Qualified Difference Sets

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Abstract

Qualified difference sets are a class of combinatorial configuration. The sets are related to the residue difference sets that were first discussed in detail in 1953 by Emma Lehmer. Qualified difference sets consist of a set of residues modulo an integer v and they possess attractive properties that suggest potential applications in areas such as image formation, signal processing and aperture synthesis. This thesis outlines the theory behind qualified difference sets and gives conditions for the existence and nonexistence of these sets in various cases.

A special case of the qualified difference sets is the qualified *residue* difference sets. These consist of the set of nth power residues of certain types of prime. Necessary and sufficient conditions for the existence of qualified residue difference sets are derived and the precise conditions for the existence of these sets are given for $n = 2$, 4 and 6. Qualified residue difference sets are proved nonexistent for $n = 8$, 10, 12, 14 and 18.

A generalisation of the qualified residue difference sets is introduced. These are the qualified difference sets composed of unions of cyclotomic classes. A cyclotomic class is defined for an integer power n and the results of an exhaustive computer search are presented for $n = 4$, 6, 8, 10 and 12. Two new families of qualified difference set were discovered in the case $n = 8$ and some isolated systems were discovered for $n = 6$, 10 and 12.

An explanation of how qualified difference sets may be implemented in physical applications is given and potential applications are discussed.

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This one's for me.

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