AN ALGORITHM FOR

GENERALISED CONVEX QUADRATIC PROGRAMMING

A Thesis

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Master of Agricultural Science

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INTRODUCTION

The purpose of this thesis is to review work carried out by Professor W. V. Candler of the Department of Agricultural Economics and Farm Management at Massey University, leading to the development of a Generalised Convex Quadratic Programming Algorithm. However the responsibility for the following manner of presenting the material and forming the arguments rests with the candidate.

The first chapter gives a brief summary of the algebra of quadratic functions which will form a background for future developments. At the end of Chapter 1 is a bibliography for further and more detailed reading.

Chapter 2 classifies the problem in the title within the frame-work of the more general mathematical programming problem.

Chapter 3 describes and developes the mathematical conditions which any successful algorithm must satisfy, and Chapter 4 developes the algorithm, in the form of three separate algorithms, as a form of presentation. The last chapter provides a brief discussion.

CHAPTER 1

NOTATION AND DEFINITIONS

I.I. Quadratic Functions

I.I.I. A Quadratic Function

$$n_i = a_i + \underline{b_i}\underline{x}' + \underline{x}B_i\underline{x}' \qquad (I)$$

defines a quadratic function where,

$\underline{x}$ is a Ixn vector of variables

a_i is a constant

$\underline{b_i}$ is a Ixn vector of constants, and

B_i is a nxn symmetric matrix of constants.

I.I.2. A Linear Function

If $B_i = 0$ then (I) reduces to a linear function of $\underline{x}$.

I.I.3. Partial Derivatives of a Quadratic Function

$$dn_{ij} = b_{ij} + \underline{2b_{ij}\underline{x}'} \qquad (2)$$

is the partial derivative of n_i with respect to x_j where,

b_{ij} is the j^{th} element of $\underline{b_i}$, and

$\underline{b_{ij}}$ is the j^{th} row of B_i

Also defined is,

$$\underline{dn_i} = \underline{b}_i' + 2\underline{B_i x}' \qquad (3)$$

where $\underline{dn_i}$ is a Ixn vector whose j^{th} element is dn_{ij}.

I.I.4. The Stationary Point of a Quadratic Function

$$\underline{dn}_i' = \underline{o}' \qquad (4)$$

defines a set of simultaneous equations in $\underline{x}$, the solution of which yields the stationary point of n_i. If B_i is of rank n, then the stationary point of n_i is unique. If B_i is of rank n-r then the stationary point is an r+I dimensional hyperplane in n-space.

I.2. Properties of Quadratic Functions

I.2.I. The Quadratic Form

The Quadratic form is defined as $\underline{x}B_i\underline{x}'$. The following terminology applies to the quadratic form.

If for all $\underline{x}$; $\underline{x} \neq \underline{o}$,

$\underline{x}B_i\underline{x}' > o$ the quadratic form is Positive Definite

$\underline{x}B_i\underline{x}' \geq o$ the quadratic form is Positive Semidefinite

$\underline{x}B_i\underline{x}' < o$ the quadratic form is Negative Definite

$\underline{x}B_i\underline{x}' \leq o$ the quadratic form is Negative Semidefinite

$-\infty \leq \underline{x}B_i\underline{x}' \leq \infty$ the quadratic form is Indefinite.

I.2.2. Latent Roots

$$\left| B_i - Iz_{ij} \right| = o \qquad (5)$$

is the characteristic equation of B_i where,

I is the nxn identity matrix, and

z_{ij} is the j^{th} root of the polynomial (5).

z_{ij} is termed the j^{th} latent root of B_i, j = I.......n . As B_i is symmetric the latent roots are real.

I.2.3. Principle Minor Determinants

$$d_{ij} = \begin{vmatrix} B_{I2}B_{I2} & \cdots\cdots & B_{Ij} \\ B_{2I}B_{22} & \cdots\cdots & B_{2j} \\ \cdot & & \\ \cdot & & \\ B_{jI}B_{j2} & \cdots\cdots & B_{jj} \end{vmatrix}$$

is the j^{th} principle minor determinant of B_i, where the B_{pq} are the elements in the p^{th} row and q^{th} coloumn of B_i.

I.2.4. Determining Quadratic Form

The quadratic form of B_i is positive definite if,

(i) $z_{ij} > o$, $j = I.....n$, or alternatively,

(ii) $d_{ij} > o$, $j = I.....n$.

The quadratic form of B_i is negative definite if,

(i) $z_{ij} < o$, $j = I.....n$, or alternatively,

(ii) $d_{ij} = k$, $j = I.....n$, where,

$k < o$ if j is odd, and,

$k > o$ if j is even.

Semidefinite forms are indicated as above, with the strict inequalitiesreplaced by the weaker $\geqslant$ or $\leqslant$ conditions. Any other situation indicates an indefinite quadratic form.

I.2.5. Nature of the Stationary Point

If we define $\underline{x} = \underline{x}^{**} + \underline{x}^{*}$, where,

$\underline{x}^{**}$ is the solution to (4), and,

$\underline{x}^{*}$ is a vector $\underline{x}$ measured from $\underline{x}^{**}$,

then (I) becomes,

$$n_i = a_i + \underline{b_i}(\underline{x}^{**} + \underline{x}^*)' + (\underline{x}^{**} + \underline{x}^*)\ B_i\ (\underline{x}^{**} + x^*)' \quad (7)$$

$$= a_i^* + \underline{dn_i^*}\underline{x}^{*'} + \underline{x}^* B_i \underline{x}^{*'} \quad (8)$$

$$= a_i^* + \underline{x}^* B_i \underline{x}^{*'} \quad (9)$$

where, a_i^* is the value of n_i at the stationary point,

and,

$\underline{dn_i^*}$ is the value of $\underline{dn_i}$ evaluate at $\underline{x}^{**}$.

From section I.2.I.and equation (9) it follows that,

(i) when the quadratic form is positive definite (or semidefinite) the stationary point is a unique minumum point (or hyperplane).

(ii) when the quadratic form is negative definite (or semidefinite) the stationary point is a unique maximum point (or hyperplane).

(iii) when the quadratic form is indefinite the stationary point is a saddle-point.

I.2.6. The Differential

The linear approximation to n_i at the point $\underline{x}^*$ is measured by,

$$dn_i = \underline{dn_i^*}\underline{dx}' \quad (10)$$

where,

$\underline{dn_i^*}$ is $\underline{dn_i}$ evaluated at the point $\underline{x}^*$, and,

$\underline{dx}$ is a Ixn vector of small differential changes in $\underline{x}$ measured away from $\underline{x}^*$.

In geometric terms, dn_i is the tangent to n_i at the point $\underline{x}^*$.

I.2.7. Roots of the Simple Quadratic Function

If the vector $\underline{x}$ in (I) is replaced by the single variable x, then a simple quadratic function is obtained as

$$n_i = a_i + b_i x + B_i x^2 \qquad (II)$$

where, a_i, b_i, and B_i are all constants. The roots of (II) are the solutions to $n_i = o$, and are given by,

$$x_o = \frac{-b_i \pm \sqrt{b_i^2 - 4B_i a_i}}{2B_i} \qquad (I2)$$

Note that roots will be real only when,

$$b_i^2 \geqslant 4B_i a_i \qquad (I3)$$

I.3. Convexity

I.3.I. Convex Sets

S is a convex set if, for any two points $s_1 \in S$ and $s_2 \in S$,

$$(\phi s_1 + (1-\phi)s_2) \in S; \qquad 0 \leq \phi \leq 1 \qquad (i4)$$

I.3.2. Convex Functions

The function $f(\underline{x}) \in$ set Q is convex if for any two values of the function $f(\underline{x}_1) \in Q$ and $f(\underline{x}_2) \in Q$,

$$f(\phi \underline{x}_1 + (1-\phi)\underline{x}_2) \in Q; \qquad 0 \leq \phi \leq 1 \qquad (15)$$

In the following discussion Q will usually be the set $\geqslant$ or $\leqslant$ k, where k is an arbitrary constant.

I.3.3. Quadratic Convexity

Let Q be the set of real numbers $\leq$ k, then if B_i has a positive definite (semidefinite) form, $n_i \in Q$ is a convex function, and the set of all $\underline{x}$ such that $n_i \in Q$ is a convex set.

Let Q be the set of real numbers $\geqslant k$, then if B_i has a negative definite (semidefinite) form, $n_i \in Q$ is a convex function, and the set of all $\underline{x}$ such that $n_i \in Q$ is a convex set.

If B_i has an indefinite form, $n_i \in Q$ is a non-convex set for both of the above definitions of Q.

As the negative of a positive definite (semidefinite) quadratic form is negative definite (semidefinite) and vice versa, it follows that if $n_i \leq (\geqslant) k$ is a convex function, then $-n_i \geqslant (\leqslant) -k$, is also a convex function.

I.4. Bibliography

The above brief summaries may be supplemented by reading from the following sources.

(i) Hadley. G. - "Linear Programming," Addison-Wesley Pub. U.S.A. 1962.

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