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BOUNDS ON THE ARITHMETIC DEGREE

A thesis presented in partial fulfilment
of the requirements for
the degree of Master of Science in
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Abstract

In this thesis we study the arithmetic degree theory of polynomial ideals. The main objectives are: (i) to show whether we can generalize a lower bound on the arithmetic degree of monomial ideals to the arithmetic degree of arbitrary homogeneous ideals; and (ii) to explain whether some known bounds for the geometric degree can be restated in terms of bounds on the arithmetic degree. We give a negative answer to all questions raised by constructing counterexamples. In some cases we provide a general method for constructing such counterexamples. Concerning properties of the arithmetic degree, we give a new Bezout-type theorem. Finally we take a brief look at open problems concerning the arithmetic degree under hypersurface sections.

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