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Preservation of Phase Space Structure in Symplectic Integration

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Abstract

This thesis concerns the study of geometric numerical integrators and how they preserve phase space structures of Hamiltonian ordinary differential equations.

We examine the invariant sets of differential equations and investigate which numerical integrators preserve these sets, and under what conditions. We prove that when periodic orbits of Hamiltonian differential equations are discretized by a symplectic integrator they are preserved in the numerical solution when the integrator step size is not resonant with the frequency of the periodic orbit.

The preservation of periodic orbits is the result of a more general theorem which proves preservation of lower dimensional invariant tori from dimension zero (fixed points) up to full dimension (the same as the number of degrees of freedom for the differential equation). The proof involves first embedding the numerical trajectory in a non-autonomous flow and then applying a KAM type theorem for flows to achieve the result. This avoids having to prove a KAM type theorem directly for the symplectic map which is generally difficult to do.

We also numerically investigate the break up of periodic orbits when the integrator's step size is resonant with the frequency of the orbit.

We study the performance of trigonometric integrators applied to highly oscillatory Hamiltonian differential equations with constant frequency. We show that such integrators may not be as practical as was first thought since they suffer from higher order resonances and can perform poorly at preserving various properties of the differential equation. We show that, despite not being intended for such systems, the midpoint rule performs no worse than many of the trigonometric integrators, and indeed, better than some.

Lastly, we present a numerical study of a Hamiltonian system consisting of two magnetic moments in an applied magnetic field. We investigate the effect of both the choice of integrator and the choice of coordinate system on the numerical solutions of the system. We show that by a good choice of integrator (in this case the generalised leapfrog method) one can preserve phase space structures of the system without having to resort to a change of coordinates that introduce a coordinate singularity.

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Contents

Abstract	iii
Acknowledgements	v
Contents	vii
List of Figures	xi
List of Tables	xiii
1 Introduction	1
1.1 Framework	2
1.2 Background & history of numerical integration	3
1.2.1 Linear multistep methods	3
1.2.2 Runge-Kutta methods	4
1.2.3 Partitioned Runge-Kutta methods	8
1.2.4 General linear methods	9
1.2.5 Splitting methods	10
1.3 Hamiltonian systems	11
1.3.1 Symplectic transformations	13
1.3.2 Integrability	16
1.3.3 Chaotic motion and KAM theory	17
1.4 Geometric numerical integration	18
1.4.1 Symplectic integrators	22
1.4.2 Backward error analysis	24
1.5 Thesis outline	29
1.6 List of original contributions	30
2 Structure Preservation: Invariant Sets	33
2.1 Fixed points & their spectra	34
2.2 Stable & unstable manifolds	43
2.3 Periodic orbits	45

2.4	Quasi-periodic orbits & invariant tori	49
2.5	Chaotic invariant sets	50
3	KAM Theory	55
3.1	Integrable Hamiltonian systems	55
3.2	Perturbation theory & Lindstedt-Poincaré series	58
3.3	Kolmogorov's iteration	61
3.4	Kolmogorov's theorem	64
4	Preservation of Periodic Orbits	71
4.1	Background to periodic orbits of Hamiltonian systems	72
4.2	Preservation of invariant tori & Shang's theorem	76
4.3	Framework & the P1 & P2 conditions	79
4.4	Embedding a map in the flow of a modified vector field	82
4.5	Main result — periodic orbits are preserved	86
4.6	Resonant periodic orbits	90
4.7	Conclusion	103
5	Highly Oscillatory Problems	105
5.1	Background to trigonometric integrators	106
5.1.1	Early trigonometric integrators	107
5.1.2	Recent trigonometric integrators	108
5.1.3	Other methods	112
5.2	The Fermi-Pasta-Ulam problem	113
5.2.1	Time scales in oscillatory problems	115
5.3	Nonlinear stability & resonances	116
5.3.1	Resonances for a planar problem	117
5.3.2	Consequences of resonance	121
5.4	Energy conservation for fixed $h\omega$	123
5.5	Slow exchange of oscillatory energy	125
5.6	Statistical properties	128
5.7	Conclusion	133
6	An Application: A Coupled Two-spin System	137
6.1	Description of the system	138
6.2	The generalised leapfrog integrator	141
6.3	Poincaré sections of the two-spin system	141
6.4	Accuracy & local error	145
6.4.1	Effect of the coordinate singularity on local error	145
6.4.2	Energy preservation	147
6.5	Conclusion	149

7 Closing Remarks and Open Questions	151
Bibliography	155

List of Figures

1.1	Calculation of the function $\gamma(\tau)$	7
1.2	Area preservation of a symplectic flow	14
1.3	Nested tori of an integrable Hamiltonian system	16
1.4	Leapfrog method in the <i>Principia Mathematica</i>	21
1.5	Schematic of backward error analysis	25
1.6	Schematic of modified Hamiltonians for symplectic integration	26
1.7	Modified Hamiltonians of the pendulum and numerical trajectories of the leapfrog method	28
2.1	Spurious chaos near the hyperbolic fixed point of the pendulum	52
2.2	Chaos in the Lorenz system, for the explicit Euler method	53
2.3	Regular and chaotic motion in the Poincaré section of the Hénon-Heiles system	54
4.1	1-parameter family of periodic orbits for the pendulum	73
4.2	1-parameter family of periodic orbits for the Hénon-Heiles system	74
4.3	Trajectory on an invariant torus winding about an elliptic periodic orbit	76
4.4	Destruction of KAM tori in the Hénon-Heiles system with increasing perturbation size	78
4.5	Possible 1-parameter families of periodic orbits for perturbed and unperturbed Hamiltonians	92
4.6	Hyperbolic and elliptic periodic orbits in the Poincaré section of the Hénon-Heiles system	95
4.7	Periodic points of a resonant discrete orbit	99
4.8	Steady state solutions of a Hamiltonian partial differential equation discretized by a multisymplectic integrator	102
5.1	The Fermi-Pasta-Ulam problem: a chain of alternating soft, nonlinear, and stiff, harmonic, springs.	114
5.2	Illustration of a planar order three resonance	118
5.3	Phase portrait and energy error for an order four resonance of the pendulum integrated with the midpoint rule	119

5.4	Energy errors due to resonances of trigonometric integrators applied to a planar problem	120
5.5	Energy errors due to resonances for trigonometric integrators applied to the Fermi-Pasta-Ulam problem	122
5.6	Evidence of a third order resonance for a trigonometric integrator applied to the Fermi-Pasta-Ulam problem	123
5.7	Oscillatory energy error for integrators applied to the Fermi-Pasta-Ulam problem	124
5.8	Order behaviour, with respect to energy conservation, of trigonometric integrators applied to the Fermi-Pasta-Ulam problem for fixed $h\omega$	125
5.9	Oscillatory energy exchange for various trigonometric integrators applied to the Fermi-Pasta-Ulam problem	126
5.10	Dependence of oscillatory energy exchange on $h\omega$ for trigonometric integrators	127
5.11	Bar graph of relative absolute differences in mean oscillatory energy for trigonometric integrators applied to the Fermi-Pasta-Ulam problem	129
5.12	Bar graph of relative absolute differences in oscillatory energy standard deviation for trigonometric integrators applied to the Fermi-Pasta-Ulam problem	129
5.13	Flattened 3D histograms showing variations in long-time distribution of oscillatory energy for various integrators applied to the Fermi-Pasta-Ulam problem	131
5.14	More flattened 3D histograms showing variations in long-time distribution of oscillatory energy for various integrators applied to the Fermi-Pasta-Ulam problem	132
5.15	Histograms showing variations in long-time distribution of oscillatory energy for various integrators applied to the Fermi-Pasta-Ulam problem.	134
6.1	Poincaré sections for the two-spin system integrated with a generalised leapfrog method	142
6.2	Poincaré sections for the two-spin system integrated with a classical “black-box” method	143
6.3	Poincaré sections for the two-spin system on the sphere	144
6.4	Excluded regions of the Poincaré sections for the two-spin system	146
6.5	Local error for the two-spin system as a function of θ_1	147
6.6	Energy jumps in the two-spin system due to a coordinate singularity	148
6.7	Long-time energy error for the two-spin system integrated in two different coordinate systems	149
6.8	Long-time energy error for the two-spin system integrated with a symplectic and a non-symplectic integrator	150

List of Tables

1.1	Trees and elementary differentials up to order three	7
4.1	Numerically calculated periodic points resulting from the symplectic discretization of elliptic and hyperbolic orbits of the Hénon-Heiles system . . .	97
5.1	Filter functions for various trigonometric integrators	111
5.2	Oscillatory energy statistics for various integrators applied to the Fermi-Pasta-Ulam problem	130
5.3	Relative errors in oscillatory energy statistics for various integrators applied to the Fermi-Pasta-Ulam problem	130
5.4	Relative normed differences for oscillatory energy probability distributions for trigonometric integrators applied to the Fermi-Pasta-Ulam problem . . .	133
5.5	Summary of the performance of various trigonometric integrators, and the midpoint rule for highly oscillatory problems	135

