{\mathbb{R}}^n}(U^{n+1})$, and a means of representing their elements when $n = 2$, we now discuss certain conjugation invariant properties of these elements and some conjugacy classes of $\mathcal{M}$.

Definition 1.1.17. *(Conjugate)*
Let g and h be distinct elements or subgroups of a group G, then g is said to be conjugate (in G) to h if and only if there exists an element p in G such that $pgp^{-1} = h$.

From a geometric viewpoint, conjugate transformations have identical actions. We shall use $\sim$ to denote conjugacy between elements.

Definition 1.1.18. *(Commutator)*
Let g and h be elements of a group G, then the commutator of g and h, denoted $[g, h]$, is defined to be

$$[g, h] = ghg^{-1}h^{-1}.$$

Definition 1.1.19. *(Trace)*
Let $M = (m_{ij})$ be any $n \times n$-square matrix, then the trace of M is defined

$$tr(M) = \sum_{i=1}^{n} m_{ii}.$$

The trace of a Möbius transformation ϕ in $\mathcal{M}$, denoted $tr(\phi)$, is defined by

$$tr(\phi) = tr(M_\phi)$$

and is determined up to sign.

Theorem 1.1.20. *Let ϕ be any element of $\mathcal{M}$, then:*

1. *$(tr(\phi))^2$, denoted $tr^2(\phi)$, is determined uniquely; and*
2. *$tr^2(\phi)$ is conjugation invariant.*

Corollary 1.1.21. *Let ϕ and ψ be elements of $\mathcal{M}$, then $tr[\phi, \psi]$ is determined independently of the choice of representatives for ϕ and ψ.*

A complex Möbius transformation acting on $\hat{\mathbb{C}}$ has either exactly one fixed point, exactly two fixed points or is the identity. This provides a basic classification of the elements of $\mathcal{M}$, we can refine this classification by observing the fixed points in $\hat{\mathbb{R}}^3$ of their Poincaré extensions. Naturally this new classification is invariant under conjugation and so it provides a classification of conjugacy classes in $\mathcal{M}$.

Definition 1.1.22. *(Standard Form)*
Let k be any non-zero complex number, then we define the Möbius transformation m_k by

$$m_1 = z + 1$$

and

$$m_k(z) = kz, \ \ (\textit{if } k \neq 1)$$

These Möbius transformations are referred to as standard forms.

The representative matrices of the standard forms are denoted M_k and have the forms

- $M_1 = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; and
- $M_k = \pm \begin{bmatrix} \sqrt{k} & 0 \\ 0 & 1/\sqrt{k} \end{bmatrix}$, where $k \neq 1$.

Corollary 1.1.23. *For all k*

$$tr^2(m_k) = k + 2 + \frac{1}{k}.$$

Theorem 1.1.24. *Let ϕ be any element of $\mathcal{M}\backslash\{I\}$, then there exists a non-zero complex number k, such that ϕ is conjugate to m_k.*

Proof:
Let ϕ be an non-identity element of $\mathcal{M}$, then either

- ϕ has exactly two fixed points ($\alpha, \beta \in \mathbb{C}$); or
- ϕ has a unique fixed point ($\alpha \in \mathbb{C}$).

In the second case we take β to be any point in $\hat{\mathbb{C}}$ other than α.

Let ψ be any element of $\mathcal{M}$ with $\psi(\alpha) = \infty$, $\psi(\beta) = 0$ and $\psi(\phi(\beta)) = 1$ if $\phi(\beta) \neq \beta$; and observe $\psi\phi\psi^{-1}(\infty) = \infty$ and $\psi\phi\psi^{-1}(0) = 0$ if $\phi(\beta) = \beta$, $\psi\phi\psi^{-1}(0) = 1$ if $\phi(\beta) \neq \beta$.

- If ϕ fixes α and β, then $\psi\phi\psi^{-1}$ fixes 0 and ∞ and so for some $k \neq 1$, we have $\psi\phi\psi^{-1} = m_k$.
- If ϕ fixes α only then $\psi\phi\psi^{-1}$ fixes ∞ only and $\psi\phi\psi^{-1}(0) = 1$: thus $\psi\phi\psi^{-1} = m_1$.

This shows that any Möbius transformation $\phi(\neq I)$ is conjugate to one of the standard forms m_k. □

And we state an important corollary to Theorems 1.1.20 and 1.1.24.

Corollary 1.1.25. *Let ϕ and ψ be non identity complex Möbius transformations, then ϕ and ψ are conjugate in $\mathcal{M}$ if and only if $tr^2(\phi) = tr^2(\psi)$.*

By these results we can separate $\mathcal{M}$ into conjugacy classes, each with its own standard form; and we now classify the conjugacy classes of $\mathcal{M}$ in terms of the fixed points in $\hat{\mathbb{R}}^3$ of their Poincaré extensions. Naturally consideration only needs to be given to the fixed points of the standard forms. Thus we have

1. $\tilde{m_1}$ fixes ∞ but no other point in $\hat{\mathbb{R}}^3$;
2. if $|k| \neq 1$, then $\tilde{m_k}$ fixes 0 and ∞ but no other points in $\hat{\mathbb{R}}^3$; and
3. if $|k| = 1, k \neq 1$, then the set of fixed points of $\tilde{m_k}$ are $\{te_3 : t \in \mathbb{R}\} \cup \{\infty\}$.

Which leads us to the following well known classification.

Definition 1.1.26. *(classification of the conjugacy classes of $\mathcal{M}$)*
Let ϕ be any element of $\mathcal{M}\backslash\{I\}$, then we say

1. *ϕ is parabolic if and only if ϕ has a unique fixed point in $\hat{\mathbb{C}}$ (equivalently $\phi \sim m_1$);*
2. *ϕ is loxodromic if and only if $\tilde{\phi}$ has exactly two fixed points in $\hat{\mathbb{R}}^3$ (equivalently $\phi \sim m_k$, where $|k| \neq 1$);*
3. *ϕ is elliptic if and only if $\tilde{\phi}$ has infinitely many fixed points in $\hat{\mathbb{R}}^3$ (equivalently $\phi \sim m_k$, where $|k| = 1$, $k \neq 1$).*

This classification is based on which standard form the complex Möbius transformation is conjugate to, by our previous results, tr^2 is conjugation invariant, so we must also be able to classify ϕ according to the value of $tr^2(\phi)$.

However before we do so we note that it is convenient to subdivide the loxodromic class by reference to invariant discs rather than just fixed points.

Definition 1.1.27. *(Hyperbolic and strictly loxodromic transformations)*
Let ϕ be a loxodromic transformation.

1. *If $\phi(D) = D$ for some open disc (or half-plane) D in $\hat{\mathbb{C}}$, then we say that ϕ is hyperbolic;*
2. *Otherwise ϕ is said to be strictly loxodromic*[2].

[2]Note that this usage is not universal, it is common for authors to use the term "loxodromic" for what we have defined to be "strictly loxodromic" and have no name for our more general "loxodromic transformations".

Our previous results combine to give the following theorem, parameterising the conjugacy classes of $\mathcal{M}$.

Theorem 1.1.28. *Let ϕ be any element of $\mathcal{M}\backslash\{I\}$, then*

1. *ϕ is parabolic if and only if $tr^2(\phi) = 4$;*
2. *ϕ is elliptic if and only if $tr^2(\phi) \in [0, 4)$;*
3. *ϕ is hyperbolic if and only if $tr^2(\phi) \in (4, +\infty)$;*
4. *ϕ is strictly loxodromic if and only if $tr^2(\phi) \notin [0, +\infty)$.*

By Definition 1.1.26 and Theorem 1.1.24 we obtain the following corollary.

Corollary 1.1.29. *Let ϕ be any element of $\mathcal{M}\backslash\{I\}$, if ϕ has finite order then ϕ is elliptic.*

Proof:
If ϕ is of finite order then there exists some p in $\mathbb{N}$ such that $\phi^p = I$. By Theorem 1.1.24 $\phi \sim m_k$ for some k and we must have that $m_k^p = I$.
$m_k^p = k^p z = z$ implies $|k| = 1$ which by Definition 1.1.26 shows m_k (and hence ϕ) is either elliptic or the identity. □

Theorem 1.1.30. *Let ϕ and ψ be any two complex Möbius transformations, then ϕ and ψ have a common fixed point in $\hat{\mathbb{C}}$ if and only if $tr[\phi, \psi] = 2$.*

Proof:
By conjugation we may assume

$$M_\phi = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, M_\psi = \begin{bmatrix} e & f \\ g & h \end{bmatrix},$$

and it follows that

$$tr[\phi, \psi] = 2 + b^2 g^2 + b(a-d)g(e-h) - (a-d)^2 fg.$$

If we assume ϕ and ψ have a fixed point, then we may choose it to be ∞, implying $g = 0$ and giving $tr[\phi, \psi] = 2$.

If $tr[\phi, \psi] = 2$, then either $b = 0$ or $b \neq 0$.

- If $b = 0$, then ϕ fixes 0 and ∞ and we have $fg = 0$ so ψ fixes either 0 or ∞.
- If $b \neq 0$ then ϕ fixes ∞ and we have $g = 0$ hence ψ must fix ∞ also.

□

Theorem 1.1.31. *Let g be an elliptic transformation of order p. Then*

$$tr^2(g) \leq 4\cos^2(\pi/p)$$

with equality if and only if g is a rotation of angle $\pm 2\pi/p$.

Proof:
Let ϕ be a complex Möbius transformation of finite order p, then by Theorem 1.1.29, ϕ is necessarily elliptic. In this case $\phi(z) \sim e^{i\theta}z$, with $\theta = 2\pi m/p$ for coprime p and m. It follows that

$$\begin{aligned} tr^2(\phi) &= 4\cos^2(\theta/2) \\ &= 2[1+\cos(2\pi m/p)]. \end{aligned}$$

Thus the value of $tr^2(\phi)$ is variable depending on the prime factors of p. Notice that the largest value of $tr^2(g)$ occurs when either $m = 1$ or $m = p-1$, in which case $\theta = \pm 2\pi/p$ and

$$tr^2(g) = 4\cos^2(\pi/p).$$

□

Having parameterised the conjugacy classes of $\mathcal{M}$ it should be clear that, as every element in $SL(2,\mathbb{C})$ induces an element of $\mathcal{M}$, we can easily extend these conjugacy classes to be defined on $SL(2,\mathbb{C})$ (and $PSL(2,\mathbb{C})$) based on the value of tr^2 and equivalently what type of transformation they induce.

Further details on the properties of Möbius transformations can be found in [1], [7], [9]; Additionally [8] details complex Möbius transformations from a more geometric viewpoint.

1.2 The Upper Half-Space Model, $\mathbb{U}^3$

Recall the upper half-space of $\hat{\mathbb{R}}^3$ is defined to be the set

$$U^3 = \{x \in \mathbb{R}^n : x_3 > 0\}.$$

We define a metric d_U, on the space U^3, by

$$d_U(x,y) = \cosh^{-1}\left(1 + \frac{|x-y|^2}{2x_3y_3}\right)$$

for all x and y in U^3.

This metric d_U is a hyperbolic metric, and the metric space (U^3, d_U) is known as the *upper half-space model* of hyperbolic 3-space, which we shall denote[3] as $\mathbb{U}^3$.

[3]It is common in literature to refer to the upper half-space as $\mathbb{H}^3$, however as we will also be discussing the hyperboloid model, we choose to use $\mathbb{U}^3$ in our work.

1.2.1 Geodesics and Isometries

The geodesics of $\mathbb{U}^3$ are the Euclidean semicircles in U^3 which are orthogonal to $\hat{\mathbb{R}}^2 \times \{0\}$, together with the vertical lines in U^3.

Consequently the geodesic hyperplanes of $\mathbb{U}^3$ are the 2-planes $P(\tilde{a}, t)$ and 2-spheres $S(\tilde{a}, r)$, as defined in Section 1.1.3, intersected with U^3. The isometries of $\mathbb{U}^3$ being compositions of reflections of U^3 in these spheres and planes.

We use $Isom(X)$ to denote the isometry group of a metric space X and $Isom^+(X)$ to denote the subgroup consisting of all orientation preserving isometries. And we have the following theorem.

Theorem 1.2.1. *Every element of $GM_{\hat{\mathbb{R}}^3}(U^3)$ restricts to a unique element of $Isom(\mathbb{U}^3)$ and every element of $Isom(\mathbb{U}^3)$ extends to a unique element of $GM_{\hat{\mathbb{R}}^3}(U^3)$.*

Equivalently we have

Corollary 1.2.2. *$Isom(\mathbb{U}^3)$ and $GM_{\hat{\mathbb{R}}^3}(U^3)$ are isomorphic.*

The boundary, or sphere at infinity, of $\mathbb{U}^3$ is $\hat{\mathbb{R}}^2$. From our previous results relating to $GM_{\hat{\mathbb{R}}^3}(U^3)$ and $GM(\hat{\mathbb{R}}^2)$ it is clear that the isometries of $\mathbb{U}^3$ are uniquely determined by their actions on the sphere at infinity.

Corollary 1.2.3. *$Isom^+(\mathbb{U}^3)$ is isomorphic to $M_{\hat{\mathbb{R}}^3}(U^3)$; hence $Isom^+(\mathbb{U}^3)$ is also isomorphic to $\mathcal{M}$.*

By these last results we have that the group $PSL(2, \mathbb{C})$ can be taken to act as a representation of the isometry group $Isom^+(\mathbb{U}^3)$.

1.3 Lorentzian n-Space

Having described the upper half-space model we now concentrate on defining the second model of hyperbolic 3-space that is of interest in our investigation, the hyperboloid model $\mathbb{H}^3$.

We find the hyperboloid model as a 3-space, with negative curvature, embedded in Lorentzian 4-space. Thus we begin with a brief description of Lorentzian n-space, under the assumption $n \geq 2$.

Definition 1.3.1. *(Lorentzian inner product)*
We define an indefinite inner product on $\mathbb{R}^n$ *by*

$$x \circ y = -x_1 y_1 + \sum_{i=2}^{n} x_i y_i,$$

where x *and* y *are any two elements of* $\mathbb{R}^n$*.*

We shall refer to this inner product as the Lorentzian inner product (on $\mathbb{R}^n$*).*

The inner product space of $\mathbb{R}^n$ with this inner product is known as *Lorentzian n-space*[4] and is denoted $\mathbb{R}^{1,n-1}$.

Definition 1.3.2. *(Lorentzian Norm)*
We define an indefinite norm on $\mathbb{R}^n$ *by*

$$\|x\|_L = (x \circ x)^{1/2}.$$

where x *is any element of* $\mathbb{R}^n$*.*

This norm is induced by the Lorentzian inner product and we refer to it as the Lorentzian norm (on $\mathbb{R}^n$*).*

Definition 1.3.3. *(Lorentzian distance)*
We define a Lorentzian distance function between two vectors x *and* y *(in* $\mathbb{R}^n$*) by*

$$d_L(x, y) = \|x - y\|_L.$$

It is worth noting that the values of $\| \cdot \|_L$, and hence d_L, can be either positive real, zero or positive imaginary.

Definition 1.3.4. *(Classification of vectors in* $\mathbb{R}^{1,n}$*)*
Let x *be any vector in* $\mathbb{R}^n$*, then*

1. x *is said to be light-like if and only if* $\|x\|_L = 0$*;*
2. x *is said to be space-like if and only if* $\|x\|_L \in \mathbb{R}^+$*;*
3. x *is said to be time-like if and only if* $\|x\|_L \in i\mathbb{R}^+$*.*

[4] The space $\mathbb{R}^{1,3}$ is also commonly referred to as Minkowski space, though we will continue to refer to it as Lorentzian 4-space.

Definition 1.3.5. *(polarity of time-like vectors)*
A time-like vector $x = (x_1, x_2, ..., x_n)$ is said to be:

- *positive if and only if $x_1 > 0$;*
- *negative if and only if $x_1 < 0$.*

The set of all light-like vectors in $\mathbb{R}^n$ form a hypercone, denoted C^{n-1}, which we refer to as the *light-cone* of $\mathbb{R}^n$. The exterior of C^{n-1} (in $\mathbb{R}^n$) is the set of all space-like vectors of $\mathbb{R}^n$; and the interior of C^{n-1} (in $\mathbb{R}^n$) is the set of all time-like vectors of $\mathbb{R}^n$. We refer to the subset of C^{n-1} containing only positive light-like vectors as the *positive light-cone*.

1.3.1 Lorentzian Transformations

Our interest lies in the specific classes of transformations of $\mathbb{R}^{1,n}$, known as positive Lorentz transformations and positive special Lorentz transformations. These transformations correlate directly with the isometry groups $Isom(\mathbb{H}^3)$ and $Isom^+(\mathbb{H}^3)$ respectively.

Definition 1.3.6. *(Lorentz orthonormal basis)*
Let $(v_1, ..., v_n)$ be a basis of $\mathbb{R}^n$, then $(v_1, ..., v_n)$ is said to be a Lorentz orthonormal basis if and only if

$$v_i \circ v_j = \begin{cases} -1 & : \quad i = j = 1 \\ \delta_{i,j} & : \quad \textit{otherwise.} \end{cases}$$

Note that the standard basis of $\mathbb{R}^n$ is a Lorentz orthonormal basis.

Definition 1.3.7. *(Lorentz Transformation)*
A function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transformation if and only if

$$\phi(x) \circ \phi(y) = x \circ y,$$

for all vectors x and y in $\mathbb{R}^n$.

We also have an equivalent definition

Theorem 1.3.8. *(Lorentz Transformation) A function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transformation if and only if ϕ is linear and the ordered set*

$$\{\phi(e_1), \phi(e_2), ..., \phi(e_n)\}$$

forms a Lorentz orthonormal basis of $\mathbb{R}^n$.

Proof:
Let ϕ be any Lorentz transformation of $\mathbb{R}^n$, then

$$\begin{aligned}\phi(e_1)\circ\phi(e_1) &= e_1\circ e_1 = -1\\ \phi(e_i)\circ\phi(e_j) &= e_i\circ e_j = \delta_{i,j} \text{ otherwise.}\end{aligned}$$

Therefore $\{\phi(e_1), ...\phi(e_n)\}$ is an orthonormal basis of $\mathbb{R}^n$.

Now let $x = \sum_{i=1}^n x_i e_i$ be an element in $\mathbb{R}^n$, then there exists coefficients c_i, such that

$$\phi(x) = \sum_{i=1}^n c_i \phi(e_i).$$

As $\{\phi(e_1), ...\phi(e_n)\}$ is a Lorentz orthonormal basis, it follows that, for $j \neq 1$

$$\begin{aligned}-c_1 &= \phi(x)\circ\phi(e_1) = x\circ e_1 = -x_1\\ c_j &= \phi(x)\circ\phi(e_j) = x\circ e_j = x_j.\end{aligned}$$

and it follows that ϕ is linear.

Conversely, suppose ϕ is linear and $\{\phi(e_1), ...\phi(e_n)\}$ is a Lorentz orthonormal basis of $\mathbb{R}^n$. Then by the following, ϕ is a Lorentz transformation.

$$\begin{aligned}\phi(x)\circ\phi(y) &= \phi(\sum_{i=1}^n x_i e_i)\circ\phi(\sum_{i=1}^n y_i e_i)\\ &= \sum_{i=1}^n\sum_{j=1}^n x_i y_i \phi(e_i)\circ\phi(e_j)\\ &= x\circ y.\end{aligned}$$

□

Thus the set of Lorentz transformations on $\mathbb{R}^n$ forms a group under function composition, referred to as the *group of Lorentz transformations* of $\mathbb{R}^n$.

Definition 1.3.9. *(Lorentzian Matrix)*
Let A be any real $n \times n$ matrix, then A is said to be Lorentzian if and only if the associated linear transformation

$$A(x) = Ax$$

is a Lorentzian transformation.

The set of all Lorentzian $n \times n$ matrices forms a group under matrix multiplication, denoted $O(1, n-1)$; and is known as the *Lorentz group* of $n \times n$ matrices.

Theorem 1.3.10. *The group of Lorentz transformations of $\mathbb{R}^n$ is isomorphic to $O(1, n-1)$.*

From Theorem 1.3.8 we have the following result.

Theorem 1.3.11. *Let A be a real $n \times n$ matrix, then the following are equivalent.*

1. *A is Lorentzian;*
2. *A satisfies the equation $A^t J A = J$;*
3. *A satisfies the equation $A J A^t = J$;*
4. *the columns of A form a Lorentz orthonormal basis of $\mathbb{R}^n$;*
5. *the rows of A form a Lorentz orthonormal basis of $\mathbb{R}^n$.*

where J is the diagonal matrix with entries $J_{1,1} = -1$ and $J_{i,j} = \delta_{i,j}$ for all other i and j.

Let A be any Lorentzian matrix, by Theorem 1.3.11 we have $A^t J A = J$. Thus $(det(A))^2 = 1$ and $det(A) = \pm 1$.

Consider the set

$$SO(1, n-1) = \{A \in O(1, n-1) : det(A) = 1\}.$$

$SO(1, n-1)$ is a subgroup of order two in $O(1, n-1)$ and consists of all the orientation preserving elements of $O(1, n-1)$. $SO(1, n-1)$ is known as the *special Lorentz group.*

Definition 1.3.12. *(polarity of Lorentz Matrices)*
A Lorentz matrix A is said to be positive if and only if A transforms positive time-like vectors into positive time-like vectors.

Similarly, a Lorentz matrix A is said to be negative if and only if A transforms positive time-like vectors into negative time-like vectors.

Let $PO(1, n-1)$ denote the set of all positive elements in $O(1, n-1)$, then $PO(1, n-1)$ is a subgroup of order two in $O(1, n-1)$; known as the *positive Lorentz group.* Similarly let $PSO(1, n-1)$ denote the set of all positive elements in $SO(1, n-1)$, then $PSO(1, n-1)$ is a subgroup of order two in $SO(1, n-1)$; known as the *positive special Lorentz group.* These positive transformations preserve the positive light cone, thus $PSO(1, n-1)$ is the group of all Lorentz transformations on $\mathbb{R}^{1,n-1}$ which preserve both orientation and the positive light cone.

Definition 1.3.13. *(Vector subspaces of $\mathbb{R}^n$)*
Let V be any vector subspace of $\mathbb{R}^n$, then V is said to be:

1. *time-like if and only if V contains a time-like vector;*

2. *space-like if and only if every element in* $V \backslash \{\boldsymbol{0}\}$ *is space-like; or*
3. *light-like otherwise.*

Theorem 1.3.14. *For each dimension $m < n$, the action of $PO(1, n-1)$ is transitive on the set of m dimensional time-like vector subspaces of $\mathbb{R}^n$.*

Theorem 1.3.15. *Let x, y be positive (negative) time-like vectors in $\mathbb{R}^n$, then*

$$x \circ y \leq \|x\|_L \|y\|_L$$

with equality if and only if x and y are linearly dependent.

Proof:
It follows from Theorem 1.3.14 that there exists an element A of $PO(1, n-1)$ such that $Ax = te_1$. As A preserves the Lorentzian inner product, we can replace x and y by Ax and Ay, such that $x = x_1 e_1$.

Thus

$$\begin{aligned} \|x\|_L^2 \|y\|_L^2 &= -x_1^2(-y_1^2 + y_2^2 ... + y_n^2) \\ &\leq x_1^2 y_1^2 = (x \circ y)^2 \end{aligned}$$

with equality if and only if $y = y_1 e_1$.

As $x \circ y = -x_1 y_1 < 0$, we have that

$$x \circ y \leq \|x\|_L \|y\|_L$$

with equality if and only if x and y are linearly dependent. □

As a direct consequence we have the following corollary.

Corollary 1.3.16. *Let x, y be any two positive (negative) time-like vectors in $\mathbb{R}^n$, then there exists a unique number $\eta(x, y) \in \mathbb{R}^*$, such that*

$$x \circ y = \|x\|_L \|y\|_L \cosh \eta(x, y).$$

This motivates the definition.

Definition 1.3.17. *(Time-like angle between Time-like vectors)*
Let x, y be positive (negative) time-like vectors in $\mathbb{R}^n$. Then the time-like angle between x and y is defined to be the number $\eta(x, y)$ such that $x \circ y = \|x\|_L \|y\|_L \cosh \eta(x, y)$.

For more details on the space $\mathbb{R}^{1,n}$ see [9].

1.4 Hyperboloid Model, $\mathbb{H}^3$

We now restrict our view to Lorentz 4-space and, within it, construct the hyperboloid model of hyperbolic 3-space. We begin by defining a 3-sphere of imaginary radius in $\mathbb{R}^{1,3}$, this requires the Lorentzian norm.

$$F^3 = \{x \in \mathbb{R}^{1,3} : \|x\|_L^2 = -1\}.$$

In $\mathbb{R}^4$ the set F^n is a hyperboloid of two sheets defined by the equation

$$x_1^2 - (x_2^2 + ... + x_{n+1}^2) = 1.$$

We concentrate on the positive sheet of F^3

$$H^3 = \{x \in \mathbb{R}^{1,3} : \|x\|_L^2 = -1, x_1 > 0\}$$

and discard the negative sheet of F^3, or equivalently, identify antipodal vectors in F^3.

We define a metric on H^3, denoted d_H, defined by

$$d_H(x, y) = \eta(x, y),$$

where $\eta(x, y)$ is the Lorentzian time-like angle between x and y.

Alternatively, by Definition 1.3.17 and as x and y are elements of H^3, we have the equation

$$d_H(x, y) = \cosh^{-1}(-x \circ y).$$

The metric d_H is a hyperbolic metric, and the metric space (H^3, d_H) is known as the *hyperboloid model* of hyperbolic 3-space, which we denote by $\mathbb{H}^3$.

1.4.1 Geodesics and Isometries

The geodesics of $\mathbb{H}^3$ are the intersections of H^3 with 2-dimensional time-like vector subspaces of $\mathbb{R}^{1,3}$. Thus every isometry is an orthogonal transformation of $\mathbb{R}^{1,3}$ which preserve the Lorentzian norm and preserve H^3. From these facts we have the following results.

Theorem 1.4.1. *Every positive Lorentz transformation of $\mathbb{R}^{1,3}$ restricts to an isometry of $\mathbb{H}^3$, and every isometry of $\mathbb{H}^3$ extends to a unique positive Lorentz transformation of $\mathbb{R}^{1,3}$.*

Corollary 1.4.2. *The group $Isom(\mathbb{H}^3)$ is isomorphic to the positive Lorentz group $PO(1, 3)$.*

As with $\mathbb{U}^3$ we are concerned with the subgroup of orientation preserving isometries, and by previous discussions in Section 1.3.1 we have

Corollary 1.4.3. *$Isom^+(\mathbb{H}^3)$ is isomorphic to $PSO(1,3)$.*

Henceforth when discussing elements of $Isom^+(\mathbb{H}^3)$ we will use elements of $PSO(1,3)$, making use of the aforementioned isomorphism.

1.5 The Conformal Ball Model, $\mathbb{B}^3$

We have described two models of hyperbolic 3-space, in this subsection we will define two functions which map both of these models into B^3, the unit ball. These mappings are isometric, and so give us an additional model of hyperbolic 3-space, but more importantly compose to give an isometry between the two models $\mathbb{U}^3$ and $\mathbb{H}^3$.

1.5.1 Mapping U^3 and H^3 into B^3

Transforming U^3 into B^3

Consider the unit 3-ball, defined by

$$B^3 = \{x \in \mathbb{R}^3 : \|x\|_E \leq 1\}.$$

The boundary of which is the unit 2-sphere, S^2.

Let $a = (0,0,1)$ and let ϕ_1 be the reflection of $\hat{\mathbb{R}}^3$ in the sphere $S(a, \sqrt{2})$, additionally let ϕ_2 be the reflection of $\hat{\mathbb{R}}^3$ in the (x_1, x_2)-plane. It is clear that

- ϕ_1 and ϕ_2 are orientation reversing elements of $GM(\hat{\mathbb{R}}^3)$;
- ϕ_1 interchanges the L^3 and B^3; and
- ϕ_2 interchanges the U^3 and L^3.

Thus, if we let $\phi = \phi_1\phi_2$, then ϕ is an element of $M(\hat{\mathbb{R}}^3)$ interchanging B^3 and U^3.

Projecting H^3 into B^3

Let ψ be the stereographic projection,

$$\psi : H^3 \to B^3$$

$$\psi : y \mapsto \left(\frac{y_2}{1+y_1}, \frac{y_3}{1+y_1}, \frac{y_4}{1+y_1}\right).$$

Then ψ is a conformal homeomorphism projecting H^3 onto B^3.

1.5.2 The Conformal Ball Model, $\mathbb{B}^3$

We can make the mapping ϕ an isometry, by defining a metric d_{B1} on B^3

$$d_{B1}(x,y) = d_U(\phi^{-1}(x), \phi^{-1}(y)).$$

Similarly we can make ψ an isometry, by defining a metric d_{B2} on B^3

$$d_{B2}(x,y) = d_H(\psi^{-1}(x), \psi^{-1}(y)).$$

By direct calculation we find that

$$d_{B1} = d_{B2}.$$

We rename this metric as d_B, hence we have

$$d_B(x,y) = \cosh^{-1}\left(1 + \frac{2\|x-y\|_E^2}{(1-\|x\|_E^2)(1-\|y\|_E^2)}\right),$$

where x and y are elements of B^3. This metric is a hyperbolic metric and the metric space (B^3, d_B) is known as the *conformal ball model* of hyperbolic 3-space, which we shall denote $\mathbb{B}^3$.

As an element of $M(\hat{\mathbb{R}}^3)$, ϕ maps spheres to spheres; and the mapping ψ maps the geodesics in $\mathbb{H}^3$ onto the parts of circles and lines (in $\mathbb{R}^3$) orthogonal to the sphere S^2 which are contained in B^3. As both these mappings are isometries it follows that the geodesics of $\mathbb{B}^3$ are the circles and lines in $\mathbb{R}^3$ orthogonal to the sphere S^2 (intersected with B^3).

Similarly the isometries of $\mathbb{B}^3$ are the reflections in spheres orthogonal to the boundary S^2, the groups $Isom(\mathbb{B}^3)$ and $Isom^+(\mathbb{B}^3)$ being isomorphic to $GM_{\hat{\mathbb{R}}^3}(\mathbb{B}^3)$ and $M_{\hat{\mathbb{R}}^3}(\mathbb{B}^3)$ respectively. Thus we have the following result.

Theorem 1.5.1. *$Isom^+(\mathbb{B}^3)$, $PSL(2,\mathbb{C})$ and $PSO(1,3)$ are all isomorphic.*

1.5.3 $\mathbb{U}^3 \leftrightarrow \mathbb{H}^3$

As both $\mathbb{H}^3$ and $\mathbb{U}^3$ are isometric to $\mathbb{B}^3$ it follows that $\mathbb{H}^3$ and $\mathbb{U}^3$ are isometric to one another.

Let $\zeta = \psi^{-1}\phi$, then ζ is an isometry between $\mathbb{U}^3$ and $\mathbb{H}^3$, thus we have

$$d_U(x,y) = d_H(\zeta(x), \zeta(y))$$

and

$$d_H(x,y) = d_U(\zeta^{-1}(x), \zeta^{-1}(y)).$$

We now have a link between the two models of hyperbolic space we are interested in and could calculate $\zeta(u)$ and $\zeta^{-1}(v)$ for generic elements u of $\mathbb{U}^3$ and v of $\mathbb{H}^3$ respectively.

Having established this mapping, we have that the isometries on one space must have actions comparable to the isometries on the other. From this it follows that the conjugacy classes of $Isom^+(\mathbb{U}^3)$ also exist in similar form, and with similar actions, in $Isom^+(\mathbb{H}^3)$. For a more in depth discussion on the $\mathbb{U}^n$, $\mathbb{H}^n$ and $\mathbb{B}^n$ and the isometries between them, see [9].

In the next chapter we will develop an explicit means of mapping the elements of $Isom^+(\mathbb{U}^3)$ onto the elements of $Isom^+(\mathbb{H}^3)$.

Chapter 2

An Isomorphism Between $PSL(2, \mathbb{C})$ and $PSO(1, 3)$

We have seen in the previous chapter that the metric spaces $\mathbb{U}^3$ and the $\mathbb{H}^3$ are isometric models of hyperbolic 3-space and that from this it must follow that there exists a isomorphism between the groups $Isom^+(\mathbb{U}^3)$ and $Isom^+(\mathbb{H}^3)$. We wish to find an explicit isomorphism between the two representative matrix groups, $PSL(2, \mathbb{C})$ and $PSO(1, 3)$.

The mapping between $\mathbb{H}^3$ and $\mathbb{U}^3$ given in the previous chapter should give rise to such an isomorphism, but to make use of this isometry to gain such a mapping we would need to:

- choose a representative matrix in $PSL(2, \mathbb{C})$;
- determine its action upon $\mathbb{U}^3$;
- map this action onto $\mathbb{H}^3$;
- determine the element of $PSO(1, 3)$ representing this action; and
- make this construction compatible with the group action.

Instead of going to all this trouble, calculating the isometries by the respective actions on each space, we make use of an alternative and known isomorphism and isometry which takes advantage of some of the particular properties in the matrix representations we are using.

2.1 Hermitian Matrices

Starting from a description of a 2×2 Hermitian matrix we show that, with an appropriate norm and inner product, the space of all such Hermitian matrices is isometric to the Lorentz 4-space $\mathbb{R}^{1,3}$.

Definition 2.1.1. *(Hermitian conjugate)*
Let M be any complex matrix, then the Hermitian conjugate of M, denoted M^*, is the

transpose of the complex conjugate of M.

$$M^* = \overline{M}^T.$$

Definition 2.1.2. *(Hermitian matrix)*
A matrix, M, is called a Hermitian matrix if and only if

$$M = M^*.$$

Trivially we get an equivalent definition of a Hermitian matrix when we restrict our view to 2×2 complex matrices.

Theorem 2.1.3. *A 2×2 complex matrix M is a Hermitian matrix if and only if*

$$M = \begin{bmatrix} a & c + di \\ c - di & b \end{bmatrix},$$

for some real numbers a, b, c and d.

Let $\mathbb{V}$ denote the set of all 2×2 complex Hermitian matrices, then it is apparent that $\mathbb{V}$ is a 4-dimensional real vector space.

We give $\mathbb{V}$ the following basis:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; B_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; B_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}; B_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now for any M in $\mathbb{V}$ we have the basis decomposition

$$M = \begin{bmatrix} \alpha & \gamma \\ \bar{\gamma} & \beta \end{bmatrix} = \frac{\alpha + \beta}{2} B_1 + \Re(\gamma) B_2 + \Im(\gamma) B_3 + \frac{\alpha - \beta}{2} B_4.$$

This leads to the following corollary.

Corollary 2.1.4. *M is an element of $\mathbb{V}$ if and only if*

$$M = \begin{bmatrix} x_1 + x_4 & x_2 + ix_3 \\ x_2 - ix_3 & x_1 - x_4 \end{bmatrix}$$

for some real numbers x_1, x_2, x_3 and x_4.

Notice that for any M in $\mathbb{V}$

$$det(M) = (x_1 + x_4)(x_1 - x_4) - (x_2 + ix_3)(x_2 - ix_3) = x_1^2 - x_2^2 - x_3^2 - x_4^2,$$

from which it follows that the determinant of any matrix in $\mathbb{V}$ is real.

Theorem 2.1.5. *Let $\|x\|_V = \sqrt{-det(x)}$, then $\|x\|_V$ defines a indefinite norm on $\mathbb{V}$.*

Proof:
$\|x\|_V^2 = -det(x) = -x_1^2 + x_2^2 + x_3^2 + x_4^2$ is an equivalent of the Lorentzian norm seen in Definition 1.3.2. □

Corollary 2.1.6. *Given any $x \in \mathbb{V}$, $\|x\|_V$ is either positive real, zero or positive imaginary.*

As with $\mathbb{R}^{1,3}$ we obtain an indefinite inner product on $\mathbb{V}$.

Theorem 2.1.7. *The norm $\|x\|_V$ on $\mathbb{V}$ induces the inner product*

$$\langle M, N\rangle_V = (1/2)(det(M) + det(N) - det(M+N)), \forall M, N \in \mathbb{V}.$$

Proof:
An indefinite inner product needs to satisfy

$$\langle M+N, M+N\rangle = \langle M, M\rangle + 2\langle M, N\rangle + \langle N, N\rangle$$

which is equivalent to

$$-det(M+N) = -det(M) + 2\langle M, N\rangle - det(N).$$

□

Relative to this inner product the basis vectors $\{B_1, B_2, B_3, B_4\}$, of $\mathbb{V}$, are mutually orthogonal and have squared norms $\{-1, +1, +1, +1\}$ respectively.

Thus we have the following theorem.

Theorem 2.1.8. *The vectors $\{B_1, B_2, B_3, B_4\}$ form a Lorentz-orthonormal basis for $\mathbb{V}$.*

From all these results the following theorem is clear.

Theorem 2.1.9. *The inner product space $\mathbb{V}$ is isometric to the space $\mathbb{R}^{1,3}$.*

In conclusion, these results show that in $\mathbb{R}^{1,3}$ and $\mathbb{V}$ we find two representations of the same inner product space. Given this fact we can use either space interchangeably and any action on one representation can be mirrored exactly in the other representation.

It follows that all of our definitions and results with respect to $\mathbb{R}^{1,3}$ in Section 1.3, carry over to equivalent results for $\mathbb{V}$. Thus we may talk about the light cone in $\mathbb{V}$ and know that the group of orientation preserving isometries of $\mathbb{V}$ preserving the positive light cone is isomorphic to $PSO(1,3)$. Furthermore it follows that we must be able to find a model of hyperbolic 3-space in $\mathbb{V}$ identical in form to $\mathbb{H}^3$.

2.2 Mapping $SL(2,\mathbb{C})$ into $Isom(\mathbb{V})$

We define a mapping which links the elements in $SL(2,\mathbb{C})$ to the actions of certain isometries of the metric space $\mathbb{V}$. This will in turn lead to a way to map the representative matrices of $Isom^+(\mathbb{U}^3)$ into the representative matrices of $Isom^+(\mathbb{H}^3)$.

Definition 2.2.1. *Define a function* Φ *by*

$$\begin{aligned}\Phi &: SL(2,\mathbb{C}) \to Isom(\mathbb{V}),\\ \Phi &: A \mapsto A\mathbb{V}A^*.\end{aligned}$$

Our interest lies in the transformation

$$\Phi(A)(X) \;\;:\;\; X \mapsto AXA^*.$$

where A can be any element of $SL(2,\mathbb{C})$. We will subsequently use $\Phi_A(X)$ to denote this transformation.

Theorem 2.2.2. *If A is any element of $SL(2,\mathbb{C})$, then Φ_A is a linear function.*

Proof:

$$\begin{aligned}\Phi_A(uX+vY) &= A(uX+vY)A^*\\ &= uAXA^* + vAYA^*\\ &= u\Phi_A(X) + v\Phi_A(Y).\end{aligned}$$

□

Theorem 2.2.3. *If A is an element of $SL(2,\mathbb{C})$, then Φ_A preserves the norm $\|\cdot\|_V$.*

Proof:
First we note that given any A in $SL(2,\mathbb{C})$, it follows trivially that A^* is also in $SL(2,\mathbb{C})$. Hence

$$det(AMA^*) = det(A)det(M)det(A^*) = det(M).$$

□

From this theorem it follows that

Corollary 2.2.4. *Let A be any matrix in $SL(2,\mathbb{C})$, then Φ_A is an isometry on the space $\mathbb{V}$.*

Not only is Φ_A an isometry but the mapping Φ preserves the group structure of $PSL(2,\mathbb{C})$, as seen in the next two results.

Theorem 2.2.5. *Φ is a homomorphism from $SL(2,\mathbb{C})$ into $Isom(\mathbb{V})$.*

Proof:
We have already shown that Φ_A is an isometry, so we need only show that Φ is a homomorphism.

Let A and A' be any elements of $SL(2,\mathbb{C})$ and let M be any element of $\mathbb{V}$, then

$$\begin{aligned}(\Phi_A \circ \Phi_{A'})(M) &= \Phi_A(\Phi_{A'}(M)) = \Phi_A(A'MA'^*) \\ &= AA'MA'^*A^* = (AA')M(AA')^* = \Phi_{AA'}(M).\end{aligned}$$

□

Theorem 2.2.6. *Let A be any element of $SL(2,\mathbb{C})$, then*

$$\Phi_A = \Phi_{-A}.$$

Proof:
Let A be any elements of $SL(2,\mathbb{C})$ and let M be any element of $\mathbb{V}$, then

$$\Phi_{-A}(M) = (-A)M(-A)^* = AMA^* = \Phi_A(M).$$

□

Thus we see that Φ maps matrices in $SL(2,\mathbb{C})$ representing the same element of $Isom^+(\mathbb{U}^3)$ to the same element of $Isom(\mathbb{V})$, preserving group structure. It is in fact true that $\Phi_A = \Phi_B$ if and only if $A = \pm B$. We can say more about the image of Φ in that for every A, the isometry Φ_A is orientation preserving. Furthermore we have the following result.

Theorem 2.2.7. *Φ induces an isomorphism of $PSL(2,\mathbb{C})$ onto W, where W is the subgroup of $Isom^+(\mathbb{V})$ preserving the light cone in $\mathbb{V}$.*

For a proof of Theorem 2.2.7 and further discussion on the function Φ see [6].

2.3 The Action of Φ_A

The function Φ is an injection of $PSL(2,\mathbb{C})$ into the isometries of a 4-dimensional vector space isometric to $\mathbb{R}^{1,3}$. We wish to explicitly determine the action of these isometries on $\mathbb{V}$, so as to be able to map from $Isom^+(\mathbb{U}^3)$ into the matrix representatives of $Isom^+(\mathbb{H}^3)$. With this in mind we detail the action of Φ_A upon $\mathbb{V}$.

Let

$$A = \pm \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

be any element of $PSL(2,\mathbb{C})$ and let

$$N = \begin{bmatrix} x_1 + x_4 & x_2 + ix_3 \\ x_2 - ix_3 & x_1 - x_4 \end{bmatrix} = x_1B_1 + x_2B_2 + x_3B_3 + x_4B_4$$

be a generic element of $\mathbb{V}$, then

$$\Phi_A(N) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

where

$$\begin{aligned}
H_{11} &= (|\alpha|^2 + |\beta|^2)x_1 + 2\Re(\alpha\bar{\beta})x_2 - 2\Im(\alpha\bar{\beta})x_3 + (|\alpha|^2 - |\beta|^2)x_4, \\
H_{12} &= (\alpha\bar{\gamma} + \beta\bar{\delta})x_1 + (\alpha\bar{\delta} + \beta\bar{\gamma})x_2 + (\alpha\bar{\delta} - \beta\bar{\gamma})ix_3 + (\alpha\bar{\gamma} - \beta\bar{\delta})x_4, \\
H_{21} &= (\bar{\alpha}\gamma + \bar{\beta}\delta)x_1 + (\bar{\alpha}\delta + \bar{\beta}\gamma)x_2 + (-\bar{\alpha}\delta + \bar{\beta}\gamma)ix_3 + (\bar{\alpha}\gamma - \bar{\beta}\delta)x_4, \\
H_{22} &= (|\gamma|^2 + |\delta|^2)x_1 + 2\Re(\gamma\bar{\delta})x_2 - 2\Im(\gamma\bar{\delta})ix_3 + (|\gamma|^2 - |\delta|^2)x_4.
\end{aligned}$$

Alternatively

$$\Phi_A(N) = y_1B_1 + y_2B_2 + y_3B_3 + y_4B_4,$$

where

$$\begin{aligned}
y_1 &= \frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2)x_1 + \Re(\alpha\bar{\beta} + \gamma\bar{\delta})x_2 \\
&\quad -\Im(\alpha\bar{\beta} + \gamma\bar{\delta})x_3 + \frac{1}{2}(|\alpha|^2 - |\beta|^2 + |\gamma|^2 - |\delta|^2)x_4, \\
y_2 &= \Re(\alpha\bar{\gamma} + \beta\bar{\delta})x_1 + \Re(\alpha\bar{\delta} + \beta\bar{\gamma})x_2 \\
&\quad -\Im(\alpha\bar{\delta} - \beta\bar{\gamma})x_3 + \Re(\alpha\bar{\gamma} - \beta\bar{\delta})x_4, \\
y_3 &= \Im(\alpha\bar{\gamma} + \beta\bar{\delta})x_1 + \Im(\alpha\bar{\delta} + \beta\bar{\gamma})x_2 \\
&\quad +\Re(\alpha\bar{\delta} - \beta\bar{\gamma})x_3 + \Im(\alpha\bar{\gamma} - \beta\bar{\delta})x_4, \\
y_4 &= \frac{1}{2}(|\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2)x_1 + \Re(\alpha\bar{\beta} - \gamma\bar{\delta})x_2 \\
&\quad -\Im(\alpha\bar{\beta} - \gamma\bar{\delta})x_3 + \frac{1}{2}(|\alpha|^2 - |\beta|^2 - |\gamma|^2 + |\delta|^2)x_4.
\end{aligned}$$

Given A, a generic element of $PSL(2,\mathbb{C})$, the exact action of Φ_A on the whole space $\mathbb{V}$ is far from clear; However we will not be interested in the generic case, subsequently we focus on $\Phi_A(N)$ for specific A and $N \in \mathbb{V}$.

2.3.1 Actions on the Basis Vectors

As Φ_A is an orthogonal transformation on a real vector space, Φ_A is determined by its action upon each of the basis vectors. Thus it is worthwhile to note the action of Φ_A upon each basis vector when A is a generic element of $PSL(2,\mathbb{C})$.

Let

$$A = \pm \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

then Φ_A acts on B_i thus:

$i = 1$

$$\Phi_A(B_1) = \begin{bmatrix} (|\alpha|^2 + |\beta|^2) & (\alpha\bar{\gamma} + \beta\bar{\delta}) \\ (\bar{\alpha}\gamma + \bar{\beta}\delta) & (|\gamma|^2 + |\delta|^2) \end{bmatrix}.$$

Alternatively

$$\Phi_A(B_1) = N_{11}B_1 + N_{12}B_2 + N_{13}N_3 + N_{14}B_4,$$

where

$$\begin{aligned} N_{11} &= \frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2), && (2.1)\\ N_{12} &= \Re(\alpha\bar{\gamma} + \beta\bar{\delta}), && (2.2)\\ N_{13} &= \Im(\alpha\bar{\gamma} + \beta\bar{\delta}), && (2.3)\\ N_{14} &= \frac{1}{2}(|\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2). && (2.4) \end{aligned}$$

$i = 2$

$$\Phi_A(B_2) = \begin{bmatrix} 2\Re(\alpha\bar{\beta}) & (\alpha\bar{\delta} + \beta\bar{\gamma}) \\ (\bar{\alpha}\delta + \bar{\beta}\gamma) & 2\Re(\gamma\bar{\delta}) \end{bmatrix}.$$

Alternatively

$$\Phi_A(B_2) = N_{21}B_1 + N_{22}B_2 + N_{23}B_3 + N_{24}B_4,$$

where

$$\begin{aligned} N_{21} &= \Re(\alpha\bar{\beta} + \gamma\bar{\delta}), && (2.5)\\ N_{22} &= \Re(\alpha\bar{\delta} + \beta\bar{\gamma}), && (2.6)\\ N_{23} &= \Im(\alpha\bar{\delta} + \beta\bar{\gamma}), && (2.7)\\ N_{24} &= \Re(\alpha\bar{\beta} - \gamma\bar{\delta}). && (2.8) \end{aligned}$$

$i = 3$

$$\Phi_A(B_3) = \begin{bmatrix} -2\Im(\alpha\bar{\beta}) & i(\alpha\bar{\delta} - \beta\bar{\gamma}) \\ -i(\bar{\alpha}\delta - \bar{\beta}\gamma) & -2i\Im(\gamma\bar{\delta}) \end{bmatrix}.$$

Alternatively

$$\Phi_A(B_3) = N_{31}B_1 + N_{32}B_2 + N_{33}B_3 + N_{34}B_4,$$

where

$$\begin{aligned} N_{31} &= -\Im(\alpha\bar{\beta} + \gamma\bar{\delta}), && (2.9)\\ N_{32} &= -\Im(\alpha\bar{\delta} - \beta\bar{\gamma}), && (2.10)\\ N_{33} &= \Re(\alpha\bar{\delta} - \beta\bar{\gamma}), && (2.11)\\ N_{34} &= -\Im(\alpha\bar{\beta} - \gamma\bar{\delta}). && (2.12) \end{aligned}$$

$i = 4$

$$\Phi_A(B_4) = \begin{bmatrix} (|\alpha|^2 - |\beta|^2) & (\alpha\bar{\gamma} - \beta\bar{\delta}) \\ (\bar{\alpha}\gamma - \bar{\beta}\delta) & (|\gamma|^2 - |\delta|^2) \end{bmatrix}$$

Alternatively

$$\Phi_A(B_4) = N_{41}B_1 + N_{42}B_2 + N_{43}B_3 + N_{44}B_4,$$

where

$$N_{41} = \frac{1}{2}(|\alpha|^2 - |\beta|^2 + |\gamma|^2 - |\delta|^2), \tag{2.13}$$

$$N_{42} = \Re(\alpha\bar{\gamma} - \beta\bar{\delta}), \tag{2.14}$$

$$N_{43} = \Im(\alpha\bar{\gamma} - \beta\bar{\delta}), \tag{2.15}$$

$$N_{44} = \frac{1}{2}(|\alpha|^2 - |\beta|^2 - |\gamma|^2 + |\delta|^2). \tag{2.16}$$

2.3.2 Standard Forms

As Φ is a isomorphism from $PSL(2,\mathbb{C})$ into W, conjugacy classes similar to those of $PSL(2,\mathbb{C})$, detailed in Section 1.1.4, exist in W. That is to say, if the matrices A and B in $PSL(2,\mathbb{C})$ represent conjugate transformations in $\mathcal{M}$, then Φ_A and Φ_B are conjugate transformations in W. Given this fact we should also be interested in the action of Φ_{M_k} as a representative of a conjugacy class of transformations in W.

Let M_k denote the matrix inducing the Möbius transformation m_k as in Section 1.1.4, then the action of Φ_{M_k} on a generic element

$$N = \begin{bmatrix} x_1 + x_4 & x_2 + ix_3 \\ x_2 - ix_3 & x_1 - x_4 \end{bmatrix} = x_1B_1 + x_2B_2 + x_3B_3 + x_4B_4.$$

of $\mathbb{V}$ can be divided into two generic cases:

- If $k = 1$, then m_k is a parabolic transformation and we have

  $$M_1 = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

  and hence

  $$\Phi_{M_1}(N) = \begin{bmatrix} 2x_1 + 2x_2 & x_1 + x_2 - x_4 + ix_3 \\ x_1 + x_2 - x_4 - ix_3 & x_1 - x_4 \end{bmatrix}.$$

  Alternatively

  $$(\Phi_{M_1})(N) = y_1B_1 + y_2B_2 + y_3B_3 + y_4B_4,$$

where

$$\begin{aligned} y_1 &= \frac{3x_1 + 2x_2 - x_4}{2}, \\ y_2 &= x_1 + x_2 - x_4, \\ y_3 &= x_3, \\ y_4 &= \frac{x_1 + 2x_2 + x_4}{2}. \end{aligned}$$

- If $k \notin \{0,1\}$, then m_k is a non-parabolic transformation and we have

$$M_k = \pm \begin{bmatrix} \sqrt{k} & 0 \\ 0 & \sqrt{k}^{-1} \end{bmatrix},$$

and hence

$$\Phi_{M_k}(N) = \begin{bmatrix} |k|(x_1 + x_4) & \frac{k}{|k|}(x_2 + ix_3) \\ \frac{\bar{k}}{|k|}(x_2 - ix_3) & |k|^{-1}(x_1 - x_4) \end{bmatrix}.$$

Alternatively

$$\Phi_{M_k}(N) = y_1 B_1 + y_2 B_2 + y_3 B_3 + y_4 B_4,$$

where

$$\begin{aligned} y_1 &= \frac{|k|(x_1 + x_4) + |k|^{-1}(x_1 - x_4)}{2}, \\ y_2 &= \Re\left(\frac{k}{|k|}\right) x_2 - \Im\left(\frac{k}{|k|}\right) x_3, \\ y_3 &= \Im\left(\frac{k}{|k|}\right) x_2 + \Re\left(\frac{k}{|k|}\right) x_3, \\ y_4 &= \frac{|k|(x_1 + x_4) - |k|^{-1}(x_1 - x_4)}{2}. \end{aligned}$$

2.4 Representative Matrices in $PSO(1,3)$

Having shown that the group $PSL(2,\mathbb{C})$ is isomorphic to the subgroup W of $Isom^+(\mathbb{V})$, corresponding to the group $Isom^+(\mathbb{H}^3)$, we can now make use of our observations, from Section 2.3.1, on the actions of Φ_A on $\mathbb{V}$. These will allow us to extend Φ to an isomorphism between the matrix groups $PSL(2,\mathbb{C})$ and $PSO(1,3)$, giving a direct link between the two isometry groups $Isom^+(\mathbb{U}^3)$ and $Isom^+(\mathbb{H}^3)$ respectively.

The function Φ, is an isomorphism mapping elements of $PSL(2,\mathbb{C})$ into W. Let Φ' be the isomorphism from W into $PSO(1,3)$, mapping x in W to $\Phi'(x)$, the element of $PSO(1,3)$ representing an action on $\mathbb{R}^{1,3}$ equivalent to the action of x upon $\mathbb{V}$. Let $\tilde{\Phi} = \Phi'\Phi$, then $\tilde{\Phi}$ is an isomorphism from $PSL(2,\mathbb{C})$ into $PSO(1,3)$.

The isometry Φ_A acts on $\mathbb{V}$ in exactly the same manner that the matrix $\tilde{\Phi}(A)$ acts on $\mathbb{R}^{1,3}$.

Now, given A, we find $\tilde{\Phi}(A)$ explicitly.

$$\begin{aligned}
\tilde{\Phi} &: PSL(2,\mathbb{C}) \to PSO(1,3), \\
\tilde{\Phi} &: \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto \begin{bmatrix} N_{11} & N_{21} & N_{31} & N_{41} \\ N_{12} & N_{22} & N_{32} & N_{42} \\ N_{13} & N_{23} & N_{33} & N_{43} \\ N_{14} & N_{24} & N_{34} & N_{44} \end{bmatrix}
\end{aligned}$$

where N_{ij} are as in equations 2.1 through 2.16.

Now we have a means of explicitly calculating the matrix $\tilde{\Phi}(A)$ from any given A. We look briefly at the results of using different representations for the entries in the matrix A.

2.4.1 Rectangular Form

If we take the entries in the matrix of $PSL(2,\mathbb{C})$ to be expressed in rectangular form, then

$$\tilde{\Phi} : \begin{bmatrix} a+ib & c+id \\ e+if & g+ih \end{bmatrix} \mapsto \begin{bmatrix} N_{11} & N_{21} & N_{31} & N_{41} \\ N_{12} & N_{22} & N_{32} & N_{42} \\ N_{13} & N_{23} & N_{33} & N_{43} \\ N_{14} & N_{24} & N_{34} & N_{44} \end{bmatrix}$$

where

$$\begin{aligned}
N_{11} &= \tfrac{1}{2}(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2), \\
N_{12} &= ae+bf+cg+dh, \\
N_{13} &= -af+be-ch+dg, \\
N_{14} &= \tfrac{1}{2}(a^2+b^2+c^2+d^2-e^2-f^2-g^2-h^2), \\
\\
N_{21} &= ac+bd+eg+fh, \\
N_{22} &= ag+bh+ce+df, \\
N_{23} &= -ah+bg-cf+de, \\
N_{24} &= ac+bd-eg-fh, \\
\\
N_{31} &= ad-bc+eh-fg, \\
N_{32} &= ah-bg-cf+de, \\
N_{33} &= ag+bh-ce-df, \\
N_{34} &= ad-bc-eh+fg, \\
\\
N_{41} &= \tfrac{1}{2}(a^2+b^2-c^2-d^2+e^2+f^2-g^2-h^2), \\
N_{42} &= ae+bf-cg-dh, \\
N_{43} &= -af+be+ch-dg, \\
N_{44} &= \tfrac{1}{2}(a^2+b^2-c^2-d^2-e^2-f^2+g^2+h^2).
\end{aligned}$$

2.4.2 Polar Form

If we take the entries in the matrix of $PSL(2,\mathbb{C})$ to be expressed in polar form, then

$$\tilde{\Phi} : \begin{bmatrix} r_1 e^{i\theta_1} & r_2 e^{i\theta_2} \\ r_3 e^{i\theta_3} & r_4 e^{i\theta_4} \end{bmatrix} \mapsto \begin{bmatrix} N_{11} & N_{21} & N_{31} & N_{41} \\ N_{12} & N_{22} & N_{32} & N_{42} \\ N_{13} & N_{23} & N_{33} & N_{43} \\ N_{14} & N_{24} & N_{34} & N_{44} \end{bmatrix}.$$

where

$$\begin{aligned}
N_{11} &= \tfrac{1}{2}(r_1^2 + r_2^2 + r_3^2 + r_4^2), \\
N_{12} &= r_1 r_3 \cos(\theta_1 - \theta_3) + r_2 r_4 \cos(\theta_2 - \theta_4), \\
N_{13} &= r_1 r_3 \sin(\theta_1 - \theta_3) + r_2 r_4 \sin(\theta_2 - \theta_4), \\
N_{14} &= \tfrac{1}{2}(r_1^2 + r_2^2 - r_3^2 - r_4^2), \\
\\
N_{21} &= r_1 r_2 \cos(\theta_1 - \theta_2) + r_3 r_4 \cos(\theta_3 - \theta_4), \\
N_{22} &= r_2 r_3 \cos(\theta_2 - \theta_3) + r_1 r_4 \cos(\theta_1 - \theta_4), \\
N_{23} &= r_2 r_3 \sin(\theta_2 - \theta_3) + r_1 r_4 \sin(\theta_1 - \theta_4), \\
N_{24} &= r_1 r_2 \cos(\theta_1 - \theta_2) - r_3 r_4 \cos(\theta_3 - \theta_4), \\
\\
N_{31} &= -r_1 r_2 \sin(\theta_1 - \theta_2) - r_3 r_4 \sin(\theta_3 - \theta_4), \\
N_{32} &= r_2 r_3 \sin(\theta_2 - \theta_3) - r_1 r_4 \sin(\theta_1 - \theta_4), \\
N_{33} &= -r_2 r_3 \cos(\theta_2 - \theta_3) + r_1 r_4 \cos(\theta_1 - \theta_4), \\
N_{34} &= -r_1 r_2 \sin(\theta_1 - \theta_2) + r_3 r_4 \sin(\theta_3 - \theta_4), \\
\\
N_{41} &= \tfrac{1}{2}(r_1^2 - r_2^2 + r_3^2 - r_4^2), \\
N_{42} &= r_1 r_3 \cos(\theta_1 - \theta_3) - r_2 r_4 \cos(\theta_2 - \theta_4), \\
N_{43} &= r_1 r_3 \sin(\theta_1 - \theta_3) - r_2 r_4 \sin(\theta_2 - \theta_4), \\
N_{44} &= \tfrac{1}{2}(r_1^2 - r_2^2 - r_3^2 + r_4^2).
\end{aligned}$$

2.4.3 Standard Forms

Carrying on from Section 2.3.2 we look at what matrices in $PSO(1,3)$ correspond to the standard forms given in Section 1.1.4. Thus we calculate $\tilde{\Phi}(M_k)$ explicitly:

- if $k = 1$, then

$$\tilde{\Phi}(M_1) = \begin{bmatrix} 3/2 & 1 & 0 & -1/2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 1/2 \end{bmatrix};$$

- if $k \notin \{0, 1\}$, then

$$\tilde{\Phi}(M_k) = \begin{bmatrix} \frac{|k|}{2} + \frac{1}{2|k|} & 0 & 0 & \frac{|k|}{2} - \frac{1}{2|k|} \\ 0 & \Re(\frac{k}{|k|}) & -\Im(\frac{k}{|k|}) & 0 \\ 0 & \Im(\frac{k}{|k|}) & \Re(\frac{k}{|k|}) & 0 \\ \frac{|k|}{2} - \frac{1}{2|k|} & 0 & 0 & \frac{|k|}{2} + \frac{1}{2|k|} \end{bmatrix}$$

These matrices $\tilde{\Phi}(M_k)$ are representatives of conjugacy classes in $PSO(1,3)$.

We shall refer to the elements of $PSO(1,3)$ as parabolic, elliptic, hyperbolic and loxodromic, dependent on which $\tilde{\Phi}(M_k)$ they are conjugate to. Thus, these terms describe the class of geometric action an element of $Isom^+(\mathbb{U}^3)$ or $Isom^+(\mathbb{H}^3)$ has, irrespective of which model the isometry acts on.

In this section we have constructed an isomorphism between the representative matrices $Isom^+(\mathbb{U}^3)$ and $Isom^+(\mathbb{H}^3)$. This gains us access to the virtues of computation in $PSO(1,3)$ as opposed to $PSL(2,\mathbb{C})$. These virtues are apparent even in the computation of $\tilde{\Phi}^{-1}(N)$ in that we can find a representative element in $GL(2,\mathbb{C})$, a scalar multiple of the desired matrix $\tilde{\Phi}^{-1}(N)$, using only real first-order linear equations on the entries in N.

For information on the computation of $\tilde{\Phi}^{-1}(N)$ and algorithms for both $\tilde{\Phi}$ and $\tilde{\Phi}^{-1}$ see [10].

Chapter 3

Discrete Groups

Definition 3.0.1. *A topological group G is said to be discrete if and only if it has the discrete topology*

It follows that to determine if a group G is discrete, it is sufficient to show that one element g of G is isolated. In the case of $g = I$ we need to show

$$\inf\{\|X - I\| : X \in G, X \neq I\} > 0,$$

where $\|\cdot\|$ is some norm on G.

The above definition covers all topological groups with a norm, if G is a subgroup of $PSL(2, \mathbb{C})$, then we have an alternate characterisation of discreteness.

Definition 3.0.2. *Let G be any subgroup of $PSL(2, \mathbb{C})$, then G is discrete if and only if for some norm $\|\cdot\|$ on G and for each positive k the set*

$$G_k = \{A \in G : \|A\| \leq k\}$$

is a finite set.

In the above definition it is clear that G is the union of all the sets G_k, from this we have the following theorem.

Theorem 3.0.3. *Let G be any discrete subgroup of $PSL(2, \mathbb{C})$, then G is countable.*

Naturally, any subgroup of a discrete group G is also discrete, as is any group that G is topologically homomorphic to. Thus if H is conjugate to G, then H must also be discrete. Most important to our work here is the fact that if G is a discrete subgroup of $PSL(2, \mathbb{C})$, then the subgroup of $\mathcal{M}$, that G can be taken to represent, is also discrete.

The study of discrete groups of $PSL(2,\mathbb{C})$ has had a distinct impact on the field of hyperbolic 3-manifolds. As any complete 3-manifold can be represented as the quotient space $\mathbb{U}^3/G$, where G is some torsion-free discrete subgroup of $Isom^+(\mathbb{U}^3)$; and any complete hyperbolic 3-orbifold can be represented as the quotient space $\mathbb{U}^3/G$ for some discrete subgroup G of $Isom^+(U)^3$. Naturally, if $\mathbb{U}^3/G$ represents a particular manifold or orbifold, then so does the quotient space $\mathbb{H}^3/\tilde{\Phi}(G)$, hence we can alternatively study the discrete groups of $PSO(1,3)$.

The study of discrete groups of $PSL(2,\mathbb{C})$ is a large and active area of modern mathematical research and we don't have space to go into any pertinent details, thus we focus on one result which we will make use of in the next chapter. For detailed discussions on discrete groups with respect to hyperbolic spaces see [1] [9] and [7].

3.1 Two Generator Discrete Groups

In regards to discrete groups, the only result we are interested in is the parameterisation of certain 2 generator discrete subgroups of $\mathcal{M}$.

Definition 3.1.1. *Let $\langle\phi,\psi\rangle$ be a 2 generator discrete subgroup of $\mathcal{M}$, then we define the parameters (or parameter set) of $\langle\phi,\psi\rangle$ to be*

$$(\beta,\beta',\gamma)$$

where

$$\beta = tr^2(\phi) - 4,$$
$$\beta' = tr^2(\psi) - 4,$$
$$\gamma = tr[\phi,\psi] - 2.$$

Note that by the results of Section 1.1.4, the parameters β, β' and γ are invariant under conjugation.

Theorem 3.1.2. *If $\gamma \neq 0$, then (β,β',γ) determines the subgroup $\langle\phi,\psi\rangle$, of $\mathcal{M}$, up to conjugacy.*

Thus if $\gamma \neq 0$, then the parameters (β,β',γ) uniquely determine a conjugacy class of 2 generator discrete subgroups of $\mathcal{M}$. A proof of Theorem 3.1.2 (for the case we will focus on) is given in [4].

By the results of Chapter 2, there must be an equivalent result to Theorem 3.1.2 in the case of discrete groups in $PSO(1,3)$.

Corollary 3.1.3. *If $\gamma \neq 0$, then (β, β', γ) determines the subgroup $\langle A, B \rangle$, of $PSO(1,3)$, up to conjugacy.*

Notice that the above parameterisation is based on the use of the trace function to parameterise the conjugacy classes of $\mathcal{M}$, as given in Section 1.1.4. These parameters have however been normalised so that the parameter set $(0,0,0)$ corresponds to the group $\langle I \rangle$.

In the following section, we will make use of results from the previous chapters to find a solution $\langle A, B \rangle$ from a particular class of parameters. This solution will represent a conjugacy class of 2 generator discrete subgroups of $PSO(1,3)$ $(Isom^+(\mathbb{H}^3))$.

Chapter 4

Parameters of 2 Generator Discrete Groups

In the previous chapters we have explored the relationship between the two groups $PSL(2, \mathbb{C})$ and $PSO(1, 3)$, viewing them as the representative groups of the isometry groups $Isom^+(\mathbb{U}^3)$ and $Isom^+(\mathbb{H}^3)$ respectively.

We have also explored various properties of the elements of $PSL(2, \mathbb{C})$ and the isometries they represent; specifically conjugacy classes and the parameterisation of certain 2 generator discrete groups. As the two groups $Isom^+(\mathbb{U}^3)$ and $Isom^+(\mathbb{H}^3)$ are isomorphic, as stated in Sections 1.5.3 and 3.1, these properties carry over to $PSO(1, 3)$.

In this section we find solutions for $\langle \phi, \psi \rangle$, or more precisely the representative group $\langle M_\phi, M_\psi \rangle$, from given parameters. We shall then focus on finding, explicitly, the matrices in $PSO(1, 3)$ corresponding to the parameter set for discrete groups with two finite order elliptic generators. Note that these solutions are actually representatives of a conjugacy class of 2 generator discrete groups, as opposed to unique solutions.

4.1 Parameters

In Theorem 3.1.2 we saw that any 2 generator discrete group

$$\langle \phi, \psi \rangle$$

is determined up to conjugacy from complex parameters (β, β', γ) whenever $\gamma \neq 0$, where

$$\beta = tr^2(\phi) - 4, \tag{4.1}$$
$$\beta' = tr^2(\psi) - 4, \tag{4.2}$$
$$\gamma = tr[\phi, \psi] - 2. \tag{4.3}$$

Notice that the values of β and β' determine which conjugacy class of $\mathcal{M}$ the isometries ϕ and ψ belong to respectively; the requirement that γ be non-zero is, by Theorem 1.1.30, equivalent to a requirement that the isometries ϕ and ψ don't share any fixed points (recall

ϕ and ψ act on $\hat{\mathbb{C}}$). This means that ϕ and ψ cannot both be standard forms, as ∞ is a fixed point for all m_k. However, as the group is determined up to conjugacy we may, and will, always assume one of either ϕ or ψ is a standard form. As $\gamma \neq 0$ is a constant requirement, we shall henceforth assume that γ is always non-zero.

In the following two sections, we briefly highlight the situation in attempting to find the group $\langle M_\phi, M_\psi \rangle$ from a given parameter set.

4.2 Representative Matrices

Let (β, β', γ) be the parameters of a 2 generator discrete subgroup $\langle \phi, \psi \rangle$ of $\mathcal{M}$, then trivially they are also the parameters of the subgroup $\langle M_\phi, M_\psi \rangle$. Henceforth our focus will be on finding the elements M_ϕ and M_ψ of $SL(2, \mathbb{C})$, given the understanding that they represent the generators of the isometry subgroup we are really trying to find. Note that we will be working in $SL(2, \mathbb{C})$ as opposed to $PSL(2, \mathbb{C})$.

We are looking for solutions up to conjugacy, so we may assume one of M_ϕ or M_ψ is a standard form. As $tr[M_\phi, M_\psi] = tr[M_\psi, M_\phi]$ the parameter sets (β, β', γ) and (β', β, γ) represent the same group; and the values of the parameter set do not impose any restriction on which transformation may be chosen to be a standard form.

The values β and β' give information on the conjugacy classes that M_ϕ and M_ψ belong to, we divide these into two cases: parabolic and non-parabolic.

- The general form of a parabolic matrix is
$$\begin{bmatrix} 1-ab & a^2 \\ -b^2 & 1+ab \end{bmatrix},$$
where a and b cannot both be 0.

- Similarly, the general form of a non-parabolic matrix, conjugate to M_k, is
$$\begin{bmatrix} (adk-bc)/\sqrt{k} & ab(1-k)/\sqrt{k} \\ cd(k-1)/\sqrt{k}) & (ad-bck)/\sqrt{k} \end{bmatrix},$$
where $ad - bc = 1$.

We will make use of the general form of parabolic transformations, but not of non-parabolic matrices as in this case it is easier to assume the transformation is represented by
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
where $ad - bc = 1$, and solve for the four variables instead of five.

In regards to standard forms, if β (or β') is equal to 0, then ϕ (or ψ) is parabolic. Suppose ϕ is non-parabolic, in which case $\phi \sim m_k$ for some $k \neq 0, 1$, and let
$$k = e^{i2x},$$

then

$$tr^2(\phi) = (e^{ix} + e^{-ix})^2 = 4\cos^2(x).$$

This allows us to determine four possible solutions for M_ϕ

$$M_\phi \in \left\{ M_k, -M_k, M_k^{-1}, -M_k^{-1} \right\}.$$

Recall that $M_k^{-1} = M_{k^{-1}}$ is conjugate to M_k.

If there is a choice between assuming a parabolic or a non-parabolic matrix to be a standard form, preference will be given to the non-parabolic matrix.

4.2.1 Matrix Generators

Prior to looking at the sets of equations required to be solved, it is prudent to discuss the types of solutions we can find. To this end, we recall several properties of matrix conjugation and 2 generator groups. We start with a simple result from group theory and a corollary to Theorem 1.1.10.

Theorem 4.2.1. *Suppose G is a 2 generator group, generated by f and g, then each of the pairs (f^{-1}, g), (f, g^{-1}) and (f^{-1}, g^{-1}) also generate G.*

Corollary 4.2.2. *Let M be an element of $SL(2, \mathbb{C})$ inducing the complex Möbius transformation ϕ, then the matrix $-M$ also induces ϕ.*

These results show that in trying to determine the subgroup $G = \langle \phi, \psi \rangle$ of $\mathcal{M}$, from $(\beta, \beta'\gamma)$, the group

$$\langle A, B \rangle$$

induces G, where

$$\begin{aligned} A &\in \{\pm M_\phi, \pm M_\phi^{-1}\}, \\ B &\in \{\pm M_\psi, \pm M_\psi^{-1}\}. \end{aligned} \tag{4.4}$$

Thus, it should be expected that there will always be multiple generating matrices for any parameter set, even once we have completely determined one of the generators.

We now consider specific cases of conjugation, a knowledge of which will prove useful in the following sections.

Let

$$\begin{aligned} A &= \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \\ B &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \\ W &= \begin{bmatrix} w & 0 \\ 0 & w^{-1} \end{bmatrix}, \\ X &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \end{aligned}$$

then

$$\begin{aligned} WAW^{-1} &= \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \\ WBW^{-1} &= \begin{bmatrix} a & w^2 b \\ w^{-2}c & d \end{bmatrix}, \\ XAX^{-1} &= \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{bmatrix}, \\ XBX^{-1} &= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}. \end{aligned}$$

Notice that conjugation in W leaves A invariant, where as conjugation in X maps A to it's inverse. However, conjugation in X and W manipulate the placement and values of the entries in B. This may allow us, pending a determination of M_ϕ (M_ψ), to make assumptions about the solutions for M_ψ (M_ϕ).

These properties give us some adaptability in our attempts to find the generators of the groups parameterised by a given β, β' and γ. And simplifies the search for solutions in the three cases we list in the following section.

4.3 The Conjugacy Classes of Generators

We briefly highlight the equations that need to be solved in the three distinct types of generator pairings.

4.3.1 Two Parabolic Generators

When the parameter set is of the form

$$(0, 0, \gamma),$$

the generators are both members of the parabolic conjugacy class. In this case we assume $\phi = m_1$ and ψ is represented by a matrix of form

$$M_\psi = \begin{bmatrix} 1-ab & a^2 \\ -b^2 & 1+ab \end{bmatrix}.$$

Thus, we are required to solve a and b subject to the conditions:

$$\gamma = b^4;$$

$$(a \neq 0) \vee (b \neq 0).$$

4.3.2 One Parabolic Generator

When the parameter set is of the form:

$$(\beta, 0, \gamma)$$

where $\beta \neq 0$; or

$$(0, \beta', \gamma)$$

where $\beta' \neq 0$.

Then one generator is a member of the parabolic conjugacy class and the other is not. We assume we are dealing with the parameter set $(\beta, 0, \gamma)$ and that, for some k, $\phi = m_k$. Hence we have

$$M_\phi = \begin{bmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{bmatrix}, M_\psi = \begin{bmatrix} 1-ab & a^2 \\ -b^2 & 1+ab \end{bmatrix}.$$

Where we are required to solve x, a and b from the following equations:

$$\begin{aligned} \beta &= -4\sin^2(x); \\ \gamma &= \beta a^2 b^2; \end{aligned}$$

$$(a \neq 0) \vee (b \neq 0).$$

4.3.3 No Parabolic Generators

When the parameter set is of the form

$$(\beta, \beta', \gamma)$$

where both β' and β are non-zero, then neither generator is a member of the parabolic conjugacy class. Hence we may assume, for some k,

$$M_\phi = \begin{bmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{bmatrix}, \tag{4.5}$$

$$M_\psi = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{4.6}$$

And we are required to find x, a, b, c and d from the following equations:

$$\begin{aligned}\beta &= -4\sin^2(x), &(4.7)\\ \beta' &= (a+d)^2-4, &(4.8)\\ \gamma &= -\beta bc, &(4.9)\\ 1 &= ad-bc. &(4.10)\end{aligned}$$

Note that in these last two cases, there has been no assumption that x is a real number and x is only real in the special case when M_ϕ is elliptic.

4.4 Finite Order Elliptic Generators

As a situation of interest we focus on the parameter set

$$\left(-4\sin^2\left(\frac{\pi}{p}\right), -4\sin^2\left(\frac{\pi}{q}\right), z\right), \tag{4.11}$$

where z is non-zero, and p and q are natural numbers greater or equal to 2.

In this case, by Theorems 1.1.28 and 1.1.31, both ϕ and ψ are rotations of angles $\frac{\pi}{p}$ and $\frac{\pi}{p}$ repectively. In fact, by Theorem 1.1.31, both ϕ and ψ are rotations with these angles if and only if (β, β', γ) is of the form given in equation 4.11.

4.4.1 $\langle M_\phi, M_\psi\rangle$

To find the representative matrices of ϕ and ψ, as in equations 4.5 and 4.6 respectively, we need to solve the system of equations 4.7, 4.8, 4.9, 4.10 which, under this parameter set, become:

$$\begin{aligned}-4\sin^2\left(\frac{\pi}{p}\right) &= -4\sin^2(x), &(4.12)\\ -4\sin^2\left(\frac{\pi}{q}\right) &= (a+d)^2-4, &(4.13)\\ z &= 4\sin^2\left(\frac{\pi}{p}\right)bc, &(4.14)\\ 1 &= ad-bc. &(4.15)\end{aligned}$$

Equation 4.12 gives possible values for x and we obtain possible solutions for M_ϕ:

$$M_\phi \in \left\{\pm\begin{bmatrix} e^{i\pi/p} & 0\\ 0 & e^{-i\pi/p}\end{bmatrix}, \pm\begin{bmatrix} e^{-i\pi/p} & 0\\ 0 & e^{i\pi/p}\end{bmatrix}\right\}.$$

Notice that all the elements of the solution set are, by the results of Section 4.2.1, effectively the same generator in $PSL(2,\mathbb{C})$ (and $\mathcal{M}$). Thus, we may choose

$$M_\phi = \begin{bmatrix} e^{i\pi/p} & 0\\ 0 & e^{-i\pi/p}\end{bmatrix}.$$

Having chosen M_ϕ, we note that conjugation by W, as defined in Section 4.2.1, does not alter M_ϕ. However, use of conjugation in matrices of the same form as W allows us to manipulate the possible values of b and c. We note that $b, c \neq 0$ as this would contradict equation 4.14 and the assumption $z \neq 0$. Thus, we may assume $b = 1$, altering equations 4.13, 4.14 and 4.15 and giving equations

$$\begin{aligned}
(a+d)^2 &= -4\sin^2\left(\frac{\pi}{q}\right) + 4 = 4\cos^2\left(\frac{\pi}{q}\right), && (4.16)\\
c &= \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right), \\
ad &= \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right) + 1. && (4.17)
\end{aligned}$$

From which we obtain the quadratic in a (there is the possibility of division by zero occuring here, a case we will deal with subsequently)

$$a^2 \pm 2a\cos\left(\frac{\pi}{q}\right) + \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right) + 1 = 0 \tag{4.18}$$

which has four solutions:

$$\begin{aligned}
a_1 &= \cos\left(\frac{\pi}{q}\right) + \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right)}; && (4.19)\\
a_2 &= \cos\left(\frac{\pi}{q}\right) - \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right)}; && (4.20)\\
a_3 &= -\cos\left(\frac{\pi}{q}\right) + \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right)}; && (4.21)\\
a_4 &= -\cos\left(\frac{\pi}{q}\right) - \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right)}. && (4.22)
\end{aligned}$$

Notice that

$$a_1a_2 = a_3a_4 = \frac{z}{4}\csc^2(\frac{\pi}{p}) + 1 = ad$$

and

$$a_1 = -a_4,$$

$$a_2 = -a_3.$$

This shows that the different solutions for a correspond to the equivalent solutions for M_ψ, as shown in equation 4.4. Therefore we may choose

$$\begin{aligned} a &= \cos\left(\frac{\pi}{q}\right) + \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right)}, \\ d &= \cos\left(\frac{\pi}{q}\right) - \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right)}. \end{aligned}$$

Thus we have found values for a, b, c and d.

In summary, we have that the parameter set

$$\left(-4\sin^2\left(\frac{\pi}{p}\right), -4\sin^2\left(\frac{\pi}{q}\right), z\right),$$

defines the conjugacy class of discrete subgroups in $PSL(2,\mathbb{C})$ containing the group

$$\langle M_\phi, M_\psi \rangle,$$

where

$$M_\phi = \begin{bmatrix} e^{i\pi/p} & 0 \\ 0 & e^{-i\pi/p} \end{bmatrix}$$

and

$$M_\psi = \begin{bmatrix} \cos\left(\frac{\pi}{q}\right) + \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right)} & 1 \\ \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right) & \cos\left(\frac{\pi}{q}\right) - \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{z}{4}\csc^2\left(\frac{\pi}{p}\right)} \end{bmatrix}.$$

This group $\langle M_\phi, M_\psi \rangle$ represents the subgroup $\langle \phi, \psi \rangle$ of $Isom^+(\mathbb{U}^3)$, which is itself a representative group of a class of discrete isometry subgroups with equivalent geometric actions upon $\mathbb{U}^3$.

4.4.2 $\left\langle \tilde{\Phi}_{M_\phi}, \tilde{\Phi}_{M_\psi} \right\rangle$

Having found solutions for the parameter set in $SL(2,\mathbb{C})$, we are now in a position to fulfill our real intention of describing the equivalent discrete subgroup of $PSO(1,3)$, explicitly in terms of the variables in the parameter set 4.11.

As we have calculated matrices M_ϕ and M_ψ, we can now make simple use of the function $\tilde{\Phi}$, as defined in Section 2.4, to obtain matrices in $PSO(1,3)$ using equations 2.1 through 2.16. Thus, we obtain a group $\left\langle \tilde{\Phi}_{M_\phi}, \tilde{\Phi}_{M_\psi} \right\rangle$ representing a conjugacy class of discrete groups in $PSO(1,3)$, whose actions on $\mathbb{H}^3$ will be geometrically equivalent to the action of the group $\langle \phi, \psi \rangle$ on $\mathbb{U}^3$.

$$\tilde{\Phi}_{M_\phi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\pi/p) & -\sin(2\pi/p) & 0 \\ 0 & \sin(2\pi/p) & \cos(2\pi/p) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tilde{\Phi}_{M_\psi} = \begin{bmatrix} N_{11} & N_{21} & N_{31} & N_{41} \\ N_{12} & N_{22} & N_{32} & N_{42} \\ N_{13} & N_{23} & N_{33} & N_{43} \\ N_{14} & N_{24} & N_{34} & N_{44} \end{bmatrix}.$$

Where

$$\begin{aligned}
N_{11} &= \frac{1}{2}\left|\cos\left(\frac{\pi}{q}\right) + \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right|^2 \\
&+\frac{1}{2}\left|\cos\left(\frac{\pi}{q}\right) - \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right|^2 \\
&+\frac{1}{32}\csc^4\left(\frac{\pi}{p}\right)|z|^2 + \frac{1}{2}, \\
N_{12} &= \frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\Re\left(z\right) \\
&+\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Re\left(\bar{z}\sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right) \\
&-\Re\left(\sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right) + \cos\left(\frac{\pi}{q}\right), \\
N_{13} &= \frac{-1}{4}\csc^2\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\Im\left(z\right) \\
&+\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Im\left(\bar{z}\sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right) \\
&+\Im\left(\sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right), \\
N_{14} &= \frac{1}{2}\left|\cos\left(\frac{\pi}{q}\right) + \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right|^2 \\
&-\frac{1}{2}\left|\cos\left(\frac{\pi}{q}\right) - \sqrt{-\sin^2\left(\frac{\pi}{q}\right) - \frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right|^2 \\
&-\frac{1}{32}\csc^4\left(\frac{\pi}{p}\right)|z|^2 + \frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
N_{21} &= \frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\Re\left(z\right)+\Re\left(\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right)\\
&\quad -\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Re\left(\bar{z}\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right)+\cos\left(\frac{\pi}{q}\right),\\
N_{22} &= -\left|\sin^2\left(\frac{\pi}{q}\right)+\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)z\right|+\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Re\left(z\right)+\cos^2\left(\frac{\pi}{q}\right),\\
N_{23} &= 2\cos\left(\frac{\pi}{q}\right)\Im\left(\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right)-\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Im\left(z\right),\\
N_{24} &= \frac{-1}{4}\csc^2\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\Re\left(z\right)+\Re\left(\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right)\\
&\quad +\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Re\left(\bar{z}\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right)+\cos\left(\frac{\pi}{q}\right),
\end{aligned}$$

$$\begin{aligned}
N_{31} &= \frac{-1}{4}\csc^2\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\Im\left(z\right)-\Im\left(\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right)\\
&\quad -\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Im\left(\bar{z}\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right),\\
N_{32} &= -2\cos\left(\frac{\pi}{q}\right)\Im\left(\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right)-\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Im\left(z\right),\\
N_{33} &= -\left|\sin^2\left(\frac{\pi}{q}\right)+\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)z\right|-\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Re\left(z\right)+\cos^2\left(\frac{\pi}{q}\right),\\
N_{34} &= \frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\Im\left(z\right)-\Im\left(\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right)\\
&\quad +\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Im\left(\bar{z}\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right),
\end{aligned}$$

$$
\begin{aligned}
N_{41} &= \frac{1}{2}\left|\cos\left(\frac{\pi}{q}\right)+\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right|^2 \\
&\quad -\frac{1}{2}\left|\cos\left(\frac{\pi}{q}\right)-\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right|^2 \\
&\quad +\frac{1}{32}\csc^4\left(\frac{\pi}{p}\right)|z|^2-\frac{1}{2}, \\
N_{42} &= \frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\Re\left(z\right) \\
&\quad +\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Re\left(\bar{z}\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right) \\
&\quad +\Re\left(\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right)-\cos\left(\frac{\pi}{q}\right), \\
N_{43} &= \frac{-1}{4}\csc^2\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\Im\left(z\right) \\
&\quad +\frac{1}{4}\csc^2\left(\frac{\pi}{p}\right)\Im\left(\bar{z}\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right) \\
&\quad -\Im\left(\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right), \\
N_{44} &= \frac{1}{2}\left|\cos\left(\frac{\pi}{q}\right)+\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right|^2 \\
&\quad +\frac{1}{2}\left|\cos\left(\frac{\pi}{q}\right)-\sqrt{-\sin^2\left(\frac{\pi}{q}\right)-\frac{1}{4}z\csc^2\left(\frac{\pi}{p}\right)}\right|^2 \\
&\quad -\frac{1}{32}\csc^4\left(\frac{\pi}{p}\right)|z|^2-\frac{1}{2}.
\end{aligned}
$$

4.4.3 Division by Zero

In the calculations of Section 4.4.1 (see equations 4.17 and 4.18) there is the possible occurrence of a division by zero and this must be addressed.

Division by zero will occur when

$$z=-4\sin^2\left(\frac{\pi}{p}\right) \tag{4.23}$$

and there is an attempt to combine the equations 4.17 and 4.16 to obtain the quadratic 4.18.

While this division by zero may occur, it won't cause any error in our solutions as

$$\frac{1}{4} z \csc^2 \left(\frac{\pi}{p} \right) = -1 \; \Rightarrow \; ad = 0.$$

Both a and d cannot be zero, as this would contradict equation 4.16, so we must choose either a or d to be zero. Conjugation in X (as defined in Section 4.2.1) shows that the choice of a or d is arbitrary with no effects on the solutions for b or c.

If we take $d = 0$, then by equation 4.16 we have

$$a = \pm 2 \cos \left(\pi / q \right);$$

and if we take $a = 0$, then we have

$$d = \pm 2 \cos \left(\pi / q \right).$$

These solutions correspond to the possible values for a_i given in equations 4.19, 4.20, 4.21 and 4.22, when we let $\frac{1}{4} z \csc^2 \left(\frac{\pi}{p} \right) = -1$. Hence our solutions to a, b, c and d, given in Section 4.4.1, will still hold.

Having shown that our calculations still hold, we note that the situation described in equation 4.23 occurs whenever the parameters γ and β are equal; this is a known and fully classified special case of parameter sets.

4.4.4 The Trace Function on $PSO(1, 3)$

We finish with a brief discussion with regard to the trace function on $PSO(1, 3)$.

The trace of a matrix in $PSO(1, 3)$ is conjugation invariant. However, there is unfortunately no result equivalent to Corollary 1.1.25 for $PSO(1, 3)$, as for any matrices M and N in $PSO(1, 3)$

$$tr(M) = tr(N) \not\Rightarrow M \text{ is conjugate to } N.$$

From our work in Section 2.4.3, each element of $PSO(1, 3)$ is part of a conjugacy class containing a standard form $\tilde{\Phi}_{M_k}$. Let M be any element of $PSO(1, 3)$, and let $\tilde{\Phi}_{M_k}$ be the standard form which is conjugate to M, now:

- If $k = 1$, then $tr(M) = tr(\tilde{\Phi}_{M_1}) = 4$; and
- If $k = re^{i\theta} \neq 1$, then $tr(M) = tr(\tilde{\Phi}_{M_k}) = r + 1/r + 2 \cos \theta$.

Notice that these trace values are comparable to the value of tr^2 given in Corollary 1.1.23. In fact, we have the following result.

Theorem 4.4.1. *Let ϕ be any element of $\mathcal{M}$, then*

$$tr(\tilde{\Phi}_{M_\phi}) = |tr(\phi)|^2.$$

Thus, given any ϕ in $Isom^+(\mathbb{U}^3)$ (using the trace function) we can easily calculate the conjugacy class containing the equivalent of transformation on $Isom^+(\mathbb{H}^3)$; and if $tr^2(\phi)$ is positive real there is equality between the value of $tr^2(\phi)$ and $tr(\tilde{\Phi}_{M_\phi})$.

However, given any $\tilde{\Phi}_{M_\phi}$ in $Isom^+(\mathbb{H}^3)$ (using the trace function) we can only identify a continuum of conjugacy classes that may contain the equivalent transformation in $\mathcal{M}$, of which only one class is not strictly loxodromic.

Hence the trace function on $PSO(1,3)$ cannot be used to parameterise the conjugacy classes of $PSO(1,3)$, nor can it be used parameterise generic 2 generator discrete subgroups of $PSO(1,3)$.

But, if we restrict our view to groups containing only parabolic, elliptic or hyperbolic transformations we can use our parameterisation of discrete groups from Section 3.1, with the additional properties

$$\beta = tr(\tilde{\Phi}_{M_\phi}) - 4,$$

$$\beta' = tr(\tilde{\Phi}_{M_\psi}) - 4,$$

$$\gamma = \sqrt{tr[\tilde{\Phi}_{M_\phi}, \tilde{\Phi}_{M_\phi}]} - 2.$$

These discrete groups contain no strictly loxodromic elements correspond to the parameter sets

$$(\beta, \beta', \gamma)$$

$$\beta, \beta' \geq -4, \ \gamma \geq -2.$$

Chapter 5

Remarks and Future Research

Our main goal in this thesis has been the explicit determination of generators, in $PSO(1,3)$, of a 2 generator discrete group from the given parameter set $(-4\sin^2(\pi/p), -4\sin^2(\pi/q), z)$. With this in mind, we have described the group $\mathcal{M}$ of complex Möbius transformations and its conjugacy classes; the isometries of $\mathbb{U}^3$ and $\mathbb{H}^3$; the isometry between the two spaces; and an isomorphism between the two groups $PSL(2,\mathbb{C})$ and $PSO(1,3)$, representing $Isom^+(\mathbb{U}^3)$ and $Isom^+(\mathbb{H}^3)$ respectively. This gave us a means to move explicitly from $Isom^+(\mathbb{U}^3)$ into $Isom^+(\mathbb{H}^3)$ and, with knowledge of discrete groups, determine the required generators.

In our future research we intend to refine this computation of generators to minimise computational errors and then implement it in computer packages based around J.Weeks' computer program SnapPea [10].

Specifically, we intend on strengthening the Dirichlet subroutines used in the construction and studying of fundamental domains of 2 generator groups. Such tools are now at their limit in the investigations we are undertaking, and we need to adapt and refine them to facilitate new investigations, particularly with regard to the classification of the 2 generator arithmetic groups.

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