The Edge Slide Graph of The n-dimensional Cube

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Abstract

The goal of this thesis is to understand the spanning trees of the n-dimensional cube Q_n by understanding their *edge slide graph*. An *edge slide* is a move that "slides" an edge of a spanning tree of Q_n across a two-dimensional face, and the edge slide graph is the graph on the spanning trees of Q_n with an edge between two trees if they are connected by an edge slide. Edge slides are a restricted form of an *edge move*, in which the edges involved in the move are constrained by the structure of Q_n, and the edge slide graph is a subgraph of the *tree graph* of Q_n given by edge moves.

The *signature* of a spanning tree of Q_n is the n-tuple $(a_1, \ldots, a_n)$, where a_i is the number of edges in the ith direction. The signature of a tree is invariant under edge slides and is therefore constant on connected components. We say that a signature is *connected* if the trees with that signature lie in a single connected component, and *disconnected* otherwise. The goal of this research is to determine which signatures are connected.

Signatures can be naturally classified as *reducible* or *irreducible*, with the reducible signatures being further divided into *strictly reducible* and *quasi-irreducible* signatures. We determine necessary and sufficient conditions for $(a_1, \ldots, a_n)$ to be a signature of Q_n, and show that strictly reducible signatures are disconnected. We conjecture that strict reducibility is the only obstruction to connectivity, and present substantial partial progress towards an inductive proof of this conjecture. In particular, we reduce the inductive step to the problem of proving under the inductive hypothesis that every irreducible signature has a "*splitting signature*" for which the upright trees with that signature and splitting signature all lie in the same component. We establish this step for certain classes of signatures, but at present are unable to complete it for all.

Hall's Theorem plays an important role throughout the work, both in characterising the signatures, and in proving the existence of certain trees used in the arguments.

Acknowledgement

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Chapter 1

Introduction

A *spanning tree* of a graph is a tree that includes every vertex in the graph. So spanning trees are subgraphs that carry a lot of information about the original graph. Spanning trees have a wide variety of applications in many areas, such as network design (see for example Yeung, Yan and Leung [13]), communication networks (see for example Hu [7]), and bioinformatics (see for example Xu, Olman and Xu [12]). The goal of this thesis is to understand the spanning trees of the n-dimensional cube by understanding their edge slide graph. The n-dimensional cube is the graph Q_n whose vertices are the subsets of $[n] = \{1, \ldots, n\}$, and two vertices are connected by an edge if they differ by adding or deleting exactly one element.

Goddard and Swart [4] define two graphs G_1 and G_2 to be related by an *edge move* if there are edges $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$ such that $G_2 = G_1 - e_1 + e_2$. Given a graph G, the graph on the spanning trees of G with an edge between two trees if they are connected by an edge move is known as the *tree graph* $T(G)$. The tree graph $T(G)$ is known to be connected for any connected graph.

An *edge slide* is a restricted form of an edge move on the spanning trees of Q_n, in which the edges involved in the operation are constrained by the structure of Q_n. The *edge slide graph* of Q_n, denoted $\mathcal{E}(Q_n)$, is defined to be the graph whose vertices are the spanning trees, with an edge between two spanning trees if they are connected by an edge slide. The edge slide graph is a subgraph of the tree graph. Edge slides were defined and used by Tuffley [11] to combinatorially count the number of spanning trees of Q_3. Tuffley was motivated to answer a question implicitly raised by Stanley [10] in the case $n = 3$. The number of spanning trees of Q_n is known by Kirchhoff's Matrix Tree Theorem to be

$$|\mathrm{Tree}(Q_n)| = 2^{2^n-n-1} \prod_{k=1}^{n} k^{\binom{n}{k}} \tag{1.1}$$

(see for example Stanley [10]), and Stanley had implicitly asked for a combinatorial proof of this fact. Tuffley's method to count the number of spanning trees of Q_3 cannot readily be extended to higher dimensions, but Stanley's question has been answered in full using different methods by Bernardi [2].

A spanning tree of Q_n can be classified by its *signature* $(a_1, \ldots, a_n)$, where a_i is the number of edges in direction i. Signatures are invariant under edge slides, so trees with different signatures belong to different components of the edge slide graph. We say that a signature is *connected* if the trees with that signature lie in a single connected component, and *disconnected* otherwise. Our goal is to determine which signatures are connected. The edge slide graph of Q_2 is easily seen to consist of two copies of K_2, one for each of the two possible signatures, and the structure of the components of $\mathcal{E}(Q_3)$ was found by Henden [6].

A useful approach is to study *upright spanning trees*. These are spanning trees of Q_n for which each edge is oriented downward, see Tuffley [11]. Upright spanning trees are relatively easy to work with, and Tuffley proved that each spanning tree is connected to at least one upright spanning tree either directly or by a sequence of edge slides. We may therefore attempt to study the connected components of the edge slide graph by determining which upright spanning trees belong to the same component. Henden used this approach to determine some components of Q_4, and we also adopt this approach here.

Signatures can be naturally classified as *reducible* or *irreducible* signatures. A signature of Q_n is *reducible* if after being permuted to increasing order it has an initial segment which is a signature of a lower dimensional cube. Otherwise it is an *irreducible* signature. We further divide reducible signatures into *strictly reducible* and *quasi-irreducible* signatures. Reducibility places tight restrictions on the available edge slides, with the consequence that the signatures of certain subtrees are also invariant under edge slides. With some additional work, we can then show that the edge slide graph of a strictly reducible signature of Q_n is disconnected. Thus, strict reducibility of $\mathcal{S}$ is an obstruction to the connectivity of the edge slide graph of $\mathcal{S}$. We conjecture that when this obstruction vanishes, the edge slide graph of $\mathcal{S}$ is connected; that is, we conjecture that a signature is connected if and only if it is irreducible or quasi-irreducible. Quasi-irreducible signatures are easily shown to be connected if an associated irreducible signature is, so the problem reduces to understanding the irreducible signatures. We present substantial partial progress towards addressing this question.

We adopt an inductive approach to answering this question. We split the n-cube into two subgraphs isomorphic to Q_{n-1} and this leads us to define *splitting signatures* of a signature $\mathcal{I}$ of Q_n. Then given a tree with signature $\mathcal{I}$, in order to use the inductive hypothesis we first transform the splitting signature to be irreducible. We then show under the inductive hypothesis that an upright spanning tree with an irreducible splitting signature can be transformed into a tree with any other ordered irreducible splitting signature. This means it is enough to connect the upright spanning trees with signature $\mathcal{I}$ and some fixed irreducible splitting signature. In order to do this, we impose conditions on the splitting signature that will assist in this. We define three such classes of amenable splitting signatures, namely *unidirectional* splitting signatures, *super rich* splitting signatures, and the (2, 2, 3) splitting signature of Q_4. We define and use *signature moves* to show that every ordered irreducible signature of Q_n has an amenable splitting signature. We then complete the step described above for two of the three types of amenable splitting signatures, by showing that the set of upright spanning trees with an irreducible signature and a fixed connected unidirectional splitting signature or the (2, 2, 3) splitting signature all lie in a single component. We conjecture under the inductive hypothesis this is also true for a super rich splitting signature, and we present partial progress towards a proof of this. Establishing this step would complete the proof of the induction step, completing the proof of our conjecture that the edge slide graph of a signature is connected if and only if it is irreducible or quasi-irreducible.

1.1 Overview of this thesis

Chapter 2 provides a summary of the concepts of graph theory which are needed for this study. Section 2.1 defines spanning trees and some useful related results. Section 2.2 defines the

n-dimensional cube, edge slides and some known results related to the n-dimensional cube.

Chapter 3 characterises and classifies signatures of spanning trees of Q_n. The main tool used to do this is Hall's Marriage Theorem, and we introduce this through two matching problems in Section 3.1.1. Section 3.1.2 discusses signatures of spanning trees of Q_n in relation to Hall's Marriage Theorem, using an example of a known signature of Q_3 to illustrate the problem. Then we use Hall's Theorem to characterise signatures of spanning trees of Q_n in Section 3.1.3. Section 3.2 classifies signatures of Q_n into reducible and irreducible signatures, and establishes the existence of certain trees with a given irreducible signature. These trees play an important role in the later chapters.

Chapter 4 introduces *local moves* on upright spanning trees of Q_n. These are moves that can be applied locally to transform one upright spanning tree into another, where by "locally" we mean the edges of the tree affected by the change all lie in a 2-dimensional face of Q_n. Sections 4.2.1 and 4.2.2 present two local moves, the *V-move* and the *path move*, which are used frequently in subsequent chapters. In addition, we define the *local move graph* of Q_n in Section 4.3.

Chapter 5 concerns splitting Q_n into two subgraphs isomorphic to Q_{n-1}. This leads us to define *splitting signatures* which are an important tool in what follows. When Q_n is split, an upright spanning tree splits into a spanning forest and a spanning tree of Q_{n-1}. Our goal is to apply the inductive hypothesis to the spanning tree of Q_{n-1}, and then we need some technique to rearrange the edges of the spanning forest. To do this we impose conditions on the splitting signature that assist in rearranging the edges. This gives us three classes of *amenable* splitting signatures, discussed in Section 5.5: *unidirectional* splitting signatures, *super rich* splitting signatures, and the (2, 2, 3) splitting signature of Q_4. To prove the existence of such an amenable splitting signature, we introduce *signature moves* in Section 5.6.

Chapter 6 addresses the connectivity of a reducible signature of Q_n. We show in Section 6.2.1 that reducibility places restrictions on the edges of an upright spanning tree, and hence on the available edge slides as we discuss in Section 6.2.3. Using these results we can show in Section 6.2.4 that the signatures of certain subtrees of a spanning tree are invariant. Combining these results we show that strictly reducible signatures are disconnected in Section 6.3.

We then turn to irreducible signatures of Q_n in Chapters 7–9. An irreducible signature has both reducible splitting signatures and irreducible splitting signatures, and we discuss irreducible splitting signatures in Chapters 7 and 8, and reducible splitting signatures in Chapter 9.

Chapter 7 discusses rearranging the edges of a tree in the upper n face. It starts with results on swapping edges of an upright spanning tree in the upper n face of the cube in Section 7.2.1. This enables us to move from one upright spanning tree to another, and leads us to introduce *settled trees* in Section 7.2.2. These are trees where all the vertices immediately below each n-edge are in direction n (the direction of splitting). Working with settled trees is easier as we know where the edges in the direction of splitting are. Section 7.2.2.1 discusses swapping the edges of a settled tree in the upper face of the cube. Section 7.3 shows upright spanning trees with either a fixed unidirectional splitting signature or the (2, 2, 3) splitting signature lie in a single component, and conjectures this is also true for a super rich splitting signature. We present partial progress towards a proof of this conjecture.

Chapter 8 shows that we can transform an upright spanning tree with an irreducible signature and an irreducible splitting signature into an upright spanning tree with any other ordered irreducible splitting signature. In order to transform a tree so it has an ordered irreducible splitting signature of our choice, we first show in Section 8.2.2 that we can transform an upright spanning tree with an irreducible splitting signature into an upright spanning tree with an ordered irreducible splitting signature, and then show in Section 8.2.3 that we can transform an upright spanning tree with an ordered irreducible splitting signature into a tree with any other ordered irreducible splitting signature. In particular, this means we can transform an upright spanning tree into a tree with an amenable splitting signature, so that we can apply the results of Chapter 7 to rearrange the edges in the upper n face.

Chapter 9 addresses reducible splitting signatures of an irreducible signature of Q_n. We show that we can transform an upright spanning tree with an irreducible signature and a reducible splitting signature into an upright spanning tree with an irreducible splitting signature. To do this, we show in Section 9.2.2 that we can always use edge slides to increase the size of the smallest reducing set of a reducible splitting signature.

Chapter 10 studies two infinite families of irreducible signatures of Q_n. Section 10.1.1 and Section 10.2.1 give a proof for the connectivity of the edge slide graph of these families using a slightly different technique from the approach we use for the general case. In these families, we only encounter signatures within the family or within closely related families that we already understand. In Section 10.1.2 and Section 10.2.2 we give formulae to count the number of upright spanning trees of Q_n with these signatures.

In Chapter 11 we combine the results from the previous chapters to prove our main result, reducing the inductive step to the problem of showing that every ordered irreducible signature $\mathcal{I}$ has an ordered irreducible splitting signature $\mathcal{D}$ such that the set of upright spanning trees with signature $\mathcal{I}$ and splitting signature $\mathcal{D}$ forms a block. We then apply this and results from Chapter 7 to prove the connectivity of the edge slide graph of an irreducible signature of Q_4, and certain irreducible signatures of Q_5.

At the end of each of Chapters 3–9 we provide a summary map to link the work between these chapters. All figures in this thesis were created using the drawing software Ipe (see `http://ipe.otfried.org`).

Chapter 2

Preliminaries

In this chapter we review some basic concepts of graph theory and give a number of definitions and examples that are used throughout the study.

2.1 Spanning trees

In this section we review the definition of spanning trees of graphs and some related results.

Definition 2.1.1. A **spanning tree** of a connected graph G is a connected subgraph of G that contains all the vertices of G and contains no cycles.

Suppose V denotes the set of vertices of G. Then a spanning tree will have $|V|$ vertices and $|V|-1$ edges (see for example Agnarsson and Greenlaw [1]). An example of a spanning tree is shown in Figure 2.1.

Definition 2.1.2. A **spanning forest** of a graph G is a subgraph of G that is a forest containing all the vertices of G.

We make use of the following lemma. A proof can be found in [3].

Lemma 2.1.3. *Let V be the set of vertices of a graph G. Let T be a subgraph of G such that T is connected, and has $|V|$ vertices and $|V|-1$ edges. Then T is a spanning tree of G.*

The following lemma is complementary to Lemma 2.1.3.

Lemma 2.1.4. *Suppose T is a subgraph of G that has no cycles, and $|V|$ vertices and $|V|-1$ edges. Then T is a spanning tree of G.*

The following lemma shows that every finite connected graph has a spanning tree. The general idea behind the proof of this result is that if G is a connected graph and has a cycle, then we can delete one edge from G without destroying any connectedness. We can repeat this until there are no more cycles to prove the existence of a spanning tree.

Lemma 2.1.5. *Suppose G is a finite connected graph. Then G has a spanning tree.*

Proof. Let G be a finite connected graph and let V be the set of vertices of G. If G has no cycle, then it is itself a spanning tree and there is nothing to prove. Suppose then that G has at least one cycle. If we remove one edge, say e from the cycle, it would not disconnect the graph

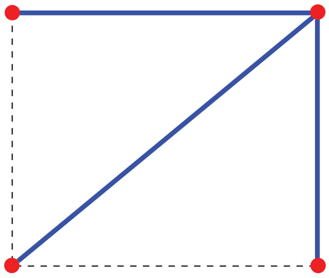


Figure 2.1: An example of a spanning tree. The blue edges are a spanning tree of the connected graph shown.

$G - e$, because we need at least two edges to disconnect a cycle. Therefore, all the vertices of the resulting graph $G - e$ are still connected by the remaining edges of the cycle. If $G_1 = G - e$ has no cycle, then G_1 is a spanning tree. If not, we can remove an edge, say e', from G_1 so that $G_2 = G_1 - e'$ is still connected and has one less edge than G_1. If the resulting graph is not a spanning tree, then we keep removing edges until no cycle is left. Then we obtain a sequence $G, G_1, G_2, \dots$ of subgraphs such that each G_i is connected, contains all the vertices of G, and G_i has one fewer edge than G_{i-1}. Since G has only finitely many edges eventually we obtain a graph G_k with $|V| - 1$ edges, and G_k is a spanning tree of G by Lemma 2.1.3. $\square$

The existence of a spanning tree in an infinite connected graph is proved using different methods such as Zorn's lemma and transfinite induction, for details see [3].

Next, we define edge moves for spanning trees and the tree graph of a connected graph.

Definition 2.1.6 (Goddard and Swart [4]). For any spanning tree T of a graph G an **edge move** is defined as adding one edge $e \in G$ to T and deleting one edge e' from T so that $T + e - e'$ is a spanning tree of G (Figure 2.2 shows an example).

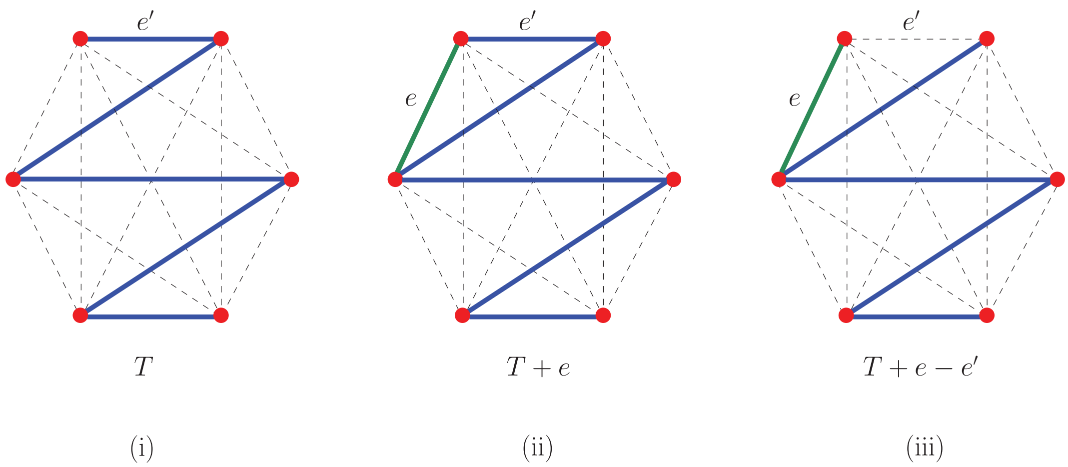


Figure 2.2: An edge move. (i) A spanning tree T of a connected graph G, in blue edges, (ii) adding an edge e to T, so $T + e$ has a cycle and (iii) deleting an edge e' from this cycle gives the tree $T + e - e'$.

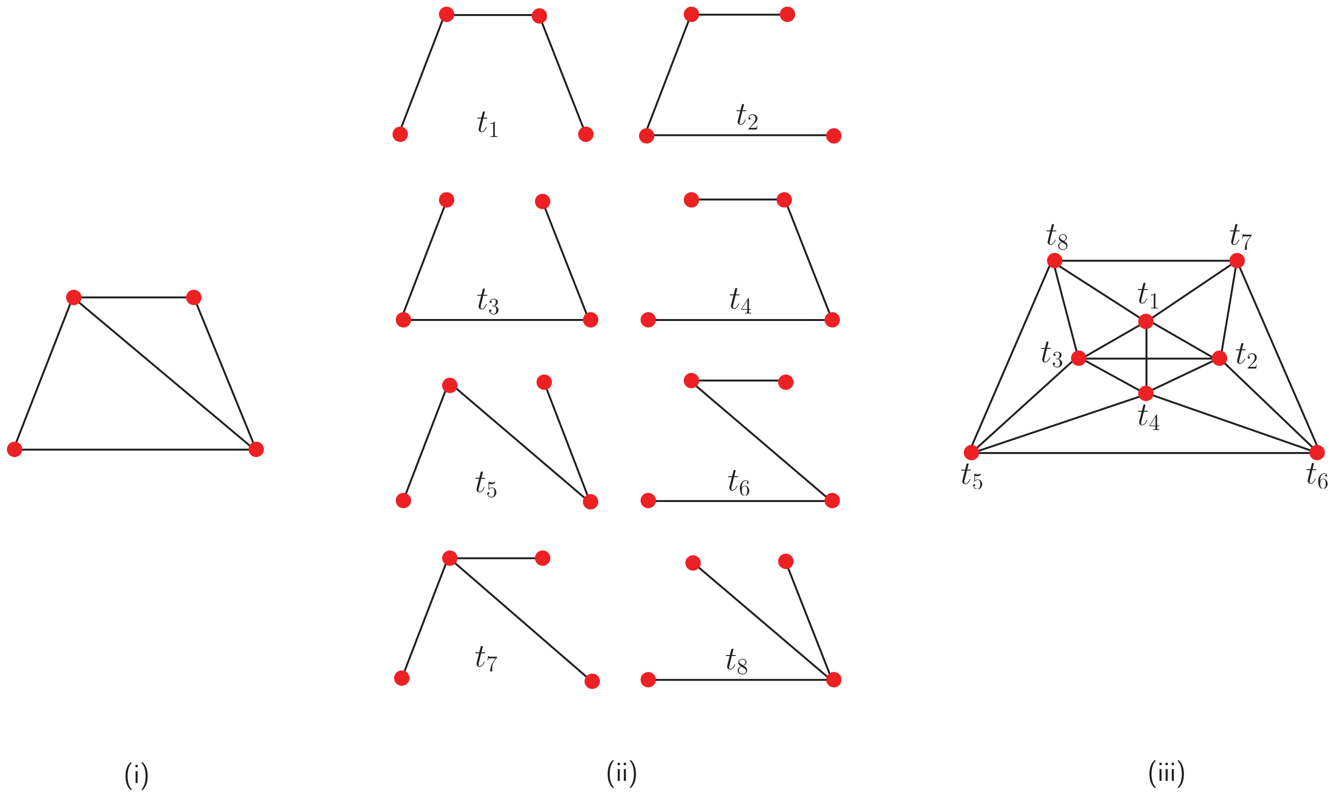


Figure 2.3: The tree graph of a connected graph. (i) A connected graph G, (ii) the spanning trees of G, and (iii) the tree graph of G.

Definition 2.1.7. The **tree graph** of a connected graph G, $T(G)$, is the graph whose vertices are the spanning trees of G, with an edge between two vertices (spanning trees) if they differ by one edge move.

For example, the graph G in Figure 2.3(i) has eight spanning trees $t_1, \ldots, t_8$, drawn in (ii), and its tree graph is shown in (iii). See Ozeki and Yamashita [9] for a survey for known results on tree graphs.

The next theorem states that the tree graph of a connected graph is connected. To prove this we show that any spanning tree can be transformed into any other spanning tree by a sequence of edge moves.

Theorem 2.1.8. *The tree graph of a connected graph G is connected.*

Proof. Let G be a connected graph and let V be the set of vertices of G. Let T and T' be two spanning trees of G. Then T and T' each have $|V|$ vertices and $|V|-1$ edges. Suppose T and T' have m edges in common, where $0 \leq m \leq |V|-1$. If $m = |V|-1$, then $T = T'$ and no further moves are needed. Otherwise, let e be an edge of T' not belonging to T. If we add e to T, then a cycle A containing e is created in $T+e$ (and $T+e$ is not a spanning tree). Since T' is a spanning tree, A contains at least one edge $e^\star$ that is not in T'. When we remove $e^\star$ from $T+e$, the resulting graph $T_1 = T+e-e^\star$ is still connected because deleting an edge from a cycle cannot disconnect it. Also, T_1 will have $|V|-1$ edges, so it must be a spanning tree. In other words, the spanning tree T is transformed into a spanning tree T_1 having $m+1$ edges in common with T', which means T_1 has one more edge in common with T' than T does.

If $T_1 = T'$ then the transformation is completed. If not, T_1 can be transformed into a spanning tree T_2 having one more edge in common with T' than T_1 does. If we repeat this process, then we obtain a sequence $T_1, T_2, T_3, \ldots, T_{|V|-1-m}$ of spanning trees such that T_{i-1} is transformed into a spanning tree T_i by an edge move and T_i has one more edge in common with T' than T_{i-1} does. The tree $T_{|V|-1-m}$ has $|V|-1$ edges in common with T', so is in fact equal to T'. Therefore, the spanning tree T can be transformed into the spanning tree T' by a sequence of edge moves.

Finally, by the definition of the tree graph $T(G)$, there is an edge from T_{i-1} to T_i, hence T and T' are in the same connected component of $T(G)$. Since T and T' are arbitrary, the tree graph $T(G)$ is connected. □

We next explain the meaning of a rooted spanning tree T of Q_n. In general, a rooted spanning tree of a graph G is defined as follows.

Definition 2.1.9. A **rooted spanning tree** of a graph G is a spanning tree of G in which one of the vertices is designated as the root.

We usually root the spanning tree of Q_n at a particular vertex at the bottom of the cube. The empty set $\varnothing$ is a vertex of Q_n and it is the one that we typically use as the root, but the root need not be at $\varnothing$. Unless stated otherwise, all spanning trees of Q_n in this study are assumed to be rooted at $\varnothing$.

2.2 The n-dimensional cube (hypercube)

In this section we define the n-dimensional cube and then define and highlight some known results related to the n-dimensional cube. Throughout this thesis we write

$$[n] = \{1, 2, \ldots, n\},$$

for a positive integer n.

Definition 2.2.1. The ***n*-dimensional cube** is the graph Q_n with 2^n vertices and $2^{n-1}n$ edges, for which the vertices of Q_n are the subsets of $[n]$, and two vertices are adjacent if the corresponding subsets differ by adding or removing exactly one element.

The graph Q_n can be constructed from two copies of the lower dimensional cube Q_{n-1}. See Figure 2.4 for the case $n = 4$.

In the next definition we define the $|S|$-dimensional cube of a set $S \subseteq [n]$.

Definition 2.2.2. For any $S \subseteq [n]$, let Q_S be the induced subgraph of Q_n with vertices the subsets of S. Observe that Q_S is an $|S|$-cube.

In the following definitions we define the cardinality of a vertex, power set of a set, and symmetric difference.

Definition 2.2.3. The **cardinality** of a vertex V of Q_n is the number of elements of the corresponding subset of $[n]$, and is denoted $|V|$. We also sometimes refer to $|V|$ as the **level** of V.

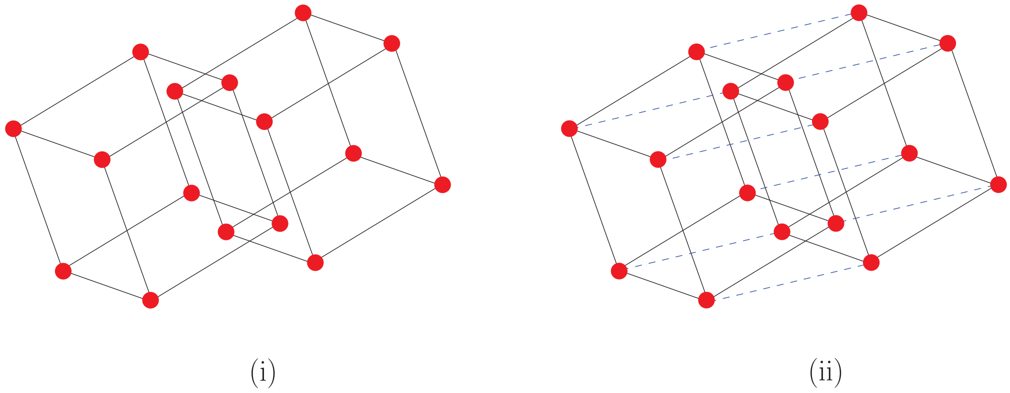


Figure 2.4: The graph Q_4 constructed from two copies of the graph Q_3. (i) Two copies of the 3-cube Q_3. (ii) Connecting each vertex of one copy of Q_3 to the corresponding vertex of the other copy of Q_3 gives the graph Q_4.

Definition 2.2.4. The **power set** of $[n]$ is the set of all subsets of $[n]$, and is denoted $\mathcal{P}([n])$.

For any $i \in [n]$, we denote by $\mathcal{P}^n_{\geq i}$ the set of all the subsets of the set $[n]$ with cardinality greater than or equal to i.

Definition 2.2.5. The **symmetric difference** of two sets V and U, denoted $V \oplus U$, is defined to be

$$\begin{aligned} V \oplus U &= (V - U) \cup (U - V) \\ &= (V \cup U) - (V \cap U). \end{aligned}$$

2.2.1 Counting spanning trees of Q_n

As given in Equation (1.1), it is known that the number of spanning trees of Q_n is

$$|\mathrm{Tree}(Q_n)| = 2^{2^n-n-1} \prod_{k=1}^{n} k^{\binom{n}{k}},$$

which can be obtained by using Kirchhoff's Matrix Tree Theorem (see for example Stanley [10]). Bernardi [2] gave two combinatorial proofs of this formula. See also Tuffley [11] for a bijective proof in the case $n = 3$ using edge slides. By evaluating the above formula (1.1) for $1 \leq n \leq 5$, we get Table 2.1. The following is a refinement of (1.1):

$$\sum_{T \in \mathrm{Tree}(Q_n)} q^{\mathrm{dir}(T)} x^{\mathrm{dd}(T)} = q_1 \cdots q_n \prod_{V \subseteq \mathcal{P}^n_{\geq 2}} \sum_{i \in V} q_i (x_i^{-1} + x_i), \tag{2.1}$$

where $q_1, \ldots, q_n, x_1, \ldots, x_n$ are certain weights carrying information about the locations and directions of the edges of a tree (see the next section for details). Each term in the expansion, and hence each spanning tree, corresponds to a choice of an element $i \in S$ and a sign ± 1 for each $S \subseteq [n]$ with $|S| \geq 2$. This was proved by Martin and Reiner [8] using a weighted version of the Matrix-Tree Theorem.

Cube	Number of spanning trees
1-cube (line)	1
2-cube (Square)	4
3-cube (Cube)	384
4-cube (Tesseract)	42467328
5-cube (Penteract)	2077601987473440768 0

Table 2.1: The number of spanning trees of Q_n, where $n = 1, 2, 3, 4$ and 5.

2.2.2 The direction and decoupled degree monomials

To define the two factors of the weight of a tree, the direction monomial $q^{\mathrm{dir}(T)}$ and the decoupled degree monomial $x^{\mathrm{dd}(T)}$, we first need to define the direction of an edge in Q_n as follows.

Definition 2.2.6. Let e be an edge in Q_n. Let V and U be the endpoints of e. Then V and U differ by one element, say i. The **direction** of e is i, denoted $\mathrm{dir}(e) = i$ (see Martin and Reiner [8]).

Definition 2.2.7 (Martin and Reiner [8])**.** Let $E(T)$ be the edge set of a spanning tree T of Q_n. Then the **direction monomial** of T is

$$q^{\mathrm{dir}(T)} = \prod_{e \in E(T)} q_{\mathrm{dir}(e)}.$$

For $V \subseteq [n]$, let $x_V = \prod_{i \in V} x_i$. Then the **decoupled degree monomial** is

$$\begin{aligned} x^{\mathrm{dd}(T)} &= \prod_{V \subseteq [n]} \left(\frac{x_V}{x_{[n] \setminus V}} \right)^{\frac{1}{2} \deg_T(V)} \\ &= \prod_{(V,U) \in E(T)} \frac{x_V x_U}{x_{[n]}}. \end{aligned}$$

Note that, for an edge $e = (V, U)$ of a spanning tree T of Q_n, if $\mathrm{dir}(e) = i$ then

$$\frac{x_V x_U}{x_{[n]}} = x_1^{\epsilon_1} \cdots x_{i-1}^{\epsilon_{i-1}} x_{i+1}^{\epsilon_{i+1}} \cdots x_n^{\epsilon_n},$$

where for $j \neq i$,

$$\epsilon_j = \begin{cases} +1, & \text{when } j \in V; \\ -1, & \text{when } j \notin V. \end{cases}$$

Then each $e \in E(T)$ such that $\mathrm{dir}(e) = i$ contributes a factor of x_j or x_j^{-1} to $x^{\mathrm{dd}(T)}$ for each $j \neq i$, where the factor is x_j if the ends of e contain j and x_j^{-1} if the ends of e do not contain j.

Tuffley [11] gives an alternative formulation of the decoupled degree monomial in terms of orientations of the edges e of T. In this formulation, e contributes a factor of $x^{\pm 1}_{\mathrm{dir}(e)}$. Orient all the edge of Q_n upwards. Root each spanning tree at the empty set $\varnothing$, and orient all the edges

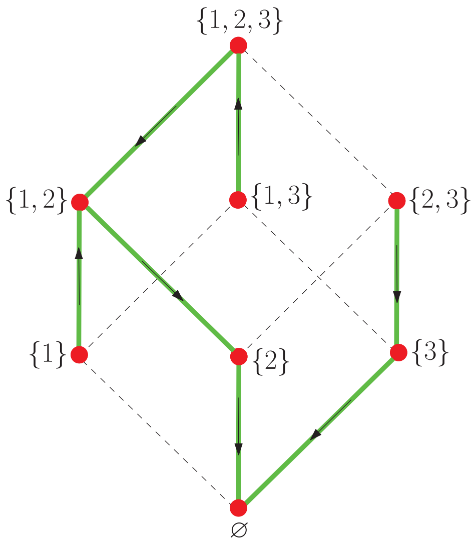


Figure 2.5: Upward edges and downward edges. The bold edges are a spanning tree of Q_3. The edges $(\{1\},\{1,2\})$ and $(\{1,3\},\{1,2,3\})$ are upward edges and the rest are downward.

of T towards $\varnothing$. So now each edge e of T has two orientations, one from the cube and one from the tree. Then we define

$$\mu(e) = \begin{cases} +1, & \text{when the orientation of } e \text{ from the cube and the tree agree;} \\ -1, & \text{otherwise.} \end{cases}$$

Then:

Lemma 2.2.8 (Tuffley [11])**.**

$$x^{\mathrm{dd}(T)} = x_1 x_2 \cdots x_n \prod_{e \in E(T)} x_{\mathrm{dir}(e)}^{\mu(e)}.$$

We define upward edges and downward edges in terms of μ as follows.

Definition 2.2.9. Let T be a spanning tree of Q_n rooted at $\varnothing$ and let $e = (X, X \cup \{i\})$ be an edge of T. Then e is an **upward edge** if $\mu(e) = +1$ and a **downward edge** if $\mu(e) = -1$. That is, when $X \cup \{i\}$ is on the path from X to $\varnothing$, then e is an upward edge and we orient e from X to $X \cup \{i\}$. When X is on the path from $X \cup \{i\}$ to $\varnothing$, then e is a downward edge and we orient e from $X \cup \{i\}$ to X.

In other words, an edge which is oriented by T in the direction of increasing cardinality is said to be an upward edge. An edge which is oriented in the direction of decreasing cardinality is said to be a downward edge. For an example, see Figure 2.5.

2.2.3 Signatures of spanning trees of Q_n

A spanning tree of Q_n can be classified by its signature.

Definition 2.2.10. For each spanning tree T of Q_n the **signature** of T is defined to be $(a_1, a_2, \ldots, a_n)$, where for each $i \in [n]$, a_i is the number of edges of T in direction i.

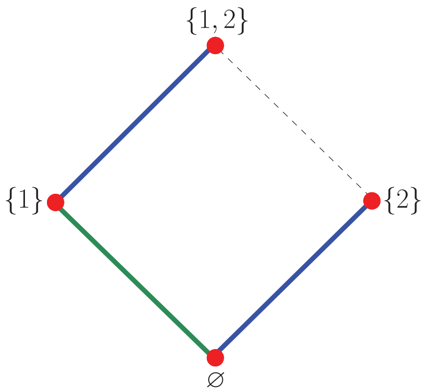


Figure 2.6: The edges from a spanning tree of Q_2, with the edges in direction 1 coloured green and the edges in direction 2 coloured blue. As we can see, there is one green edge and two blue edges. Therefore, the signature is (1, 2).

An example appears in Figure 2.6. The sum of the a_i over $1 \leq i \leq n$ is equal to the number of edges of T, and so the signature of a spanning tree of Q_n satisfies

$$\sum_{i=1}^{n} a_i = 2^n - 1.$$

Each a_i must be at least 1 since every tree must have at least one edge in each direction in order to connect the vertices, and it cannot be bigger than 2^{n-1} because the number of edges of Q_n in each direction is 2^{n-1}. Therefore $1 \leqslant a_i \leqslant 2^{n-1}$ for all i. Note that these conditions are not sufficient conditions for an n-tuple $(a_1, a_2, \dots a_n)$ to be a signature of Q_n. We obtain necessary and sufficient conditions in Chapter 3.

Observation 2.2.11. *For a spanning tree T of Q_n with signature $(a_1, a_2, \dots, a_n)$, the direction monomial $q^{\mathrm{dir}(T)}$ is $q^{a_1}q^{a_2}\dots q^{a_n}$, so the signature and $q^{\mathrm{dir}(T)}$ carry the same information.*

Definition 2.2.12. A signature $(a_1, a_2, \dots, a_n)$ is said to be **ordered** if $a_1 \leq a_2 \leq \dots \leq a_n$.

It is clear that the n-dimensional cube, Q_n, has lots of symmetries. We can permute the n coordinates and we can also do reflections σ_i in each direction i, where $i \in [n]$. Using the symmetric difference $\oplus$, the reflection σ_i is defined to be

$$\sigma_i(V) = V \oplus \{i\} = \begin{cases} V - \{i\}, & \text{if} \quad i \in V; \\ V \cup \{i\}, & \text{if} \quad i \notin V, \end{cases}$$

where $V \subseteq [n]$. Together these generate the symmetry group $\mathrm{Aut}(Q_n) \cong \mathbb{Z}_2^n \rtimes S_n$, of order $2^n n!$.

Since each permutation π of $[n]$ induces an automorphism that carries a spanning tree with signature $(a_1, a_2, \dots, a_n)$ to a spanning tree with signature $(a_{\pi^{-1}(1)}, a_{\pi^{-1}(2)}, \dots, a_{\pi^{-1}(n)})$, we say that the two signatures $(a_1, a_2, \dots, a_n)$ and $(a_{\pi^{-1}(1)}, a_{\pi^{-1}(2)}, \dots, a_{\pi^{-1}(n)})$ are the same up to permutation. For example, there are three different signatures of Q_3 up to permutation: (1, 2, 4), (1, 3, 3) and (2, 2, 3), (see Henden [6]).

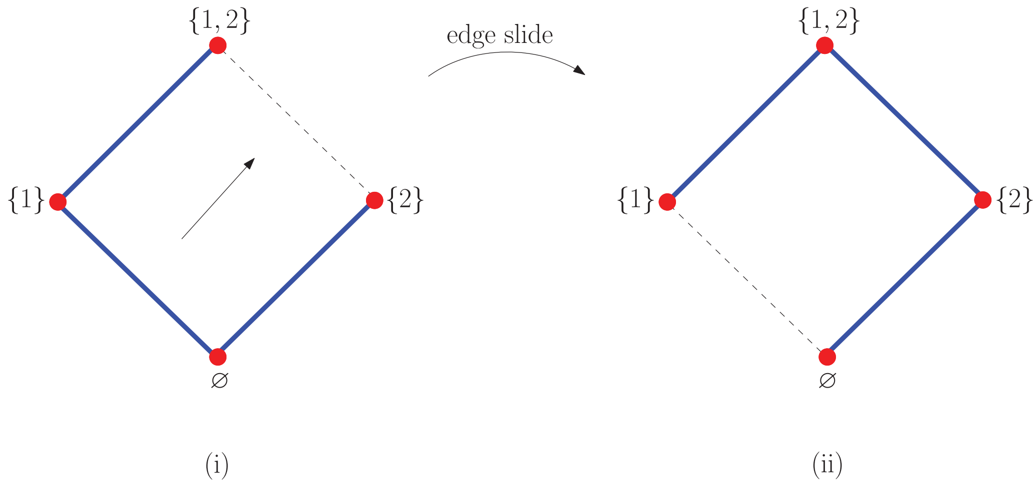


Figure 2.7: An edge slide between two spanning trees of Q_2.

2.2.4 Edge slides

Edge slides are a specialisation of Goddard and Swart's edge move [4] to the spanning trees of Q_n, in which the edges involved are constrained by the structure of the cube. Note that the definition of edge slide used here differs from that of Goddard and Swart's definition.

Before defining edge slides, we define the upper and lower faces of Q_n.

Definition 2.2.13. Let $i \in [n]$. The **upper face** $\mathcal{F}_n^{i+}$ of Q_n with respect to direction i is the subgraph induced by the subsets of $[n]$ that include i. The **lower face** $\mathcal{F}_n^{i-}$ of Q_n with respect to direction i is the subgraph induced by the subsets of $[n]$ that do not include i. Hereafter we will refer to $\mathcal{F}_n^{i+}$ and $\mathcal{F}_n^{i-}$ as **upstairs** and **downstairs**.

Note that the reflection σ_i of Q_n induces a symmetry that swaps the upper and lower faces of Q_n with respect to direction i. Now, we define an edge slide as follows.

Definition 2.2.14 (Tuffley [11])**.** Let T_1 and T_2 be two spanning trees of Q_n. Then we say T_1 is transformed into T_2 by an **edge slide** if there exists an edge $e \in T_1$ and direction $i \in [n]$ with $i \neq \mathrm{dir}(e)$ such that deleting e and replacing it with its reflection $\sigma_i(e)$ gives the spanning tree T_2. An edge that can be slid in direction i is called i**-slidable** or **slidable in direction** i.

In other words, it is an operation of sliding an edge of one spanning tree across a two dimensional face of the cube to get another spanning tree. An example of an edge slide appears in Figure 2.7.

Note that for a spanning tree of Q_n, there are at least $a_i - 1$ edges that may be slid in direction i, where a_i is the number of edges in direction i, see Tuffley [11, Corollary 13] for a proof.

Edge slides can be classified as **upward edge slides** or **downward edge slides** as follows.

Definition 2.2.15. An edge slide is said to be **upward**, if the number of elements of each subset corresponding to an endpoint of the edge is increased. If the number of elements of each subset corresponding to an endpoint of the edge is decreased, then the edge slide is **downward**.

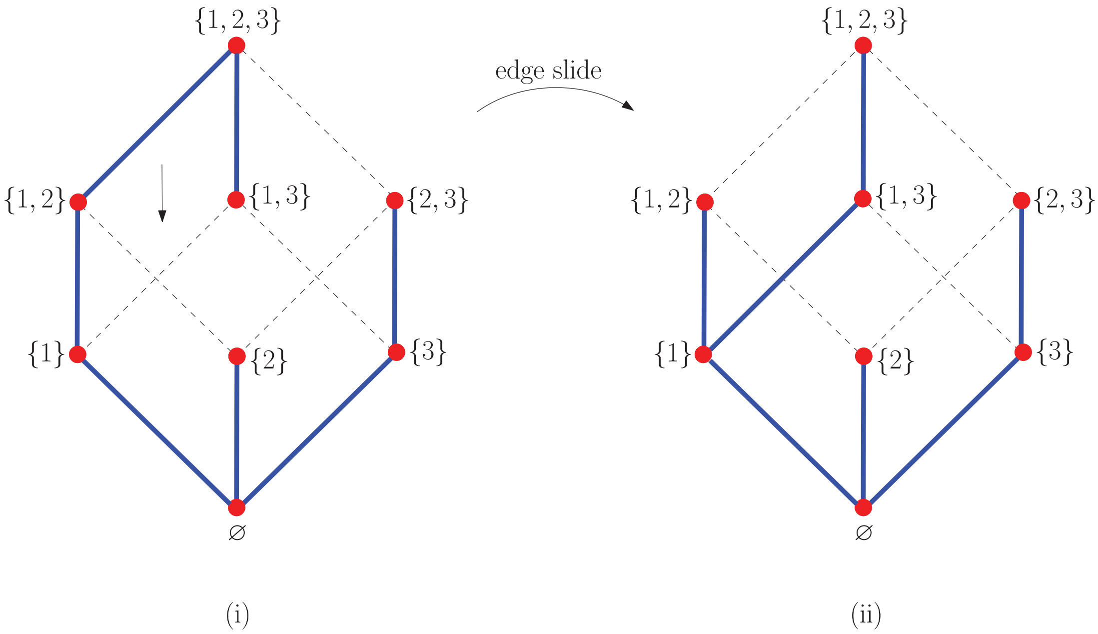


Figure 2.8: An edge slide between two spanning trees of Q_3. Bold edges represent the spanning trees. (i) The slide in this picture is downward, since the corresponding endpoints of the edge $(\{1,3\},\{1\})$ have fewer elements than the corresponding endpoints of the edge $(\{1,2,3\},\{1,2\})$ as we can see in (ii).

Figure 2.8 shows an example of downward edge slide. Note that for a spanning tree T of Q_n with u_i and d_i upward and downward edges in direction i there are at least u_i edges that may be slide downwards and at least $d_i - 1$ that may be slid upwards (Tuffley [11, Corollary 13]). The effect of an edge slide in direction i on $x^{\mathrm{dd}(T)}$ is to multiply it by $x_i^{\pm 2}$: by x_i^2 if the edge is slid up and x_i^{-2} is the edge is slid down. Therefore, the number of upward edges in direction i must be changed by ± 1, and there can be no change in the number of upward edges in other directions.

Definition 2.2.16 (Tuffley [11])**.** The **edge slide graph** of Q_n, denoted $\mathcal{E}(Q_n)$, is the graph on the spanning trees of Q_n, with an edge between two spanning trees if they are related by an edge slide.

The 1-cube, Q_1, has a unique spanning tree so its edge slide graph consists of a single vertex. Spanning trees of Q_2 and their edge slide graph are shown in Figure 2.9. The edge slide graph is a subgraph of the tree graph. Note that edge slides do not change the signature of a spanning tree, so spanning trees with different signatures belong to different components of the edge slide graph.

Next, we give a definition of the edge slide graph of a signature $(a_1, a_2, \ldots, a_n)$.

Definition 2.2.17. Let $\mathcal{S} = (a_1, a_2, \ldots, a_n)$ be a signature of a spanning tree of Q_n. Then the **edge slide graph of signature** $\mathcal{S}$, denoted $\mathcal{E}(\mathcal{S})$, is defined to be the subgraph of $\mathcal{E}(Q_n)$ induced by the trees with signature $\mathcal{S}$.

Since spanning trees with different signatures belong to different components of $\mathcal{E}(Q_n)$, the edge slide graph $\mathcal{E}(\mathcal{S})$ is comprised of one or more connected components of $\mathcal{E}(Q_n)$. We make the following definition:

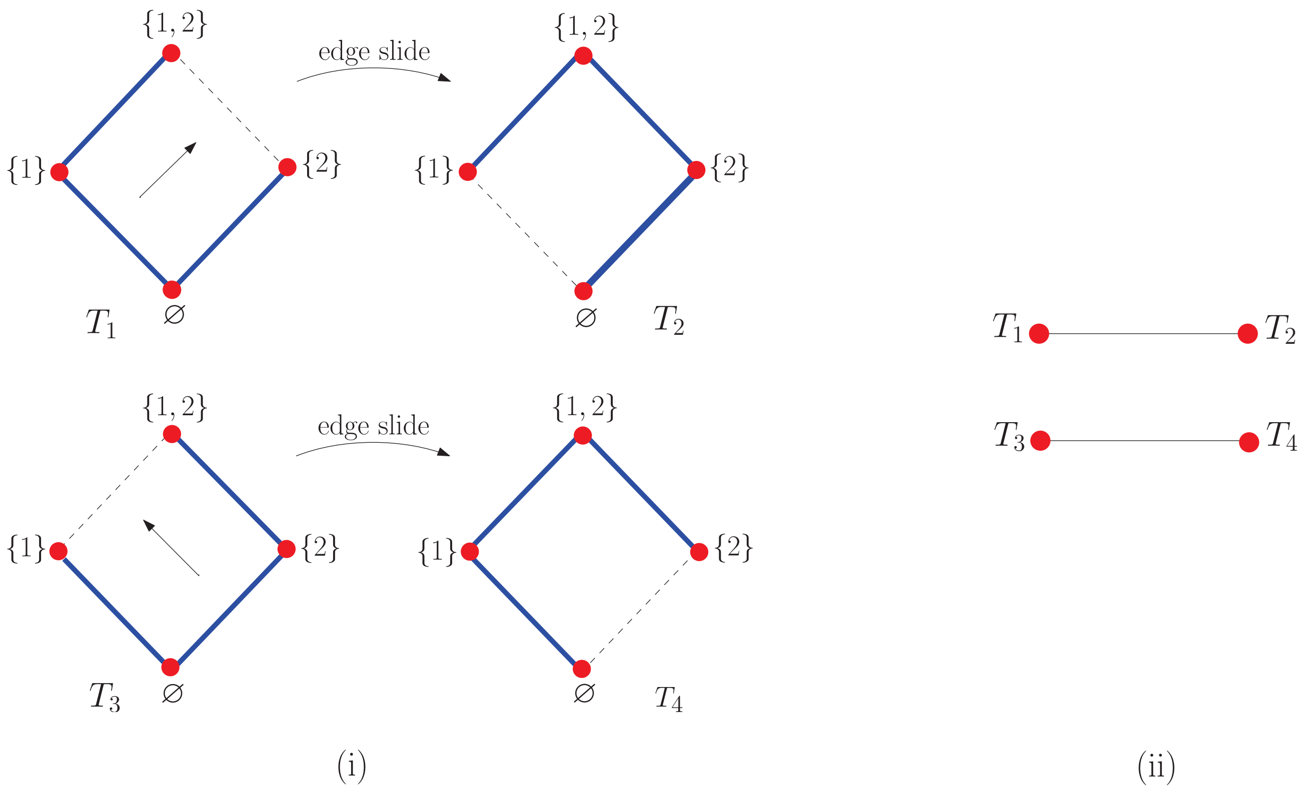


Figure 2.9: Spanning trees of Q_2 with signatures (1, 2) and (2, 1), and the edge slide graph of Q_2. (i) Spanning trees of Q_2 with edge slides. (ii) The edge slide graph of Q_2.

Definition 2.2.18. A signature is **connected** if its edge slide graph is connected, and **disconnected** otherwise.

The goal of this research is to answer the following question:

Question 2.2.19. Which signatures are connected?

The following lemma states that the edge slide graphs of two signatures are isomorphic, if the signatures are related by a permutation.

Lemma 2.2.20 (Henden [6])**.** *When two signatures* $(a_1, a_2, \dots, a_n)$ *and* $(b_1, b_2, \dots, b_n)$ *are the same up to permutation, then their edge slide graphs,* $\mathcal{E}(a_1, a_2, \dots, a_n)$ *and* $\mathcal{E}(b_1, b_2, \dots, b_n)$ *are isomorphic.*

Proof. A symmetry of Q_n induced by a permutation of $[n]$ induces a symmetry of $\mathcal{E}(Q_n)$. Such a symmetry carries a spanning tree T of Q_n with signature $(a_1, a_2, \dots, a_n)$ to a spanning tree of Q_n with the signature $(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \dots, a_{\sigma^{-1}(n)}) = (b_1, b_2, \dots, b_n)$. Therefore, the edge slide graphs of the two signatures are isomorphic. □

2.2.5 Upright spanning trees

We define an upright spanning tree of Q_n as follows.

Definition 2.2.21 (Tuffley [11])**.** An **upright spanning tree** of Q_n is a spanning tree such that all its edges are oriented downward.

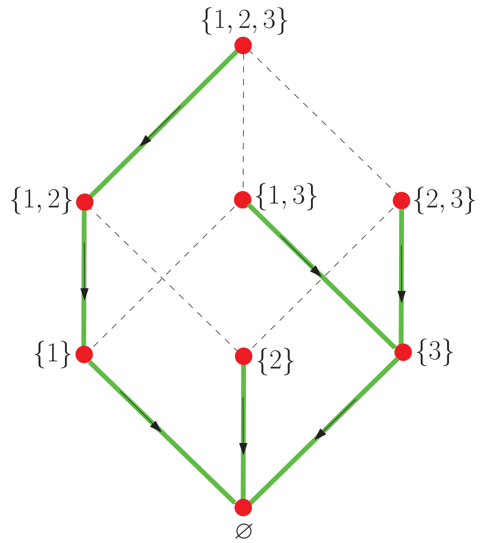


Figure 2.10: An upright spanning tree of Q_3 with signature (2, 3, 2), in bold edges. This corresponds to choosing direction 3 from $\{1,2,3\}$, direction 2 from $\{2,3\}$, direction 1 from $\{1,3\}$, direction 2 from $\{1,2\}$, and directions 1, 2 and 3 from $\{1\}, \{2\}$ and $\{3\}$ respectively.

An example of an upright spanning tree of Q_3 is shown in Figure 2.10.

Definition 2.2.22. An **upright spanning forest** of Q_n is a spanning forest such that all its edges are oriented downward.

The following corollary shows that each spanning tree of Q_n is connected to an upright spanning tree by a sequence of edge slides.

Theorem 2.2.23. *(Tuffley [11, Corollary 15]) Let T be a spanning tree of Q_n. Then there is a sequence of edge slides that transforms T into an upright spanning tree.*

2.2.5.1 Encoding spanning trees of the graph Q_n

For the purpose of this study, we label a spanning tree of Q_n rooted at $\varnothing$ as follows. For each vertex $V \in V(Q_n)$, there is a path from V to $\varnothing$. We label each vertex V with the direction of the first edge on the path from V to $\varnothing$. An example is shown in Figure 2.11.

In the case $n = 3$, Tuffley showed that upright spanning trees of Q_3 are in bijection with the sections of $\mathcal{P}^3_{\geq 1}$, (see Tuffley [11, Lemma 4.2]). The method used by Tuffley is not limited to the case $n = 3$; it can be applied to all n and shows that there is a bijection between the upright spanning trees of Q_n and the sections of $\mathcal{P}^n_{\geq 1}$ as follows.

Construction 2.2.24. Given a spanning tree T of Q_n, we define a function $\psi_T : \mathcal{P}^n_{\geq 1} \to [n]$ as follows. The first edge on the path from a nonempty vertex V to the root is in a direction $\psi_T(V)$ belonging to $[n]$.

If T is an upright spanning tree, then all the edges of T are oriented downwards. So, the first edge on the path from a nonempty vertex to the root must be in a direction $\psi_T(V) \in V$. We call such a function, $\psi : \mathcal{P}^n_{\geq 1} \to [n]$ such that $\psi(V) \in V$, a *section* of $\mathcal{P}^n_{\geq 1}$.

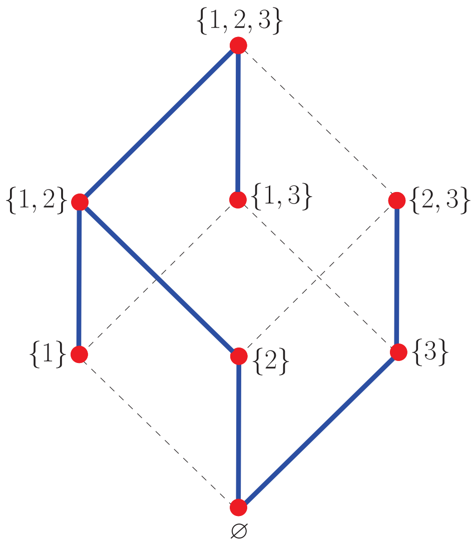


Figure 2.11: The first edge in the path from $\{1,2,3\}$ to the root $\varnothing$ is in direction 3. Therefore, the label at $\{1,2,3\}$ is 3. The first edges on the paths from $\{1,2\}$ to $\varnothing$, $\{1,3\}$ to $\varnothing$, $\{2,3\}$ to $\varnothing$ and $\{1\}$ to $\varnothing$ are in directions 1, 2, 2 and 2 respectively. Therefore, the labels at $\{1,2\},\{1,3\},\{2,3\}$ and $\{1\}$ are 1, 2, 2 and 2 respectively. The first edge in the paths from $\{2\}$ to $\varnothing$ and $\{3\}$ to $\varnothing$ are in directions 2 and 3 respectively. So then the labels at $\{2\}$ and $\{3\}$ are 2 and 3 respectively.

A function $\psi : \mathcal{P}^n_{\geq 1} \to [n]$ corresponds to a choice of label $\psi(V)$ from the set $[n]$ at each nonempty vertex V of Q_n. Given such a function, we define G_ψ to be the subgraph of Q_n containing all the vertices, and all edges of the form $\{\{V, V \oplus \{\psi(V)\}\} : V \in \mathcal{P}^n_{\geq 1}\}$. Recall that

$$(V, V \oplus \{\psi(V)\}) = \begin{cases} (V, V - \{\psi(V)\}), & \text{if} \quad \psi(V) \in V; \\ (V, V \cup \{\psi(V)\}), & \text{if} \quad \psi(V) \notin V. \end{cases}$$

This subgraph has at most $2^n - 1$ edges, and may or may not be a tree.

The following lemma shows that a spanning tree of Q_n corresponds to G_{ψ_T}, and moreover G_{ψ_T} corresponds to the spanning tree that we started with.

Lemma 2.2.25. *If T is a spanning tree, then $G_{\psi_T} = T$.*

Proof. Every edge of T is on a path from one of the $2^n - 1$ nonempty vertices of T to the root. Let $e = (V, U)$ be an edge of T, and without loss of generality, suppose V is the endpoint further from the root. Then e is the first edge on the path from V to the root, and the direction of e is $\psi_T(V)$. Therefore, the edge e belongs to G_{ψ_T}. Since G_{ψ_T} has at most $2^n - 1$ edges, the edge set $E(G_{\psi_T}) = E(T)$ and therefore $G_{\psi_T} = T$. □

The next lemma shows that when a function corresponds to a spanning tree, then this spanning tree corresponds to the same function that we started with.

Lemma 2.2.26. *Let $\psi : \mathcal{P}^n_{\geq 1} \to [n]$, and suppose $G_\psi = T$ for some spanning tree T. Then $\psi_T = \psi$.*

Proof. Suppose $G_\psi = T$ for some spanning tree T. Let V be a vertex of Q_n. Then starting at $V = V_0, V_1, V_2, \ldots$, we successively construct a path by leaving V_i in the direction $\psi(V_i)$. So we get a path P belonging to the graph G_ψ because all these edges on P belong to G_ψ. Since the graph consists of a finite set of vertices, this path P has to terminate either at a vertex that it has visited before, or reach the root, which it cannot leave. If the path visits a vertex twice, then we get a cycle. Therefore, G_ψ is not a tree, and this contradicts our hypothesis. So the path P from V ends at the root. Since $G_\psi = T$, the path P from V to the root must be the unique such path in T, and therefore $\psi_T(V)$ is the direction of the first edge on the path from V to the root. So $\psi_T(V) = \psi(V)$ and therefore $\psi_T = \psi$. □

Let $\mathcal{U}_n$ be the set of upright spanning trees of Q_n rooted at $\varnothing$. Let $\mathcal{S}_n$ be the set of sections of $\mathcal{P}^n_{\geq 1}$. We define the map $\Theta : \mathcal{U}_n \to \mathcal{S}_n$, by setting $\Theta(T) = \psi_T$.

Lemma 2.2.27. *The map Θ is a bijection between $\mathcal{U}_n$ and $\mathcal{S}_n$.*

Proof. We showed earlier that the upright spanning tree $T \in \mathcal{U}_n$ corresponds to section ψ_T, and it is shown by using Lemma 2.2.25 that ψ_T corresponds to the upright spanning tree that we started with.

Conversely, given a section $\psi \in \mathcal{S}_n$, the graph G_ψ is the subgraph of Q_n containing all the 2^n vertices and has at most $2^n - 1$ edges

$$\{\{V, V - \{\psi(V)\}\} : V \in \mathcal{P}^n_{\geq 1}\}.$$

At each vertex V of $Q_n \setminus \{\varnothing\}$, there is a specific edge e in G_ψ connecting V to a vertex $V - \{\psi(V)\}$. Joining the vertices by those specific edges gives us a path in G_ψ from V to the root $\varnothing$. It follows that G_ψ is connected. Since G_ψ is connected and contains all the vertices of Q_n, it has at least $2^n - 1$ edges, and thus exactly $2^n - 1$ edges. Therefore the graph G_ψ is a spanning tree. Furthermore, these paths in G_ψ show that G_ψ is an upright spanning tree, and it is shown by using Lemma 2.2.26 that G_ψ corresponds to the section ψ. □

The number of upright spanning trees of Q_n is

$$\prod_{k=1}^{n} k^{\binom{n}{k}},$$

which appears in the formula (1.1) that we used for counting spanning trees of Q_n.

2.2.5.2 Notation for upright spanning trees of Q_n

Building on the existing method used by Henden [6] to label the four upright spanning trees of Q_3 with the same signature (2, 2, 3), we label the upright spanning trees of Q_n as follows. Since each upright spanning tree of Q_n corresponds to a choice of direction $i \in V$ at each vertex $V \in V(Q_n)$, we may specify a tree by simply listing the directions that are chosen at each vertex V such that $|V| \geq 2$. To do this we need to agree on the order of the vertices of Q_n. One way is to order them by decreasing cardinality and then lexicographic order. The label would then be represented as a string of length $2^n - n - 1$, listing the chosen directions from the vertices of Q_n in the agreed order of those vertices. An example is shown in Figure 2.12.

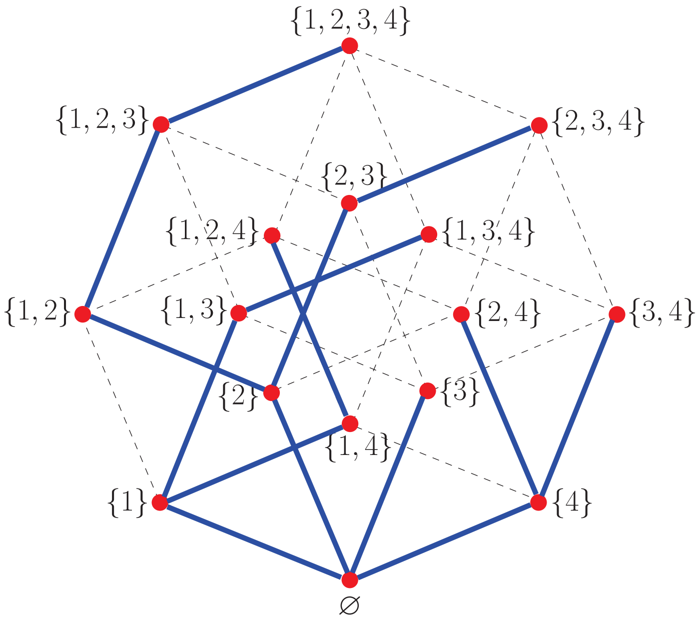


Figure 2.12: An upright spanning tree of Q_4 with signature (2, 3, 5, 5) in bold edges. The label for this upright spanning tree using the vertex ordering of Section 2.2.5.2 is 43244134323. The first 4 means we leave $\{1, 2, 3, 4\}$ in direction 4, then the 3244 that follows specifies the the directions at $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$ respectively. Then 134323 specifies the directions at $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$ and $\{3, 4\}$ respectively. We omit the directions at vertices $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$, since these are completely determined by the condition $\psi(V) \in V$, where V is a nonempty vertex of Q_n.

Chapter 3

Signatures of spanning trees of Q_n

In this chapter we obtain necessary and sufficient conditions for an n-tuple $(a_1, a_2, \dots, a_n)$ to be a signature of Q_n, and classify signatures as reducible and irreducible. Then we establish some lemmas on the existence of certain trees with a given irreducible signature.

3.1 Characterising signatures of spanning trees of Q_n

In this section the main goal is to determine the n-tuples $(a_1, a_2, \dots, a_n)$ that are the signatures of the spanning trees of Q_n. We use Hall's Theorem to prove the result. Hall's Theorem gives a simple necessary and sufficient condition for the existence of a perfect (complete) matching of two finite sets if they have the same size. We introduce a matching problem that leads us to Hall's Theorem which we use to prove our result. Then we discuss our problem in relation to Hall's Theorem.

Before we discuss the matching problems we define a matching in a graph and then we define a perfect matching in bipartite graph.

Definition 3.1.1. Let G be a graph. Then a **matching** in G is a subgraph of G such that each vertex has degree at most one. A matching is **perfect** if it contains all the vertices of G.

3.1.1 Matchings and Hall's Marriage Theorem

To illustrate the ideas of perfect matching and Hall's Marriage Theorem we start this section with the following two problems:

Problem 1

Suppose four women and four men want to find love so they join a dating agency. The agency compares applications and tries to find common interests. Two women, Jo and Mary, enjoy skiing, travelling, musical theatre and music. Kate likes basketball and Sue likes travelling, but hates musical theatre. Tom likes skiing and sport, John likes travelling and basketball, but hates other sports, and Steve and Ralph like reading, musical theatre, swimming and skiing. Is it possible to match the men and women into couples so that each couple share a common interest?

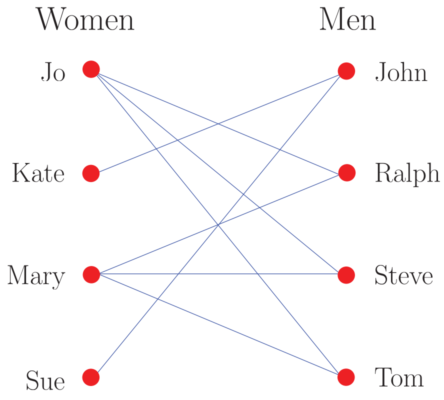


Figure 3.1: A bipartite graph describing the common interests between the four women and the four men. On the left, the set of vertices represents the women and on the right the set of vertices represents the men. Two vertices are joined by an edge if the people they represent share a common interest.

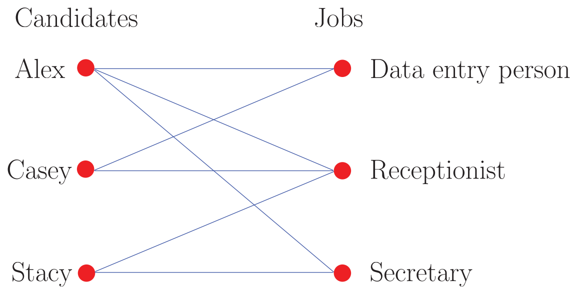


Figure 3.2: A bipartite graph describing the work experience of the candidates. The two sets of vertices represent the candidates and jobs. Two vertices are joined by an edge if the candidate has experience in that job.

Problem 2

A company needs to hire a secretary, data entry person and receptionist. There are three candidates who have experience and qualifications for some or all of these jobs: Alex has worked as a receptionist, data entry person and secretary. Casey has experience in data entry and reception, and Stacy has experience as both a secretary and a receptionist. The company has one position for each job. Is it possible to match each candidate to a job they have experience with so all of them are hired?

We may model each of the situations above using a bipartite graph. For the first problem, the two sets of vertices represent the women and men respectively, and the edges represent a shared interest, see Figure 3.1. For the second problem, let the two sets of vertices represent the candidates and jobs respectively, and the edges represent experience in that job, see Figure 3.2. The problem is then to pair the women and men, and the candidates and jobs, so each pair is joined by an edge.

In order to solve these kinds of matching problems and find a perfect matching, Hall gave us a necessary and sufficient condition in a bipartite graph to determine whether a perfect matching is possible. Note that Hall's condition applies when the two sets have the same size. It is easy to see that the condition is necessary, so the main mathematical content of the theorem is that the condition is also sufficient. Hall's Theorem is also commonly called the "Marriage Theorem" because it is frequently stated in terms of matching men and women, as we have done above.

Theorem 3.1.2 (Hall 1935)**.** *Let G be a bipartite graph with parts A and B such that $|A| = |B|$. The following are equivalent:*

1. *There is a perfect matching from A to B.*
2. *For all $H \subseteq A$, we have $|H| \leq |N(H)|$ (Hall's condition), where $N(H) \subseteq B$ is the set of vertices in B adjacent to a vertex in H.*

If the stronger condition $|H| < |N(H)|$ holds for all H, then for any $a \in A$ and $X \in N(A)$ there exists a perfect matching such that a is matched with X. More generally, if $|H| < |N(H)| - r$ for all H then we expect to be able to extend a partial matching at any r vertices of A to a matching in all of A.

Hall's Theorem was first proved by Philip Hall [5] in 1935. Hall's condition is clearly necessary because, if there is a subset H of A for which $|N(H)| < |H|$, then there are not enough vertices of B available to match the vertices in H. Note that Hall's Theorem does not tell us how to find a perfect matching.

Now, we illustrate Hall's Theorem using the problems outlined above.

Problem 1

Consider the graph shown in Figure 3.1. To find out whether the agency is able to match all of the applicants into couples that share a common interest, we need to check that for every subset H of the women, the neighbourhood $N(H)$ is at least as large as H. However, for

$$H = \{\text{Kate, Sue}\} \subseteq \{\text{Jo, Kate, Mary, Sue}\},$$

we have

$$|N(H)| = |\{\text{John}\}| = 1 < 2 = |\{\text{Kate, Sue}\}| = |H|.$$

Therefore, we can conclude that by Hall's Theorem, the agency is not able to match the four women with the four men so that they share a common interest.

Problem 2

Consider the graph shown in Figure 3.2. We again need to check that for every subset H of the candidates, the neighbourhood $N(H)$ is at least as large as H. For example, for

$$H = \{\text{Casey, Stacy}\} \subseteq \{\text{Alex, Casey, Stacy}\},$$

we have

$$|N(H)| = |\{\text{Secretary, Data entry person, Receptionist}\}| = 3 > 2 = |\{\text{Casey, Stacy}\}| = |H|.$$

By exhaustively checking the remaining six nonempty subsets $H \subseteq A$, we find that in each case, the neighbourhood $N(H)$ is at least as large as H. Therefore, by Hall's Theorem, the company is able to hire all the candidates in a position they are experienced in. Figure 3.3 shows one possible match.

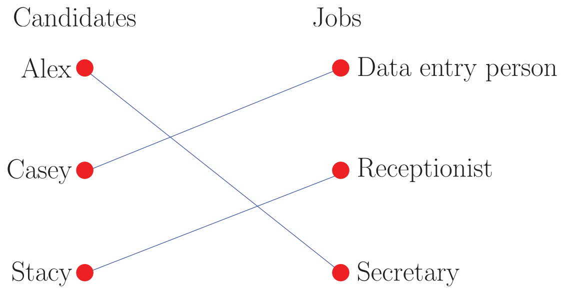


Figure 3.3: A bipartite graph describing one possible matching from candidates to jobs.

3.1.2 Hall's Marriage Theorem and signatures of spanning trees of Q_n

In this section we discuss our problem in relation to Hall's Theorem. Since every spanning tree is connected to at least one upright spanning tree either directly or by a sequence of edge slides (Tuffley [11]), and upright spanning trees are relatively easy to work with, we study the signatures of spanning trees of Q_n using upright spanning trees.

An upright spanning tree T of Q_n is a choice of index at each vertex of Q_n except the root. There is some restriction as to what indices we can choose. In other words, the $2^n - 1$ nonempty vertices of Q_n need to be matched to $2^n - 1$ edges of T. Each vertex must be matched to exactly one edge such that the direction of this edge belongs to the subset corresponding to that vertex. The n-tuple $(a_1, a_2, \ldots, a_n)$ is a signature if and only if there is a perfect matching from the $2^n - 1$ nonempty vertices of Q_n to $2^n - 1$ edges of which a_1 are labelled 1, a_2 are labelled 2, etc, such that each vertex V is matched to a label i where $i \in V$ for all $\varnothing \neq V \subseteq [n]$.

We illustrate our problem by giving an example of an upright spanning tree from Q_3, for which we already know the signatures. We consider the signature (1, 3, 3). Let $\mathcal{P}_{\geq 1}([3])$ be the set of nonempty vertices of Q_3, namely $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$, and $\{1,2,3\}$. Let X be a set of $2^3 - 1 = 7$ vertices: one is labelled 1, three are labelled 2 and the remaining three are labelled 3. As shown in Figure 3.4 (a), $\{1,2\}$ can be matched to the vertex labelled 1 and the three vertices labelled 2; $\{1,3\}$ can be matched to the vertex labelled 1 and the three vertices labelled 3; and so on. We need to match the seven vertices of $\mathcal{P}_{\geq 1}([3])$ to the seven vertices of X, such that each vertex V of $\mathcal{P}_{\geq 1}([3])$ is matched with a vertex whose label i belongs to V. We can use Hall's Theorem to determine that it is possible to match all the vertices of $\mathcal{P}_{\geq 1}([3])$ to the vertices which are labelled 1, 2 and 3, showing that (1, 3, 3) is a signature of Q_3. For example, consider the subset $Y = \{\{2\}, \{3\}, \{2,3\}\} \subseteq \mathcal{P}_{\geq 1}([3])$. To find the size of the neighbourhood we must determine the number of vertices which are labelled 2 and 3. There are three vertices which are labelled 2 and three vertices which are labelled 3. This gives us a total of 6 vertices which are labelled 2 or 3. Therefore,

$$|N(Y)| = 6 > |Y| = |\{2\}, \{3\}, \{2,3\}| = 3$$

(see Figure 3.4 (b)). Similarly we can check all the subsets of $\mathcal{P}_{\geq 1}([3])$ and show that Hall's condition is satisfied. Then, by Hall's Theorem there exists a perfect matching from the vertices of $\mathcal{P}_{\geq 1}([3])$, namely $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$, and $\{1,2,3\}$, to the seven vertices: one

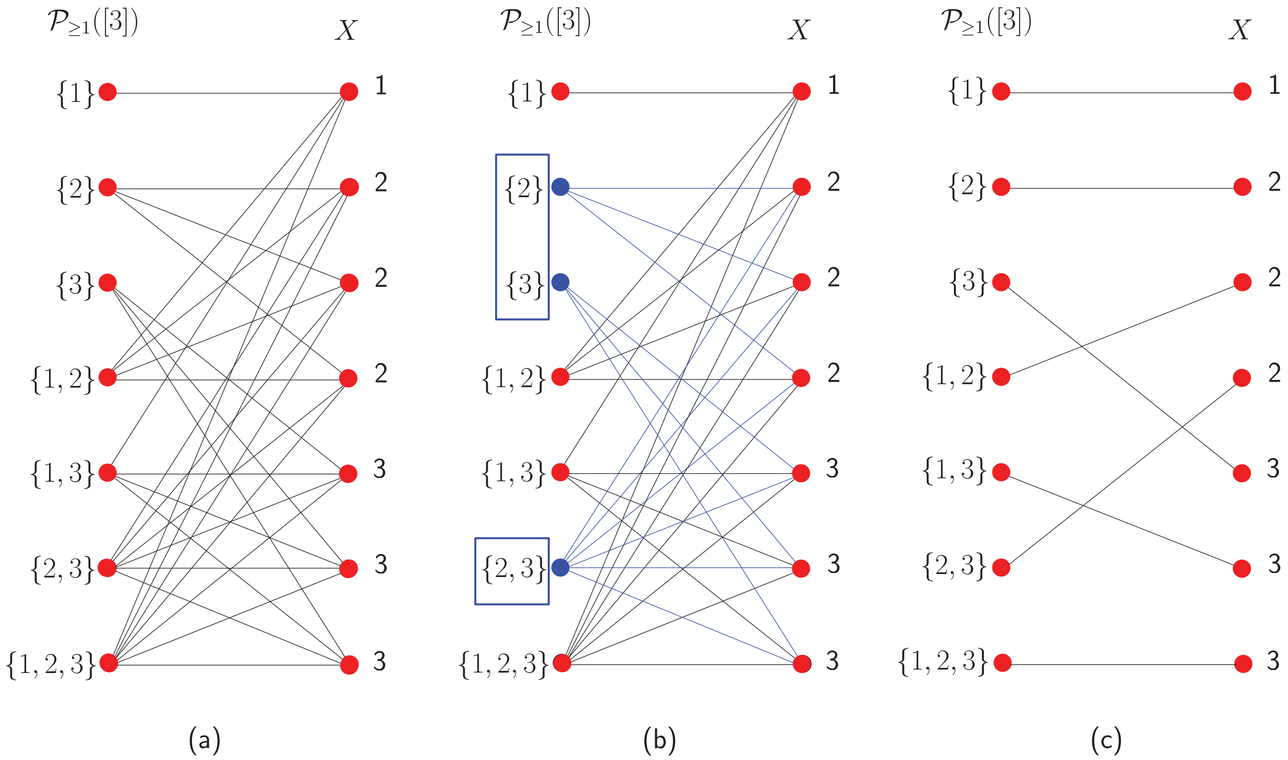


Figure 3.4: (a) On the left, $\mathcal{P}_{\geq 1}([3])$ is the set of nonempty vertices of Q_3, namely $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$. On the right, X is a set of seven nodes: one node is labelled 1, three are labelled 2 and the remaining three are labelled 3. Then we draw an edge matching each nonempty vertex of Q_3 with the possible choices that can be taken from X coloured in blue. (b) This graph shows a subset $Y = \{\{2\}, \{3\}, \{2,3\}\}$ of $\mathcal{P}_{\geq 1}([3])$ and the possible choices that can be taken from X coloured blue. (c) One possible perfect matching between the nonempty vertices of Q_n and the nodes labelled 1, 2 and 3, corresponding to an upright spanning tree, so (1, 3, 3) is a signature of Q_3.

labelled 1, three labelled 2 and the remaining three labelled 3 (see Figure 3.4 (c)). Therefore (1, 3, 3) is a signature of an upright spanning tree of Q_3.

Figure 3.5 shows the upright spanning tree 3232 of Q_3 with signature (1, 3, 3), where 2 is chosen at $\{1,2\}$ and $\{2,3\}$, and 3 is chosen at $\{1,2,3\}$ and $\{1,3\}$.

3.1.3 Main result

The following theorem is our main result of this chapter:

Theorem 3.1.3. *Let $\mathcal{S} = (a_1, a_2, \ldots, a_n)$, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and $\sum_{i=1}^{n} a_i = 2^n - 1$. Then $\mathcal{S}$ is a signature of a spanning tree of Q_n if and only if $\sum_{j=1}^{k} a_j \geq 2^k - 1$, for all $k \leq n$.*

Proof. Since every spanning tree of Q_n is connected to at least one upright spanning tree by a series of edge slides (Tuffley [11]), and edge slides preserve the signature, it suffices to show $\mathcal{S}$ is a signature of an upright spanning tree if and only if $\sum_{j=1}^{k} a_j \geq 2^k - 1$, for all $k \leq n$.

Let A be $\mathcal{P}_{\geq 1}([n])$, the set of $2^n - 1$ nonempty vertices of Q_n, and let B be a set of $2^n - 1$ vertices of which a_i are labelled i, for each $i \in [n]$. From each vertex V of A we draw an edge

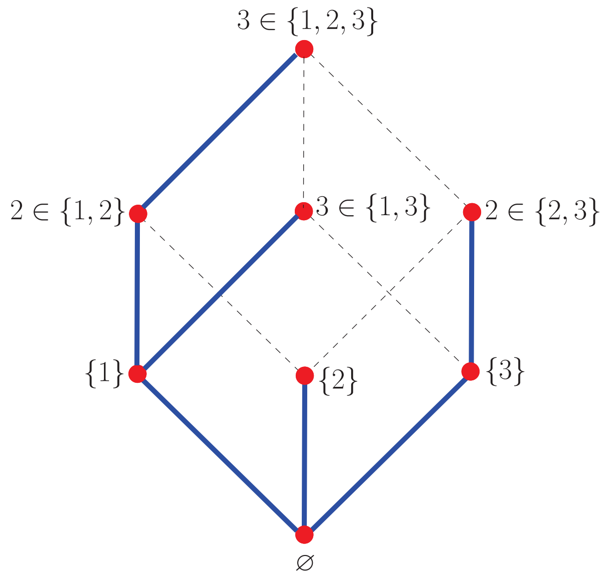


Figure 3.5: The upright spanning tree 3232 of Q_3 with signature (1, 3, 3), where 2 is chosen at $\{1,2\}$ and $\{2,3\}$, and 3 is chosen at $\{1,2,3\}$ and $\{1,3\}$.

to every vertex of B, labelled i, for which $i \in V$. Let $G_{\mathcal{S}}$ be the resulting bipartite graph with bipartition (A, B) (see Figure 3.6). An upright spanning tree of Q_n with signature $\mathcal{S}$ corresponds to a perfect matching in $G_{\mathcal{S}}$. We show there is a perfect matching in $G_{\mathcal{S}}$ if and only if the signature condition $\sum_{j=1}^{k} a_j \geq 2^k - 1$ is satisfied for all $k \leq n$.

Let H be any subset of A, and let

$$Y = \bigcup_{V \in H} V = \{i_1, i_2 \ldots, i_k\},$$

where $i_1 < \cdots < i_k$. Then

$$\begin{aligned} |N(H)| = \sum_{i \in Y} a_i &= \sum_{j=1}^{k} a_{i_j} \\ &\geq \sum_{j=1}^{k} a_j \qquad\qquad \text{(as } a_{i_j} \geq a_j\text{, because } i_j \geq j\text{)}, \end{aligned}$$

with equality if $Y = \{1, \ldots, k\}$, where $N(H) \subseteq B$. Also

$$|H| \leq |\mathcal{P}_{\geq 1}(Y)| = 2^k - 1,$$

with equality if $H = \mathcal{P}_{\geq 1}(Y)$. Then we conclude that $|N(H)| \geq |H|$ for all $H \subseteq A$ if and only if $\sum_{j=1}^{k} a_j \geq 2^k - 1$ for all $k \leq n$. Thus, by Hall's Theorem, and since $|A| = |B|$, there exists a perfect matching if and only if $\sum_{j=1}^{k} a_j \geq 2^k - 1$, for all k. □

Using the fact that each spanning tree is connected to an upright spanning tree, each upright spanning tree corresponds to a section and each section corresponds to a matching in a matching graph, we showed that signatures of Q_n correspond to n-tuples such that the corresponding graph has a perfect matching. The other way to show that signatures of Q_n correspond to

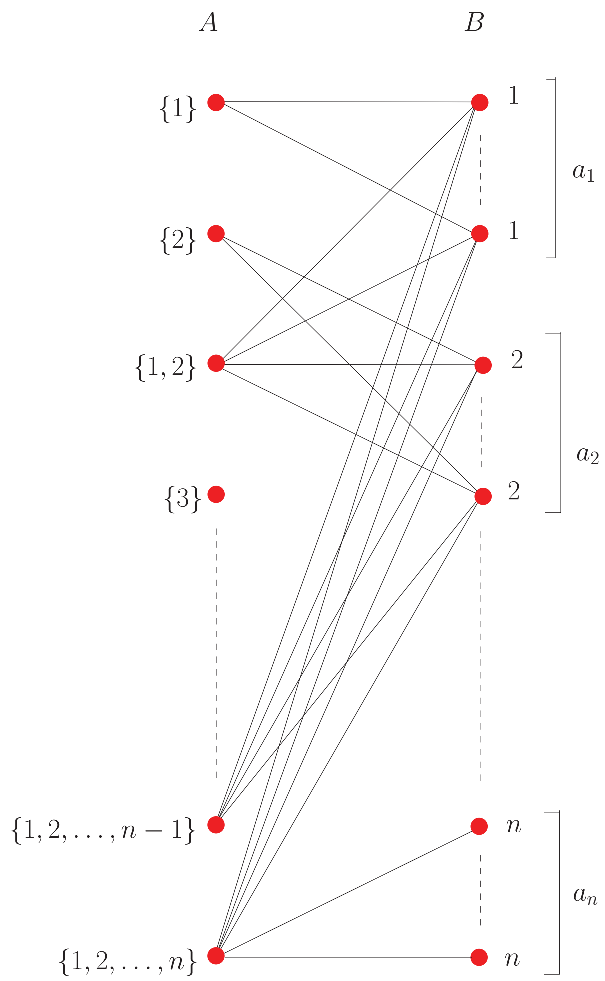


Figure 3.6: This graph shows the set $\mathcal{P}_{\geq 1}([n])$ with $2^n - 1$ nodes corresponding to the nonempty vertices of Q_n, and a set B with $2^n - 1$ vertices of which a_i are labelled i, for each $i \in [n]$.

n-tuples such that the corresponding graph $G_{\mathcal{S}}$ has a perfect matching is using Martin and Reiner's [8] weighted count. We recall their formula (2.1),

$$\sum_{T \in \mathrm{Tree}(Q_n)} q^{\mathrm{dir}(T)} x^{\mathrm{dd}(T)} = q_1 \cdots q_n \prod_{V \subseteq \mathcal{P}^n_{\geq 2}} \sum_{i \in V} q_i (x_i^{-1} + x_i).$$

Set $x_i = 1$ for all i. Then

$$\begin{aligned} \sum_{T \in \mathrm{Tree}(Q_n)} q^{\mathrm{dir}(T)} &= q_1 \cdots q_n \prod_{V \subseteq \mathcal{P}^n_{\geq 2}} \sum_{i \in V} 2q_i. \\ &= \frac{1}{2^n} \prod_{V \subseteq \mathcal{P}^n_{\geq 1}} \sum_{i \in V} 2q_i. \end{aligned}$$

If T has signature $(a_1, \ldots, a_n)$ then $q^{\mathrm{dir}(T)} = q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n}$, and each term in the expansion corresponds to a choice of $i \in V$ for each $V \neq \varnothing$, and hence to a section. Note that $\prod_{V \subseteq \mathcal{P}^n_{\geq 1}} \sum_{i \in V} q_i$ is the generating function for sections of $\mathcal{P}^n_{\geq 1}$, and therefore for the number of upright spanning trees of Q_n also.

There are 18 signatures of spanning trees of Q_4 up to permutation namely:

(1, 2, 4, 8)	(1, 2, 5, 7)	(1, 3, 6, 6)	(2, 2, 4, 7)	(2, 3, 4, 6)	(3, 3, 3, 6)
(1, 3, 3, 8)	(1, 2, 6, 6)	(1, 4, 4, 6)	(2, 2, 5, 6)	(2, 3, 5, 5)	(3, 3, 4, 5)
(2, 2, 3, 8)	(1, 3, 4, 7)	(1, 4, 5, 5)	(2, 3, 3, 7)	(2, 4, 4, 5)	(3, 4, 4, 4)

The first column from the left contains the signatures of Q_3 followed by $2^3 = 8$, which we will define to be *saturated* signatures in Chapter 6. In the next section, we will define the signatures in the first three columns to be *reducible* signatures and the signatures in the last three columns to be *irreducible* signatures.

3.2 Classification of signatures of spanning trees of Q_n

We classify signatures of Q_n as reducible and irreducible as follows.

Definition 3.2.1. Let $\mathcal{S} = (a_1, \ldots, a_n)$ be a signature of a spanning tree of Q_n. Then $\mathcal{S}$ is **reducible** if there exists $\varnothing \neq R \subseteq [n]$ with $|R| \leq n-1$ such that $\sum_{i \in R} a_i = 2^{|R|} - 1$. Moreover we call R a **reducing** set corresponding to $\mathcal{S}$. Otherwise $\mathcal{S}$ is **irreducible**.

The 3-dimensional cube has three signatures up to permutation, namely (1, 2, 4), (1, 3, 3) and (2, 2, 3). The signatures (1, 2, 4) and (1, 3, 3) are reducible signatures with reducing set $\{1\}$, and the signature (2, 2, 3) is irreducible. In Chapter 6 we will divide reducible signatures into *strictly reducible* and *quasi-irreducible* signatures, and we will see that (1, 3, 3) is strictly reducible and (1, 2, 4) is quasi-irreducible. In addition, the signature (1, 2, 4) belongs to the family of *supersaturated signatures*.

Note that if $\mathcal{S}$ is ordered then $\mathcal{S}$ is reducible if and only if $\sum_{i=1}^{r} a_i = 2^r - 1$ for some $r \leq n-1$. In which case an ordered reducible signature has an initial segment that is a signature of a lower dimensional cube. For example, (2, 2, 3, 9, 15, 32, 64, 130, 259, 507) is a reducible signature of Q_{10}, as seen by taking $r = 3, 5, 6$ or 7, whereas (2, 3, 3, 9, 16, 72, 74, 76, 343, 425) is an irreducible signature of Q_{10}.

Lemma 3.2.2. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered signature of Q_n. Then $i \leq a_i$ for all $i \in [n]$.*

Proof. We use the fact easily proved by induction that $m(m-1) < 2^m - 1$ for all $m \geq 1$. Let $j < i$. Since $\mathcal{I}$ is ordered and $j < i$, we have $a_j \leq a_i$ and therefore $2^i - 1 \leq \sum_{t=1}^{i} a_t \leq i a_i$.

Suppose that $i > a_i$. Then $a_i \leq i-1$ and so $2^i - 1 \leq i(i-1)$, contradicting the fact that $i(i-1) < 2^i - 1$. Therefore $i > a_i$ is impossible, so $i \leq a_i$. □

3.2.1 The excess of a signature of Q_n

We define the excess of a signature of Q_n as follows.

Definition 3.2.3. Let $\mathcal{S} = (a_1, \ldots, a_n)$ be an ordered signature of Q_n. We define the **excess** at k, $\varepsilon_k^{\mathcal{S}}$, of signature $\mathcal{S}$ to be

$$\varepsilon_k^{\mathcal{S}} = \sum_{i=1}^{k} a_i - (2^k - 1),$$

where $k \leq n-1$.

In other words, the excess at k is the minimum quantity by which a set of k directions exceeds the marriage condition. Observe that, if $\mathcal{S}$ is reducible, then $\varepsilon_k^{\mathcal{S}} = 0$ for some $k \leq n-1$. If $\mathcal{S}$ irreducible, then $\varepsilon_k^{\mathcal{S}} \geq 1$ for all $k \leq n-1$.

In the following chapters we will frequently require the existence of an upright spanning tree with a given signature and a specified direction at a certain vertex or vertices. The following lemma shows that when the signature is irreducible, we can arbitrarily specify the direction of an arbitrary vertex. This follows directly from the strengthened form of Hall's Theorem discussed immediately after Theorem 3.1.2 and our proof below is actually the proof of the strengthened form in the special case of a signature. More generally, if the signature $\mathcal{S}$ has excess $\varepsilon_k^{\mathcal{S}} \geq r$ for all $k \leq n-1$, we expect to be able to arbitrarily specify the directions of r vertices. Lemma 3.2.7 and Lemma 3.2.8 show that, under certain conditions, we can specify the directions of two vertices even when we do not have $\varepsilon_k^{\mathcal{S}} \geq 2$ for all k. To prove these results we require Lemma 3.2.5, which shows that when a_k and a_{k+1} are close enough, the excess at k must be at least 2.

Lemma 3.2.4. *Let $\mathcal{I} = (a_1, \dots, a_n)$ be an irreducible signature of Q_n. Let $j \in [n]$ and let $X \in V(Q_n)$ be such that $j \in X$. Then there exists an upright spanning tree T of Q_n with signature $\mathcal{I}$ where $\psi_T(X) = j$.*

Proof. Let $G_{\mathcal{I}}$ be the matching graph with bipartition (A, B) constructed in the proof of Theorem 3.1.3. Match the vertex X with a vertex V in B labelled j. Let $G'_{\mathcal{I}}$ be the matching graph with the two vertices X and V, and all the edges meeting them removed. We show there exists a perfect matching in $G'_{\mathcal{I}}$.

Let $A' = A \setminus \{X\}$ and let $B' = B \setminus \{V\}$. Let $\varnothing \neq Y \subseteq A'$ and let S be the union of the sets of Y. Then

$$|Y| \leq 2^{|S|} - 1.$$

Since $N(Y) \subseteq B'$, we have

$$|N(Y)| = \sum_{i \in S} a'_i,$$

where

$$a'_i = \begin{cases} a_i, & i \neq j; \\ a_i - 1 & i = j. \end{cases}$$

Without loss of generality we may assume $\mathcal{I}$ is ordered. Then

$$\begin{aligned} |N(Y)| = \sum_{i \in S} a'_i &\geq \left(\sum_{i \in S} a_i \right) - 1 \\ &\geq \left(\sum_{i=1}^{|S|} a_i \right) - 1 \\ &\geq 2^{|S|} - 1 && \text{(by irreducibility)} \\ &\geq |Y|. \end{aligned}$$

Thus, the condition $|N(Y)| \geq |Y|$ holds, so an upright spanning tree T of Q_n with signature $\mathcal{I}$ where $\psi_T(X) = j$ does exist. □

Lemma 3.2.5. *Let $n \geq 4$ and let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered irreducible signature of Q_n. Suppose that, for some $i \in \{2, \ldots, n-1\}$, we have $a_{i+1} - a_i \leq 1$. Then $\varepsilon_i^{\mathcal{I}} \geq 2$.*

Proof. Since $\mathcal{I}$ is irreducible we necessarily have $\varepsilon_i^{\mathcal{I}} \geq 1$. Suppose that $\varepsilon_i^{\mathcal{I}} = 1$. Then

$$\sum_{j=1}^{i} a_j = 2^i.$$

Since $\mathcal{I}$ is an irreducible signature of Q_n, we have

$$\sum_{j=1}^{i-1} a_j \geq 2^{i-1}$$

and therefore $a_i \leq 2^{i-1}$. Then

$$\sum_{j=1}^{i} a_j + a_{i+1} \geq 2^{i+1} - 1,$$

which implies $a_{i+1} \geq 2^i - 1$, so $a_{i+1} > a_i + 1$. But $a_{i+1} - a_i \leq 1$ by assumption, so it must be the case that $\varepsilon_i^{\mathcal{I}} \geq 2$. □

Observation 3.2.6. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered irreducible signature of Q_n. If $\varepsilon_2^{\mathcal{I}} \geq 2$ and $a_1 = a_2$, then $\varepsilon_1^{\mathcal{I}} \geq 2$.*

Proof. Suppose that $\varepsilon_2^{\mathcal{I}} \geq 2$. Since $\mathcal{I}$ is an ordered irreducible signature, we have $a_1 + a_2 \geq 5$. Then $a_2 \geq 3$ and therefore if $a_1 = a_2$ then $a_1 \geq 3$. So $\varepsilon_1^{\mathcal{I}} \geq 2$. □

Lemma 3.2.7. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered irreducible signature of Q_n. Let $i, j \in [n]$ be such that $i \neq j$ and let $X_1, X_2 \in V(Q_n)$ be such that $i \in X_1$ and $j \in X_2$. Suppose that*

$$\varepsilon_k^{\mathcal{I}} \geq 2,$$

for all $k \geq \max\{i, j\}$. Then there exists an upright spanning tree T of Q_n with signature $\mathcal{I}$ such that $\psi_T(X_1) = i$ and $\psi_T(X_2) = j$.

Proof. Let $G_{\mathcal{I}}$ be the matching graph with bipartition (A, B) constructed in the proof of Theorem 3.1.3. Match the vertices X_1 and X_2 in A with vertices V and U in B labelled i and j respectively. Let $G'_{\mathcal{I}}$ be the matching graph with the vertices X_1, X_2, U and V, and all incident edges removed. We show there exists a perfect matching in $G'_{\mathcal{I}}$.

Let $A' = A \setminus \{X_1, X_2\}$ and let $B' = B \setminus \{V, U\}$. Let $\varnothing \neq Y \subseteq A'$ and let $S = \bigcup_{W \in Y} W$ be the union of the sets of Y. Then

$$|Y| \leq 2^{|S|} - 1.$$

Since $N(Y) \subseteq B'$, we have

$$|N(Y)| = \sum_{\ell \in S} a'_\ell,$$

where

$$a'_\ell = \begin{cases} a_\ell, & \ell \neq i, j; \\ a_\ell - 1 & \ell = i, j. \end{cases}$$

We distinguish the following cases according to whether $|\{i, j\} \cap S| \leq 1$ or $|\{i, j\} \cap S| = 2$.

a) Suppose $|\{i,j\} \cap S| \leq 1$. Then

$$\begin{aligned} |N(Y)| = \sum_{\ell \in S} a'_\ell &\geq \left(\sum_{\ell \in S} a_\ell\right) - 1 \\ &\geq \left(\sum_{\ell=1}^{|S|} a_\ell\right) - 1 \\ &\geq 2^{|S|} - 1 && \text{(by irreducibility)} \\ &\geq |Y|. \end{aligned}$$

b) Suppose $|\{i,j\} \cap S| = 2$. Then

$$\begin{aligned} |N(Y)| = \sum_{\ell \in S} a'_\ell &= \left(\sum_{\ell \in S} a_\ell\right) - 2 \\ &\geq 2^{|S|} - 2 && \text{(by irreducibility).} \end{aligned}$$

We consider the following cases according to whether or not $X_1, X_2 \subseteq S$.

I) Suppose that $X_1, X_2 \subseteq S$. Then

$$|Y| \leq 2^{|S|} - 1 \underbrace{-1}_{\text{for } X_1} \underbrace{-1}_{\text{for } X_2} = 2^{|S|} - 3.$$

Then we conclude $|N(Y)| > |Y|$.

II) Suppose that either $X_1 \subseteq S$ or $X_2 \subseteq S$. Then

$$|Y| \leq 2^{|S|} - 1 - 1 = 2^{|S|} - 2.$$

Then we conclude $|N(Y)| \geq |Y|$.

III) Suppose that $X_1, X_2 \nsubseteq S$. Let $S = \{\ell_1, \ell_2, \ldots, \ell_k\}$ where $\ell_1 < \ell_2 < \cdots < \ell_k$, and note that $k \geq 2$ because $\{i,j\} \subseteq S$. We distinguish the following cases according to whether $|S| \geq \max\{i,j\}$ or $|S| < \max\{i,j\}$.

i) Suppose that $|S| \geq \max\{i,j\}$. Then

$$\begin{aligned} |N(Y)| = \sum_{\ell \in S} a'_\ell &= \left(\sum_{\ell \in S} a_\ell\right) - 2 \\ &\geq \left(\sum_{\ell=1}^{|S|} a_\ell\right) - 2 \\ &\geq 2^{|S|} - 1 && \left(\text{since } \sum_{\ell=1}^{|S|} a_\ell \geq 2^{|S|} + 1 \text{ for all } |S| \geq \max\{i,j\}\right) \\ &\geq |Y|. \end{aligned}$$

ii) Suppose that $|S| < \max\{i, j\}$. Since $X_1, X_2 \nsubseteq S$, there exists some $r_1 \in X_1$ and $r_2 \in X_2$ such that $r_1, r_2 \notin S$. Since $\max\{i, j\} \in S$ and $|S| < \max\{i, j\}$, we have S is not a consecutive set of directions. In particular, $\ell_k > k$. We show that

$$\sum_{i \in S} a_i \geq 2^k + 1,$$

so that

$$|N(Y)| = \sum_{\ell \in S} a'_\ell = \left(\sum_{\ell \in S} a_\ell \right) - 2 \geq 2^k - 1.$$

Since $\ell_t \geq t$ for all t and $\ell_k > k$ we have $a_{\ell_t} \geq a_t$ and $a_{\ell_k} \geq a_{k+1}$. Then

$$\begin{aligned} \sum_{t \in S} a_t &= \sum_{t=1}^{k} a_{\ell_t} \\ &\geq \sum_{t=1}^{k-1} a_t + a_{k+1} \\ &\geq \sum_{t=1}^{k} a_t \geq 2^k && \text{(by irreducibility).} \end{aligned}$$

If $\sum_{t \in S} a_t > 2^k$ does not hold, then

$$\sum_{t=1}^{k-1} a_t + a_{k+1} = \sum_{t=1}^{k} a_t = 2^k,$$

so

$$a_{k+1} = 2^k - \sum_{t=1}^{k-1} a_t = a_k.$$

But by Lemma 3.2.5, if $a_k = a_{k+1}$ then we must have $\sum_{t=1}^{k} a_t > 2^k$, a contradiction. So $\sum_{t \in S} a_t > 2^{|S|}$ must in fact hold.

Thus, in all cases the condition $|N(Y)| \geq |Y|$ holds, so an upright spanning tree T of Q_n with signature $\mathcal{I}$ where $\psi_T(X_1) = i$ and $\psi_T(X_2) = j$ does exist. □

Lemma 3.2.8. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered irreducible signature of Q_n. For any $i, j \in [n]$, let $X, X \oplus \{j\} \in V(Q_n)$ such that $i \in X$ and $j \in X \oplus \{j\}$. Suppose either $i = \max X$ or $j = \max(X \oplus \{j\})$. Then there exists an upright spanning tree T of Q_n with signature $\mathcal{I}$, where $\psi_T(X) = i$ and $\psi_T(X \oplus \{j\}) = j$.*

Proof. Let $G_{\mathcal{I}}$ be the matching graph with bipartition (A, B) constructed in the proof of Theorem 3.1.3. Match the vertices X and $X \oplus \{j\}$ in A with vertices V and U in B labelled i and j respectively. Let $G'_{\mathcal{I}}$ be the matching graph with the vertices $X, X \oplus \{j\}, V$ and U, and all incident edges removed. We show there exists a perfect matching in $G'_{\mathcal{I}}$.

Let $A' = A \setminus \{X, X \oplus \{j\}\}$ and let $B' = B \setminus \{V, U\}$. Let $\varnothing \neq Y \subseteq A'$ and let $S = \bigcup_{W \in Y} W$ be the union of the sets in Y. Then $\varnothing \neq V \subseteq S$ for all $V \in Y$, so

$$|Y| \leq 2^{|S|} - 1.$$

Since $N(Y) \subseteq B'$, we have

$$|N(Y)| = \sum_{\ell \in S} a'_\ell,$$

where

$$a'_\ell = \begin{cases} a_\ell, & \ell \neq i, j; \\ a_\ell - 1, & \ell = i, j. \end{cases}$$

We distinguish the following cases according to whether $|\{i, j\} \cap S| \leq 1$ or $|\{i, j\} \cap S| = 2$.

a) Suppose $|\{i, j\} \cap S| \leq 1$. Then

$$\begin{aligned} |N(Y)| = \sum_{\ell \in S} a'_\ell &\geq \left(\sum_{\ell \in S} a_\ell \right) - 1 \\ &\geq \left(\sum_{\ell \in 1}^{|S|} a_\ell \right) - 1 \\ &\geq 2^{|S|} - 1 && \text{(by irreducibility)} \\ &\geq |Y|. \end{aligned}$$

b) Suppose $|\{i, j\} \cap S| = 2$. Then

$$|N(Y)| = \sum_{\ell \in S} a'_\ell = \left(\sum_{\ell \in S} a_\ell \right) - 2 \geq 2^{|S|} - 2.$$

We consider the following cases according to whether or not $X \subseteq S$.

I) If $X \subseteq S$, then $X \oplus \{j\} \subseteq S$ and therefore

$$|Y| \leq 2^{|S|} - 1 \underbrace{-1}_{\text{for } X} \underbrace{-1}_{\text{for } X \oplus \{j\}} = 2^{|S|} - 3.$$

Then we conclude $|N(Y)| > |Y|$.

II) If $X \nsubseteq S$, then also $X \oplus \{j\} \nsubseteq S$. Let $S = \{\ell_1, \ell_2, \ldots, \ell_k\}$ where $\ell_1 < \ell_2 < \cdots < \ell_k$, and note that $2 \leq k \leq n - 1$ because $\{i, j\} \subseteq S$. Since $X \nsubseteq S$, there exists some $r \in X$ such that $r \notin S$. Then

$$r \leq \max X \leq \max\{i, j\} \in S$$

which implies S is not a consecutive set of directions. In particular, $\ell_k > k$. We show that

$$\sum_{\ell \in S} a_\ell \geq 2^k + 1,$$

so that

$$|N(Y)| = \sum_{\ell \in S} a'_\ell = \sum_{\ell \in S} a_\ell - 2 \geq 2^k - 1.$$

Since $\ell_t \geq t$ for all ℓ and $\ell_k > k$ we have $a_{\ell_t} \geq a_t$ and $a_{\ell_k} \geq a_{k+1}$. Then

$$\begin{aligned}\sum_{t\in S} a_t = \sum_{t=1}^{k} a_{\ell_t} &\geq \sum_{t=1}^{k-1} a_t + a_{k+1} \\ &\geq \sum_{t=1}^{k} a_t \\ &\geq 2^k.\end{aligned}$$

If $\sum_{t\in S} a_t \geq 2^k + 1$ does not hold then

$$\sum_{t=1}^{k-1} a_t + a_{k+1} = \sum_{t=1}^{k} a_j = 2^k,$$

so

$$a_{k+1} = 2^k - \sum_{t=1}^{k-1} a_t = a_k.$$

But by Lemma 3.2.5, if $a_k = a_{k+1}$ then we must have $\sum_{t=1}^{k} a_t > 2^k$, a contradiction.

Thus, in all cases the condition $|N(Y)| \geq |Y|$ holds, so an upright spanning tree T of Q_n with signature $\mathcal{I}$ where $\psi_T(X) = i$ and $\psi_T(X \oplus \{j\}) = j$ does exist. □

Observation 3.2.9. *Note that if $i = \max X$ and $j = \max X \oplus \{j\}$, then we can have an upright spanning tree with signature $\mathcal{I}$ where $\psi_T(X) = i$ and $\psi_T(X \oplus \{j\}) = j$.*

3.3 Summary map

In this chapter we obtained necessary and sufficient conditions for an n-tuple to be a signature. Then we classified signatures into reducible and irreducible signatures. In Chapter 6 we study the reducible signatures in detail and prove that the edge slide graph of a strictly reducible signature of Q_n is disconnected. For irreducible signatures, we showed that under certain conditions we can specify the directions of up to two vertices. Hall's Theorem plays an important role in characterising the signatures, and also in showing the existence of certain trees which are used in rearranging the labels of upright spanning trees of Q_n with irreducible signatures. We study irreducible signatures and use these results in Chapters 7–10.

Chapter 4

Local moves on upright spanning trees of Q_n

4.1 Introduction

In this chapter we define a local move, which is an operation on upright spanning trees of Q_n. We prove the existence of two different local moves, and then define the local move graph of Q_n. Then we define the local move graph of a signature, and finally we prove that the local move graph of the irreducible signature (2, 2, 3) of Q_3 is connected.

Since every spanning tree is connected to at least one upright spanning tree directly or by a series of edge slides (Tuffley [11]), and upright spanning trees are relatively easy to work with, we try to understand the components of the edge slide graph of Q_n by defining and using "local" moves on the upright spanning trees of Q_n to determine which upright spanning trees are connected by edge slides.

The idea of a local move is to carry out a change that affects a small part of a configuration, while leaving the rest of that configuration unchanged. So the effect is "local" rather than "global". Local moves are used with this same meaning in knot theory to talk about, for example, the Reidemeister moves. However, our local moves are not the same as those used in knot theory.

4.2 Local moves

We define a local move on upright spanning trees of Q_n as follows.

Definition 4.2.1. A **local move** is a series of edge slides that transforms one upright spanning tree T of Q_n into another spanning tree T', where all the vertices V such that $\psi_T(V) \neq \psi_{T'}(V)$ lie in a single 2-dimensional face F of Q_n. Moreover, we require that we can tell whether the local move is possible simply by looking at the 2-dimensional face F of Q_n.

We can construct a graph where its vertices are the upright spanning trees of Q_n and, instead of having edge slides connecting these vertices, we have local moves connecting the vertices. We call such a graph a local move graph of Q_n. So, we try to use the idea of local moves to show upright spanning trees of Q_n that have the same signature $(a_1, a_2, \ldots, a_n)$, are connected by local moves and belong to the same component of the local move graph of Q_n, and therefore belong to the same component of the edge slide graph of Q_n.

Before presenting the local moves, we need to determine which edges are i-slidable. The following lemma identifies an edge that can be slid in direction i.

Lemma 4.2.2. *Let T be a spanning tree of Q_n rooted at $\varnothing$, and let $e = \{X, X \oplus \{j\}\} \in T$. Suppose the path in T from $X \oplus \{i, j\}$ to the root $\varnothing$ passes through $X \oplus \{j\}$ and X, and the path from $X \oplus \{i\}$ to the root does not pass through $X \oplus \{j\}$. Then the edge e of T is i-slidable.*

Proof. Referring to Figure 4.1(i), since the path from $X \oplus \{i, j\}$ to the root $\varnothing$ passes through $X \oplus \{j\}$ and X, and the path from $X \oplus \{i\}$ to the root $\varnothing$ does not pass through $X \oplus \{j\}$, the two paths must meet at some vertex Y on the path from X to the root, which does not involve e. The cycle created by adding $\sigma_i(e)$ to T contains both e and $\sigma_i(e)$ because both paths meet at Y, as seen in Figure 4.1(ii). Deleting e from the cycle does not disconnect the graph, (as we can see in Figure 4.1(iii)). Then we conclude that $T + \sigma_i(e) - e$ is still a spanning tree, and therefore e is i-slidable. The path from $X \oplus \{j\}$ to the root now passes through $X \oplus \{i, j\}$ and $X \oplus \{i\}$, as shown in Figure 4.1(iii). □

The following lemma shows the effect of an edge slide on the orientations of the edges of a spanning tree, for a proof see Tuffley [11, Lemma 3].

Lemma 4.2.3. *(Tuffley [11, Lemma 3]) Let e be an edge of a spanning tree T that is slidable in direction i and let C be the cycle created by adding $\sigma_i(e)$ to T. Suppose T_- and T_+ are the components of $T - e$ such that T_- contains the root, and T_+ does not. Then sliding the edge e in direction i reverses the orientation of only those edges belong to $C \cap T_+$.*

4.2.1 The V-move

The following lemma concerns a particular type of local move, that we call a V**-move**, which we can apply inside an upright spanning tree T of Q_n to swap certain labels of T, to get a different upright spanning tree of Q_n. The effect of the move is shown at Figure 4.2.

Lemma 4.2.4 (V-move)**.** *Let T be an upright spanning tree of Q_n. Suppose there is a 2-dimensional face F of Q_n which is labelled by T as in Figure 4.2(a), where $i \neq k \neq \ell \neq i$ and $j \neq k \neq \ell \neq j$. Note that we allow the possibility that i and j may be equal. Then there is a sequence of four edge slides that transforms T into the upright spanning tree T' in which F is labelled as in Figure 4.2(b), and all other labels of T' are the same as T.*

Proof. The case $i \neq j$ is shown in Figure 4.3, and the case $i = j$ is shown in Figure 4.4. We consider first the case $i \neq j$. Given an upright spanning tree T of Q_n, suppose there is a face in Q_n with different labels i, j, k and ℓ, as in Figure 4.3(i) on page 38. The path from $X \cup \{i, j, k\}$ to the root goes through $X \cup \{i, j\}$, and $X \cup \{i\}$, and the path from $X \cup \{i, k\}$ to the root does not go through $X \cup \{i, j\}$ because T is upright so the path must decrease cardinality. Therefore by Lemma 4.2.2, the edge $(X \cup \{i, j\}, X \cup \{i\})$ can be slid up in direction k. Sliding the edge in direction k swaps the two labels j and k, as shown in Figure 4.3(ii).

Then by Lemma 4.2.3 the only upward edge is in direction k, namely $(X \cup \{i, j\}, X \cup \{i, j, k\})$. The path from $X \cup \{i, j, k, \ell\}$ to the root goes through $X \cup \{i, j, k\}$ and $X \cup \{i, k\}$. The two vertices $X \cup \{i, j, k\}$ and $X \cup \{i, k, \ell\}$ have the same cardinality, so we need to go down in direction j and then up in direction ℓ to go from $X \cup \{i, j, k\}$ to $X \cup \{i, k, \ell\}$. Since the only

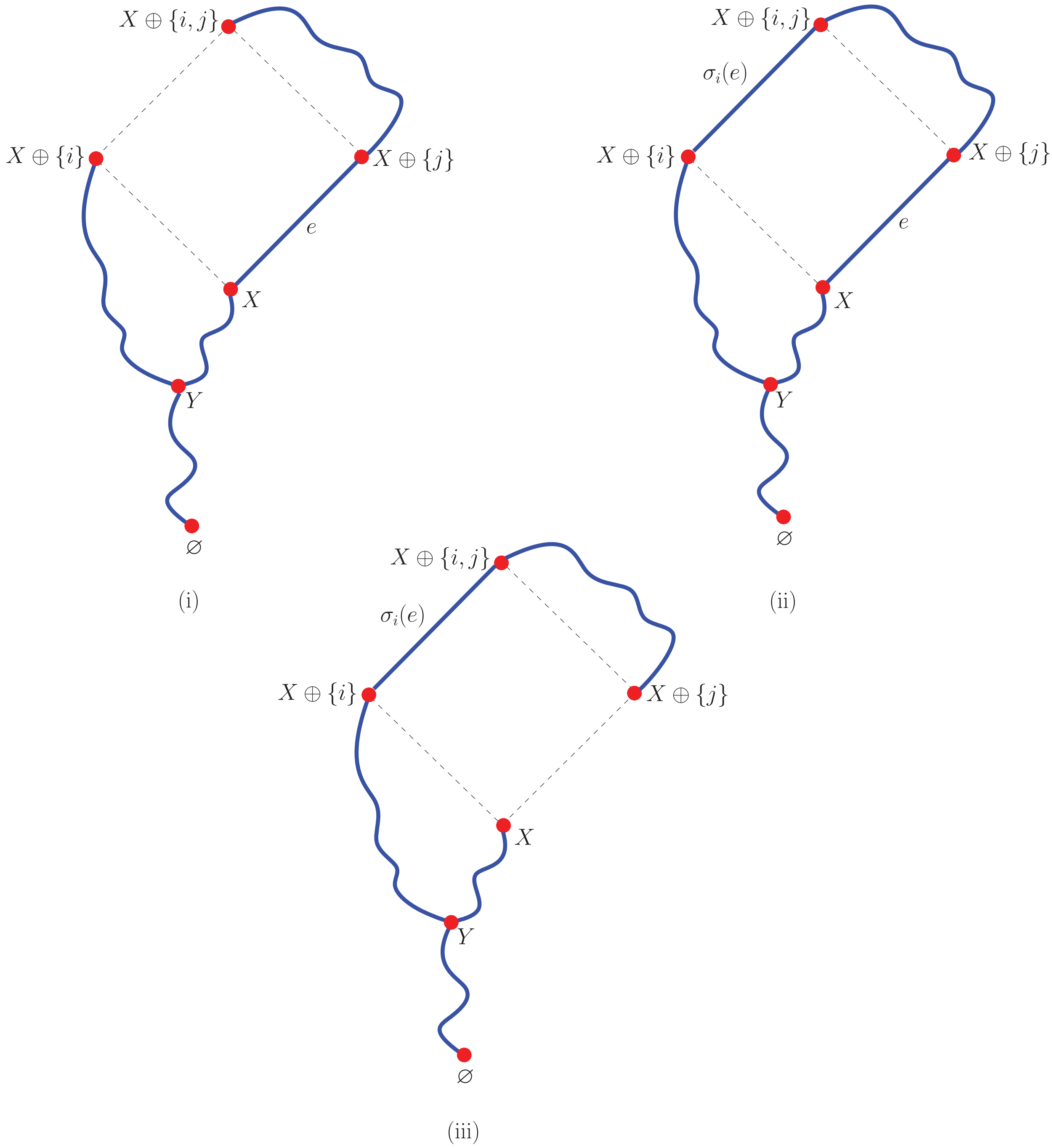


Figure 4.1: Diagram for the proof of Lemma 4.2.2 on page 35. The process of sliding an edge in direction i. (i) Shows two paths in a spanning tree T. (ii) Adding $\sigma_i(e)$ to T creates a cycle in $T + \sigma_i(e)$, and (iii) deleting e from this cycle gives the spanning tree $T + \sigma_i(e) - e$.

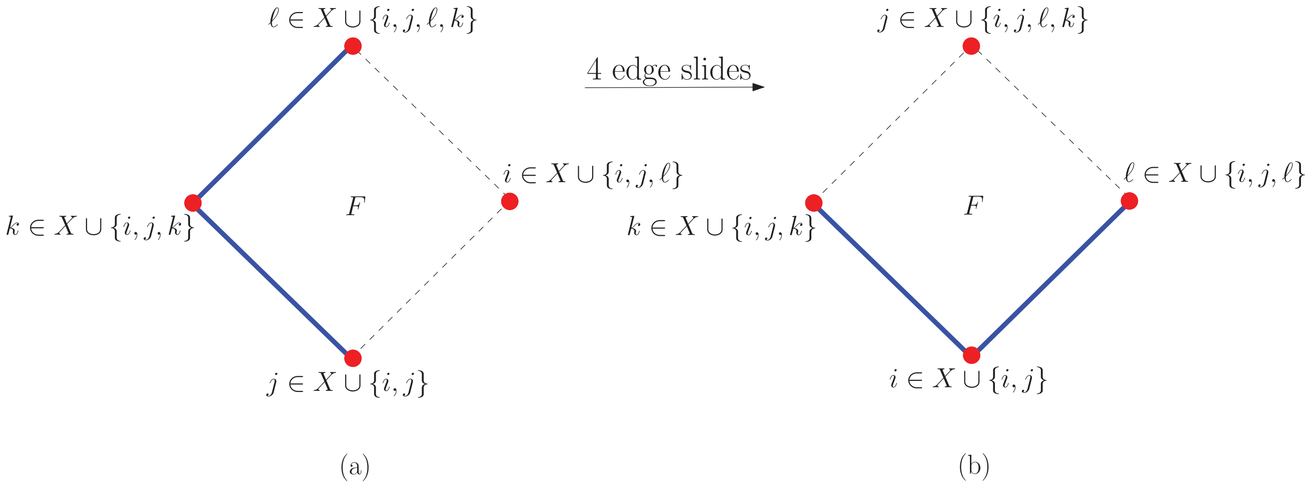


Figure 4.2: The V-move of Lemma 4.2.4. (a) A face of Q_n with different labels i, j, k and ℓ of an upright spanning tree. (b) The face of Q_n with the transformed labels gives another upright spanning tree.

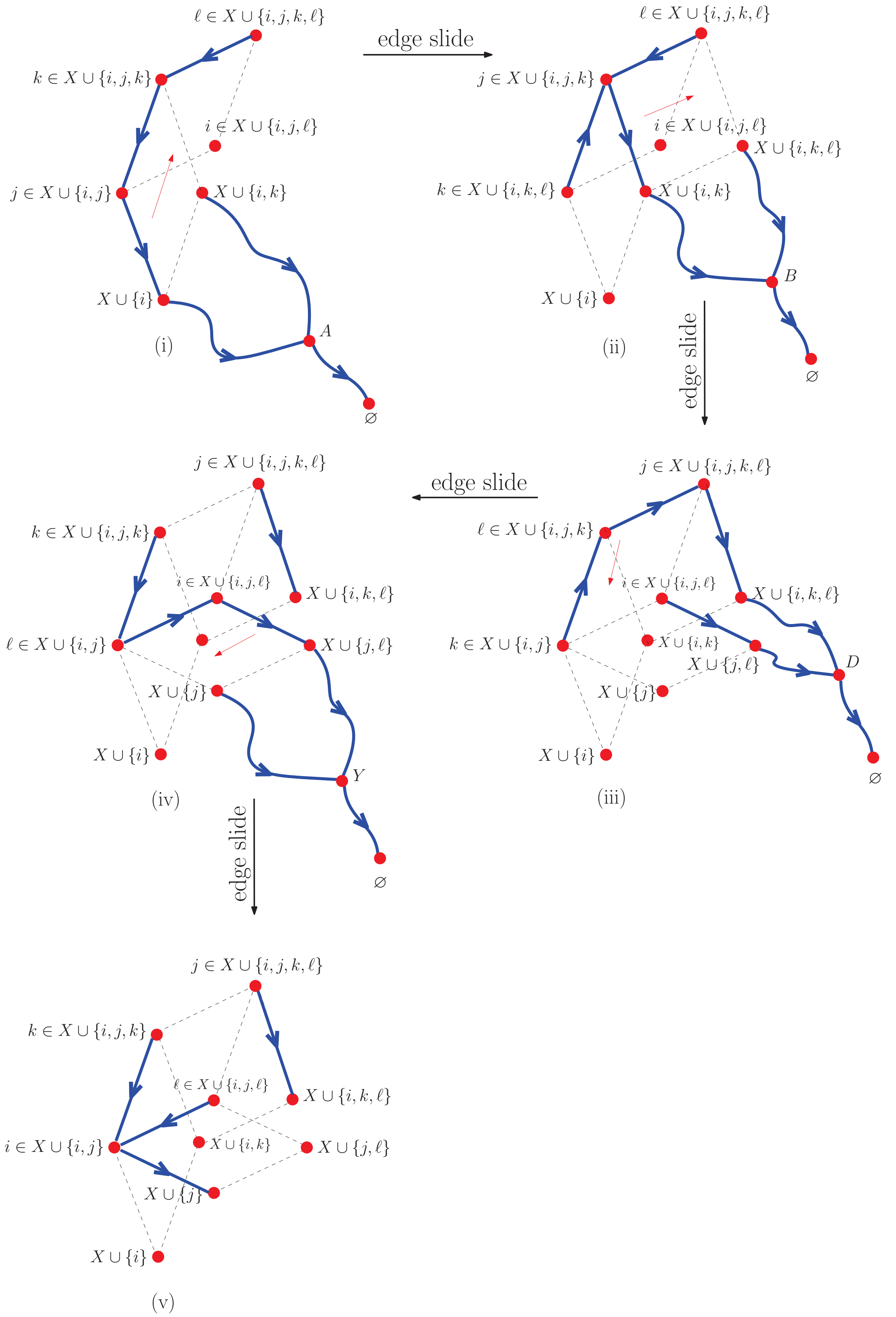


Figure 4.3: Diagram for the proof of Lemma 4.2.4 on page 35 where $i \neq j$. The process of the V-move. The bold edges belong to the spanning trees of Q_n. A series of edge slides that shift the labels of a face of Q_n to transform one upright spanning tree of Q_n to another.

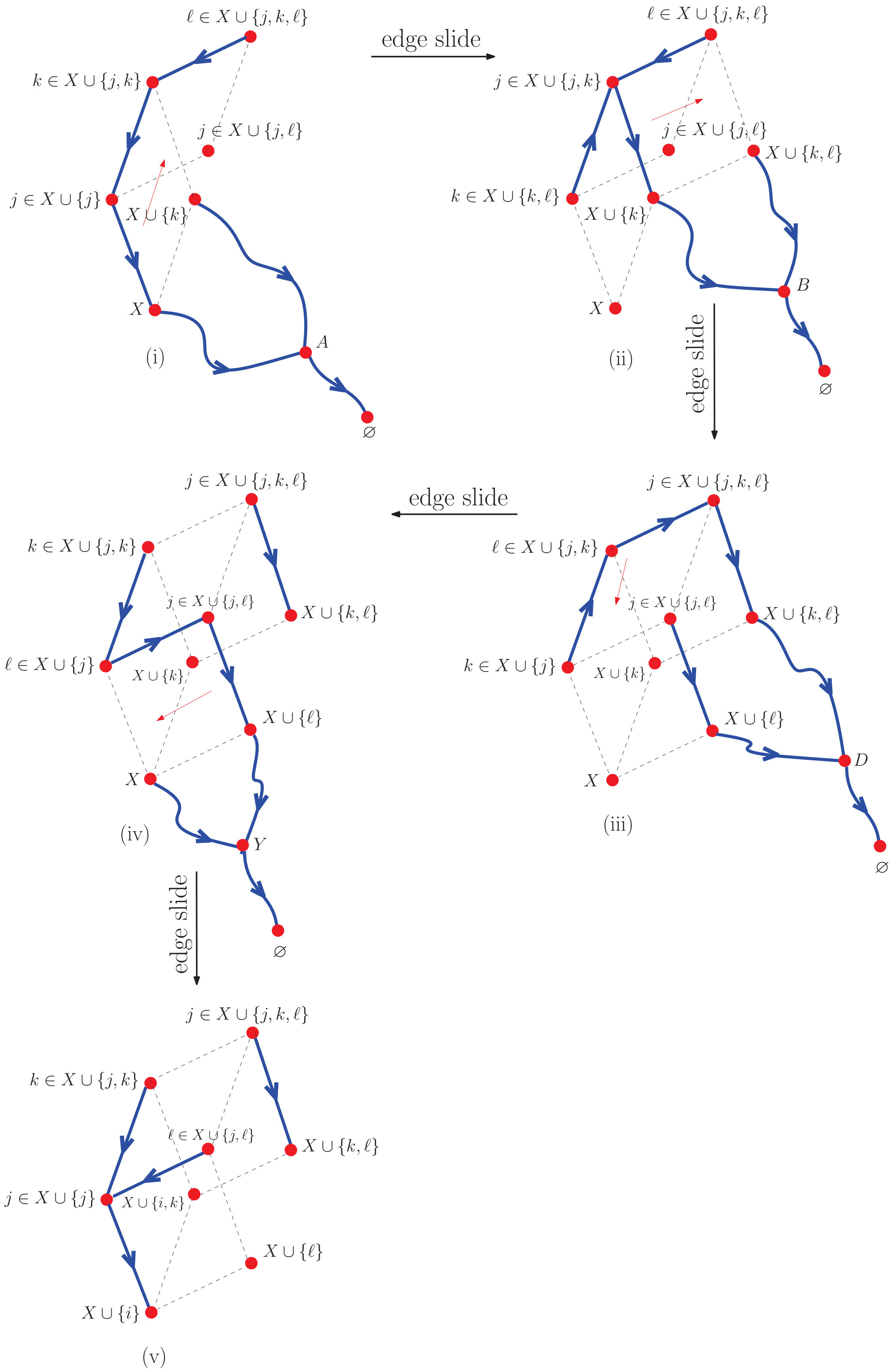


Figure 4.4: Diagram for the proof of Lemma 4.2.4 on page 35 where $i = j$. The process of the V-move in the case $i = j$. The bold edges belong to the spanning trees of Q_n. A series of edge slides that shift the labels of a face of Q_n to transform one upright spanning tree of Q_n to another.

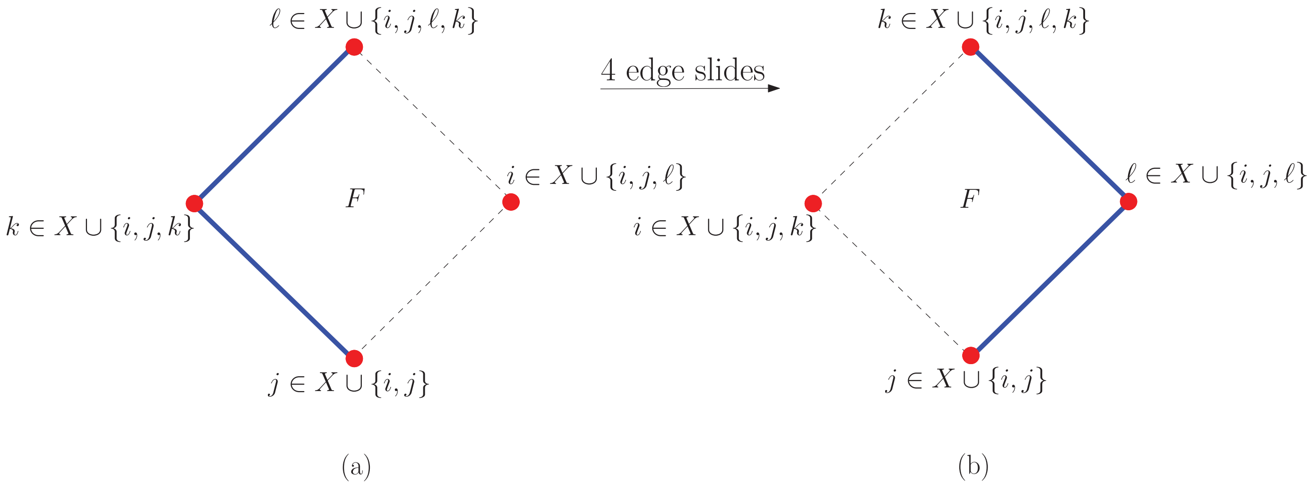


Figure 4.5: The path move of Lemma 4.2.5. (a) A face of Q_n with different labels i, j, k and ℓ of an upright spanning tree. (b) The face of Q_n with the transformed labels gives another upright spanning tree.

upward edge is in direction k, we cannot go up by direction ℓ. So the path from $X \cup \{i, k, \ell\}$ to the root does not go through $X \cup \{i, j, k\}$. Therefore, the edge $(X \cup \{i, j, k\}, X \cup \{i, k\})$ can be slid up in direction ℓ (by Lemma 4.2.2). Then the labels j and ℓ are swapped, see Figure 4.3(iii).

Then by Lemma 4.2.3 the only upward edges are now in directions k and ℓ and all other edges are downward edges, as presented in Figure 4.3(iii). The path from $X \cup \{i, j, \ell\}$ to the root goes down in direction i and passes through $X \cup \{j, \ell\}$. The only upward edge in direction k is at $X \cup \{i, j\}$, so for the path from $X \cup \{j, \ell\}$ to the root to pass through $X \cup \{i, j, k\}$ it would have to go down in direction ℓ and up in direction i. Since the only upward edges are in directions k and ℓ and $i \neq k, \ell$ this is impossible, so the path from $X \cup \{i, j, \ell\}$ to the root does not pass through $X \cup \{i, j, k\}$. Therefore the edge $(X \cup \{i, j, k\}, X \cup \{i, j, k, \ell\})$ can be slid down in direction k (by Lemma 4.2.2). Then the labels k and ℓ are swapped, as shown in Figure 4.3(iv).

Now, we want to reverse the edge $(X \cup \{i, j, \ell\}, X \cup \{j, \ell\})$, so we get an upright spanning tree. Since the only upward edge is in direction ℓ and and since i is not equal to ℓ (by hypothesis), the path from $X \cup \{j\}$ to the root cannot go up in direction i to get to $X \cup \{i, j, \ell\}$ as shown in Figure 4.3(iv). So the path from $X \cup \{j\}$ to the root does not go through $X \cup \{i, j, \ell\}$, and therefore the edge $(X \cup \{i, j, \ell\}, X \cup \{j, \ell\})$ can be slid down in direction ℓ (by Lemma 4.2.2). Then the labels i and ℓ are swapped as in Figure 4.2(v), completing the proof in the case $i \neq j$.

In the case $i = j$, we apply the same process but the edge slides now take place in 3-dimensional face of Q_n instead of a 4-dimensional face. As a result the edge involved in the slides are different, but the underlying argument is otherwise unchanged. See Figure 4.4 on page 39 for the process of the V-move in the case $i = j$.

Therefore, an upright spanning tree of Q_n with the face labels as in Figure 4.2(a) may be transformed using the V-move to another upright tree of Q_n with the face labels as in Figure 4.2(b). $\square$

4.2.2 The path move

The following move that we call a **path move** provides a local move that is different from the one in Lemma 4.2.4. In order to prove Lemma 4.2.5, we start with the same labels and conditions on a face of Q_n as in Lemma 4.2.4 (Figure 4.2(a)), and then we apply different edge slides. The effect of the move is shown at Figure 4.5. The reason we called this move a 'path' move is because the path in direction ℓ followed by k is pushed across the face F to become the path in direction k followed by ℓ.

Lemma 4.2.5 (Path move)**.** *Let T be an upright spanning tree of Q_n. Suppose there is a 2-dimensional face F of Q_n which is labelled by T as in Figure 4.5(a) on page 40, where $i \neq k \neq \ell \neq i$ and $j \neq k \neq \ell \neq j$. Note that we allow the possibility that i and j may be equal. Then there is a sequence of four edge slides that transforms T into the upright spanning tree T' in which F is labelled as in Figure 4.5(b), and all other labels of T' are the same as T.*

Proof. Given an upright spanning tree of Q_n, suppose there is a face of Q_n with different labels i, j, k and ℓ, as shown in Figure 4.6(i) on page 42. The path from $X \cup \{i, j, k, \ell\}$ to the root goes through $X \cup \{i, j, k\}$ and $X \cup \{i, j\}$, and the path from $X \cup \{i, j, \ell\}$ to the root does not go through $X \cup \{i, j, k\}$ because T is upright so path must decrease cardinality. Therefore, the edge $(X \cup \{i, j, k\}, X \cup \{i, j\})$ can be slid up in direction ℓ (by Lemma 4.2.2), and then the labels k and ℓ are swapped, as shown in Figure 4.6(ii).

The path from $X \cup \{i, j, k, \ell\}$ to the root goes through $X \cup \{i, j, \ell\}$ and $X \cup \{j, \ell\}$. Since the only upward edge is in direction ℓ (by Lemma 4.2.3), namely $(X \cup \{i, j, k\}, X \cup \{i, j, k, \ell\})$, the path from $X \cup \{j, k, \ell\}$ to the root does not go through $X \cup \{i, j, \ell\}$. Then by Lemma 4.2.2 the edge $(X \cup \{i, j, \ell\}, X \cup \{j, \ell\})$ can be slid up in direction k, and then the labels i and k are swapped, see Figure 4.6(iii).

The only upward edge in direction ℓ is at $X \cup \{i, j, k\}$, so for the path from $X \cup \{j, k\}$ to the root to pass through $X \cup \{i, j, k, \ell\}$ it would have to go up in direction i. Since the only upward edges are in directions k and ℓ (by Lemma 4.2.3), and $i \neq k, \ell$, this is impossible so the path from $X \cup \{j, k\}$ to the root does not go through $X \cup \{i, j, k, \ell\}$, and therefore the edge $(X \cup \{i, j, k, \ell\}, X \cup \{j, k, \ell\})$ can be slid down in direction ℓ (by Lemma 4.2.2). Then the labels i and ℓ are swapped, as presented in Figure 4.6(iv).

Now, we want to reverse the edge $(X \cup \{i, j, k, \ell\}, X \cup \{i, j, k\})$, so we get an upright spanning tree. The only upward edge in directions k is at $X \cup \{i, j, \ell\}$, so for the path from $X \cup \{i, j\}$ to the root to pass through $X \cup \{i, j, k, \ell\}$ it would have to go up in direction ℓ. Since the only upward edge is in direction k and $\ell \neq k$ (by hypothesis), this is impossible so the path from $X \cup \{i, j\}$ to the root does not pass through $X \cup \{i, j, k, \ell\}$. Therefore the edge $(X \cup \{i, j, k, \ell\}, X \cup \{i, j, k\})$ can be slid down in direction k (by Lemma 4.2.2). Then the labels k and ℓ are swapped, as shown in Figure 4.6(v).

Therefore, an upright spanning tree of Q_n with the face labels as in Figure 4.5(a) may be transformed using the path move to another upright tree of Q_n with the face labels as in Figure 4.5(b). □

The following lemma is an observation on the path and the V-move. It shows that if direction i is chosen at a vertex of level α with $\alpha \geq 3$, but not at any vertex of level $\alpha - 1$, then using either the path or the V-move, the i may be moved one level down.

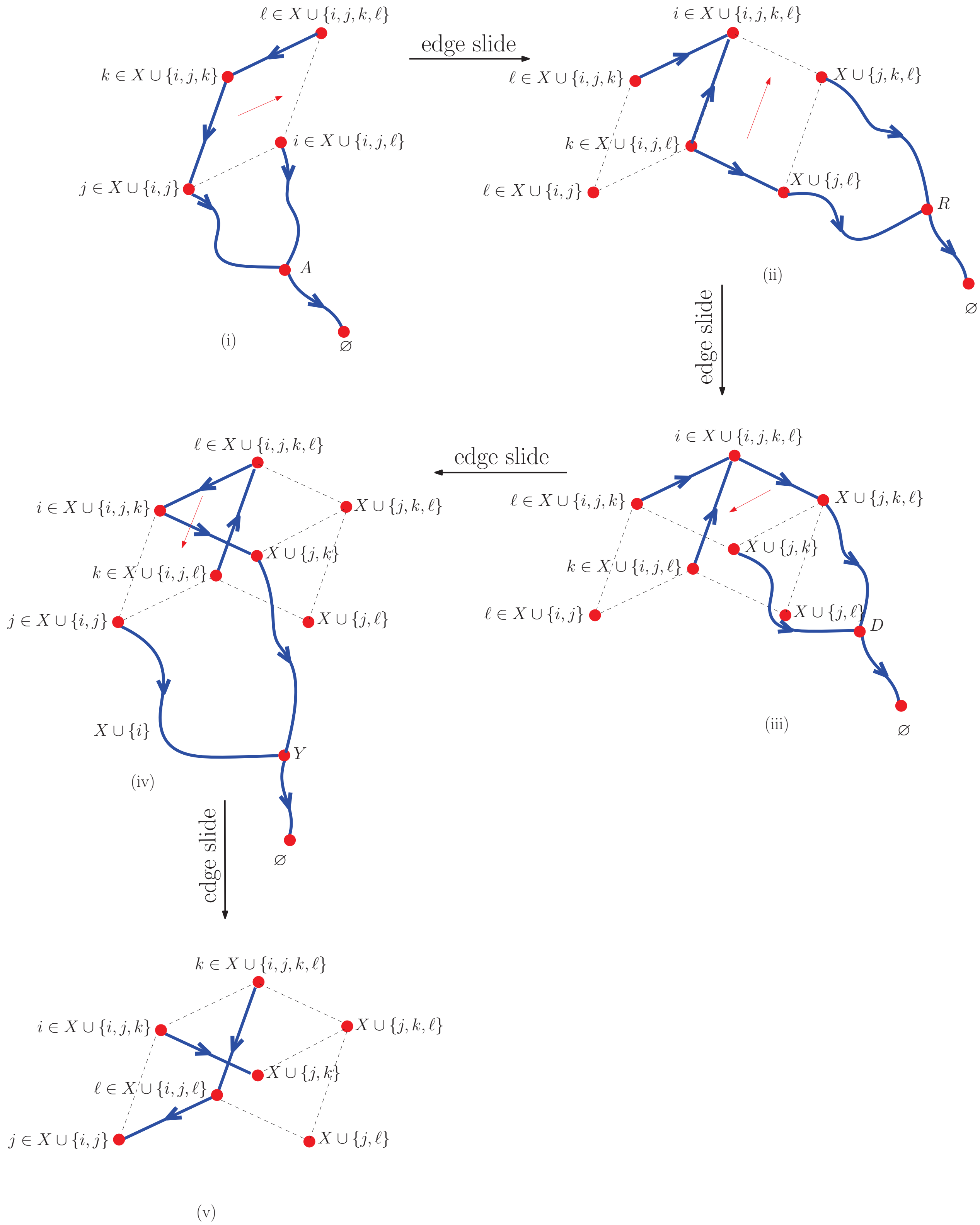


Figure 4.6: Diagram for the proof of Lemma 4.2.5 on page 41. The process of the path move. The bold edges belong to the spanning trees of Q_n. A sequence of four edge slides that transformed one upright tree to another.

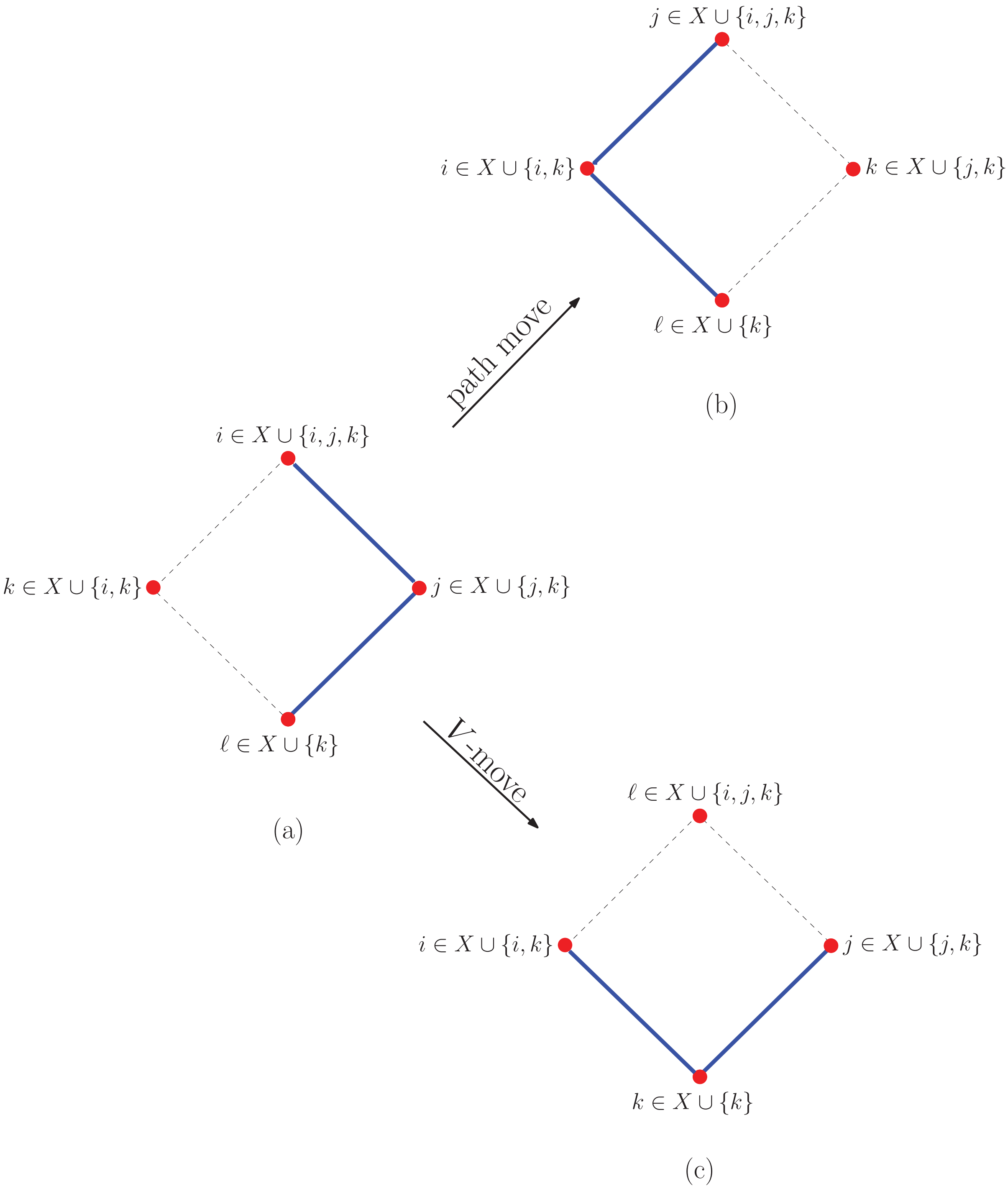


Figure 4.7: Diagram for the proof of Lemma 4.2.6. (a) Labels i, j, k and ℓ from the path and V-move. (b) After applying the path move label k moves to $X \cup \{j,k\}$, label j moves to $X \cup \{i,j,k\}$ and label i moves to $X \cup \{i,k\}$. (c) After applying the V-move on (a) label ℓ moves to $X \cup \{i,j,k\}$, label i moves to $X \cup \{i,k\}$ and label k moves to $X \cup \{k\}$.

Lemma 4.2.6. *Let T be an upright spanning tree of Q_n. Let Y be a vertex Y of Q_n of level α such that $\alpha \geq 3$. Suppose that $\psi_T(Y) = i$ and $\psi_T(Z) \neq i$ for all Z such that $|Z| = \alpha - 1$. Let $j = \psi_T(Y \setminus \{i\})$. Then there is a local move that moves the i at Y to $Y \setminus \{j\}$.*

Proof. Since i is not chosen at any vertex of level $\alpha - 1$, we have $\psi_T(Y \setminus \{j\}) = k$ for some $k \neq i$. Let $X \subseteq [n] \setminus \{i, j, k\}$ be such that $Y = X \cup \{i, j, k\}$. Let $\psi_T(X \cup \{k\}) = \ell$, where ℓ could be equal to k. As can be seen in Figure 4.7(a), the labels i, j, k and ℓ are from the path move and the V-move. Applying either of these moves takes the label i to $X \cup \{i, k\}$, as can be seen in Figures 4.7(b) and (c). Therefore the label i can be moved from level α to level $\alpha - 1$ by a local move. □

4.3 The local move graph of Q_n

In this section we define the local move graph of Q_n using the V-move and the path move that we introduced in this chapter. Then we define the local move graph of a signature of Q_n.

Definition 4.3.1. The **local move graph** of Q_n is the graph $\mathcal{L}(Q_n)$ whose vertices are the upright spanning trees, with an edge between two vertices (upright spanning trees) if they are related by either the V-move or the path move.

Definition 4.3.2. Let $\mathcal{S} = (a_1, \ldots, a_n)$ be a signature of a spanning tree of Q_n. Then the **local move graph of signature** $\mathcal{S}$, denoted $\mathcal{L}(\mathcal{S})$, is defined to be the subgraph of $\mathcal{L}(Q_n)$ produced by the trees with signature $(a_1, \ldots, a_n)$. Note that $\mathcal{L}(a_1, \ldots, a_n)$ is always a union of components of $\mathcal{L}(Q_n)$.

4.3.1 The local move graph of the irreducible signature (2, 2, 3)

In this section we count the number of upright spanning trees of Q_3 with signature (2, 2, 3), and then we show that the local move graph of signature (2, 2, 3) is connected. Before working with signature (2, 2, 3) we establish the following lemma.

Lemma 4.3.3. *Let $\mathcal{U}_i^{(1)} = (0, \ldots, \underbrace{1}_{i^{th}}, 0, \ldots, 0)$, where $i \in [n]$. Then the number of upright spanning forests of Q_n with signature $\mathcal{U}_i^{(1)}$ is 2^{n-1}.*

Proof. An upright spanning forest of Q_n with signature $\mathcal{U}_i^{(1)}$ consists of $2^n - 2$ trivial trees and a rooted upright tree with a single edge in direction i. Since i can be taken from 2^{n-1} vertices of Q_n, we have 2^{n-1} upright rooted spanning forests of Q_n with signature $\mathcal{U}_i^{(1)}$. □

Observation 4.3.4. *There are $n2^{n-1}$ upright spanning forests of Q_n with signature*

$$\mathcal{U}_1^{(1)} = (1, 0, \ldots, 0)$$

up to permutation.

Lemma 4.3.5. *There are 4 upright spanning trees of Q_3 with signature (2, 2, 3).*

Proof. An upright spanning tree of Q_3 with signature (2, 2, 3) consists of an upright spanning forest of Q_2 in $\mathcal{F}_3^{3+}$ with signature (1, 0) or signature (0, 1), and an upright spanning tree of Q_2 in $\mathcal{F}_3^{3-}$ with signature (1, 2) or signature (2, 1) respectively. The positions of the 3-edges are forced by the choice of edge in $\mathcal{F}_3^{3+}$. There are two upright spanning forests of Q_2 with signature (1, 0) (by Lemma 4.3.3) and one upright spanning tree of Q_2 with signature (1, 2) up to permutation. Thus, altogether there are a total of 2(1)+2(1)= 4 upright spanning trees of Q_3 with signature (2, 2, 3). $\square$

Note that the above lemma can also be proved by observing that the monomial $q_1^2q_2^2q_3^3$ appears with coefficient 4 in the generating function

$$q_1q_2q_3(q_1+q_2)(q_1+q_3)(q_2+q_3)(q_1+q_2+q_3)$$

for sections of $\mathcal{P}^3_{\geq 1}$.

Lemma 4.3.6. *The edge slide graph of the irreducible signature (2, 2, 3) is connected.*

Proof. Since every spanning tree of Q_3 is connected to at least one upright spanning tree by a series of edge slides (Tuffley [11]), it suffices to show that the local move graph of the signature (2, 3, 3) is connected. We label the trees as discussed in Section 2.2.5.2, and show that any upright spanning tree of Q_3 with the irreducible signature (2, 2, 3) can be transformed into the upright spanning tree 1233 (Figure 4.8(iv) on page 46) by a local move.

Consider the tree 3213, shown in Figure 4.8(i). Applying the V-move on the face $\mathcal{F}_3^{1+}$(green edges), the labels 1 and 3 are swapped. Therefore the tree 3213 is transformed to the tree 1233 (Figure 4.8(iv)). Note that the tree 3213 can also be transformed to the tree 2133 (Figure 4.8(iii)) by applying the path move on the same $\mathcal{F}_3^{1+}$.

Consider the tree 3132, shown in Figure 4.8(ii). Applying the path move on the face $\mathcal{F}_3^{2+}$ (green edges), label 2 moves to $\{1,2\}$, label 1 moves to $\{1,2,3\}$ and label 3 moves to $\{2,3\}$. Therefore the tree 3132 is transformed to the tree 1233 (Figure 4.8(iv)). Note that the tree 3132 can also be transformed to the tree 2133 (Figure 4.8(iii)) by applying the V-move on $\mathcal{F}_3^{2+}$.

Consider the tree 2133, shown in Figure 4.8(iii). Applying the V-move on the face $\mathcal{F}_3^{1+}$ (green edges), the labels 1 and 2 are swapped. Therefore the tree 2133 is transformed to the tree 1233 (Figure 4.8(iv)).

By the definition of the local move graph $\mathcal{L}(2,2,3)$, there is a local move from each upright spanning tree 3213, 3132 and 2133 to the upright spanning tree 1233, hence all four upright spanning trees are in the same connected component of $\mathcal{L}(2,2,3)$. Therefore the local move graph $\mathcal{L}(2,2,3)$ is connected.

Since every spanning tree of Q_3 is connected to an upright spanning tree, spanning trees of Q_3 with signature (2, 2, 3) lie in a single component of the edge slide graph of Q_3 and therefore $\mathcal{E}(2,2,3)$ is connected. $\square$

Figure 4.9 on page 47 shows the local move graph of the irreducible signature (2, 2, 3).

4.4 Summary map

As stated in this chapter, upright spanning trees with the same signature which belong to the same connected component of the local move graph of Q_n, also belong to the same connected component of the edge slide graph of Q_n. In the following chapters we use local moves, the V-move and the path move, to connect upright spanning trees.

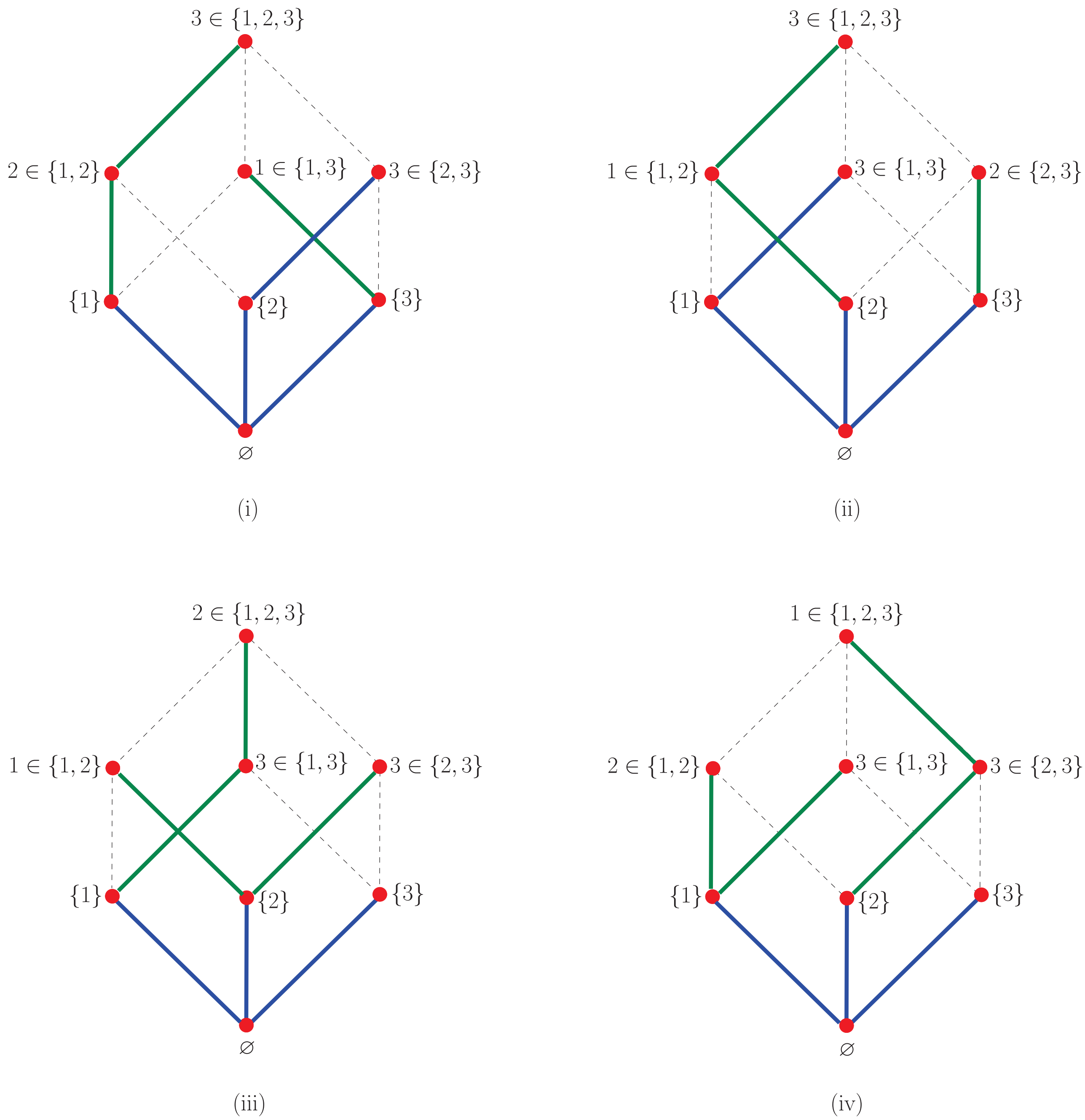


Figure 4.8: The four upright spanning trees (bold edges) of Q_3 with the irreducible signature (2, 2, 3), where the possible V-moves and path moves (green) are shown. The labels are (i) 3213, (ii) 3132, (iii) 2133, (iv) 1233.

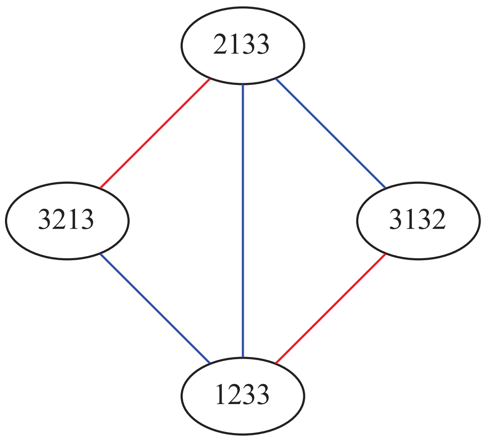


Figure 4.9: The local move graph of signature (2, 2, 3), with the V-move (blue) and the path move (red).

Chapter 5

Splitting spanning trees and signatures of Q_n

5.1 Introduction

In this chapter we start with the definition of splitting the Q_n graph and then provide some related results. Then we study the splitting of the upright spanning trees of Q_4 with signature (2, 3, 3, 7), which we use to prove that the local move graph $\mathcal{L}(2,3,3,7)$ is connected. This result implies the edge slide graph $\mathcal{E}(2,3,3,7)$ is connected. The proof of the connectivity of the edge slide graph of signature (2, 3, 3, 7) forms a model for the way in which we study the connectivity of the edge slide graph of an irreducible signature $\mathcal{I} = (a_1, \ldots, a_n)$ of Q_n in general.

If T is an upright spanning tree of Q_n with signature $\mathcal{S}$, then the signature of $T \cap \mathcal{F}_n^{i-}$ is called a **splitting signature** of $\mathcal{S}$. One of the steps we use to study the connectivity of the edge slide graph $\mathcal{E}(\mathcal{I})$ of Q_n is to split the signature in such a way that the resulting splitting signature in $\mathcal{F}_n^{i-}$ satisfies certain conditions. To find such a splitting, we introduce signature moves in Section 5.6.

5.2 Splitting the Q_n graph

Splitting the Q_n graph is defined as follows.

Definition 5.2.1. Deleting all the edges in direction i divides Q_n into two subgraphs $\mathcal{F}_n^{i+}$ and $\mathcal{F}_n^{i-}$, where $\mathcal{F}_n^{i+}$ and $\mathcal{F}_n^{i-}$ are the upper and lower faces of Q_n consisting of the vertices that do and do not contain i respectively. We refer to this as **splitting the graph Q_n in direction i**.

Note that both subgraphs $\mathcal{F}_n^{i+}$ and $\mathcal{F}_n^{i-}$ are isomorphic to Q_{n-1}. Splitting Q_n in direction i splits each spanning tree T into spanning forests $T_{i+} = T \cap \mathcal{F}_n^{i+}$ and $T_{i-} = T \cap \mathcal{F}_n^{i-}$ of Q_{n-1} (hereafter referred to as T_{i+} and T_{i-}). Throughout this thesis we usually split in direction n.

5.3 Upright spanning forests

We define a **rooted spanning forest** of a graph G to be a subgraph of G that contains all the vertices, and each connected component is a tree with a vertex designated as the root of that

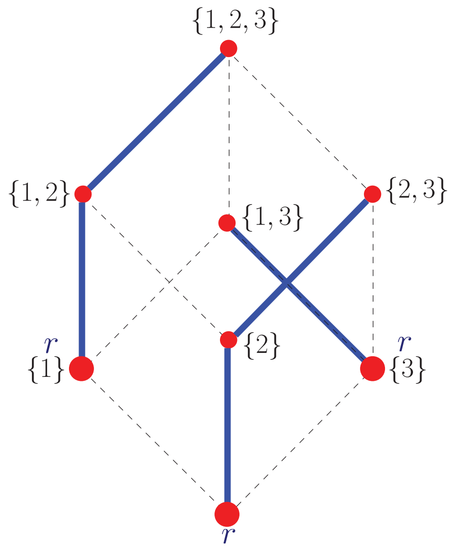


Figure 5.1: A rooted upright spanning forest of Q_3. The root of each component is presented by the big vertex and labelled r.

tree. We call a rooted spanning forest of Q_n **upright** if all the edges in each component are oriented downwards when they are oriented towards the root. Figure 5.1 shows an example of an upright rooted spanning forest of Q_3.

5.3.1 Signatures of rooted spanning forests

The **signature** of a rooted spanning forest of Q_n is defined to be $(b_0, b_1, \ldots, b_n)$, where b_i is the number of edges in direction i for $i \geq 1$ and b_0 corresponds to the number of roots which is the number of components. It can be understood in the same way as the signature of a spanning tree of Q_n with $\sum_{i=0}^{n} b_i = 2^n$, where $0 \leq b_i \leq 2^{n-1}$.

We can label the rooted upright spanning forests of Q_n using the same method that is used to label the upright spanning trees of Q_n in Section 2.2.5.2, except we specify the label at every vertex of Q_n, and we put a 'r' every time we have a root. The label can be represented as a string of length 2^n. An example appears in Figure 5.2.

5.3.2 Rooted spanning forests of $\mathcal{F}_n^{i+}$ and $\mathcal{F}_n^{i-}$

When we split a spanning tree of Q_n with signature $\mathcal{S} = (a_1, \ldots, a_n)$ in direction i, we get a spanning forest T_{i+} of Q_{n-1} in $\mathcal{F}_n^{i+}$, and a spanning forest T_{i-} of Q_{n-1} in $\mathcal{F}_n^{i-}$. We root each component at the vertex where the path from the component to the root of T leaves $\mathcal{F}_n^{i+}$ and $\mathcal{F}_n^{i-}$ in direction i. In other words, the root of each component of T_{i+} and T_{i-} is the vertex closest to the root of T and is the start of an i-edge. Sliding an i-edge of T moves the root of each component it meets and it can be thought of as sliding the root.

We define the signatures of T_{i+} and T_{i-} to be $\mathcal{U} = (u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_n)$ and $\mathcal{D} = (d_1, d_2, \ldots, d_{i-1}, d_i, d_{i+1}, \ldots, d_n)$ respectively, where u_i is the number of roots in T_{i+} and d_i is the number of roots in T_{i-}, excluding the root of T at $\varnothing$. Note that $u_i + d_i = a_i$ corresponds to the total number of roots in both T_{i+} and T_{i-}, again excluding the root of T at $\varnothing$.

In the case of an upright spanning tree T of Q_n the forest T_{i-} is in fact an upright spanning

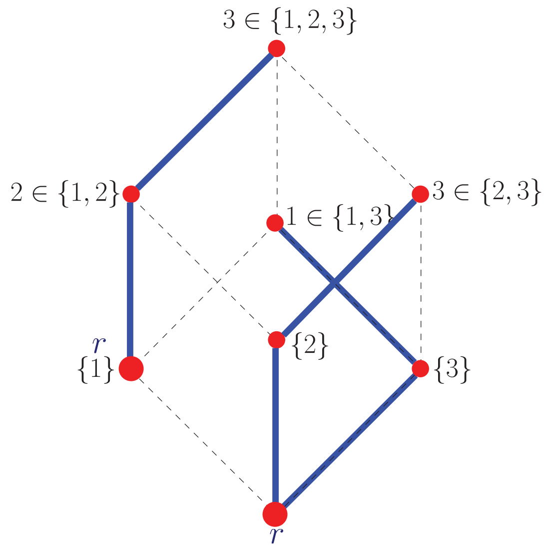


Figure 5.2: An upright spanning forest of Q_3 with signature (1, 2, 3). The label is $3213r23r$.

tree of Q_{n-1}, because T corresponds to a section of $\mathcal{P}^n_{\geq 1}$ and when we restrict to $\mathcal{F}^{i-}_n$ we still have a section of $\mathcal{P}([n] \setminus \{i\})$. So, we have $u_i = a_i$ and $d_i = 0$ and therefore the signatures can be written as $\mathcal{U} = (u_1, u_2, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_n)$ and $\mathcal{D} = (d_1, d_2, \dots, d_{i-1}, 0, d_{i+1}, \dots, d_n)$.

Definition 5.3.1. Let $\mathcal{S} = (a_1, \dots, a_n)$ be a signature of Q_n and let

$$\mathcal{D} = (d_1, \dots, d_{i-1}, 0, d_{i+1}, \dots, d_n)$$

be such that $\mathcal{D}' = (d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n)$ is a signature of Q_{n-1}. Then $\mathcal{D}$ is a **splitting signature** of $\mathcal{S}$ with respect to i if there is an upright spanning tree T of Q_n with $\mathcal{S}$ such that T_{i-} has signature $\mathcal{D}$.

If n is the direction of the split, then there is no need to put a zero in the signature $\mathcal{D}$ of T_{n-}, and we simply write $\mathcal{D} = (d_1, \dots, d_{n-1})$.

Convention 5.3.2. *If $\mathcal{D} = (d_1, d_2, \dots, d_{n-1})$ is a signature of Q_{n-1} and $\mathcal{U} = (u_1, u_2, \dots, u_{n-1}, u_n)$ is a signature of $\mathcal{F}^{n+}_n$, we will allow ourselves to write*

$$\mathcal{S} = \mathcal{D} + \mathcal{U} = (d_1 + u_1, d_2 + u_2, \dots, d_{n-1} + u_{n-1}, u_n) = (a_1, a_2, \dots, a_{n-1}, a_n).$$

In other words, we allow ourselves to write $\mathcal{S} = \mathcal{D} + \mathcal{U}$ even though $\mathcal{D}$ and $\mathcal{U}$ have different lengths, with the understanding that the absent nth entry of $\mathcal{D}$ is zero.

Definition 5.3.3. Let $\mathcal{B}$ be a set of spanning trees of Q_n with an irreducible signature $\mathcal{I} = (a_1, \dots, a_n)$. We say $\mathcal{B}$ is a **block** if any tree in $\mathcal{B}$ can be transformed into any other tree in $\mathcal{B}$ by edge slides. We do not require the trees to remain in $\mathcal{B}$ throughout the transformation. Thus, $\mathcal{B}$ is a block if it is contained in some connected component of $\mathcal{E}(Q_n)$.

5.3.3 Hall's Theorem and signatures of spanning forests of $\mathcal{F}^{n+}_n$

In this section we obtain necessary and sufficient conditions for an n-tuple to be a signature of an upright rooted spanning forest of $\mathcal{F}^{n+}_n$.

Lemma 5.3.4. *Let $\mathcal{U} = (u_1, \ldots, u_{n-1}, u_n)$, where $\sum_{t=1}^{n} u_t = 2^{n-1}$. Then $\mathcal{U}$ is a signature of an upright rooted spanning forest of $\mathcal{F}_n^{n+}$ if and only if $(\sum_{j\in A} u_j) + u_n \geq 2^{|A|}$ for all $A \subseteq [n-1]$.*

Proof. Let B be the set of 2^{n-1} vertices of $\mathcal{F}_n^{n+}$, and let C be a set of 2^{n-1} vertices of which u_i are labelled i for each $i \in [n]$. For each $V \in B$ we draw an edge to every vertex of C labelled i, for which $i \in V$. Note that every vertex in B is connected to every vertex in C labelled n. Let $G_{\mathcal{U}}$ be the resulting bipartite graph with bipartition (B, C). An upright rooted spanning forest of $\mathcal{F}_n^{n+}$ with signature $\mathcal{U}$ corresponds to a perfect matching in $G_{\mathcal{U}}$. We show there is a perfect matching in $G_{\mathcal{U}}$ if and only if the signature condition $(\sum_{j\in A} u_j) + u_n \geq 2^{|A|}$ for all $A \subseteq [n-1]$.

Let H be any subset of B, and let

$$Y = \bigcup_{V\in H} V.$$

Write $Y = A \cup \{n\}$. Then

$$|N(H)| = \sum_{i\in Y} u_i = (\sum_{j\in A} u_j) + u_n.$$

Also

$$|H| \leq |\mathcal{P}(Y)| = 2^{|Y|},$$

with equality if $H = \mathcal{P}(Y)$. Then we conclude that $|N(H)| \geq |H|$ for all $H \subseteq B$ if $(\sum_{j\in A} u_j) + u_n \geq 2^{|A|}$ for all $A \subseteq [n-1]$. Thus by Hall's Theorem, and since $|B| = |C|$, there exists a perfect matching if $(\sum_{j\in A} u_j) + u_n \geq 2^{|A|}$ for all $A \subseteq [n-1]$. $\square$

As we mentioned earlier, edge slides do not change the signature of a spanning tree, however, edge slides in direction i change the signatures of the spanning forests of Q_{n-1} in $\mathcal{F}_n^{i+}$ and $\mathcal{F}_n^{i-}$.

Example 5.3.5. Splitting the upright spanning tree 3213 of Q_3 with the irreducible signature (2, 2, 3) in direction 3 gives the upright spanning forest of Q_2 with signature $(1, 0)$ in F_3^{3+}, where direction 1 is chosen at $\{1, 3\}$, and the upright spanning tree of Q_2 with signature $(1, 2)$ in F_3^{3-}, as shown in Figure 5.3(a). The path from $\{1, 2, 3\}$ to the root goes in direction 3 and then direction 2. Then using the path move the 1-edge moves to $\{1, 2\}$, the 2-edge moves to $\{1, 2, 3\}$ and the 3-edge moves to $\{1, 3\}$. Therefore the upright spanning tree 3213 is transformed into the upright spanning tree 2133 where the upright spanning forest of Q_2 has signature $(0, 1)$ in $\mathcal{F}_3^{3+}$ and the upright spanning tree of Q_2 has signature $(2, 1)$ in $\mathcal{F}_3^{3-}$, as shown in Figure 5.3(b).

5.4 The edge slide graph of the irreducible signature (2, 3, 3, 7)

In this section we start with showing that there are 40 upright spanning trees of Q_4 with signature (2, 3, 3, 7), and then we show that the edge slide graph of (2, 3, 3, 7) is connected by showing that the local move graph of (2, 3, 3, 7) is connected. The strategy for proving the connectivity of the local move graph (2, 3, 3, 7) is by showing that each upright spanning tree of Q_4 with signature (2, 3, 3, 7) that has a reducible splitting signature in $\mathcal{F}_4^{4-}$ can be transformed into an upright spanning tree that has an irreducible splitting signature in $\mathcal{F}_4^{4-}$ using a sequence of edge slides. Then we show that upright spanning trees of Q_4 with signature (2, 3, 3, 7) that have an irreducible splitting signature in $\mathcal{F}_4^{4-}$ form one block. Finally, we show

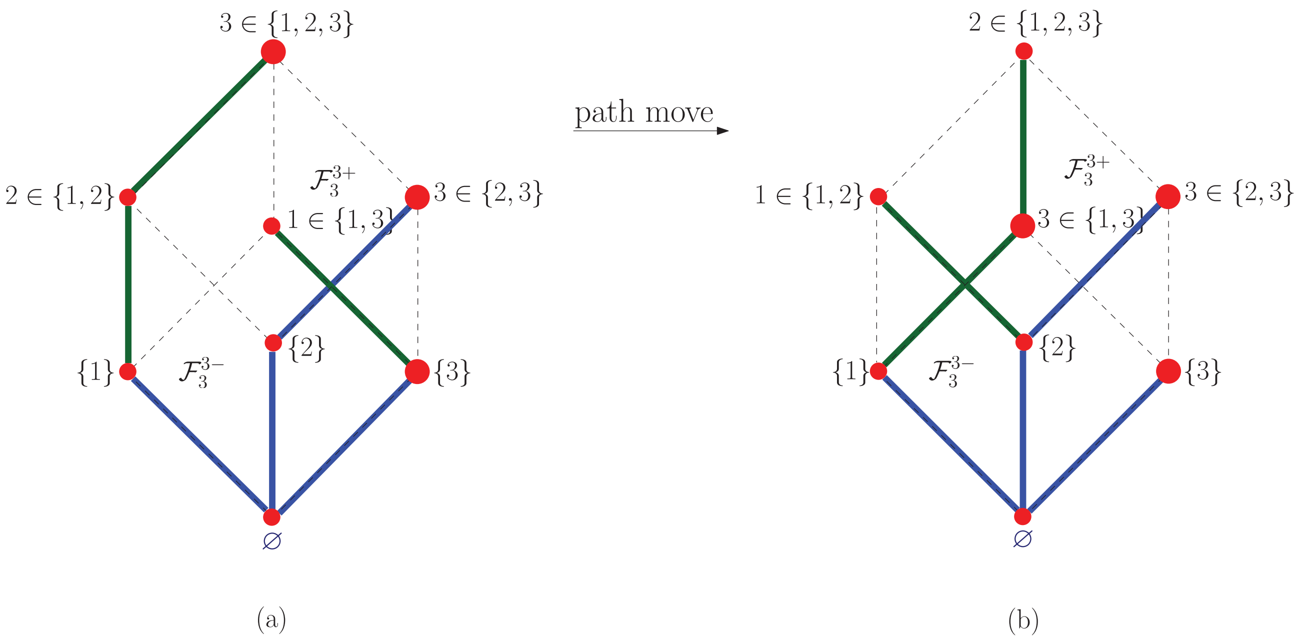


Figure 5.3: (a) The upright spanning tree 3213 of Q_3 with the irreducible signature (2, 2, 3). (b) Then using the path move on the face with the green edges the tree 3213 is transformed into the upright spanning tree 2133.

we can transform a tree with an irreducible splitting signature in $\mathcal{F}_4^{4-}$ into a tree with any other irreducible splitting signature by a sequence of edge slides. So the blocks of trees with irreducible signatures are connected.

Lemma 5.4.1. *There are 40 upright spanning trees of Q_4 with the irreducible signature (2, 3, 3, 7).*

Proof. Splitting Q_4 in direction 4, upright spanning trees of Q_4 with the irreducible signature (2, 3, 3, 7) consist of upright spanning forests of Q_3 in $\mathcal{F}_4^{4+}$ with signature $(1,0,0), (0,1,0)$ or $(0,0,1)$, and upright spanning trees of Q_3 in $\mathcal{F}_4^{4-}$ with signature (1, 3, 3), (2, 2, 3) or (2, 3, 2) respectively. For any permutation of (1, 0, 0) there are four upright spanning forests of Q_3 with that signature (by Lemma 4.3.3). There are two upright spanning trees of Q_3 with signature (1, 3, 3), and for any permutation of (2, 2, 3) there are four upright spanning trees of Q_3 with that signature. Thus, altogether there are a total of $4(2) + 4(4) + 4(4) = 8 + 16 + 16 = 40$ upright spanning trees of Q_4 with signature (2, 3, 3, 7). □

Proposition 5.4.2. *The edge slide graph of the irreducible signature (2, 3, 3, 7) is connected.*

Proof. Since every spanning tree of Q_4 is connected to at least one upright spanning tree by a series of edge slides (Tuffley [11]), it suffices to show that the local move graph of the signature (2, 3, 3, 7) is connected. Splitting upright spanning trees of Q_4 in direction 4 gives us upright spanning forests of Q_3 with signature $(1,0,0,7), (0,1,0,7)$ or $(0,0,1,7)$ in $\mathcal{F}_4^{4+}$, and upright spanning trees of Q_3 with signature $(1,3,3), (2,2,3)$ or $(2,3,2)$ in $\mathcal{F}_4^{4-}$ respectively.

We first show any upright spanning tree T of Q_4 with signature (2, 3, 3, 7) such that T_{4-} has signature $(1,3,3)$ can be transformed by local moves into a tree $T^{\star}$ such that $T^{\star}_{4-}$ has either signature (2, 2, 3) or (2, 3, 2).

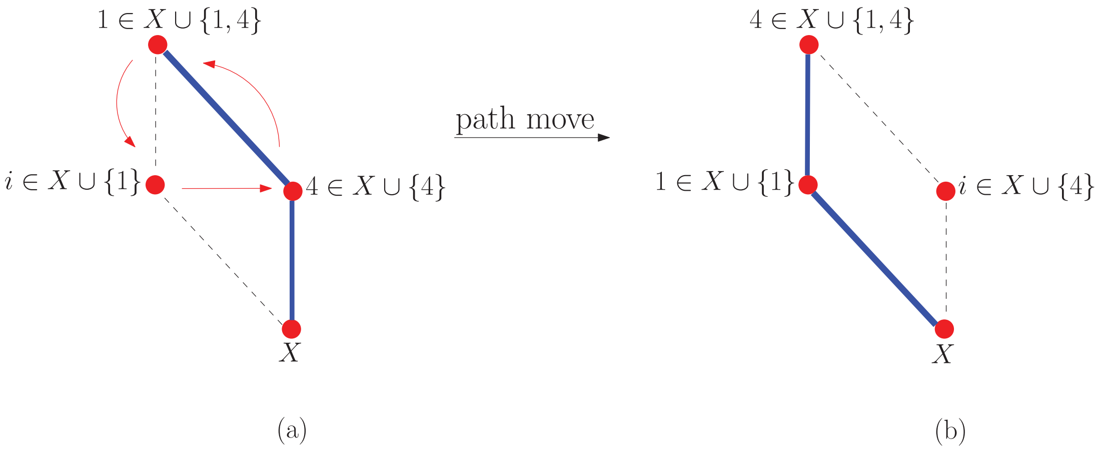


Figure 5.4: (a) Directions 1 and 4 are chosen at $X \cup \{1,4\}$ and $X \cup \{4\}$ respectively, and direction $i \neq 1$ is chosen at $X \cup \{1\}$. (b) The i-edge moves to $X \cup \{4\}$, the 4-edge moves to $X \cup \{1,4\}$ and the 1-edge moves to $X \cup \{1\}$ using the path move.

Let $X \subseteq \{2,3\}$ be such that $Y = X \cup \{1,4\}$ is the unique vertex of $\mathcal{F}_4^{4+}$ that has the 1-edge of T_{4+}. We consider the following cases according to whether or not $X = \varnothing$.

Suppose that $X \neq \varnothing$. Since all the vertices of $\mathcal{F}_4^{4+}$ apart from $X \cup \{1,4\}$ are in direction 4, we have $\psi_T(X \cup \{4\}) = 4$. The vertex $X \cup \{1\}$ cannot be in direction 1 because the only 1-edge of T_{4-} is chosen at $\{1\}$, so $\psi_T(X \cup \{1\}) = i$ for some $i \in X$. As shown in Figure 5.4(a), using the path move the i-edge moves to $X \cup \{4\}$, the 4-edge moves to $X \cup \{1,4\}$ and the 1-edge moves to $X \cup \{1\}$ as shown in Figure 5.4(b). Therefore T is transformed into a tree $T^\star$ such that $T^\star_{4-}$ has either signature (2, 2, 3) (if $i = 2$) or (2, 3, 2) (if $i = 3$).

Suppose on the other hand that $X = \varnothing$. Then $X \oplus \{1,4\} = \{1,4\}$ and therefore the 1-edge of T_{4+} is chosen at $\{1,4\}$. Choose $i \in \{2,3\}$. Since all the vertices of $\mathcal{F}_4^{4+}$ apart from $\{1,4\}$ are in direction 4, we have $\psi_T(\{1,i,4\}) = 4$. The vertex $\{1,i\}$ cannot be in direction 1 because the only 1-edge of T_{4-} is chosen at $\{1\}$, so $\psi_T(\{1,i\}) = i$. As shown in Figure 5.5(a), using the path move the 1-edge moves to $\{1,i\}$, the i-edge moves to $\{1,i,4\}$ and the 4-edge moves to $\{1,4\}$ as shown in Figure 5.5(b). Therefore T is transformed into a tree $T^\star$ such that $T^\star_{4-}$ has either signature (2, 2, 3) (if $i = 2$) or (2, 3, 2) (if $i = 3$).

We have now completed the proof that we can transform T such that T_{4-} has signature (1, 3, 3) into a tree $T^\star$ such that $T^\star_{4-}$ has signature (2, 2, 3) or (2, 3, 2). We next show that all upright spanning trees T of Q_4 with signature (2, 3, 3, 7) such that T_{4-} has signature (2, 2, 3) are connected and form one block by a sequence of local moves.

There are sixteen such trees, determined by choosing one of the four spanning trees of $\mathcal{F}_4^{4-}$ with signature (2, 2, 3), and one of the four spanning forests of $\mathcal{F}_4^{4+}$ with signature (0, 1, 0, 7). By Lemma 4.3.6 the local move graph $\mathcal{L}(2,2,3)$ is connected, so we can freely change T_{4-} using local moves. It therefore suffices to show that we can use local moves to move the 2-edge upstairs from one vertex containing 2 to any other.

Let $T^\star$ be an upright spanning tree of Q_4 with signature (2, 3, 3, 7) such that $T^\star_{4+}$ has signature (0, 1, 0, 7) and the 2-edge is chosen at the vertex $\{2,4\}$. Let T be any other upright spanning tree with signature (2, 3, 3, 7) such that T_{4+} has signature (0, 1, 0, 7). We show that there is a sequence of local moves that transforms T into $T^\star$.

Let $X \subseteq \{1,3\}$ be such that $Y = X \cup \{2,4\}$ is the vertex of $\mathcal{F}_4^{4+}$ that has the 2-edge of T_{4+}. If $X = \varnothing$, then $X \cup \{2,4\} = \{2,4\}$ and therefore $T = T^\star$. Otherwise, let $i = \max X$. Since all

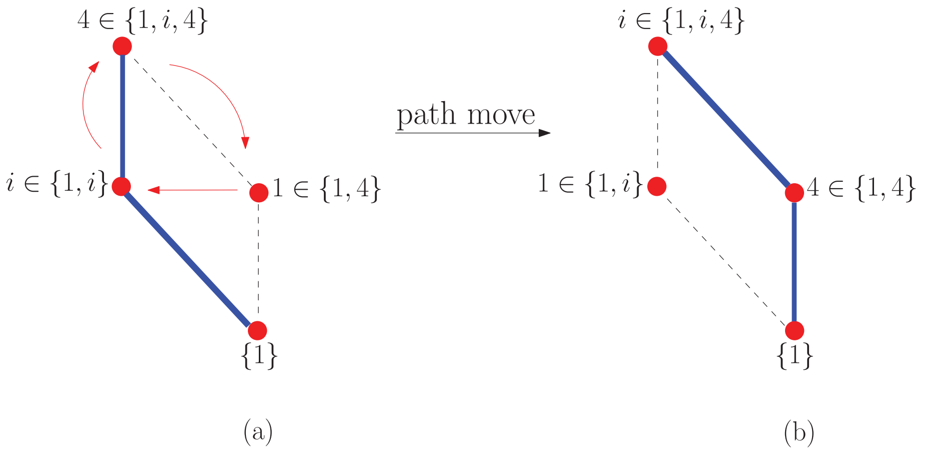


Figure 5.5: (a) Directions 4 and $i \neq 1$ are chosen at $\{1, i, 4\}$ and $\{1, i\}$ respectively, and direction 1 is chosen at $\{1, 4\}$. (b) The 1-edge moves to $\{1, i\}$, the i-edge moves to $\{1, i, 4\}$ and the 4-edge moves to $\{1, 4\}$ using the path move.

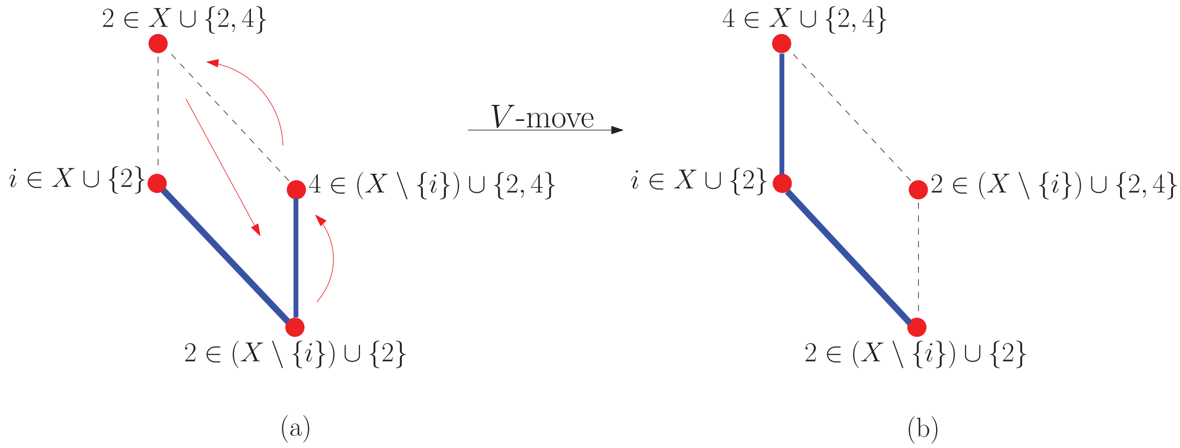


Figure 5.6: (a) Direction 2 is chosen at $X \cup \{2, 4\}$ and $(X \setminus \{i\}) \cup \{2\}$, and directions i and 4 are chosen at $X \cup \{2\}$ and $(X \setminus \{i\}) \cup \{2, 4\}$ respectively. (b) The 2-edge and 4-edge are swapped using the V-move.

the vertices of $\mathcal{F}_4^{4+}$ apart from $X \cup \{2, 4\}$ are in direction 4, we have $\psi_T((X \setminus \{i\}) \cup \{2, 4\}) = 4$. Since (2, 2, 3) is irreducible, there exists a tree T'_{4-} of Q_3 with signature (2, 2, 3) where $\psi_{T'}(X \cup \{2\}) = i$ and $\psi_{T'}((X \setminus \{i\}) \cup \{2\}) = 2$ (by Lemma 3.2.8). Since the local move graph $\mathcal{L}(2, 2, 3)$ is connected, we can move from T_{4-} to T'_{4-} using local moves.

As seen in Figure 5.6(a), using the V-move, the 2-edge and the 4-edge are swapped as seen in Figure 5.6(b). Therefore we reach a tree T'' with $\psi_{T''}((X \setminus \{i\}) \cup \{2, 4\}) = 2$ and all other labels of T'' the same as the labels of T'. If $(X \setminus \{i\}) \cup \{2\} = \{2, 4\}$, then $T'' = T^\star$ and no further move is needed. Otherwise we repeat the above process once more to reach a tree where the 2-edge in $\mathcal{F}_4^{4+}$ is chosen at $\{2, 4\}$. By the definition of the local move graph of (2, 3, 3, 7), there is a sequence of edge slides from T to $T^\star$ such that T_{n-} and $T^\star_{n-}$ have signature (2, 2, 3), hence T and $T^\star$ are in the same connected component of $\mathcal{L}(2, 3, 3, 7)$. Since T is arbitrary, upright spanning trees that have signature (2, 2, 3) in $\mathcal{F}_4^{4-}$ form a single block.

By symmetry, the upright spanning trees T such that T_{4-} has signature (2, 3, 2) also form a single block.

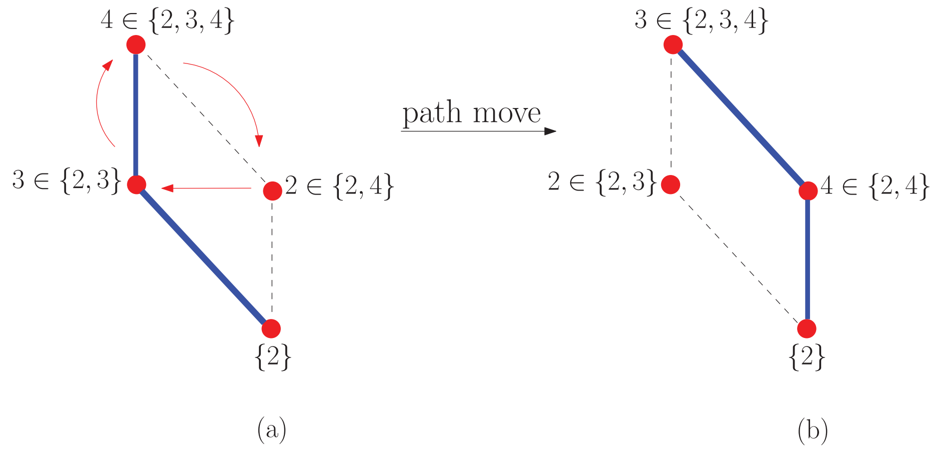


Figure 5.7: (a) Directions 4 and 3 are chosen at $\{2,3,4\}$ and $\{2,3\}$ respectively, and direction 2 is chosen at $\{2,4\}$. Using the path move the 2-edge moves to $\{2,3\}$, the 3-edge moves to $\{2,3,4\}$ and the 4-edge moves to $\{2,4\}$ (b).

We have now completed the proof that upright spanning trees that have signature (2, 2, 3) in $\mathcal{F}_4^{4-}$ form a single block, and that upright spanning trees that have signature (2, 3, 2) in $\mathcal{F}_4^{4-}$ also form a single block. We next show the two blocks are connected by a sequence of local moves. Let $T^\star$ be as above. We show that $T^\star$ can be transformed into a tree $T^{\star\star}$ such that $T^{\star\star}_{4-}$ has signature (2, 3, 2).

Since all the vertices of $\mathcal{F}_4^{4+}$ apart from $\{2,4\}$ are in direction 4, we have $\psi_{T^\star}(\{2,3,4\}) = 4$. By Lemma 3.2.4 there exists a tree T'_{4-} of Q_3 with signature (2, 2, 3) such that $\psi_{T'}(\{2,3\}) = 3$ because (2, 2, 3) is irreducible. Since the local move graph $\mathcal{L}(2,2,3)$ is connected, we can move from $T^\star_{4-}$ to T'_{4-} using local moves. Then $T^\star$ is transformed into T' with $\psi_{T'}(\{2,3\}) = 3$ and all labels of T'_{4+} the same as the labels of $T^\star_{4+}$.

As shown in Figure 5.7(a), using the path move the 2-edge moves to $\{2,3\}$, the 3-edge moves to $\{2,3,4\}$ and the 4-edge moves to $\{2,4\}$ as shown in Figure 5.7(b). Therefore T' is transformed into T'' such that T''_{4-} has signature (2, 3, 2).

By the definition of the local move graph of (2, 3, 3, 7), there is a sequence of local moves from each upright spanning tree of Q_4 with signature (2, 3, 3, 7) that has signature (1, 3, 3) in $\mathcal{F}_4^{4-}$ to a tree with either signature (2, 2, 3) or (2, 3, 2) in $\mathcal{F}_4^{4-}$, hence these trees are in the same connected component of $\mathcal{L}(2,3,3,7)$. There is also a sequence of local moves from each upright spanning tree of Q_4 with signature (2, 3, 3, 7) that has signature (2, 2, 3) in $\mathcal{F}_4^{4-}$ to a tree with (2, 3, 2) in $\mathcal{F}_4^{4-}$, hence these trees are in the same connected component of $\mathcal{L}(2,3,3,7)$. All trees with either signature (2, 2, 3) or (2, 3, 2) in $\mathcal{F}_4^{4-}$ are connected by a sequence of local moves and belong to the same connected component of $\mathcal{L}(2,3,3,7)$. Therefore the local move graph $\mathcal{L}(2,3,3,7)$ is connected.

Since every spanning tree of Q_4 is connected to an upright spanning tree, spanning trees of Q_4 with signature (2, 3, 3, 7) lie in a single component of the edge slide graph of Q_4 and therefore $\mathcal{E}(2,3,3,7)$ is connected. □

Observation 5.4.3. *Observe that the final step in the above proof of showing that we can go from a tree with signature (2, 2, 3) in $\mathcal{F}_4^{4-}$ to a tree with (2, 3, 2) was not strictly necessary because we showed that from a tree with signature (1, 3, 3) in $\mathcal{F}_4^{4-}$ we could choose to go to a tree with either signature (2, 2, 3) or (2, 3, 2). So the trees with signature (1, 3, 3) connect them. However, in the general case we need to show the final step of showing we can change* *the splitting signature to go from any irreducible splitting signature to any other.*

5.5 Amenable splitting signatures

We start this section with the definition of three types of splitting signatures which play a crucial role in understanding the connectivity of the edge slide graph of an irreducible signature of Q_n.

Definition 5.5.1. Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be an ordered irreducible splitting signature of $\mathcal{I}$ with respect to n. Let $u_t = a_t - d_t$ for all $t \in [n]$. Signature $\mathcal{D}$ is a **unidirectional splitting signature** if there exists $i \in [n-1]$ such that $u_i \neq 0$ and $u_j = 0$ for all $j \neq i, n$. Signature $\mathcal{D}$ is a **rich splitting signature** if $\varepsilon^{\mathcal{D}}_{\mu} \geq 2$ for each $\mu \in [n-1]$ such that there exists $i \leq \mu$ with $u_i \neq 0$. A rich splitting signature $\mathcal{D}$ is called a **super rich splitting signature** if for each i such that $u_i \neq 0$ we have $u_i < a_n - \nu$, where $\nu = \lfloor \log_2 n \rfloor$.

Next, we define an amenable splitting signature of an irreducible signature of Q_n as follows.

Definition 5.5.2. Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be an ordered irreducible splitting signature of $\mathcal{I}$ with respect to n. Then $\mathcal{D}$ is an **amenable splitting signature** if it is either a unidirectional splitting signature, a super rich splitting signature, or the splitting signature (2, 2, 3) of Q_4.

These signatures are amenable in the sense that these properties can be used to rearrange the labels of the trees with these splitting signatures. In particular, we show that upright spanning trees with a fixed connected unidirectional splitting signature or the (2, 2, 3) splitting signature form a block, and we conjecture under an inductive hypothesis this is also true for a super rich splitting signature.

Observation 5.5.3. *Note that there is no rich splitting signature of any irreducible signature of Q_4, because (2, 2, 3) is the only ordered irreducible signature of Q_3. So we consider the splitting signature (2, 2, 3) of Q_4 as an amenable splitting signature.*

The following lemma shows that an irreducible signature of Q_n with excess at most two at $n-2$ has a unidirectional splitting signature.

Lemma 5.5.4. *Let $n \geq 5$ and let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered irreducible signature of Q_n with $\varepsilon^{\mathcal{I}}_{n-2} \leq 2$. Then $\mathcal{I}$ has a unidirectional splitting signature.*

Proof. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ and $\mathcal{U} = (u_1, \ldots, u_n)$ where

$$d_i = \begin{cases} a_i, & \text{when } 1 \leq i \leq n-2; \\ a_{n-1} - \varepsilon^{\mathcal{I}}_{n-1}, & \text{when } i = n-1, \end{cases}$$

and

$$u_i = \begin{cases} 0, & \text{when } 1 \leq i \leq n-2; \\ \varepsilon^{\mathcal{I}}_{n-1}, & \text{when } i = n-1; \\ a_n, & \text{when } i = n. \end{cases}$$

Then $\mathcal{U} + \mathcal{D} = \mathcal{I}$ and $\sum_{i=1}^{n-1} d_i = 2^{n-1} - 1$. We show that $\mathcal{D}$ is an ordered irreducible signature of Q_{n-1} and $\mathcal{U}$ is a signature of $\mathcal{F}^{n+}_n$.

We first show that $\mathcal{D}$ is ordered, by showing that $d_{n-1} \geq d_{n-2}$. Since $\varepsilon^{\mathcal{I}}_{n-2} \leq 2$, we have

$$\sum_{j=1}^{n-2} a_j \leq 2^{n-2} + 1.$$

Then

$$a_{n-1} + a_n \geq 2^{n-1} + 2^{n-2} - 2.$$

Since

$$a_n + \varepsilon^{\mathcal{I}}_{n-1} = 2^{n-1},$$

we have

$$\begin{aligned} a_{n-1} + a_n &\geq a_n + \varepsilon^{\mathcal{I}}_{n-1} + 2^{n-2} - 2 \\ a_{n-1} - \varepsilon^{\mathcal{I}}_{n-1} &\geq 2^{n-2} - 2. \end{aligned}$$

Since

$$\sum_{j=1}^{n-3} a_j \geq 2^{n-3} \quad \text{(by irreducibility)}$$

and

$$\sum_{j=1}^{n-2} a_j \leq 2^{n-2} + 1,$$

we have

$$a_{n-2} \leq 2^{n-3} + 1.$$

Then

$$\begin{aligned} a_{n-1} - \varepsilon^{\mathcal{I}}_{n-1} - a_{n-2} &\geq 2^{n-2} - 2 - (2^{n-3} + 1) \\ &= 2^{n-3} - 3 \\ &\geq 0 && \text{(for all } n \geq 5). \end{aligned}$$

Therefore

$$a_{n-2} \leq a_{n-1} - \varepsilon^{\mathcal{I}}_{n-1}.$$

Therefore $\mathcal{D}$ is ordered. The signature condition and irreducibility of $\mathcal{D}$ now follows from the signature condition and irreducibility of $\mathcal{I}$.

We now show that $\mathcal{U}$ is a signature of $\mathcal{F}^{n+}_n$. Since $a_n \geq 2^{n-2} + 2^{n-3} - 1 > 2^{n-2}$, we have $\sum_{j \in A} u_j + a_n \geq 2^{|A|}$ for all $A \subseteq [n-1]$. Therefore $\mathcal{U}$ is a signature of $\mathcal{F}^{n+}_n$. Since $u_{n-1} > 0$ and $u_t = 0$ for all $t \neq n-1, n$ and since $\mathcal{D}$ is an ordered irreducible signature of Q_{n-1}, by Definition 5.5.1 we conclude that $\mathcal{D}$ is a unidirectional splitting signature of $\mathcal{I}$. □

5.6 Signature moves

We start this section with the definition of a signature move. Then we show that a particular irreducible signature can be transformed into any other irreducible signature using signature moves. Finally we show using signature moves that every irreducible signature has an amenable splitting signature.

Definition 5.6.1. Let $\mathcal{S} = (a_1, \ldots, a_n)$ be an ordered signature of Q_n. Suppose there exists $i, j \in [n]$ with $i < j$ and $a_j - a_i \geq 2$ such that

$$a_i < a_{i+1} \quad \text{and} \quad a_{j-1} < a_j.$$

Let $\mathcal{S}' = (a'_1, \ldots, a'_n)$ be such that $a'_i = a_i + 1$, $a'_j = a_j - 1$ and $a'_k = a_k$, for all $k \neq i, j$. We say that $\mathcal{S}$ is transformed into $\mathcal{S}'$ by a **signature move**. Note that when $j - i \geq 2$ the condition $a_j - a_i \geq 2$ is immediate from $a_i < a_{i+1} \leq a_{j-1} < a_j$. This condition is required when $j = i + 1$ to ensure $\mathcal{S}'$ is ordered.

The following lemma shows that $\mathcal{S}'$ is a signature of Q_n. Also it shows that if the first signature is irreducible, then so is the second.

Lemma 5.6.2. *The result $\mathcal{S}'$ of applying a signature move to an ordered signature $\mathcal{S}$ is an ordered signature. Moreover, if $\mathcal{S}$ is irreducible then so is $\mathcal{S}'$.*

Proof. In the notation of definition 5.6.1, since

$$a_1 \leq \cdots \leq a_{i-1} \leq a_i < a_{i+1} \leq \cdots \leq a_{j-1} < a_j \leq a_{j+1} \leq \cdots \leq a_n,$$

we have

$$a'_1 \leq \cdots \leq a'_{i-1} < a'_i \leq a'_{i+1} \leq \cdots \leq a'_j < a'_{j+1} \leq \cdots \leq a'_n.$$

So $\mathcal{S}'$ is in increasing order, and

$$\sum_{h=1}^{k} a'_h = \begin{cases} \sum_{h=1}^{k} a_h + 1 \geq 2^k, & \text{when } i \leq k < j; \\ \sum_{h=1}^{k} a_h \geq 2^k - 1, & \text{otherwise.} \end{cases}$$

Thus, by the signature condition in Lemma 3.1.3 and the definition of an ordered signature we conclude that $\mathcal{S}'$ is an ordered signature of Q_n.

If in addition $\mathcal{S}$ is irreducible, then

$$\sum_{h=1}^{k} a'_h = \begin{cases} \sum_{h=1}^{k} a_h + 1 \geq 2^k + 1, & \text{when } i \leq k < j; \\ \sum_{h=1}^{k} a_h \geq 2^k, & \text{otherwise.} \end{cases}$$

Then, by Definition 3.2.1 we conclude that $\mathcal{S}$ is irreducible. □

The next observation follows by similar reasoning to Lemma 5.6.2.

Observation 5.6.3. *Observe that $\varepsilon_\mu^{S'} \geq \varepsilon_\mu^{S}$ for all μ, because*

$$\varepsilon_\mu^{S'} = \begin{cases} (\varepsilon_\mu^{S}) + 1, & \textit{when } i \leq \mu < j; \\ \varepsilon_\mu^{S}, & \textit{otherwise.} \end{cases}$$

Next, we define an infinite family of irreducible signatures of Q_n.

Definition 5.6.4. For $n \geq 3$, $\mathcal{I}_n^{(-1)} = (a_1, \ldots, a_n)$ is the ordered irreducible signature of Q_n where

$$a_i = \begin{cases} 2, & \text{if } i = 1; \\ 2^{n-1} - 1, & \text{if } i = n; \\ 2^{i-1}, & \text{otherwise.} \end{cases}$$

Note that signature $\mathcal{I}_n^{(-1)}$ has excess $\varepsilon_\mu^{\mathcal{I}_n^{(-1)}} = 1$ for all μ.

n	$\mathcal{I}_n^{(-1)} = (a_1, \ldots, a_n)$
3	$\mathcal{I}_3^{(-1)} = (2, 2, 3)$
4	$\mathcal{I}_4^{(-1)} = (2, 2, 4, 7)$
5	$\mathcal{I}_5^{(-1)} = (2, 2, 4, 8, 15)$
6	$\mathcal{I}_6^{(-1)} = (2, 2, 4, 8, 16, 31)$

Table 5.1: The first four members of the infinite family $\mathcal{I}_n^{(-1)}$.

The following lemma is used to prove Corollary 5.6.14.

Lemma 5.6.5. *For $n \geq 4$, the signature $\mathcal{I}_n^{(-1)}$ can be transformed into any other ordered irreducible signature using signature moves.*

Proof. Let $\hat{\mathcal{I}} = (\hat{a}_1, \ldots, \hat{a}_n)$ be an ordered irreducible signature of Q_n. Write

$$\mathcal{I}_n^{(-1)} = (a_1, a_2, a_3, a_4, \ldots, a_n) = (2, 2, 4, 8, \ldots, 2^{n-1} - 1).$$

We show there exists a sequence of signature moves that transforms $\mathcal{I}_n^{(-1)}$ to $\hat{\mathcal{I}}$.

Let $\delta\left(\hat{\mathcal{I}}, \mathcal{I}_n^{(-1)}\right) = ||\mathcal{I}_n^{(-1)} - \hat{\mathcal{I}}|| = \sum_{t=1}^{n-1} |\hat{a}_t - a_t|$. Note that $\sum_{t=1}^{n} |\hat{a}_t - a_t| = 2r$ where $r \geq 0$, because the a_t and the $\hat{a}_t$ sum to the same value, so the sum here without the absolute value signs is zero. Adding the absolute value signs leaves each term unchanged or changes it by an even number, so the resulting sum must have the same parity and so is even. If $\sum_{t=1}^{n} |\hat{a}_t - a_t| = 0$, then $\hat{\mathcal{I}} = \mathcal{I}_n^{(-1)}$ and no further moves are needed. Otherwise there exists $|\hat{a}_t - a_t| \neq 0$ for some t.

Let i be the least index such that $|\hat{a}_i - a_i| \neq 0$. Since $\varepsilon_\mu^{\mathcal{I}_n^{(-1)}} = 1$ for all μ and since $\hat{\mathcal{I}}$ is an ordered irreducible signature such that $|a_i - \hat{a}_i| \neq 0$, we must have $\varepsilon_i^{\hat{\mathcal{I}}} \geq 2$ and therefore $a_i < \hat{a}_i$. Since i is the least index such that $a_i < \hat{a}_i$ and since $\hat{\mathcal{I}}$ is ordered, we have $\hat{a}_{i-1} = a_{i-1} \leq a_i < \hat{a}_i$. Since $\sum_{t=1}^{n} a_t = \sum_{t=1}^{n} \hat{a}_t$ and $a_i < \hat{a}_i$, there must exist f such that $\hat{a}_f < a_f$. Let j be the least index such that $\hat{a}_j < a_j$, and choose k to be the largest index such that $\hat{a}_k = \hat{a}_j$. Then we either have $\hat{a}_k < \hat{a}_{k+1}$ or $k = n$. By Lemma 3.2.5, we have $\varepsilon_\mu^{\hat{\mathcal{I}}} \geq 2$ for all $j \leq \mu < k$.

Since $\varepsilon_i^{\hat{\mathcal{I}}} \geq 2$ where i is the least index such that $a_i < \hat{a}_i$, and since j is the least index such that $\hat{a}_j < a_j$, we have $\varepsilon_\mu^{\hat{\mathcal{I}}} \geq 2$ for all $i \leq \mu < j$. In other words, since $\varepsilon_\mu^{\hat{\mathcal{I}}} > \varepsilon_\mu^{\mathcal{I}_n^{(-1)}}$ for $i \leq \mu < j$ and since $\varepsilon_\mu^{\mathcal{I}_n^{(-1)}} = 1$ for all μ, we have $\varepsilon_\mu^{\hat{\mathcal{I}}} \geq 2$ for all $i \leq \mu < j$. We therefore have $\varepsilon_\mu^{\hat{\mathcal{I}}} \geq 2$ for all $i \leq \mu < k$.

Let $\mathcal{I}' = (a_1', \ldots, a_n')$ with $a_i' = \hat{a}_i - 1, a_k' = \hat{a}_k + 1$ and $a_t' = \hat{a}_t$ for all $t \neq i, k$. Since $\hat{a}_{i-1} < \hat{a}_i$ and $\hat{a}_k < \hat{a}_{k+1}$, we have

$$a_1' \leq \cdots \leq a_{i-1}' \leq a_i' < a_{i+1}' \leq \cdots \leq a_{k-1}' < a_k' \leq a_{k+1}' \leq \cdots \leq a_n',$$

so $\mathcal{I}'$ is in increasing order. Since $\hat{a}_i \leq \hat{a}_k$, we have $a_i' < \hat{a}_i \leq \hat{a}_k < a_k'$ and therefore $a_k' - a_i' \geq 2$. Since the excess $\varepsilon_\mu^{\hat{\mathcal{I}}} \geq 2$ for $i \leq \mu < k$, we have

$$\varepsilon_\mu^{\mathcal{I}'} = \begin{cases} (\varepsilon_\mu^{\hat{\mathcal{I}}}) - 1 \geq 1, & \text{for } i \leq \mu < k; \\ \varepsilon_\mu^{\hat{\mathcal{I}}} & \text{otherwise.} \end{cases}$$

So $\mathcal{I}'$ is irreducible and therefore $\hat{\mathcal{I}}$ is the result of a signature move on $\mathcal{I}'$. We have $\delta\left(\mathcal{I}', \mathcal{I}_n^{(-1)}\right) = \delta\left(\hat{\mathcal{I}}, \mathcal{I}_n^{(-1)}\right) - 2$. If $\sum_{t=1}^{n} |a_t - a'_t| = 0$, then $\mathcal{I}' = \mathcal{I}_n^{(-1)}$ and therefore the transformation is completed. Otherwise $\sum_{t=1}^{n} |a_t - a'_t| \neq 0$ and then we repeat the above process and decrease $\delta\left(\mathcal{I}', \mathcal{I}_n^{(-1)}\right)$ by two until we reach the ordered irreducible signature $\mathcal{I}_n^{(-1)}$. □

The following lemma is used to prove Lemma 5.6.8.

Lemma 5.6.6. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered irreducible signature of Q_n and let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be an ordered irreducible splitting signature of $\mathcal{I}$ with respect to n. Let $u_i = a_i - d_i$ for $1 \leq i \leq n-1$. Let $A \subseteq \{1, \ldots, n-1\}$ be such that $|A| \leq n-2$ and $2^{|A|} \geq a_n$. Then*

$$\sum_{i \in A} u_i + a_n > 2^{|A|}. \tag{5.1}$$

Proof. Write $|A| = n - j$ where $j \geq 2$. Since $\sum_{i=1}^{n} a_i = 2^n - 1$, we have

$$\begin{aligned}\sum_{i=1}^{n-j} a_i &= 2^n - 1 - \sum_{i=(n-j)+1}^{n} a_i \\ &\geq 2^n - ja_n - 1 && \text{(since } a_i \leq a_n \text{ for all } i).\end{aligned}$$

Since $\mathcal{I}$ is ordered, we have

$$\sum_{i \in A} a_i \geq \sum_{i=1}^{n-j} a_i$$

and therefore

$$\sum_{i \in A} a_i \geq 2^n - ja_n - 1.$$

Similarly, since $\sum_{i=1}^{n-1} d_i = 2^{n-1} - 1$, we have

$$\begin{aligned}\sum_{i=j}^{n-1} d_i &= 2^{n-1} - 1 - \sum_{i=1}^{j-1} d_i \\ &\leq 2^{n-1} - 2^{j-1} - 1 && \left(\text{since } \sum_{i=1}^{j-1} d_i \geq 2^{j-1} \quad \text{by the irreducibility of } \mathcal{D}\right).\end{aligned}$$

Since $\mathcal{D}$ is ordered, we have

$$\sum_{i \in A} d_i \leq \sum_{i=j}^{n-1} d_i$$

and therefore

$$\sum_{i \in A} d_i \leq 2^{n-1} - 2^{j-1} - 1.$$

Then

$$\begin{aligned}\sum_{i \in A} u_i &= \sum_{i \in A} a_i - \sum_{i \in A} d_i \\ &\geq (2^n - ja_n - 1) - (2^{n-1} - 2^{j-1} - 1) \\ &= 2^{n-1} + 2^{j-1} - ja_n.\end{aligned}$$

Thus the left hand side of (5.1) satisfies

$$\sum_{i \in A} u_i + a_n \geq 2^{n-1} + 2^{j-1} - (j-1)a_n.$$

Since $2^{|A|} = 2^{n-j}$, the left hand side of (5.1) minus the right hand side of (5.1) satisfies

$$\begin{aligned}
\sum_{i \in A} u_i + a_n - 2^{|A|} &\geq 2^{n-1} + 2^{j-1} - (j-1)a_n - 2^{n-j} \\
&\geq 2^{n-1} + 2^{j-1} - j2^{n-j} && \text{(since } a_n \leq 2^{n-j} \text{ by hypothesis)} \\
&= 2^{n-1}\left(1 - \frac{2j}{2^j}\right) + 2^{j-1}.
\end{aligned}$$

Since $j \geq 2$, we have

$$\left(1 - \frac{2j}{2^j}\right) \geq 0,$$

and therefore $\sum_{i \in A} u_i + a_n > 2^{|A|}$ as required. □

A signature move on a signature of $\mathcal{F}_n^{n+}$ is defined as follows.

Definition 5.6.7. Let $\mathcal{U} = (u_1, \ldots, u_n)$ be a signature of $\mathcal{F}_n^{n+}$. For $i, j \in [n]$, suppose that $u_j > 0$. Let $\mathcal{U}' = (u'_1, \ldots, u'_n)$ be such that $u'_i = u_i + 1$, $u'_j = u_j - 1$ and $u'_k = u_k$, for all $k \neq i, j$. We say that $\mathcal{U}$ is transformed into $\mathcal{U}'$ by a **signature move**.

The following lemma shows that $\mathcal{U}'$ is a signature of $\mathcal{F}_n^{n+}$.

Lemma 5.6.8. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered irreducible signature of Q_n and let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be an ordered splitting signature of $\mathcal{I}$ with respect to n. Let $\mathcal{U} = \mathcal{I} - \mathcal{D}$. Then the result $\mathcal{U}'$ of applying a signature move on $\mathcal{U}$ is a signature of $\mathcal{F}_n^{n+}$.*

Proof. Given $A \subseteq [n-1]$, we show that

$$\sum_{t \in A} u'_t + u'_n \geq 2^{|A|}. \tag{5.2}$$

In the notation of Definition 5.6.7, since $u'_i = u_i + 1$, $u'_j = u_j - 1$ and $u'_k = u_k$, for all $k \neq i, j$ we have

$$\sum_{t \in A} u'_t + u'_n = \begin{cases} (\sum_{t \in A} u_t + u_n) + 1 & \text{when } i \in A \cup \{n\}, \quad \text{but} \quad j \notin A \cup \{n\}; \\ (\sum_{t \in A} u_t + u_n) - 1 & \text{when } j \in A \cup \{n\}, \quad \text{but} \quad i \notin A \cup \{n\}; \\ \sum_{t \in A} u_t + u_n & \text{otherwise.} \end{cases}$$

If $u_n > 2^{|A|}$, then Equation (5.2) holds. If $u_n \leq 2^{|A|}$, then by Lemma 5.6.6 we have $\sum_{t \in A} u_t + u_n > 2^{|A|}$ and therefore Equation (5.2) holds. If $|A| = n-1$, then $\sum_{t=1}^{n-1} u'_t + u'_n = 2^{n-1}$ and therefore Equation (5.2) holds.

Thus, by the signature condition in Lemma 5.3.4 we conclude that $\mathcal{U}'$ is a signature of $\mathcal{F}_n^{n+}$ □

The following two lemmas are required to prove Lemma 5.6.11.

Lemma 5.6.9. *Let* $\nu = \lfloor \log_2 n \rfloor$. *Then* $\nu \leq \frac{n}{2}$ *for all* $n \geq 4$.

Proof. We use the fact easily proved by induction that $n^2 \leq 2^n$ for all $n \geq 4$. Then

$$\begin{aligned} 2\log_2 n &\leq n \\ \log_2 n &\leq \frac{n}{2} \\ \lfloor \log_2 n \rfloor &\leq \left\lfloor \frac{n}{2} \right\rfloor \\ &\leq \frac{n}{2}. \end{aligned}$$

Therefore $\nu \leq \frac{n}{2}$. □

Lemma 5.6.10. *Let* $n \geq 5$, *and suppose that* $\mathcal{S} = (a_1, \ldots, a_n)$ *is a signature of* Q_n *such that* $a_n \geq a_t$ *for all* t. *Let* $\nu = \lfloor \log_2 n \rfloor$. *Then* $a_n - \nu > 0$.

Proof. The average number of edges in each direction is $\frac{2^n-1}{n}$ and since $a_n \geq a_t$ for all t, we have $a_n \geq \frac{2^n-1}{n}$.

We use the fact easily proved by induction that $2^n - 1 > n^2$ for all $n \geq 5$. Then

$$\begin{aligned} a_n &\geq \frac{2^n - 1}{n} \\ &> n \\ &> \frac{n}{2} \\ &\geq \nu && \text{(by Lemma 5.6.9).} \end{aligned}$$

Therefore $a_n - \nu > 0$ for all $n \geq 5$. □

The following lemma shows that each irreducible signature of Q_n with a rich splitting signature has a super rich splitting signature.

Lemma 5.6.11. *Let* $n \geq 5$, *and let* $\mathcal{I} = (a_1, \ldots, a_n)$ *be an irreducible signature of* Q_n *such that* $a_n \geq a_t$ *for all* t. *Suppose that* $\mathcal{I}$ *has a rich splitting signature. Then* $\mathcal{I}$ *has a super rich splitting signature.*

Proof. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be a rich splitting signature of $\mathcal{I}$ and let $u_i = a_i - d_i$. Then $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for each μ such that there exists $i \leq \mu$ with $u_i \neq 0$. Let

$$\begin{aligned} Z(\mathcal{D}) &= \{t | u_t \geq a_n - \nu\}, \\ \lambda(\mathcal{D}) &= \sum_{t \in Z(\mathcal{D})} u_t. \end{aligned}$$

Among all rich splitting signatures of $\mathcal{I}$, let $\mathcal{D}$ be such that $\lambda(\mathcal{D})$ is minimal. If $\lambda(\mathcal{D}) = 0$, then $Z(\mathcal{D}) = \varnothing$ and therefore $\mathcal{D}$ is a super rich splitting signature. Suppose then that $\lambda(\mathcal{D}) \neq 0$, and let $i = \max Z(\mathcal{D})$. Note that

$$\begin{aligned} d_i &= a_i - u_i \\ &\leq a_n - u_i && \text{(since } a_n \geq a_t \text{ for all } t\text{)} \\ &\leq a_n - (a_n - \nu) && \text{(since } u_i \geq a_n - \nu\text{)} \\ &= \nu. \end{aligned}$$

We show that there is $j > i$ such that we can do signature moves from j to i on $\mathcal{D}$ and from i to j on $\mathcal{U}$ so as to decrease $\lambda(\mathcal{D})$.

To this end, suppose that $d_i = d_{i+1}$. Then $u_{i+1} + d_i = u_{i+1} + d_{i+1} = a_{i+1} \geq a_i = u_i + d_i$, so $u_{i+1} \geq u_i \geq a_n - \nu$. Therefore $i + 1 \in Z(\mathcal{D})$, contradicting the choice of i. So $d_{i+1} > d_i$.

We now show how to choose j. Note that for all $n \geq 5$, we have

$$\nu + 1 \leq \frac{n}{2} + 1 < n - 1.$$

Then by Lemma 3.2.2, we have $d_{n-1} \geq \nu + 2$, so there exists k such that $d_k \geq \nu + 2$. Let j be the least index such that $d_j \geq \nu + 2$. Then $d_j > d_{j-1}$, and

$$d_j - d_i \geq (\nu + 2) - \nu = 2.$$

Therefore all the conditions in Definition 5.6.1 hold. So we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_i = d_i + 1, \quad \hat{d}_j = d_j - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

for all $t \neq i, j$.

Since $u_i \geq a_n - \nu > 0$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_j = u_j + 1, \quad \hat{u}_i = u_i - 1 \geq 0 \quad \text{and} \quad \hat{u}_t = u_t$$

for all $t \neq i, j$. Then $\hat{\mathcal{D}} + \hat{\mathcal{U}} = \mathcal{I}$ and therefore $\hat{\mathcal{D}}$ is an ordered irreducible splitting signature of $\mathcal{I}$. Moreover $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}}$ for all μ and $\hat{u}_t > 0$ only if $u_t > 0$ or $t = j > i$, and therefore $\hat{\mathcal{D}}$ is rich.

Now we have $\hat{d}_j = d_j - 1 \geq (\nu + 2) - 1 > \nu$. Therefore we either have $Z(\hat{\mathcal{D}}) = Z(\mathcal{D})$ or $Z(\hat{\mathcal{D}}) = Z(\mathcal{D}) \setminus \{i\}$. Then

$$\begin{aligned}\lambda(\hat{\mathcal{D}}) = \sum_{t \in Z(\hat{\mathcal{D}})} \hat{u}_t &\leq \sum_{t \in Z(\mathcal{D})} \hat{u}_t \\ &= \lambda(\mathcal{D}) - 1.\end{aligned}$$

This contradicts the choice of $\mathcal{D}$. Then $\mathcal{D}$ must in fact satisfy $\lambda(\mathcal{D}) = 0$ and therefore $Z(\mathcal{D}) = \varnothing$, so $\mathcal{D}$ is super rich. $\square$

The next lemma is used in the process of proving Proposition 5.6.13.

Lemma 5.6.12. *Let $n \geq 5$, and let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be a rich splitting signature of $\mathcal{I}$. Suppose there exists $f, b \in [n-1]$ with $f < b$ such that*

$$d_f < d_{f+1} \quad \text{and} \quad d_{b-1} < d_b.$$

Suppose there is $\mu \geq f$ such that $\varepsilon_\mu^{\mathcal{D}} = 1$. Then there is $\ell \geq b$ such that there is a signature move on $\mathcal{D}$ from ℓ to f and the resulting signature $\hat{\mathcal{D}}$ satisfies $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq f$.

Proof. Let ℓ be the least index greater than or equal to b such that $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $\mu \geq \ell$. If there is no such index, then let $\ell = n - 1$. We claim that $d_{\ell-1} < d_\ell$. We consider the following cases according to whether $b = \ell < n - 1$; $b < \ell < n - 1$; or $\ell = n - 1$.

Suppose that $b = \ell < n-1$. Then $d_{\ell-1} < d_\ell$ by hypothesis.

Suppose that $b < \ell < n-1$. Since ℓ is the least index greater than or equal to b such that $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $\mu \geq \ell$, we must have $d_{\ell-1} < d_\ell$ by Lemma 3.2.5.

Suppose that $\ell = n-1$. Then $\varepsilon_{n-2}^{\mathcal{D}} = 1$ and therefore $d_{n-2} \leq 2^{n-3}$, so $d_\ell = d_{n-1} = 2^{n-2} = 2^{\ell-1}$ and $d_{\ell-1} \leq 2^{\ell-2}$. So $d_{\ell-1} < d_\ell$.

In all cases we showed we have $d_{\ell-1} < d_\ell$. We now claim that $d_\ell - d_f \geq 2$. If $\ell > f+1$, then $d_\ell - d_f \geq 2$ is immediate from $d_f < d_{f+1}$ and $d_{\ell-1} < d_\ell$.

Suppose $\ell = b = f+1$. Since there exists $\mu \geq f$ such that $\varepsilon_\mu^{\mathcal{D}} = 1$ by hypothesis, and since $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $\mu \geq \ell = f+1$, we must have $\varepsilon_f^{\mathcal{D}} = 1$. Then by Lemma 3.2.5 we have $d_f < d_{f+1} - 1$ and therefore $d_\ell - d_f = d_{f+1} - d_f \geq 2$.

In all cases we showed $d_\ell - d_f \geq 2$. Then all the conditions in Definition 5.6.1 hold and therefore we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_f = d_f + 1, \quad \hat{d}_\ell = d_\ell - 1 \quad \text{and} \quad \hat{d}_t = d_t$$

for all $t \neq f, \ell$. Now we have $\varepsilon_\mu^{\hat{\mathcal{D}}} = \varepsilon_\mu^{\mathcal{D}} + 1 \geq 2$ for $f \leq \mu < \ell$ and $\varepsilon_\mu^{\hat{\mathcal{D}}} = \varepsilon_\mu^{\mathcal{D}} \geq 2$ for $\mu \geq \ell$ by the choice of ℓ, and so $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq f$. □

The following lemma plays an important role in an inductive proof that every irreducible signature has an amenable splitting signature.

Proposition 5.6.13. *Let $\mathcal{I} = (a_1, \dots, a_n)$ be an irreducible signature of Q_n and let $\hat{\mathcal{I}} = (\hat{a}_1, \dots, \hat{a}_n)$ be the result of a signature move on $\mathcal{I}$. Suppose that $\mathcal{I}$ has an amenable splitting signature $\mathcal{D} = (d_1, \dots, d_{n-1})$. Then $\hat{\mathcal{I}}$ has an amenable splitting signature $\hat{\mathcal{D}} = (\hat{d}_1, \dots, \hat{d}_{n-1})$.*

Proof. Let $\mathcal{U} = (u_1, \dots, u_{n-1}, u_n)$ be such that $u_t = a_t - d_t$ for all $t \in [n]$. Let $i < j$ with $a_j - a_i \geq 2$ be the indices involved in the signature move. Then by Definition 5.6.1, we have $a_i < a_{i+1}$ and $a_{j-1} < a_j$.

By Lemma 5.6.11 it suffices to show that if $\mathcal{I}$ has a unidirectional or rich splitting signature then so does $\hat{\mathcal{I}}$. We consider the following cases according to whether $\mathcal{D}$ is a unidirectional splitting signature or a rich splitting signature.

1. Suppose that $\mathcal{D}$ is a unidirectional splitting signature with $u_k \neq 0$. Since $\mathcal{I}$ is irreducible, we have $u_n \leq 2^{n-1} - 1$ and therefore $k \neq n$. We distinguish the following cases according to whether or not $k \in \{i+1, j\}$.

   (a) Suppose that $k \notin \{i+1, j\}$. **Suppose first that** $j \neq n$. Then

   $$d_i \leq a_i < a_{i+1} = d_{i+1} \quad \text{and} \quad d_{j-1} \leq a_{j-1} < a_j = d_j.$$

   Since $a_j - a_i \geq 2$ and since $d_i \leq a_i$ and $d_j = a_j$, we have $d_j - d_i \geq 2$ and therefore all the conditions in Definition 5.6.1 hold. Then we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

   $$\hat{d}_i = d_i + 1, \quad \hat{d}_j = d_j - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

   for all $t \neq i, j$. Let $\hat{\mathcal{U}} = \mathcal{U}$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\mathcal{U}$ is unchanged, we have $\hat{\mathcal{D}}$ is a unidirectional splitting signature. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

Suppose now that $j = n$**. Suppose first that** either $k = i$, or $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $\mu \geq i$ (if $i < k$), or $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $\mu \geq k$ (if $k < i$). Since $u_n > 0$, all the conditions in Definition 5.6.7 hold. Then we may apply a signature move on $\mathcal{U}$ to transform it into $\hat{\mathcal{U}}$ with

$$\hat{u}_i = u_i + 1, \quad \hat{u}_n = u_n - 1 \quad \text{and} \quad \hat{u}_t = u_t$$

for all $t \neq i, n$. Let $\hat{\mathcal{D}} = \mathcal{D}$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. If $k \neq i$, then $\hat{\mathcal{D}}$ is rich splitting signature because $u_t \neq 0$ for $t = i, k, n$ and $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq i$ (if $i < k$) or $\mu \geq k$ (if $k < i$). If $k = i$, then $\hat{\mathcal{D}}$ is a unidirectional splitting signature because $u_t = 0$ for $t \neq i, n$. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

Suppose now that $k \neq i$ and $\varepsilon_\mu^{\mathcal{D}} = 1$ for some $\mu \geq \min\{i, k\}$. We distinguish the following cases according to whether or not $k < i$

Suppose first that $k > i$. Since $d_k < a_k \leq a_{k+1} = d_{k+1}$ and $d_i = a_i < a_{i+1} = d_{i+1}$, we have $d_k < d_{k+1}$ and $d_i < d_{i+1}$ and therefore all the conditions in Lemma 5.6.12 hold with $f = i$ and $b = k+1$. Choose $\ell \geq k+1$ as in Lemma 5.6.12. Then applying the lemma signature $\mathcal{D}$ can be transformed into $\hat{\mathcal{D}}$ where $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq i$.

Since $u_n > 0$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_\ell = u_\ell + 1, \quad \hat{u}_n = u_n - 1 \quad \text{and} \quad \hat{u}_t = u_t$$

for all $t \neq \ell, n$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\ell \neq k$, we have $\hat{u}_t = 0$ for all $t \neq \ell, k, n$ and therefore $\hat{\mathcal{D}}$ is a rich splitting signature, because $k, \ell > i$. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

Suppose now that $k < i$**.** Since $d_k < a_k \leq a_{k+1} = d_{k+1}$ and $d_i = a_i < a_{i+1} = d_{i+1}$, we have $d_k < d_{k+1}$ and $d_i < d_{i+1}$ and therefore all the conditions in Lemma 5.6.12 hold with $f = k$ and $b = i+1$. Choose $\ell \geq i+1$ as given in Lemma 5.6.12. Then applying the lemma $\mathcal{D}$ can be transformed into $\hat{\mathcal{D}}$ with

$$\hat{d}_k = d_k + 1, \quad \hat{d}_\ell = d_\ell - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

for all $t \neq k, \ell$. Then $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq k$.

Since $u_k > 0$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\mathcal{U}'$ with

$$u'_\ell = u_\ell + 1, \quad u'_k = u_k - 1 \quad \text{and} \quad u'_t = u_t$$

for all $t \neq k, \ell$. Observe that $\mathcal{U}' + \hat{\mathcal{D}} = \mathcal{I}$, so $\hat{\mathcal{D}}$ is a splitting signature for $\mathcal{I}$.

Since $u_n > 0$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}'$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_i = u_i + 1, \quad \hat{u}_n = u_n - 1 \quad \text{and} \quad \hat{u}_t = u_t$$

for all $t \neq i, n$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\hat{u}_t = 0$ for all $t \notin \{i, \ell, k, n\}$, $k < i < \ell$ and $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq k$, $\hat{\mathcal{D}}$ is a rich splitting signature. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

(b) Suppose that $k = i+1$ and $i+1 \neq j$. Then

$$a_i = d_i \leq d_{i+1} < a_{i+1} \leq d_{j-1} = a_{j-1} < a_j = d_j,$$

and therefore $d_j - d_i \geq 2$. We distinguish the following cases according to whether or not $d_i = d_{i+1}$.

(I) Suppose that $d_i < d_{i+1}$. Since $d_i < d_{i+1}$, $d_{j-1} < d_j$ and $d_j - d_i \geq 2$, all the conditions in Definition 5.6.1 hold. Then we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_i = d_i + 1, \quad \hat{d}_j = d_j - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

for all $t \neq i, j$. Let $\hat{\mathcal{U}} = \mathcal{U}$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$, so $\hat{\mathcal{D}}$ is a splitting signature for $\hat{\mathcal{I}}$. Since $\mathcal{U}$ is unchanged and $\hat{\mathcal{D}}$ is an ordered irreducible splitting signature, we have $\hat{\mathcal{D}}$ is a unidirectional splitting signature. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

(II) Suppose that $d_i = d_{i+1}$. We distinguish the following cases according to whether or not $u_{i+1} = 1$.

(i) Suppose that $u_{i+1} = 1$. Since $a_i = d_i = d_{i+1} < a_{i+1} \leq a_{i+2} = d_{i+2}$, we have $d_{i+1} < d_{i+2}$. Since $d_i = d_{i+1}$ and $d_j - d_i = a_j - a_i \geq 2$, we have $d_j - d_{i+1} \geq 2$. Then all the conditions in Definition 5.6.2 hold. Therefore we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_{i+1} = d_{i+1} + 1, \quad \hat{d}_j = d_j - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

for all $t \neq i+1, j$.

Since $u_{i+1} = 1$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_{i+1} = u_{i+1} - 1 = 0, \quad \hat{u}_i = u_i + 1 = 1 \quad \text{and} \quad \hat{u}_t = u_t$$

for all $t \neq i, i+1$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\hat{u}_t = 0$ for all $t \neq i, n$, $\hat{\mathcal{D}}$ is a unidirectional splitting signature. Therefore $\hat{\mathcal{D}}$ is an amenable splitting signature for $\hat{\mathcal{I}}$.

(ii) Suppose that $u_{i+1} > 1$. We distinguish the following cases according to whether $i \neq 1$; $i = 1$ and $d_1 = d_2 \geq 3$; or $i = 1$ and $d_1 = d_2 = 2$.

(A) Suppose that $i \neq 1$; or $i = 1$ and $d_1 = d_2 \geq 3$. If $i = 1$ and $d_1 = d_2 \geq 3$, then $\varepsilon_1^{\mathcal{D}} \geq 2$, and otherwise $\varepsilon_i^{\mathcal{D}} \geq 2$ by Lemma 3.2.5. In either case, $\varepsilon_i^{\mathcal{D}} \geq 2$. Since $d_{i+1} = d_i = a_i < a_{i+1} \leq a_{i+2} = d_{i+2}$, we have $d_{i+1} < d_{i+2}$. Since $d_{j-1} = a_{j-1} < a_j = d_j$, we have $d_{j-1} < d_j$. Since $d_{i+1} < d_{i+2}$ and $d_{j-1} < d_j$, all the conditions in Lemma 5.6.12 hold with $f = i+1$ and $b = j$. Choose $\ell \geq j$ as given in Lemma 5.6.12. Then applying the lemma $\mathcal{D}$ can be transformed into $\mathcal{D}'$ with

$$d'_{i+1} = d_{i+1} + 1, \quad d'_\ell = d_\ell - 1 \quad \text{and} \quad d'_t = d_t,$$

for all $t \neq i+1, \ell$. Therefore $\varepsilon_\mu^{\mathcal{D}'} \geq 2$ for all $\mu \geq i+1$. Since $\varepsilon_i^{\mathcal{D}'} = \varepsilon_i^{\mathcal{D}} \geq 2$, we have $\varepsilon_\mu^{\mathcal{D}'} \geq 2$ for all $\mu \geq i$.

Suppose first that $\ell = j$. Since $u_{i+1} > 1$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_i = u_i + 1, \quad \hat{u}_{i+1} = u_{i+1} - 1 \quad \text{and} \quad \hat{u}_t = u_t,$$

for all $t \neq i, i+1$. Then $\hat{\mathcal{U}} + \mathcal{D}' = \hat{\mathcal{I}}$. Since $\hat{u}_t = 0$ unless $t = i, i+1, n$ and $\varepsilon_\mu^{\mathcal{D}'} \geq 2$ for all $\mu \geq i$, $\mathcal{D}'$ is a rich splitting signature. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

Suppose now that $\ell \neq j$. Since $u_{i+1} > 1$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_{i+1} = u_{i+1} - 1, \quad \hat{u}_\ell = u_\ell + 1 \quad \text{and} \quad \hat{u}_t = u_t,$$

for all $t \neq i+1, \ell$. Observe that $\mathcal{D}' + \hat{\mathcal{U}} = \mathcal{I}$, so $\mathcal{D}'$ is a splitting signature for $\mathcal{I}$.

Since $d'_i = d_i = d_{i+1} < d'_{i+1} \leq d'_{j-1} = d_{j-1} < d_j = d'_j$, we have $d'_j - d'_i \geq 2$ and therefore all the conditions in Definition 5.6.1 hold. Therefore we may apply a signature move on $\mathcal{D}'$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_i = d'_i + 1, \quad \hat{d}_j = d'_j - 1 \quad \text{and} \quad \hat{d}_t = d'_t,$$

for all $t \neq i, j$. Therefore $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}'} \geq 2$ for all $\mu \geq i$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\hat{u}_t = 0$ unless $t = i+1, \ell, n$ and $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq i$, $\hat{\mathcal{D}}$ is a rich splitting signature. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

(B) Suppose that $i = 1$ and $d_1 = d_2 = 2$. Then $\varepsilon_1^{\mathcal{D}} = \varepsilon_2^{\mathcal{D}} = 1$. Since $\mathcal{D}$ is irreducible and $d_2 = 2$, we have $d_3 \geq 4$. Then $d_2 < d_3 - 1$, and therefore all the conditions in Lemma 5.6.12 hold with $f = 2$ and $b = 3$. Choose $\ell \geq 3$ as in Lemma 5.6.12. Then applying the lemma $\mathcal{D}$ can be transformed into $\mathcal{D}'$ with

$$d'_2 = d_2 + 1 = 3, \quad d'_\ell = d_\ell - 1 \quad \text{and} \quad d'_t = d_t,$$

for all $t \neq 2, \ell$. Therefore $\varepsilon_\mu^{\mathcal{D}'} \geq 2$ for all $\mu \geq 2$.

Suppose first that $\ell = j$. Since $d'_{j+1} = d_{j+1} \geq d_j > d'_j$; $d'_1 = d_1 = 2 < 3 = d'_2$ and $d_j - d_1 \geq 2$, we have $d'_{j+1} - d'_1 \geq 2$. Then all the conditions in Definition 5.6.1 hold. Therefore we may apply a signature move on $\mathcal{D}'$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_1 = d'_1 + 1, \quad \hat{d}_{j+1} = d'_{j+1} - 1 \quad \text{and} \quad \hat{d}_t = d'_t,$$

for all $t \neq 1, j+1$.

Since $u_2 > 1$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_2 = u_2 - 1, \quad \hat{u}_{j+1} = u_{j+1} + 1 \quad \text{and} \quad \hat{u}_t = u_t,$$

for all $t \neq 2, j+1$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}'}$ for all μ and since $\varepsilon_1^{\hat{\mathcal{D}}} = 2$ and $\varepsilon_\mu^{\mathcal{D}'} \geq 2$ for all $\mu \geq 2$, we have $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq 1$. Thus $\hat{\mathcal{D}}$ is a rich splitting signature. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

Suppose now that $\ell \neq j$. Since $u_2 > 1$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_2 = u_2 - 1, \quad \hat{u}_\ell = u_\ell + 1 \quad \text{and} \quad \hat{u}_t = u_t,$$

for all $t \neq 2, \ell$. Observe that $\mathcal{D}' + \hat{\mathcal{U}} = \mathcal{I}$, so $\mathcal{D}'$ is a splitting signature for $\mathcal{I}$.
Since $d'_1 = d_1 = d_2 < d'_2$ and $d'_{j-1} = d_{j-1} < d_j = d'_j$, we have $d'_j - d'_1 \geq 2$ and therefore all the conditions in Definition 5.6.1 hold. Therefore we may apply a signature move on $\mathcal{D}'$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_1 = d'_1 + 1 = 3, \quad \hat{d}_j = d'_j - 1 \quad \text{and} \quad \hat{d}_t = d'_t,$$

for all $t \neq 1, j$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}'}$ for all μ and since $\varepsilon_1^{\hat{\mathcal{D}}} = 2$ and $\varepsilon_\mu^{\mathcal{D}'} \geq 2$ for all $\mu \geq 2$, we have $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq 1$. Thus $\hat{\mathcal{D}}$ is a rich splitting signature. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

(c) Suppose that $k = j$, allowing the possibility that $j = k = i + 1$. We distinguish the following cases according to whether or not $u_j = 1$.

(I) Suppose that $u_j = 1$. Then the condition in Definition 5.6.7 holds. Therefore we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_i = u_i + 1, \quad \hat{u}_j = u_j - 1 \quad \text{and} \quad \hat{u}_t = u_t,$$

for all $t \neq i, j$. Let $\mathcal{D} = \hat{\mathcal{D}}$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

(II) Suppose that $u_j > 1$. **Suppose first that** either $j \neq i + 1$, or $j = i + 1$ such that $d_i < d_{i+1}$. Since $d_i < d_{i+1}$ (by assumption when $j = i + 1$, and because $d_i = a_i < a_{i+1} = d_{i+1}$ otherwise) and since $d_j < d_{j+1}$ because $d_j < a_j \leq a_{j+1} = d_{j+1}$, all the conditions in Lemma 5.6.12 hold with $f = i$ and $b = j + 1$. Choose $\ell \geq j + 1$ as in Lemma 5.6.12. Then applying the lemma $\mathcal{D}$ is transformed into $\hat{\mathcal{D}}$ with

$$\hat{d}_i = d_i + 1, \quad \hat{d}_\ell = d_\ell - 1 \quad \text{and} \quad \hat{d}_t = d_t$$

for all $t \neq i, \ell$. Therefore $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq i$.
Since $u_j > 1$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_j = u_j - 1, \quad \hat{u}_\ell = u_\ell + 1 \quad \text{and} \quad \hat{u}_t = u_t,$$

for all $t \neq j, \ell$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq 2$ for all $\mu \geq i$ and $\hat{u}_t = 0$ unless $t \in \{j, \ell, n\}$, we have $\hat{\mathcal{D}}$ is a rich splitting signature. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.
Suppose now that $j = i + 1$ such that $d_i = d_{i+1}$. Since $d_{i+1} < a_{i+1} \leq a_{i+2} = d_{i+2}$, we have $d_{i+1} < d_{i+2} - 1$ and therefore all the conditions in Lemma 5.6.12 hold with $f = i + 1$ and $b = i + 2$. Choose $\ell \geq i + 2$ as in Lemma 5.6.12. Then applying the lemma $\mathcal{D}$ can be transformed into $\mathcal{D}'$ with

$$d'_{i+1} = d_{i+1} + 1, \quad d'_\ell = d_\ell - 1 \quad \text{and} \quad d'_t = d_t$$

for all $t \neq i + 1, \ell$. Therefore $\varepsilon_\mu^{\mathcal{D}'} \geq 2$ for all $\mu \geq i + 1$.

Since $u_{i+1} > 1$, the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\mathcal{U}'$ with

$$u'_{i+1} = u_{i+1} - 1, \quad u'_\ell = u_\ell + 1 \quad \text{and} \quad u'_t = u_t,$$

for all $t \neq i+1, \ell$. Then $\mathcal{U}' + \hat{\mathcal{D}} = \mathcal{I}$, so $\hat{\mathcal{D}}$ is a rich splitting signature for $\mathcal{I}$ and therefore we can move to Case 2 below.

2. Suppose that $\mathcal{D}$ is a rich splitting signature. Let k be the first index such that $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $\mu \geq k$. Note that such k exists because $\mathcal{D}$ is rich. We distinguish the following cases according to whether $k \leq i$ or $k > i$.

(a) Suppose that $k \leq i$. We distinguish the following cases according to whether $u_j = 0$.

(I) Suppose that $u_j \neq 0$. Then the condition in Definition 5.6.7 holds. Therefore we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_i = u_i + 1, \quad \hat{u}_j = u_j - 1 \quad \text{and} \quad \hat{u}_t = u_t,$$

for all $t \neq i, j$. Let $\mathcal{D} = \hat{\mathcal{D}}$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$, so $\hat{\mathcal{D}}$ is a rich splitting signature for $\hat{\mathcal{I}}$. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

(II) Suppose that $u_j = 0$. Then $j < n$ and

$$d_{j-1} \leq a_{j-1} < a_j = d_j.$$

Since $d_i \leq a_i < a_j = d_j$ and since $a_j - a_i \geq 2$, we have $d_j - d_i \geq 2$. We distinguish the following cases according to whether or not $d_i = d_{i+1}$.

(i) Suppose that $d_i < d_{i+1}$. Since $d_i < d_{i+1}$, $d_{j-1} < d_j$ and $d_j - d_i \geq 2$, all the conditions in Definition 5.6.1 hold. Then we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_i = d_i + 1, \quad \hat{d}_j = d_j - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

for all $t \neq i, j$. Then $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}}$ for all μ. Let $\hat{\mathcal{U}} = \mathcal{U}$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\mathcal{U}$ is unchanged and $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}}$ for all μ, we have $\hat{\mathcal{D}}$ is a rich splitting signature for $\hat{\mathcal{I}}$. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

(ii) Suppose that $d_i = d_{i+1}$. Let $\ell \geq i$ be the largest index such that $d_\ell = d_i$. Then $d_\ell < d_{\ell+1}$. Since $d_\ell = d_i$ and $d_j - d_i \geq 2$, we have $d_j - d_\ell \geq 2$. Since $d_\ell < d_{\ell+1}$, $d_{j-1} < d_j$ and $d_j - d_\ell \geq 2$, all the conditions in Definition 5.6.1 hold. Then we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_\ell = d_\ell + 1, \quad \hat{d}_j = d_j - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

for all $t \neq j, \ell$. Then $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}}$ for all μ.
Since $a_i < a_{i+1} \leq \cdots \leq a_\ell$ and $d_i = d_\ell$, we have $u_\ell > 0$ and therefore the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_i = u_i + 1 \quad , \hat{u}_\ell = u_\ell - 1 \quad \text{and} \quad \hat{u}_t = u_t,$$

for all $t \neq i, \ell$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\hat{u}_t$ is unchanged unless $t = i, \ell \geq k$ and since $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}}$ for all μ, we have $\hat{\mathcal{D}}$ is a rich splitting signature for $\hat{\mathcal{I}}$. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

(b) Suppose that $k > i$. Then $\varepsilon_\mu^{\mathcal{D}} = 1$ for some $i \le \mu < k$. Since $\mathcal{D}$ is rich and $\varepsilon_\mu^{\mathcal{D}} = 1$ for some $i \le \mu < k$, we have $d_{k-1} < d_k - 1$ by Lemma 3.2.5.

Suppose first that $j = n$. Then $k \ne j$. Since $d_i \le d_{k-1} < d_k - 1$ and $i < k$, we have $d_k - d_i \ge 2$. Since $\mathcal{D}$ is rich and $k > i$ is the least index such that $\varepsilon_\mu^{\mathcal{D}} \ge 2$ for all $\mu \ge k$, we have $u_\ell = 0$ for $\ell < k$ and therefore $d_\ell = a_\ell$ for $\ell < k$. If $i < k-1$, then $d_i = a_i < a_{i+1} = d_{i+1}$. If $i = k-1$, then $d_i < d_{i+1}$ by the choice of k. Since $d_i < d_{i+1}$, $d_{k-1} < d_k$ and $d_k - d_i \ge 2$, all the conditions in Definition 5.6.1 hold. Then we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_i = d_i + 1, \quad \hat{d}_k = d_k - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

for all $t \ne i, k$. Then $\varepsilon_\mu^{\hat{\mathcal{D}}} \ge \varepsilon_\mu^{\mathcal{D}}$ for all μ. Since $u_n > 0$, the condition in Definition 5.6.1 holds. Then may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_k = u_k + 1, \quad \hat{u}_n = u_n - 1 \quad \text{and} \quad \hat{u}_t = u_t,$$

for all $t \ne k, n$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\varepsilon_\mu^{\hat{\mathcal{D}}} \ge \varepsilon_\mu^{\mathcal{D}}$ for all μ and $\hat{u}_t$ is unchanged unless $t = k$, $\hat{\mathcal{D}}$ is a rich splitting signature for $\hat{\mathcal{I}}$. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

Suppose now that $j \ne n$. We distinguish the following cases according to whether $k \le j$ or $k > j$.

(i) Suppose that $k \le j$. Since k is the first index such that $\varepsilon_\mu^{\mathcal{D}} \ge 2$ for all $\mu \ge k$, we have $d_k - d_{k-1} \ge 2$ by Lemma 3.2.5. Since $i < k \le j$, we have $d_j - d_i \ge 2$. We distinguish the following cases according to where $d_{j-1} < d_j$ or $d_{j-1} = d_j$.

(A) Suppose that $d_{j-1} < d_j$. Since $d_i < d_{i+1}$, $d_{j-1} < d_j$ and $d_j - d_i \ge 2$ all the conditions in Definition 5.6.1 hold. Then we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_i = d_i + 1, \quad \hat{d}_j = d_j - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

for all $t \ne i, j$. Then $\varepsilon_\mu^{\hat{\mathcal{D}}} \ge \varepsilon_\mu^{\mathcal{D}}$ for all μ. Let $\hat{\mathcal{U}} = \mathcal{U}$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\mathcal{U}$ is unchanged and $\varepsilon_\mu^{\hat{\mathcal{D}}} \ge \varepsilon_\mu^{\mathcal{D}}$ for all μ, we have $\hat{\mathcal{D}}$ is a rich splitting signature for $\hat{\mathcal{I}}$. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

(B) Suppose that $d_{j-1} = d_j$. Let ℓ be the least index such that $d_\ell = d_j$. Then $\ell \ge k$ because by Lemma 3.2.5 we have $\varepsilon_\mu^{\mathcal{D}} \ge 2$ for all $\ell \le \mu < j$ and $k \le j$ is the least index such that $\varepsilon_\mu^{\mathcal{D}} \ge 2$ for all $\mu \ge k$. Since ℓ is the least index such that $d_\ell = d_j$, we have $d_{\ell-1} < d_\ell$. Since $d_\ell = d_j$ and $d_j - d_i \ge 2$, we have $d_\ell - d_i \ge 2$. Since $d_i < d_{i+1}$, $d_{\ell-1} < d_\ell$ and $d_\ell - d_i \ge 2$ all the conditions in Definition 5.6.1 hold. Then we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_i = d_i + 1, \quad \hat{d}_\ell = d_\ell - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

for all $t \ne i, \ell$. Then $\varepsilon_\mu^{\hat{\mathcal{D}}} \ge \varepsilon_\mu^{\mathcal{D}}$ for all μ.

Since $d_j = d_{j-1} \le a_{j-1} < a_j$, we have $u_j > 0$ and therefore the condition in Definition 5.6.7 holds. Then we may apply a signature move on $\mathcal{U}$ to transform it into the signature $\hat{\mathcal{U}}$ with

$$\hat{u}_\ell = u_\ell + 1, \quad \hat{u}_j = u_j - 1 \quad \text{and} \quad \hat{u}_t = u_t,$$

for all $t \neq \ell, j$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\hat{u}_t$ is unchanged unless $t = j, \ell \geq k$ and $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}}$ for all μ, we have $\hat{\mathcal{D}}$ is a rich splitting signature for $\hat{\mathcal{I}}$. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

(ii) Suppose that $k > j$. Then $a_t = d_t$ for all $t < k$. Therefore $d_i = a_i < a_{i+1} = d_{i+1} \leq \cdots \leq d_{j-1} = a_{j-1} < a_j = d_j$. Since $a_j - a_i \geq 2$, $d_i = a_i$ and $d_j = a_j$, we have $d_j - d_i \geq 2$. Than all the conditions in Definition 5.6.1 hold. Then we may apply a signature move on $\mathcal{D}$ to transform it into the ordered irreducible signature $\hat{\mathcal{D}}$ with

$$\hat{d}_i = d_i + 1, \quad \hat{d}_j = d_j - 1 \quad \text{and} \quad \hat{d}_t = d_t,$$

for all $t \neq i, j$. Then $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}}$ for all μ. Let $\hat{\mathcal{U}} = \mathcal{U}$. Then $\hat{\mathcal{U}} + \hat{\mathcal{D}} = \hat{\mathcal{I}}$. Since $\mathcal{U}$ is unchanged and $\varepsilon_\mu^{\hat{\mathcal{D}}} \geq \varepsilon_\mu^{\mathcal{D}}$ for all μ, we have $\hat{\mathcal{D}}$ is a rich splitting signature for $\hat{\mathcal{I}}$. Therefore $\hat{\mathcal{I}}$ has an amenable splitting signature.

In all cases we showed that $\hat{\mathcal{I}}$, which is the result of a signature move on $\mathcal{I}$, has an amenable splitting signature as required. □

The following theorem follows from Lemma 5.5.4, Lemma 5.6.5 and Proposition 5.6.13.

Theorem 5.6.14. *For $n \geq 5$ every ordered irreducible signature of Q_n has an amenable splitting signature.*

Proof. By Lemma 5.5.4, signature $\mathcal{I}_n^{(-1)}$ has a unidirectional splitting signature and hence an amenable splitting signature. The signature $\mathcal{I}_n^{(-1)}$ can be transformed into any other ordered irreducible signature by Lemma 5.6.5. So given any irreducible signature $\mathcal{I}'$ there is a sequence of signature moves transforming $\mathcal{I}_n^{(-1)}$ into $\mathcal{I}'$. By Proposition 5.6.13 if an irreducible signature $\mathcal{I}$ has an amenable splitting signature, then signature $\hat{\mathcal{I}}$, which is the result of a signature move on $\mathcal{I}$, has an amenable splitting signature. Therefore we conclude that every ordered irreducible signature of Q_n has an amenable splitting signature. □

5.7 Summary map

In this chapter, we proved the connectivity of the edge slide graph $\mathcal{E}(2, 3, 3, 7)$ which we use in Chapter 10 to prove the connectivity of the edge slide graph of a family of irreducible signatures of Q_n. Also in Chapter 10 we study in detail the signature $\mathcal{I}_n^{(-1)}$. We proved that every ordered irreducible signature has an amenable splitting signature. In Chapter 7, we give a complete proof that the set of upright spanning trees of Q_n with a fixed connected unidirectional splitting signature or the (2, 2, 3) splitting signature of Q_4 forms a block. We conjecture under an inductive hypothesis this is also true for a super rich splitting signature and give partial progress towards the conjecture. These results along with Theorem 5.6.14 are used to understand the connectivity of the edge slide graph of an irreducible signature of Q_n. In the following chapters we split the signatures of Q_n in direction n unless stated otherwise.

Chapter 6

Reducible signatures of Q_n

6.1 Introduction

In this chapter we study reducible signatures, which were introduced in Section 3.2, in detail in order to understand the structure of the edge slide graph associated with this type of signature.

Reducible signatures of Q_n can be divided into two types: **strictly reducible** and **quasi-irreducible** signatures (defined below). We first introduce the notion of **saturated** and **unsaturated** signatures as follows.

Definition 6.1.1. Let $\mathcal{S} = (a_1, \ldots, a_n)$ be an ordered signature of Q_n. Let r be the largest index such that $a_r \neq 2^{r-1}$. Then $(a_1, \ldots, a_r)$ is necessarily a signature of Q_r, and is the **unsaturated part** of $\mathcal{S}$. If $r = n$ then $\mathcal{S}$ is **unsaturated signature**, and otherwise $\mathcal{S}$ is **saturated above direction** r. In other words, all signatures have an unsaturated part, which could be the entire signature. A (not-necessarily ordered) signature is saturated or unsaturated according to whether its ordered permutation is saturated or unsaturated.

If the ordered signature $\mathcal{S}$ is saturated above direction r, then for $r \leq s \leq n$ we have

$$\begin{aligned}\sum_{i=1}^{s} a_i &= (2^n - 1) - \sum_{i=s+1}^{n} a_i \\ &= (2^n - 1) - \sum_{i=s+1}^{n} 2^{i-1} \\ &= 2^s - 1.\end{aligned}$$

It follows that $(a_1, \ldots, a_s)$ is a signature of Q_s, and hence $[s]$ is a reducing set for $r \leq s \leq n$. A saturated signature is therefore necessarily reducible. Observe that the unsaturated part of S is in fact an unsaturated signature.

Example 6.1.2. The signature (2, 2, 3, 8, 16) is a saturated signature of Q_5, because the first three entries form a signature of Q_3 followed by $a_4 = 8 = 2^{4-1}$ and $a_5 = 16 = 2^{5-1}$. The signature (2, 2, 3, 9, 15) is an unsaturated signature of Q_5, because $a_5 \neq 2^{5-1}$.

In other words, if $\mathcal{S}$ is ordered, then $\mathcal{S}$ is saturated if and only if it consists of a signature of a lower dimensional cube followed by an increasing sequence of consecutive powers of two.

Reducible signatures can be divided into quasi-irreducible and strictly reducible signatures as follows.

Definition 6.1.3. Let $\mathcal{S} = (a_1, \dots, a_n)$ be an ordered reducible signature of Q_n. Then $\mathcal{S}$ is said to be **quasi-irreducible** if the unsaturated part $(a_1, \dots, a_r)$ is irreducible. Otherwise $\mathcal{S}$ is **strictly reducible**.

Example 6.1.4. The signature (2, 3, 3, 8, 15, 32, 64) is a quasi-irreducible signature of Q_7 because the unsaturated part consists of the first five entries, which form an irreducible signature of Q_5. The signature (2, 2, 3, 9, 15, 32, 64) is a strictly reducible signature of Q_7 because the unsaturated part consists of the first five entries, which form a reducible signature of Q_5.

We define a **supersaturated** signature as follows.

Definition 6.1.5. A signature $(a_1, \dots, a_n)$ is **supersaturated** if it is a permutation of the signature $(1, 2, 4, 8, \dots, 2^{n-1})$. It is quasi-irreducible by taking $r = \{1\}$ as an irreducible signature of Q_1 followed by consecutive powers of two.

In other words, a signature that is saturated at each direction is called a supersaturated signature. Observe that the supersaturated signature has excess $\varepsilon_k = 0$ for all k.

The following theorem is the main result of this chapter.

Theorem 6.1.6. *Let $\mathcal{S} = (a_1, \dots, a_n)$ be a strictly reducible signature of Q_n. Then the edge slide graph $\mathcal{E}(\mathcal{S})$ is disconnected.*

The idea is to construct an invariant on trees with signature $\mathcal{S}$ that is preserved by edge slides but is not constant on all trees with that signature. Lemma 6.2.9 shows that if $\mathcal{S} = (a_1, \dots, a_n)$ is a reducible signature of a tree T and $R \subseteq [n]$ is a reducing set for $\mathcal{S}$ then deleting the edges in directions R breaks T into $2^{|R|}$ spanning trees of copies of $Q_{n-|R|}$. Lemma 6.2.11 shows the signatures of these subtrees are invariant under edge slides, and Lemma 6.2.13 shows that these can be chosen to be different.

6.2 Preparatory lemmas

To prove Theorem 6.1.6, we use the lemmas given in the following sections.

6.2.1 Labels of spanning trees of Q_n with a reducible signature

The first lemma describes the directions of edges of an upright spanning tree of Q_n with a reducible signature.

Lemma 6.2.1. *Let $\mathcal{S} = (a_1, \dots, a_n)$ be a reducible signature of Q_n and suppose R is a corresponding reducing set. Let T be an upright spanning tree of Q_n with signature $\mathcal{S}$ and let W be a vertex of Q_n. Then $\psi_T(W) \in R$ if and only if $W \subseteq R$.*

Proof. The 'if' part is immediate from the definition of an upright spanning tree.

To prove the contrapositive of the 'only if' part, suppose without loss of generality that $\mathcal{S}$ is ordered. Then for $r = |R| \leq n - 1$ we have $\sum_{i=1}^{r} a_i = 2^r - 1$. Suppose that $W \not\subseteq [r]$. Every edge of T in Q_r is a direction in $[r]$. Since Q_r has $2^r - 1$ nonempty vertices, and T has $2^r - 1$ edges in directions $1, \dots, r$, this accounts for all $2^r - 1$ edges of T in directions $1, \dots, r$, so no edge in these directions is left over to place at W. Therefore $\psi_T(W) \notin [r]$. □

The following corollary follows immediately from Lemma 6.2.1 and the definition of a saturated signature.

Corollary 6.2.2. *Let $\mathcal{S} = (a_1, \ldots, a_n)$ be an ordered signature. If $\mathcal{S}$ is saturated above direction r and $W \not\subseteq [r]$, then $\psi_T(W) = \max W$.*

Proof. Since $\mathcal{S}$ is saturated above direction r, the sets $[s-1]$ and $[s]$ are reducing sets for each $s > r$. If $\max W = s$, then $W \subseteq [s]$ but $W \not\subseteq [s-1]$. Therefore $\psi_T(W)$ belongs to $[s]$ but not $[s-1]$, and therefore $\psi_T(W) = s$. □

The next corollary follows from the definition of a supersaturated signature and Corollary 6.2.2.

Corollary 6.2.3. *There is only one upright spanning tree of Q_n with the supersaturated signature $(1, 2, 4, 8, \ldots, 2^{n-1})$.*

Proof. Let T be an upright spanning tree of Q_n with the signature $(1, 2, 4, 8, \ldots, 2^{n-1})$. Since the supersaturated signature $(1, 2, 4, 8, \ldots, 2^{n-1})$ is saturated above direction 1, by Corollary 6.2.2 we have $\psi_T(W) = \max W$ for all $W \subseteq [n]$ such that $W \not\subseteq [1]$. In addition, when $W = \{1\}$ we must have $\psi_T(W) = 1$. Therefore there is only one upright spanning tree of Q_n with signature $(1, 2, 4, 8, \ldots, 2^{n-1})$. □

The following corollary follows from Corollary 6.2.2.

Corollary 6.2.4. *Let the ordered signature $\mathcal{S} = (a_1, \ldots, a_n)$ of Q_n be saturated above direction r. Then the number of upright spanning trees of Q_n with signature $\mathcal{S}$ is equal to the number of upright spanning trees of Q_r with signature the unsaturated part $(a_1, \ldots, a_r)$.*

Proof. Let T be an upright spanning tree of Q_n with signature $\mathcal{S}$. Let $W \not\subseteq [r]$. Then $\psi_T(W) = \max W$ by Corollary 6.2.2, so T is completely determined at each $W \not\subseteq [r]$. Let $G_{\mathcal{S}}$ be the matching graph constructed in the proof of Theorem 3.1.3. Since each $W \not\subseteq [r]$ is matched to a vertex labelled by its maximum, we are left with the matching graph of the unsaturated part. Therefore the number of matchings and hence the number of upright spanning trees is the same as the number of matchings of the unsaturated part. □

The following lemma shows that the edge slide graph of an ordered saturated signature is connected if the edge slide graph of the unsaturated part is connected.

Lemma 6.2.5. *Let the ordered signature $\mathcal{S} = (a_1, \ldots, a_n)$ be saturated above direction r. Then $\mathcal{E}(\mathcal{S})$ is connected if $\mathcal{E}(a_1, \ldots, a_r)$ is connected.*

Proof. We have $\mathcal{S} = (a_1, \ldots, a_{n-1}, 2^{n-1})$. By induction it suffices to consider the case $r = n-1$.

Suppose that $\mathcal{E}(a_1, \ldots, a_{n-1})$ is connected. Let T_1 and T_2 be any upright spanning trees of Q_n with signature $\mathcal{S}$. Observe that for $i = 1, 2$, T_i consists of an upright spanning tree $\bar{T}_i$ of Q_{n-1} with signature $(a_1, \ldots, a_{n-1})$ in $\mathcal{F}_n^{n-}$, together with an edge in direction n at each vertex of $\mathcal{F}_n^{n+}$.

Since $\mathcal{E}(a_1, \ldots, a_{n-1})$ is connected, there is a series of edge slides in Q_{n-1} from $\bar{T}_1$ to $\bar{T}_2$. These edge slides can all be carried out in Q_n. Therefore the edge slides in Q_{n-1} that converting $\bar{T}_1$ to $\bar{T}_2$ convert T_1 to T_2 in Q_n.

By the definition of the edge slide graph $\mathcal{E}(\mathcal{S})$, there is a sequence of edge slides from T_1 to T_2, hence T_1 and T_2 are in the same connected component of $\mathcal{E}(\mathcal{S})$. Since T_1 and T_2 are arbitrary, the edge slide graph $\mathcal{E}(\mathcal{S})$ is connected. □

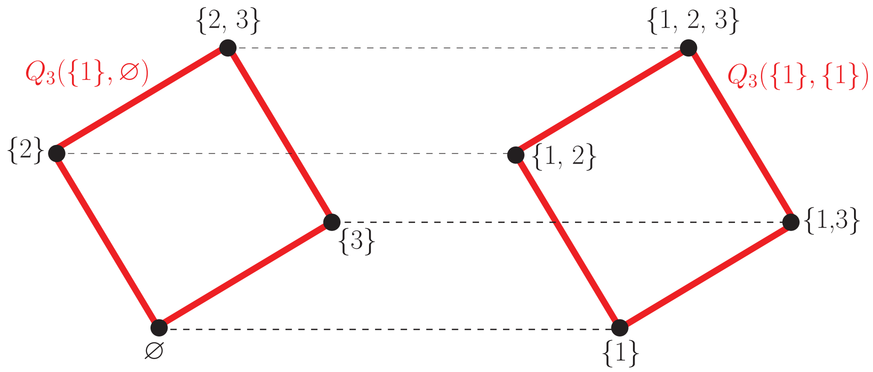


Figure 6.1: The graph Q_3. The induced subgraphs, $Q_3(\{1\},\varnothing)$ and $Q_3(\{1\},\{1\})$, of Q_3 are coloured in red.

In particular, applying the above lemma to a quasi-irreducible signature and the associated irreducible signature given by its unsaturated part we obtain the following:

Corollary 6.2.6. *Let $\mathcal{S} = (a_1,\dots,a_n)$ be a quasi-irreducible signature of Q_n, and let $\mathcal{I} = (a_1,\dots,a_r)$ be its unsaturated part. Then the edge slide graph $\mathcal{E}(\mathcal{S})$ is connected if $\mathcal{E}(\mathcal{I})$ is connected.*

6.2.2 Partitioning the n-dimensional cube

For any $R \subseteq [n]$, the n-dimensional cube can be partitioned into $2^{|R|}$ cubes of dimension $n-|R|$. Here we explain how this is done using the $2^{|R|}$ subsets of R.

Definition 6.2.7. For any $R \subseteq [n]$ and $X \subseteq R$ (possibly the empty set), let $Q_n(R,X)$ be the induced subgraph of Q_n with vertices

$$\{W \in \mathcal{P}([n]) | W \cap R = X\},$$

where $\mathcal{P}([n])$ denotes the power set of $[n]$, that is, the set of vertices of Q_n. Observe that $Q_n(R,X) = (Q_{[n]-R}) \oplus X$, and so is an $(n-|R|)$-cube. Note that every edge of Q_n in a direction $i \notin R$ belongs to $Q_n(R,X)$ for some X. For any $X_1, X_2 \subseteq R$ such that $X_1 \neq X_2$ we have

$$Q_n(R,X_1) \cap Q_n(R,X_2) = \varnothing.$$

.

Example 6.2.8. Let $n=3$ and $R=\{1\}$. The subsets of R are $\varnothing$ and $\{1\}$. Then

$$\varnothing, \{2\}, \{3\} \quad \text{and} \quad \{2,3\}$$

are the vertices of $Q_3(\{1\},\varnothing)$. Similarly

$$\{1\}, \{1,2\}, \{1,3\} \quad \text{and} \quad \{1,2,3\}$$

are the vertices of $Q_3(\{1\},\{1\})$. The subgraphs $Q_3(\{1\},\{1\})$ and $Q_3(\{1\},\varnothing)$ are shown in Figure 6.1.

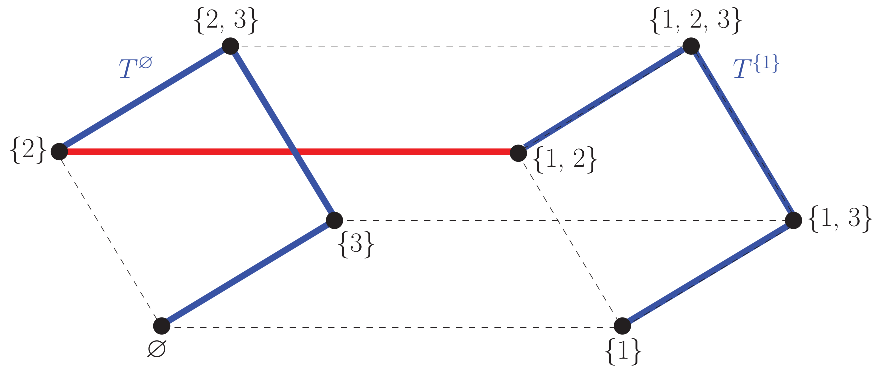


Figure 6.2: The blue edges and the red edge together form a spanning tree of Q_3 with the reducible signature $(1, 2, 4)$. The blue edges represent spanning trees $T^{\varnothing}$ and $T^{\{1\}}$ of $Q_3(\{1\}, \varnothing)$ and $Q_3(\{1\}, \{1\})$ respectively.

Lemma 6.2.9. *Let $\mathcal{S} = (a_1, \dots, a_n)$ be a reducible signature of Q_n and let R be a corresponding reducing set. Let T be a spanning tree of Q_n with signature $\mathcal{S}$, and for $X \subseteq R$ let $T^X = T \cap Q_n(R, X)$. Then for every $X \subseteq R$, the graph T^X is a spanning tree of $Q_n(R, X)$.*

Note that in Lemma 6.2.9 we do not require T to be upright.

Proof. For any $X \subseteq R$, let C_X be the number of edges of the forest T^X. Each C_X satisfies

$$C_X \leq 2^{n-|R|} - 1, \tag{6.1}$$

because $Q_n(R, X)$ is a cube of dimension $n - |R|$, and T is a tree. Since $\bigcup_{X \subseteq R} Q_n(R, X)$ contains all edges of Q_n in directions $[n] \setminus R$, all edges of T in those directions belong to $\bigcup_{X \subseteq R} T^X$. Therefore

$$\begin{aligned}\sum_{X \subseteq R} C_X = \sum_{i \notin R} a_i &= 2^n - 1 - \sum_{i \in R} a_i \\ &= 2^n - 1 - (2^{|R|} - 1) \\ &= 2^n - 2^{|R|} \\ &= 2^{|R|}(2^{n-|R|} - 1).\end{aligned}$$

In view of the inequality (6.1), this implies $C_X = 2^{n-|R|} - 1$ for all $X \subseteq R$. Since T^X has no cycles and $2^{n-|R|}$ vertices and $2^{n-|R|} - 1$ edges, T^X is a spanning tree of $Q_n(R, X)$ (by Lemma 2.1.4). □

Example 6.2.10. For the case $n = 3$, $\mathcal{S} = (1, 2, 4)$ and $R = \{1\}$ with $\varnothing$ and $\{1\}$ (the subsets of R), spanning trees $T^{\varnothing}$ and $T^{\{1\}}$ are shown in Figure 6.2.

6.2.3 Edge slides of spanning trees of Q_n with reducible signatures

In this section we study edge slides on spanning trees of Q_n with reducible signatures. As previously stated, edge slides do not change the signature of a spanning tree of Q_n. In the next lemma we start by looking at the effect of edge slides on spanning trees of $Q_n(R, X)$. Also, we determine the possible directions for sliding an edge of a spanning tree of Q_n with a reducible signature.

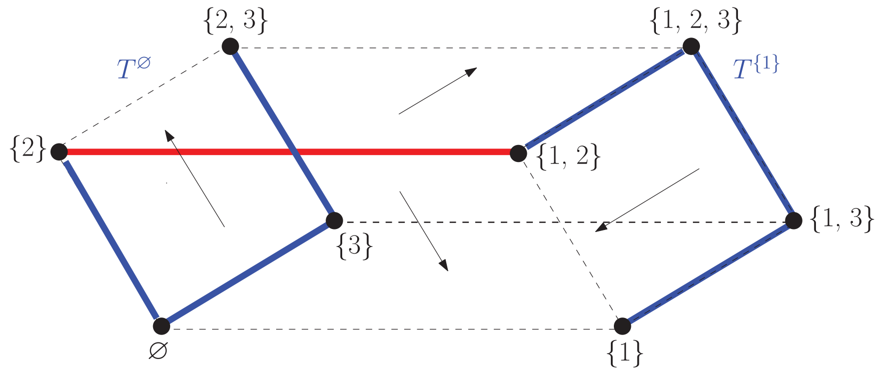


Figure 6.3: The blue edges and the red edge together form a spanning tree T of Q_3 with reducible signature (1, 3, 3). The blue edges represent spanning trees $T^{\varnothing}$ and $T^{\{1\}}$ of $Q_3(\{1\}, \varnothing)$ and $Q_3(\{1\}, \{1\})$ respectively. The 3-edge in $T^{\varnothing}$ can only be slid in direction 2, and the 2-edge in $T^{\{1\}}$ can only be slide in direction 3. The unique 1-edge (red) joining $T^{\varnothing}$ and $T^{\{1\}}$ can be slid in either direction 2 or 3.

Lemma 6.2.11. *Let $\mathcal{S} = (a_1, \ldots, a_n)$ be a reducible signature of Q_n and suppose R is a corresponding reducing set. Let T be a spanning tree of Q_n with signature $\mathcal{S}$ and let $T^X = T \cap Q_n(R, X)$ for each $X \subseteq R$. Then*

1. *No edge of T^X is slidable in any direction $j \in R$. Consequently, the signature of T^X is an invariant of the component of $\mathcal{E}(\mathcal{S})$ containing T.*

2. *Any edge of T in a direction $j \in R$ can be slid in any direction $\ell \notin R$.*

Proof.

1. Let i be a direction belonging to $[n] \setminus R$ and let e be an edge of T^X in direction i. Suppose for a contradiction that e is j-slidable. Then $\sigma_j(e) \notin T$, and the cycle in $T \cup \sigma_j(e)$ created by adding $\sigma_j(e)$ to T contains both e and $\sigma_j(e)$, and is broken by deleting e. However, since $T^{X \oplus \{j\}}$ is a spanning tree of $Q_n(R, X \oplus \{j\})$, the cycle created by adding $\sigma_j(e)$ to T lies entirely in $Q_n(R, X \oplus \{j\})$, which does not contain e. Therefore the cycle is not broken when e is deleted. Thus we have a contradiction and so e is not j-slidable.

2. Let $e = (Y, Y \oplus \{j\})$ be an edge of T in direction j and let $X = Y \cap R$. Since the vertices Y and $Y \oplus \{\ell\}$ belong to $Q_n(R, X)$ and T^X is a spanning tree of $Q_n(R, X)$, there is a path P^X in T^X from Y to $Y \oplus \{\ell\}$. Similarly, since the vertices $Y \oplus \{j\}$ and $Y \oplus \{j, \ell\}$ belong to $Q_n(R, X \oplus \{j\})$ and $T^{X \oplus \{j\}}$ is a spanning tree of $Q_n(R, X \oplus \{j\})$, there is a path $P^{X \oplus \{j\}}$ in $T^{X \oplus \{j\}}$ from $Y \oplus \{j\}$ and $Y \oplus \{j, \ell\}$. Then P^X, $P^{X \oplus \{j\}}$, $\sigma_\ell(e)$ and e form a cycle. This cycle is broken by deleting e, so e is ℓ-slidable.

$\square$

Example 6.2.12. The case $n = 3$, $\mathcal{S} = (1, 3, 3)$ with $R = \{1\}$ is illustrated in Figure 6.3. The subsets of R are $\varnothing$ and $\{1\}$, and the spanning trees $T^{\varnothing}$ and $T^{\{1\}}$ are joined by an edge in direction 1. No edge slide in direction 1 can be made, while $T^{\varnothing}$ and $T^{\{1\}}$ each have one possible edge slide. The 1-edge joining $T^{\varnothing}$ and $T^{\{1\}}$ may be slid in either direction 2 or 3.

6.2.4 The existence of subtrees with different signatures of spanning trees with reducible signatures

In this section we prove the existence of subtrees with different signatures of a spanning tree with a reducible signature.

Lemma 6.2.13. *Let $\mathcal{S} = (a_1, \ldots, a_n)$ be an ordered strictly reducible signature with unsaturated part $(a_1, \ldots, a_s)$, where $s \leq n$. Let r be the largest index such that $r < s$ and $\sum_{i=1}^{r} a_i = 2^r - 1$, and for any spanning tree T of Q_n with signature $\mathcal{S}$ and $X \subseteq [r]$ let $T^X = T \cap Q_n([r], X)$. Then for any distinct $X, Y \subseteq [r]$, there exists a spanning tree T of Q_n with signature $\mathcal{S}$ such that T^X and T^Y have different signatures.*

Proof. Let T be a spanning tree of Q_n with signature $\mathcal{S}$. If there exists $Z \subseteq [r]$ such that T^X and T^Z have different signatures then we can construct the required tree by (if necessary) swapping T^Y and T^Z. So suppose that this is not the case. Then the subtrees T^Z have the same signature for all $Z \subseteq [r]$. Let $\mathcal{U} = (e_{r+1}, \ldots, e_n)$ be this common signature.

For any $i \in \{r+1, ..., n\}$ each edge of T in direction i lies in T^Z for some $Z \subseteq [r]$, and since each such tree contains e_i edges in direction i we have $a_i = 2^r e_i$. It follows that $\mathcal{U}$ is ordered. Suppose that $e_{r+1} = 1$. Then $a_{r+1} = 2^r$, so $r + 1 \neq s$ because $a_s \neq 2^{s-1}$ by the choice of s. Hence $r + 1 < s$. Moreover

$$\sum_{i=1}^{r+1} a_i = \sum_{i=1}^{r} a_i + a_{r+1} = 2^r - 1 + 2^r = 2^{r+1} - 1,$$

so $[r + 1]$ is a reducing set. This contradicts the choice of r, so we must have $e_{r+1} \geq 2$, which forces $e_{r+2} \geq 2$ also because $\mathcal{U}$ is ordered.

Consider

$$\mathcal{U}_1 = (e_{r+1} - 1, e_{r+2} + 1, e_{r+3}, \ldots, e_n),$$
$$\mathcal{U}_2 = (e_{r+1} + 1, e_{r+2} - 1, e_{r+3}, \ldots, e_n),$$

and note that $\mathcal{U}_1 + \mathcal{U}_2 = 2\mathcal{U}$. We show $\mathcal{U}_1$ and $\mathcal{U}_2$ are signatures of Q_{n-r}, hence there are spanning trees T_1 and T_2 of Q_{n-r} with signatures $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively. For simplicity, we let $f_i = e_{r+i}$ for $i = 1, 2, \ldots, n - r$. We consider $\mathcal{U}_1$ and $\mathcal{U}_2$ separately.

For $\mathcal{U}_1$, we distinguish the following cases according to whether or not $f_2 < f_3$.

1. Suppose $f_1 \leq f_2 < f_3 \leq \cdots \leq f_{n-r}$. Then we have $f_1 - 1 < f_2 + 1 \leq f_3$. Write

$$\mathcal{U}_1 = (f_1 - 1, f_2 + 1, f_3, \ldots, f_{n-r}) = (f_1', f_2', f_3', \ldots, f_{n-r}').$$

Then

$$f_1' < f_2' \leq f_3' \leq \cdots \leq f_{n-r}'$$

is in increasing order, and

$$\sum_{i=1}^{k} f_i' = \begin{cases} f_1 - 1 \geq 1, & \text{for } k = 1; \\ \sum_{i=1}^{k} f_i \geq 2^k - 1, & \text{for } k \geq 2. \end{cases}$$

So we conclude $\mathcal{U}_1$ is a signature of Q_{n-r}, by Theorem 3.1.3.

2. Suppose $f_1 \leq f_2 = f_3 = \cdots = f_p < f_{p+1} \leq \cdots \leq f_{n-r}$ for some p, with $3 \leq p \leq n-r$. Then we have

$$f_1 - 1 < f_3 = \cdots = f_p < f_2 + 1 \leq f_{p+1} \leq \cdots \leq f_{n-r}.$$

Let

$$\mathcal{U}_1' = (f_1', f_2', \ldots, f_p', f_{p+1}', \ldots, f_{n-r}') = (f_1 - 1, f_3, \ldots, f_p, f_2 + 1, f_{p+1}, \ldots, f_{n-r}).$$

Then $\mathcal{U}_1'$ is an ordered permutation of $\mathcal{U}_1$, so it suffices to show $\mathcal{U}_1'$ is a signature. The only sums $\sum_{i=1}^k f_i'$ that are not equal to the corresponding sum $\sum_{i=1}^k f_i$ are

$$\sum_{i=1}^{j} f_i' = f_1 - 1 + (j-1)f_2,$$

for $1 \leq j < p$. So we need to check the value of $f_1 - 1 + (j-1)f_2$. For all $1 \leq y \leq p$ let

$$f(y) = \sum_{i=1}^{y} f_i' = f_1 + (y-1)f_2,$$

and let

$$g(y) = 2^y - 1.$$

Then

$$f(1) = f_1 \geq 2 > 1 = g(1),$$

and

$$f(p) = \sum_{i=1}^{p} f_i \geq 2^p - 1 = g(p).$$

To verify that $\mathcal{U}_1$ is a signature of Q_{n-r}, it remains to show that $g(j) < f(j)$, for all $1 < j < p$. Since g is convex, for any $0 \leq t \leq 1$, we have

$$g((1-t) + tp) \leq (1-t)g(1) + tg(p).$$

Let $t = \frac{j-1}{p-1}$. Then for $1 \leq j \leq p$ we have $0 \leq t \leq 1$ and $1 - t + tp = j$. So

$$\begin{aligned}
g(j) &\leq \left(1 - \frac{j-1}{p-1}\right) g(1) + \frac{j-1}{p-1} g(p) \\
&< \left(1 - \frac{j-1}{p-1}\right) f(1) + \frac{j-1}{p-1} f(p) \\
&= \left(1 - \frac{j-1}{p-1}\right) f_1 + \frac{j-1}{p-1}(f_1 + (p-1)f_2) \\
&= f_1 - \frac{j-1}{p-1} f_1 + \frac{j-1}{p-1} f_1 + (j-1)f_2 \\
&= f_1 + (j-1)f_2 \\
&= f(j),
\end{aligned}$$

where the second inequality follows since $g(1) < f(1)$ and $g(p) \leq f(p)$, and the first equality follows since $f(1) = f_1$ and $f(p) = f_1 + (p-1)f_2$. Therefore

$$\sum_{i=1}^{j} f_i' = \sum_{i=1}^{j} f_i - 1 \geq 2^j - 1,$$

which satisfies the signature condition.

For $\mathcal{U}_2$, we consider the following cases according to whether $f_2 = f_1$, $f_2 = f_1 + 1$ or $f_2 > f_1 + 1$.

1. If $f_1 = f_2$, then $\mathcal{U}_2$ is the permutation of $\mathcal{U}_1$ obtained by swapping the first two entries. Therefore $\mathcal{U}_2$ is a signature of Q_{n-r}.
2. If $f_1 + 1 = f_2$, then $\mathcal{U}_2$ is the permutation of $\mathcal{U}$ obtained by swapping the first two entries. Therefore $\mathcal{U}_2$ is a signature of Q_{n-r}.
3. If $f_1 < f_2 - 1$, then $f_1 + 1 \leq f_2 - 1$. Let

   $$\mathcal{U}_2 = (f_1 + 1, f_2 - 1, f_3, \ldots, f_{n-r}) = (f_1'', f_2'', f_3'', \ldots, f_{n-r}'').$$

   Then
   $$f_1'' \leq f_2'' < f_3'' \leq \cdots \leq f_n''$$
   is in increasing order and

   $$\sum_{i=1}^{k} f_i'' = \begin{cases} f_1 + 1 \geq 3 & \text{for } k = 1; \\ \sum_{i=1}^{k} f_i \geq 2^k - 1, & \text{for } k \geq 2. \end{cases}$$

   Therefore the signature condition is satisfied and we conclude $\mathcal{U}_2$ is a signature of Q_{n-r} (by Theorem 3.1.3).

Since $\mathcal{U}_1$ and $\mathcal{U}_2$ are signatures of Q_{n-r}, there are spanning trees T_1 and T_2 of Q_{n-r} with signatures $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively. Let T' be the tree obtained from T by replacing T^X with T_1, and T^Y with T_2. Then T' has signature $\mathcal{S}$, and the subtrees $(T')^X$ and $(T')^Y$ have different signatures, as required. □

Example 6.2.14. The spanning tree T of Q_3 in Figure 6.3 with strictly reducible signature (1, 3, 3) consists of the spanning tree $T^{\varnothing}$ of $Q_3(\{1\}, \varnothing)$ with signature $(2, 1)$, and the spanning tree $T^{\{1\}}$ of $Q_3(\{1\}, \{1\})$ with signature $(1, 2)$, joined by an edge in direction 1.

6.3 The edge slide graph of a reducible signature of Q_n

In this section we give a complete proof that the edge slide graph of a reducible signature is disconnected. We recall the statement of Theorem 6.1.6 on page 73.

Theorem 6.1.6. *Let $\mathcal{S} = (a_1, \ldots, a_n)$ be a strictly reducible signature of Q_n. Then the edge slide graph $\mathcal{E}(\mathcal{S})$ is disconnected.*

Proof. Without loss of generality, we may assume $\mathcal{S}$ is ordered with unsaturated part $(a_1, \ldots, a_s)$. Let r be the largest index such that $r < s$ and $\sum_{i=1}^{r} a_i = 2^r - 1$. By Lemma 6.2.13, for any distinct $X, Y \subseteq [r]$ there exists a spanning tree T of Q_n with signature $\mathcal{S}$ such that the subtrees $T^X = T \cap Q_n([r], X)$ and $T^Y = T \cap Q_n([r], Y)$ have different signatures. Let T' be the spanning tree obtained from T by swapping T^X and T^Y. Since the signatures of T^X and T^Y are invariant under edge slides by Lemma 6.2.11, the trees T and T' lie in different components of the edge slide graph $\mathcal{E}(\mathcal{S})$. This shows that the edge slide graph $\mathcal{E}(\mathcal{S})$ is disconnected, as required. $\square$

The above theorem shows that strict reducibility is an obstruction to being connected. We conjecture that this is the only obstruction to connectivity:

Conjecture 6.3.1. *Let $\mathcal{S} = (a_1, \ldots, a_n)$ be a signature of Q_n. Then the edge slide graph $\mathcal{E}(S)$ is connected if and only if $\mathcal{S}$ is irreducible or quasi-irreducible.*

By Corollary 6.2.6 it suffices to consider the irreducible case only.

6.4 Summary map

In this chapter we divided reducible signatures into strictly reducible and quasi-irreducible signatures. To help understand these signatures we defined saturated and unsaturated signatures. We showed that a signature that is saturated above direction r behaves the same as an associated signature $(a_1, \ldots, a_r)$. We proved that the edge slide graph of a strictly reducible signature of Q_n is disconnected. In Chapter 9 we show under the inductive hypothesis that an upright spanning tree of Q_n with an irreducible signature $\mathcal{I}$ and a reducible splitting signature can be transformed into an upright spanning tree with signature $\mathcal{I}$ and an irreducible splitting signature using edge slides. This result plays an important role in understanding the connectivity of the edge slide graph of an irreducible signature of Q_n. In the following chapters Lemma 6.2.1 and Corollary 6.2.2 are used to determine the labels of an upright spanning tree of Q_n with a reducible signature.

Chapter 7

Irreducible signatures of Q_n: irreducible splitting signatures

7.1 Introduction

In Chapters 7–9 we study irreducible signatures of Q_n in detail, with the goal of proving Conjecture 6.3.1. This is done under the inductive hypothesis that the edge slide graph of every irreducible signature of Q_k, where $k < n$, is connected. An irreducible signature has both reducible splitting signatures and irreducible splitting signatures, and we discuss irreducible splitting signatures in Chapters 7 and 8, and reducible splitting signatures in Chapter 9.

The idea is to construct a block of connected trees. From any given tree we show we can always use edge slides to reach a tree with a splitting signature of our choice. Chapter 9 shows that a tree of Q_n with an irreducible signature and a reducible splitting signature can be transformed into a tree with an irreducible splitting signature. Chapter 8 shows that a tree with an irreducible signature and an irreducible splitting signature can be transformed into a tree with an irreducible splitting signature of our choice. Since every irreducible signature has an amenable splitting signature, this means we can move from any tree to a tree with a fixed amenable splitting signature.

Let T be a tree of Q_n with an irreducible signature $\mathcal{I} = (a_1, \ldots, a_n)$ that has an irreducible splitting signature in $\mathcal{F}_n^{n-}$. In this chapter we develop tools to rearrange the labels of T_{n+} in $\mathcal{F}_n^{n+}$ using edge slides. The building blocks of these are local moves in 2-dimensional faces with two vertices in $\mathcal{F}_n^{n+}$ and two vertices in $\mathcal{F}_n^{n-}$. In order to make the local moves possible we need to specify the directions of the vertices in $\mathcal{F}_n^{n-}$. To do this we use the inductive hypothesis in conjunction with Lemma 3.2.7 or Lemma 3.2.8 from Section 3.2.1. Each result has two versions: one using Lemma 3.2.8 and the other using Lemma 3.2.7, which gives a stronger result but requires the stronger hypothesis of a rich splitting signature in $\mathcal{F}_n^{n-}$.

Our building block local moves also require an n at one of the vertices in $\mathcal{F}_n^{n+}$. A useful tool for applying these is the notion of a **settled tree**, which we define in Section 7.2.2. Let T be an upright spanning tree and let ξ be a vertex of $\mathcal{F}_n^{n+}$ such that $\psi_T(\xi) = n$. If for each vertex $\eta \subseteq \xi$ of $\mathcal{F}_n^{n+}$ we have $\psi_T(\eta) = n$, then T is called a settled tree. We show that every upright spanning tree with an irreducible splitting signature can be transformed into a settled tree using edge slides. Working with settled trees is easier, because we know the locations of the labels n and then we can use these to rearrange the labels of T_{n+} using the tools discussed above. This allows us to move from one settled tree to another by swapping labels in $\mathcal{F}_n^{n+}$.

In Section 7.3 we applying the tools developed above to the problem of showing that the set of upright spanning trees of Q_n with irreducible signature $\mathcal{I}$ and a fixed amenable splitting signature $\mathcal{D} = (d_1, \ldots, d_{n-1})$ in $\mathcal{F}_n^{n-}$ forms a block. We show that the set of trees with a fixed unidirectional splitting signature of an irreducible signature of Q_n or the splitting signature (2, 2, 3) of Q_4 forms a block, and we conjecture that this is also true for a super rich splitting signature. We present substantial partial progress towards a proof of this conjecture.

In this chapter we do not require the full strength of the inductive hypothesis, and prove our results under either the weaker inductive hypothesis that the edge slide graph of every irreducible signature of Q_{n-1} is connected, or under the assumption that the edge slide graph of a particular irreducible signature is connected. The strong form of the inductive hypothesis is required in Chapter 9.

7.2 Preparatory lemmas

7.2.1 Swapping labels in the upper n face

In this section let T be a spanning tree of Q_n. We show that under certain conditions labels of T at adjacent and nonadjacent vertices in $\mathcal{F}_n^{n+}$ can be swapped.

The first lemma shows that if a lowest i-edge in $\mathcal{F}_n^{n+}$ is not followed by an n-edge (the direction of splitting) then there exists a sequence of edge slides that moves the i-edge down, until it is followed by an n-edge. This lemma is used as a tool to move an edge from $\mathcal{F}_n^{n+}$ to $\mathcal{F}_n^{n-}$.

Lemma 7.2.1. *Let $\mathcal{S} = (a_1, \ldots, a_n)$ be a signature of Q_n. Let T be an upright spanning tree of Q_n with signature $\mathcal{S}$. Let $X \subseteq [n-1] \setminus \{i\}$ be such that $Y = X \cup \{i, n\}$ is a lowest vertex of $\mathcal{F}_n^{n+}$ that has an i-edge. Let $\psi_T(X \cup \{n\}) = j$. If $j \neq n$, then there is a sequence of edge slides in $\mathcal{F}_n^{n+}$ that transforms T into an upright spanning tree T' where, for some $X' \subseteq X$, $X' \cup \{i, n\}$ is a lowest vertex of $\mathcal{F}_n^{n+}$ that has an i-edge and moreover $\psi_T(X' \cup \{n\}) = n$. In addition, $T_{n-} = T'_{n-}$ because the edge slides were only applied in $\mathcal{F}_n^{n+}$.*

Proof. The vertex $(X \cup \{i, n\}) \setminus \{j\}$ cannot be in direction i because we assumed $X \cup \{i, n\}$ is a lowest vertex of $\mathcal{F}_n^{n+}$ that has an i-edge, so $\psi_T((X \setminus \{j\}) \cup \{i, n\}) = k$ for some $k \neq i$, as shown in Figure 7.1(a) on page 85. Then using the path move the k-edge moves to $X \cup \{n\}$, the j-edge moves to $X \cup \{i, n\}$ and the i-edge moves to $(X \setminus \{j\}) \cup \{i, n\}$, as shown in Figure 7.1(b). The vertex $(X \setminus \{j\}) \cup \{i, n\}$ is now the lowest vertex that has an i-edge, and note that $X \setminus \{j\} \subseteq X$. If the vertex $(X \setminus \{j\}) \cup \{n\}$ is not in direction n, then we repeat the above process until we reach a tree $T^\star$ where, for some $X' \subseteq X$, the vertex $X' \cup \{i, n\}$ is the lowest vertex of $\mathcal{F}_n^{n+}$ that has an i-edge and moreover $\psi_{T^\star}(X' \cup \{n\}) = n$, as shown in Figure 7.1(c). Note that we reach such a tree because each path from a vertex of $\mathcal{F}_n^{n+}$ to the root of the tree must pass through an n-edge. $\square$

The following lemma shows that two labels of a tree at adjacent vertices in $\mathcal{F}_n^{n+}$ can be swapped using edge slides. We require one of the labels to be in the direction of splitting, and the tree to have an ordered connected irreducible splitting signature in $\mathcal{F}_n^{n-}$. In this lemma we use Lemma 3.2.8 to specify the directions of two vertices.

Lemma 7.2.2. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be a signature of Q_n and let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be an ordered connected irreducible splitting signature of $\mathcal{I}$ with respect to n. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Let ξ be a vertex of $\mathcal{F}_n^{n+}$ with $\psi_T(\xi) \neq n$. Let $i = \psi_T(\xi)$ and let $m = \max(\xi \setminus \{n\})$.*

1. *Suppose that $i = m$ and suppose that, for some $j \neq m, n$, we have $\psi_T(\xi \setminus \{j\}) = n$. Then there exists a sequence of edge slides that transforms T into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$, $\psi_{\hat{T}}(\xi \setminus \{j\}) = m$ and all other labels of $\hat{T}$ the same as the labels of T.*
2. *Suppose that $i \neq m$ and suppose $\psi_T(\xi \setminus \{m\}) = n$. Then there exists a sequence of edge slides that transforms T into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$, $\psi_{\hat{T}}(\xi \setminus \{m\}) = i$ and all other labels of $\hat{T}$ the same as the labels of T.*

Proof.

1. Since $\mathcal{D}$ is an ordered irreducible signature of Q_{n-1} and since $m = \max(\xi \setminus \{n\})$, by Lemma 3.2.8 there exists an upright spanning tree T'_{n-} of Q_{n-1} with signature $\mathcal{D}$ where $\psi_{T'}(\xi \setminus \{n\}) = j$ and $\psi_{T'}(\xi \setminus \{j, n\}) = \max(\xi \setminus \{j, n\}) = m$. Since the edge slide graph of signature $\mathcal{D}$ is connected, we can move from T_{n-} to T'_{n-} using edge slides. Then T is transformed into T' with $\psi_{T'}(\xi \setminus \{n\}) = j$, $\psi_{T'}(\xi \setminus \{j, n\}) = m$ and all labels of T'_{n+} the same as the labels of T_{n+}.

   As shown in Figure 7.2, using the V-move the labels in directions m and n are swapped. Let T'' be the resulting tree with $\psi_{T''}(\xi) = n$, $\psi_{T''}(\xi \setminus \{j\}) = m$ and all other labels of T'' the same as the labels of T'.

   Note that the sequence of edge slides that has been done to T_{n-} to get T'_{n-} can be reversed to get back to T_{n-}. Therefore we can conclude that T is transformed into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$, $\psi_{\hat{T}}(\xi \setminus \{j\}) = m$ and all other labels of $\hat{T}$ the same as the labels of T.

2. Since $\mathcal{D}$ is an ordered irreducible signature of Q_{n-1} and since $m = \max(\xi \setminus \{n\})$, by Lemma 3.2.8 there exists an upright spanning tree T'_{n-} of Q_{n-1} with signature $\mathcal{D}$ where $\psi_{T'}(\xi \setminus \{n\}) = m$ and $\psi_{T'}(\xi \setminus \{m, n\}) = i$. Since the edge slide graph of signature $\mathcal{D}$ is connected, we can move from T_{n-} to T'_{n-} using edge slides. Then T is transformed into T' with $\psi_{T'}(\xi \setminus \{n\}) = m$, $\psi_{T'}(\xi \setminus \{m, n\}) = i$ and all labels of T'_{n+} the same as the labels of T_{n+}.

   As shown in Figure 7.3, using the V-move the labels in directions i and n are swapped. Let T'' be the resulting tree with $\psi_{T''}(\xi) = n$, $\psi_{T''}(\xi \setminus \{m\}) = i$ and all other labels of T'' the same as the labels of T'.

   Note that the sequence of edge slides that has been done to T_{n-} to get T'_{n-} can be reversed to get back to T_{n-}. Therefore we can conclude that T is transformed into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$, $\psi_{\hat{T}}(\xi \setminus \{m\}) = i$ and all other labels of $\hat{T}$ the same as the labels of T.

□

We define useful paths in $\mathcal{F}_n^{n+}$ as follows.

Definition 7.2.3. Let $\xi = \xi_0$ be a vertex of Q_n. Let $P = (\xi_0, \ldots, \xi_\alpha)$ be a path in Q_n such that $|\xi_{t+1}| = |\xi_t| - 1$ for all $1 \leq t \leq \alpha$, where $\alpha \leq |\xi_0|$. Then P is a **descending path** in Q_n from ξ_0 to ξ_α.

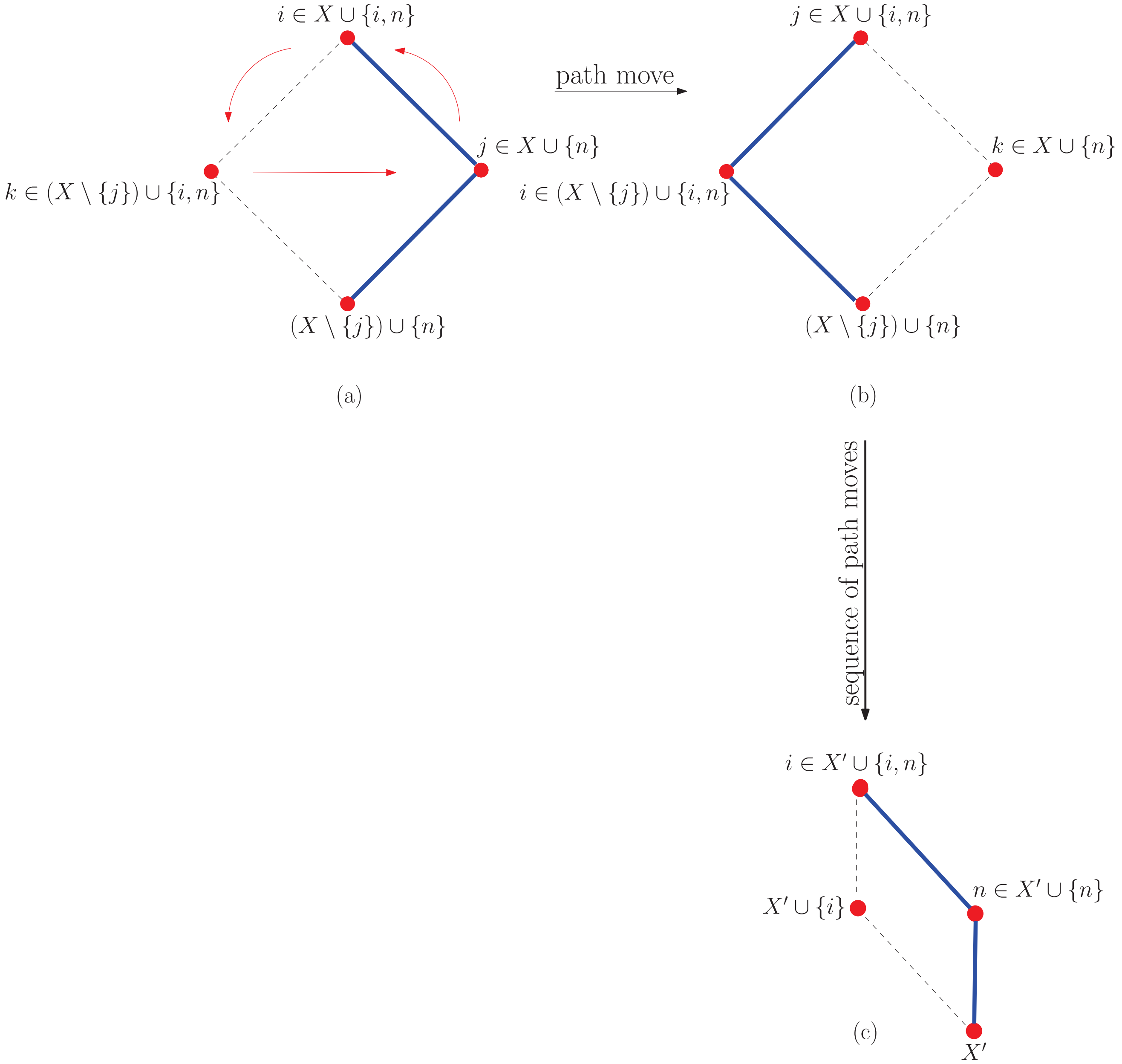


Figure 7.1: Diagram for the proof of Lemma 7.2.1 on page 83. (a) Direction i is chosen at $X \cup \{i,n\}$ and direction j and direction k are chosen at $X \cup \{n\}$ and $(X \cup \{i,n\}) \setminus \{j\}$ respectively. (b) After the path move the k-edge moves to $X \cup \{n\}$, the i-edge and the j-edge move to $(X \cup \{i,n\}) \setminus \{j\}$ and $X \cup \{i,n\}$ respectively. (c) After a sequence of path moves, for some $X' \subseteq X$, the path from $X' \cup \{i,n\}$ to the root goes in direction i and then direction n.

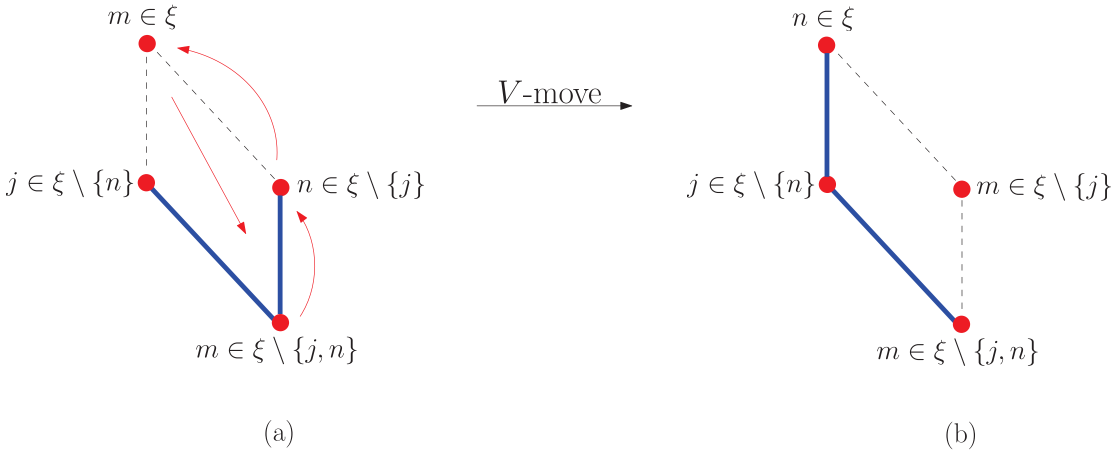


Figure 7.2: Diagram for Case 1 of Lemma 7.2.2. (a) Direction m is chosen at ξ and $\xi \setminus \{j, n\}$, direction j is chosen at $\xi \setminus \{n\}$ and direction n is chosen at $\xi \setminus \{j\}$. (b) After applying the V-move the labels m and n are swapped.

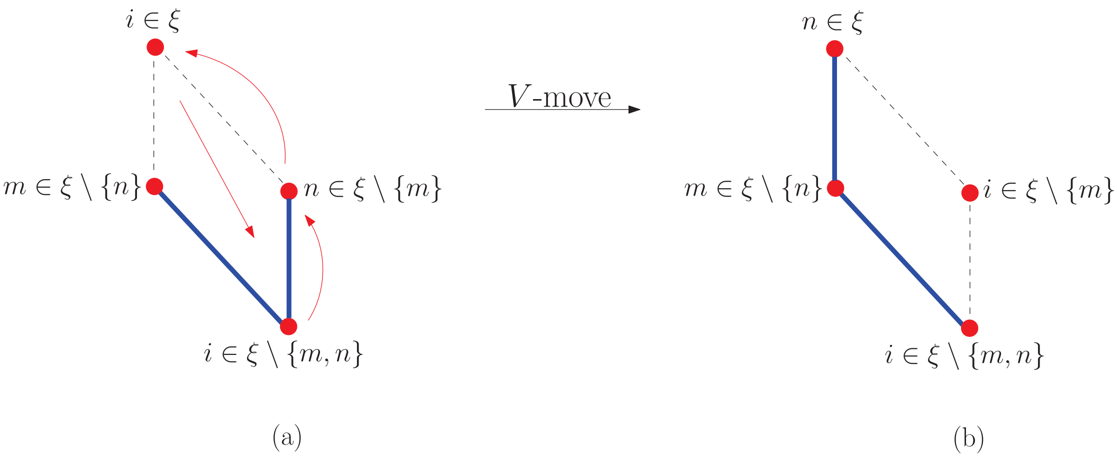


Figure 7.3: Diagram for Case 2 of Lemma 7.2.2. (a) Direction i is chosen at ξ and $\xi \setminus \{m, n\}$, direction m is chosen at $\xi \setminus \{n\}$ and direction n is chosen at $\xi \setminus \{m\}$. After applying the V-move the labels i and n are swapped.

Definition 7.2.4. Let ξ be a vertex of $\mathcal{F}_n^{n+}$. Let $\xi_0 = \xi$, $\alpha = |\xi| - 1$, and for $a = 0, \dots, \alpha - 1$ let $m_a = \max(\xi_a \setminus \{n\})$ and $\xi_{a+1} = \xi_a \setminus \{m_a\}$. Then $P_{\mathcal{F}_n^{n+}}(\xi) = (\xi_0, \xi_1 \dots, \xi_\alpha)$ is the **max removing path** in $\mathcal{F}_n^{n+}$ from ξ_0 to $\xi_\alpha = \{n\}$. If $i \notin \xi$ then for $\eta = \xi \cup \{i\}$ we define $P_{\mathcal{F}_n^{n+}}(\eta, i) = \sigma_i(P_{\mathcal{F}_n^{n+}}(\xi))$ to be the ***i*-retaining max removing path** $(\eta_0, \eta_1 \dots, \eta_\alpha) = (\xi_0 \cup \{i\}, \xi_1 \cup \{i\} \dots, \xi_\alpha \cup \{i\})$ in $\mathcal{F}_n^{n+}$ from $\eta_0 = \xi_0 \cup \{i\}$ to $\eta_\alpha = \xi_\alpha \cup \{i\} = \{i, n\}$. Note that $\eta_{a+1} = \eta_a \setminus \{\ell_a\}$, where $\ell_a = \max(\eta_a \setminus \{i, n\})$. Note that both $P_{\mathcal{F}_n^{n+}}(\xi)$ and $P_{\mathcal{F}_n^{n+}}(\eta, i)$ are descending paths.

By applying Lemma 7.2.2 repeatedly we can move an i (a direction not equal to n, the direction of splitting) down along a suitably chosen path in $\mathcal{F}_n^{n+}$, as shown in the next lemma.

Lemma 7.2.5. *Let $\mathcal{I}$, $\mathcal{D}$ and T be as defined in Lemma 7.2.2 on page 84. Let ξ be a vertex of $\mathcal{F}_n^{n+}$ such that $\psi_T(\xi) = i$ for some $i \neq n$. Let $P_{\mathcal{F}_n^{n+}}(\xi, i) = (\xi_0, \xi_1 \dots, \xi_\alpha)$ be the i-retaining max removing path from $\xi_0 = \xi$ to $\xi_\alpha = \{i, n\}$. Suppose that $\psi_T(\xi_a) = n$ for $a = 1, \dots, \alpha$. Then there exists a sequence of edge slides that transforms T into T_α with $\psi_{T_\alpha}(\xi_\alpha) = i$, $\psi_{T_\alpha}(\xi_a) = n$ for $a = 0, \dots, \alpha - 1$ and all other labels of T_α the same as the labels of T.*

Proof. For $a = 0, \dots, \alpha$, let T_a be the tree such that $\psi_{T_a}(\xi_a) = i$, $\psi_{T_a}(\xi_b) = n$ for $b \neq a$, and all other labels of T_a are the same as the labels of T. Note that these trees exist because $T_0 = T$ exists, and i and n both belong to ξ_a for all a. If $m_a \neq \max(\xi_a \setminus \{n\})$, then $i = \max(\xi_a \setminus \{n\})$. Since $i = \max(\xi_a \setminus \{n\})$ and $\psi_T(\xi_a \setminus \{m_a\}) = n$, all the conditions in Case 1 of Lemma 7.2.2 are satisfied. Otherwise $m_a = \max(\xi_a \setminus \{n\})$. Since $i \neq m_a$ and $\psi_T(\xi_a \setminus \{m_a\}) = n$, all the conditions in Case 2 of Lemma 7.2.2 are satisfied. Applying the applicable case, T_a can be transformed into T_{a+1} and therefore by induction $T = T_0$ can be transformed into T_α. $\square$

The following corollary follows from Lemma 7.2.5. This corollary shows that, under certain conditions, two labels of non-adjacent vertices in $\mathcal{F}_n^{n+}$ can be swapped. We require one of the labels to be in the direction of splitting. This corollary is used in proving Lemma 7.2.13, Lemma 7.3.2 and Lemma 9.2.2.

Corollary 7.2.6. *Let $\mathcal{I}$, $\mathcal{D}$, T, ξ and $P_{\mathcal{F}_n^{n+}}(\xi, i)$ be defined as in Lemma 7.2.5. Let ξ' be a vertex of $\mathcal{F}_n^{n+}$ such that $\{i, n\} \subseteq \xi' \cap \xi$. Let $P_{\mathcal{F}_n^{n+}}(\xi', i) = (\xi'_0, \xi'_1 \dots, \xi'_{\alpha'})$ be the i-retaining max removing path from $\xi'_0 = \xi'$ to $\xi'_{\alpha'} = \{i, n\}$. Suppose $\psi_T(\xi'_a) = n$ for $a = 0, \dots, \alpha'$. Then there exists a sequence of edge slides that transforms T into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$, $\psi_{\hat{T}}(\xi') = i$ and all other labels of $\hat{T}$ the same as the labels of T.*

Proof. Consider the tree $\hat{T}$, which exists because it is obtained from T by swapping the labels at ξ and ξ', and these labels belong to $\xi \cap \xi'$. We show that $\hat{T}$ can be transformed into the resulting tree T_α of Lemma 7.2.5. Since all the labels of $\hat{T}$ are the same as the labels of T apart from $\psi_{\hat{T}}(\xi) = n$ and $\psi_{\hat{T}}(\xi') = i$, and since $\psi_T(\xi'_a) = n$ for $a = 1, \dots, \alpha'$, we have $\psi_{\hat{T}}(\xi'_a) = n$ for $a = 1, \dots, \alpha'$. Applying Lemma 7.2.5 $\hat{T}$ is transformed into $\hat{T}_{\alpha'}$ with $\psi_{\hat{T}_{\alpha'}}(\xi'_{\alpha'}) = i$, $\psi_{\hat{T}_{\alpha'}}(\xi'_0) = n$ and all other labels of $\hat{T}_{\alpha'}$ the same as the labels of $\hat{T}$. Then all the labels of $\hat{T}_{\alpha'}$ apart from $\psi_{\hat{T}_{\alpha'}}(\xi'_{\alpha'}) = i$ and $\psi_{\hat{T}}(\xi) = n$ are the same of the labels of T because all the labels of $\hat{T}$ apart form $\psi_{\hat{T}}(\xi) = n$ and $\psi_{\hat{T}}(\xi') = i$ are the same as the labels of T. Since $\xi_\alpha = \xi'_{\alpha'} = \{i, n\}$, we have $\hat{T}_{\alpha'} = T_\alpha$.

Since both trees T and $\hat{T}$ can reach T_α, we conclude that T can be transformed into $\hat{T}$ by a sequence of edge slides. $\square$

The next lemma provides a similar result to Lemma 7.2.2 in terms of swapping two labels in directions i and n at adjacent vertices in $\mathcal{F}_n^{n+}$. The difference is that in this lemma we

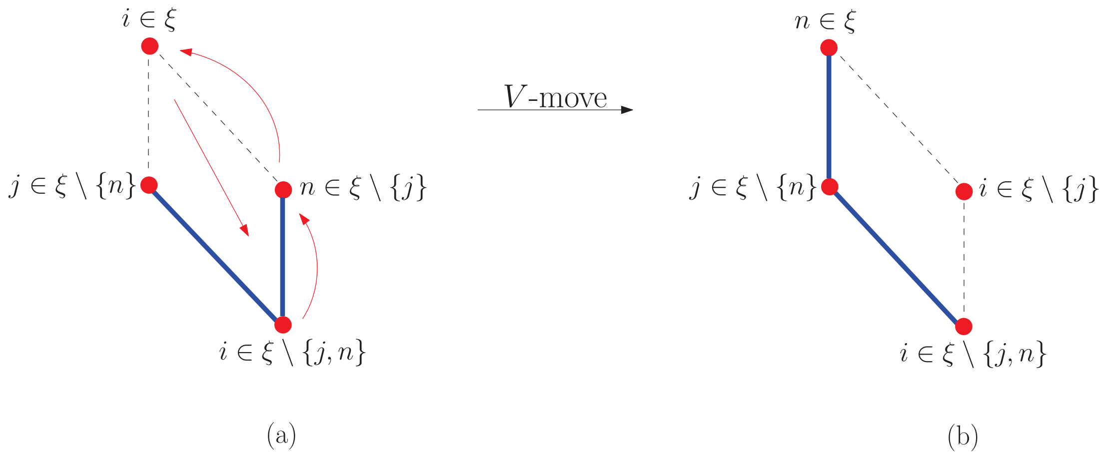


Figure 7.4: Diagram for the proof of Lemma 7.2.7 on page 88. (a) Direction i is chosen at ξ and $\xi \setminus \{j, n\}$, direction j is chosen at $\xi \setminus \{n\}$ and direction n is chosen at $\xi \setminus \{j\}$. (b) After applying the V-move the labels i and n are swapped.

specifically have a rich splitting signature in $\mathcal{F}_n^{n-}$ and we use Lemma 3.2.7 to specify the directions of two vertices. Having the stronger hypothesis of a rich splitting signature in $\mathcal{F}_n^{n-}$ gives a stronger result than Lemma 7.2.2.

Lemma 7.2.7. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n and let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be a connected rich splitting signature of $\mathcal{I}$ with respect to n. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Let ξ be a vertex of $\mathcal{F}_n^{n+}$ with $\psi_T(\xi) = i$ for some $i \neq n$. Suppose that $\psi_T(\xi \setminus \{j\}) = n$ for some $j \neq i$. Then there exists a sequence of edge slides that transforms T into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$, $\psi_{\hat{T}}(\xi \setminus \{j\}) = i$ and all other labels of $\hat{T}$ the same as the labels of T.*

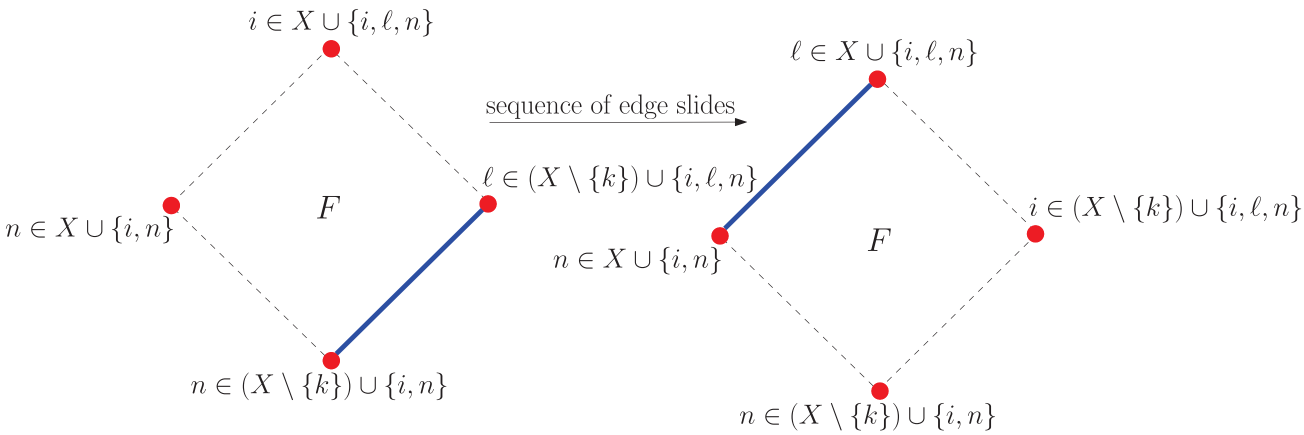
Proof. Let $u_t = a_t - d_t$ for all t. Since $\mathcal{D}$ is a rich splitting signature and since $u_i \neq 0$, we have $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $\mu \geq i$. Then by Lemma 3.2.7 there exists a tree T'_{n-} of Q_{n-1} with signature $\mathcal{D}$ where $\psi_{T'}(\xi \setminus \{n\}) = j$ and $\psi_{T'}(\xi \setminus \{j, n\}) = i$. Since the edge slide graph of signature $\mathcal{D}$ is connected, we can move from T_{n-} to T'_{n-} using edge slides. Then T is transformed into T' with $\psi_{T'}(\xi \setminus \{n\}) = j$, $\psi_{T'}(\xi \setminus \{j, n\}) = i$ and all labels of T'_{n+} the same as the labels of T_{n+}.

As shown in Figure 7.4, using the V-move the labels in directions i and n are swapped. Let T'' be the resulting tree with $\psi_{T''}(\xi) = n$, $\psi_{T''}(\xi \setminus \{j\}) = i$ and all other labels of T'' the same as the labels of T'.

Note that the sequence of edge slides that has been done to T_{n-} to get T'_{n-} can be reversed to get back to T_{n-}. Therefore we can conclude that T is transformed into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$, $\psi_{\hat{T}}(\xi \setminus \{j\}) = i$ and all other labels of $\hat{T}$ the same as the labels of T. □

By applying Lemma 7.2.7 repeatedly we can move an i (a direction not equal to n, the direction of splitting) down along a suitably chosen path in $\mathcal{F}_n^{n+}$, as shown in the next lemma. This lemma is similar to Lemma 7.2.5, but the stronger hypothesis allows us to freely choose the descending path from ξ to $\{i, n\}$.

Lemma 7.2.8. *Let $\mathcal{I}$, $\mathcal{D}$ and T be as defined in Lemma 7.2.7 on page 88. Let ξ be a vertex of $\mathcal{F}_n^{n+}$ such that $\psi_T(\xi) = i$ for some $i \neq n$. Let $P_{\mathcal{F}_n^{n+}} = (\xi_0, \xi_1 \ldots, \xi_\alpha)$ be a descending path in $\mathcal{F}_n^{n+}$ from $\xi_0 = \xi$ to $\xi_\alpha = \{i, n\}$. Suppose that $\psi_T(\xi_a) = n$ for $a = 1, \ldots, \alpha$. Then there*



(a) (b)

Figure 7.5: Diagram for Corollary 7.2.10. (a) A face F of $\mathcal{F}_n^{n+}$ with different labels in directions i, j and n of an upright spanning tree. (b) The face F of $\mathcal{F}_n^{n+}$ after swapping the labels i and j by a sequence of edge slides.

exists a sequence of edge slides that transforms T into T_α with $\psi_{T_\alpha}(\xi_\alpha) = i$, $\psi_{T_\alpha}(\xi_a) = n$ for $a = 0, \ldots, \alpha - 1$ and all other labels of T_α the same as the labels of T.

Proof. For $a = 0, \ldots, \alpha$, let T_a be the tree such that $\psi_{T_a}(\xi_a) = i$, $\psi_{T_a}(\xi_b) = n$ for $b \neq a$, and all other labels of T_a are the same as the labels of T. Note that these trees exist because $T_0 = T$ exists, and i and n both belong to ξ_a for all a. Since $\mathcal{D}$ is rich, $\psi_{T_a}(\xi_a) = i$ and $\psi_{T_a}(\xi_{a+1}) = n$, all the conditions in Lemma 7.2.7 hold. Then by applying this lemma, T_a can be transformed into T_{a+1} and therefore by induction $T = T_0$ can be transformed into T_α. □

The following corollary follows from Lemma 7.2.8. This corollary shows that, under certain conditions, two labels of non-adjacent vertices of $\mathcal{F}_n^{n+}$ can be swapped. We require one of the labels to be in the direction of splitting.

Corollary 7.2.9. *Let $\mathcal{I}$, $\mathcal{D}$, T, ξ and $P_{\mathcal{F}_n^{n+}}$ be defined as in Lemma 7.2.8. Let ξ' be a vertex of $\mathcal{F}_n^{n+}$ such that $\{i, n\} \subseteq \xi' \cap \xi$. Let $P'_{\mathcal{F}_n^{n+}} = (\xi'_0, \xi'_1 \ldots, \xi'_{\alpha'})$ be a descending path in $\mathcal{F}_n^{n+}$ from ξ'_0 to $\xi'_{\alpha'} = \{i, n\}$. Suppose $\psi_T(\xi'_a) = n$ for $a = 0, \ldots, \alpha'$. Then there exists a sequence of edge slides that transforms T into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$, $\psi_{\hat{T}}(\xi') = i$ and all other labels of $\hat{T}$ the same as the labels of T.*

Proof. Consider the tree $\hat{T}$, which exists because it is obtained from T by swapping the labels at ξ and ξ', and these labels belong to $\xi \cap \xi'$. We show that $\hat{T}$ can be transformed into the resulting tree T_α of Lemma 7.2.5. Since all the labels of $\hat{T}$ are the same as the labels of T apart from $\psi_{\hat{T}}(\xi) = n$ and $\psi_{\hat{T}}(\xi') = i$, and since $\psi_T(\xi'_a) = n$ for $a = 1, \ldots, \alpha'$, we have $\psi_{\hat{T}}(\xi'_a) = n$ for $a = 1, \ldots, \alpha'$. Applying Lemma 7.2.8, $\hat{T}$ is transformed into $\hat{T}_{\alpha'}$ with $\psi_{\hat{T}_{\alpha'}}(\xi'_{\alpha'}) = i$, $\psi_{\hat{T}_{\alpha'}}(\xi'_0) = n$ and all other labels of $\hat{T}_{\alpha'}$ the same as the labels of $\hat{T}$. Then all the labels of $\hat{T}_{\alpha'}$ apart from $\psi_{\hat{T}_{\alpha'}}(\xi'_{\alpha'}) = i$ and $\psi_{\hat{T}}(\xi) = n$ are the same of the labels of T because all the labels of $\hat{T}$ apart from $\psi_{\hat{T}}(\xi) = n$ and $\psi_{\hat{T}}(\xi') = i$ are the same as the labels of T. Since $\xi_\alpha = \xi'_{\alpha'} = \{i, n\}$, we have $\hat{T}_{\alpha'} = T_\alpha$.

Since both trees T and $\hat{T}$ can reach T_α, we conclude that T can be transformed into $\hat{T}$ by a sequence of edge slides. □

The following corollary follows from applying Lemma 7.2.7 four times to swap two labels at adjacent vertices in $\mathcal{F}_n^{n+}$ that are not equal to n, the direction of splitting.

Corollary 7.2.10. *Let $\mathcal{I}$, $\mathcal{D}$ and T be as defined in Lemma 7.2.7. Suppose there is a face F in $\mathcal{F}_n^{n+}$ of Q_n which is labelled by T as in Figure 7.5(a), where i, k, ℓ and n are all different. Then there is a sequence of edge slides that transforms T into $\hat{T}$ in which i and j in F are swapped as in Figure 7.5(b), and all other labels of $\hat{T}$ are the same as the labels of T.*

Proof. Refer to Figure 7.6, where (i) shows the initial labels on face F. Since $\mathcal{D}$ is a connected rich splitting signature, $\psi_T(X \cup \{i, \ell, n\}) = i$ and $\psi_T(X \cup \{i, n\}) = n$, all the conditions in Lemma 7.2.7 are satisfied at the vertices $X \cup \{i, \ell, n\}$ and $X \cup \{i, n\}$. Then T is transformed into T^1 with $\psi_{T^1}(X \cup \{i, \ell, n\}) = n$, $\psi_{T^1}(X \cup \{i, n\}) = i$, and all other labels of T^1 the same as the labels of T. The face F is now labelled as in Figure 7.6 (ii).

Applying Lemma 7.2.7 at the vertices $X \cup \{i, \ell, n\}$ and $(X \setminus \{k\}) \cup \{i, \ell, n\}$, the tree T^1 is transformed into T^2 with $\psi_{T^2}(X \cup \{i, \ell, n\}) = \ell$, $\psi_{T^2}((X \setminus \{k\}) \cup \{i, \ell, n\}) = n$, and all other labels of T^2 the same as the labels of T^1. The face F is now labelled as in Figure 7.6 (iii).

Note that since all the labels apart from $\psi_{T^1}(X \cup \{i, \ell, n\}) = n$ and $\psi_{T^1}(X \cup \{i, n\}) = i$ are the same as the labels of T, all the labels of T^2 apart from $\psi_{T^2}(X \cup \{i, \ell, n\}) = \ell$, $\psi_{T^2}(X \setminus \{k\}) \cup \{i, \ell, n\}) = n$ and $\psi_{T^2}(X \cup \{i, n\}) = i$ are the same as the labels of T.

Applying Lemma 7.2.7 at the vertices $X \cup \{i, n\}$ and $(X \setminus \{k\}) \cup \{i, n\}$, the tree T^2 is transformed into T^3 with $\psi_{T^3}(X \cup \{i, n\}) = n$, $\psi_{T^3}((X \setminus \{k\}) \cup \{i, n\}) = i$, and all other labels of T^3 the same as the labels of T^2. The face F is now labelled as in Figure 7.6 (iv).

Note that since all the labels of T^2 apart from $\psi_{T^2}(X \cup \{i, \ell, n\}) = \ell$, $\psi_{T^2}((X \setminus \{k\}) \cup \{i, \ell, n\}) = n$ and $\psi_{T^2}(X \cup \{i, n\}) = i$ are the same as the labels of T, all the labels of T^3 apart from $\psi_{T^3}(X \cup \{i, \ell, n\}) = \ell$, $\psi_{T^3}((X \setminus \{k\}) \cup \{i, \ell, n\}) = n$ and $\psi_{T^3}((X \setminus \{k\}) \cup \{i, n\}) = i$ are the same as the labels of T.

Applying Lemma 7.2.7 at the vertices $(X \setminus \{k\}) \cup \{i, n\}$ and $(X \setminus \{k\}) \cup \{i, \ell, n\}$, the tree T^3 is transformed into $\hat{T}$ with $\psi_{\hat{T}}(X \cup \{i, n\}) = n$, $\psi_{\hat{T}}((X \setminus \{k\}) \cup \{i, n\}) = i$, and all other labels of $\hat{T}$ the same as the labels of T^3. The face F is now labelled as in Figure 7.6 (v).

Note that since all the labels of T^3 apart from $\psi_{T^3}(X \cup \{i, \ell, n\}) = \ell$, $\psi_{T^3}((X \setminus \{k\}) \cup \{i, \ell, n\}) = n$ and $\psi_{T^3}((X \setminus \{k\}) \cup \{i, n\}) = i$ are the same as the labels of T, and since $\psi_{\hat{T}}(X \cup \{i, n\}) = \psi_T(X \cup \{i, n\}) = n$, all the labels of $\hat{T}$ apart from $\psi_{\hat{T}}(X \cup \{i, \ell, n\}) = \ell$ and $\psi_{\hat{T}}((X \setminus \{k\}) \cup \{i, \ell, n\}) = i$ are the same as the labels of T, as required. □

7.2.2 Settled trees

In this section we define a settled tree of Q_n and show that every upright spanning tree of Q_n that has a connected irreducible splitting signature can be transformed into a settled tree using edge slides. Also, we provide some related definitions and results. We start with the definition of a settled vertex.

Definition 7.2.11. Let T be an upright spanning tree of Q_n with signature $\mathcal{S} = (a_1, \ldots, a_n)$. A **settled vertex** of T is a vertex ξ of $\mathcal{F}_n^{n+}$ with $\psi_T(\xi) = n$ and $\psi_T(\xi') = n$ for all $\xi' \subseteq \xi$ with $|\xi'| = |\xi| - 1$ and $\xi' \in \mathcal{F}_n^{n+}$. If $\psi_T(\xi) = n$ and some vertex of $\mathcal{F}_n^{n+}$ immediately below ξ is not in direction n, then ξ is an **unsettled vertex**.

Definition 7.2.12. A **settled tree** T with respect to n is an upright spanning tree of Q_n such that each vertex ξ of $\mathcal{F}_n^{n+}$ with $\psi_T(\xi) = n$ is settled. Note that if T is settled and $\psi_T(\xi) = n$, then $\psi_T(\eta) = n$ for all $\{n\} \subseteq \eta \subseteq \xi$.

An example of a settled tree of Q_4 is shown in Figure 7.7. We use the word "settled" in the sense of suspended particles descending until they come to rest at the bottom. We regard the labels in direction n (the direction of splitting) as the particles descending until they settle at the bottom of $\mathcal{F}_n^{n+}$. Working with settled trees is easier as we know where the labels in the direction of splitting are and then we can swap labels using the previous lemmas. This allows us to move from one settled tree to another by swapping labels in $\mathcal{F}_n^{n+}$.

The following lemma shows that every upright spanning tree of Q_n that has a connected irreducible signature in the lower n face $\mathcal{F}_n^{n-}$ is connected to a settled tree by edge slides.

Lemma 7.2.13. *Let $\mathcal{S} = (a_1, \ldots, a_n)$ be a signature of Q_n and let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be a connected irreducible splitting signature of $\mathcal{S}$ with respect to n. Let T be an upright spanning tree of Q_n with signature $\mathcal{S}$ such that T_{n-} has signature $\mathcal{D}$. Then there exists a sequence of edge slides that transforms T into a settled tree $\hat{T}$ with all labels of $\hat{T}_{n-}$ the same as the labels of T_{n-}. This can be done through a series of label swaps in which the labels in direction n only move downwards.*

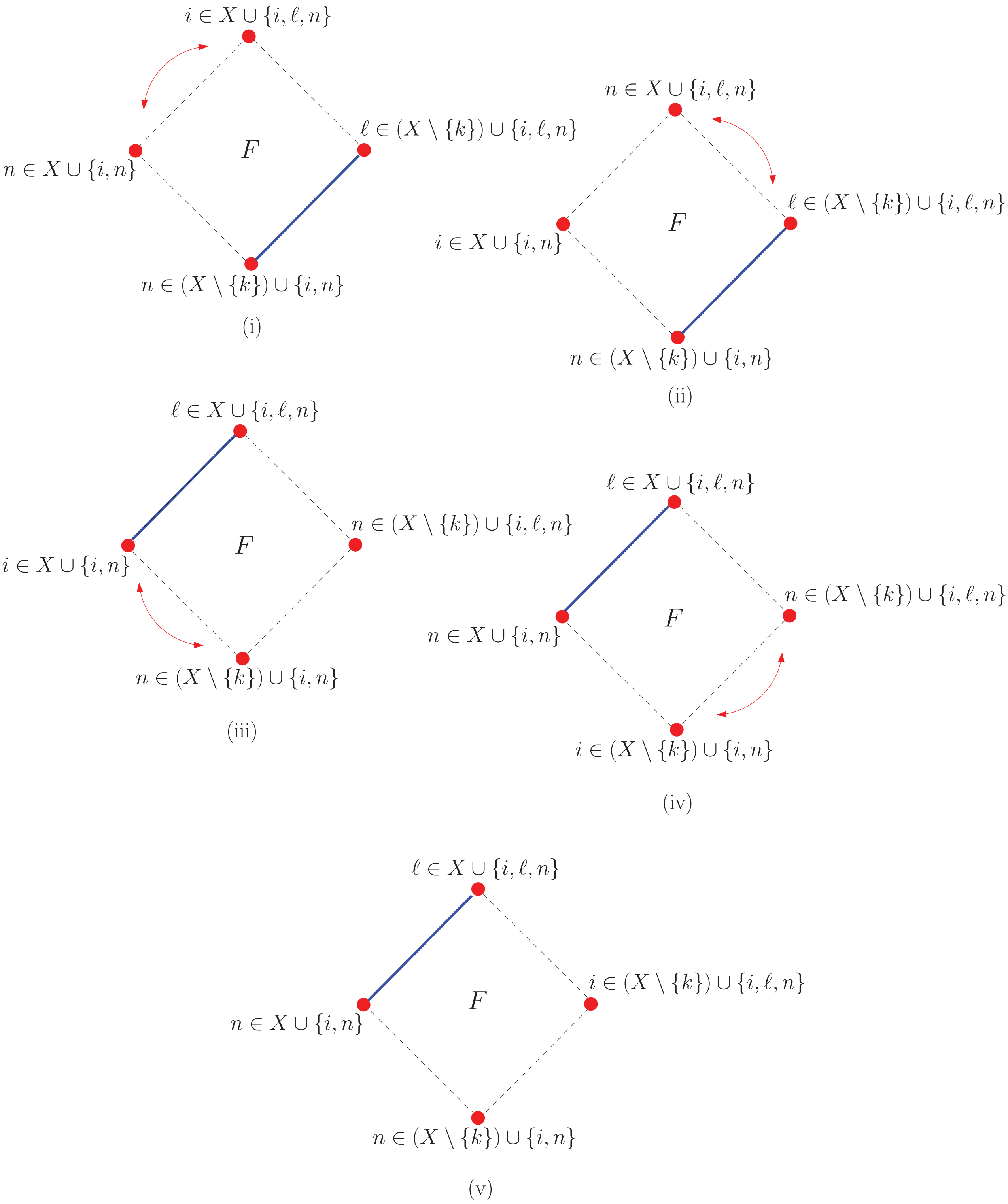


Figure 7.6: Diagram for the proof of Corollary 7.2.10 on page 90. A series of edge slides that swaps two labels at adjacent vertices in a face F of $\mathcal{F}_n^{n+}$ to transform one upright spanning tree into another.

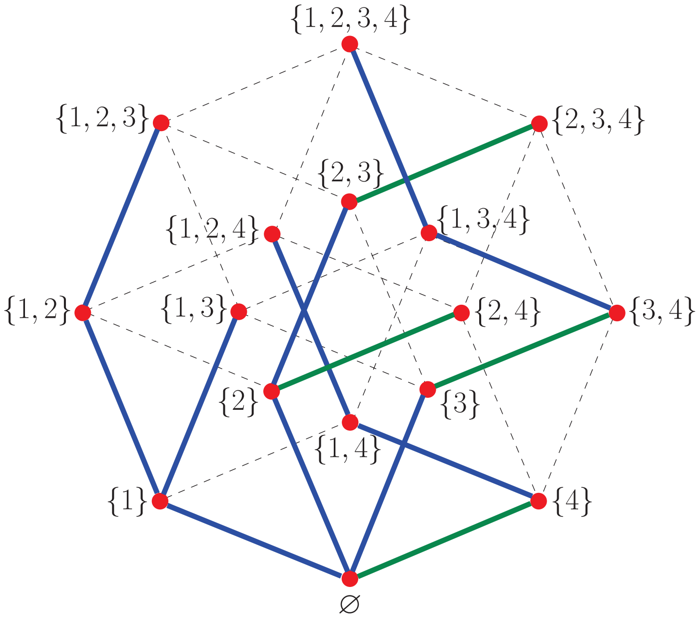


Figure 7.7: A settled tree of Q_4 with signature (3, 4, 4, 4) in bold edges. As we can see all vertices of $\mathcal{F}_4^{4+}$ immediately below a 4-edge are in direction 4 (green edges).

Proof. Let L be the sum of the levels of the vertices with label n. We show that if there is some vertex of $\mathcal{F}_n^{n+}$ with direction n such that some vertex below it is not in direction n, then we can move the n down one level using a sequence of edge slides, without moving any other n up. So L is decreased. However, L cannot decrease indefinitely, so we must eventually reach a settled tree.

Without loss of generality we may assume $\mathcal{D}$ is ordered. Suppose ξ is a lowest unsettled vertex of $\mathcal{F}_n^{n+}$. Let $m = \max(\xi \setminus \{n\})$. We distinguish the following cases according to whether or not $\psi_T(\xi \setminus \{m\}) = n$.

(a) Suppose that $\psi_T(\xi \setminus \{m\}) \neq n$. Then $\psi_T(\xi \setminus \{m\}) = i$ for some $i \neq n$. Let T' be the upright spanning tree obtained by swapping $\psi_T(\xi)$ and $\psi_T(\xi \setminus \{m\})$; in other words, the tree with $\psi_{T'}(\xi) = i$, $\psi_{T'}(\xi \setminus \{m\}) = n$ and all other labels of T' the same as the labels of T. This tree exists because $n \in \xi \setminus \{m\}$, and $i \in \xi \setminus \{m\}$ implies $i \in \xi$. Since $m = \max(\xi \setminus \{n\})$, we can apply Case 2 of Lemma 7.2.2 to transform T' into T. Therefore applying those edge slides in reverse T can be transformed into the required tree T'. Then L has been decreased by one, as required.

(b) Suppose that $\psi_T(\xi \setminus \{m\}) = n$. Let $j = \max\{\ell \in (\xi \setminus \{n\}) | \psi_T(\xi \setminus \{\ell\}) \neq n\}$. Note that j exists because of the choice of ξ. We distinguish the following cases according to whether or not $\psi_T(\xi \setminus \{j\}) = m$.

(I) Suppose that $\psi_T(\xi \setminus \{j\}) = m$. Let T' be the tree with $\psi_{T'}(\xi) = m$, $\psi_{T'}(\xi \setminus \{j\}) = n$ and all other labels of $\hat{T}$ the same as the labels of T. This tree exists because $n \in \xi \setminus \{j\}$, and $m \in \xi \setminus \{j\}$ implies $m \in \xi$. Applying Case 1 of Lemma 7.2.2, T' can be transformed into T. Then L has been decreased by one, as required.

(II) Suppose that $\psi_T(\xi \setminus \{j\}) \neq m$. Then $\psi_T(\xi \setminus \{j\}) = i$ for some $i \neq m, n$. Let $P_{\mathcal{F}_n^{n+}}(\xi \setminus \{j\}, i) = (\xi'_0, \xi'_1 \ldots, \xi'_\alpha)$ be the i-retaining max removing path from $\xi'_0 = \xi \setminus \{j\}$ to $\xi'_\alpha = \{i, n\}$. Since $m \neq j$ and $m \in \xi \setminus \{j\}$, we have $m'_0 = \max(\xi'_0 \setminus \{i, n\}) = m$ and therefore ξ_a for all $a = 1, \ldots, \alpha'$ are below the vertex $\xi \setminus \{m\}$. Since $\psi_T(\xi \setminus \{m\}) = n$ and ξ is a lowest unsettled vertex, we have $\psi_T(\xi'_a) = n$ for $a = 1, \ldots, \alpha'$.

Let $P_{\mathcal{F}_n^{n+}}(\xi, i) = (\xi_0, \ldots, \xi_\alpha)$ be the i-retaining max removing path from $\xi_0 = \xi$ to $\xi_\alpha = \{i, n\}$. Since ξ is a lowest unsettled vertex of $\mathcal{F}_n^{n+}$ and since $\psi_T(\xi \setminus \{m\}) = n$, we have $\psi_T(\xi_a) = n$ for $a = 0, \ldots, \alpha$.

Since $\psi_T(\xi \setminus \{j\}) = i$, $\psi_T(\xi) = n$, $P_{\mathcal{F}_n^{n+}}(\xi \setminus \{j\}, i)$ and $P_{\mathcal{F}_n^{n+}}(\xi, i)$ satisfy the hypotheses in Corollary 7.2.6, the tree T can be transformed into T' with $\psi_{T'}(\xi) = i$, $\psi_{T'}(\xi \setminus \{j\}) = n$ and all other labels of T' the same as the labels of T. Then L has been decreased by one, as required.

In all cases we showed we can move the n down one level without moving any other n up. So L is decreased by one, as required. □

7.2.2.1 Swapping labels of a settled tree of Q_n in the upper n face

In this section we show that under certain conditions labels of a settled tree of Q_n in the upper n face can be swapped.

The first lemma concerns swapping labels of a settled tree in $\mathcal{F}_n^{n+}$, where the tree has an ordered connected irreducible splitting signature in $\mathcal{F}_n^{n-}$. We use Lemma 7.2.2 to swap the labels.

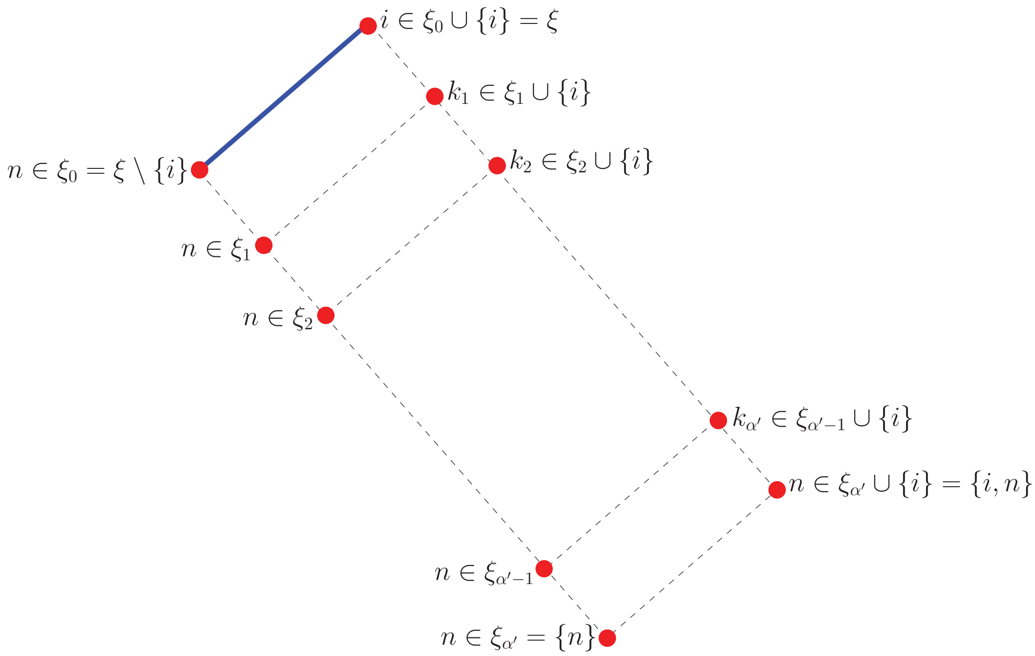


Figure 7.8: Diagram for Case 1 for the proof of Lemma 7.2.14. The paths $P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\}) = (\xi_0, \ldots, \xi_{\alpha'})$ and $\sigma_i(P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\})) = (\xi_0 \cup \{i\}, \ldots, \xi_{\alpha'} \cup \{i\})$ in $\mathcal{F}_n^{n+}$ with their vertex labels.

Lemma 7.2.14. *Let $\mathcal{S} = (a_1, \ldots, a_n)$ be a signature of Q_n and let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be an ordered connected irreducible splitting signature of $\mathcal{S}$ with respect to n. Let T be an upright spanning tree with signature $\mathcal{S}$ such that T_{n-} has signature $\mathcal{D}$. Let ξ be a vertex of $\mathcal{F}_n^{n+}$ of level α with $\psi_T(\xi) = i$. Suppose that all the vertices in Q_ξ with an n-edge are settled and suppose that $\psi_T(\{i, n\}) = n$.*

1. *Suppose that $\psi_T(\xi \setminus \{i\}) = n$. Then there exists a sequence of edge slides in Q_ξ that transforms T into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$ and all other labels of $\hat{T}$ apart from the labels of $\hat{T} \cap (Q_\xi \cap \mathcal{F}_n^{n+})$ the same as the labels of T.*

2. *Suppose that $\psi_T(\xi \setminus \{i\}) \neq n$. Then there exists a sequence of edge slides in Q_ξ that transforms T into $\hat{T}$ with $\psi_{\hat{T}}(\xi) \neq i$ and all other labels of $\hat{T}$ apart from the labels of $\hat{T} \cap (Q_\xi \cap \mathcal{F}_n^{n+})$ the same as the labels of T.*

Proof.

1. Let $P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\}) = (\xi_0, \ldots, \xi_{\alpha'})$ be the max removing path in $\mathcal{F}_n^{n+}$ from $\xi_0 = \xi \setminus \{i\}$ to $\xi_{\alpha'} = \{n\}$. Since all the vertices in Q_ξ are settled and $\psi_T(\xi \setminus \{i\}) = n$, we have $\psi_T(\xi_a) = n$ for all a. Consider the i-retaining max removing path $\sigma_i(P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\})) = (\xi_0 \cup \{i\}, \ldots, \xi_{\alpha'} \cup \{i\})$ from $\xi_0 \cup \{i\} = \xi$ to $\xi_{\alpha'} \cup \{i\} = \{i, n\}$. Let $k_a = \psi_T(\xi_a \cup \{i\})$ for $a = 0, \ldots, \alpha'$ as shown in Figure 7.8. Then $k_0 = i$. Let h be the least index in $0 \leq a \leq \alpha'$ such that $\psi_T(\xi_h \cup \{i\}) = n$. Note that such h exists because $\psi_T(\{i, n\}) = n$. Then $k_{h-1} \neq n$. We distinguish the following cases according to whether or not $k_{h-1} = \max(\xi_{h-1} \setminus \{n\})$.

   (a) Suppose that $k_{h-1} \neq \max(\xi_{h-1} \setminus \{n\})$. Then applying Case 1 of Lemma 7.2.2, T can be transformed into T' with $\psi_{T'}(\xi_{h-1} \cup \{i\}) = n$, $\psi_{T'}(\xi_h \cup \{i\}) = k_{h-1}$, and all other labels of T' the same as the labels of T.

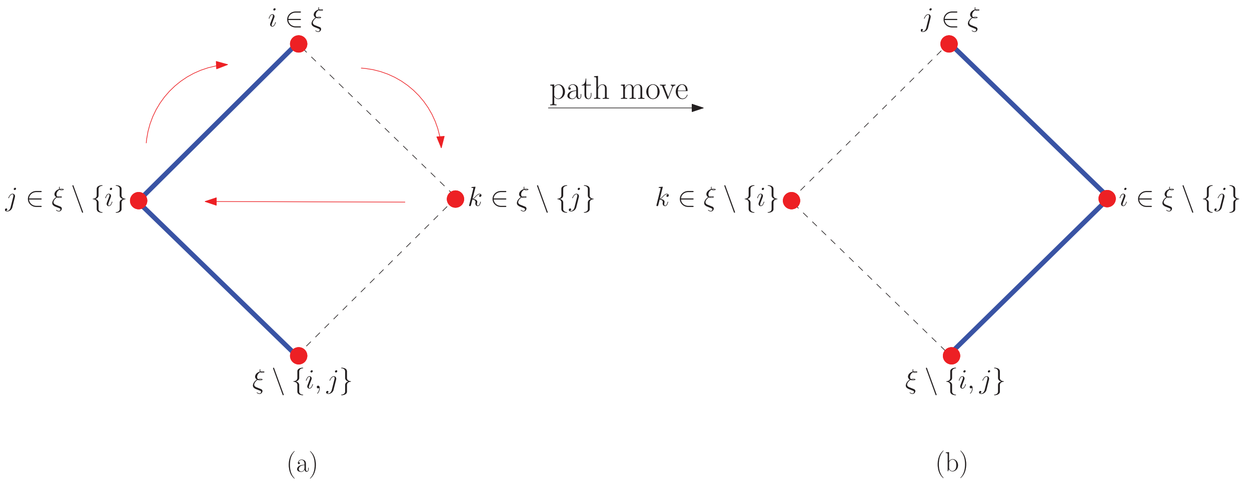


Figure 7.9: Diagram for Case 2(a) for the proof of Lemma 7.2.14. (a) A two dimensional face of $\mathcal{F}_n^{n+}$ where direction i is chosen at ξ, direction j is chosen at $\xi \setminus \{i\}$ and direction k is chosen at $\xi \setminus \{j\}$. (b) After applying the path move direction k moves to $\xi \setminus \{i\}$, direction j moves to $\xi \setminus \{i\}$ and direction i moves to $\xi \setminus \{j\}$.

(b) Suppose that $k_{h-1} = \max(\xi_{h-1} \setminus \{n\})$. Then applying Case 2 of Lemma 7.2.2, T can be transformed into T' with $\psi_{T'}(\xi_{h-1} \cup \{i\}) = n$, $\psi_{T'}(\xi_{h-1}) = k_{h-1}$, and all other labels of T' the same as the labels of T.

So now $h-1$ is the least index in $0 \leq a \leq \alpha'$ such that $\xi_{h-1} \cup \{i\}$ has an n-edge. If $\xi_{h-1} \cup \{i\} \neq \xi \cup \{i\}$, then we repeat the above process of moving an n-edge up until eventually we reach a tree where $\xi \cup \{i\}$ is in direction n.

2. Let $\psi_T(\xi \setminus \{i\}) = j$ for some $j \neq n$ and let $\psi_T(\xi \setminus \{j\}) = k$ for some k. We distinguish the following cases according to whether or not $k = i$.

(a) Suppose that $k \neq i$. As shown in Figure 7.9, using the path move the label with direction k moves to $\xi \setminus \{i\}$, direction j moves to ξ and direction i moves to $\xi \setminus \{j\}$. Then T is transformed into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = j$, $\psi_{\hat{T}}(\xi \setminus \{i\}) = k$, $\psi_{\hat{T}}(\xi \setminus \{j\}) = i$, and all other labels of $\hat{T}$ the same as the labels of T. So we reach the required tree $\hat{T}$ where $\psi_{\hat{T}}(\xi) \neq i$.

(b) Suppose that $k = i$. Let $\xi_0 = \xi \setminus \{i\}$ and for $a = 0, \ldots, \alpha - 2$ let $\ell_a = \psi_T(\xi_a)$ and $\xi_{a+1} = \xi_a \setminus \{\ell_a\}$. Let $P_T(\xi \setminus \{i\}) = (\xi_0, \ldots, \xi_\alpha)$ be the path in T from $\xi_0 = \xi \setminus \{i\}$ to $\xi_\alpha = \varnothing$. Let $\alpha' \leq \alpha - 2$ be such that $\xi_{\alpha'}$ is a vertex in $\mathcal{F}_n^{n+}$ with $\psi_T(\xi_{\alpha'}) = n$. Note that α' exists because every path in T from a vertex of $\mathcal{F}_n^{n+}$ to the root $\varnothing$ of T must pass through an n-edge. Consider $\sigma_i(P_T(\xi \setminus \{i\})) = (\xi_0 \cup \{i\}, \ldots, \xi_{\alpha'} \cup \{i\})$ and let $k_a = \psi_T(\xi_a \cup \{i\})$ for $a = 0, \ldots, \alpha'$, as shown in Figure 7.10. Let h be the least index in $0 \leq a \leq \alpha'$ such that $\psi_T(\xi_h \cup \{i\}) \neq i$. Note that if $\psi_T(\xi_a \cup \{i\}) = i$ for all $a = 0, \ldots, \alpha'$ then by Part 1 of Lemma 7.2.14 after a sequence of edge slides in $Q_{\xi_{\alpha'} \cup \{i\}}$ we have $\psi_T(\xi_{\alpha'} \cup \{i\}) = n$. As shown in Figure 7.11, using the path move the k_h-edge moves to ξ_{h-1}, the ℓ_{h-1}-edge moves to $\xi_{h-1} \cup \{i\}$ and the i-edge moves

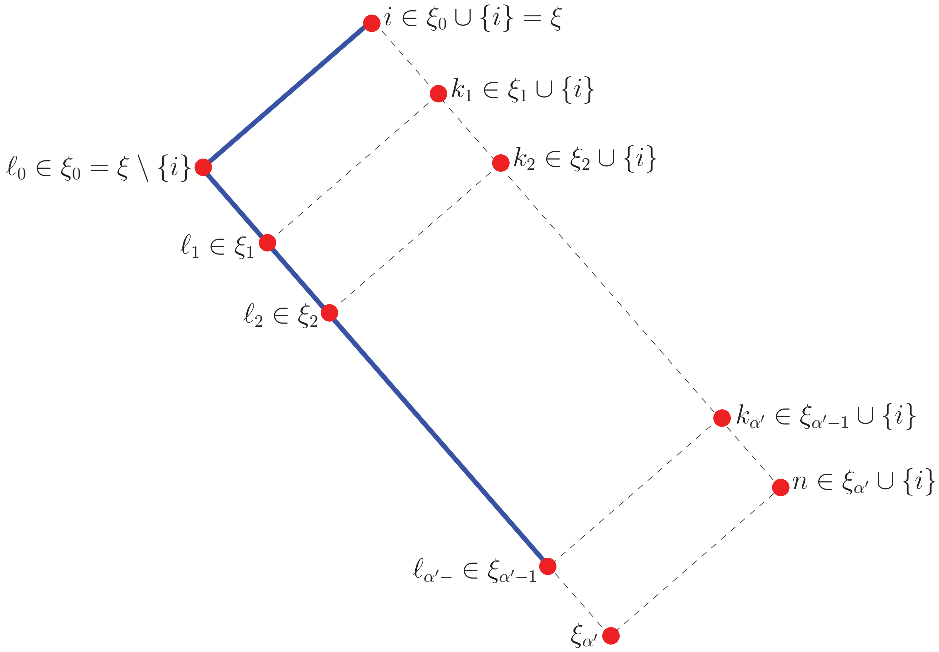


Figure 7.10: Diagram 1 for Case 2(b) for the proof of Lemma 7.2.14 on page 95. The paths $P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\}) = (\xi_0, \dots, \xi_{\alpha'})$ and $\sigma_i(P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\})) = (\xi_0 \cup \{i\}, \dots, \xi_{\alpha'} \cup \{i\})$ in $\mathcal{F}_n^{n+}$ with their vertex labels.

to $\xi_h \cup \{i\}$. Then T is transformed into T' where

$$\psi_{T'}(V) = \begin{cases} k_h, & \text{if } V = \xi_{h-1}; \\ \ell_{h-1}, & \text{if } V = \xi_{h-1} \cup \{i\}; \\ i, & \text{if } V = \xi_h \cup \{i\}; \\ \psi_T(V), & \text{otherwise.} \end{cases}$$

So now $h-1$ is the least index in $0 \leq a \leq \alpha'$ such that $\psi_{T'}(\xi_{h-1} \cup \{i\}) \neq i$. If $\xi_{h-1} \cup \{i\} \neq \xi \cup \{i\}$, then we repeat using the path move until eventually we reach the tree $\hat{T}$ with $\psi_{\hat{T}}(\xi_0 \cup \{i\} = \xi) \neq i$ and all other labels of $\hat{T}$ apart from $\hat{T} \cap (Q_\xi \cap \mathcal{F}_n^{n+})$ the same as the labels of T. Note that we can move to such a tree because all the labels of vertices above the vertex $\xi_{h-1} \cup \{i\}$ in $\sigma_i(P_T(\xi))$ were unchanged. This means all the labels of higher levels than the level of $\xi_{h-1} \cup \{i\}$ were unchanged, and therefore we can use the path move in the same way we used it above.

In all cases we showed we can move to the tree $\hat{T}$ with $\psi_{\hat{T}}(\xi) \neq i$ and all other labels apart from the labels of $\hat{T} \cap (Q_\xi \cap \mathcal{F}_n^{n+})$ the same as the labels of T. $\square$

The following lemma shows that a settled tree with $\{j, n\}$ in direction j for some j can be transformed into a settled tree where $\{j, n\}$ is in direction n.

Lemma 7.2.15. *Let $\mathcal{I} = (a_1, \dots, a_n)$ be an irreducible signature of Q_n with $a_n \geq a_i$ for all i and let $\mathcal{D} = (d_1, \dots, d_{n-1})$ be an ordered connected irreducible splitting signature of $\mathcal{I}$ with respect to n. Let T be a settled tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Let $j \in [n-1]$ and suppose that $\psi_T(\{j, n\}) = j$. Then there exists a sequence of edge slides that*

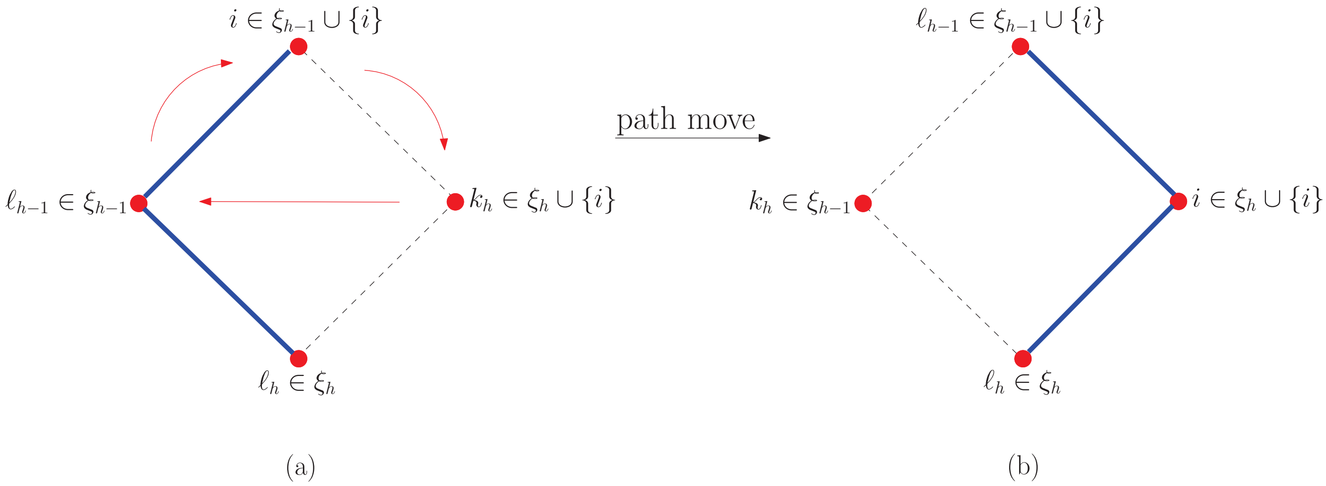


Figure 7.11: Diagram 2 for Case 2(b) for the proof of Lemma 7.2.14 on page 95. (a) A two dimensional face of $\mathcal{F}_n^{n+}$ where direction i is chosen at $\xi_{h-1} \cup \{i\}$, direction ℓ_{h-1} is chosen at ξ_{h-1}, direction ℓ_h is chosen at ξ_h and k_h is chosen at $\xi_h \cup \{i\}$. (b) After applying the path move direction k moves to $\xi \setminus \{i\}$, direction j moves to $\xi \setminus \{i\}$ and direction i moves to $\xi \setminus \{j\}$.

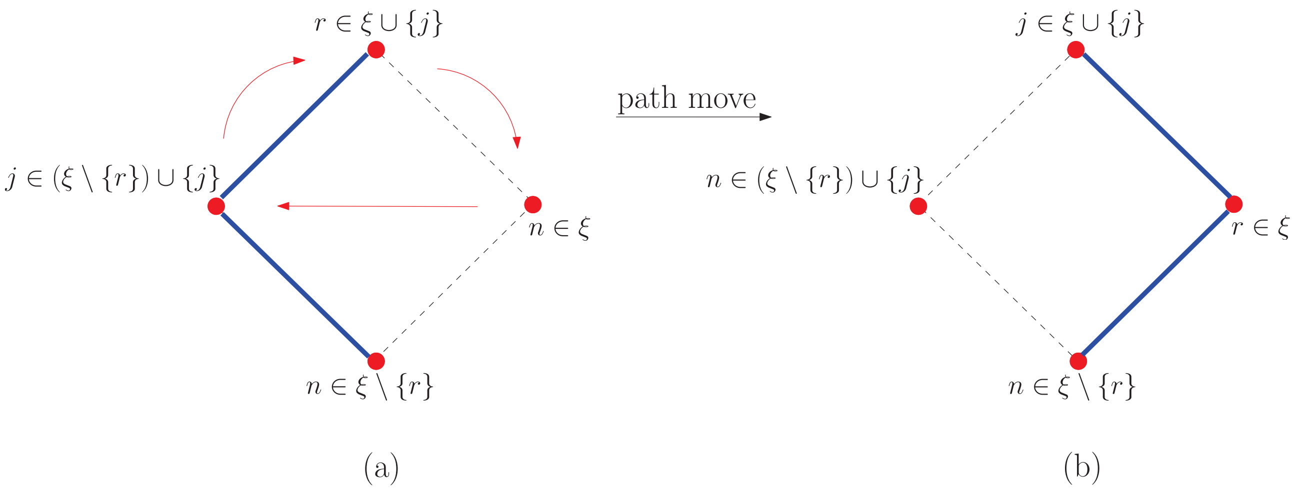


Figure 7.12: Diagram for the proof of Lemma 7.2.15 on page 97. (a) Direction r is chosen at ξ, direction j is chosen at $\xi \setminus \{r\}$ and direction n is chosen at $\xi \setminus \{j\}$ and $\xi \setminus \{j, r\}$. (b) After the path move the n-edge moves to $\xi \setminus \{r\}$, the j-edge moves to ξ and the r-edge moves to $\xi \setminus \{j\}$.

transforms T into a settled tree $\hat{T}$ such that $\psi_{\hat{T}}(\{j, n\}) = n$ and all the labels of $\hat{T}_{n-}$ are the same as the labels of T_{n-}.

Proof. Let X be the set of the vertices of $\mathcal{F}_n^{n+}$. Let Y be the set of all the vertices of $\mathcal{F}_n^{n+}$ that have an n-edge and let S be the union of the sets in Y. Since T is settled with $\psi_T(\{j, n\}) = j$, no vertex containing j in $\mathcal{F}_n^{n+}$ is in direction n.

Consider $\xi' \cup \{j\}$ for each $\xi' \in Y$ except $\{n\}$. Then there are $a_n - 1$ such vertices and none of these vertices is in direction n because they all contain j. Since $u_j \leq a_n - 2$, there must exist a vertex $\xi' \in Y$ such that $\psi_T(\xi' \cup \{j\}) \neq j$. Let $\xi \in Y$ be a lowest vertex in Y such that $\psi_T(\xi \cup \{j\}) \neq j$. Then $\psi_T(\xi \cup \{j\}) = r$ for some $r \in S$. So $\xi \setminus \{r\} \in Y$ (because $\xi \in Y$ and T is settled) and $\psi_T((\xi \setminus \{r\}) \cup \{j\}) = j$. Note that $\psi_T(\xi) = n$ by the choice of ξ.

As seen in Figure 7.12, using the path move the n-edge moves to $(\xi \setminus \{r\}) \cup \{j\}$, the j-edge moves to $\xi \cup \{j\}$, and the r-edge moves to ξ. Let T' be the resulting tree with $\psi_{T'}(\xi \cup \{j\}) = j$, $\psi_{T'}((\xi \setminus \{r\}) \cup \{j\}) = n$, $\psi_{T'}(\xi) = r$ and all other labels of T' the same as the labels of T. Since the path move was applied in $\mathcal{F}_n^{n+}$, we have $T_{n-} = T'_{n-}$.

So now $\xi \setminus \{r\} \in Y$ is the lowest vertex in Y such that $\psi_{T'}((\xi \setminus \{r\}) \cup \{j\}) \neq j$. Let $P_T((\xi \setminus \{r\}) \cup \{j\}, j) = (\xi_0, \ldots, \xi_\alpha)$ be the j-retaining max removing path from $\xi_0 = (\xi \setminus \{r\}) \cup \{j\}$ to $\xi_\alpha = \{j, n\}$. We show now that T' can be transformed into T_α with $\psi_{T_\alpha}(\{j, n\}) = n$, $\psi_{T_\alpha}((\xi \setminus \{r\}) \cup \{j\}) = j$, and all other labels of T_α the same as the labels of T'. The argument is similar to the proof of Lemma 7.2.5, with the roles of i and n taken by n and j.

Since $\xi \setminus \{r\} \in Y$ is the lowest vertex in Y such that $\psi_{T'}((\xi \setminus \{r\}) \cup \{j\}) \neq j$, we have $\psi_{T'}(\xi_a \cup \{j\}) = j$ for $a = 1, \ldots, \alpha$. For $a = 0, \ldots, \alpha$, let T_a be the tree such that $\psi_{T_a}(\xi_a \cup \{j\}) = n$, $\psi_{T_a}(\xi_b \cup \{j\}) = j$ for $b \neq a$, and all other labels of T_a are the same as the labels of T'. Note that these trees exist because $T_0 = T'$ exists, and j and n both belong to $\xi_a \cup \{j\}$ for all a and we have $T_0 = T'$.

Since $\mathcal{D}$ is an irreducible signature and $m_a = \max \xi_a \setminus \{n\}$, by Lemma 3.2.7 there exists a tree T''_{n-} of Q_{n-1} with signature $\mathcal{D}$ where $\psi_{T''}((\xi_a \setminus \{n\}) \cup \{j\}) = m_a$ and $\psi_{T''}((\xi_{a+1} \setminus \{n\}) \cup \{j\}) = j$. Since the edge slide graph of signature $\mathcal{D}$ is connected, we can move from T'_{n-} to T''_{n-} using edge slides. Then T' is transformed into T'' where $\psi_{T''}((\xi_a \setminus \{n\}) \cup \{j\}) = m_a$, $\psi_{T''}((\xi_{a+1} \setminus \{n\}) \cup \{j\}) =$

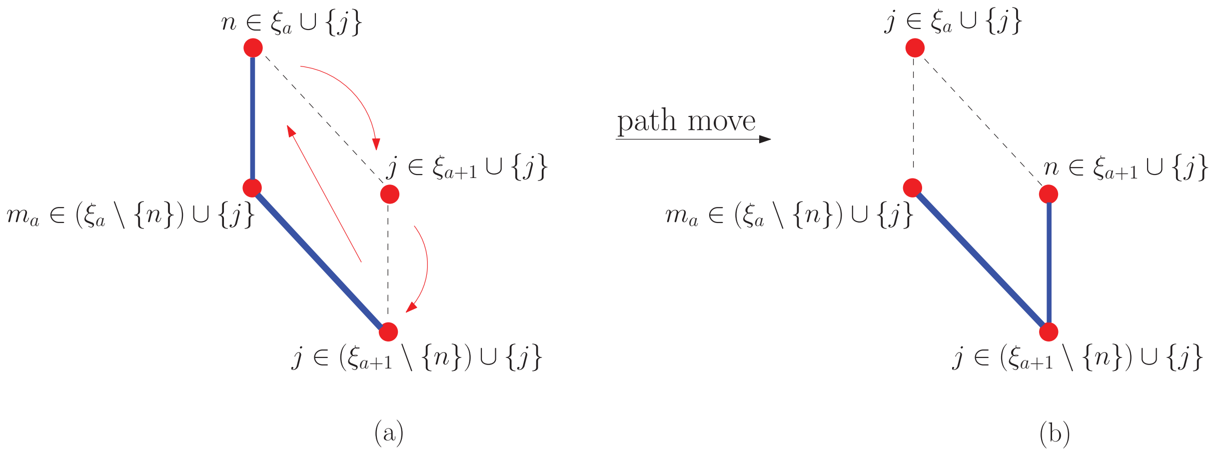


Figure 7.13: Diagram for the proof of Lemma 7.2.15 on page 97. (a) Direction n is chosen at ξ_a, direction m_a is chosen at $\xi_a \setminus \{n\}$ and direction j is chosen at ξ_{a+1} and $\xi_{a+1} \setminus \{n\}$. (b) After applying the V-move labels n and j are swapped.

j and all labels of T''_{n+} are the same as the labels of T'_{n+}.

As shown in Figure 7.13 using the V-move the labels in directions n and j are swapped. Note that the sequence of edge slides applied to T'_{n-} to get to T''_{n-} can be reversed to get back to T'_{n-}. Then T_a is transformed into T_{a+1} with $\psi_{T_{a+1}}(\xi_{a+1} \cup \{j\}) = n$, $\psi_{T_{a+1}}(\xi_b \cup \{j\}) = j$, and all other labels of T_{a+1} the same as the labels of T'. Since $T_{n-} = T'_{n-}$, we have $(T_{a+1})_{n-} = T'_{n-}$. Therefore by induction $T_0 = T'$ can be transformed into T_α such that $\psi_{T_\alpha}(\xi_\alpha = \{j, n\}) = n$ and all the labels of $(T_\alpha)_{n-}$ are the same as the labels of T_{n-}.

Suppose that T_α is not settled. Then applying Lemma 7.2.13 T_α can be transformed into a settled tree $\hat{T}$. Since settling only moves the n-edges down, the vertex $\{j, n\}$ is still in direction n. □

Example 7.2.16. Let T be a settled tree of Q_4 with signature (3, 4, 4, 4) such that T_{4-} has signature (2, 2, 3). Then the forest T_{4+} has signature (1, 2, 1, 4). Altogether there are nine settled forests with signature (1, 2, 1, 4): five settled forests where $\{t, 4\}$ is not in direction 4 for some $t \in [3]$, and four settled forests where $\{t, 4\}$ is in direction 4 for all t. These are shown in Figure 7.14 and Figure 7.15 respectively.

Suppose that direction 4 is chosen at $\{2, 4\}$, $\{3, 4\}$ and $\{2, 3, 4\}$, direction 1 is chosen at $\{1, 4\}$, direction 2 is chosen at $\{1, 2, 4\}$ and $\{1, 2, 3, 4\}$ and direction 3 is chosen at $\{1, 3, 4\}$, as shown in Figure 7.16(i). Apply the path move on the face $\{\{1, 2, 4\}, \{2, 4\}, \{4\}, \{1, 4\}\}$. Then T is transformed into T' with $\psi_{T'}(\{1, 2, 4\}) = 1$, $\psi_{T'}(\{1, 4\}) = 4$, $\psi_{T'}(\{2, 4\}) = 2$ and all other labels of T' the same as the labels of T. Therefore $\psi_T(\{2, 3, 4\}) = \psi_{T'}(\{2, 3, 4\}) = 4$. Figure 7.16(ii) shows T'_{4+}.

Since (2, 2, 3) is irreducible, by Lemma 3.2.4 there exists a tree T''_{4-} of Q_3 with signature (2, 2, 3) where 3 is chosen at $\{2, 3\}$. Since the edge slide graph of signature (2, 2, 3) is connected (by Henden [6] or Lemma 4.3.6), we can move from T' to such a tree using edge slides in $Q_{[3]}$. As shown in Figure 7.16(iii), applying the V-move the labels 2 and 4 are swapped as shown in Figure 7.16(iv). Therefore T' is transformed into $\hat{T}$ with $\psi_{\hat{T}}(\{2, 3, 4\}) = 2$, $\psi_{\hat{T}}(\{2, 4\}) = 4$ and all other labels of $\hat{T}$ the same as the labels of T'. Since $\psi_{\hat{T}}(\{1, 4\}) = \psi_{\hat{T}}(\{2, 4\}) = \psi_{\hat{T}}(\{3, 4\})$, and since there is no 4 chosen at any vertex of cardinality 3 or 4, we conclude that $\hat{T}$ is a settled tree with $\psi_{\hat{T}}(\{t, 4\}) = 4$ for all $t \in [3]$. Figure 7.16 shows $\hat{T}_{4+}$.

In a similar way, we can show that we can move from any settled tree of Q_4 with signature (3, 4, 4, 4) such that the tree has the splitting signature (2, 2, 3) to a settled tree where $\{t, 4\}$ is in direction 4 for all $t \in [3]$. There are four settled forests with signature (1, 2, 1, 4) such that $\{t, n\}$ is in direction 4 for all t, and the trees have the splitting signature (2, 2, 3) in $\mathcal{F}_4^{4-}$.

Lemma 7.2.17 along with Lemma 7.2.18 are used to show that for $n \geq 5$, there exists a settled tree of Q_n with an irreducible signature and a super rich splitting signature where $\{t, n\}$ is in direction n for all $t \in [n-1]$.

Lemma 7.2.17. *Let $n \geq 5$, and let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered irreducible signature of Q_n with $a_n \geq a_t$ for all $t \in [n-1]$. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be an ordered irreducible splitting signature of $\mathcal{I}$. Let T be a settled tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Let Y be the set of all the vertices of $\mathcal{F}_n^{n+}$ that have an n-edge and let S be the union of the sets in Y. Let $Z = [n] \setminus S$. Let $B \subseteq S$ be the set of directions t such that there is only one vertex $\xi \in Y$ where $t \in \xi$. Let $|B| = b$, $|Z| = z$ and let $\nu = \lfloor \log_2 n \rfloor$. Then $b + z \leq \nu$, and in particular $b, z \leq \nu$.*

Proof. Let $A \subseteq S$ be the set of directions $t \in [n-1]$ such that there are at least two vertices containing t with an n-edge. In other words, $A = S \setminus (B \cup \{n\})$. Let $|A| = a$. Then $a + b + z = n - 1$.

Every n lies above Q_A or at $\{t, n\}$ for some $t \in B$. So $a_n \leq |Q_A| + |B| = 2^a + b$. Since $\mathcal{I}$ is irreducible with $a_n \geq a_t$ for all t, we have $\frac{2^n-1}{n} \leq a_n$, and so $\left\lceil \frac{2^n-1}{n} \right\rceil \leq 2^a + b$.

For $n = 5$, this inequality is $7 \leq 2^a + b$ with $a + b + z = 4$. Suppose that $a \leq 2$. Then we must have $b \geq 3$. Thus we have a contradiction and so $a \geq 3$ and $b + z \leq 1 < 2 = \nu$.

For $n \geq 6$, the inequality is

$$\begin{aligned}
\frac{2^n - 1}{n} &\leq 2^a + b \\
2^n &\leq n2^a + nb + 1 && \text{(multiplied by } n\text{)} \\
&\leq n2^a + n(n-1) + 1 && \text{(since } b \leq n-1\text{)} \\
&= n2^a + n^2 - n + 1 \\
&< 2^{\nu+1}2^a + 2^{n-1} && \text{(since } n^2 - n + 1 < 2^{n-1} \text{ for all } n \geq 6\text{, and } n < 2^{\nu+1}\text{)} \\
2^{n-1} &< 2^{a+\nu+1}.
\end{aligned}$$

So $n-1 < a+\nu+1$ and therefore $n \leq a+\nu+1$. Since $a+b+z = n-1$, we have $a+1 = n-b-z$. Then $n \leq n - b - z + \nu$ and therefore $b + z \leq \nu$. □

Lemma 7.2.18. *Let $n \geq 5$ and let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered irreducible signature of Q_n with $a_n \geq a_t$ for all t. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be a super rich splitting signature of $\mathcal{I}$. Let T be a settled tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Then there exists a sequence of edge slides that transforms T into $\hat{T}$ such that $\psi_{\hat{T}}(\{t, n\}) = n$ for all t.*

Proof. With notation as in Lemma 7.2.17, if $z = 0$, then $S = [n]$ and therefore $\psi_T(\{t, n\}) = n$ for all t. Otherwise there exists $j \in Z$. Since $\mathcal{D}$ is a super rich signature, we have $u_j < a_n - \nu$ where $\nu = \lfloor \log_2 n \rfloor$.

So we have $b + 1$ edges in direction n located at $\{n\}$ and $\{t, n\}$ for all $t \in B$, and therefore $a_n - (b + 1) \geq a_n - \nu$ edges in direction n located in $Q_A \cup \{n\}$.

Since $u_j < a_n - \nu$, there exists a vertex $\varnothing \neq \zeta \subseteq A$ such that $\psi_T(\zeta \cup \{j, n\}) \neq j$ and $\psi_T(\zeta \cup \{n\}) = n$. Then the vertex ξ and direction r in the proof of Lemma 7.2.15 may be chosen such that $\xi = \zeta \cup \{n\} \subseteq A \cup \{n\}$ and $r = \psi_T(\xi \cup \{j\}) \in A$. Following the argument given there we reach a tree T' such that $\psi_{T'}(\{j,n\}) = n$ and $\psi_{T'}(V) = \psi_T(V)$ for all $V \in \mathcal{F}_n^{n-}$. Since the direction r belongs to A there were at least two vertices containing r with label n, after settling T' (if necessary) we can ensure we still have $\psi_{T'}(\{r,n\}) = n$. Therefore the number of directions t such that $\psi_T(\{t,n\}) = n$ has increased by one. Repeating this process we eventually reach a settled tree $\hat{T}$ such that $\psi_{\hat{T}}(\{t,n\}) = n$ for all t. □

The next lemma shows that the labels of a settled tree in $\mathcal{F}_n^{n+}$ can be swapped using edge slides. We require the tree to have a connected rich splitting signature in $\mathcal{F}_n^{n-}$ and we use Lemma 7.2.7 to swap the labels.

Lemma 7.2.19. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n and let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be a connected rich splitting signature of $\mathcal{I}$ with respect to n. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Let ξ be a vertex of $\mathcal{F}_n^{n+}$ of level α such that $\psi_T(\xi) \neq n$ and let $i \in \xi$. Suppose that all the vertices in Q_ξ with an n-edge are settled and suppose $\psi_T(\{i,n\}) = n$.*

1. *Suppose that $\psi_T(\xi \setminus \{i\}) = n$. Then there exists a sequence of edge slides in Q_ξ that transforms T into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$ and all other labels of $\hat{T}$ apart from the labels of $\hat{T} \cap (Q_\xi \cap \mathcal{F}_n^{n+})$ the same as the labels of T.*
2. *Suppose that $\psi_T(\xi) = i$ and $\psi_T(\xi \setminus \{i\}) \neq n$. Then there exists a sequence of edge slides that transforms T into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$ and all other labels of $\hat{T}$ apart from the labels of $\hat{T} \cap (Q_\xi \cap \mathcal{F}_n^{n+})$ the same as the labels of T.*

Proof.

1. We distinguish the following cases according to whether or not $\psi_T(\xi) = i$.

   (a) Suppose that $\psi_T(\xi) \neq i$. Then $\psi_T(\xi) = j$ for some $j \neq i$. Since $\mathcal{D}$ is a connected rich splitting signature, $\psi_T(\xi) = j$ and $\psi_T(\xi \setminus \{i\}) = n$, all conditions in Lemma 7.2.7 are satisfied at the vertices ξ and $\xi \setminus \{i\}$. Then T can be transformed into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$, $\psi_{\hat{T}}(\xi \setminus \{i\}) = j$ and all other labels of $\hat{T}$ the same as the labels of T.
   Figure 7.17 (a) shows the 2-dimensional face of $\mathcal{F}_n^{n+}$ where two vertices ξ and $\xi \setminus \{i\}$ are labelled j and n respectively by T and (b) shows the same face after edge slides as labelled by $\hat{T}$.

   (b) Suppose that $\psi_T(\xi) = i$. We distinguish the following cases according to whether or not $\psi_T(\xi \setminus \{j\}) = i$ for some $j \neq i$.

   (I) Suppose that $\psi_T(\xi \setminus \{j\}) \neq i$ for some $j \neq i$. Then $\psi_T(\xi \setminus \{j\}) = k$ for some $k \neq i$. We consider the following cases according to whether or not $k = n$.

   (i) Suppose that $k = n$. Since $\mathcal{D}$ is a connected rich splitting signature, $\psi_T(\xi) = i$ and $\psi_T(\xi \setminus \{j\}) = n$, all conditions in Lemma 7.2.7 are satisfied at the vertices ξ and $\xi \setminus \{j\}$. So T can be transformed into $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$, $\psi_{\hat{T}}(\xi \setminus \{j\}) = i$ and all other labels of $\hat{T}$ the same as the labels of T.
   Figure 7.18 (a) shows the 2-dimensional face of $\mathcal{F}_n^{n+}$ labelled by T and (b) shows the same face after edge slides as labelled by $\hat{T}$.

(ii) Suppose that $k \neq n$. Since all the vertices of Q_ξ with an n-edge are settled and since $\psi_T(\xi \setminus \{i\}) = n$, we have $\psi_T(\xi \setminus \{i, j\}) = n$. Since signature $\mathcal{D}$ is a connected rich splitting signature, $\psi_T(\xi \setminus \{j\}) = k$ and $\psi_T(\xi \setminus \{i, j\}) = n$, all conditions in Lemma 7.2.7 are satisfied at the vertices $\xi \setminus \{j\}$ and $\xi \setminus \{i, j\}$. Then T can be transformed into T' with $\psi_{T'}(\xi \setminus \{j\}) = n$, $\psi_{T'}(\xi \setminus \{i, j\}) = k$ and all other labels of T' the same as the labels of T. Now since $\psi_{T'}(\xi) = i$ and $\psi_{T'}(\xi \setminus \{j\}) = n$, all the conditions in Lemma 7.2.7 are satisfied at the vertices ξ and $\xi \setminus \{j\}$. Then T' can be transformed into T'' with $\psi_{T''}(\xi) = n$, $\psi_{T''}(\xi \setminus \{j\}) = i$ and all other labels of T'' the same as the labels of T'.
Figure 7.19 shows the 2-dimensional face labelled by T in (a), the same face labelled by T' in (b) and the same face labelled by T'' in (c).

(II) Suppose that $\psi_T(\xi \setminus \{j\}) = i$ for every $j \neq i$. Let $\xi_0 = \xi \setminus \{i\}$, $\alpha' = \alpha - 1$ and for $a = 0, \dots, \alpha' - 1$ let $P_{\mathcal{F}_n^{n+}} = (\xi_0, \xi_1, \dots, \xi_{\alpha'})$ be a descending path in $\mathcal{F}_n^{n+}$ from ξ_0 to $\xi_{\alpha'} = \{n\}$. Since all the vertices of Q_ξ with an n-edge are settled and $\psi_T(\xi \setminus \{i\}) = n$, we have $\psi_T(\xi_a) = n$ for all a. Consider $\sigma_i(P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\})) = (\xi_0 \cup \{i\}, \dots, \xi_{\alpha'} \cup \{i\})$ and let $k_a = \psi_T(\xi_a \cup \{i\})$ for $a = 0, \dots, \alpha'$. Since $\psi_T(\xi) = i$ and $\psi_T(\{i, n\}) = n$, we have $k_0 = i$ and $k_{\alpha'} = n$ as shown in Figure 7.20. Let h be the least index in $0 < h \leq \alpha'$ such that $k_h \neq i$. Note that h exists because $\psi_T(\{i, n\}) = n$. We show that there exists a sequence of edge slides to move an n up in $\sigma_i(P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\}))$ so T transforms into a tree where $\xi_0 \cup \{i\} = \xi$ is in direction n.
Since h is the least index in $0 < h \leq \alpha'$ such that $\psi_T(\xi_h \cup \{i\}) \neq i$, we have $\psi_T(\xi_{h-1} \cup \{i\}) = i$.
Suppose first that $k_h \neq n$, as shown in Figure 7.21 . Since $\mathcal{D}$ is a connected rich splitting signature, $\psi_T(\xi_h \cup \{i\}) = k_h \neq i$ and $\psi_T(\xi_h) = n$, all conditions in Lemma 7.2.7 are satisfied at the vertices $\xi_h \cup \{i\}$ and ξ_h. Then T can be transformed into T' with $\psi_{T'}(\xi_h \cup \{i\}) = n$, $\psi_T(\xi_h) = k_h$ and all other labels of T' the same as the labels of T. Therefore we move to the following case.
Suppose now that $k_h = n$, as shown in Figure 7.22 or Figure 7.21 (b) if we came from the case $k_h \neq n$. Since $\mathcal{D}$ is a connected rich splitting signature, $\psi_T(\xi_{h-1} \cup \{i\}) = i$ and $\psi_T(\xi_h \cup \{i\}) = n$, all conditions in Lemma 7.2.7 are satisfied at the vertices $\xi_{h-1} \cup \{i\}$ and $\xi_h \cup \{i\}$. Then T can be transformed into T' with $\psi_{T'}(\xi_{h-1} \cup \{i\}) = n$, $\psi_{T'}(\xi_h \cup \{i\}) = i$, and all other labels of T' the same as the labels of T.
So now $h - 1$ is the least index in $0 \leq a \leq \alpha'$ such that $\psi_{T'}(\xi_{h-1} \cup \{i\}) \neq i$. Note that the labels that changed belonged to the paths $P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\})$ and $\sigma_i(P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\}))$, which lie in Q_ξ.

If $\xi_{h-1} \cup \{i\} \neq \xi$, then we repeat the above process of moving up an n in $\sigma_i(P_{\mathcal{F}_n^{n+}}(\xi \setminus \{i\}))$ until we reach the tree $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$ and all other labels of $\hat{T}$ the same as the labels of T.

2. Let $\xi_0 = \xi \setminus \{i\}$ and for $a = 0, \dots, \alpha - 1$ let $k_a = \psi_T(\xi_a)$ and $\xi_{a+1} = \xi_a \setminus \{k_a\}$. Let $P_T(\xi \setminus \{i\}) = (\xi_0, \dots, \xi_{\alpha-1})$ be the path in T from ξ_0 to $\xi_{\alpha-1} = \varnothing$. Let $\beta \leq \alpha - 2$ be such that $\psi_T(\xi_\beta) = n$. Note that β exists because every path in T from a vertex in $\mathcal{F}_n^{n+}$ to the root of T must pass through an edge in direction n. Consider $\sigma_i(P_T(\xi \setminus \{i\})) = (\xi_0 \cup \{i\}, \dots, \xi_{\alpha-1} \cup \{i\})$. Let $\ell_a = \psi_T(\xi_a \cup \{i\})$ for $0 \leq a \leq \beta$. If $\ell_\beta \neq n$, then applying

Part 1 of this lemma (which we have just proved above) we get $\psi_T(\xi_\beta \cup \{i\}) = n$, as shown in Figure 7.23.

Let f be the largest index in $0 \le f \le \beta - 1$ such that $\psi_T(\xi_f \cup \{i\}) = i$. Then $\psi_T(\xi_{f+1} \cup \{i\}) = \ell_{f+1}$ for some $\ell_{f+1} \ne i$. We first show that the lowest i-edge in $\sigma_i(P_T(\xi \setminus \{i\}))$ can be moved down to $\xi_\beta \cup \{i\}$ using a sequence of path moves. As shown in Figure 7.24, using the path move the ℓ_{f+1}-edge moves to ξ_f, the k_f-edge moves to $\xi_f \cup \{i\}$ and the i-edge moves to $\xi_{f+1} \cup \{i\}$. So T is transformed into T' with $\psi_{T'}(\xi_f \cup \{i\}) = k_f$, $\psi_{T'}(\xi_f) = \ell_{f+1}$, $\psi_{T'}(\xi_{f+1} \cup \{i\}) = i$ and all other labels of T' the same as the labels of T.

So now $f+1$ is the largest index in $0 \le a \le \beta - 1$ such that $\xi_{f+1} \cup \{i\}$ is in direction i. If $\xi_{f+1} \cup \{i\} \ne \xi_\beta \cup \{i\}$, then we repeat the above process of moving the i-edge down until eventually we reach a tree T^2 where

$$\psi_{T^2}(V) = \begin{cases} k_a, & \text{if } V = \xi_a \cup \{i\} \text{ for } f \le a \le \beta - 1; \\ \ell_{a+1}, & \text{if } V = \xi_a \text{ for } f \le a \le \beta - 2; \\ i, & \text{if } V = \xi_\beta \cup \{i\}; \\ n, & \text{if } V = \xi_{\beta-1}; \\ \psi_{T'}(V) & \text{otherwise.} \end{cases}$$

Figure 7.25 shows the paths $P_{T^2}(\xi \setminus \{i\})$ and $\sigma_i(P_{T^2}(\xi \setminus \{i\}))$ as labelled by T^2.

We now show that the n-edge at $\xi_{\beta-1}$ can be moved up to the vertex ξ_f using a sequence of edge slides. Note that $\ell_j \ne k_{j-1}$ for any $1 \le j \le \alpha - 1$ because $k_{j-1} \in \xi_{j-1}$, but $k_{j-1} \notin \xi_j$. Since $\mathcal{D}$ is a connected rich splitting signature, $\psi_{T^2}(\xi_{\beta-2}) = \ell_{\beta-1}$ and $\psi_{T^2}(\xi_{\beta-1}) = n$, all conditions in Lemma 7.2.7 are satisfied at the vertices $\xi_{\beta-2}$ and $\xi_{\beta-1}$. Then T^2 can be transformed into T^3 with $\psi_{T^3}(\xi_{\beta-2}) = n$, $\psi_{T^3}(\xi_{\beta-1}) = \ell_{\beta-1}$ and all other labels of T^3 the same as the labels of T^2. So we repeat the above process until eventually we reach a tree T^4 where

$$\psi_{T^4}(V) = \begin{cases} n, & \text{if } V = \xi_f; \\ \ell_a, & \text{if } V = \xi_a \text{ for } f - 1 \le a \le \beta - 1; \\ \psi_{T^3}(V), & \text{otherwise.} \end{cases}$$

Figure 7.26 shows the paths $P_{T^4}(\xi \setminus \{i\})$ and $\sigma_i(P_{T^4}(\xi \setminus \{i\}))$ as labelled by T^4.

Since $\mathcal{D}$ is a connected rich splitting signature, $\psi_{T^4}(\xi_f \cup \{i\}) = k_f$ and $\psi_{T^4}(\xi_f) = n$, all conditions in Lemma 7.2.7 are satisfied at the vertices $\xi_f \cup \{i\}$ and ξ_f. Then T^4 can be transformed into T^5 where

$$\psi_{T^5}(V) = \begin{cases} n, & \text{if } V = \xi_f \cup \{i\}; \\ k_f, & \text{if } V = \xi_f; \\ \psi_{T^4}(V), & \text{otherwise.} \end{cases}$$

Figure 7.27 shows the paths $P_{T^5}(\xi \setminus \{i\})$ and $\sigma_i(P_{T^5}(\xi \setminus \{i\}))$ as labelled by T^5.

If $f \ne 0$, then there exists an index h such that $0 \le h < f$ and $\xi_h \cup \{i\}$ is in direction i. Then we repeat the above process starting with the largest such h until eventually we reach the required tree $\hat{T}$ where

$$\psi_{\hat{T}}(V) = \begin{cases} n, & \text{when } V = \xi_0 \cup \{i\}; \\ \psi_T(V), & \text{when } V \ne \xi_a,\ V \ne \xi_a \cup \{i\} \text{ for } 0 \le a \le \beta. \end{cases}$$

In all cases we showed we can move to the tree $\hat{T}$ with $\psi_{\hat{T}}(\xi) = n$ and all other labels apart from the labels of $\hat{T} \cap (Q_\xi \cap \mathcal{F}_n^{n+})$ the same as the labels of T. $\square$

7.3 Rearranging the labels in the upper n face

In this section we rearrange the labels (directions) of upright spanning trees of Q_n with an irreducible signature in $\mathcal{F}_n^{n+}$ such that the trees have a connected amenable signature in $\mathcal{F}_n^{n-}$. There are three types of amenable splitting signatures, namely the (2, 2, 3) splitting signature of Q_4, unidirectional splitting signatures and super rich splitting signatures. We show that the set of upright spanning trees of Q_n with an irreducible signature and the (2, 2, 3) splitting signature of Q_4 or a fixed connected unidirectional splitting signature forms a block. We conjecture under the inductive hypothesis this is also true for a super rich splitting signature, and present partial progress towards a proof of this.

7.3.1 The (2, 2, 3) splitting signature of an irreducible signature of Q_4

Altogether there are nine ordered irreducible signatures of Q_4, namely

$$\begin{array}{ccc} (2,\ 2,\ 4,\ 7) & (2,\ 3,\ 4,\ 6) & (3,\ 3,\ 3,\ 6) \\ (2,\ 2,\ 5,\ 6) & (2,\ 3,\ 5,\ 5) & (3,\ 3,\ 4,\ 5) \\ (2,\ 3,\ 3,\ 7) & (2,\ 4,\ 4,\ 5) & (3,\ 4,\ 4,\ 4). \end{array}$$

Since Q_3 has a unique irreducible signature up to permutation, there is a unique ordered irreducible splitting signature, namely (2, 2, 3). For signatures in the first column this is a unidirectional splitting signature and it can be handled using the result in Section 7.3.2. In addition, signatures (2, 2, 4, 7) and (2, 3, 3, 7) belong to the families $\mathcal{I}_n^{(-1)}$ and $\mathcal{I}_{(3,n)}^{(-1,-1)}$ respectively, where a proof for the connectivity of the edge slide graph of each family is given in Chapter 10, and we proved the connectivity of the edge slide graph $\mathcal{E}(2, 3, 3, 7)$ in Chapter 5. For columns two and three, we get multiple directions in $\mathcal{F}_4^{4+}$. In the following theorem we show for each of these six signatures that the set of upright spanning trees of Q_4 with the (2, 2, 3) splitting signature forms a block.

Theorem 7.3.1. *Let $\mathcal{I} \in \{(2, 3, 4, 6), (2, 3, 5, 5), (2, 4, 4, 5), (3, 3, 3, 6), (3, 3, 4, 5), (3, 4, 4, 4)\}$ be an ordered irreducible signature of Q_4. Then the set of upright spanning trees of Q_4 with signature $\mathcal{I}$ and the splitting signature (2, 2, 3) in $\mathcal{F}_4^{4-}$ forms a block.*

Proof. Since we always can move from an upright spanning tree to a settled tree where all the vertices of $\mathcal{F}_4^{4+}$ of level two are in direction 4, it suffices to show that we can move from any settled tree with signature $\mathcal{I}$ where all the vertices of $\mathcal{F}_4^{4+}$ of level two are in direction 4 to any other. For $\mathcal{I} \neq (3, 4, 4, 4)$, there are at least five 4-edges and so the condition $\psi_T(\{t, 4\}) = 4$ holds for all $t \in [3]$ for any settled tree. In the case (3, 4, 4, 4) we can move to such a tree as shown in Example 7.2.16. Since the edge slide graph of signature (2, 2, 3) is connected, we can move from any tree with signature (2, 2, 3) to any other using edge slides. So it suffices to show we can move from any settled forest in $\mathcal{F}_4^{4+}$ to any other. We do this using the path move, the V-move, or Corollary 7.2.6 applied to a pair of vertices. In Figures 7.28 to 7.33 we show for each signature in turn that there is a collection of such moves that suffices to connect the trees.

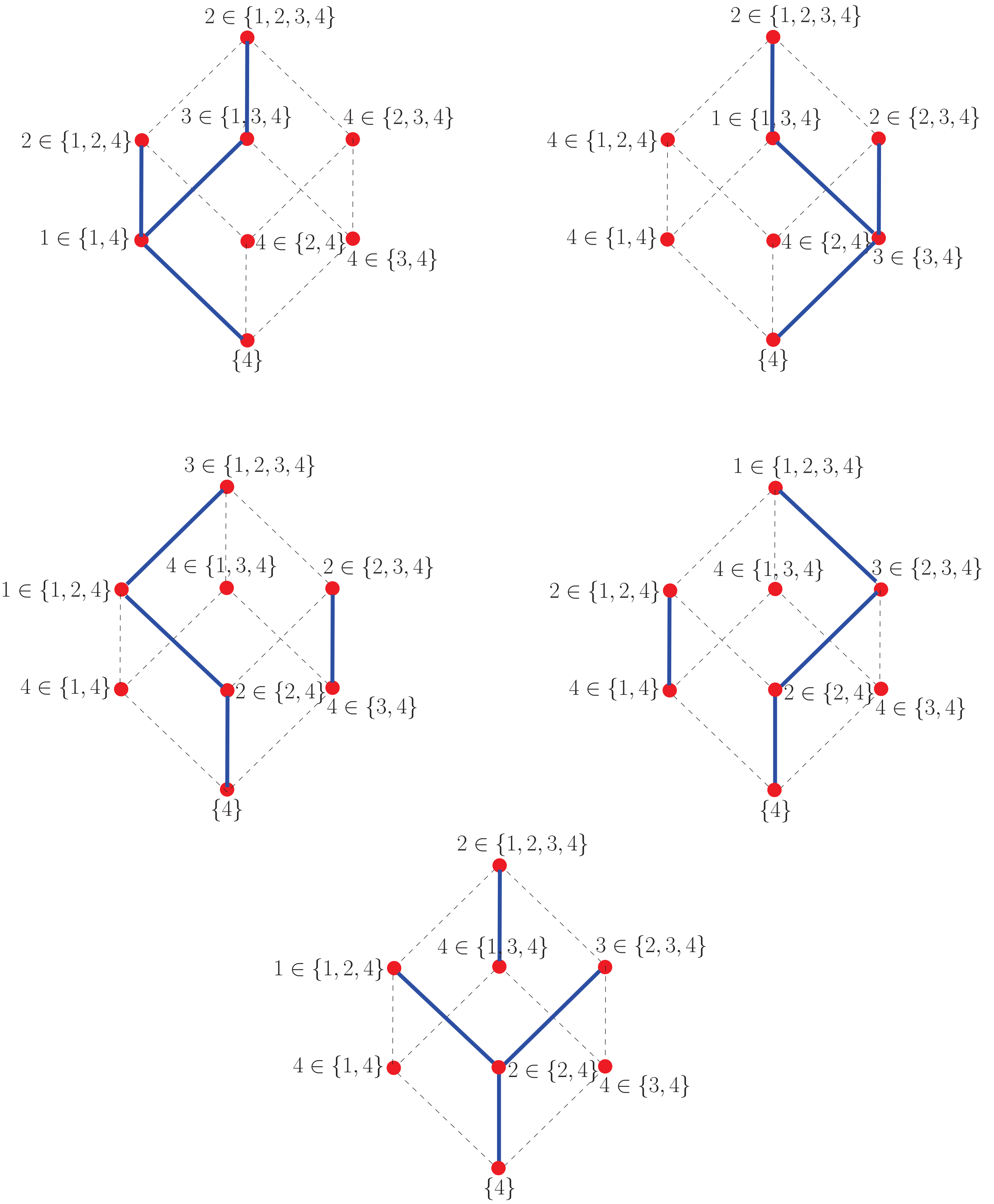


Figure 7.14: Diagram 1 for Example 7.2.16 on page 100. The five settled forests with signature (1, 2, 1, 4) where $\{t, 4\}$ is not in direction 4 for some $t \in [3]$.

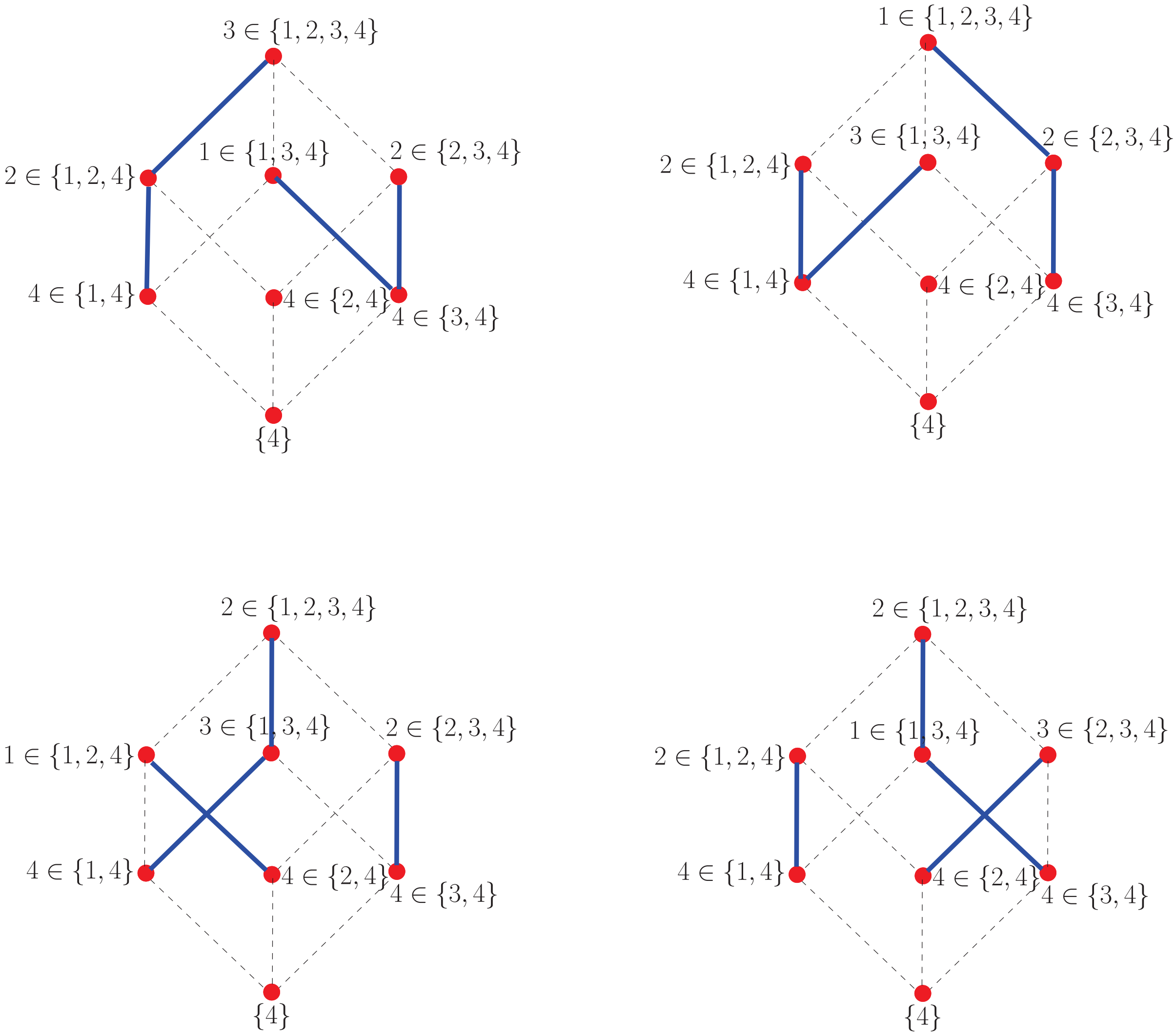


Figure 7.15: Diagram 2 for Example 7.2.16 on page 100. The four settled forests with signature (1, 2, 1, 4) where $\{t, 4\}$ is in direction 4 for all $t \in [3]$.

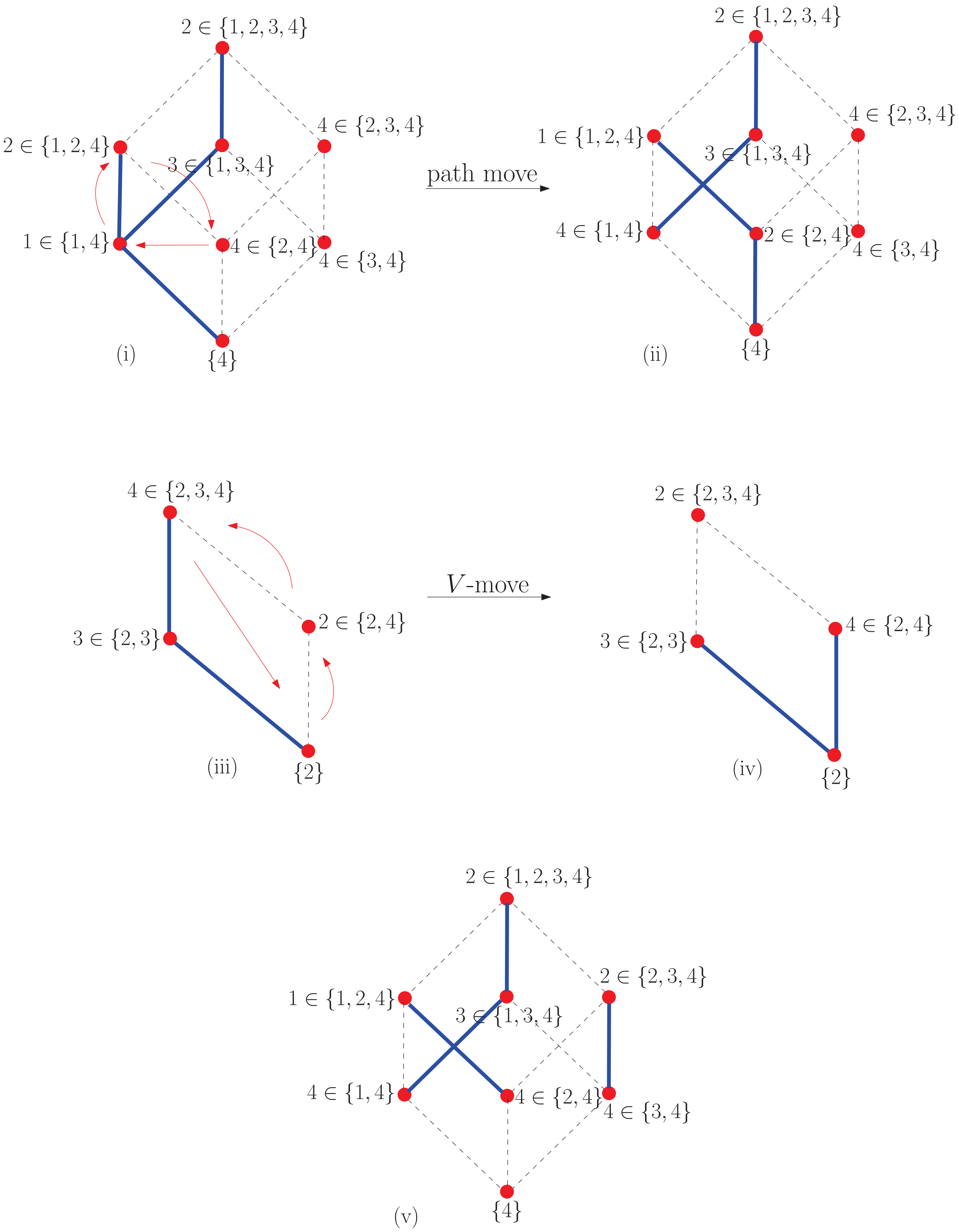


Figure 7.16: Diagram 3 for Example 7.2.16 on page 100. (i) Shows T_{4+} with signature (1, 2, 1, 4) where direction 4 is chosen at $\{2,4\}$, $\{3,4\}$ and $\{2,3,4\}$, direction 1 is chosen at $\{1,4\}$ direction 2 is chosen at $\{1,2,4\}$ and $\{1,2,3,4\}$ and direction 3 is chosen at $\{1,3,4\}$. (ii) After applying the path move on the face $\{\{1,2,4\},\{2,4\},\{4\},\{1,4\}\}$ direction 4 moves to $\{1,4\}$ direction 1 moves to $\{1,2,4\}$ and direction 2 moves to $\{2,4\}$. (iii) Shows the face $\{\{2,3,4\},\{2,3\},\{2,4\},\{4\}$. (iv) After applying the V-move the labels 2 and 4 are swapped. (v) Shows a settled tree with all the vertices of cardinality two in direction 4.

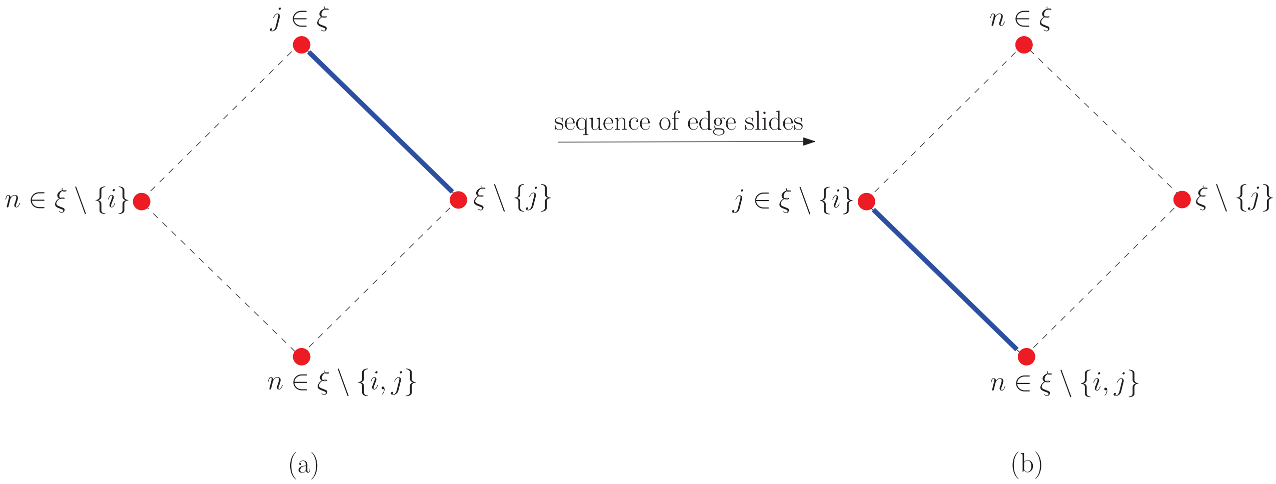


Figure 7.17: Diagram for Case 1(a) for the proof of Lemma 7.2.19 on page 102. (a) A two dimensional face of Q_n with different labels j and n by T. (b) The same face with the labels at the vertices ξ and $\xi \setminus \{i\}$ swapped.

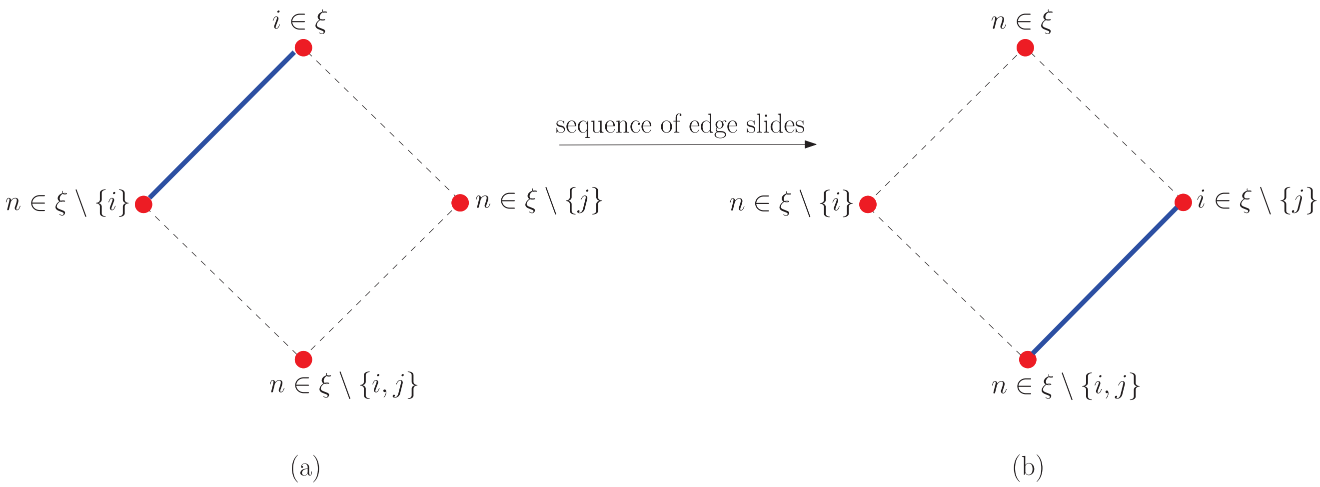


Figure 7.18: Diagram for Case 1(b(I(i))) for the proof of Lemma 7.2.19 on page 102. (a) A two dimensional face of Q_n with different labels i and n by T. (b) The same face with the labels at the vertices ξ and $\xi \setminus \{j\}$ swapped.

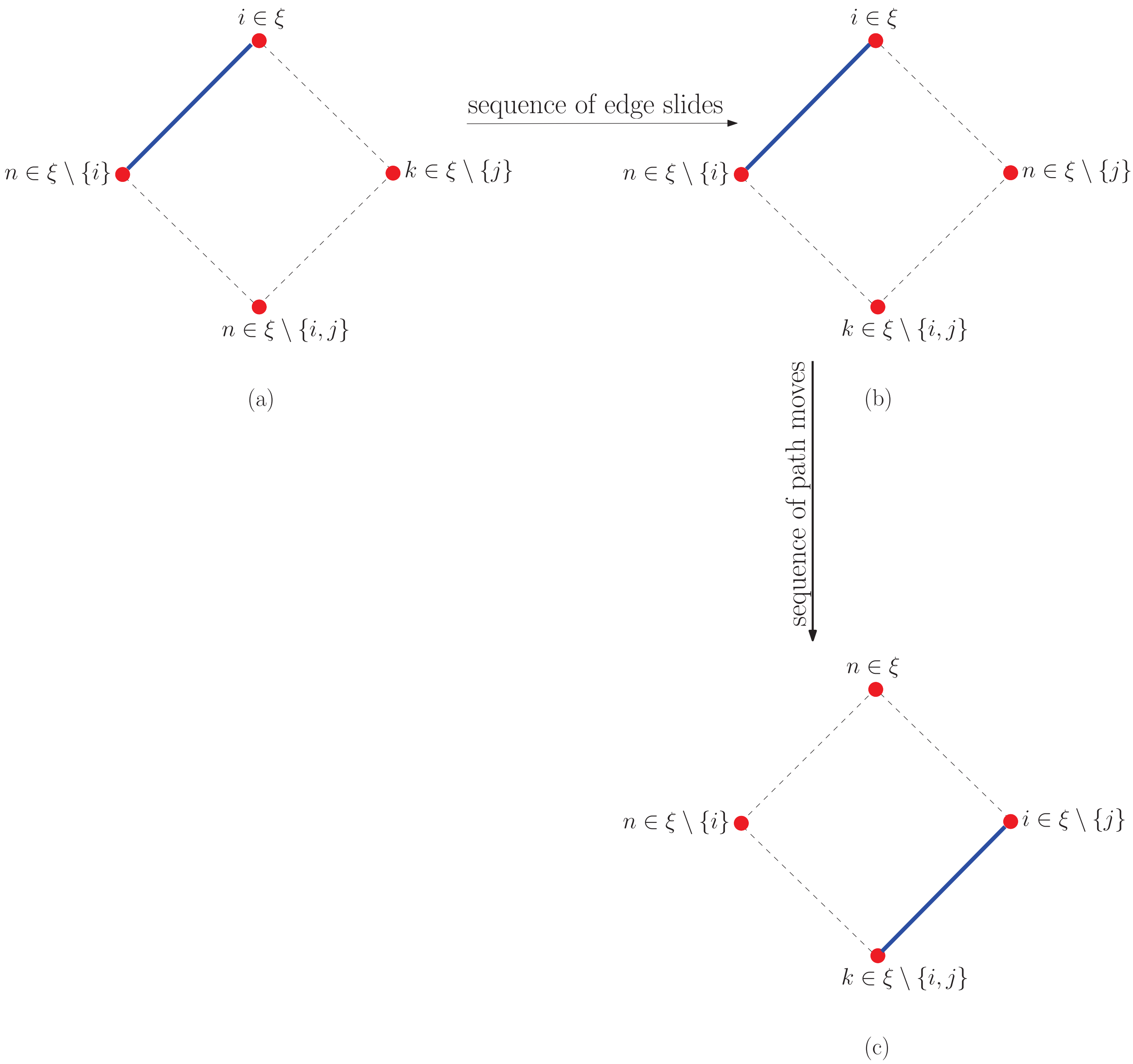


Figure 7.19: Diagram for Case 1(b(I(ii))) for the proof of Lemma 7.2.19 on page 102. (a) A two dimensional face of Q_n with different labels i, k and n of T. (b) The face with the labels at the vertices $\xi \setminus \{j\}$ and $\xi \setminus \{i, j\}$ swapped. (c) The face with the labels at the vertices ξ and $\xi \setminus \{j\}$ swapped.

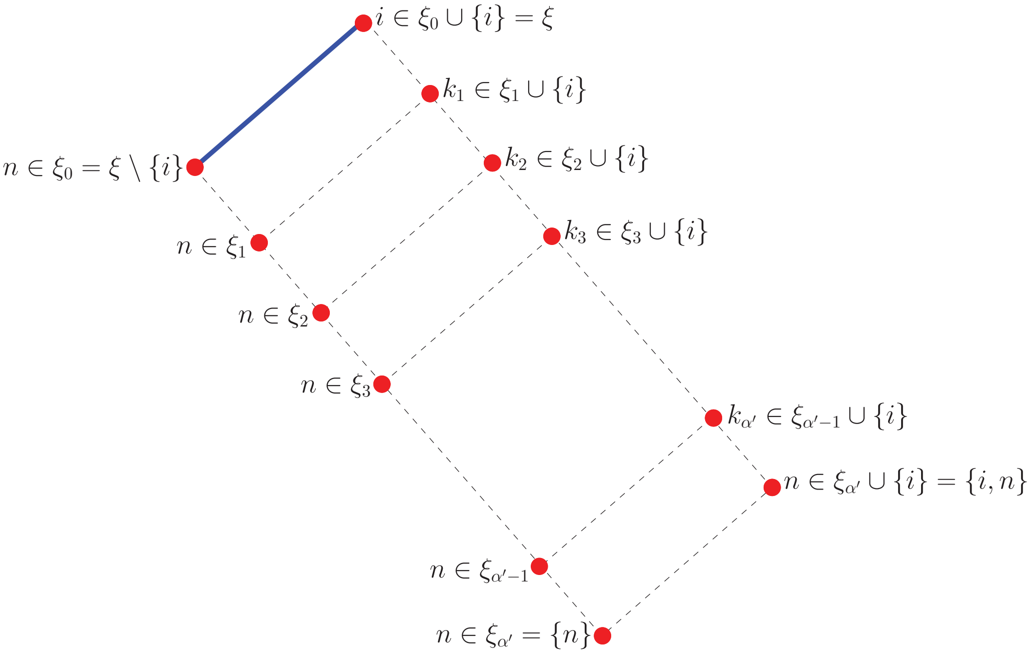


Figure 7.20: Diagram 1 for Case 1(b(II)) for the proof of Lemma 7.2.19 on page 102. The paths $P_{\mathcal{F}_n^{n+}}(\xi) = (\xi_0, \ldots, \xi_{\alpha'})$ and $\sigma_i(P_{\mathcal{F}_n^{n+}}(\xi)) = (\xi_0 \cup \{i\}, \ldots, \xi_{\alpha'} \cup \{i\})$ in $\mathcal{F}_n^{n+}$ with their vertex labels.

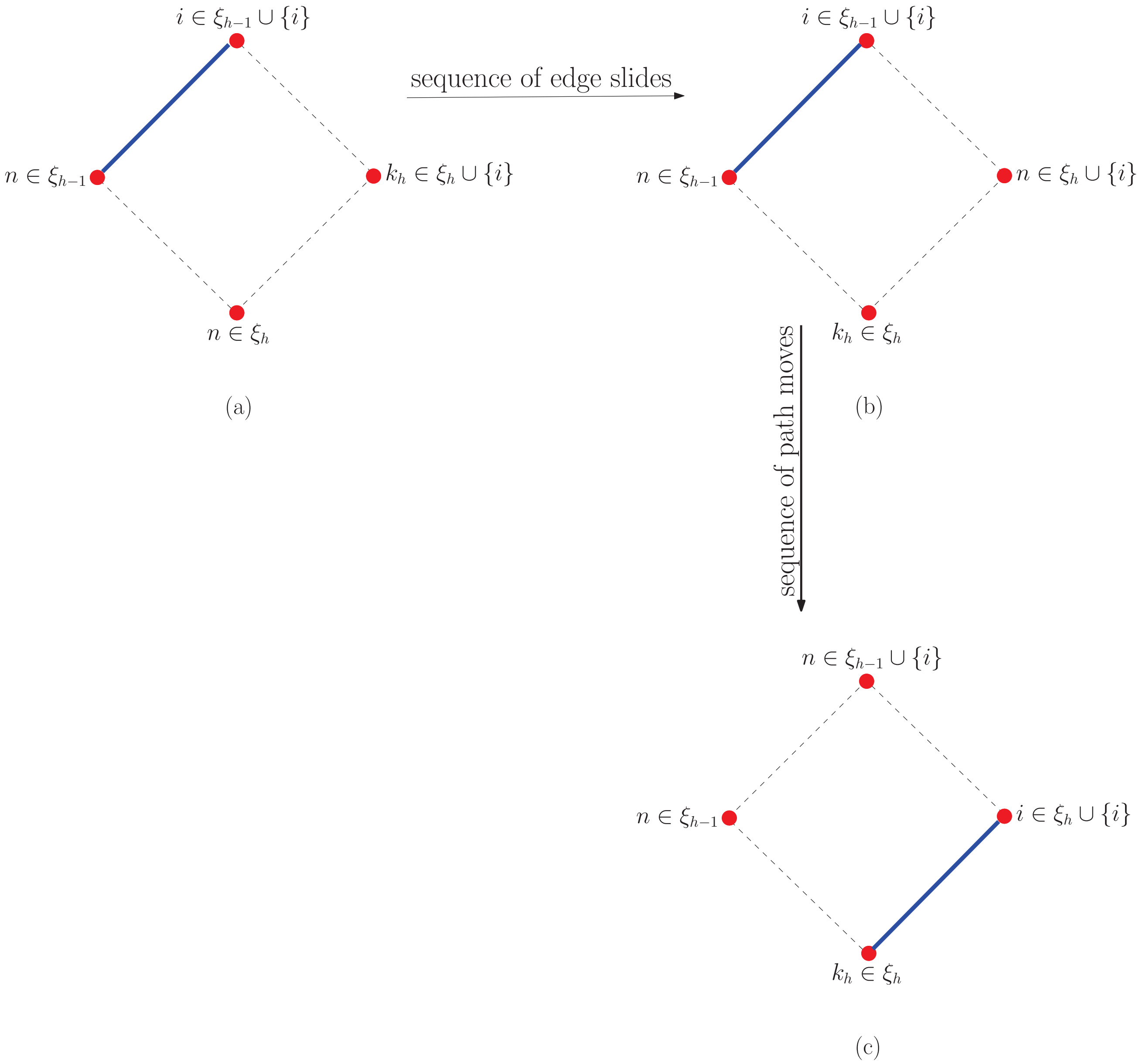


Figure 7.21: Diagram 2 for Case 1(b(II)) for the proof of Lemma 7.2.19 on page 102. (a) A two dimensional face of Q_n with different labels i, k and n of T. (b) The same face with the labels at the vertices $\xi_h \cup \{i\}$ and ξ_h swapped. (c) The same face with the labels at the vertices $\xi_{h-1} \cup \{i\}$ and $\xi_h \cup \{i\}$ swapped.

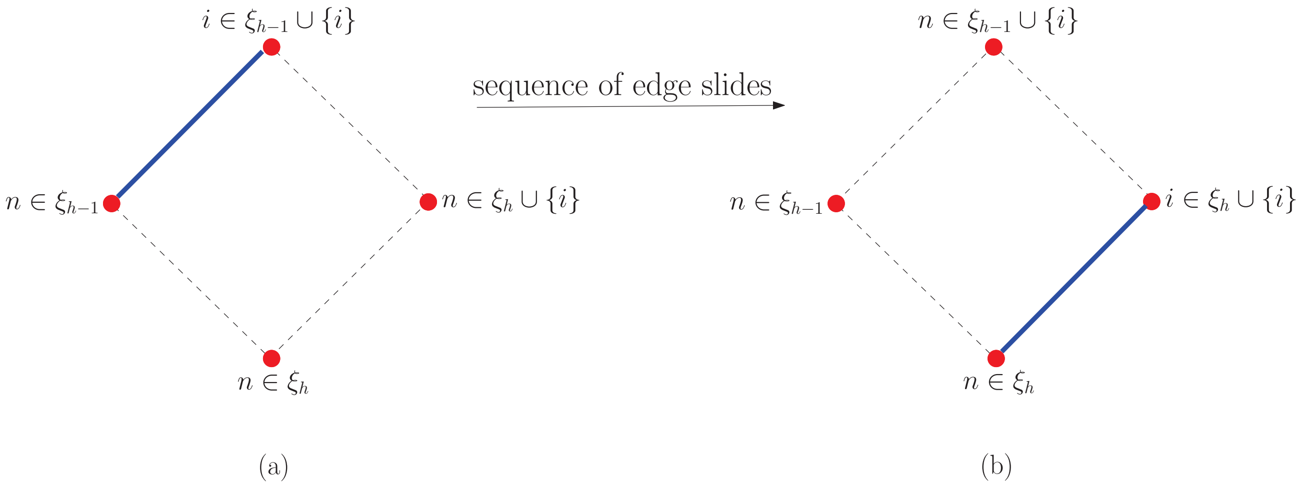


Figure 7.22: Diagram 3 for Case 1(b(II)) for the proof of Lemma 7.2.19 on page 102. (a) A two dimensional face of Q_n labelled i and n by T. (b) The same face with the labels at the vertices $\xi_{h-1} \cup \{i\}$ and $\xi_h \cup \{i\}$ swapped.

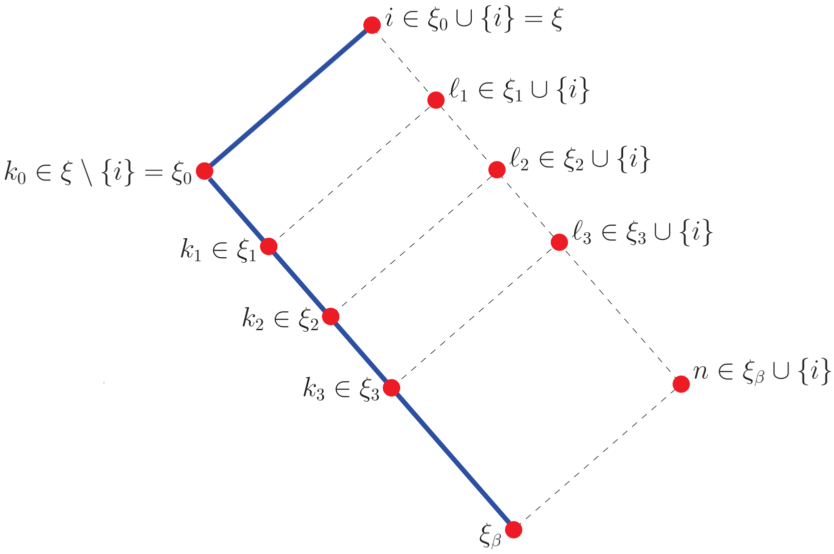


Figure 7.23: Diagram 1 for Case 2 for the proof of Lemma 7.2.19 on page 102. The paths $P_T(\xi \setminus \{i\}) = (\xi_0, \dots, \xi_{\alpha-1})$ and $\sigma_i(P_T(\xi \setminus \{i\})) = (\xi_0 \cup \{i\}, \dots, \xi_{\alpha-1} \cup \{i\})$ in $\mathcal{F}_n^{n+}$ as labelled by T.

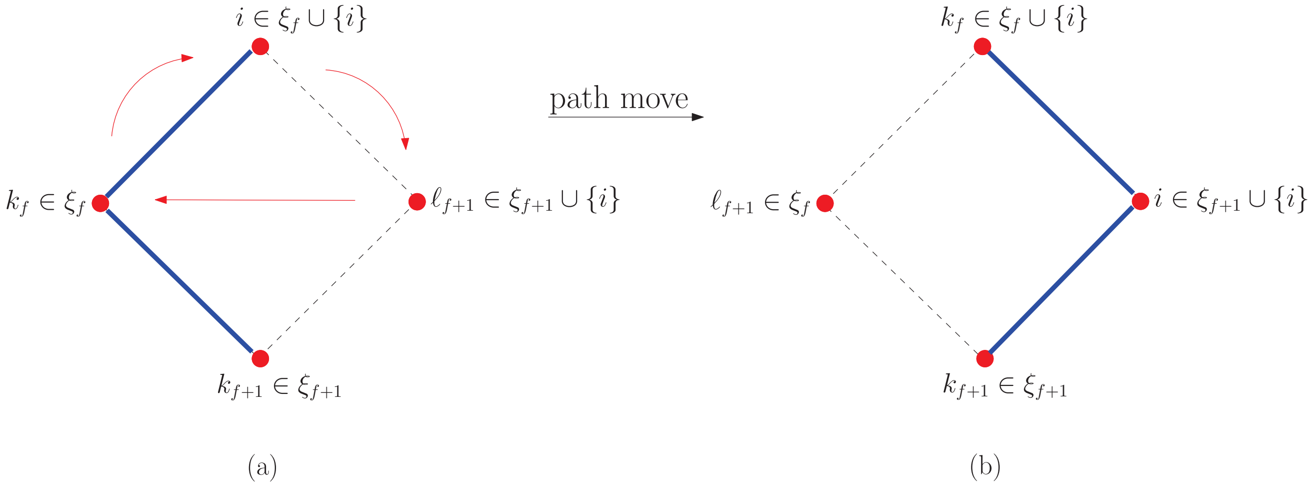


Figure 7.24: Diagram 2 for Case 2 for the proof of Lemma 7.2.19 on page 102. (a) Direction i is chosen at $\xi_f \cup \{i\}$, directions k_f, k_{f+1} and ℓ_{f+1} are chosen at ξ_f, ξ_{f+1} and $\xi_{f+1} \cup \{i\}$ respectively. (b) After the path move direction ℓ_{f+1} moves to ξ_f, direction k_f moves to $\xi_f \cup \{i\}$ and direction i moves to $\xi_{f+1} \cup \{i\}$.

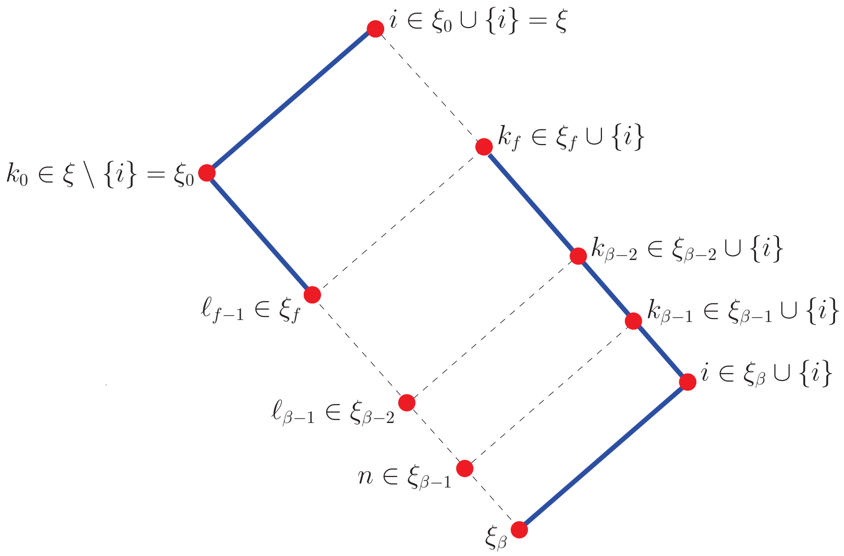


Figure 7.25: Diagram 3 for Case 2 for the proof of Lemma 7.2.19 on page 102. The paths $P_{T^2}(\xi \setminus \{i\})$ and $\sigma_i(P_{T^2}(\xi \setminus \{i\}))$ as labelled by T^2.

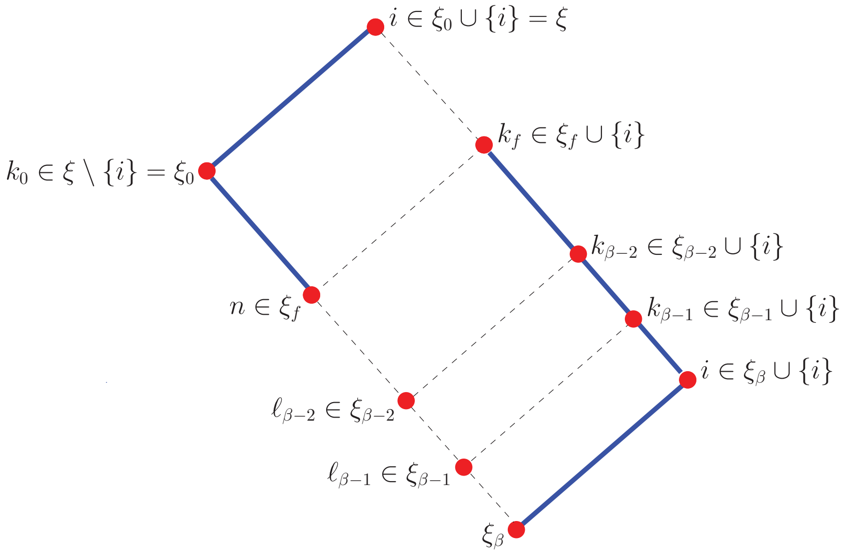


Figure 7.26: Diagram 4 for Case 2 for the proof of Lemma 7.2.19 on page 102. The paths $P_{T^4}(\xi \setminus \{i\})$ and $\sigma_i(P_{T^4}(\xi \setminus \{i\}))$ as labelled by T^4.

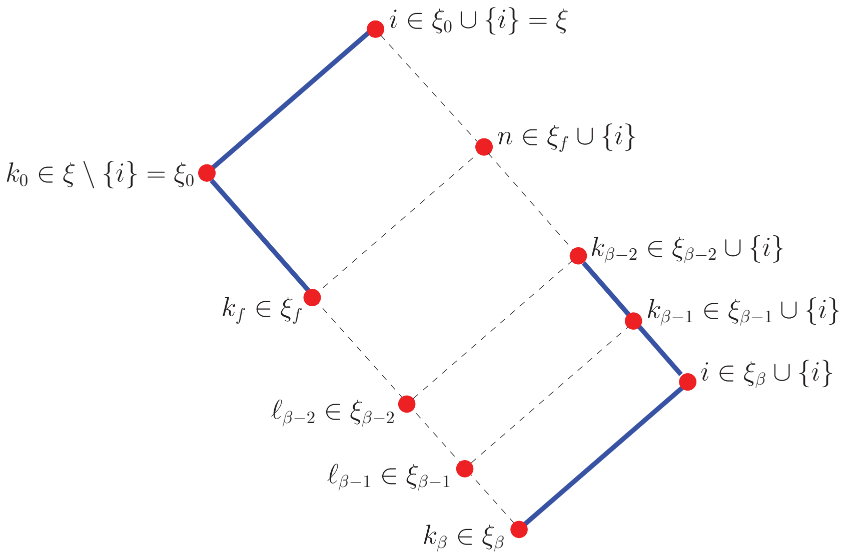


Figure 7.27: Diagram 5 for Case 2 for the proof of Lemma 7.2.19 on page 102. The paths $P_{T^5}(\xi \setminus \{i\})$ and $\sigma_i(P_{T^5}(\xi \setminus \{i\}))$ as labelled by T^5.

We give the proof in detail below for the signature (3, 4, 4, 4). The proofs for the other signatures are similar, and are detailed in the respective Figure 7.28 to 7.33 on pages 123 to 128.

Since the edge slide graph of signature (2, 2, 3) is connected, we can move from any tree with signature (2, 2, 3) to any other using edge slides. So it suffices to show we can move from any settled forest where $\{t, 4\}$ is in direction 4 for all $t \in [3]$ to any other using either the path move or the V-move. In this case we do not need to use Corollary 7.2.6. Using the notation of Section 2.2.5.2, let $T_1 = 22134443213$, $T_2 = 12324443213$, $T_3 = 21324443213$ and $T_4 = 32124443213$, which all have the tree 3213 shown in Figure 4.8(i) in $\mathcal{F}_4^{4-}$.

For the tree T_1, the forest $(T_1)_{4+}$ is shown in Figure 7.33(i). Applying the path move on the face $\{\{1,2,3,4\},\{1,3,4\},\{3,4\},\{2,3,4\}\}$ the label 3 moves to $\{1,3,4\}$, the label 1 moves to $\{1,2,3,4\}$ and the label 2 moves to $\{2,3,4\}$ and therefore T_1 is transformed into T_2. Figure 7.33(ii) shows $(T_2)_{4+}$. Applying the V-move on the face $\{\{1,2,3,4\},\{1,3,4\},\{1,4\},\{1,2,4\}\}$ the label 1 moves to $\{1,4\}$, the label 4 moves to $\{1,2,4\}$ and the label 2 moves to $\{1,2,3,4\}$ and therefore T_2 is transformed into the tree $T' = 24321443213$. Figure 7.33(iii) shows $(T')_{4+}$. Note that T' is an unsettled tree. Applying the V-move on the face $\{\{1,2,4\},\{1,2\},\{1\},\{1,4\}\}$, the labels 1 and 4 are swapped and therefore T' is transformed into the settled tree T_3. Figure 7.33(iv) shows $(T_3)_{4+}$. Applying the path move on the face $\{\{1,2,3,4\},\{1,3,4\},\{1,4\},\{1,2,4\}\}$, the label 1 moves to $\{1,3,4\}$, the label 3 moves to $\{1,2,3,4\}$ and the label 2 moves to $\{1,2,4\}$ and therefore T_3 is transformed into T_4. Figure 7.33(v) shows $(T_4)_{4+}$.

By the definition of the edge slide graph of signature (3, 4, 4, 4) there is a sequence of edge slides from $(T_i)_{4+}$ to $(T_4)_{4+}$ for $i = 1, 2, 3$. Therefore T_1, T_2, T_3 and T_4 are in the same connected component of $\mathcal{E}(3,4,4,4)$. Hence the result for (3, 4, 4, 4) follows, by our discussion above. □

7.3.2 Unidirectional splitting signatures

The following lemma shows that the set of upright spanning trees of Q_n with an irreducible signature and a fixed connected unidirectional splitting signature in $\mathcal{F}_n^{n-}$ forms a block.

Theorem 7.3.2. *Let $n \geq 4$, and let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n with $a_n \geq a_t$ for all $t \in [n-1]$. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be a connected unidirectional splitting signature of $\mathcal{I}$ with respect to n. Let $\mathcal{B}$ be the set of upright spanning trees of Q_n with signature $\mathcal{I}$ and signature $\mathcal{D}$ in $\mathcal{F}_n^{n-}$. Then $\mathcal{B}$ forms a block.*

Proof. Given trees T and T' belonging to $\mathcal{B}$, we show that there are sequences of edge slides transforming (T, T') into $(\hat{T}, \hat{T}')$ such that $\hat{T} = \hat{T}'$. Note that since $\mathcal{D}$ is a connected signature, we can always move from T_{n-} to T'_{n-} using edge sides, so it suffices to show that we can rearrange the labels of T and T' in $\mathcal{F}_n^{n+}$ to bring them into agreement. We first settle the trees using Lemma 7.2.13, and then proceed to rearrange the labels in $\mathcal{F}_n^{n+}$ level by level, starting at the top and working towards the bottom. At each stage we ensure all labels at the current level are in agreement before proceeding to the next.

If $T_{n+} \neq T'_{n+}$, then let α be the highest level where T_{n+} and T'_{n+} disagree. Then T and T' have all edges of level $\alpha+1$ and above in common in $\mathcal{F}_n^{n+}$, for some $1 \leq \alpha \leq n$. In other words, we have $\psi_T(V) = \psi_{T'}(V)$ for all V in $\mathcal{F}_n^{n+}$ such that $|V| \geq \alpha + 1$ for some $1 \leq \alpha \leq n$. Since $\mathcal{I}$ is an irreducible signature, we must have $a_n \leq 2^{n-1} - 1$. Let i be the direction other than n occurring in $\mathcal{F}_n^{n+}$. Since the only labels occurring in $\mathcal{F}_n^{n+}$ are i and n and since T and T' are settled, we must have $\psi_T([n]) = \psi_{T'}([n]) = i$. Therefore $\alpha \leq n-1$.

Let η be a vertex of $\mathcal{F}_n^{n+}$ of level α where T and T' disagree (have different labels). Since the only labels that occur upstairs are i and n, we may suppose without loss of generality that $\psi_T(\eta) = n$ and $\psi_{T'}(\eta) = i$. Since T and T' are upright spanning trees of Q_n with signature $\mathcal{I}$ such that both trees have the connected unidirectional splitting signature $\mathcal{D}$ in $\mathcal{F}_n^{n-}$, both trees have the same number of i-edges in $\mathcal{F}_n^{n+}$. Therefore there must exist an i-edge of T_{n+} of level α', where $2 \leq \alpha' \leq \alpha$ because all edges of T_{n+} and T'_{n+} of level $\alpha + 1$ and above agree. Note that $\alpha' \geq 2$ because $\psi_T(\{n\}) = \psi_{T'}(\{n\}) = n$.

Let ξ be a lowest vertex of $\mathcal{F}_n^{n+}$ with $\psi_T(\xi) = i$. If $|\xi| = \alpha$, then level α is the lowest level that has label i in T and therefore all the vertices of levels lower than level α must be in direction n, because T is settled. In this case choose ξ to be a vertex where the trees disagree (have different labels), so that $\psi_T(\xi) = i$ and $\psi_{T'}(\xi) = n$. Note that such a vertex exists because both trees have the same number of i-edges in $\mathcal{F}_n^{n+}$ at level α and below, and level α is the lowest level of T with an i-edge. So there must be a vertex of level α other than η where the trees disagree. Otherwise $|\xi| < \alpha$ and then we can freely choose ξ to be any lowest vertex of $\mathcal{F}_n^{n+}$ with $\psi_T(\xi) = i$, because we are rearranging the labels from the top to the bottom.

Let $P_{\mathcal{F}_n^{n+}}(\xi, i) = (\xi_0, \ldots, \xi_{\alpha'_1})$ be the i-retaining max removing path from $\xi_0 = \xi$ to $\xi_{\alpha'_1} = \{i, n\}$. Since T is settled, all the labels of T in $\mathcal{F}_n^{n+}$ are in directions i and n, and since ξ is a lowest vertex of $\mathcal{F}_n^{n+}$ with $\psi_T(\xi) = i$, we have $\psi_T(\xi_a) = n$ for all $a = 1, \ldots, \alpha'_1$. Let $P_{\mathcal{F}_n^{n+}}(\eta, i) = (\eta_0, \ldots, \eta_{\alpha-1})$ be the i-retaining max removing path from $\eta = \eta_0$ to $\eta_{\alpha-1} = \{i, n\}$. Since T is a settled tree and $\psi_T(\eta) = n$, we have $\psi_T(\eta_a) = n$ for all $a = 0, \ldots, \alpha - 1$. Then all the conditions in Corollary 7.2.6 are satisfied at the vertices ξ and η and therefore T can be transformed into $\bar{T}$ with $\psi_{\bar{T}}(\eta) = i$, $\psi_{\bar{T}}(\xi) = n$ and $\psi_{\bar{T}}(V) = \psi_T(V)$ for all $V \neq \eta, \xi$. Then $\psi_{\bar{T}}(V) = \psi_{T'}(V)$ whenever $|V| \geq \alpha + 1$; $|V| = \alpha$ and $\psi_T(V) = \psi_{T'}(V)$; for $V = \eta$; and additionally at $V = \xi$ if $|\xi| = \alpha$. Thus $\bar{T}$ and T' have at least one more edge in common at level α and above.

We claim that $\bar{T}$ is settled. Since $\psi_{\bar{T}}(V) = \psi_T(V)$ for all V in $\mathcal{F}_n^{n+}$ apart from $\psi_{\bar{T}}(\xi) = n$ and $\psi_{\bar{T}}(\eta) = i$, we need only check that ξ and all vertices immediately above η in $\mathcal{F}_n^{n+}$ are settled. Since ξ was a lowest vertex of T with an i-edge, all vertices immediately below ξ in $\bar{T}$ must be in direction n. Therefore ξ is a settled vertex of $\bar{T}$. Since T' is settled with $\psi_{T'}(\eta) = i$, we must have $\psi_{T'}(\eta \cup \{k\}) = i$ for all $k \notin \eta$. But then $|\eta \cup \{k\}| = \alpha + 1$ so $\psi_{\bar{T}}(\eta \cup \{k\}) = \psi_T(\eta \cup \{k\}) = \psi_{T'}(\eta \cup \{k\}) = i$, and all such vertices are settled. Therefore $\bar{T}$ is settled.

Let $T' = \bar{T}'$. Then (T, T') is transformed into $(\bar{T}, \bar{T}')$ with at least one more edge at level α in common in $\mathcal{F}_n^{n+}$. If $\bar{T} = \bar{T}'$, then the transformation is completed. Otherwise, repeating this process we eventually reach a pair of trees such that all the vertices of level α and above are in agreement. We then proceed to rearrange the labels of level $\alpha - 1$ using the same process. At each level we ensure all labels at that level are in agreement before proceeding to the next, and eventually we reach a pair $(\hat{T}, \hat{T}')$ where $\hat{T} = \hat{T}'$.

Thus, by the definition of the edge slide graph $\mathcal{E}(\mathcal{I})$, there are sequences of edge slides transforming (T, T') into $(\hat{T}, \hat{T}')$, and hence a sequence of edge slides transforming T into T'. Therefore T and T' are in the same connected component of $\mathcal{E}(\mathcal{I})$. Since T and T' are arbitrary, we conclude $\mathcal{B}$ forms a block. □

7.3.3 Super rich splitting signatures

In this section we conjecture under the inductive hypothesis that upright spanning trees with an irreducible signature and a fixed super rich splitting signature form a block. We present partial progress towards a proof of this.

Conjecture 7.3.3. *Let $n \geq 5$, and suppose that every irreducible signature of Q_{n-1} is connected. Let $\mathcal{I} = (a_1, \dots, a_n)$ be an irreducible signature of Q_n with $a_n \geq a_t$ for all $t \in [n-1]$. Let $\mathcal{D} = (d_1, \dots, d_{n-1})$ be a super rich splitting signature of $\mathcal{I}$ with respect to n. Let $\mathcal{B}$ be the set of upright spanning trees of Q_n with signature $\mathcal{I}$ and signature $\mathcal{D}$ in $\mathcal{F}_n^{n-}$. Then $\mathcal{B}$ forms a block.*

Strategy 7.3.4. Given $T, T' \in \mathcal{B}$, we aim to show that there are sequences of edge slides transforming (T, T') into $(\hat{T}, \hat{T}')$ such that $\hat{T} = \hat{T}'$. Note that since $\mathcal{D}$ is a connected signature, we can always move from T_{n-} to T'_{n-} using edge sides, so it suffices to show that we can rearrange the labels of T and T' in $\mathcal{F}_n^{n+}$ to bring them into agreement. We first settle the trees, and moreover we may assume that $\psi_T(\{t, n\}) = \psi_{T'}(\{t, n\}) = n$ for all t using Lemma 7.2.13 and Lemma 7.2.18. Then we proceed to rearrange the labels in $\mathcal{F}_n^{n+}$ level by level, starting at the top and working towards the bottom.

Let α be the highest level where T and T' disagree. Then $3 \leq \alpha \leq n$ and $\psi_T(V) = \psi_{T'}(V)$ for all V in $\mathcal{F}_n^{n+}$ such that $|V| > \alpha$. Let $X \subseteq [n-1]$ be such that $Y = X \cup \{n\}$ is a vertex of level α that has different labels in T and T'. Without loss of generality we may assume that $\psi_T(Y) = j$ and $\psi_{T'}(Y) = i$ where $i \neq n$. Since T and T' are upright spanning trees of Q_n with signature $\mathcal{I}$, where they have all edges of level $\alpha + 1$ and above in common in $\mathcal{F}_n^{n+}$ and signature $\mathcal{D}$ in $\mathcal{F}_n^{n-}$, there must exist an i-edge of T_{n+} of level α', where $3 \leq \alpha' \leq \alpha$. Let $X' \subseteq [n-1] \setminus \{i\}$ be such that $Y' = X' \cup \{i, n\}$ is a vertex of $\mathcal{F}_n^{n+}$ with $\psi_T(Y') = i$ and $|Y'| = \alpha'$, where $3 \leq \alpha' \leq \alpha$.

Suppose that there is no such vertex Y' where $Y' \subseteq Y$. Let $X'' \subseteq [n-1] \setminus \{i\}$ be such that $Y'' = X'' \cup \{i, n\}$ is a lowest vertex of $\mathcal{F}_n^{n+}$ with $\psi_T(Y'') = i$. If $|Y''| = \alpha$ then we additionally require that $\psi_{T'}(Y'') \neq i$. Note that such a vertex exists because T and T' have the same number of i-edges in $\mathcal{F}_n^{n+}$ at level α and below, and T' has an i at Y where T does not. Otherwise $|Y''| < \alpha$, and then we can freely choose Y'' to be any lowest vertex of $\mathcal{F}_n^{n+}$ with $\psi_T(Y'') = i$. By applying Lemma 7.2.1 (if necessary) we may assume $\psi_T(Y'' \setminus \{i\}) = n$.

The following lemma shows that T can be transformed into a tree where the i-edge is chosen at a vertex that is a subset of Y.

Lemma 7.3.5. *Let T, Y and Y'' be as above. Let $Y_0 = Y''$, $\beta = |Y''| - 1$ and let $P = (Y_0, \dots, Y_\beta)$ be a descending path in $\mathcal{F}_n^{n+}$ from $Y_0 = Y''$ to $Y_\beta = \{i, n\}$. Let $k_a = \psi_T(Y_a)$ for $a = 0, \dots, \beta$. Then for $1 \leq a \leq \beta$ there is a sequence of edge slides transforming T into the tree T_a, where for $0 \leq a \leq \beta$ we define*

$$\psi_{T_a}(V) = \begin{cases} i, & \textit{when } V = Y_a; \\ k_{b-1}, & \textit{when } V = Y_b, \textit{ for } 0 \leq b < a; \\ \psi_T(V), & \textit{otherwise.} \end{cases}$$

Note that tree T_a might be unsettled, but if this is the case the only unsettled vertices are immediately above the vertex Y_a.

Proof. Observe that $T_0 = T$. Note that $k_a \neq i$ for all $a = 1, \dots, \beta$, because we assumed Y'' is a lowest vertex of $\mathcal{F}_n^{n+}$ such that $\psi_T(Y'') = i$. Since $\psi_T(\{t, n\}) = n$ for all $t \in [n-1]$, we have $k_\beta = n$. Let $P' = (Y_0 \setminus \{i\}, \dots, Y_\beta \setminus \{i\})$ be the descending path from $Y_0 \setminus \{i\} = Y'' \setminus \{i\}$

to $Y_\beta \setminus \{i\} = \{n\}$. Since T is settled and $\psi_T(Y'' \setminus \{i\}) = n$, we have $\psi_T(Y_a \setminus \{i\}) = n$ for all $a = 0, \ldots, \beta$.

Suppose first that $k_a \neq n$. Then applying Corollary 7.2.10, T_a can be transformed into T_{a+1}. If T_a is settled, then T_{a+1} is settled too because T_{a+1} is obtained by swapping $\psi_{T_a}(Y_a)$ and $\psi_{T_a}(Y_{a+1})$ where none of these labels is equal to n.

Suppose now that $k_a = n$. Let t be the least index such that $k_t = n$. Then $\psi_T(Y_a) = n$ for $t \leq a \leq \beta$. Then applying Lemma 7.2.7, T_a can be transformed into T_{a+1}. Note that T_{a+1} is unsettled, but the only unsettled vertices are immediately above the vertex Y_{a+1}.

In all cases T_a can be transformed into T_{a+1}, so by induction $T = T_0$ can be transformed into T_a for $1 \leq a \leq \beta$. □

Next, we show that we can increase the number of common edges of the two settled trees T and T' at level α and above by one. We address the case $j = n$ and $j \neq n$ separately. When $j \neq n$, we assume without loss of generality that $j < i$. The idea in this case is to transform T into $\bar{T}$ where $\psi_{\bar{T}}(Y) = i$ and $\psi_{\bar{T}}(V) = \psi_T(V)$ for all other V such that $|V| \geq \alpha$, and moreover $\bar{T}_{n-}$ has signature $\mathcal{D}'$ where

$$d'_i = d_i - 1, \quad d'_j = d_j + 1, \quad \text{and} \quad d'_t = d_t$$

for all $t \neq i, j$. So $\psi_{\bar{T}}(Y) = \psi_{T'}(Y) = i$ and the signature in $\mathcal{F}_n^{n-}$ is changed from $\mathcal{D}$ into $\mathcal{D}'$. Then we show that there exists a sequence of edge slides that transforms $\bar{T}$ into $\hat{T}$ such that $\psi_{\hat{T}}(Y) = \psi_{T'}(Y) = i$ and $\hat{T}_{n-}$ has signature $\mathcal{D}$, while preserving any agreement between labels at levels α and above. So we are able to return the signature in $\mathcal{F}_n^{n-}$ to $\mathcal{D}$.

Lemma 7.3.6. *Let T, T' and Y be as described above in Strategy 7.3.4. Suppose that $i > j$ such that $i, j \neq n$. There exists a sequence of edge slides that transforms T into $\bar{T}$ where $\psi_{\bar{T}}(Y) = i$, $\psi_{\bar{T}}(V) = \psi_T(V)$ for all V such that $|V| > \alpha$, and $\psi_{\bar{T}}(V) = \psi_T(V)$ whenever $|V| = \alpha$ and $\psi_{T'}(V) = \psi_T(V)$. Moreover, $\bar{T}_{n-}$ has a rich signature $\mathcal{D}'$ where*

$$d'_i = d_i - 1, \quad d'_j = d_j + 1, \quad \textit{and} \quad d'_t = d_t$$

for all $t \neq i, j$. Then there exists a sequence of edge slides that transform $\bar{T}$ into $\hat{T}$ such that $\psi_{\hat{T}}(Y) = i$ and $\psi_{\hat{T}}(V) = \psi_{T'}(V)$ for all V such that $|V| \geq \alpha$, and $\psi_{\bar{T}}(V) = \psi_T(V)$ whenever $|V| = \alpha$ and $\psi_{T'}(V) = \psi_T(V)$. Moreover $\hat{T}_{n-}$ has signature $\mathcal{D}$.

Note that this lemma does not guarantee the resulting tree $\hat{T}$ is settled. We can apply Lemma 7.2.13 to settle the tree, but doing so risks moving labels already positioned in bringing the trees into agreement. When an n is moved down this may cause vertices immediately above it to become unsettled, with the result that ns at higher levels may have to move down too. This could lead to a cascading effect affecting the ns at levels α and above, decreasing the number of labels in agreement at these levels.

Proof. We first show that we can replace label j at Y with an i through a change of splitting signature to $\mathcal{D}'$. As the first step towards this, we need the vertex $Y \setminus \{j\}$ to be in direction n. If this is not already the case let $\ell = \psi_T(Y \setminus \{j\})$. If $\psi_T(Y \setminus \{j, \ell\})) = n$, then applying Case 1 of Lemma 7.2.19 T can be transformed into T_1 with $\psi_{T_1}(Y \setminus \{j\}) = n$ and all other labels of T_1 apart from the labels of $T_1 \cap (Q_{Y\setminus\{j\}} \cap \mathcal{F}_n^{n+})$ the same as the labels of T. If $\psi_T(Y \setminus \{j, \ell\}) \neq n$, then applying Case 2 of Lemma 7.2.19 T can be transformed into T_1 with $\psi_{T_1}(Y \setminus \{j\}) = n$ and all other labels of T_1 apart from the labels of $T_1 \cap (Q_{Y\setminus\{j\}} \cap \mathcal{F}_n^{n+})$ the same as the labels of T. So after a series of zero or more edge slides we reach a tree T_1 where $Y \setminus \{j\}$ is in direction n.

Since $\mathcal{D}$ is an irreducible signature, by Lemma 3.2.7 there exists an upright spanning tree $(T_2)_{n-}$ of Q_{n-1} with signature $\mathcal{D}$ where $\psi_{T_2}(Y \setminus \{n\}) = i$. Since the edge slide graph of signature $\mathcal{D}$ is connected by the inductive hypothesis, we can move from $(T_1)_{n-}$ to $(T_2)_{n-}$ using edge slides. Then T_1 is transformed into T_2 with $\psi_{T_2}(Y \setminus \{n\}) = i$ and all the labels of $(T_2)_{n+}$ the same as the labels of $(T_1)_{n+}$.

Applying the path move the i-edge at $Y \setminus \{n\}$ moves to $Y \setminus \{j\}$, the n-edge at $Y \setminus \{j\}$ moves to Y and the j-edge at Y moves to $Y \setminus \{i\}$. Therefore T_2 is transformed into T_3 with $\psi_{T_3}(Y) = n$, $\psi_{T_3}(Y \setminus \{n\}) = j$, $\psi_{T_3}(Y \setminus \{j\}) = i$ and all other labels of T_3 the same as the labels of T_2. Then $(T_3)_{n-}$ has signature $\mathcal{D}'$ where

$$d'_i = d_i - 1, \quad d'_j = d_j + 1 \quad \text{and} \quad d'_t = d_t$$

for all $t \neq i, j$. Since $i > j$, we have

$$\varepsilon_\mu^{\mathcal{D}'} = \begin{cases} (\varepsilon_\mu^{\mathcal{D}}) + 1, & \text{when } j \leq \mu < i; \\ \varepsilon_\mu^{\mathcal{D}}, & \text{otherwise.} \end{cases}$$

Therefore $\mathcal{D}'$ is an irreducible signature, and is still rich.

Let T_4 be the tree obtained by swapping $\psi_{T_3}(Y)$ and $\psi_{T_3}(Y \setminus \{j\})$; in other words, the tree with $\psi_{T_4}(Y) = i$, $\psi_{T_4}(Y \setminus \{j\}) = n$ and all other labels of T_4 the same as the labels of T_3. Since $\mathcal{D}'$ is rich, we can apply Lemma 7.2.7 to transform T_4 to T_3. Therefore applying those edge slides in reverse T_3 can be transformed into T_4. Observe that all labels of $(T_4)_{n+}$ are the same as the labels of $(T_1)_{n+}$ apart from $\psi_{T_4}(Y) = i$. Therefore we may apply the label swaps in $Q_{Y \setminus \{j\}} \cap \mathcal{F}_n^{n+}$ transforming T_{n+} into $(T_1)_{n+}$ in reverse to reach a settled tree $\bar{T}$ such that $\psi_{\bar{T}}(Y) = i$, $\psi_{\bar{T}}(\{t, n\}) = n$ for all $t \in [n-1]$, and $\bar{T}_{n-}$ has signature $\mathcal{D}'$. Therefore T is transformed into $\bar{T}$ where $\psi_{\bar{T}}(Y) = i$, $\psi_{\bar{T}}(V) = \psi_T(V)$ for all $Y \neq V \in \mathcal{F}_n^{n+}$ and $\bar{T}_{n-}$ has signature $\mathcal{D}'$.

We now show there exists a sequence of edge slides that transforms $\bar{T}$ back into a tree with splitting signature $\mathcal{D}$, while preserving $\psi_{\bar{T}}(Y) = i$, and any agreement between labels at levels α and above.

Let $\tilde{X} \subseteq [n-1] \setminus \{i\}$ be such that $\tilde{Y} = \tilde{X} \cup \{i, n\}$ is a lowest vertex of $\mathcal{F}_n^{n+}$ with $\psi_{\bar{T}}(\tilde{Y}) = i$. If $|\tilde{Y}| = \alpha$ then we additionally require that $\psi_{T'}(\tilde{Y}) \neq i$. Note that such a vertex exists because $\bar{T}_{n+}$ has more i-edges at level α and below than T'_{n+}. Otherwise $|\tilde{Y}| < \alpha$, and then we can freely choose $\tilde{Y}$ to be any lowest vertex of $\mathcal{F}_n^{n+}$ with $\psi_{\bar{T}}(\tilde{Y}) = i$. By applying Lemma 7.2.1 (if necessary) we may assume $\psi_{\bar{T}}(\tilde{Y} \setminus \{i\}) = n$.

We distinguish the following cases according to whether or not $j \in \tilde{Y}$.

(a) Suppose that $j \in \tilde{Y}$. Since $\mathcal{D}'$ is an irreducible signature, by Lemma 3.2.4 there exists an upright spanning tree $(T_5)_{n-}$ of Q_{n-1} such that $\psi_{T_5}(\tilde{Y} \setminus \{n\}) = j$. Since the edge slide graph of signature $\mathcal{D}'$ is connected by the inductive hypothesis, we can move from $\bar{T}_{n-}$ to $(T_5)_{n-}$ using edge slides. Then $\bar{T}$ is transformed into T_5 with $\psi_{T_5}(\tilde{Y} \setminus \{n\}) = j$ and all the labels of $(T_5)_{n+}$ the same as the labels of $\bar{T}_{n+}$.

Applying the path move the j-edge at $\tilde{Y} \setminus \{n\}$ moves to $\tilde{Y} \setminus \{i\}$, the n-edge at $\tilde{Y} \setminus \{i\}$ moves to $\tilde{Y}$ and the i-edge at $\tilde{Y}$ moves to $\tilde{Y} \setminus \{n\}$. Therefore T_5 is transformed into T_6 with $\psi_{T_6}(\tilde{Y}) = n$, $\psi_{T_6}(\tilde{Y} \setminus \{n\}) = i$, $\psi_{T_6}(\tilde{Y} \setminus \{i\}) = j$ and all other labels of T_5 the same as the labels of T_5. Then $(T_6)_{n-}$ has the rich splitting signature $\mathcal{D}$.

Let $\hat{T}$ be the tree obtained by swapping $\psi_{T_6}(\tilde{Y})$ and $\psi_{T_6}(\tilde{Y} \setminus \{i\})$; in other words, the tree with $\psi_{\hat{T}}(\tilde{Y}) = j$, $\psi_{\hat{T}}(\tilde{Y} \setminus \{i\}) = n$ and all other labels of $\hat{T}$ the same as the labels of

T_6. Since $\mathcal{D}$ is rich, we can apply Lemma 7.2.7 to transform $\hat{T}$ to T_6. Therefore applying those edge slides in reverse T_6 can be transformed into $\hat{T}$. The tree $\hat{T}$ is settled with $\psi_{\hat{T}}(\{t,n\}) = n$ for all $t \in [n-1]$, because the net effect of the transformation is that no n has moved.

Therefore $\hat{T}$ and T' have at least one more edge in common at level α and above than T and T'.

(b) Suppose that $j \notin \tilde{Y}$. Then $\psi_{\bar{T}}(\{i,j,n\}) \neq i$, because then we be in the case (a) above with $\tilde{Y} = \{i,j,n\}$. Consider the tree $\bar{T} \cap Q_{\{i,j,n\}}$. Since $\psi_{\bar{T}}(\{t,n\}) = n$ for all t, we have both $\{i,n\}$ and $\{j,n\}$ in direction n.

Since $\psi_{\bar{T}}(\tilde{Y}) = i$ and $\psi_{\bar{T}}(\tilde{Y} \setminus \{i\}) = n$, applying Lemma 7.3.5 $\bar{T}$ can be transformed into $\tilde{T}$ with $\psi_{\tilde{T}}(\{i,n\}) = i$. Observe that this has no effect on $\{i,j,n\}$, since $j \neq \tilde{Y}$. Therefore $\psi_{\tilde{T}}(\{i,j,n\}) = \psi_{\bar{T}}(\{i,j,n\})$ and $\psi_{\tilde{T}}(\{t,n\}) = n$ for all $t \neq i$.

Suppose that $\psi_{\tilde{T}}(\{i,j,n\}) = j$. Since $\mathcal{D}'$ is rich and $\psi_{\tilde{T}}(\{j,n\}) = n$, applying Lemma 7.2.7 $\tilde{T}$ can be transformed into T_5 where $\psi_{T_5}(\{i,j,n\}) = n$, $\psi_{T_5}(\{j,n\}) = j$ and all other labels of T_5 are the same as the labels of $\tilde{T}$.

So perhaps after some edge slides we may assume $\psi_{\tilde{T}}(\{i,j,n\}) = n$. Since $\mathcal{D}'$ is an irreducible signature, by Lemma 3.2.4 there exists an upright spanning tree $(T_6)_{n-}$ of Q_{n-1} such that $\psi_{T_6}(\{i,j\}) = j$. Since $\mathcal{D}'$ is connected by the inductive hypothesis, we can move from $(\tilde{T})_{n-}$ to $(T_6)_{n-}$ using edge slides. Then $\tilde{T}$ is transformed into T_6 with $\psi_{T_6}(\{i,j\}) = j$ and all the labels of $(T_6)_{n+}$ the same as the labels of $(\tilde{T})_{n+}$.

Applying the path move the i-edge at $\{i,n\}$ moves to $\{i,j\}$, the j-edge at $\{i,j\}$ moves to $\{i,j,n\}$ and the n-edge at $\{i,j,n\}$ moves to $\{i,n\}$. Therefore T_6 is transformed into T_7 with $\psi_{T_7}(\{i,j,n\}) = j$, $\psi_{T_7}(\{i,j\}) = i$, $\psi_{T_7}(\{i,n\}) = n$ and all other labels of T_7 the same as the labels of T_6. Therefore $(T_7)_{n-}$ has the rich splitting signature $\mathcal{D}$.

Case (a) results in a settled tree, but Case (b) does not necessarily result in a settled tree. As discussed after the statement of the lemma, applying Lemma 7.2.13 to settle the tree might disrupt the labels already positioned in bringing the trees into agreement. In other words, this method for bringing the trees into agreement may fail because when an n is moved down any ns immediately above this may also have to move down. So we can get a cascading effect affecting the ns at levels $\alpha+1$ and above. We also need $\{t,n\}$ to be in direction n for all t at each step of the transformation, and these labels may sometimes move during the transformation above. Applying Lemma 7.2.15 to restore this may also involve a step to settle the tree. At present, in some cases, we are unable to get to a settled tree without affecting the progress already made in bringing the trees into agreement. □

The next lemma handles the case where $j = n$. We show that when $\psi_T(Y) = n$ and $\psi_{T'}(Y) = i$, T can be transformed into $\hat{T}$ with $\psi_{\hat{T}}(Y) = i$, while preserving any labels already in agreement at levels α and above. So $\hat{T}$ and T' have at least one more edge in common at level α and above than T and T'.

Lemma 7.3.7. *Let T, T', Y and Y'' be as described in Strategy 7.3.4. Suppose that $\psi_T(Y) = n$ and $\psi_{T'}(Y) = i$. Then there exists an sequence of edge slides that transforms T into $\hat{T}$ where $\psi_{\hat{T}}(Y) = i$, $\psi_{\hat{T}}(V) = \psi_T(V)$ for all V such that $|V| > \alpha$ and and $\psi_{\hat{T}}(V) = \psi_T(V)$ whenever $|V| = \alpha$ and $\psi_{T'}(V) = \psi_T(V)$. Therefore $\hat{T}$ and T' have at least one more edge in common at level α and above than T and T'.*

As with Lemma 7.3.6 this lemma does not guarantee the resulting tree $\hat{T}$ is settled, and applying Lemma 7.2.13 to settle it may also result in a cascading effect affecting the ns at levels α and above.

Proof. By Lemma 7.3.5 T is transformed into $\bar{T}$ such that $\psi_{\bar{T}}(\tilde{Y}) = i$, where $\tilde{Y} \subseteq Y$ and the only unsettled vertices are immediately above $\tilde{Y}$. Let $P = (Y_0, \ldots, Y_\beta)$ be a descending path from $Y_0 = Y$ to $Y_\beta = \tilde{Y}$. Then $\psi_{\bar{T}}(Y_a) = n$ for $a = 0, \ldots, \beta - 1$, because T is settled with $\psi_T(Y) = n$, and the labels of Y_a have not been changed for $a = 0, \ldots, \beta - 1$. Since $\mathcal{D}$ is rich, applying Corollary 7.2.9 at vertices Y and $\tilde{Y}$ the tree $\bar{T}$ can be transformed into $\hat{T}$ with $\psi_{\hat{T}}(Y) = i$, $\psi_{\hat{T}}(\tilde{Y}) = n$ and all other labels of $\hat{T}$ the same as the labels of $\bar{T}$.

We now check that $\hat{T}$ is settled. Since $\hat{T}$ was obtained from $\bar{T}$ by swapping only the labels of Y and $\tilde{Y}$ and since the only unsettled vertices of $\bar{T}$ were above $\tilde{Y}$, we only need to check that the vertices above Y and $\tilde{Y}$ are settled . Since T' is settled with $\psi_{T'}(Y) = i$, we must have $\psi_{T'}(Y \cup \{k\}) \neq n$ for all $k \notin Y$. But $|Y \cup \{k\}| = \alpha + 1$, so $\psi_{\hat{T}}(Y \cup \{k\}) = \psi_{\bar{T}}(Y \cup \{k\}) = \psi_{T'}(Y \cup \{k\}) \neq n$, and all such vertices are settled. Since only the vertices of $\bar{T}$ immediately above $\tilde{Y}$ were unsettled and $\psi_{\hat{T}}(\tilde{Y}) = n$, all vertices of $\hat{T}$ immediately above $\tilde{Y}$ are settled. Therefore $\hat{T}$ is settled. □

7.4 Summary map

In Chapter 5 we showed that each irreducible signature has an amenable splitting signature using signature moves, and in this chapter we showed that upright spanning trees of Q_n with an irreducible signature and a fixed connected unidirectional splitting signature or the (2, 2, 3) splitting signature of Q_4 form a block. We conjecture under the inductive hypothesis that this is also true for a super rich splitting signature, and present partial progress towards the conjecture. In Chapter 8 we show under the inductive hypothesis that an upright spanning tree of Q_n with an irreducible signature that has an irreducible splitting signature can be transformed into a tree with any other irreducible splitting signature. This result allows us to reduce the problem of determining the connectivity of the edge slide graph of an irreducible signature to the problem considered here of showing that the trees with an irreducible signature and a fixed irreducible splitting signature form a block.

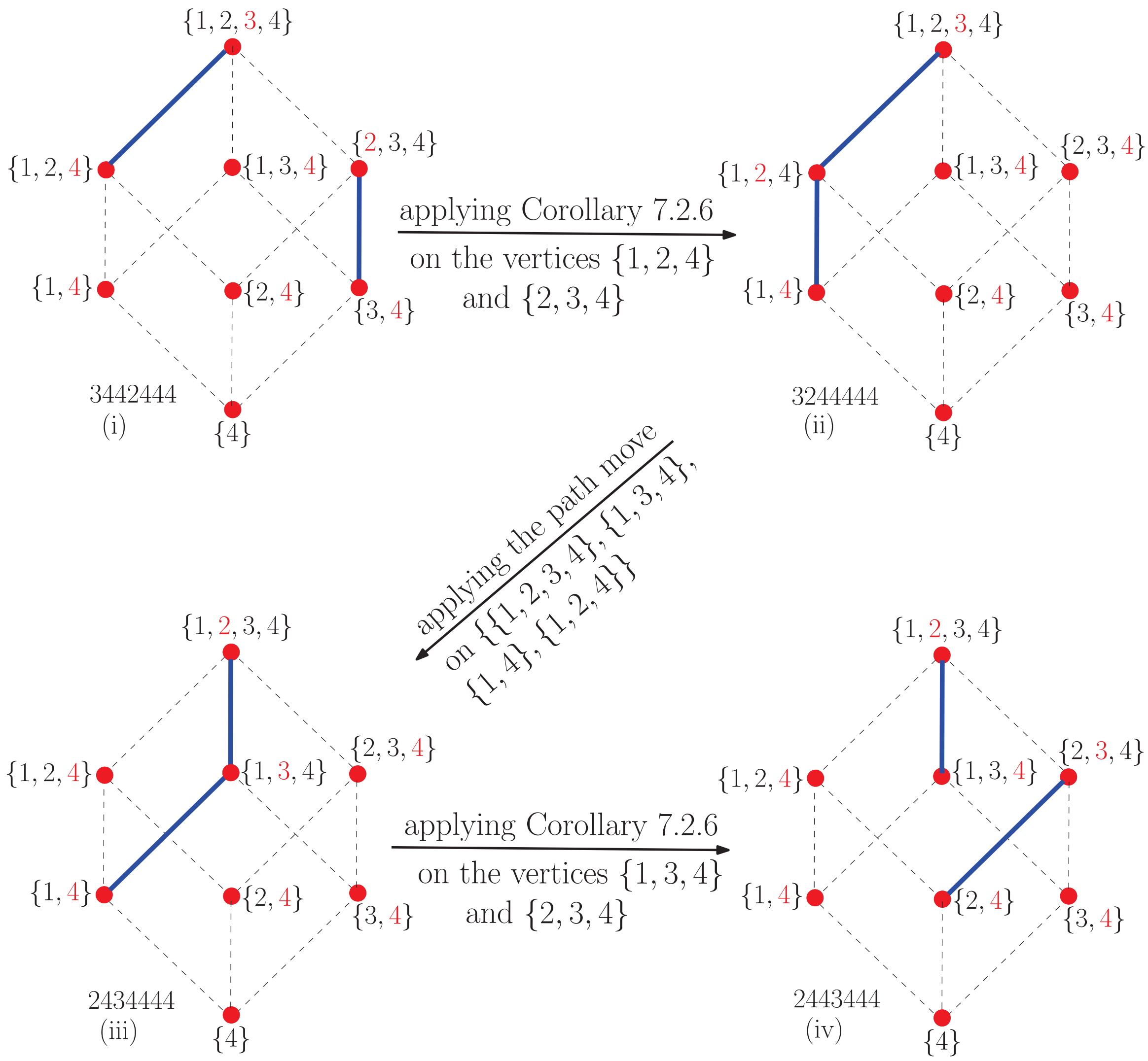


Figure 7.28: Diagram 1 for the proof of Theorem 7.3.1 on page 105 for signature (2, 3, 4, 6). The four settled forests in $\mathcal{F}_4^{4+}$ with signature (0, 1, 1, 6) such that $\{t, 4\}$ is in direction 4 for all t. The direction chosen at each vertex is coloured in red. (i) Shows the settled forest 3442444. (ii) After applying Corollary 7.2.6 on the vertices $\{1, 2, 4\}$ and $\{2, 3, 4\}$, the labels 2 and 4 at these vertices are swapped and therefore we get the settled forest 3244444. (iii) After applying the path move on the face $\{\{1, 2, 3, 4\}, \{1, 3, 4\}, \{1, 4\}, \{1, 2, 4\}\}$ the label 4 moves to $\{1, 2, 4\}$, the label 2 moves to $\{1, 2, 3, 4\}$ and the label 3 moves to $\{1, 3, 4\}$, and therefore we get the settled forest 2434444. (iv) After applying Corollary 7.2.6 on the vertices $\{1, 3, 4\}$ and $\{2, 3, 4\}$, the labels 3 and 4 at these vertices are swapped and therefore we get the settled forest 3244444. As can be seen there exists a sequence of edge slides from every settled forest to the settled forest 2443444 in (iv).

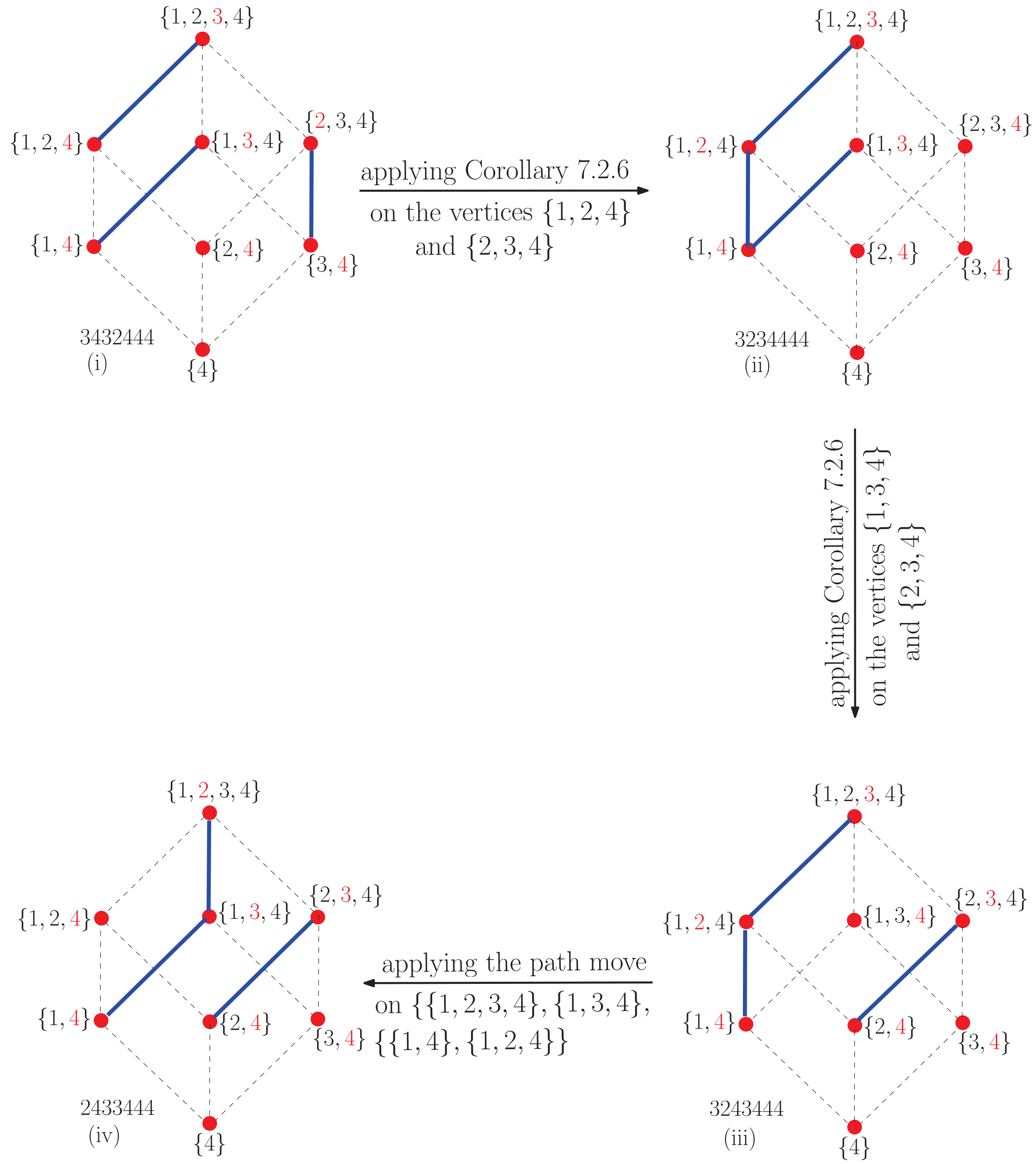


Figure 7.29: Diagram 2 for the proof of Theorem 7.3.1 on page 105 for signature (2, 3, 5, 5).The four settled forests in $\mathcal{F}_4^{4+}$ with signature (0, 1, 2, 5) such that $\{t, 4\}$ is in direction 4 for all t. The direction chosen at each vertex is coloured in red. (i) Shows the settled forest 3234444. (ii) After applying Corollary 7.2.6 on the vertices $\{1, 2, 4\}$ and $\{2, 3, 4\}$, the labels 2 and 4 at these vertices are swapped and therefore we get the settled forest 3234444. (iii) After applying Corollary 7.2.6 on the vertices $\{1, 3, 4\}$ and $\{2, 3, 4\}$, the labels 3 and 4 at these vertices are swapped and therefore we get the settled forest 3243444. (iv) After applying the path move on the face $\{\{1, 2, 3, 4\}, \{1, 3, 4\}, \{1, 4\}, \{1, 2, 4\}\}$ the label 4 moves to $\{1, 2, 4\}$, the label 2 moves to $\{1, 2, 3, 4\}$ and the label 3 moves to $\{1, 3, 4\}$, and therefore we get the settled forest 2433444. Therefore we can move from any settled forest to the forest 2433444 in (iv).

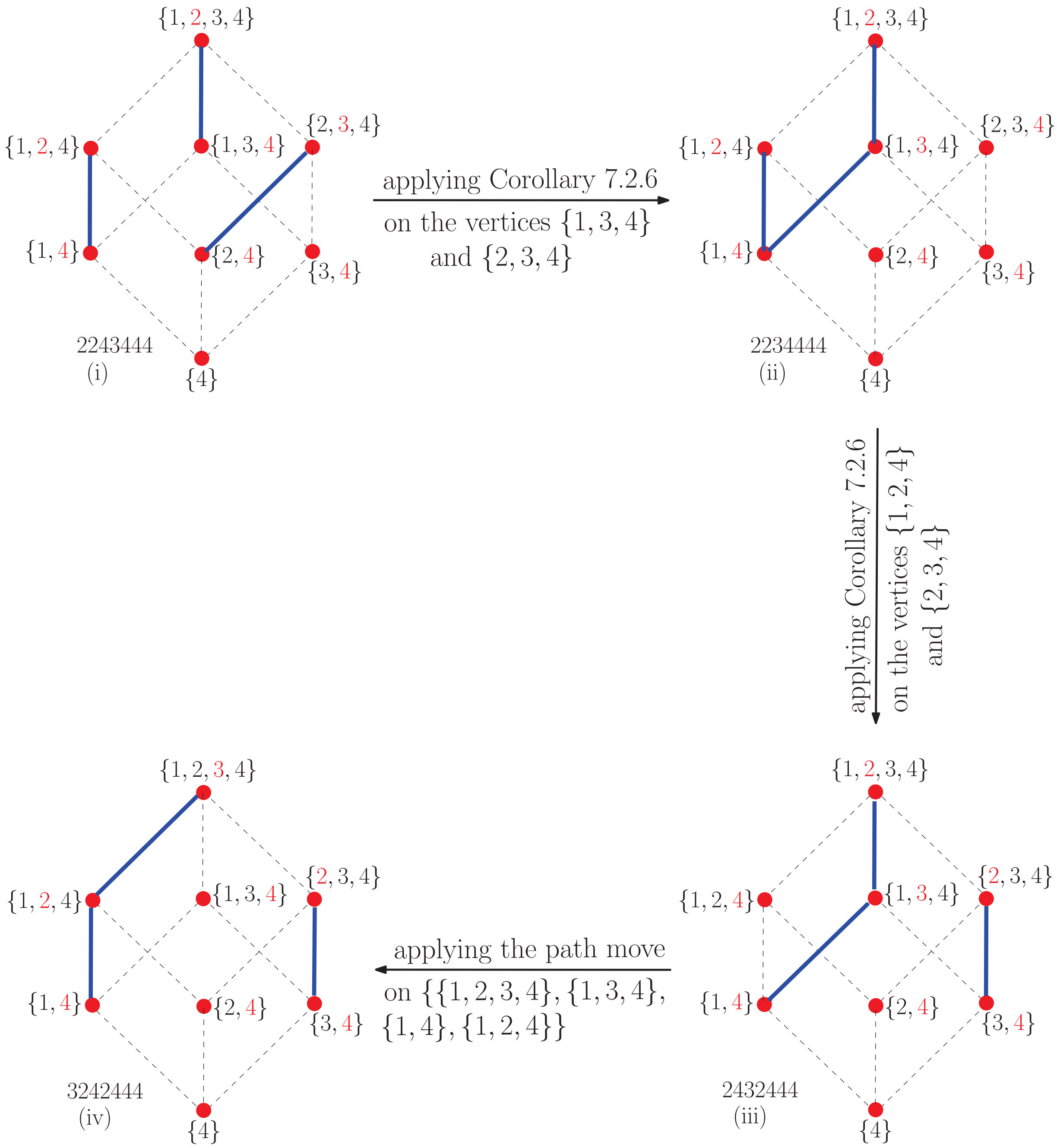


Figure 7.30: Diagram 3 for the proof of Theorem 7.3.1 on page 105 for signature (2, 4, 4, 5). The four settled forests in $\mathcal{F}_4^{4+}$ with signature (0, 2, 1, 5) such that $\{t, 4\}$ is in direction 4 for all t. The direction chosen at each vertex is coloured in red. (i) Shows the settled forest 2243444. (ii) After applying Corollary 7.2.6 on the vertices $\{1, 3, 4\}$ and $\{2, 3, 4\}$, the labels 3 and 4 at these vertices are swapped and therefore we get the settled forest 2234444. (iii) After applying Corollary 7.2.6 on the vertices $\{1, 2, 4\}$ and $\{2, 3, 4\}$, the labels 2 and 4 at these vertices are swapped and therefore we get the settled forest 2432444. (iv) After applying the path move on the face $\{\{1, 2, 3, 4\}, \{1, 3, 4\}, \{1, 4\}, \{1, 2, 4\}\}$ the label 4 moves to $\{1, 3, 4\}$, the label 3 moves to $\{1, 2, 3, 4\}$ and the label 2 moves to $\{1, 2, 4\}$, and therefore we get the settled forest 3242444. Therefore we can move from any settled forest to the forest 3242444 in (iv).

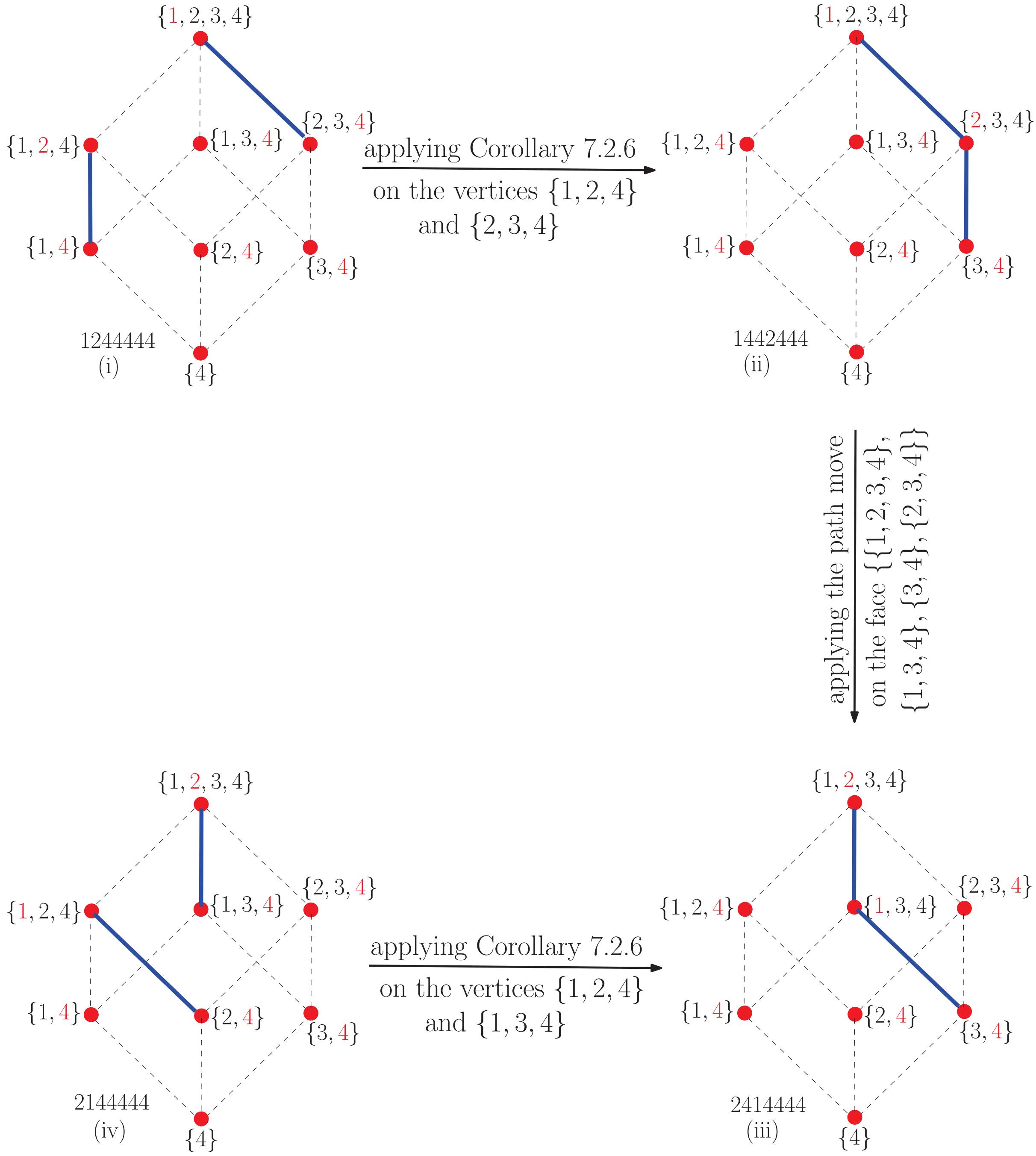


Figure 7.31: Diagram 4 for the proof of Theorem 7.3.1 on page 105 for signature (3, 3, 3, 6). The four settled forests in $\mathcal{F}_4^{4+}$ with signature (1, 1, 0, 6) such that $\{t, 4\}$ is in direction 4 for all t. The direction chosen at each vertex is coloured in red. (i) Shows the settled forest 1244444. (ii) After applying Corollary 7.2.6 on the vertices $\{1, 2, 4\}$ and $\{2, 3, 4\}$, the labels 2 and 4 at these vertices are swapped and therefore we get the settled forest 1442444. (iii) After applying the path move on the face $\{\{1, 2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}, \{2, 3, 4\}\}$ the label 4 moves to $\{2, 3, 4\}$, the label 2 moves to $\{1, 2, 3, 4\}$ and the label 1 moves to $\{1, 3, 4\}$, and therefore we get the settled forest 2414444. (iv) After applying Corollary 7.2.6 on the vertices $\{1, 2, 4\}$ and $\{1, 3, 4\}$, the labels 1 and 4 at these vertices are swapped and therefore we get the settled forest 2144444. Therefore we can move from any settled forest to the forest 2414444 in (iii).

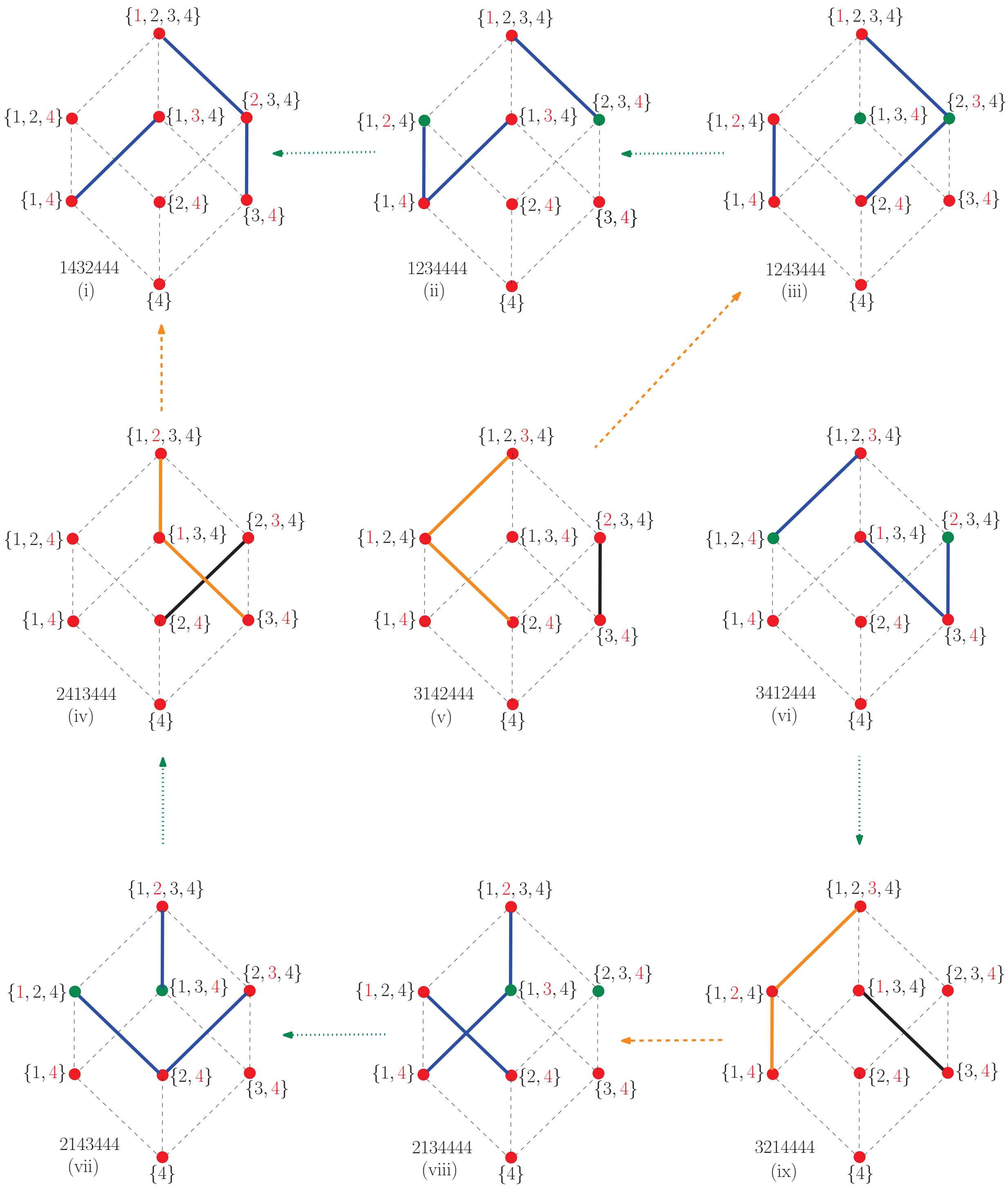


Figure 7.32: Diagram 5 for the proof of Theorem 7.3.1 on page 105 for signature (3, 3, 4, 5). The nine settled forests in $\mathcal{F}_4^{4+}$ with signature (1, 1, 1, 5) such that $\{t, 4\}$ is in direction 4 for all t. The direction chosen at each vertex is coloured in red. Dotted green arrows represent applying Corollary 7.2.6 at the vertices coloured in green. Dashed orange arrows represent applying the path move on the face with the orange edges. As can be seen all settled forests in $\mathcal{F}_4^{4+}$ with signature (1, 1, 1, 5) are connected to the settled forest 1432444 in (i).

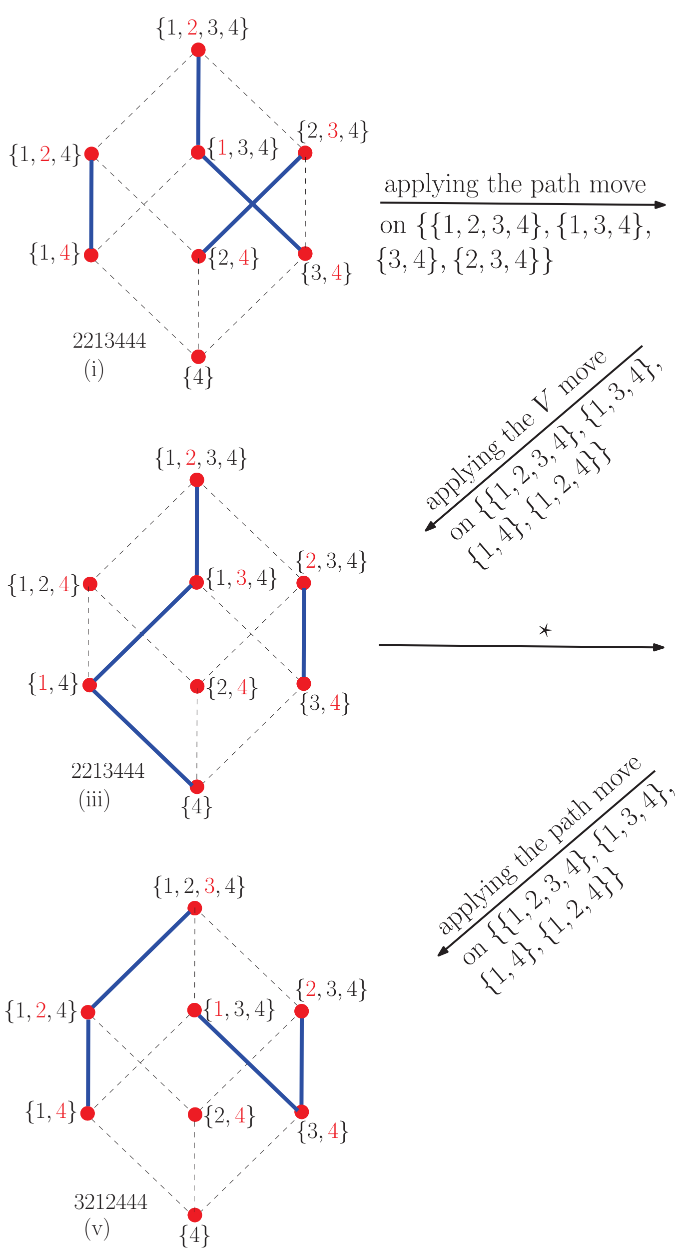


Figure 7.33: Diagram 6 for the proof of Theorem 7.3.1 on page 105 for signature (3, 4, 4, 4). The four settled forests in $\mathcal{F}_4^{4+}$ with signature (1, 2, 1, 4) such that $\{t, 4\}$ is in direction 4 for all t. The direction chosen at each vertex is coloured in red. (i) Shows the settled forest 2213444. (ii) After applying the path move on the face $\{\{1,2,3,4\},\{1,3,4\},\{3,4\},\{2,3,4\}\}$ the label 3 moves to $\{1,3,4\}$, the label 1 moves to $\{1,2,3,4\}$ and the label 2 moves to $\{2,3,4\}$, and therefore we get the settled forest 1232444. (iii) $\star$ After applying the V-move on the face $\{\{1,2,3,4\},\{1,3,4\},\{1,4\},\{1,2,4\}\}$ the label 1 moves to $\{1,4\}$, the label 4 moves to $\{1,2,4\}$ and the label 2 moves to $\{1,2,3,4\}$, and therefore we get the settled forest 2213444. (iv) After applying the V-move on the face $\{\{1,2,4\},\{1,2\},\{1\},\{1,4\}\}$ the labels 1 and 4 are swapped and therefore get the settled forest 2132444. (v) After applying the path move on the face $\mathcal{F}_4^{1+}$ the label 1 moves to $\{1,3,4\}$, the label 3 moves to $\{1,2,3,4\}$ and the label 2 moves to $\{1,2,4\}$ and therefore we get the settled forest 3212444.

Chapter 8

Irreducible signatures of Q_n: transforming irreducible splitting signatures

8.1 Introduction

Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n. The main goal of this chapter is to show that under the inductive hypothesis that the edge slide graph of an irreducible signature of Q_{n-1} is connected, an irreducible splitting signature of $\mathcal{I}$ can be transformed into any other such splitting signature by a sequence of edge slides. This result is used to show that we can move from a tree with an irreducible splitting signature to a tree with a fixed amenable splitting signature. This is an important step to understanding the connectivity of an irreducible signature of Q_n.

8.2 Preparatory lemmas

8.2.1 The main technical tool of this chapter

The following lemma is the main technical tool we use to transform a tree of Q_n with an irreducible signature and an irreducible splitting signature in $\mathcal{F}_n^{n-}$ into a tree with any other splitting signature in $\mathcal{F}_n^{n-}$. We prove this lemma under the inductive hypothesis that the edge slide graph of an irreducible signature of Q_{n-1} is connected. This lemma is used to prove Lemma 8.2.2 and Lemma 8.2.3.

Lemma 8.2.1. *Suppose that every irreducible signature of Q_{n-1} is connected and let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n and let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be an irreducible splitting signature of $\mathcal{I}$ with respect to n. Let T be a settled tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Let $i, j \in [n-1]$ where $i \neq j$ and $u_i = a_i - d_i \neq 0$. Then there is a sequence of edge slides that transforms T into $\hat{T}$ such that $\hat{T}_{n-}$ has signature $\hat{\mathcal{D}} = (\hat{d}_1, \ldots, \hat{d}_{n-1})$, where*

$$\hat{d}_i = d_i + 1, \quad \hat{d}_j = d_j - 1, \quad \text{and} \quad \hat{d}_k = d_k,$$

for all $k \neq i, j$.

Proof. Without loss of generality we may assume $\mathcal{D}$ is ordered; moreover, we may choose the order so that $d_i < d_{i+1}$. Applying Lemma 7.2.15 (if necessary), we may assume that

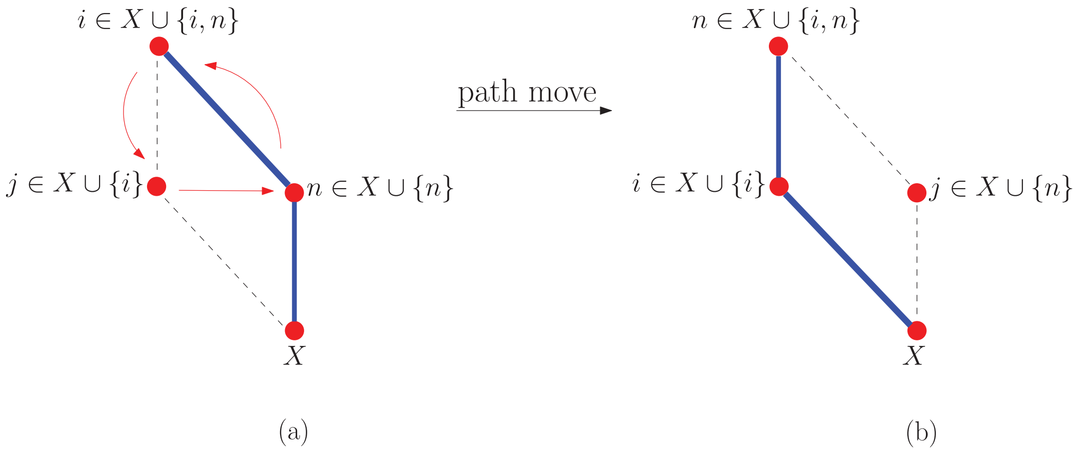


Figure 8.1: Diagram of the proof of Lemma 8.2.1 on page 129. (a) Direction i is chosen at $X \cup \{i, n\}$, direction n is chosen at $X \cup \{n\}$ and direction j is chosen at $X \cup \{i\}$. (b) After applying the path move, direction j moves to $X \cup \{n\}$, direction n moves to $X \cup \{i, n\}$, and direction i moves to $X \cup \{i\}$.

$\psi_T(\{j, n\}) = n$. Let $X \subseteq [n-1] \setminus \{i\}$ be such that $X \cup \{i, n\}$ is a lowest vertex of $\mathcal{F}_n^{n+}$ that has an i-edge. After applying Lemma 7.2.1 (if necessary), we may assume that $\psi_T(X \cup \{n\}) = n$. We distinguish the following cases according to whether or not $j \in X$.

1. Suppose that $j \in X$. Since $\mathcal{D}$ is irreducible, by Lemma 3.2.4 there exists a tree T'_{n-} of Q_{n-1} with signature $\mathcal{D}$ such that $\psi_{T'}(X \cup \{i\}) = j$. Since the edge slide graph of signature $\mathcal{D}$ is connected (by the inductive hypothesis), we can move from T_{n-} to such a tree using edge slides. Then T can be transformed into T' with $\psi_T(X \cup \{i\}) = j$. Since the edge slides were only applied in $\mathcal{F}_n^{n-}$, we have $T'_{n+} = T_{n+}$. So $\psi_{T'}(X \cup \{i, n\}) = i$ and $\psi_{T'}(X \cup \{n\}) = j$.

   As shown in Figure 8.1 on page 130, applying the path move direction j moves to $X \cup \{n\}$, direction n moves to $X \cup \{i, n\}$, and direction i moves to $X \cup \{i\}$. Then T' is transformed into $\hat{T}$ such that $\hat{\mathcal{D}} = (\hat{d}_1, \ldots, \hat{d}_{n-1})$, where

   $$\hat{d}_i = d_i + 1, \quad \hat{d}_j = d_j - 1 \quad \text{and} \quad \hat{d}_k = d_k,$$

   for all $k \neq i, j$ as required.

2. Suppose that $j \notin X$. We distinguish the following cases according to whether or not $\psi_T(\{i, n\}) = i$.

   (a) Suppose that $\psi_T(\{i, n\}) \neq i$. Then $\psi_T(\{i, n\}) = n$. Let $P_{\mathcal{F}_n^{n+}}(X \cup \{n\}) = (\xi_0, \ldots, \xi_\alpha)$ be the max removing path in $\mathcal{F}_n^{n+}$ from $\xi_0 = X \cup \{n\}$ to $\xi_\alpha = \{n\}$. Since T is settled and $\psi_T(X \cup \{n\}) = n$, we have $\psi_T(\xi_a) = n$ for all a. Consider $\sigma_i(P_{\mathcal{F}_n^{n+}}(X \cup \{n\})) = (\xi_0 \cup \{i\}, \ldots, \xi_\alpha \cup \{i\})$ and let $\ell_a = \psi_T(\xi_a \cup \{i\})$ for $a = 0, \ldots, \alpha$. Since $X \cup \{i, n\}$ is a lowest vertex of $\mathcal{F}_n^{n+}$ that has an i-edge, we have $\ell_a \neq i$ for $a = 1, \ldots, \alpha$. We distinguish the following cases according to whether or not $\ell_a = n$ for all $a = 1, \ldots, \alpha$.

   (I) Suppose that $\ell_a \neq n$ for some $a = 1, \ldots, \alpha$. We distinguish the following cases according to whether or not $i = \max X \cup \{i\}$.

(i) Suppose that $i \neq \max X \cup \{i\}$. Then choose $m > i$ such that $m \in X$. Since $d_i < d_{i+1}$ and $m > i$, we have $d_i < d_m$. After reordering the indices j such that $d_j = d_m$ (if necessary) we can assume that $d_{m-1} < d_m$. Since $\mathcal{D}$ is irreducible, by Lemma 3.2.4 there exists a tree T'_{n-} of Q_{n-1} with signature $\mathcal{D}$ such that $\psi_{T'}(X \cup \{i\}) = m$. Since the edge slide graph of signature $\mathcal{D}$ is connected (by the inductive hypothesis), we can move from T_{n-} to such a tree using edge slides. Then T can be transformed into T' with $\psi_{T'}(X \cup \{i\}) = m$. Since the edge slides were only applied in $\mathcal{F}_n^{n-}$, we have $T'_{n+} = T_{n+}$.

As shown in Figure 8.2 on page 137, applying the path move direction m moves to $X \cup \{n\}$, direction n moves to $X \cup \{i, n\}$, and direction i moves to $X \cup \{i\}$. Let T^2 be the resulting tree and let $\mathcal{D}' = (d'_1, \ldots, d'_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$d'_i = d_i + 1, \quad d'_m = d_m - 1 \quad \text{and} \quad d'_k = d_k,$$

for $k \neq i, m$. If $m = i+1$ and $d_m = d_{i+1} = d_i + 1$ then $\mathcal{D}'$ is the permutation of $\mathcal{D}$ obtained by swapping the ith and $(i+1)$th entries, and is therefore irreducible. Otherwise, since $d_i < d_{i+1}$ and $d_{m-1} < d_m$, we have

$$d'_i \leq d'_{i+1} \leq \cdots \leq d'_{m-1} \leq d'_m < d'_{m+1} \leq \cdots \leq d'_{n-1}.$$

So $\mathcal{D}'$ is in increasing order, and

$$\varepsilon_\mu^{\mathcal{D}'} = \begin{cases} (\varepsilon_\mu^{\mathcal{D}}) + 1, & \text{when } i \leq \mu < m; \\ \varepsilon_\mu^{\mathcal{D}}, & \text{otherwise.} \end{cases}$$

So T is transformed into T^2 with $\psi_{T^2}(X \cup \{i, n\}) = n, \psi_{T^2}(X \cup \{n\}) = m, \psi_{T^2}(X \cup \{i\}) = i$ and all other labels of T^2 the same as the labels of T'. Since $T'_{n+} = T_{n+}$, we have all labels of T^2_{n+} apart from $\psi_{T^2}(X \cup \{i, n\})$ and $\psi_{T^2}(X \cup \{n\})$ are the same as the labels of T_{n+}.

Let $P_{\mathcal{F}_n^{n+}}(X \cup \{n\}, m) = (\xi'_0, \ldots, \xi'_{\alpha-1})$ be the m-retaining max removing path from $\xi'_0 = X \cup \{n\}$ to $\xi'_{\alpha-1} = \{m, n\}$. Since all labels of T^2 apart from $\psi_{T^2}(X \cup \{i, n\})$ and $\psi_{T^2}(X \cup \{n\})$ are the same as the labels of T_{n+}, and since T is settled with $\psi_T(X \cup \{n\}) = n$, we have $\psi_{T^2}(\xi'_a) = n$ for all $a = 1, \ldots, \alpha-1$. All the conditions in Lemma 7.2.5 are satisfied at the vertex $X \cup \{n\}$. Then T^2 can be transformed into $T^2_{\alpha-1}$ with $\psi_{T^2_{\alpha-1}}(\xi'_{\alpha-1}) = m$, $\psi_{T^2_{\alpha-1}}(\xi'_a) = n$ for $a = 0, \ldots, \alpha-2$, and all other labels of $(T^2_{\alpha-1})_{n+}$ the same as the labels of T^2. Since all labels of T^2_{n+} apart from $\psi_{T^2}(X \cup \{i, n\})$ and $\psi_{T^2}(X \cup \{n\})$ are the same as the labels of T_{n+}, we have all the labels of $(T^2_{\alpha-1})_{n+}$ apart from $\psi_{T^2_{\alpha-1}}(X \cup \{i, n\})$ and $\psi_{T^2_{\alpha-1}}(\xi'_{\alpha-1} = \{m, n\})$ the same as the labels of T_{n+}.

Since $\mathcal{D}'$ is irreducible, by Lemma 3.2.4 there exists an upright spanning tree T^3_{n-} of Q_{n-1} with signature $\mathcal{D}'$ such that $\psi_{T^3}(\{j, m\}) = j$. Since the edge slide graph of signature $\mathcal{D}'$ is connected, we can move from $T^2_{\alpha-1}$ to such a tree using edge slides. Then $T^2_{\alpha-1}$ is transformed into T^3 with $\psi_{T^3}(\{j, m\}) = j$. Since the edge slides were only applied in $\mathcal{F}_n^{n-}$, we have $T^3_{n+} = (T^2_{\alpha-1})_{n+}$. So $\psi_{T^3}(\{m, n\}) = m$ and $\psi_{T^3}(\{j, n\}) = n$.

Let S be the signature of $T^3 \cap Q_{\{j,m,n\}}$. Since $\psi_{T^3}(\{m,n\}) = m$; $\psi_{T^3}(\{j,n\}) = n$ and $\psi_{T^3}(\{j,m\}) = j$, we have $S \in \{(2,2,3),(2,3,2),(3,2,2)\}$, depending on the direction μ of the vertex $\{j,m,n\}$. As shown in Figure 8.3, there exists a tree of $Q_{\{j,m,n\}}$ with signature S where direction m is chosen at $\{j,m\}$, direction j is chosen at $\{j,n\}$, and direction n is chosen at $\{m,n\}$. Since the edge slide graph of signature S is connected (by Henden [6] or Lemma 4.3.6), we can move from $T^3 \cap \{j,m,n\}$ to such a tree using edge slides. Let $\hat{T}$ be the resulting tree and let $\hat{\mathcal{D}} = (\hat{d}_1, \ldots, \hat{d}_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$\hat{d}_i = d_i' = d_i + 1, \quad \hat{d}_j = d_j' = d_j - 1, \quad \hat{d}_m = d_m + 1 = d_m \quad \text{and} \quad \hat{d}_k = d_k,$$

for all $k \neq i, j, m$ as required.

(ii) Suppose that $i = \max X \cup \{i\}$. There exists $\beta \leq \alpha$ such that $\psi_T(\xi_\beta \cup \{i\}) = n$. Note that β exists because $\xi_\alpha \cup \{i\} = \{i,n\}$ and $\psi_T(\{i,n\}) = n$. We show there exists a sequence of edge slides that swaps n and ℓ_a for all $a = 1, \ldots, \beta - 1$ so T can be transformed into T' where

$$\psi_{T'}(V) = \begin{cases} n, & \text{if } V = \xi_a \cup \{i\} \text{ and } a = 1, \ldots, \beta - 1; \\ \ell_a, & \text{if } V = \xi_a \text{ and } a = 1, \ldots, \beta - 1; \\ \psi_T(V), & \text{otherwise.} \end{cases}$$

Since $i = \max(\xi_a \setminus \{n\}) \cup \{i\}$, $\psi_T(\xi_a) = n$ and $\ell_a \neq i$ for all $a = 1, \ldots, \beta - 1$, all conditions in Case 2 of Lemma 7.2.2 hold. Therefore we can swap ℓ_1 and n to get T_1 where

$$\psi_{T_1}(V) = \begin{cases} n, & \text{if } V = \xi_1 \cup \{i\}, \text{ or } V = \xi_a \text{ for } a = 2, \ldots, \beta - 1; \\ \ell_1, & \text{if } V = \xi_1; \\ \ell_a, & \text{if } V = \xi_a \cup \{i\} \text{ for } a = 2, \ldots, \beta - 1; \\ \psi_T(V), & \text{otherwise.} \end{cases}$$

Then we repeat the above process of swapping labels n and ℓ_a for $a = 2, \ldots, \beta - 1$ until eventually we reach T' and therefore we can move to Case (II).

(II) Suppose that $\ell_a = n$ for $a = 1, \ldots, \alpha$. All the conditions in Lemma 7.2.5 are satisfied at the vertex $X \cup \{i,n\}$. Then T can be transformed into T_α with $\psi_{T_\alpha}(\xi_\alpha \cup \{i\}) = i$, $\psi_{T_\alpha}(\xi_a \cup \{i\}) = n$ for $a = 0, \ldots, \alpha - 1$ and all other labels of T_α the same as the labels of T. Since $\xi_\alpha \cup \{i\} = \{i,n\}$, we have $\psi_{T_\alpha}(\{i,n\}) = i$ and therefore we can move to Case (b).

(b) Suppose that $\psi_T(\{i,n\}) = i$. Since $\mathcal{D}$ is irreducible, by Lemma 3.2.4 there exists a tree T' of Q_{n-1} with signature $\mathcal{D}$ such that $\psi_T(\{i,j\}) = j$. Since the edge slide graph of signature $\mathcal{D}$ is connected (by the inductive hypothesis), we can move from T_{n-} to such a tree using edge slides. Then T is transformed into T' with $\psi_T'(\{i,j\}) = j$. Since the edge slides were only applied in $\mathcal{F}_n^{n-}$, we have $T_{n+} = T'_{n+}$. So $\psi_T(\{j,n\}) = n$ and $\psi_T(\{i,n\}) = i$.

Let S be the signature of $T' \cap Q_{\{i,j,n\}}$. Since $\psi_T(\{i,j\}) = j$; $\psi_T(\{j,n\}) = n$ and $\psi_T(\{i,n\}) = i$, we have $S \in \{(2,2,3),(2,3,2),(3,2,2)\}$ depending on the direction

μ of the vertex $\{i, j, n\}$. As shown in Figure 8.4, there exists a tree of $Q_{\{i,j,n\}}$ with signature S where direction i is chosen at $\{i, j\}$ and direction j is chosen at $\{j, n\}$. Since the edge slide graph of signature S is connected (by Henden [6] or Lemma 4.3.6), we can move to such a tree using edge slides. Let $\hat{T}$ be the resulting tree and let $\hat{\mathcal{D}} = (\hat{d}_1, \dots, \hat{d}_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$\hat{d}_i = d_i + 1,\ \hat{d}_j = d_j - 1 \text{ and } \hat{d}_k = d_k,$$

for all $k \neq i, j$, as required.

In all cases above we succeeded in transforming T into $\hat{T}$ such that $\hat{T}_{n-}$ has signature $\hat{\mathcal{D}} = (\hat{d}_1, \dots, \hat{d}_{n-1})$, where

$$\hat{d}_i = d_i + 1, \quad \hat{d}_j = d_j - 1, \quad \text{and} \quad \hat{d}_k = d_k,$$

for all $k \neq i, j$. $\square$

8.2.2 Ordering irreducible splitting signatures

The next lemma shows that an upright spanning tree of Q_n with an ordered irreducible signature and irreducible splitting signature in $\mathcal{F}_n^{n-}$ can be transformed into a tree with an ordered irreducible splitting signature in $\mathcal{F}_n^{n-}$. We prove this lemma under the inductive hypothesis that every irreducible signature of Q_{n-1} is connected.

Lemma 8.2.2. *Suppose that every irreducible signature of Q_{n-1} is connected and let $\mathcal{I} = (a_1, \dots, a_n)$ be an ordered irreducible signature of Q_n. Let $\mathcal{D} = (d_1, \dots, d_{n-1})$ be an irreducible splitting signature of $\mathcal{I}$ with respect to n. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Suppose that $\mathcal{D}$ is not ordered. Then there exists a sequence of edge slides that transforms T into $\hat{T}$ such that $\hat{T}_{n-}$ has an ordered irreducible signature $\hat{\mathcal{D}} = (\hat{d}_1, \dots, \hat{d}_{n-1})$.*

Proof. Given a signature $\bar{\mathcal{D}}$ we define $\kappa(\bar{\mathcal{D}}) = \sum_{h>\ell} \max\{0, d_\ell - d_h\} \geq 0$. Then $\bar{\mathcal{D}}$ is ordered if and only if $\kappa(\bar{\mathcal{D}}) = 0$. Let $S = \{k | d_k = \min_{i \geq k}\{d_i\}\}$. If $S = [n-1]$, then $\mathcal{D}$ is ordered. Otherwise, let $k = \min([n-1] \setminus S)$. Then $d_i \leq d_k$ for all $i \leq k$, but there is $j > k$ such that $d_j < d_k$. Let j be such that d_j is as large as possible with $j > k$ and $d_j < d_k$, and among all such indices choose j to be as large as possible as well. Since $\mathcal{I}$ is ordered and since $j > k$, we have $d_j < d_k \leq a_k \leq a_j$. So $d_j < a_j$ and therefore $a_j - d_j = u_j > 0$. Then using Lemma 8.2.1 T can be transformed into T' such that T'_{n-} has signature $\mathcal{D}' = (d'_1, \dots, d'_{n-1})$, where

$$d'_j = d_j + 1,\ d'_k = d_k - 1, \text{ and } d'_i = d_i,$$

for $i \neq j, k$.

We now check that $\mathcal{D}'$ is still an irreducible signature. Write

$$d_{i_1} \leq d_{i_2} \leq \cdots \leq d_{i_{n-1}},$$

with $i_a \neq i_b$ if $a \neq b$ and $i_a = a$ for $a < k$.

Suppose that $k = i_t$. Then by our choice of j and k we can choose the ordering such that $i_{t-1} = j$. If $d_k = d_j + 1$, then $\mathcal{D}'$ is the permutation of $\mathcal{D}$ obtained by swapping d_j and d_k and therefore $\mathcal{D}'$ is irreducible. Otherwise $d_j < d_k - 1$. So

$$d'_j = d_j + 1 \leq d_k - 1 = d'_k$$

and

$$d'_{i_1} \le d'_{i_2} \le \cdots \le d'_{i_{n-1}}.$$

Then

$$\sum_{a=1}^{s} d'_{i_a} = \begin{cases} \sum_{a=1}^{s} d_{i_a}, & \text{when } s \neq t-1; \\ (\sum_{a=1}^{s} d_{i_a}) + 1, & \text{when } s = t-1, \end{cases}$$

and therefore $\mathcal{D}'$ is an irreducible signature.

We now show that $\kappa(\mathcal{D}') < \kappa(\mathcal{D})$. Since $d'_t = d_t$ unless $t \in \{j,k\}$, we need only consider the terms $d'_\ell - d'_h$ when $\{\ell, h\} \cap \{j,k\} \neq \varnothing$. We consider the following cases according to whether $h = j$, $h = k$, $\ell = j$, $\ell = k$, or $(h,\ell) = (j,k)$.

(a) Suppose that $h = j$ with $\ell \neq k$. Then

$$\begin{aligned} d'_\ell - d'_h &= d'_\ell - d'_j \\ &= d_\ell - d_j - 1 \\ &< d_\ell - d_j \\ &\le \max\{0, d_\ell - d_j\}. \end{aligned}$$

Hence $\max\{0, d'_\ell - d'_h\} \le \max\{0, d_\ell - d_j\}$.

(b) Suppose that $h = k$. Then $\ell < k$ so $\ell \in S$ and therefore $d_\ell \le d_j < d_k$. Therefore $d'_\ell = d_\ell \le d_k - 1 = d'_k$ and hence $\max\{0, d'_\ell - d'_k\} = 0 = \max\{0, d_\ell - d_k\}$.

(c) Suppose that $\ell = j$. Then $j < h$ and so $d_h \neq d_j$ by our choice of j. If $d_h > d_j$, then $d'_j \le d'_h$ and therefore $d'_j - d'_h \le 0$. Otherwise $d_h < d_j$. Then $d_h < d_j < d_k$, and we consider this term together with $d_k - d_h$. We have $d'_h < d'_j$ and $d'_h \le d'_k$ so

$$\begin{aligned} \max\{0, d'_j - d'_h\} + \max\{0, d'_k - d'_h\} &= (d'_j - d'_h) + (d'_k - d'_h) \\ &= (d_j + 1 - d_h) + (d_k - 1 + d_h) \\ &= (d_j - d_h) + (d_k - d_h) \\ &= \max\{0, d_j - d_h\} + \max\{0, d_k - d_h\}. \end{aligned}$$

Therefore the net contribution from these terms is unchanged.

(d) Suppose that $\ell = k$ with $h \neq j$. If $(h,\ell) = (h,k)$ has not already been considered in (c) then

$$\begin{aligned} d'_\ell - d'_h &= d'_k - d'_h \\ &= d_k - 1 - d_h \\ &< d_k - d_h \\ &\le \max\{0, d_k - d_h\}. \end{aligned}$$

Hence $\max\{0, d'_k - d'_h\} \le \max\{0, d_k - d_h\}$.

(e) Suppose that $(h,\ell) = (j,k)$. Then $d_k - d_j > 0$ so

$$\begin{aligned} d'_\ell - d'_h &= d'_k - d'_j \\ &= d_k - 1 - d_j - 1 \\ &= d_k - d_j - 2 \\ &< d_k - d_j \\ &= \max\{0, d_k - d_j\}. \end{aligned}$$

Therefore $\max\{0, d'_k - d'_j\} < \max\{0, d_k - d_j\}$.

From all cases we conclude that $\kappa(\mathcal{D}') < \kappa(\mathcal{D})$ and therefore $\mathcal{D}$ is transformed into $\mathcal{D}'$ such that $\kappa(\mathcal{D}') < \kappa(\mathcal{D})$. Let $S' = \{k | d'_k = \min_{i \geq k}\{d_i\}\}$. If $S' = [n-1]$, then $\mathcal{D}'$ is ordered. Otherwise we repeat the above process and decrease $\kappa(\mathcal{D}')$ until we reach an ordered signature. □

8.2.3 Connecting irreducible splitting signatures

The following lemma shows that a tree with an ordered irreducible splitting signature of an irreducible signature can be transformed into a tree with any other ordered irreducible splitting signature. We prove this lemma under the inductive hypothesis that every irreducible signature of Q_{n-1} is connected.

Lemma 8.2.3. *Suppose that every irreducible signature of Q_{n-1} is connected and let $\mathcal{I} = (a_1, \ldots, a_n)$ be an ordered irreducible signature of Q_n. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ and $\mathcal{D}' = (d'_1, \ldots, d'_{n-1})$ be ordered irreducible splitting signatures of $\mathcal{I}$ with respect to n. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Then there exists a sequence of edge slides that transforms T into T' such that T'_{n-} has signature $\mathcal{D}'$.*

Proof. Let $\delta(\mathcal{D}, \mathcal{D}') = ||\mathcal{D} - \mathcal{D}'|| = \sum_{t=1}^{n-1} |d_t - d'_t|$. Note that $\sum_{t=1}^{n} |d_t - d'_t| = 2r$ for some $r \geq 0$, because the d_t and the d'_t sum to the same value, so the sum here without the absolute value signs is zero. Adding the absolute value signs leaves each term unchanged or changes it by an even number, so the resulting sum must have the same parity and so is even.

If $\sum_{t=1}^{n-1} |d_t - d'_t| = 0$, then $\mathcal{D} = \mathcal{D}'$ and no further moves are needed. Otherwise there exists $|d_t - d'_t| \neq 0$ for some t. Choose h to be the least index such that $|d_h - d'_h| \neq 0$. We consider the following cases according to whether $d_h < d'_h$ or $d_h > d'_h$.

(a) Suppose that $d_h < d'_h$. Choose $i \geq h$ to be the largest index such that $d_i = d_h$. Then $d_i < d_{i+1}$ and also $d_i = d_h < d'_h \leq d'_i$. Since $d_i < d'_i \leq a_i$, we have $u_i = a_i - d_i > 0$. Since $\sum_{t=1}^{n-1} d_t = \sum_{t=1}^{n-1} d'$ and $d_i < d'_i$, there must exist $f > i$ such that $d_f > d'_f$. Let j be the least such f. Then $d_j > d'_j \geq d'_{j-1} \geq d_{j-1}$. Applying Lemma 8.2.1, T can be transformed into $\hat{T}$ such that $\hat{T}_{n-}$ has signature $\hat{\mathcal{D}}$ where

$$\hat{d}_i = d_i + 1,\ \hat{d}_j = d_j - 1 \text{ and } \hat{d}_k = d_k,$$

for all $k \neq i, j$.

Since $d_i < d'_i \leq d'_j < d_j$, we have $d_j - d_i \geq 2$. Since $\mathcal{D}$ is ordered, $d_i < d_{i+1}$, $d_{j-1} < d_j$, and $d_j - d_i \geq 2$, all the conditions in Definition 5.6.1 hold. Therefore $\hat{\mathcal{D}}$ is the result of applying a signature move to $\mathcal{D}$, and is therefore ordered irreducible.

(b) Suppose that $d_h > d'_h$. Since $\sum_{t=1}^{h-1} |d_t - d'_t| = 0$, and since $d_h > d'_h \geq d'_{h-1} = d_{h-1}$, we must have $d_{h-1} < d_h$. Since $\sum_{t=1}^{n-1} d_t = \sum_{t=1}^{n-1} d'_t$ and $d_h > d'_h$, there must exist $f > h$ such that $d_f < d'_f$. Let $g > h$ be the least such f. Since $\mathcal{D}$ and $\mathcal{D}'$ are ordered irreducible, and $d_i \geq d'_i$ for $1 \leq i < g$ with $d_h > d'_h$, we have $\varepsilon_\mu^{\mathcal{D}} > \varepsilon_\mu^{\hat{\mathcal{D}}} \geq 1$ for $h \leq \mu < g$. Hence $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $h \leq \mu < g$.

Choose $j \geq g$ to be the largest index such that $d_j = d_g$. Then we either have $d_j < d_{j+1}$ or $j = n-1$. By Lemma 3.2.5, we have $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $g \leq \mu < j$. Since $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $h \leq \mu < g$ and $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $g \leq \mu < j$, we have $\varepsilon_\mu^{\mathcal{D}} \geq 2$ for all $h \leq \mu < j$.

Since $d_j = d_g < d'_g \leq d'_j \leq a_j$, we have $u_j > 0$. Applying Lemma 8.2.1, T can be transformed into $\hat{T}$ such that $\hat{T}_{n-}$ has signature $\hat{\mathcal{D}}$ where

$$\hat{d}_h = d_h - 1,\ \hat{d}_j = d_j + 1 \text{ and } \hat{d}_k = d_k,$$

for all $k \neq i, j$. Since $\mathcal{D}$ is ordered, $d_{h-1} < d_h$ and $d_j < d_{j+1}$, and since $h < j$, we have

$$\hat{d}_1 \leq \cdots \leq \hat{d}_{h-1} \leq \hat{d}_h \leq \cdots \leq \hat{d}_j \leq \hat{d}_{j+1} \leq \cdots \leq \hat{d}_{n-1},$$

and therefore $\hat{D}$ is ordered. Since $h < j$, we have

$$\varepsilon_\mu^{\hat{\mathcal{D}}} = \begin{cases} (\varepsilon_\mu^{\mathcal{D}}) - 1 \geq 1, & \text{when } h \leq \mu < j; \\ \varepsilon_\mu^{\mathcal{D}}, & \text{otherwise,} \end{cases}$$

and therefore $\hat{\mathcal{D}}$ is an irreducible signature of Q_{n-1}.

In either case $\mathcal{D}$ is transformed into $\hat{\mathcal{D}}$ such that $\delta(\hat{\mathcal{D}}, \mathcal{D}') = \delta(\mathcal{D}, \mathcal{D}') - 2$. If $\sum_{t=1}^{n-1} |\hat{d}_t - d'_t| = 0$, then $\hat{\mathcal{D}} = \mathcal{D}'$ and therefore the transformation is completed. Otherwise we repeat the above process and decrease $\delta(\hat{\mathcal{D}}, \mathcal{D}')$ by two until we reach the required irreducible splitting signature $\mathcal{D}'$. $\square$

8.3 Transforming an irreducible splitting signature of a tree

By combining Lemmas 8.2.2 and 8.2.3 we obtain the following corollary:

Corollary 8.3.1. *Suppose that every irreducible signature of Q_{n-1} is connected. Let $\mathcal{I}$ be an ordered irreducible signature of Q_n, and let $\mathcal{D}$ be an ordered irreducible splitting signature of $\mathcal{I}$ with respect to n. Let T be an upright spanning tree with signature $\mathcal{I}$ such that T_{n-} has an irreducible signature. Then there is a sequence of edge slides that transforms T into T' such that T'_{n-} has signature $\mathcal{D}$.*

8.4 Summary map

In this chapter we showed that under the inductive hypothesis an upright spanning tree of Q_n with an irreducible signature that has an irreducible splitting signature can be transformed into a tree with any other irreducible splitting signature. This result is used to show that a tree with an irreducible splitting signature can be transformed into a tree with an amenable splitting signature. In Chapter 9, we show that an upright spanning tree of Q_n with an irreducible signature such that the tree has a reducible splitting signature in $\mathcal{F}_n^{n-}$ is connected to a tree with an irreducible splitting signature in $\mathcal{F}_n^{n-}$.

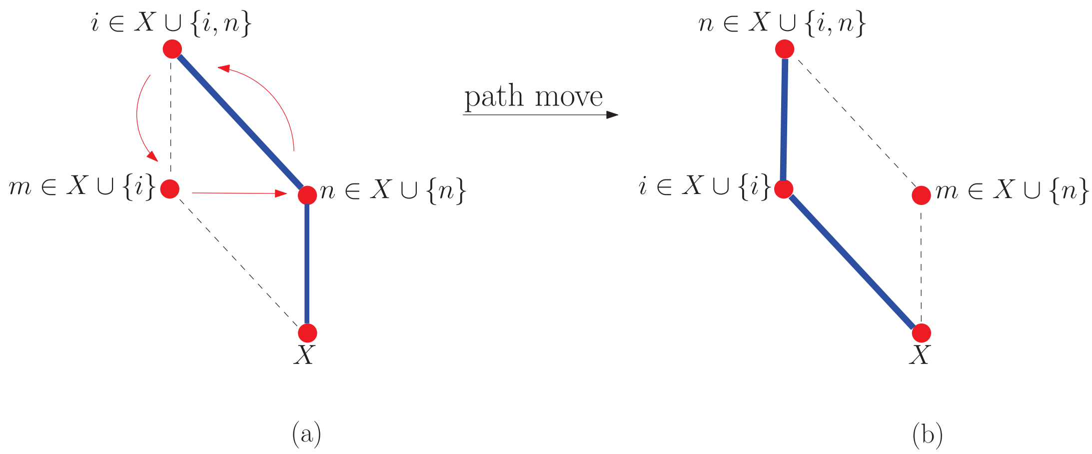


Figure 8.2: Diagram of the proof of Lemma 8.2.1 on page 129. (a) Direction i is chosen at $X \cup \{i,n\}$, direction n is chosen at $X \cup \{n\}$ and direction m is chosen at $X \cup \{i\}$. (b) After applying the path move, direction m moves to $X \cup \{n\}$, direction n moves to $X \cup \{i,n\}$ and direction i moves to $X \cup \{i\}$.

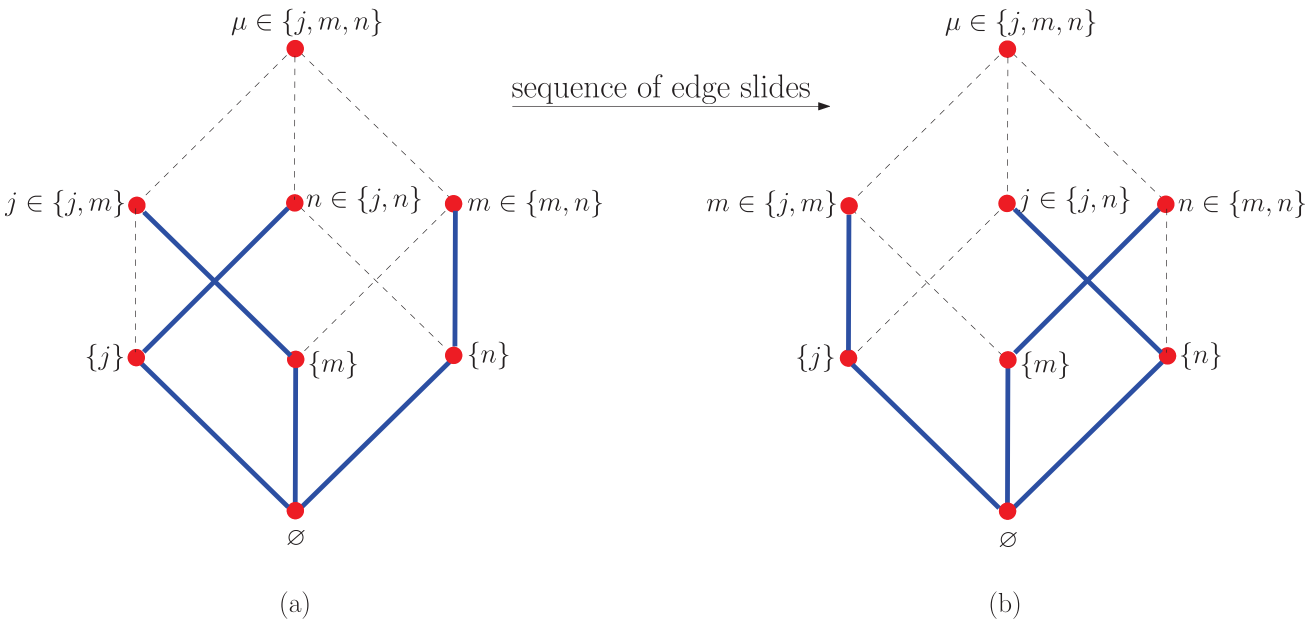


Figure 8.3: Diagram 2 for Case (a(I(i))) for the proof of Lemma 8.2.1. Upright spanning trees of $Q_{\{j,m,n\}}$ with signature $S \in \{(2,2,3),(2,3,2),(3,2,2)\}$, depending on the direction μ of the vertex $\{j,m,n\}$. (a) Direction j is chosen at $\{j,m\}$, direction n is chosen at $\{j,n\}$ and direction m is chosen at $\{m,n\}$. (b) After a sequence of edge slides direction m moves to $\{j,m\}$, direction j moves to $\{j,n\}$ and direction n moves to $\{m,n\}$.

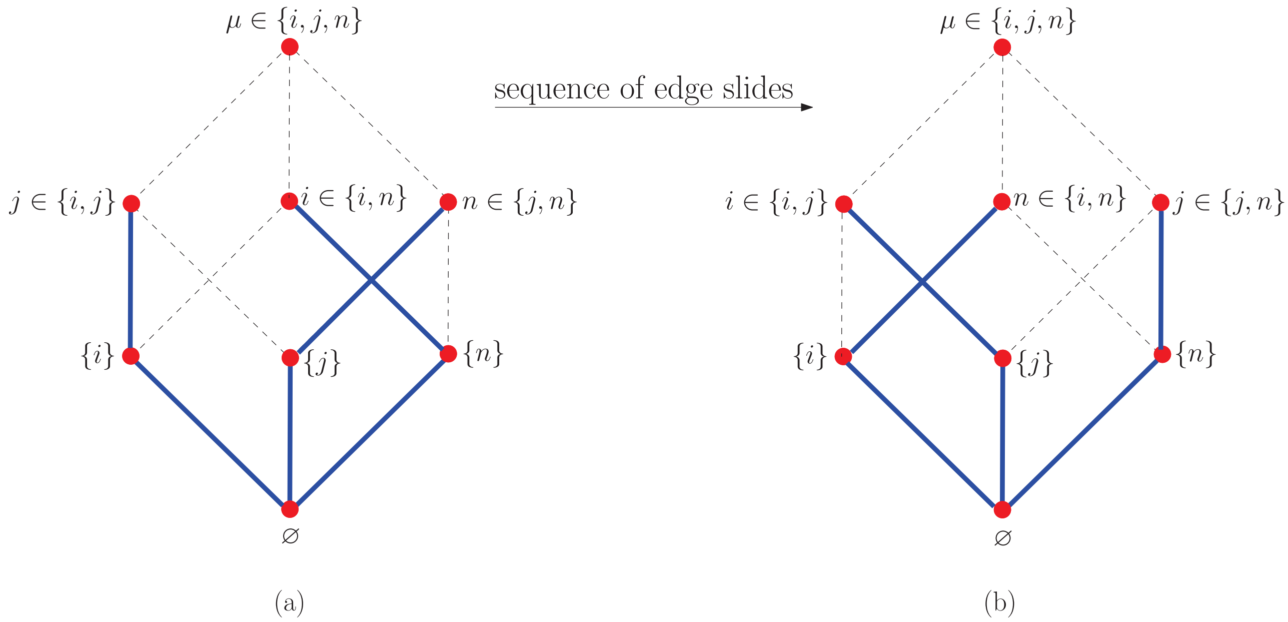


Figure 8.4: Diagram for Case (b) for the proof of Lemma 8.2.1. Upright spanning trees of $Q_{\{i,j,n\}}$ with signature $S \in \{(2,2,3),(2,3,2),(3,2,2)\}$, depending on the direction μ of the vertex $\{i,j,n\}$. (a) Direction i is chosen at $\{i,n\}$, direction j is chosen at $\{i,j\}$ and direction n is chosen at $\{j,n\}$. (b) After a sequence of edge slides direction i moves to $\{i,j\}$, direction j moves to $\{j,n\}$ and direction n moves to $\{i,n\}$.

Chapter 9

Irreducible signatures of Q_n: reducible splitting signatures

9.1 Introduction

In this chapter we study reducible splitting signatures of an irreducible signature of Q_n in detail. The main goal is to show that under the inductive hypothesis each tree of Q_n with an irreducible signature but a reducible splitting signature in $\mathcal{F}_n^{n-}$ can be transformed into a tree with an irreducible splitting signature in $\mathcal{F}_n^{n-}$. This is an important step to understand the connectivity of the edge slide graph of an irreducible signature of Q_n.

First we introduce notation for the size of the smallest reducing set.

Definition 9.1.1. Let $\mathcal{S} = (a_1, \ldots, a_n)$ be a signature of Q_n. We define $\omega(\mathcal{S})$ to be the size of the smallest set $R \subseteq [n]$ such that $\sum_{i \in R} a_i = 2^{|R|} - 1$. Then $\omega(\mathcal{S}) = n$ if and only if $\mathcal{S}$ is irreducible, and when $\mathcal{S}$ is reducible $\omega(\mathcal{S})$ is the size of the smallest reducing set.

Our strategy is to increase the size of the smallest reducing set $\omega(\mathcal{S})$ for all reducible signatures using edge slides. The main technical tool for this is Lemma 9.2.2, which addresses the case where $d_t \geq 3$ for all t outside the reducing set. The case where $d_1 = 1$ and $d_2 = 2$, where both [1] and [2] are reducing sets, is handled in Lemma 9.2.3. In this case we risk moving to a signature with $d_1 = 2$ and $d_2 = 1$, swapping the problem from direction 1 to direction 2, and the lemma shows that we can avoid this.

The following theorem is the main result of this chapter:

Theorem 9.1.2. *Suppose that every irreducible signature of Q_k is connected for all $k < n$. Let $n \geq 4$, and let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n with $a_n \geq a_\ell$ for all ℓ. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be a reducible splitting signature of $\mathcal{I}$ with respect to n. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Then there exists a sequence of edge slides that transforms T into $\bar{T}$ such that $\bar{T}_{n-}$ has an irreducible signature.*

9.2 Preparatory lemmas

In this section we establish lemmas that are used to prove Theorem 9.1.2.

9.2.1 The existence of a vertex with label n that does not lie above the reducing set

Note that if $\mathcal{S}$ is ordered then $\omega(\mathcal{S})$ is the least r such that $\sum_{t=1}^{r} a_t = 2^r - 1$. For example, $\mathcal{S} = (2,2,3,9,15,32)$ is a reducible signature of Q_6, as seen by taking $\sum_{t=1}^{r} a_t = 2^r - 1$ for $r = 3$ and 5. So $\omega(\mathcal{S}) = 3$.

Let R be a reducing set. The following lemma shows the existence of a vertex ξ in $\mathcal{F}_n^{n+}$ with label n such that $\xi \not\subseteq R \cup \{n\}$.

Lemma 9.2.1. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n with $a_n \geq a_\ell$ for all ℓ. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be a reducible splitting signature of $\mathcal{I}$ with respect to n and let R be a corresponding reducing set. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Then there exists a vertex ξ of $\mathcal{F}_n^{n+}$ such that $\xi \setminus \{n\} \not\subseteq R$ and $\psi_T(\xi) = n$.*

Proof. Without loss of generality we may assume $\mathcal{D}$ is ordered, and therefore for $r = |R| \leq n-2$ we have $\sum_{i=1}^{r} d_i = 2^r - 1$. From the signature condition applied to $\mathcal{D}$ we have $\sum_{i=1}^{r+1} d_i \geq 2^{r+1} - 1$, which implies

$$d_{r+1} \geq (2^{r+1} - 1) - (2^r - 1) = 2^r.$$

Therefore, for $r + 1 \leq j \leq n$ we have

$$a_n \geq a_j \geq d_j \geq d_{r+1} \geq 2^r. \tag{9.1}$$

If $a_n > 2^r$, then since $\sigma_n(Q_r)$ has only 2^r vertices, there must necessarily be a vertex ξ of $\mathcal{F}_n^{n+}$ such that $\psi_T(\xi) = n$ and $\xi \setminus \{n\} \not\subseteq [r]$. Otherwise, by (9.1) we have

$$a_n = a_j = d_j = 2^r$$

for all $r + 1 \leq j \leq n - 1$. In particular, all vertices in direction $n - 1$ must lie in $\mathcal{F}_n^{n-}$, so direction n must be chosen at the vertex $\xi = \{n-1, n\}$. But now observe that $\xi \not\subseteq R$, because $R = [r]$ and $r \leq n - 2$. □

9.2.2 Increasing the size of the smallest reducing set of reducible splitting signatures

The aim of this section is to show that under the inductive hypothesis we can increase the size of the smallest reducing set of a reducible splitting signature by edge slides. This major step shows we can move from a reducible splitting signature to an irreducible splitting signature, which is the main theorem of this chapter. We include flowcharts to assist understanding the proofs of the lemmas in this section.

The first lemma shows that under certain conditions an upright spanning tree of Q_n with an irreducible signature where the tree has a reducible splitting signature in $\mathcal{F}_n^{n-}$ can be transformed into an upright spanning tree where the size of the smallest reducing set is increased. To prove this lemma we use the inductive hypothesis that the edge slide graph of an irreducible signature of Q_k is connected for all $k < n$.

Lemma 9.2.2. *Suppose that every irreducible signature of Q_k is connected for all $k < n$. Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n with $a_n \geq a_t$ for all t and let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be a reducible splitting signature of $\mathcal{I}$ with respect to n. Let S be a reducing set*

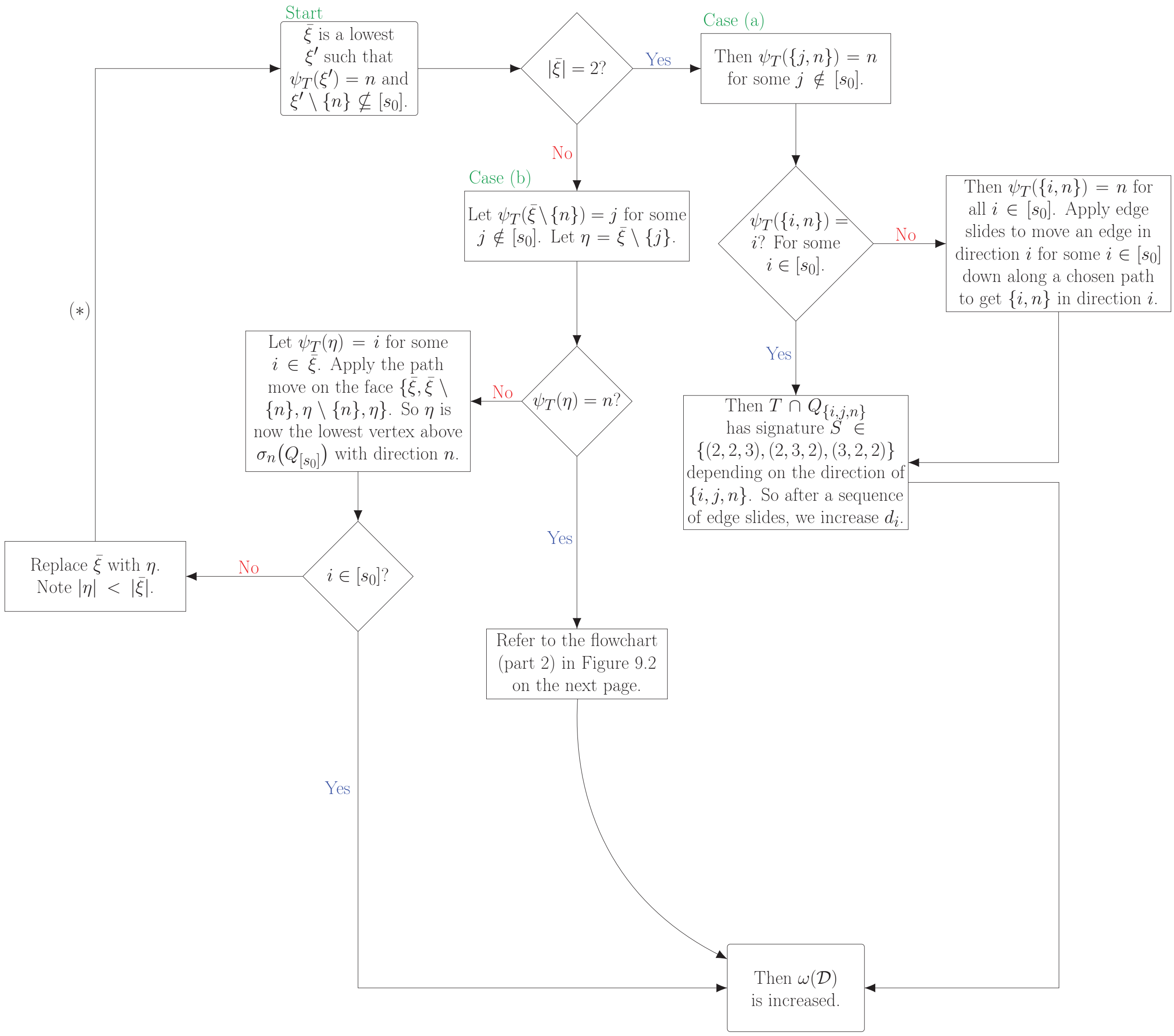


Figure 9.1: Flowchart (part 1) for the proof of Lemma 9.2.2. Note that we can travel around the loop containing edge $(*)$ only finitely many times, because the size of $\bar{\xi}$ decreases each time.

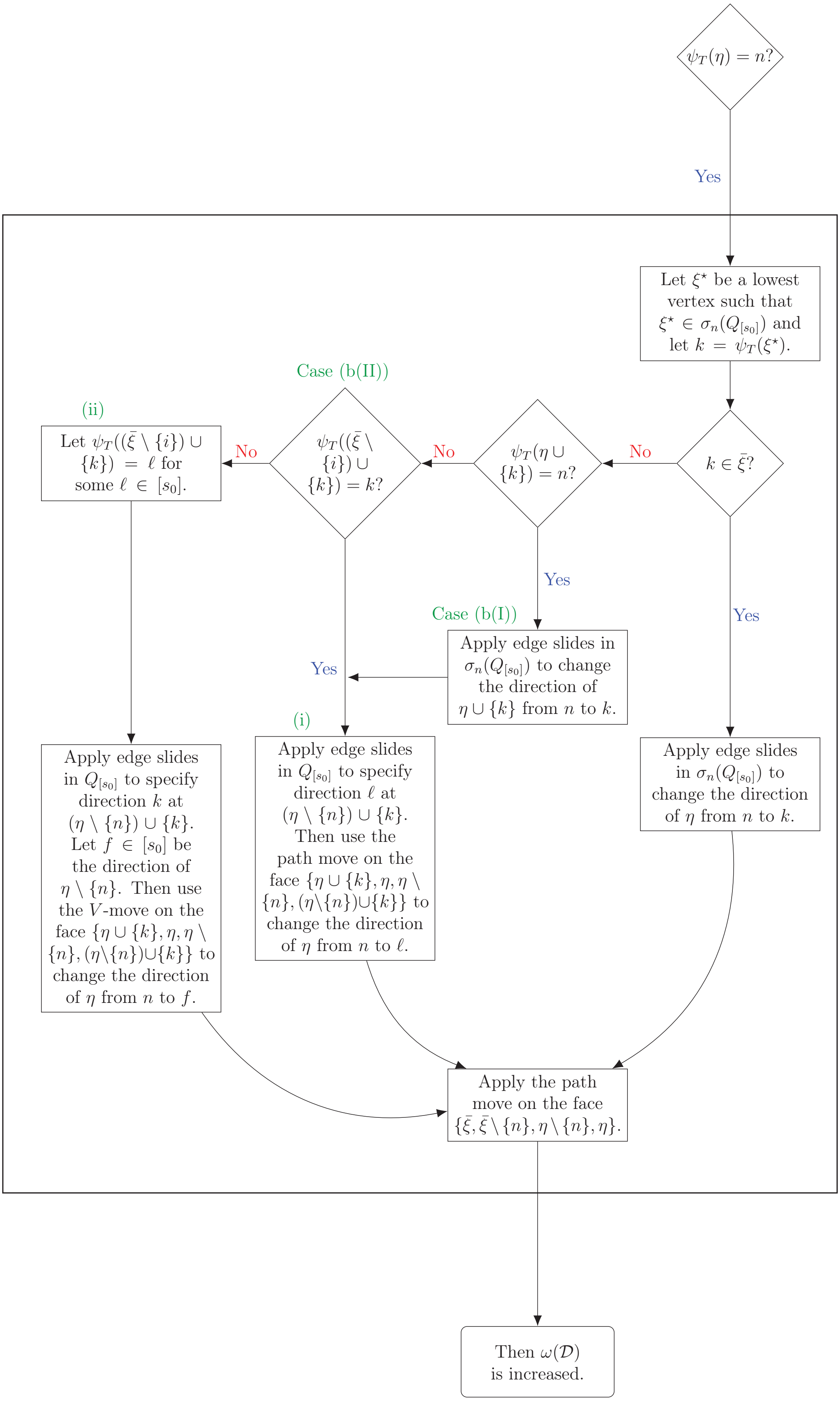


Figure 9.2: Flowchart (part 2) for the proof of Lemma 9.2.2.

such that $|S| = s_0 = \omega(\mathcal{D})$. *Let* T *be an upright spanning tree of* Q_n *with signature* $\mathcal{I}$ *such that* T_{n-} *has signature* $\mathcal{D}$. *Suppose there exists a vertex* ξ *of* $\mathcal{F}_n^{n+}$ *such that* $\xi \setminus \{n\} \subseteq S$ *and* $\psi_T(\xi) \in S$. *Suppose that* $d_t \geq 3$ *for all* $t \notin S$. *Then there exists a sequence of edge slides that transforms* T *into* $\hat{T}$ *such that* $\hat{T}_{n-}$ *has a splitting signature* $\hat{\mathcal{D}} = (\hat{d}_1, \ldots, \hat{d}_{n-1})$ *where* $\omega(\hat{\mathcal{D}}) > \omega(\mathcal{D})$.

The hypothesis $d_t \geq 3$ for all $t \notin S$ rules out the case where $d_1 = 1$ and $d_2 = 2$, where both $[1]$ and $[2]$ are reducing sets. In this case we risk moving to a signature with $\hat{d}_1 = 2$ and $\hat{d}_2 = 1$, swapping the problem from direction 1 to direction 2. This case is handled by Lemma 9.2.3 below.

Proof. Without loss of generality we may assume $\mathcal{D}$ is ordered. Then $S = [s_0]$, and s_0 is the smallest index s such that $\sum_{t=1}^{s} d_t = 2^s - 1$. Our definition of s_0 implies that $(d_1, \ldots, d_{s_0})$ is an irreducible signature of Q_{s_0}. It is therefore a connected signature, by the inductive hypothesis. Therefore by Lemma 7.2.13 after a series of edge slides in $Q_{[s_0]\cup\{n\}}$ we may assume the tree $T \cap Q_{[s_0]\cup\{n\}}$ is settled, and all the labels of T apart from the labels of $T \cap \sigma_n(Q_{s_0})$ are unchanged. In other words, all vertices of $\sigma_n(Q_{s_0})$ immediately below an n in $\sigma_n(Q_{s_0})$ are in direction n, and the labels of T apart from the labels of $T \cap \sigma_n(Q_{s_0})$ are unchanged. In the process of settling the tree, the labels of $T \cap \sigma_n(Q_{s_0})$ may move around, but they still lie above Q_{s_0}. Therefore the edge slides used to settle the tree do not affect the existence of the vertex ξ given in the hypothesis of the lemma.

We show that we can increase d_i by one for some $1 \leq i \leq s_0$, while leaving d_j unchanged for all other directions $j \in [s_0]$ and decreasing d_j by at most one for each $j > s_0$. This increases the size of the smallest reducing set, as we now explain. Since $\sum_{t=1}^{s_0} d_t = 2^{s_0} - 1$, we have $d_t \leq 2^{s_0} - 1$ for $t \leq s_0$ (with equality possible only if $s_0 = 1$), and we must have $d_j \geq d_{s_0+1} \geq 2^{s_0}$ for all $j > s_0$, in order for $\mathcal{D}$ to be a signature. In addition, in the case $s_0 = 1$ we have $d_1 = 1$ and $d_j \geq 3$ for all $j \geq 2$, by hypothesis. Therefore we still have $\hat{d}_k \leq \hat{d}_j$ whenever $k \leq s_0 < j$. If $\hat{\mathcal{D}}$ is not ordered, then

$$\hat{d}_{k_1} \leq \hat{d}_{k_2} \leq \cdots \leq \hat{d}_{k_{n-1}}$$

where k_1 to k_{s_0} are 1 to s_0 in some order, and k_{s_0+1} to k_{n-1} are $s_0 + 1$ to $n - 1$ in some order. So for $s < s_0$ we have

$$\sum_{t=1}^{s} \hat{d}_{k_t} \geq \sum_{t=1}^{s} d_t \geq 2^s$$

and

$$\sum_{t=1}^{s_0} \hat{d}_{k_t} \geq \sum_{t=1}^{s_0} d_t + 1 = 2^{s_0},$$

so $\hat{\mathcal{D}}$ has no reducing set of size $s \leq s_0$.

Since $\mathcal{D}$ is a reducible splitting signature of Q_{n-1} and $[s_0]$ is a corresponding reducing set, by Lemma 9.2.1 there exists a vertex ξ' of $\mathcal{F}_n^{n+}$ with $\psi_T(\xi') = n$ where $\xi' \setminus \{n\} \nsubseteq [s_0]$. Let $\bar{\xi}$ be a lowest such ξ'. We distinguish the following cases according to whether or not $|\bar{\xi}| = 2$.

(a) Suppose that $|\bar{\xi}| = 2$. Then $\bar{\xi} = \{j, n\}$ for some $j \notin [s_0]$. **Suppose first that** $\psi_T(\{i, n\}) = n$ for all $i \in [s_0]$. Let $\xi^\star$ be a lowest vertex of $\mathcal{F}_n^{n+}$ such that $\xi^\star \setminus \{n\} \subseteq [s_0]$ and $\psi_T(\xi^\star) \neq n$. Note that $\xi^\star$ exists by the hypothesis that there exists ξ such that $\xi \setminus \{n\} \subseteq [s_0]$ and $\psi_T(\xi) \in [s_0]$. Let $i = \psi_T(\xi^\star)$ and let $P_{\mathcal{F}_n^{n+}}(\xi^\star, i) = (\xi_0, \ldots, \xi_\alpha)$ be the i-retaining max removing path from $\xi_0 = \xi^\star$ to $\xi_\alpha = \{i, n\}$. Since $\xi^\star$ is a lowest vertex of $\mathcal{F}_n^{n+}$ such

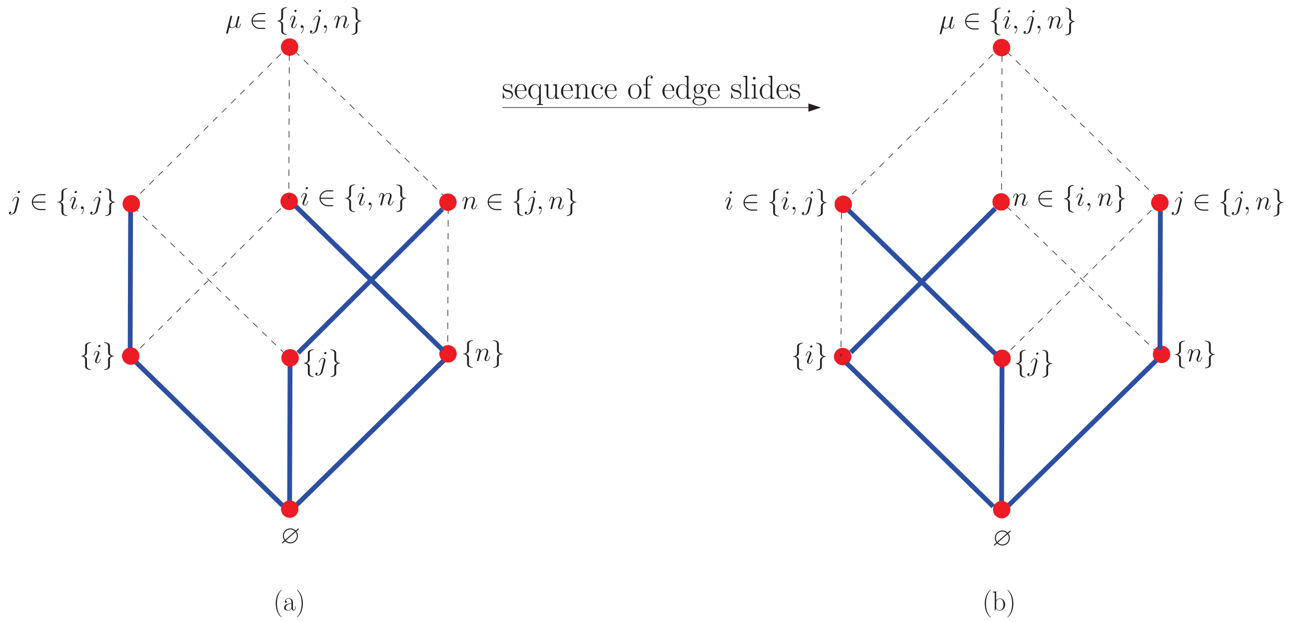


Figure 9.3: Diagram for Case (a) for the proof of Lemma 9.2.2. Upright spanning trees of $Q_{\{i,j,n\}}$ with signature $S \in \{(2,2,3),(2,3,2),(3,2,2)\}$, depending on the direction μ of the vertex $\{i,j,n\}$. (a) Direction i is chosen at $\{i,n\}$, direction j is chosen at $\{i,j\}$ and direction n is chosen at $\{j,n\}$. (b) After a sequence of edge slides direction i moves to $\{i,j\}$, direction j moves to $\{j,n\}$ and direction n moves to $\{i,n\}$.

that $\xi^\star \setminus \{n\} \subseteq [s_0]$ and $\psi_T(\xi^\star) \neq n$, we have $\psi_T(\xi_a) = n$ for $a = 1,\ldots,\alpha$. Then all the conditions in Lemma 7.2.5 are satisfied at the vertex $\xi^\star$, and therefore T can be transformed into T_α with $\psi_{T_\alpha}(\{i,n\}) = i$, $\psi_{T_\alpha}(\xi^\star) = n$ and all other labels of T_α the same as the labels of T. Since $\psi_{T_\alpha}(\{i,n\}) = i$ and all other labels of T_α apart from $\psi_{T_\alpha}(\xi^\star)$ are the same as the labels of T, we can move to the following case.

Suppose now that $\psi_T(\{i,n\}) = i$ for some $i \in [s_0]$, and refer to Figure 9.3(a). Since $j \notin [s_0]$, we have $\psi_T(\{i,j\}) = j$ by Lemma 6.2.1. Let S be the signature of $T \cap Q_{\{i,j,n\}}$. Then S is equal to one of (2, 2, 3), (2, 3, 2) and (3, 2, 2), depending on the direction μ of the vertex $\{i,j,n\}$. There exists a tree of $Q_{\{i,j,n\}}$ with signature S where direction i is chosen at $\{i,j\}$, and since the edge slide graph of signature S is connected (by Henden [6] or Lemma 4.3.6), we can move from T to such a tree using edge slides in $Q_{\{i,j,n\}}$ as shown in Figure 9.3(b). Let $\hat{T}$ be the resulting tree in Q_n and let $\hat{\mathcal{D}} = (\hat{d}_1,\ldots,\hat{d}_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$\hat{d}_i = d_i + 1,\ \hat{d}_j = d_j - 1 \text{ and } \hat{d}_k = d_k$$

for all $k \neq i,j$. Since $i \in [s_0]$, we are done because d_i has been increased where $i \in [s_0]$ and d_j has been decreased where $j \notin [s_0]$.

(b) Suppose that $|\bar{\xi}| > 2$. Since $\bar{\xi} \setminus \{n\} \nsubseteq [s_0]$, by Lemma 6.2.1 we have $\psi_T(\bar{\xi} \setminus \{n\}) \notin [s_0]$. Let $\psi_T(\bar{\xi} \setminus \{n\}) = j$ for some $j \notin [s_0]$. Since $j \notin [s_0]$, we have $j > s_0$. Let $\eta = \bar{\xi} \setminus \{j\}$.

Suppose first that $\psi_T(\eta) \neq n$. Let $\psi_T(\eta) = i$ for some $i \in \eta$. As shown in Figure 9.4, using the path move the i-edge moves to $\bar{\xi} \setminus \{n\}$, the j-edge moves to $\bar{\xi}$ and the n-edge moves to η. Let T' be the resulting tree and let $\mathcal{D}' = (d'_1,\ldots,d'_{n-1})$ be its signature in

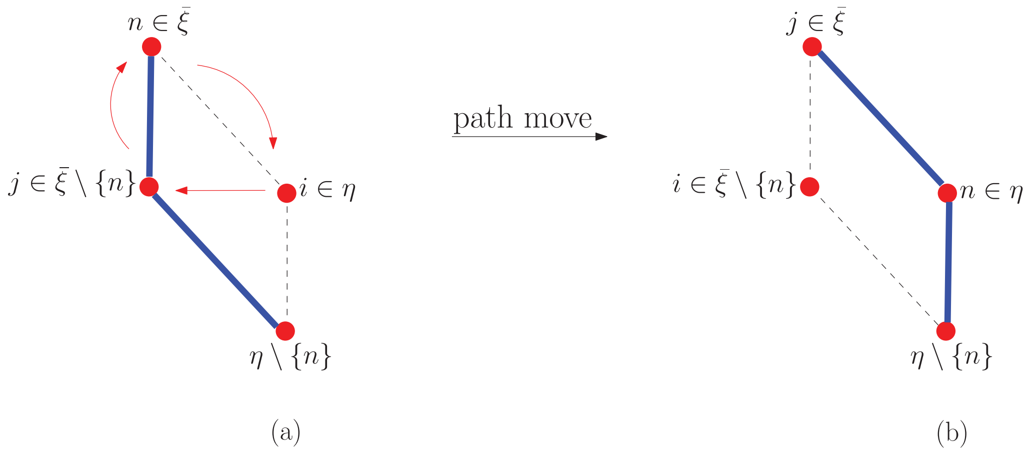


Figure 9.4: Diagram 1 for Case (b) for the proof of Lemma 9.2.2. (a) Direction n is chosen at $\bar{\xi}$, direction j is chosen at $\bar{\xi} \setminus \{n\}$ and direction i is chosen at η. (b) After applying the path move direction i moves to $\bar{\xi} \setminus \{n\}$, direction j moves to $\bar{\xi}$ and direction n moves to η.

$\mathcal{F}_n^{n-}$. Then

$$d_i' = d_i + 1,\ d_j' = d_j - 1,\ \text{and}\ d_k' = d_k$$

for all $k \neq i, j$. If $i \in [s_0]$, then we are done because d_i has been increased and d_j has been decreased where $j > s_0$. Otherwise $i \notin [s_0]$ and therefore $i > s_0$ and $\sum_{t \in S'} \hat{d}_t = \sum_{t \in S'} d_t$ for all $S' \subseteq [s_0]$. So $|\bar{\xi}| > |\eta| \geq 2$ and η is now the lowest vertex of $\mathcal{F}_n^{n+}$ such that $\psi_{T'}(\eta) = n$ and $\eta \setminus \{n\} \nsubseteq [s_0]$, because $i \in \eta$, but $i \notin [s_0]$. If $|\eta| = 2$, then we move to Case (a) above. Otherwise $|\eta| > 2$. Then we repeat the above process using the path move to move the n-edge down until we reach a tree T'' where the n-edge is chosen at a vertex $\tilde{\xi} \subseteq \eta \setminus \{i\}$ such that $\psi_{T''}(\tilde{\xi} \setminus \psi_{T''}(\tilde{\xi} \setminus \{n\})) \in [s_0] \cup \{n\}$. Let $k = \psi_{T''}(\tilde{\xi} \setminus \{n\})$. Then we either have $|\tilde{\xi}| = 2$, or $\psi_{T''}(\tilde{\xi} \setminus \{k\}) \in [s_0]$, or $\psi_{T''}(\tilde{\xi} \setminus \{k\}) = n$. If $|\tilde{\xi}| = 2$, then we move to Case (a) above with $\bar{\xi}$ replaced with $\tilde{\xi}$. If $\psi_{T''}(\tilde{\xi} \setminus \{k\}) = \ell \in [s_0]$, then at the next step d_ℓ is increased as shown above, increasing $\omega(\mathcal{D})$. If $\psi_{T''}(\tilde{\xi} \setminus \{k\}) = n$, then we move to the case below with $\bar{\xi}$ replaced by $\tilde{\xi}$. Note that the direction $j > s_0$ such that d_j is decreased is different at each step because the n moves from a vertex containing j to one that does not. In what follows the direction j that is decreased will belong to $\tilde{\xi}$, and so no d_j will be decreased twice.

Suppose now that $\psi_T(\eta) = n$. Since $|\bar{\xi}| > 2$ and $|\bar{\xi} \setminus \{j\} = \eta| \geq 2$ we have $|\eta \setminus \{n\}| \geq 1$, and $\eta \setminus \{n\} \subseteq [s_0]$ by the choice of $\bar{\xi}$. Let $\xi^\star$ be a lowest vertex of $\sigma_n(Q_{[s_0]})$ with $\psi_T(\xi^\star) \neq n$. Let $\psi_T(\xi^\star) = k$ for some $k \in [s_0]$.

Suppose first that $k \in \bar{\xi}$. Then $\{k, n\} \subseteq \xi^\star \cap \eta$. Let $P_{\mathcal{F}_n^{n+}}(\xi^\star, k) = (\xi_0, \ldots, \xi_\alpha)$ be the k-retaining max removing path from $\xi_0 = \xi^\star$ to $\xi_\alpha = \{k, n\}$. Since $\xi^\star$ is a lowest vertex of $\sigma_n(Q_{[s_0]})$ with $\psi_T(\xi^\star) \neq n$, we have $\psi_T(\xi_a) = n$ for $a = 1, \ldots, \alpha$. Let $P_{\mathcal{F}_n^{n+}}(\eta, k) = (\eta_0, \ldots, \eta_{\alpha'})$ be the k-retaining max removing path from $\eta_0 = \eta$ to $\eta_{\alpha'} = \{k, n\}$. Since $\psi_T(\eta) = n$ and $T \cap \sigma_n(Q_{[s_0]})$ is settled, we have $\psi_T(\eta_a) = n$ for all $a = 0, \ldots, \alpha'$. Then all the conditions in Corollary 7.2.6 are satisfied at the vertices $\xi^\star$ and η. Therefore T can be transformed into T' with $\psi_{T'}(\eta) = k$, $\psi_{T'}(\xi^\star) = n$ and all other labels of T' the same as the labels of T.

As shown in Figure 9.5, using the path move the k-edge moves to $\bar{\xi} \setminus \{n\}$, the j-edge moves to $\bar{\xi}$ and the n-edge moves to η. Let $\hat{T}$ be the resulting tree and let $\hat{\mathcal{D}} = (\hat{d}_1, \ldots, \hat{d}_{n-1})$

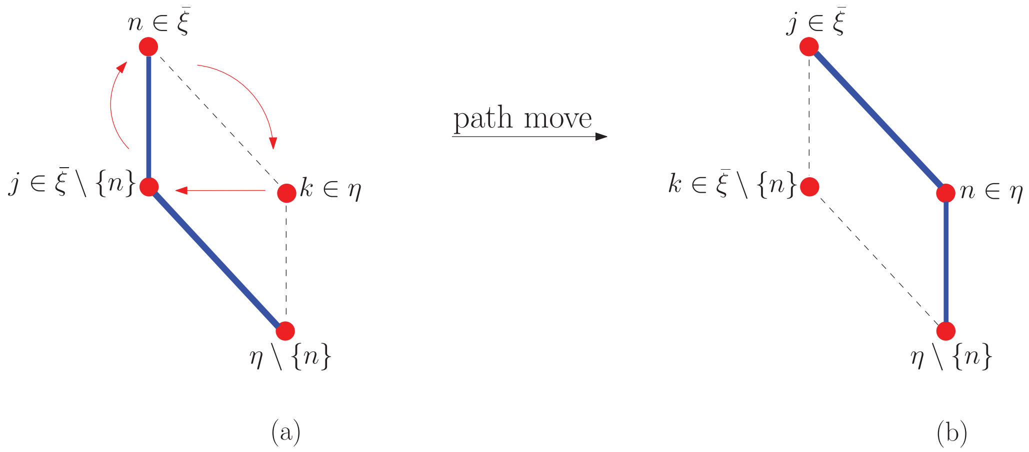


Figure 9.5: Diagram 2 for Case (b) for the proof of Lemma 9.2.2. (a) Direction n is chosen at $\bar{\xi}$, direction j is chosen at $\bar{\xi} \setminus \{n\}$ and direction k is chosen at η. (b) After applying the path move direction k moves to $\bar{\xi} \setminus \{n\}$, direction j moves to $\bar{\xi}$ and direction n moves to η.

be its signature in $\mathcal{F}_n^{n-}$. Then

$$\hat{d}_k = d_k + 1,\ \hat{d}_j = d_j - 1 \text{ and } \hat{d}_\ell = d_\ell,$$

for all $\ell \neq j, k$. Therefore we are done because d_k has been increased where $k \in [s_0]$ and d_j has been decreased where $j > s_0$.

Suppose now that $k \notin \bar{\xi}$. We distinguish the following cases according to whether or not $\psi_T(\eta \cup \{k\}) = n$.

(I) Suppose that $\psi_T(\eta \cup \{k\}) = n$. Then $\{k, n\} \subseteq \xi^\star \cap (\eta \cup \{k\})$. Let $P_{\mathcal{F}_n^{n+}}(\xi^\star, k) = (\xi_0, \dots, \xi_\alpha)$ be the k-retaining max removing path from $\xi_0 = \xi^\star$ to $\xi_\alpha = \{k, n\}$. Since $\xi^\star$ is a lowest vertex of $\sigma_n(Q_{[s_0]})$ with $\psi_T(\xi^\star) \neq n$, we have $\psi_T(\xi_a) = n$ for $a = 1, \dots, \alpha$. Let $P_{\mathcal{F}_n^{n+}}(\eta \cup \{k\}, k) = (\eta_0 \cup \{k\}, \dots, \eta_{\alpha'} \cup \{k\})$ be the k-retaining max removing path from $\eta_0 \cup \{k\} = \eta \cup \{k\}$ to $\eta_{\alpha'} \cup \{k\} = \{k, n\}$. Since $\psi_T(\eta \cup \{k\}) = n$ and $T \cap \sigma_n(Q_{[s_0]})$ is settled, we have $\psi_T(\eta_a \cup \{k\}) = n$ for $a = 0, \dots, \alpha'$. Then all the conditions in Corollary 7.2.6 are satisfied at the vertices $\xi^\star$ and η. Therefore T can be transformed into T' with $\psi_{T'}(\eta \cup \{k\}) = k$, $\psi_{T'}(\xi^\star) = n$ and all other labels of T' the same as the labels of T. Since $\psi_{T'}(\eta \cup \{k\}) = k$ and all other labels apart from $\psi_{T'}(\xi^\star)$ are the same as the labels of T, we can move to the following Case (II(i)).

(II) Suppose that $\psi_T(\eta \cup \{k\}) \neq n$. We distinguish the following cases according to whether or not $\psi_T(\eta \cup \{k\}) = k$.

(i) Suppose that $\psi_T(\eta \cup \{k\}) = k$. Choose $\ell \in \eta \setminus \{n\}$, which exists because $|\eta \setminus \{n\}| \geq 1$. Since $(d_1, \dots, d_{s_0})$ is an irreducible signature and $|\eta \setminus \{n\} \cup \{k\}| \geq 2$ because $|\eta \setminus \{n\}| \geq 1$, by Lemma 3.2.4 there exists a tree $T' \cap Q_{[s_0]}$ with signature $(d_1, \dots, d_{s_0})$ where $\psi_{T'}((\eta \setminus \{n\}) \cup \{k\}) = \ell$. Since the edge slide graph of signature $(d_1, \dots, d_{s_0})$ is connected by the inductive hypothesis, we can move from $T \cap Q_{[s_0]}$ to such a tree using edge slides in $Q_{[s_0]}$. Then T is transformed into T' with $\psi_{T'}((\eta \setminus \{n\}) \cup \{k\}) = \ell$ and all the labels of T' apart from the labels of $T' \cap Q_{[s_0]}$ the same as the labels of T. Note that T'_{n-} has signature $\mathcal{D}$ because edge slides were only applied in $\mathcal{F}_n^{n-}$ and edge slides do not change the signature.

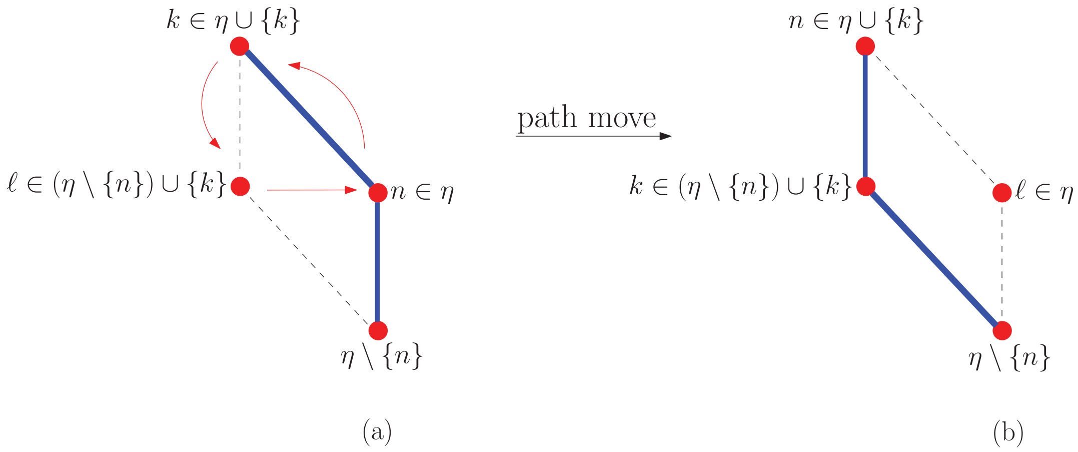


Figure 9.6: Diagram 1 for Case (b(II(i))) for the proof of Lemma 9.2.2. (a) Direction k is chosen at $\eta \cup \{k\}$, direction n is chosen at η and direction ℓ is chosen at $(\eta \setminus \{n\}) \cup \{k\}$. (b) After applying the path move direction ℓ moves to η, direction n moves to $\eta \cup \{k\}$ and direction k moves to $(\eta \setminus \{n\}) \cup \{k\}$.

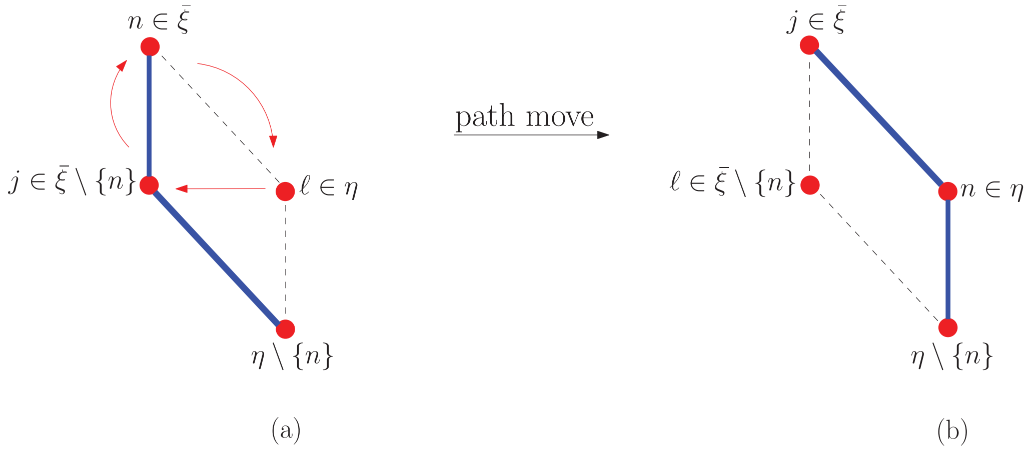


Figure 9.7: Diagram 2 for Case (b(II(i))) for the proof of Lemma 9.2.2. (a) Direction n is chosen at $\bar{\xi}$, direction j is chosen at $\bar{\xi} \setminus \{n\}$ and direction ℓ is chosen at η. (b) After applying the path move direction ℓ moves to $\bar{\xi} \setminus \{n\}$, direction j moves to $\bar{\xi}$ and direction n moves to η.

As shown in Figure 9.6, using the path move the ℓ-edge moves to $\bar{\xi} \setminus \{j\}$, the n-edge moves to $\eta \cup \{k\}$ and the k-edge moves to $(\eta \setminus \{n\}) \cup \{k\}$. Let T'' be the resulting tree and let $\mathcal{D}'' = (d''_1, \ldots, d''_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$d''_k = d_k + 1,\ d''_\ell = d_\ell - 1, \text{ and } d''_r = d_r$$

for all $r \neq k, \ell$.
As shown in Figure 9.7, using the path move the ℓ-edge moves to $\bar{\xi} \setminus \{n\}$, the j-edge moves to $\bar{\xi}$ and the n-edge moves to η. Let $\hat{T}$ be the resulting tree and let $\hat{\mathcal{D}} = (\hat{d}_1, \ldots, \hat{d}_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$\hat{d}_\ell = d''_\ell + 1 = d_\ell,\ \hat{d}_j = d''_j - 1 = d_j - 1, \hat{d}_k = d''_k = d_k + 1, \text{ and } \hat{d}_r = d''_r \geq d_r$$

for all $r \neq j, \ell$. Therefore we are done because d_k has been increased where $k \in [s_0]$ and d_j has been decreased where $j \notin [s_0]$.

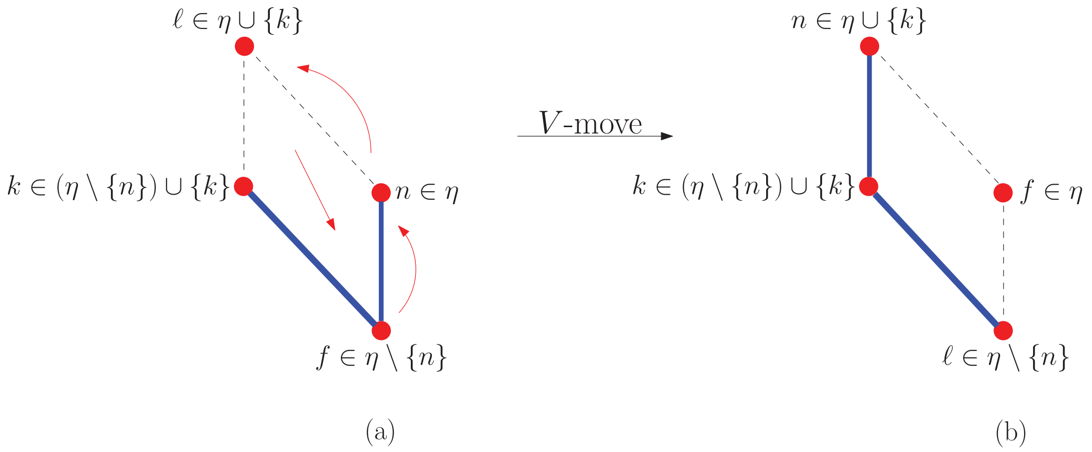


Figure 9.8: Diagram 1 for Case (b(II(ii))) for the proof of Lemma 9.2.2. (a) Direction ℓ is chosen at $\eta \cup \{k\}$, direction n is chosen at η, f is chosen at $\eta \setminus \{n\}$ and direction k is chosen at $(\eta \setminus \{n\}) \cup \{k\}$. (b) After applying the V-move direction ℓ moves to $\eta \setminus \{n\}$, direction f moves to η and direction n moves to $\eta \cup \{k\}$.

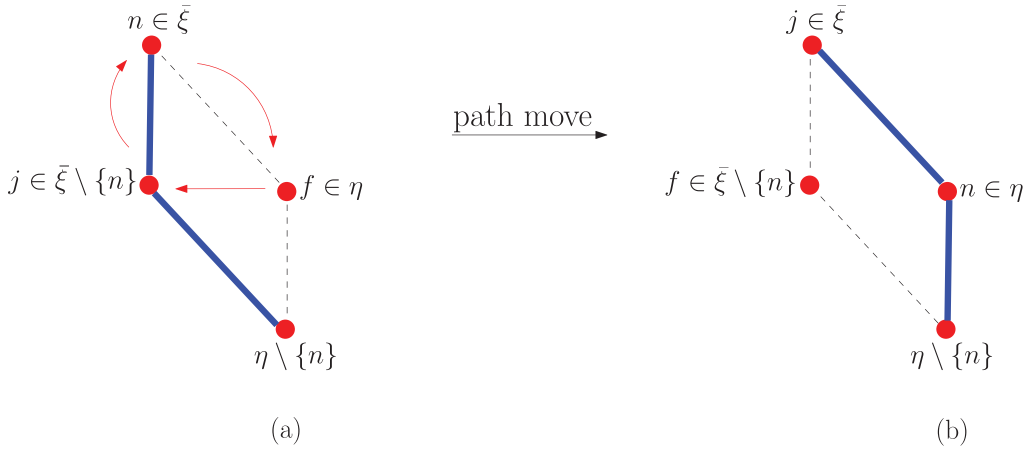


Figure 9.9: Diagram 2 for Case (b(II(ii))) for the proof of Lemma 9.2.2. (a) Direction n is chosen at $\bar{\xi}$, direction j is chosen at $\bar{\xi} \setminus \{n\}$ and direction f is chosen at η. (b) After applying the path move direction f moves to $\bar{\xi} \setminus \{n\}$, direction j moves to $\bar{\xi}$ and direction n moves to η.

(ii) Suppose that $\psi_T(\eta \cup \{k\}) \neq k$. Let $\psi_T(\eta \cup \{k\}) = \ell$ for some $\ell \in [s_0]$. Since $(d_1, \ldots, d_{s_0})$ is an irreducible signature, by Lemma 3.2.4 there exists a tree $T' \cap Q_{[s_0]}$ with signature $(d_1, \ldots, d_{s_0})$ where $\psi_{T'}((\eta \setminus \{n\}) \cup \{k\}) = k$. Since the edge slide graph of signature $(d_1, \ldots, d_{s_0})$ is connected by the inductive hypothesis, we can move from $T \cap Q_{[s_0]}$ to such a tree using edge slides in $Q_{[s_0]}$. Then T can be transformed into T' with $\psi_{T'}((\eta \setminus \{n\}) \cup \{k\}) = k$, and all the labels of T' apart from the labels of $T' \cap Q_{[s_0]}$ the same as the labels of T.
Let $f = \psi_{T'}(\eta \setminus \{n\}) \in [s_0]$. As shown in Figure 9.8, using the V-move the ℓ-edge moves to $\eta \setminus \{n\}$, the f-edge moves to η and the n-edge moves to $\eta \cup \{k\}$. Let T'' be the resulting tree and let $\mathcal{D}'' = (d''_1, \ldots, d''_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$d''_\ell = d_\ell + 1, \; d''_f = d_f - 1, \text{ and } d''_r = d_r,$$

for all $r \neq f, \ell$.

As shown in Figure 9.9, using the path move the f-edge moves to $\bar{\xi} \setminus \{n\}$, the j-edge moves to $\bar{\xi}$ and the n-edge moves to η. Let $\hat{T}$ be the resulting tree and let $\hat{\mathcal{D}} = (\hat{d}_1, \ldots, \hat{d}_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$\hat{d}_f = d''_f + 1 = d_f,\ \hat{d}_j = d''_j - 1 = d_j - 1, \hat{d}_\ell = d''_\ell = d_\ell + 1 \text{ and } \hat{d}_r = d''_r \geq d_r$$

for all $r \neq j, f$. Therefore we are done because d_ℓ has been increased where $\ell \in [s_0]$ and d_j has been decreased where $j \notin [s_0]$.

In all cases we showed that we can increase d_h by one for some $h \in [s_0]$ while decreasing d_j by at most one for each $j \notin [s_0]$, and therefore the size of the smallest reducing set is increased. □

The following lemma is used to show that an upright spanning tree of Q_n with an irreducible signature that has a reducible splitting signature in $\mathcal{F}_n^{n-}$ with the reducing set [2] can be transformed into an upright spanning tree where the size of the reducing set is increased. The difference between Lemma 9.2.2 and Lemma 9.2.3 is that the reducing set [2] is not the smallest reducing set.

Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be an ordered reducible splitting signature of Q_{n-1} with the reducing set [2]. Let T be a spanning tree of Q_n with $\mathcal{D}$ in $\mathcal{F}_n^{n-}$ and suppose that a lowest 1-edge in $\mathcal{F}_n^{n+}$ is chosen at either $\{1, 2, n\}$ or $\{1, n\}$. Then the signature of $T \cap Q_{\{1,2,n\}}$ is $S \in \{(2, 4, 1), (3, 3, 1), (2, 2, 3), (2, 3, 2), (3, 2, 2)\}$. We introduce the following lemma to show that we can increase d_1 without decreasing d_2.

Lemma 9.2.3. *Let $\mathcal{I} = (a_1, \ldots, a_n)$ be an irreducible signature of Q_n with $a_n \geq a_t$ for all t and let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ be an ordered reducible splitting signature of $\mathcal{I}$ with respect to n. Let [2] be a reducing set. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Suppose that either $\psi_T(\{1, 2, n\}) = 1$ or $\psi_T(\{1, n\}) = 1$. Then there exists a sequence of edge slides that transforms T into $\hat{T}$ such that $\hat{T}_{n-}$ has signature $\hat{\mathcal{D}} = (\hat{d}_1, \ldots, \hat{d}_{n-1})$ where $\hat{d}_t \geq 2$ for all t.*

Proof. Since [2] is a reducing set, we have $d_1 = 1$ and $d_2 = 2$. So in order for $\mathcal{D}$ to be a signature we must have $d_i \geq 4$ for all $i \geq 3$. The unique 1-edge in T_{n-} is at $\{1\}$, and the two 2-edges in T_{n-} are chosen at $\{1, 2\}$ and $\{2\}$. Let $S = (s_1, s_2, s_3)$ be the signature of $T \cap Q_{\{1,2,n\}}$. Then $2 \leq s_1 \leq 3$ and $s_2 \geq 2$ and therefore $S \in \{(2, 4, 1), (3, 3, 1), (2, 2, 3), (2, 3, 2), (3, 2, 2)\}$.

Since $\mathcal{I}$ is an irreducible signature and $\mathcal{D}$ is a reducible signature with a reducing set [2], by Lemma 9.2.1 there exists a vertex $\hat{\xi}$ of $\mathcal{F}_n^{n+}$ with $\psi_T(\hat{\xi}) = n$ and $\hat{\xi} \setminus \{n\} \nsubseteq \{1, 2\}$. Let ξ be a lowest such $\hat{\xi}$. We distinguish the following cases according to whether or not $|\xi| = 2$.

(a) Suppose that $|\xi| = 2$. Then $\xi = \{i, n\}$ for some $i \notin \{1, 2\}$. **Suppose first that** $\psi_T(\{1, n\}) \neq 1$. Then $\psi_T(\{1, n\}) = n$, and therefore $\psi_T(\{1, 2, n\}) = 1$ by our assumption that $\{1, 2, n\}$ or $\{1, n\}$ is in direction 1. So $S \in \{(2, 2, 3), (2, 3, 2)\}$. As shown in Figure 9.12, using the V-move the 1-edge and the n-edge are swapped. Then T is transformed into T' where $\psi_{T'}(\{1, 2, n\}) = n$, $\psi_{T'}(\{1, n\}) = 1$ and all other labels of T' are the same as the labels of T and therefore we move to the following case.

Suppose now that $\psi_T(\{1, n\}) = 1$. Let S' be the signature of $T \cap Q_{\{1,i,n\}}$. Since the only 1-edge in T_{n-} is chosen at $\{1\}$, we have $\psi_T(\{1, i\}) = i$. So $S' \in \{(2, 2, 3), (2, 3, 2), (3, 2, 2)\}$, depending on $\mu = \psi_T(\{1, i, n\})$. As shown in Figure 9.13, there exists a tree of $Q_{\{1,i,n\}}$ with signature S' where direction 1 is chosen at $\{1, i\}$ and since the edge slide graph of

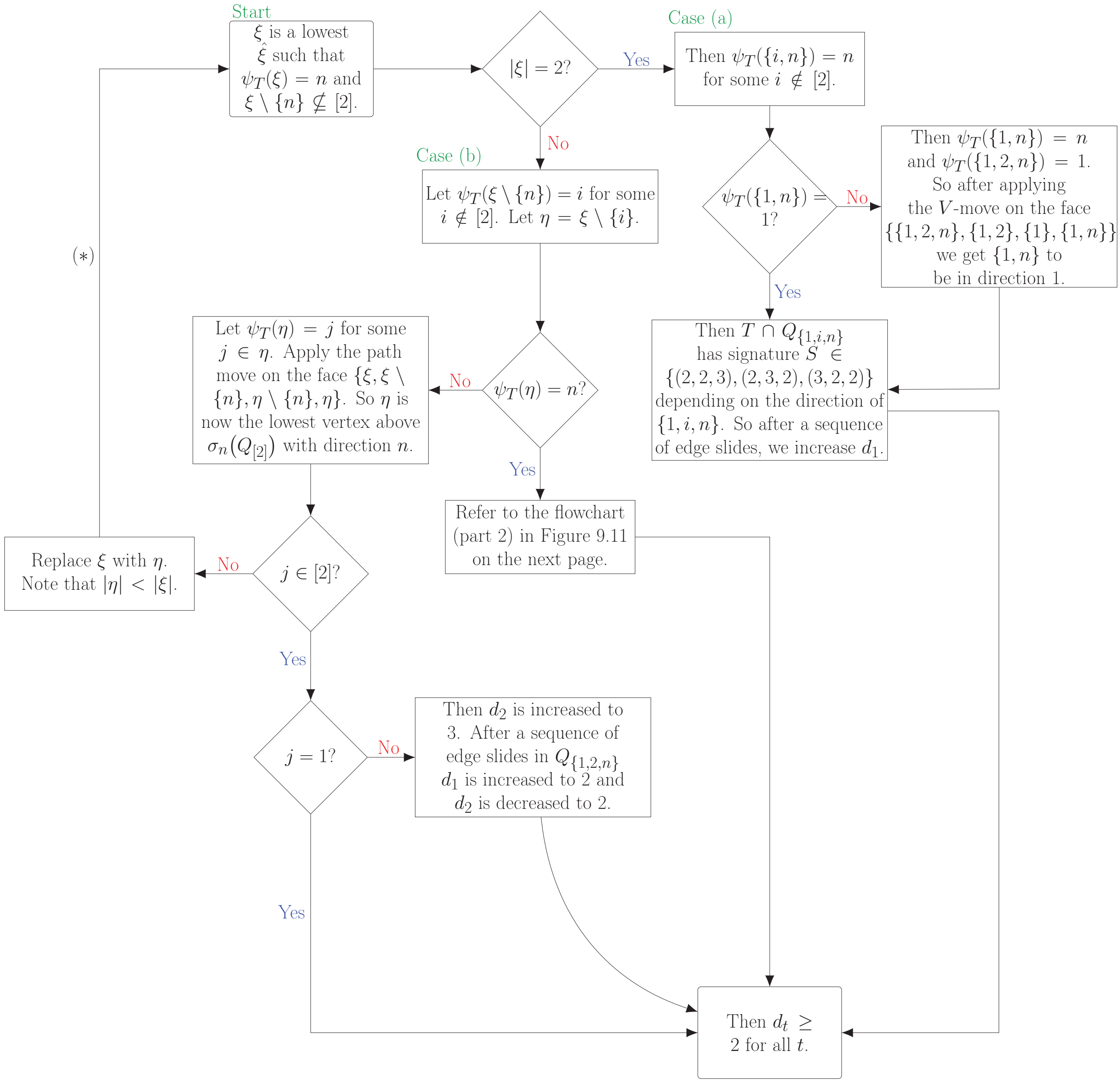


Figure 9.10: Flowchart (part 1) for the proof of Lemma 9.2.3. Note that we can travel around the loop containing edge $(*)$ only finitely many times, because the size of ξ decreases each time. This flowchart is continued on the next page.

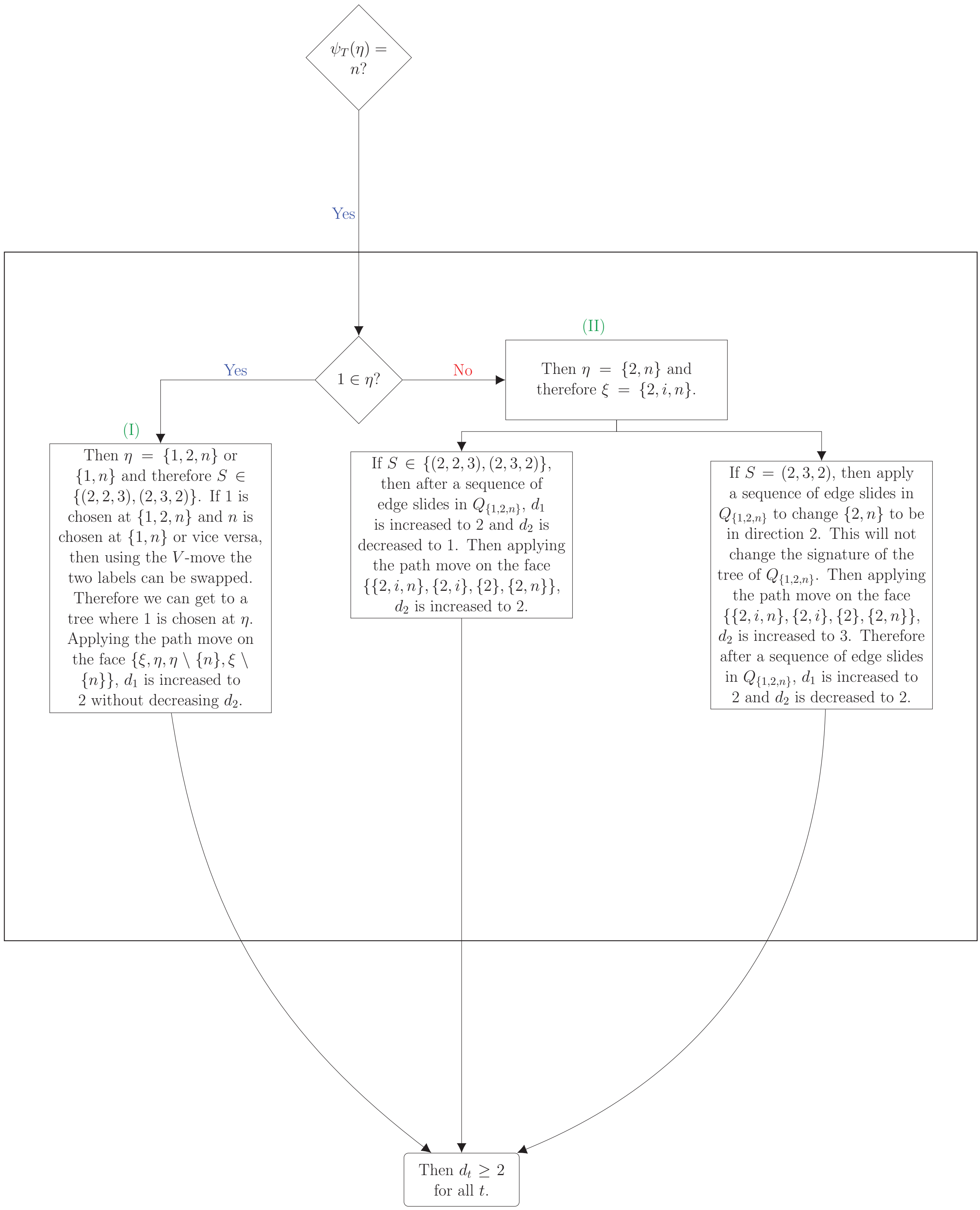


Figure 9.11: Flowchart (part 2) for the proof of Lemma 9.2.3.

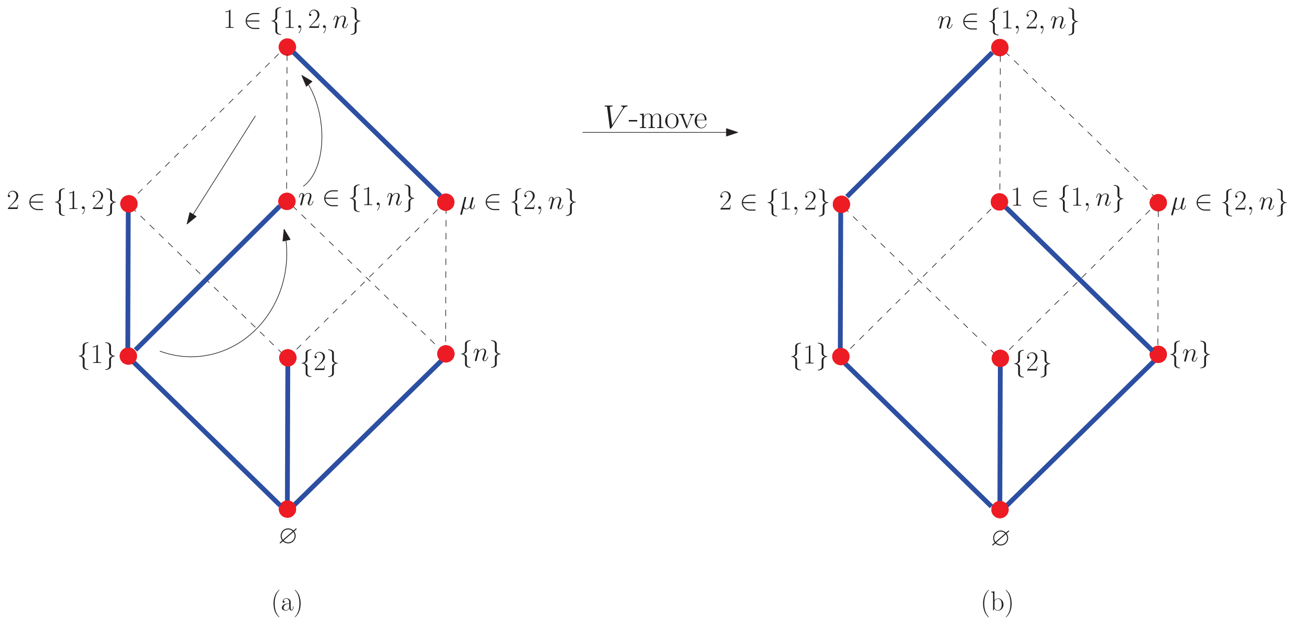


Figure 9.12: Diagram 1 for Case (a) for the proof of Lemma 9.2.3. Upright spanning trees of $Q_{\{1,2,n\}}$ with signature $S \in \{(2,2,3),(2,3,2)\}$, depending on the direction μ of the vertex $\{2,n\}$. (a) Direction n is chosen at $\{1,n\}$, direction 2 is chosen at $\{1,2\}$ and direction 1 is chosen at $\{1,2,n\}$. (b) After applying the V-move direction 1 moves to $\{1,n\}$ and direction n moves to $\{1,2,n\}$.

signature S' is connected (by Henden [6] or Lemma 4.3.6) we can move from T to such a tree using edge slides in $Q_{\{1,i,n\}}$. Let $\hat{T}$ be the resulting tree and let $\hat{D} = (\hat{d}_1, \ldots, \hat{d}_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$\hat{d}_1 = d_1 + 1 = 2,\ \hat{d}_i = d_i - 1 \geq 3 \text{ and } \hat{d}_j = d_j,$$

for all $j \neq 1, i$. So d_1 has been increased without decreasing d_2 and therefore $\hat{d}_t \geq 2$ for all t.

(b) Suppose that $|\xi| > 2$. Since $\xi \setminus \{n\} \not\subseteq [2]$, by Lemma 6.2.1 we have $\psi_T(\xi \setminus \{n\}) = i$ for some $i \notin [2]$. **Suppose first that** $\psi_T(\xi \setminus \{i\}) \neq n$. Let $\psi_T(\xi \setminus \{i\}) = j$ for some $j \neq n$. Then as shown in Figure 9.14, using the path move the j-edge moves to $\xi \setminus \{n\}$, the i-edge moves to ξ and the n-edge moves to $\xi \setminus \{i\}$. Let T' be the resulting tree and let $\mathcal{D}' = (d'_1, \ldots, d'_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$d'_i = d_i - 1 \geq 3,\ d'_j = d_j + 1 \text{ and } d'_k = d_k,$$

for $k \neq i, j$. In other words, T is transformed into T' with $\psi_{T'}(\xi) = i$, $\psi_{T'}(\xi \setminus \{n\}) = 2$, $\psi_{T'}(\xi \setminus \{i\}) = n$ and all other labels of T' the same as the labels of T.

If $j = 1$, then d_1 has increased without decreasing d_2 because $i \notin [2]$, and therefore $d'_t \geq 2$ for all t.

If $j = 2$, then $d'_2 = d_2 + 1 = 3$ and $d'_1 = d_1 = 1$ because $i \notin [2]$. Since either $\psi_T(\{1,n\}) = 1$ or $\psi_T(\{1,2,n\}) = 1$ by our assumption and since all the labels of T' apart from $\psi_{T'}(\xi)$, $\psi_{T'}(\xi \setminus \{n\})$, $\psi_{T'}(\xi \setminus \{i\})$ are the same as the labels of T, we have either $\psi_{T'}(\{1,n\}) = 1$ or $\psi_{T'}(\{1,2,n\}) = 1$. If $\psi_{T'}(\{1,2,n\}) = 1$ and $\psi_{T'}(\{1,n\}) = n$,

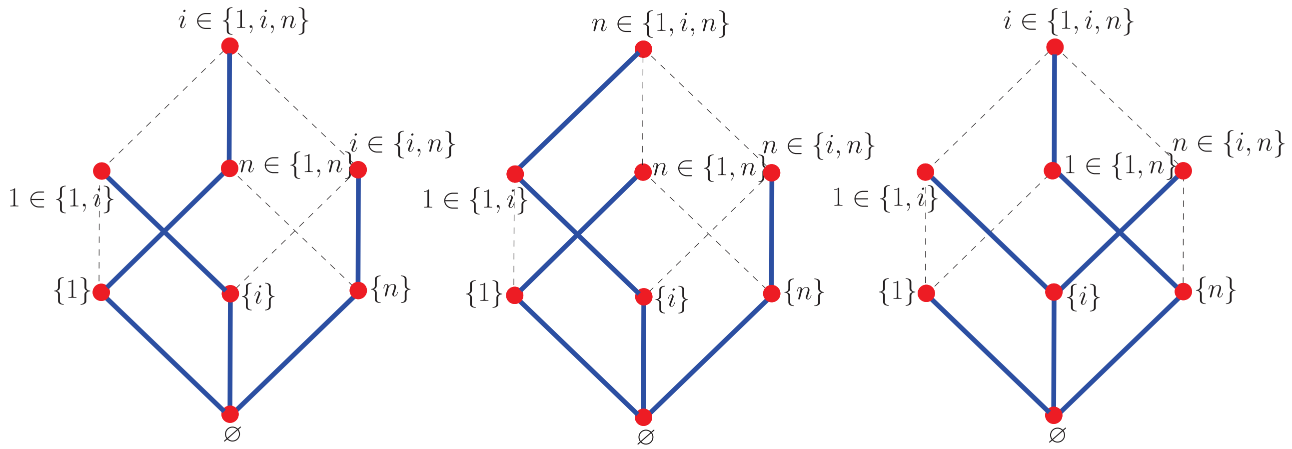


Figure 9.13: Diagram 2 for Case (b) for the proof of Lemma 9.2.3. The possible trees of $Q_{\{1,i,n\}}$ with signature $S \in \{(2,3,2),(2,2,3),(3,2,2)\}$ where 1 is chosen at $\{1,i\}$.

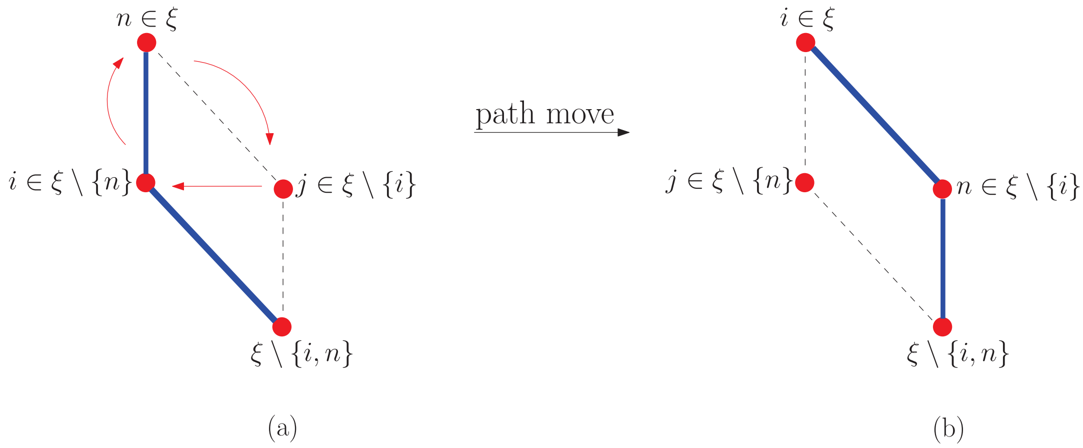


Figure 9.14: Diagram 1 for Case (b) for the proof of Lemma 9.2.3. (a) Direction n is chosen at ξ, direction i is chosen at $\xi \setminus \{n\}$ and direction j is chosen at $\xi \setminus \{i\}$. (b) After applying the path move direction j moves to $\xi \setminus \{n\}$, direction i moves to ξ and direction n moves to $\xi \setminus \{i\}$.

then as shown in Figure 9.15 using the V-move the labels 1 and n are swapped. Therefore T' is transformed into T'' with $\psi_{T''}(\{1,2,n\}) = n$, $\psi_{T''}(\{1,n\}) = 1$ and all other labels of T'' the same as the labels of T'. Since the V-move was applied in $\mathcal{F}_n^{n+}$, the signature in $\mathcal{F}_n^{n-}$ is unchanged so T''_{n-} has signature $\mathcal{D}'$. Then we can apply Lemma 9.2.2 to increase d'_1, and therefore T'' can be transformed into $\hat{T}$ where $\hat{d}_t \geq 2$ for all t.

Otherwise $j \notin [2]$, and therefore $d'_1 = d_1$ and $d'_2 = d_2$. Let $\eta = \xi \setminus \{i\}$. So η is now the lowest vertex of $\mathcal{F}_n^{n+}$ such that $\psi_{T'}(\eta) = n$ and $\eta \setminus \{n\} \not\subseteq [2]$, because $j \in \eta$ but $j \notin [s_0]$. If $|\eta| = 2$, then $\eta = \{j,n\}$ and we move to Case (a) above and increase d_1 by decreasing d_j. Otherwise $|\eta| > 2$ and $\eta \setminus \{j,n\} \not\subseteq [2]$, and we repeatedly apply the path move as above until eventually we reach a tree T'' where the n-edge is chosen at a vertex $\hat{\xi} \subseteq \eta \setminus \{j\}$ such that $\psi_{T''}(\hat{\xi} \setminus \psi_{T''}(\hat{\xi} \setminus \{n\})) \in [2] \cup \{n\}$. Let $k = \psi_{T''}(\hat{\xi} \setminus \{n\})$. Then we either have $|\hat{\xi}| = 2$, or $\psi_{T''}(\hat{\xi} \setminus \{k\}) \in [2]$, or $\psi_{T''}(\hat{\xi} \setminus \{k\}) = n$. If $|\hat{\xi}| = 2$, then we move to Case (a) above. If $\psi_{T''}(\hat{\xi} \setminus \{k\}) \in [2]$, then at the next step either d_1 or d_2 is increased as shown above. Otherwise $\psi_{T''}(\hat{\xi} \setminus \{k\}) = n$, and we move to the case below. Note that the direction i such that d_i is decreased is different at each step because the n moves from a

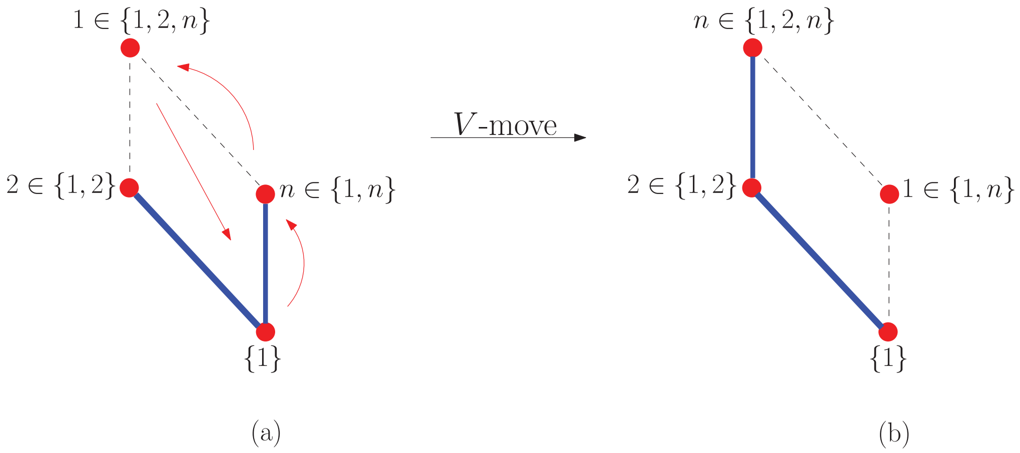


Figure 9.15: Diagram 2 for Case (b) for the proof of Lemma 9.2.3. (a) Direction 1 is chosen at $\{1, 2, n\}$, direction n is chosen at $\{1, n\}$ and direction 2 is chosen at $\{1, 2\}$. (b) After applying the V-move directions 1 and n are swapped.

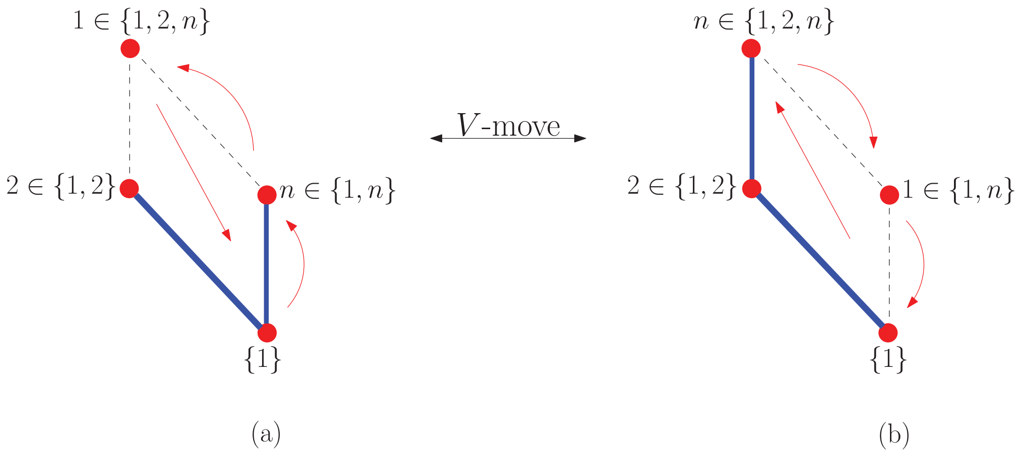


Figure 9.16: Diagram 1 for Case (b(I)) for the proof of Lemma 9.2.3. (a) Direction 1 is chosen at $\{1, 2, n\}$, direction n is chosen at $\{1, n\}$ and direction 2 is chosen at $\{1, 2\}$. (b) After applying the V-move directions 1 and n are swapped. Applying the V-move to (b) we can move to (a).

vertex containing i to one that does not. In what follows the direction i that is decreased will belong to $\hat{\xi}$, and so no d_i will be decreased twice.

Suppose now that $\psi_T(\eta = \xi \setminus \{i\}) = n$. Then $\eta \setminus \{n\} \subseteq [2]$ by the choice of ξ. We consider the following cases according to whether or not $1 \in \eta$.

(I) Suppose that $1 \in \eta$. Then $\eta = \{1, 2, n\}$ or $\{1, n\}$ and therefore $S \in \{(2, 2, 3), (2, 3, 2)\}$ depending on the direction of the vertex $\{2, n\}$. As shown in Figure 9.16, wherever the n-edge and the 1-edge are chosen, using the V-move the two labels can be swapped. So T can be transformed into T' with $\psi_{T'}(\eta) = 1$ and all other labels of T' apart from $\psi_{T'}(\{1, 2, n\})$ and $\psi_{T'}(\{1, n\})$ the same as the labels of T. Note that $T_{n-} = T'_{n-}$ because the labels of T' apart from $\psi_{T'}(\{1, 2, n\})$ and $\psi_{T'}(\{1, n\})$ were unchanged.

As shown in Figure 9.17, using the path move the 1-edge moves to $\xi \setminus \{n\}$, the i-edge moves to ξ and the n-edge moves to η. Let $\hat{T}$ be the resulting tree and let

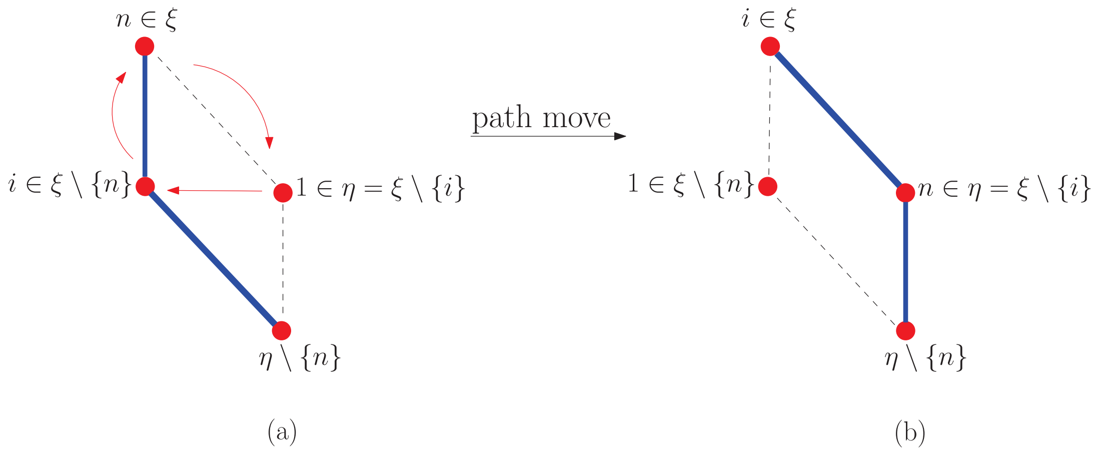


Figure 9.17: Diagram 2 for Case (b(I)) for the proof of Lemma 9.2.3. (a) Direction n is chosen at ξ, direction i is chosen at $\xi \setminus \{n\}$ and direction 1 is chosen at $\eta = \xi \setminus \{i\}$. (b) After applying the path move directions 1 moves to $\xi \setminus \{n\}$, direction i moves to ξ and n moves to $\eta = \xi \setminus \{i\}$.

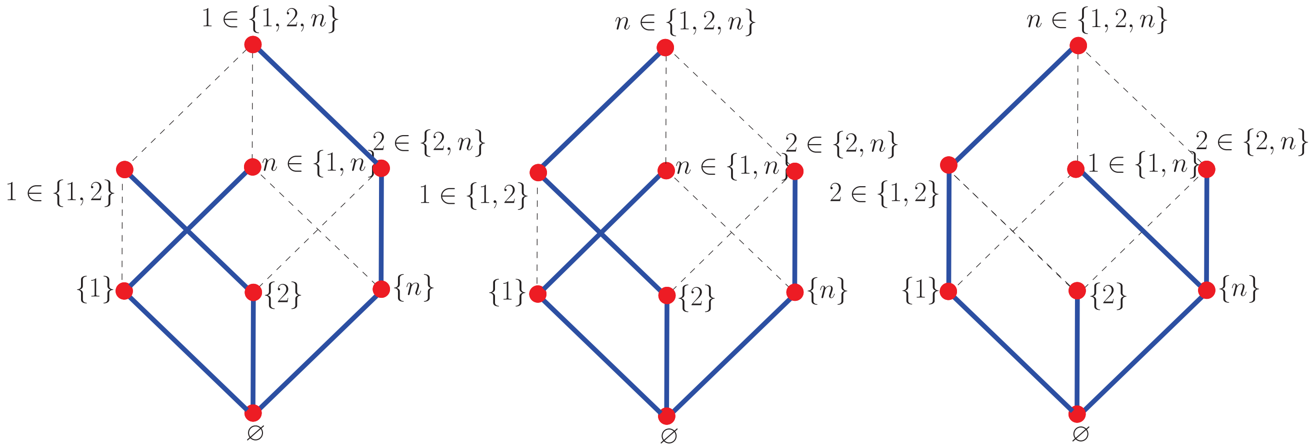


Figure 9.18: Diagram 1 for Case (b(II)) for the proof of Lemma 9.2.3. The possible trees of $Q_{\{1,2,n\}}$ with signature $S \in \{(3, 2, 2), (2, 2, 3), (2, 3, 2)\}$ where 2 is chosen at $\{2, n\}$.

$\hat{\mathcal{D}} = (\hat{d}_1, \ldots, \hat{d}_{n-1})$ in $\mathcal{F}_n^{n-}$. Then

$$\hat{d}_1 = d_1 + 1 = 2,\ \hat{d}_2 = d_2 = 2,\ \hat{d}_i = d_i - 1 \geq 3 \text{ and } \hat{d}_k = d_k,$$

for $k \neq 1, i$ and therefore $\hat{d}_t \geq 2$ for all t.

(II) Suppose that $1 \notin \eta = \xi \setminus \{i\}$. Then $\xi \setminus \{i\} = \eta = \{2, n\}$ and therefore $\xi = \{2, i, n\}$ and $S \in \{(2, 2, 3), (2, 3, 2), (3, 2, 2)\}$. Since S is irreducible, by Lemma 3.2.4 there exists a tree $T' \cap Q_{\{1,2,n\}}$ with signature $S \in \{(2, 2, 3), (2, 3, 2), (3, 2, 2)\}$ where $\psi_{T'}(\{2, n\}) = 2$ as shown in Figure 9.18. Since the edge slide graph of signature S is connected (by Henden [6] or Lemma 4.3.6), we can move from $T \cap Q_{\{1,2,n\}}$ to such a tree using edge slides. So T can be transformed into T' with $\psi_{T'}(\{2, n\}) = 2$ and all other labels of T' apart from the labels of $T' \cap Q_{\{1,2,n\}}$ the same as the labels of T. Let $\mathcal{D}' = (d'_1, \ldots, d'_{n-1})$ be the signature of T'_{n-}.

If $S \in \{(2, 2, 3), (3, 2, 2)\}$, then as seen in the figure we have

$$d'_1 = d_1 + 1 = 2,\ d'_2 = d_2 - 1 = 1 \text{ and } d'_k = d_k \geq 3,$$

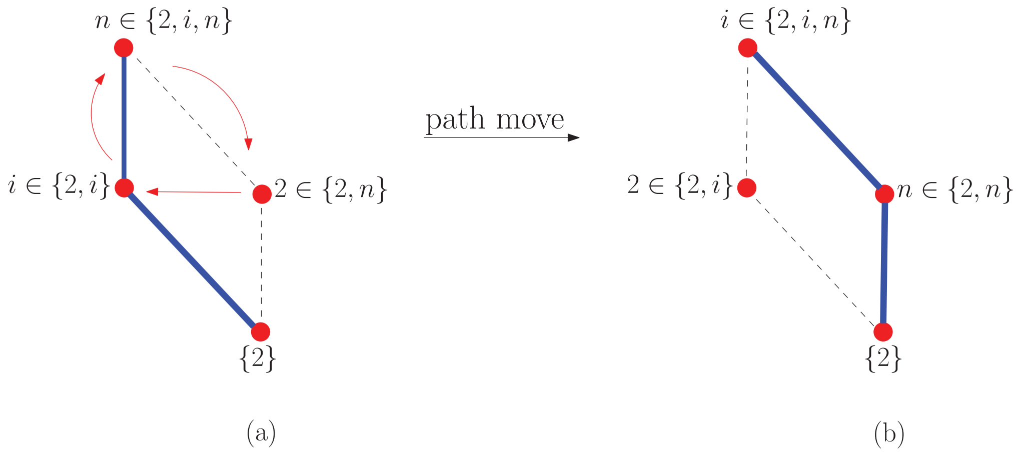


Figure 9.19: Diagram 2 for Case (b(II)) for the proof of Lemma 9.2.3. (a) Direction n is chosen at $\{2, i, n\}$, direction i is chosen at $\{2, i\}$ and direction 2 is chosen at $\{2, n\}$. (b) After applying the path move direction 2 moves to $\{2, i\}$, direction i moves to $\{2, i, n\}$ and n moves to $\{2, n\}$.

for $k \neq 1, 2$. So d_1 has been increased, but d_2 has been decreased. So now we show that d_2 can be increased without decreasing d_1.

As shown in Figure 9.19, using the path move the 2-edge at $\{2, n\}$ moves to $\{2, i\}$, the i-edge at $\{2, i\}$ moves to $\{2, i, n\}$ and the n-edge at $\{2, i, n\}$ moves to $\{2, n\}$. Let $\hat{T}$ be the resulting tree and let $\hat{\mathcal{D}} = (\hat{d}_1, \dots, \hat{d}_{n-1})$ in $\mathcal{F}_n^{n-}$. Then

$$\hat{d}_2 = d'_2 + 1 = d_2 = 2,\ \hat{d}_i = d'_i - 1 = d_i - 1 \geq 3, \hat{d}_1 = d'_1 = d_1 + 1 = 2$$

and $\hat{d}_k = d'_k \geq d_k$, for $k \neq 2, i$ and therefore $\hat{d}_t \geq 2$ for all t.

If $S = (2, 3, 2)$, then $T_{n-} = T'_{n-}$ because all the labels apart from $\psi_{T'}(\{1, 2, n\})$, $\psi_{T'}(\{1, n\})$ and $\psi_{T'}(\{2, n\})$ were unchanged.

As shown in Figure 9.19, using the path move the 2-edge at $\{2, n\}$ moves to $\{2, i\}$, the i-edge moves to $\{2, i, n\}$ and the n-edge moves to $\{2, n\}$. Then T' is transformed into T'' with $\psi_{T''}(\{2, i, n\}) = i$, $\psi_{T''}(\{2, n\}) = n$, $\psi_{T''}(\{2, i\}) = 2$ and all other labels of T'' the same as the labels of T'. Let $\mathcal{D}'' = (d''_1, \dots, d''_{n-1})$ be the signature of T''_{n-}. Then

$$d''_2 = d'_2 + 1 = d_2 + 1 = 3,\ d''_i = d_i - 1 \geq 3, d''_1 = d'_1 = d_1 = 1$$

and $d''_k = d_k$, for $k \neq 2, i$. So d_2 has been increased and therefore $d''_k \geq 3$ for all $k \neq 1$ because $d_k \geq 4$ for all $k \geq 3$ and $i > 2$. Since $\psi_{T'}(\{1, n\}) = 1$ and since all the labels of T'' apart from $\psi_{T''}(\{2, i, n\})$, $\psi_{T''}(\{2, n\})$, and $\psi_{T''}(\{2, i\})$ are the same as the labels of T', we have $\psi_{T''}(\{1, n\}) = 1$. Since $d''_k \geq 3$ for all $k \geq 2$ and $\psi_{T''}(\{1, n\}) = 1$, we can apply Lemma 9.2.2 to increase d''_1, and therefore T'' can be transformed into $\hat{T}$ where $\hat{d}_t \geq 2$ for all t.

In all cases we showed that we can always increase d_1 without decreasing $d_2 = 2$, and therefore T can be transformed into $\hat{T}$ with $\hat{d}_t \geq 2$ for all t. □

9.3 Transforming a tree with a reducible splitting signature into a tree with an irreducible splitting signature

We recall the main result of this chapter Theorem 9.1.2. It shows that under the inductive hypothesis, for $n \geq 4$ each upright spanning tree of Q_n with an irreducible signature such that the tree has a reducible signature in $\mathcal{F}_n^{n-}$ can be transformed by a sequence of edge slides into an upright spanning tree with an irreducible signature in $\mathcal{F}_n^{n-}$:

Theorem 9.1.2. *Suppose that every irreducible signature of Q_k is connected for all $k < n$. Let $n \geq 4$, and let $\mathcal{I} = (a_1, \dots, a_n)$ be an irreducible signature of Q_n with $a_n \geq a_\ell$ for all ℓ. Let $\mathcal{D} = (d_1, \dots, d_{n-1})$ be a reducible splitting signature of $\mathcal{I}$ with respect to n. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}$ such that T_{n-} has signature $\mathcal{D}$. Then there exists a sequence of edge slides that transforms T into $\bar{T}$ such that $\bar{T}_{n-}$ has an irreducible signature.*

Proof. Without loss of generality, we may assume $\mathcal{D}$ is ordered, and therefore there exists $s \leq n-2$ such that $\sum_{i=1}^{s} d_i = 2^s - 1$. Let $s_0 = \omega(\mathcal{D})$ be the least such s. Note that s_0 cannot be equal to 2, because in order for $s_0 = 2$, we need $d_1 + d_2 = 1 + 2 = 3$ as $\mathcal{D}$ is ordered. Since we defined s_0 to be the least s, in this case we would have $s_0 = 1$. We show that $\omega(\mathcal{D})$ can always be increased using edge slides. We consider the cases $s_0 = 1$ and $s_0 \geq 3$ separately.

1. Suppose that $s_0 = 1$. Since $\mathcal{I}$ is irreducible and $s_0 = 1$, we must have $d_1 = 1$, and the unique edge of T_{n-} in direction 1 is chosen at $\{1\}$. We show there exists a sequence of edge slides that transforms T into an upright spanning tree $T^\star$ such that $T^\star_{n-}$ has signature $\mathcal{D}^\star = (d^\star_1, \dots, d^\star_{n-1})$, where
$$d^\star_i \geq d_i \geq 2$$
for all i. It follows that $\omega(\mathcal{D}^\star) \geq 3$.

   Since $\mathcal{I}$ is an irreducible signature and $d_1 = 1$, there exists at least one edge of T_{n+} in direction 1. Let $X \subseteq \{2, \dots, n-1\}$ be such that $Y = X \cup \{1, n\}$ is a lowest vertex of $\mathcal{F}_n^{n+}$ that has a 1-edge. After applying Lemma 7.2.1 (if necessary), we may assume that $\psi_T(X \cup \{n\}) = n$.

   To increase $\omega(\mathcal{D})$ we must increase $d_1 = 1$ while decreasing d_j by at most one for any $j \neq 1$. We distinguish cases according to whether or not $X = \varnothing$.

   (a) $[X \neq \varnothing]$: If $X \neq \varnothing$, then $|X \cup \{1, n\}| \geq 3$. The vertex $X \cup \{1\}$ cannot be in direction 1, because the only 1-edge in T_{n-} is at $\{1\}$. So $\psi_T(X \cup \{1\}) = j$ for some $j \neq 1$. **Suppose first that** $d_j \geq 3$. As shown in Figure 9.20(a), using the path move the 1-edge moves to $X \cup \{1\}$, the j-edge moves to $X \cup \{n\}$ and the n-edge moves $X \cup \{1, n\}$ as shown in Figure 9.20(b). Let $\hat{T}$ be the resulting tree, and let $\hat{\mathcal{D}} = (\hat{d}_1, \dots, \hat{d}_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then
$$\hat{d}_1 = d_1 + 1 = 2, \quad \hat{d}_j = d_j - 1 \quad \text{and} \quad \hat{d}_i = d_i,$$
for all $i \neq 1, j$. So $\hat{d}_t \geq 2$ for all t because d_1 has been increased and $d_j \geq 3$ has been decreased by one.

Suppose now that $d_j = 2$. Then we must have $j = 2$ because $\mathcal{D}$ is an ordered signature. So $X = \{2\}$ and therefore the vertex $\{1, 2, n\}$ is in direction 1. Then applying Lemma 9.2.3, T can be transformed into $\hat{T}$ such that $\hat{T}_{n-}$ has signature $\hat{\mathcal{D}} = (\hat{d}_1, \dots, \hat{d}_{n-1})$ in $\mathcal{F}_n^{n-}$ where $\hat{d}_t \geq 2$ for $t \geq 1$.

(b) $[X = \varnothing]$: If $X = \varnothing$, then $X \cup \{1, n\} = \{1, n\}$. **Suppose first that** $d_j \geq 3$ for all $j \geq 2$. Since $[1]$ is the smallest reducing set, $d_j \geq 3$ for all $j \geq 2$, and 1 is chosen at $\{1, n\}$, all the conditions in Lemma 9.2.2 are satisfied. Then T can be transformed into $\hat{T}$ with signature $\hat{\mathcal{D}} = (\hat{d}_1, \dots, \hat{d}_{n-1})$ where $\hat{d}_t \geq 2$ for all t.

Suppose now that $d_2 = 2$. Since $d_1 = 1$ and the vertex $\{1, n\}$ is in direction 1, all the condition in Lemma 9.2.3 are satisfied. Then T can be transformed into $\hat{T}$ such that $\hat{T}_{n-}$ has signature $\hat{\mathcal{D}} = (\hat{d}_1, \dots, \hat{d}_{n-1})$ in $\mathcal{F}_n^{n-}$ where $\hat{d}_i \geq 2$ for $i \geq 1$.

In all cases we reached a signature $\hat{\mathcal{D}}$ such that $\hat{d}_t \geq 2$ for all t. Therefore $\omega(\hat{\mathcal{D}}) \geq 3$. If $\omega(\hat{\mathcal{D}}) = n - 1$, then $\hat{\mathcal{D}}$ is an irreducible signature. Otherwise $3 \leq \omega(\hat{\mathcal{D}}) \leq n - 2$ and therefore we move to Case 2.

2. Suppose that $s_0 \geq 3$. Then $(d_1, \dots, d_{s_0})$ is an irreducible signature of Q_{s_0} because s_0 is the least index s such that $\sum_{t=1}^{s} d_i = 2^s - 1$. Let $u_i = a_i - d_i$ for all $i \in [n-1]$. Since $\mathcal{I}$ is irreducible and $\sum_{t=1}^{s_0} d_t = 2^{s_0} - 1$, we must have $u_i \neq 0$ for some $i \in [s_0]$. We show that we can increase d_i while decreasing d_f by at most 1 for all $f \in \{s_0 + 1, \dots, n - 1\}$ using a sequence of edge slides, so $\omega(\mathcal{D})$ is increased. Let $X \subseteq [n-1] \setminus \{i\}$ be such that $Y = X \cup \{i, n\}$ is a lowest vertex of $\mathcal{F}_n^{n+}$ that has an i-edge. After applying Lemma 7.2.1 (if necessary), we may assume that $\psi_T(X \cup \{n\}) = n$. We consider the following cases according to whether or not $X \subseteq [s_0]$.

   (a) Suppose that $X \nsubseteq [s_0]$. Then by Lemma 6.2.2 the vertex $X \cup \{i\}$ cannot be in any direction belonging to $[s_0]$, so the vertex $X \cup \{i\}$ is in direction j for some $j \notin [s_0]$. As shown in Figure 9.21(a), using the path move the j-edge moves to $X \cup \{n\}$, the n-edge moves to $X \cup \{i, n\}$ and the i-edge moves to $X \cup \{i\}$ as shown in Figure 9.21(b). Let T' be the resulting tree and let $D' = (d'_1, \dots, d'_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$, where

   $$d'_i = d_i + 1,\ d'_j = d_j - 1 \text{ and } d'_k = d_k,$$

   for all $k \neq i, j$. Therefore d_i has been increased where $i \in [s_0]$ while d_j has been decreased where $j \notin [s_0]$. So $\omega(\mathcal{D}') > \omega(\mathcal{D})$.

   (b) Suppose that $X \subseteq [s_0]$. Since $[s_0]$ is the smallest reducing set where $s_0 \geq 3$ and the vertex $X \cup \{i, n\}$ is a vertex of $\mathcal{F}_n^{n+}$ such that $X \subseteq [s_0]$, all the conditions in Lemma 9.2.2 are satisfied. Then T can be transformed into T' with signature $\mathcal{D}' = (d'_1, \dots, d'_{n-1})$ in $\mathcal{F}_n^{n-}$ where $\omega(\mathcal{D}') > \omega(\mathcal{D})$.

In all cases we showed that we can always increase $\omega(\mathcal{D})$. Since $\omega(\mathcal{D})$ cannot increase indefinitely, we must eventually reach a tree where $\omega(\mathcal{D}) = n - 1$. Then we conclude that any upright spanning tree with a reducible splitting signature can be transformed into a tree with an irreducible splitting signature. $\square$

9.4 Summary map

In this chapter we showed that under the inductive hypothesis each upright spanning tree of Q_n with an irreducible signature such that the tree has a reducible splitting signature in $\mathcal{F}_n^{n-}$ is connected to a tree with an irreducible splitting signature in $\mathcal{F}_n^{n-}$. Previously in Chapter 5, we showed that every irreducible signature has an amenable splitting signature using signature moves. In Chapter 7 we showed under the inductive hypothesis that the set of upright spanning trees of Q_n with an irreducible signature and a fixed unidirectional splitting signature or (2, 2, 3) splitting signature in $\mathcal{F}_n^{n-}$ forms one block. In Chapter 8, we showed that each upright spanning tree of Q_n with an irreducible signature such that the tree has an irreducible splitting signature in $\mathcal{F}_n^{n-}$ is connected to a tree with any other irreducible splitting signature in $\mathcal{F}_n^{n-}$. In Chapter 11 we show that if an irreducible signature has an irreducible splitting signature such that the set of upright spanning trees with this splitting signature forms a block then the edge slide graph of the irreducible signature is connected.

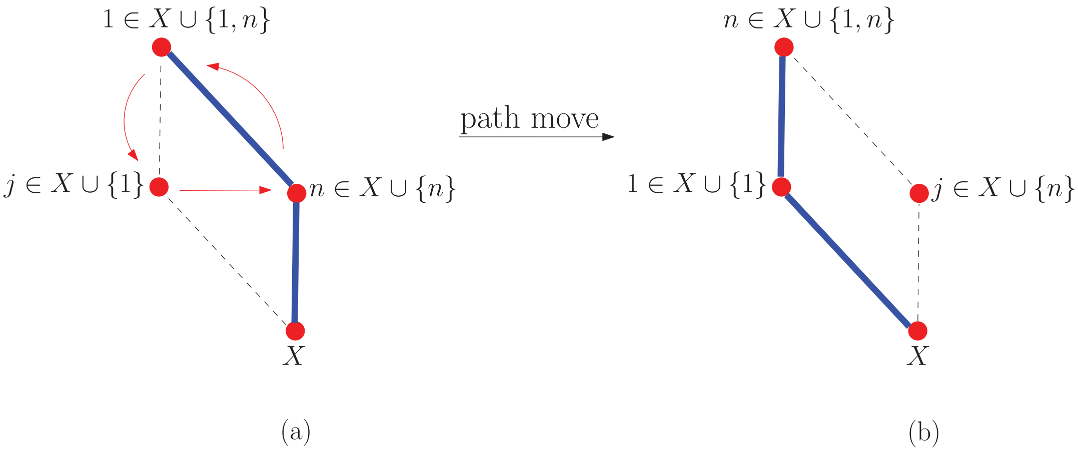


Figure 9.20: Diagram 1 for the proof of Theorem 9.1.2. (a) Direction 1 is chosen at $X \cup \{1, n\}$ and direction n and direction j are chosen at $X \cup \{n\}$ and $X \cup \{1\}$ respectively. (b) Under the path move the j-edge moves to $X \cup \{n\}$, the n-edge moves to $X \cup \{1, n\}$ and 1-edge moves to $X \cup \{1\}$.

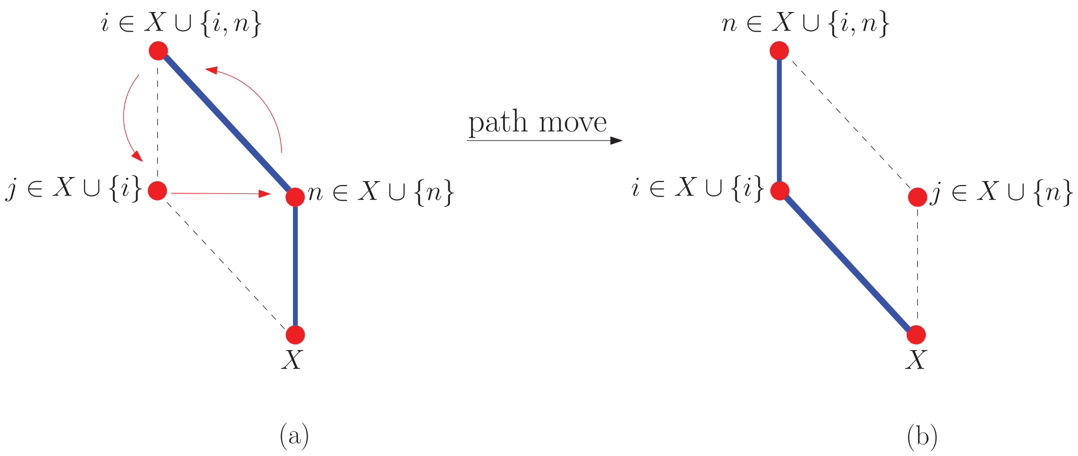


Figure 9.21: Diagram 2 for the proof of Theorem 9.1.2. (a) Direction i is chosen at $X \cup \{i, n\}$ and direction n and direction j are chosen at $X \cup \{n\}$ and $X \cup \{i\}$ respectively. (b) Under the path move the j-edge moves to $X \cup \{n\}$, the n-edge moves to $X \cup \{i, n\}$ and i-edge moves to $X \cup \{i\}$.

Chapter 10

Two infinite families of irreducible signatures of Q_n.

In this chapter we study two infinite families of irreducible signatures of Q_n. We give a proof for the connectivity of the edge slide graph of each family using a slightly different technique from the approach we use for the general case. In these families, we only encounter signatures within the family or within closely related families that we already understand. Also, we give formulae to determine the number of upright spanning trees of Q_n with these signatures.

10.1 The irreducible signature $\mathcal{I}_n^{(-1)}$

In this section we start by showing that the edge slide graph of this signature is connected. We also count the number of upright spanning trees of Q_n with signature $\mathcal{I}_n^{(-1)}$.

Recall from Definition 5.6.4 that the ordered irreducible signature $\mathcal{I}_n^{(-1)} = (a_1, \dots, a_n)$ of Q_n is defined by

$$a_i = \begin{cases} 2, & \text{if } i = 1; \\ 2^{n-1} - 1, & \text{if } i = n; \\ 2^{i-1}, & \text{otherwise.} \end{cases}$$

The signature $\mathcal{I}_n^{(-1)}$ is the only irreducible signature of Q_n with excess $\varepsilon_k^{\mathcal{I}_n^{(-1)}} = 1$ for all $k \leq n-1$. Given an upright spanning tree of Q_n with signature $\mathcal{I}_n^{(-1)}$, since the signature $\mathcal{I}_n^{(-1)}$ is symmetric with respect to directions 1 and 2, choosing either direction in the upper n face $\mathcal{F}_n^{n+}$ gives a tree of Q_{n-1} with a supersaturated signature in the lower n face $\mathcal{F}_n^{n-}$. So for upright spanning trees of Q_n with signature $\mathcal{I}_n^{(-1)}$, choosing either direction 1 or 2 in $\mathcal{F}_n^{n+}$ determines the tree of Q_{n-1} in $\mathcal{F}_n^{n-}$ because the supersaturated signature has only one upright spanning tree. Therefore there are 2^{n-2} unique upright spanning trees of Q_n with signature $\mathcal{I}_n^{(-1)}$ in which direction i is chosen in $\mathcal{F}_n^{n+}$ for each $i \in \{1, 2\}$. Choosing direction i in $\mathcal{F}_n^{n+}$, where $i \in \{3, \dots, n-2\}$, gives a tree of Q_{n-1} with a quasi-irreducible signature $(\mathcal{I}_i^{(-1)}, 2^i, \dots, 2^{n-2})$ in $\mathcal{F}_n^{n-}$. Also, choosing $n-1$ in $\mathcal{F}_n^{n+}$ gives a tree with the irreducible signature $\mathcal{I}_{n-1}^{(-1)}$ of Q_{n-1} in $\mathcal{F}_n^{n-}$.

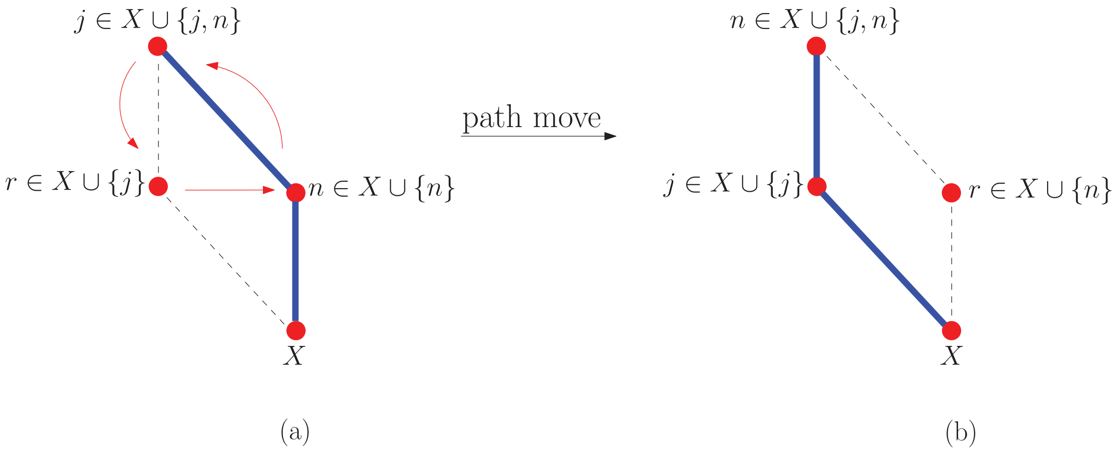


Figure 10.1: Diagram 1 for the proof of Theorem 10.1.1. (a) A face of Q_n with direction j is chosen at $X \cup \{j, n\}$, direction r is chosen at $X \cup \{j\}$, and direction n is chosen at $X \cup \{n\}$. (b) Using the path move, the label r goes to $X \cup \{n\}$ and labels n and j go to $X \cup \{j, n\}$ and $X \cup \{j\}$ respectively.

10.1.1 The edge slide graph of signature $\mathcal{I}_n^{(-1)}$ of Q_n

The following theorem states that the edge slide graph $\mathcal{E}(\mathcal{I}_n^{(-1)})$ is connected. Let $T^\star$ be the unique upright spanning tree of Q_n with signature $\mathcal{I}_n^{(-1)}$ where 1 is chosen at $\{1, n\}$. To prove this theorem we show that any spanning tree of Q_n with signature $\mathcal{E}(\mathcal{I}_n^{(-1)})$ can be transformed by a sequence of edge slides into the upright spanning tree $T^\star$.

Theorem 10.1.1. *The edge slide graph $\mathcal{E}(\mathcal{I}_n^{(-1)})$ is connected for all $n \geq 3$.*

Proof. We prove the result by induction on n.

Henden [6] as well as Lemma 4.3.6 proved that the edge slide graph $\mathcal{E}(2, 2, 3)$ is connected, hence the result is true for $n = 3$.

Suppose the result is true for all $3 \leq k < n$.

Since every spanning tree of Q_n is connected to at least one upright spanning tree by a series of edge slides (Tuffley [11]), it suffices to prove the result for upright spanning trees of Q_n with signature $\mathcal{I}_n^{(-1)}$. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}_n^{(-1)}$. We show that there exists a sequence of edge slides that transforms T into the tree $T^\star$.

Since T has $2^{n-1} - 1$ edges in direction n, there exists a unique edge of T_{n+} with direction not equal to n. Let j be the direction of this unique edge, and let $X \subseteq [n-1] \setminus \{j\}$ be such that $W = X \cup \{j, n\}$ is the unique vertex of $\mathcal{F}_n^{n+}$ that is not in direction n. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ denote the signature of T_{n-}. We now distinguish the following cases according to whether or not $X \subseteq [j-1]$.

Suppose first that $X \nsubseteq [j-1]$. We show there exists a sequence of edge slides that transforms T into a tree where $r = \max X > j$ is chosen at a vertex $X' \cup \{r, n\}$ of $\mathcal{F}_n^{n+}$ such that $X' \subseteq [r-1]$. Since $\mathcal{D}$ is saturated above direction j and $X \nsubseteq [j-1]$, the vertex $X \cup \{j\}$ must be in direction r (by Lemma 6.2.1), as shown in Figure 10.1(a). Then using the path move the r-edge moves to $X \cup \{n\}$, the n-edge moves to $X \cup \{j, n\}$, and the j-edge moves to $X \cup \{j\}$, as shown in Figure 10.1(b). Let T' be the resulting tree and let $\mathcal{D}' = (d'_1, \ldots, d'_{n-1})$

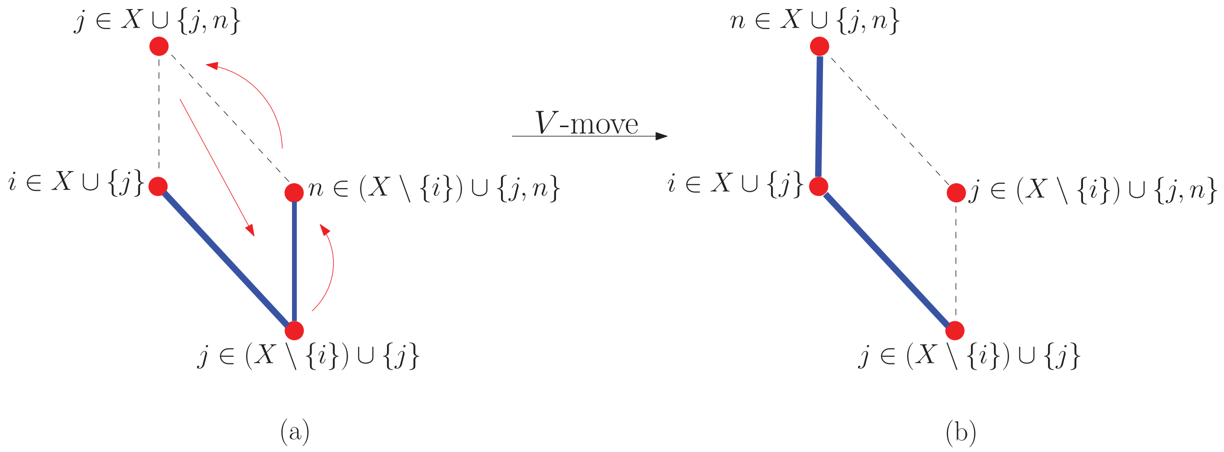


Figure 10.2: Diagram 2 for the proof of Theorem 10.1.1. (a) A face of Q_n with direction j is chosen at the vertex $X \cup \{j,n\}$ and $(X \cup \{j\}) \setminus \{i\}$, direction i is chosen at $X \cup \{j\}$ and n is chosen at $(X \setminus \{i\}) \cup \{j,n\}$. (b) After applying the V-move, labels j and n are swapped.

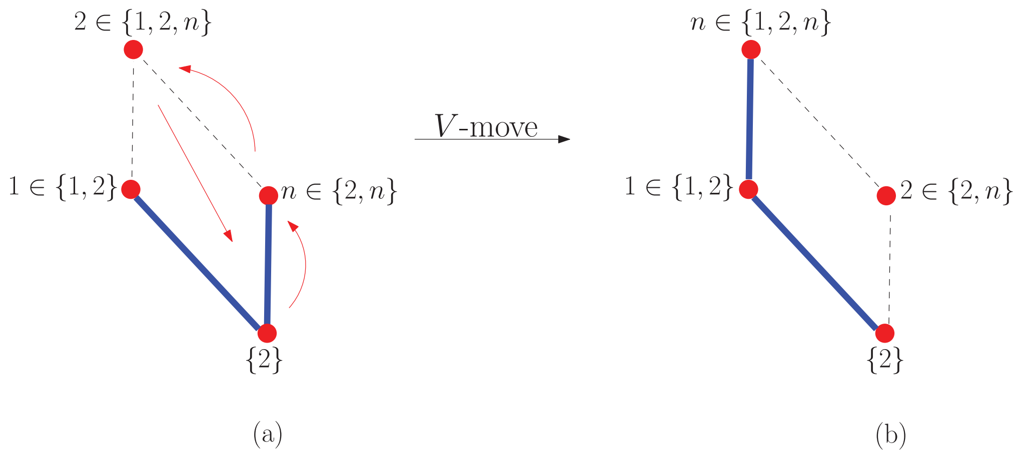


Figure 10.3: Diagram 3 for the proof of Theorem 10.1.1. (a) A face of Q_n with direction 2 is chosen at the vertex $\{1,2,n\}$, direction 1 is chosen at $\{1,2\}$ and $\{1\}$, and direction n is chosen at $\{2,n\}$. (b) After applying the V-move, labels 2 and n are swapped.

be its signature in $\mathcal{F}_n^{n-}$. Then

$$d'_j = d_j + 1, \quad d'_r = d_r - 1 \quad \text{and} \quad d'_\ell = d_\ell,$$

for all $\ell \neq j, r$. Let

$$\begin{aligned} X' &= (X \cup \{n\}) \setminus \{r,n\} \\ &= X \setminus \{r\} \\ &\subseteq [r-1] \qquad\qquad (\text{since } r = \max X > j). \end{aligned}$$

Therefore T is transformed to T' such that r is chosen in $\mathcal{F}_n^{n+}$ at $X' \cup \{r,n\}$, where $X' \subseteq [r-1]$, and we may apply the case that follows.

Suppose now that $X \subseteq [j-1]$. The signature $\mathcal{D} = (\mathcal{I}_j^{(-1)}, 2^j, \ldots, 2^{n-2})$ with the unsaturated part equal to $\mathcal{I}_j^{(-1)}$ for all $3 \leq j \leq n-1$. We show that there exists a sequence of edge slides that transforms T to a tree where direction j is chosen at $\{j,n\}$. Let $T_j = T_{n-} \cap Q_j$. Then T_j has signature $\mathcal{I}_j^{(-1)}$. Since $\mathcal{I}_j^{(-1)}$ is an irreducible signature of Q_j, by Lemma 3.2.4 given $i \in X$ there exists a tree $\hat{T}_j$ of Q_j with signature $\mathcal{I}_j^{(-1)}$ such that $\psi_{\hat{T}_j}(X \cup \{j\}) = i$. Since the edge slide graph $\mathcal{E}(\mathcal{I}_j^{(-1)})$ is connected (by the inductive hypothesis), we can move from T_j to such a tree using edge slides. Then T is transformed into $\hat{T}$ where $\hat{T}_j = \hat{T} \cap Q_j$ and all labels of $\hat{T}$ apart from the labels of $\hat{T}_j$ are the same as the labels of T. Since all the vertices of $\mathcal{F}_j^{j+}$ other than $X \cup \{j\}$ are in direction j, the vertex $(X \setminus \{i\}) \cup \{j\}$ is in direction j. Since all the vertices of $\mathcal{F}_n^{n+}$ other than $X \cup \{j,n\}$ are in direction n, the vertex $(X \setminus \{i\}) \cup \{j,n\}$ is in direction n, as shown in Figure 10.2(a). Then using the V-move the j-edge and the n-edge are swapped, as shown in Figure 10.2(b). Let $\tilde{T}$ be the resulting tree. Then T is transformed into $\tilde{T}$ where, for $\tilde{X} = X \setminus \{i\}$, the j-edge is chosen at $\tilde{X} \cup \{j,n\}$. Note that $|\tilde{X}| < |X|$. If $\tilde{X} \cup \{j,n\} \neq \{j,n\}$, then we repeat this process until we eventually reach the vertex $\{j,n\}$. This must necessarily occur because the cardinality of the vertex where j is chosen decreases by one at each step. Therefore T is transformed to a tree T'' that has the same signature as T in $\mathcal{F}_n^{n-}$, and the j-edge in $\mathcal{F}_n^{n+}$ is chosen at $\{j,n\}$.

Suppose that $j = 2$. Then $T \cap Q_2$ has signature (2, 1) and the two 1-edges are chosen at $\{1,2\}$ and $\{1\}$. If 2 is chosen at $\{1,2,n\}$, then as shown in Figure 10.3 using the V-move the label 2 moves to $\{2,n\}$ and the label n moves to $\{1,2,n\}$ and therefore we move to the following case.

We now show that there exists a sequence of edge slides that transforms T'' into the tree $T^\star$. Since $\mathcal{I}_j^{(-1)}$ is an irreducible signature, there exists a tree T_j''' of Q_j with signature $\mathcal{I}_j^{(-1)}$ where $\psi_{T_j'''}(\{1,j\}) = 1$ (by Lemma 3.2.4). Since the edge slide graph $\mathcal{E}(\mathcal{I}_j^{(-1)})$ is connected (by the inductive hypothesis), we can move from T_j'' to such a tree using edge slides. Since all the vertices of $\mathcal{F}_n^{n+}$ other than $\{j,n\}$ are in direction n, the vertices $\{1,j,n\}$ and $\{1,n\}$ are in direction n, as shown in Figure 10.4(a). Therefore we have a tree of $Q_{\{1,j,n\}}$ with signature (2, 2, 3) where its edge slide graph is connected (by Henden [6] or Lemma 4.3.6). As shown in Figure 10.4(b) and (c), applying the path move and then the V-move the 1-edge moves to $\{1,n\}$, the j-edge moves to $\{1,j\}$ and the n-edge moves to $\{j,n\}$. Therefore T'' is transformed into the tree $T^\star$.

Finally, by the definition of the edge slide graph $\mathcal{E}(\mathcal{I}_n^{(-1)})$, there is a sequence of edge slides from T to $T^\star$, hence T and $T^\star$ are in the same connected component of $\mathcal{E}(\mathcal{I}_n^{(-1)})$. Since T is arbitrary, the edge slide graph $\mathcal{E}(\mathcal{I}_n^{(-1)})$ is connected. □

10.1.2 Counting the number of upright spanning trees of Q_n with signature $\mathcal{I}_n^{(-1)}$

In this section we give a formula to count the number of upright spanning trees of Q_n with signature $\mathcal{I}_n^{(-1)}$. We start with the following definition.

Definition 10.1.2. Let $\mathcal{S} = (a_1, \dots, a_n)$ be a signature of Q_n. Then $\tau(\mathcal{S})$ is the number of upright spanning trees of Q_n with signature $\mathcal{S}$.

The following lemma gives a recurrence relation for the number of upright spanning trees of Q_n with signature of the form $(a_1, \dots, a_{n-2}, 2^{n-2}, 2^{n-1} - 1)$, in terms of the number of upright spanning trees of Q_{n-1} with signature $(a_1, \dots, a_{n-2}, 2^{n-2} - 1)$. This lemma is used to prove Theorem 10.1.4 and Theorem 10.2.2.

Lemma 10.1.3. *Let $\mathcal{S} = (a_1, \dots, a_{n-2}, a_{n-1})$ be a signature of Q_{n-1} such that $a_{n-1} = 2^{n-2} - 1$. Then $\mathcal{S}' = (a_1, \dots, a_{n-2}, 2^{n-2}, 2^{n-1} - 1)$ is a signature of Q_n and*

$$\tau(\mathcal{S}') = (2^{n-2} + 2)\tau(\mathcal{S}).$$

Proof. The signature conditions for $\mathcal{S}'$ hold, because they follow from the conditions for $\mathcal{S}$. To count the upright spanning trees we first show that from any upright spanning tree of Q_{n-1} with signature $\mathcal{S}$, we can construct $2^{n-2} + 2$ different upright spanning trees of Q_n with signature $\mathcal{S}'$. Let T be an upright spanning tree of Q_{n-1} with signature $\mathcal{S}$. We start by constructing 2^{n-2} different upright spanning trees from T.

Let T' be the upright spanning tree of Q_n with $T'_{n-} = T$, direction $n-1$ chosen at a vertex ξ of $\mathcal{F}_n^{n+}$ and all other vertices of $\mathcal{F}_n^{n+}$ in direction n. Then T' has signature $\mathcal{S}'$. Since there are 2^{n-2} choices for ξ, there are 2^{n-2} such trees and they all satisfy $T'_{n-} = T$.

Now we construct the other two trees from T. Let $X \subseteq [n-2]$ be such that $Y = X \cup \{n-1\}$ is the unique vertex of $\mathcal{F}_{(n-1)}^{(n-1)+}$ with $\psi_T(X \cup \{n-1\}) \neq n-1$. Note that $X \cup \{n-1\}$ exists and is unique because T has signature $\mathcal{S}$. Let $\psi_T(X \cup \{n-1\}) = i$. Let T' be the upright spanning tree of Q_n such that $T'_{n-} = T$ with the i in $\mathcal{F}_{(n-1)}^{(n-1)+}$ replaced by an n, i chosen at ξ for some $\xi \in \{X \cup \{n\}, X \cup \{n-1, n\}\}$, and all other vertices of $\mathcal{F}_n^{n+}$ in direction n. Since there are two choices for ξ, we get two different trees and these are both different to the 2^{n-2} trees constructed above. Moreover, T can be recovered from T'_{n-} and the location of the i-edge in T'_{n+}. Therefore we have constructed $2^{n-2} + 2$ different upright spanning trees of Q_n with signature $\mathcal{S}'$ from T.

Now we show that each tree of Q_n with signature $\mathcal{S}'$ derives from a unique tree of Q_{n-1} with signature $\mathcal{S}$ by this process.

Let T' be an upright spanning tree of Q_n with signature $\mathcal{S}'$ and let $X \subseteq [n-1]$ be such that $Y = X \cup \{n\}$ is the unique vertex of $\mathcal{F}_n^{n+}$ with $\psi_T(X \cup \{n\}) \neq n$.

Suppose first that $\psi_T(X \cup \{n\}) = n-1$. Then $T = T'_{n-}$ has signature $\mathcal{S}$, and T' is one of the 2^{n-2} trees obtained from T' by placing $n-1$ in $\mathcal{F}_n^{n+}$.

Suppose now that $\psi_T(X \cup \{n\}) = i$ for some $i \neq n-1$. Then all 2^{n-2} edges of T' in direction $n-1$ must be chosen at all the vertices of $\mathcal{F}_n^{n-}$ that contain $n-1$. Let T be T'_{n-}, with the label $n-1$ at vertex $X \cup \{n-1\}$ replaced by i. Then T has signature $\mathcal{S}$, and T' is one of the two trees obtained from T by placing the direction $i \neq n-1$ occurring in $\mathcal{F}_{n-1}^{(n-1)+}$ in $\mathcal{F}_n^{n+}$.

It follows that every tree with signature $\mathcal{S}'$ is derived from a unique tree with signature $\mathcal{S}$ by the above process, and the recurrence relation follows. □

The following result provides the number of upright spanning trees of Q_n with signature $\mathcal{I}_n^{(-1)}$.

Theorem 10.1.4. *Let $\tau(\mathcal{I}_n^{(-1)})$ be the number of upright spanning trees of Q_n with signature $\mathcal{I}_n^{(-1)}$. Then*

$$\tau(\mathcal{I}_n^{(-1)}) = 2^{n-2} \prod_{j=0}^{n-3} (2^j + 1), \tag{10.1}$$

for all $n \geq 3$.

Proof. Our proof is by induction on n.

We first show the result is true for the base case $n = 3$. In this case

$$\begin{aligned} 2^{n-2}\prod_{j=0}^{n-3}(2^j+1) &= 2^{3-2}\times(2^0+1) \\ &= 4. \end{aligned}$$

Since $\tau(\mathcal{I}_3^{(-1)}) = 4$ by Henden [6] or Lemma 4.3.6, Equation (10.1) is true for $n = 3$.

We now assume that Equation (10.1) holds for $n = k$ for some $k \geq 3$, that is

$$\tau(\mathcal{I}_k^{(-1)}) = 2^{k-2}\prod_{j=0}^{k-3}(2^j+1).$$

We prove that Equation (10.1) holds for $n = k + 1$, that is

$$\tau(\mathcal{I}_{k+1}^{(-1)}) = 2^{(k+1)-2}\prod_{j=0}^{(k+1)-3}(2^j+1).$$

Since

$$\tau(\mathcal{I}_k^{(-1)}) = 2^{k-2}\prod_{j=0}^{k-3}(2^j+1),$$

by Lemma 10.1.3 we have

$$\begin{aligned} \tau(\mathcal{I}_{k+1}^{(-1)}) &= (2^{(k+1)-2}+2)\tau(\mathcal{I}_k^{(-1)}) \\ &= (2^{(k+1)-2}+2)(2^{k-2}\prod_{j=0}^{k-3}(2^j+1)) \\ &= (2^{k-1}+2)(2^{k-2}\prod_{j=0}^{k-3}(2^j+1)) \\ &= 2(2^{k-2}+1)(2^{k-2}\prod_{j=0}^{k-3}(2^j+1)) \\ &= 2^{k-1}\prod_{j=0}^{k-2}(2^j+1). \end{aligned}$$

□

10.2 The irreducible signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$

In this section we start by defining the irreducible signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$, where $n \geq 4$, and then we show that the edge slide graph of this signature is connected. We also count the number of upright spanning trees of Q_n with signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$.

For $n \geq 4$, consider the ordered irreducible signature $\mathcal{I}_{(3,n)}^{(-1,-1)} = (a_1, \ldots, a_n)$ of Q_n, where

$$a_i = \begin{cases} 2, & \text{if } i = 1; \\ 3, & \text{if } i = 2; \\ 2^{i-1} - 1, & \text{if } i = 3, n; \\ 2^{i-1}, & \text{otherwise.} \end{cases}$$

Unlike the signature $\mathcal{I}_n^{(-1)}$, the signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$ has excess

$$\varepsilon_k^{\mathcal{I}_{(3,n)}^{(-1,-1)}} = \begin{cases} 2, & \text{if } k = 2; \\ 1, & \text{otherwise.} \end{cases}$$

For an upright spanning tree of Q_n with signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$, choosing direction 1 in the upper n face $\mathcal{F}_n^{n+}$ gives a tree of Q_{n-1} with the strictly reducible signature $(1, 3, 3, 8, 16, \ldots, 2^{n-1})$ in the lower n face $\mathcal{F}_n^{n-}$. Since the signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$ is symmetric with respect to directions 2 and 3, choosing either direction in $\mathcal{F}_n^{n+}$ gives a tree of Q_{n-1} with a quasi-irreducible signature $(2, 2, 3, 8, 16, \ldots, 2^{n-1})$ or $(2, 3, 2, 8, 16, \ldots, 2^{n-1})$ in $\mathcal{F}_n^{n-}$. Choosing a direction i in $\mathcal{F}_n^{n+}$, where $i \in \{4, \ldots, n-2\}$, gives a tree of Q_{n-1} with a quasi-irreducible signature $(\mathcal{I}_{(3,i)}^{(-1,-1)}, 2^i, \ldots, 2^{n-2})$ in $\mathcal{F}_n^{n-}$. Also, choosing $n-1$ in $\mathcal{F}_n^{n+}$ gives a tree with the irreducible signature $\mathcal{I}_{(3,n-1)}^{(-1,-1)}$ of Q_{n-1} in $\mathcal{F}_n^{n-}$.

10.2.1 The edge slide graph of signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$ of Q_n

The next theorem states that the edge slide graph $\mathcal{E}(\mathcal{I}_{(3,n)}^{(-1,-1)})$ is connected. Let $T^\star$ be the upright spanning tree of Q_n with signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$ where 2 is chosen at $\{2, n\}$ and $\{1, 2, 3\}$, and 1 is chosen at $\{1, 2\}$. The proof of this theorem is similar to the proof of Theorem 10.1.1 in terms of showing that any spanning tree of Q_n with signature $\mathcal{E}(\mathcal{I}_{(3,n)}^{(-1,-1)})$ can be transformed by a sequence of edge slides into the upright spanning tree $T^\star$.

Theorem 10.2.1. *The edge slide graph $\mathcal{E}(\mathcal{I}_{(3,n)}^{(-1,-1)})$ is connected for all $n \geq 4$.*

Proof. We prove the result by induction on n.

Lemma 5.4.2 proved that the edge slide graph $\mathcal{E}(2, 3, 3, 7)$ is connected, hence the result is true for $n = 4$.

Suppose the result is true for all $4 \leq k < n$.

Since every spanning tree of Q_n is connected to at least one upright spanning tree by a series of edge slides (Tuffley [11]), it suffices to prove the result for upright spanning trees of Q_n with signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$. Let T be an upright spanning tree of Q_n with signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$. We show that there exists a sequence of edge slides that transforms T to the tree $T^\star$.

Since T has $2^{n-1} - 1$ edges in direction n, there exists a unique edge of T_{n+} with direction not equal to n. Let j be the direction of this unique edge, and let $X \subseteq [n-1] \setminus \{j\}$ be such that $W = X \cup \{j, n\}$ is the unique vertex of $\mathcal{F}_n^{n+}$ that is not in direction n. Let $\mathcal{D} = (d_1, \ldots, d_{n-1})$ denote the signature of T_{n-}. We now distinguish the following cases according to whether or not $X \subseteq [j-1]$.

Suppose first that $X \not\subseteq [j-1]$. We show there exists a sequence of edge slides that transforms T to a tree where $r = \max X > j$ is chosen at a vertex $X' \cup \{r, n\}$ of $\mathcal{F}_n^{n+}$ such that $X' \subseteq [r-1]$. Since $\mathcal{D}$ is saturated above direction j and $X \not\subseteq [j-1]$, the vertex $X \cup \{j\}$ must be in direction r (by Lemma 6.2.1), as shown in Figure 10.5(a). Then using the path move the r-edge moves to $X \cup \{n\}$, the n-edge moves to $X \cup \{j, n\}$, and the j-edge moves to $X \cup \{j\}$, as shown in Figure 10.5(b). Let T' be the resulting tree and let $\mathcal{D}' = (d'_1, \ldots, d'_{n-1})$ be its signature in $\mathcal{F}_n^{n-}$. Then

$$d'_j = d_j + 1, \quad d'_r = d_r - 1 \quad \text{and} \quad d'_\ell = d_\ell,$$

for all $\ell \neq j, r$. Let

$$\begin{aligned} X' &= (X \cup \{n\}) \setminus \{r, n\} \\ &= X \setminus \{r\} \\ &\subseteq [r-1] & \text{(since } r = \max X > j). \end{aligned}$$

Therefore T is transformed to T' such that r is chosen in $\mathcal{F}_n^{n+}$ at $X' \cup \{r, n\}$, where $X' \subseteq [r-1]$, and we may apply the case that follows.

Suppose now that $X \subseteq [j-1]$. We show that there exists a sequence of edge slides that transformed T into a tree where direction j is chosen at $\{j, n\}$. We consider the cases $4 \leq j \leq n-1$ and $j = 3$ separately.

Suppose that $4 \leq j \leq n-1$. Then the signature $\mathcal{D} = (\mathcal{I}_{(3,j)}^{(-1,-1)}, 2^j, \ldots, 2^{n-2})$ with the unsaturated part equal to $\mathcal{I}_{(3,j)}^{(-1,-1)}$. The edge slide graph $\mathcal{E}(\mathcal{I}_{(3,j)}^{(-1,-1)})$ is connected (by the inductive hypothesis). Let $P_{\mathcal{F}_n^{n+}}(X \cup \{j, n\}, j) = (X_0 \cup \{j, n\}, \ldots, X_\alpha \cup \{j, n\})$ be the j-retaining max remover path from $X_0 \cup \{j, n\} = X \cup \{j, n\}$ to $X_\alpha \cup \{j, n\} = \{j, n\}$. Since T has $2^{n-1} - 1$ edges in direction n and since $\psi_T(X \cup \{j, n\}) = j$, all the vertices of $\mathcal{F}_n^{n+}$ apart from $X \cup \{j, n\}$ are in direction n and therefore $\psi_T(X_a \cup \{j, n\}) = n$ for $a = 1, \ldots, \alpha$. Applying Lemma 7.2.5 at the vertex $X \cup \{j, n\}$, T can be transformed into T_α with $\psi_{T_\alpha}(X_\alpha \cup \{j, n\}) = j$, $\psi_{T_\alpha}(X_a \cup \{j, n\}) = n$ for $a = 0, \ldots, \alpha - 1$ and all other labels of T_α the same as the labels of T.

If $j = 3$, then $\mathcal{D} = (2, 3, 2, 8, \ldots, 2^{n-2})$ where (2, 3, 2) is an irreducible signature of Q_3. Then 3 is chosen at either $\{3, n\}$ or $\{\ell, 3, n\}$ for some $\ell \in \{1, 2\}$. If 3 is chosen at $\{3, n\}$, then no further moves are needed. Suppose that 3 is chosen at $\{\ell, 3, n\}$. Then all the vertices of $\mathcal{F}_n^{n+}$ apart from $\{\ell, 3, n\}$ are in direction n. Since (2, 3, 2) is irreducible, by Lemma 3.2.4 there exists a tree T'_3 of Q_3 with signature (2, 3, 2) where $\psi_{T'_3}(\{\ell, 3\}) = \ell$. Since the edge slide graph $\mathcal{E}(2, 3, 2)$ is connected (by Henden [6] or Lemma 4.3.6), we can move from $T_3 = T \cap Q_3$ to T'_3 using edge slides. Therefore T is transformed into T' with $\psi_{T'}(\{\ell, 3\}) = \ell$ and all the labels of T' apart from the labels of T'_3 the same as the labels of T. As shown in Figure 10.6, applying the V-move the labels n and 3 are swapped.

Therefore, for $j \geq 3$, we can always move from T to T_α with $\psi_{T_\alpha}(X_\alpha \cup \{j, n\}) = j$, $\psi_{T_\alpha}(X_a \cup \{j, n\}) = n$ for $a = 0, \ldots, \alpha - 1$ and all other labels of T_α the same as the labels of T.

We next show that there exists a sequence of edge slides that transforms T_α to a tree $\tilde{T}$ where $\tilde{T} \cap Q_{\{2,j,n\}}$ has signature (2, 2, 3). Since $(d_1, \ldots, d_j)$ is an irreducible signature, by Lemma 3.2.4 there exists a tree $T' \cap Q_j$ of Q_j with signature $(d_1, \ldots, d_j)$ where $\psi_{T'_j}(\{2, j\}) = 2$. Since the edge slide graph $\mathcal{E}(d_1, \ldots, d_j)$ is connected (either by the inductive hypothesis if $4 \leq j \leq n-1$, or by Henden [6] or Lemma 4.3.6 if $j = 3$), we can move from $T' \cap Q_j$ to such a tree using edge slides. Since all the vertices of $\mathcal{F}_n^{n+}$ apart from $\{j, n\}$ are in direction n, the vertices $\{2, j, n\}$ and $\{2, n\}$ are in direction n. Then $T \cap Q_{\{2,j,n\}}$ has signature (2, 2, 3).

Note that if $j = 1$ and is chosen at $\{1, n\}$, then T' has signature $(1, 3, 3, 8, 16, \ldots, 2^{n-1})$ in $\mathcal{F}_n^{n-}$ and therefore the only 1-edge in T_{n-} is chosen at $\{1\}$. Then 2 is chosen at $\{1, 2\}$ by Lemma 6.2.1. Since all the vertices of $\mathcal{F}_n^{n+}$ apart from $\{1, n\}$ are in direction n, the vertices $\{1, 2, n\}$ and $\{2, n\}$ are in direction n. Therefore the tree $T' \cap Q_{\{1,2,n\}}$ has signature (2, 2, 3).

As shown above we can always move to $\tilde{T}$ where $\tilde{T} \cap Q_{\{2,j,n\}}$ has signature (2, 2, 3). As shown in Figure 10.7 using edge slides we can move to a tree $T'' \cap Q_{\{2,j,n\}}$ where 2 is chosen at $\{2, n\}$, j is chosen at $\{2, j\}$, n is chosen at $\{j, n\}$. Therefore $\tilde{T}$ is transformed into T'' where 2 is chosen at $\{2, n\}$, j is chosen at $\{2, j\}$, n is chosen at $\{j, n\}$, and all other labels of T'' are the same as the labels of $\tilde{T}$.

We now show that using edge slides T'' can be transformed into $T^\star$. Since $\psi_{T''}(\{2, n\}) = 2$, the tree $T'' \cap Q_3$ has signature (2, 2, 3). Since the edge slide graph of signature (2, 2, 3) is connected, we can move from $T'' \cap Q_3$ to the tree $T^\star \cap Q_3$ where 2 is chosen at $\{1, 2, 3\}$, 1 is chosen at $\{1, 2\}$ and 3 is chosen $\{1, 3\}$ and $\{2, 3\}$. Then T'' is transformed into $T^\star$ where 2 is chosen at $\{1, 2, 3\}$, 1 is chosen at $\{1, 2\}$, 3 is chosen $\{1, 3\}$ and $\{2, 3\}$ and all other labels of $T^\star$ are the same as the labels of T''.

Finally, by the definition of the edge slide graph $\mathcal{E}(\mathcal{I}_{(3,n)}^{(-1,-1)})$, there is a sequence of edge slides from T to $T^\star$, hence T and $T^\star$ are in the same connected component of $\mathcal{E}(\mathcal{I}_{(3,n)}^{(-1,-1)})$. Since T is arbitrary, the edge slide graph $\mathcal{E}(\mathcal{I}_{(3,n)}^{(-1,-1)})$ is connected. □

10.2.2 Counting the number of upright spanning trees of Q_n with signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$

The following result provides the number of upright spanning trees of Q_n with signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$.

Theorem 10.2.2. *Let $\tau(\mathcal{I}_{(3,n)}^{(-1,-1)})$ be the number of upright spanning trees of Q_n with signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$. Then*

$$\tau(\mathcal{I}_{(3,n)}^{(-1,-1)}) = (2^{n-1} \times 5) \prod_{j=2}^{n-3} (2^j + 1), \tag{10.2}$$

for all $n \geq 4$.

Proof. Our proof is by induction on n.

We first show the result is true for the base case $n = 4$. In this case

$$\begin{aligned}(2^{n-1} \times 5) \prod_{j=2}^{n-3} (2^j + 1) &= 2^{4-1} \times 5 \\ &= 40.\end{aligned}$$

Since $\tau(\mathcal{I}_{(3,4)}^{(-1,-1)}) = 40$ by Lemma 5.4.1, Equation (10.2) is true for $n = 4$.

We now assume that Equation (10.2) holds for $n = k$ for some $k \geq 4$, that is

$$\tau(\mathcal{I}_{(3,k)}^{(-1,-1)}) = (2^{k-1} \times 5) \prod_{j=2}^{k-3} (2^j + 1).$$

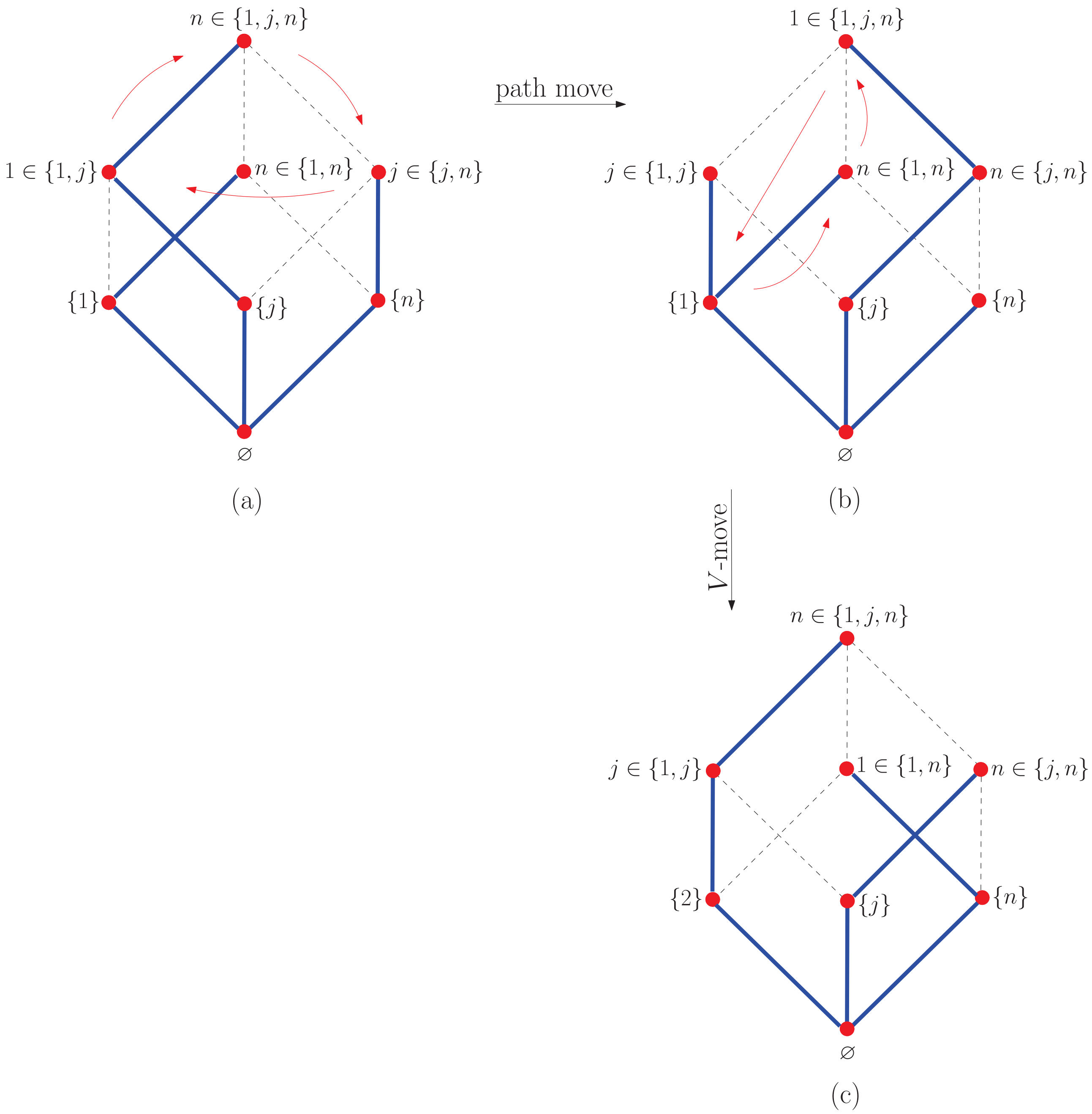


Figure 10.4: Diagram 4 for the proof of Theorem 10.1.1. (a) A tree of $Q_{\{1,j,n\}}$ with signature (2, 2, 3) where direction n is chosen at $\{1, j, n\}$ and $\{1, n\}$, direction j is chosen at $\{j, n\}$ and direction 1 is chosen at $\{1, j\}$. (b) After applying the path move, the label j goes to $\{1, j\}$, label 1 goes to $\{1, j, n\}$ and the label n goes to $\{j, n\}$. (c) After applying the V-move the labels n and 1 are swapped.

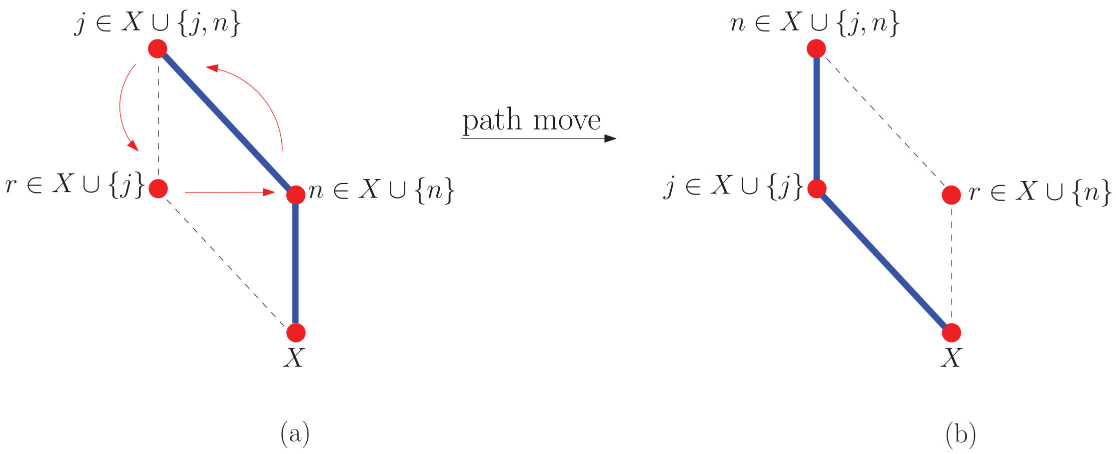


Figure 10.5: Diagram 1 for the proof of Theorem 10.2.1. (a) A face of Q_n with direction j is chosen at $X \cup \{j, n\}$, direction r is chosen at $X \cup \{j\}$, and direction n is chosen at $X \cup \{n\}$. (b) After applying the path move, the label r goes to $X \cup \{n\}$, the label n goes to $X \cup \{j, n\}$ and the label j goes to $X \cup \{j\}$.

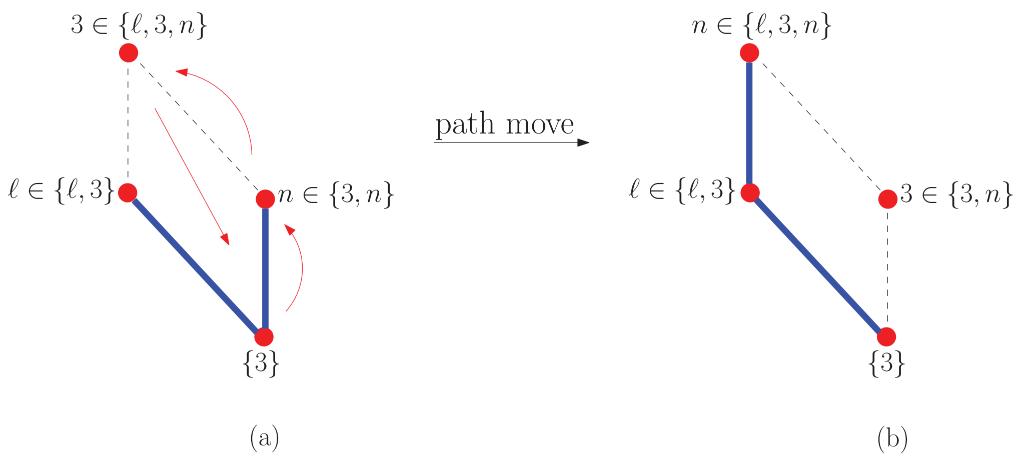


Figure 10.6: Diagram 2 for the proof of Theorem 10.2.1. (a) A face of Q_n with direction 3 is chosen at the vertex $\{\ell, 3, n\}$, direction ℓ is chosen at $\{\ell, 3\}$ and n is chosen at $\{3, n\}$. (b) After applying the V-move, the labels 3 and n are swapped.

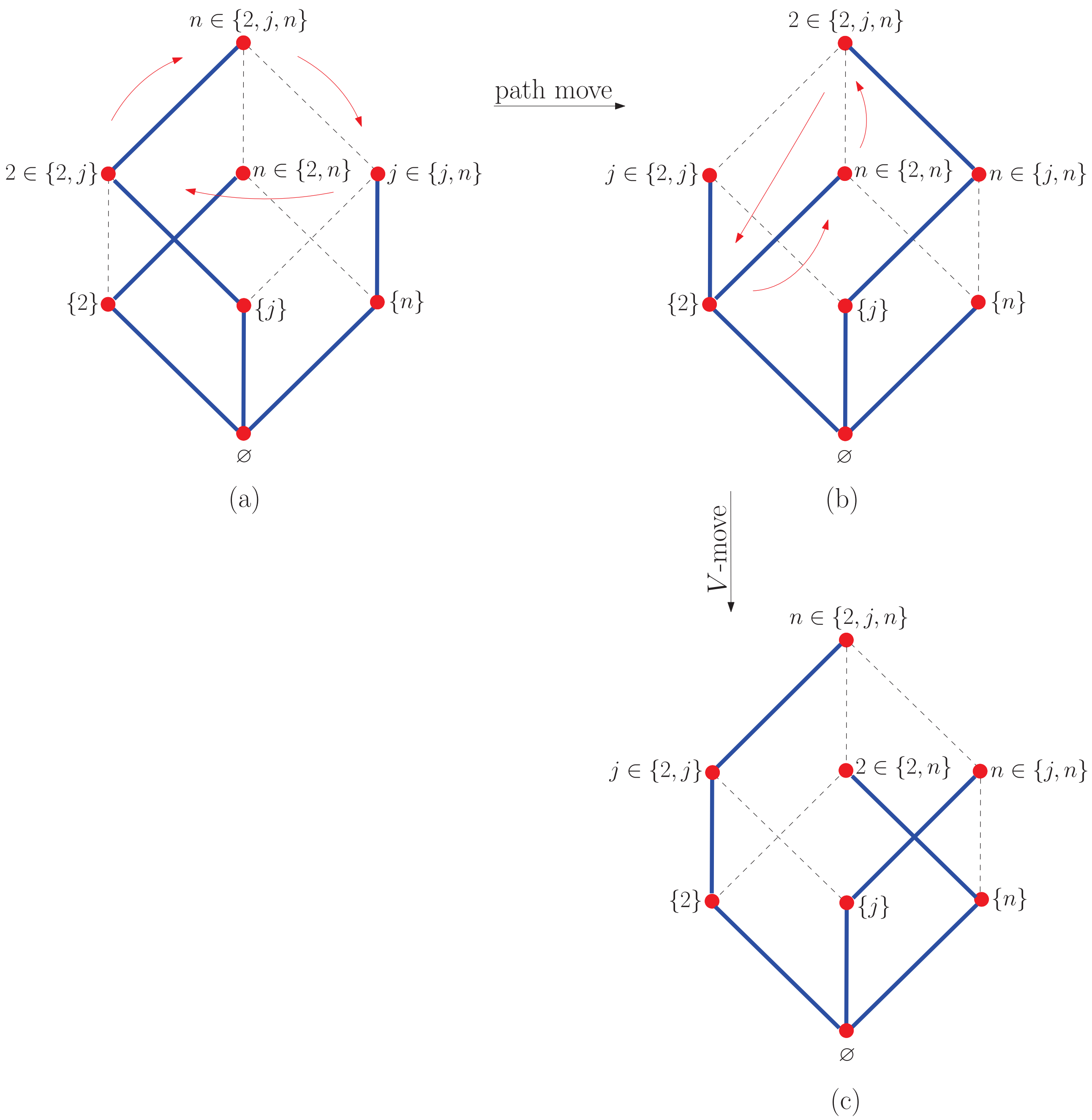


Figure 10.7: Diagram 3 for the proof of Theorem 10.2.1. (a) A tree of $Q_{\{2,j,n\}}$ with signature (2, 2, 3) where direction j is chosen at $\{j, n\}$, direction n is chosen at $\{2, j, n\}$ and $\{2, n\}$, and direction 2 is chosen at $\{2, j\}$. (b) After applying the path move, the label j goes to $\{2, j\}$, label 2 goes to $\{2, j, n\}$ and the label n goes to $\{j, n\}$. (c) After applying the V-move the labels n and 2 are swapped.

We prove that Equation (10.2) holds for $n = k+1$, that is

$$\tau(\mathcal{I}^{(-1,-1)}_{(3,k+1)}) = (2^{(k+1)-1} \times 5) \prod_{j=2}^{(k+1)-3} (2^j + 1).$$

Since

$$\tau(\mathcal{I}^{(-1,-1)}_{(3,k)}) = (2^{k-1} \times 5) \prod_{j=2}^{k-3} (2^j + 1),$$

by Lemma 10.1.3 we have

$$\begin{aligned}
\tau(\mathcal{I}^{(-1,-1)}_{(3,k+1)}) &= (2^{(k+1)-2} + 2)\tau(\mathcal{I}^{(+1,-1)}_{(2,k)}) \\
&= (2^{(k+1)-2} + 2)(2^{k-1} \times 5) \prod_{j=2}^{k-3} (2^j + 1) \\
&= (2^{k-1} + 2)(2^{k-1} \times 5) \prod_{j=2}^{k-3} (2^j + 1) \\
&= 2(2^{k-2} + 1)(2^{k-1} \times 5) \prod_{j=2}^{k-3} (2^j + 1) \\
&= (2^k \times 5) \prod_{j=2}^{k-2} (2^j + 1).
\end{aligned}$$

□

Chapter 11

Discussion

In this thesis we study the edge slide graph of Q_n with the goal of proving the following conjecture:

Conjecture 6.3.1. *Let $\mathcal{S} = (a_1, \dots, a_n)$ be a signature of Q_n. Then the edge slide graph $\mathcal{E}(S)$ is connected if and only if $\mathcal{S}$ is irreducible or quasi-irreducible.*

Chapter 6 establishes the 'only if' direction and Chapters 7–9 present substantial partial progress toward the 'if' direction. By Corollary 6.2.6 it suffices to consider the irreducible case. The underlying strategy is inductive with the base case given by the signatures (1) and (2, 2, 3), which up to permutation are the only irreducible signatures for $n \leq 3$. The edge slide graph of signature (1) consists of a single vertex, and the connectivity of signature (2, 2, 3) was established by Henden [6] as well as Lemma 4.3.6 in this thesis. Our main result is reducing the inductive step to the problem of proving that each irreducible signature has a splitting signature for which the upright spanning trees of that signature form a block, as we now explain.

Given an irreducible signature $\mathcal{I}$ of Q_n, we may suppose without loss of generality $\mathcal{I}$ is ordered. Suppose that signature $\mathcal{I}$ has an ordered irreducible splitting signature $\mathcal{D}$ such that the set of upright spanning trees with signature $\mathcal{I}$ and splitting signature $\mathcal{D}$ forms a block. Then under the inductive hypothesis the edge slide graph of signature $\mathcal{I}$ is connected, as shown in the following theorem.

Theorem 11.1. *Let $n \geq 4$ and let $\mathcal{I} = (a_1, \dots, a_n)$ be an ordered irreducible signature of Q_n. Suppose that every irreducible signature of Q_k is connected for all $k < n$. Suppose that $\mathcal{I}$ has an ordered irreducible splitting signature $\mathcal{D} = (d_1, \dots, d_{n-1})$ such that the set of upright spanning trees of Q_n with signature $\mathcal{I}$ and splitting signature $\mathcal{D}$ forms a block. Then the edge slide graph of signature $\mathcal{I}$ is connected.*

Proof. Let $\hat{T}$ be an upright spanning tree such that $\hat{T}_{n-}$ has the splitting signature $\mathcal{D}$. Let T be a spanning tree of Q_n with signature $\mathcal{I}$. After a sequence of edge slides (if necessary) we may assume T is upright. Moreover, by Theorem 9.1.2 after a series of edge slides we may further assume that T_{n-} has an irreducible signature. If T_{n-} has an irreducible splitting signature, then by Corollary 8.3.1 we can apply edge slides until we reach signature $\mathcal{D}$. Let T'' be the resulting tree where T''_{n-} has signature $\mathcal{D}$. If $T'' \neq \hat{T}$, then T'' can be transformed into $\hat{T}$ by our hypothesis that the upright spanning trees with signature $\mathcal{I}$ and splitting signature $\mathcal{D}$ form a block.

Finally, by the definition of the edge slide graph $\mathcal{E}(\mathcal{I})$, there is a sequence of edge slides from T to $\hat{T}$, hence T and $\hat{T}$ are in the same connected component of $\mathcal{E}(\mathcal{I})$. Since T was arbitrary, the edge slide graph $\mathcal{E}(\mathcal{I})$ is connected. □

To prove the inductive step it therefore suffices to prove the condition of Theorem 11.1 for all irreducible signatures:

Theorem 11.2. *Under the inductive hypothesis of Theorem 11.1 suppose that every ordered irreducible signature of Q_n has an ordered irreducible splitting signature such that the set of upright spanning trees with this splitting signature forms a block. Then the edge slide graph of every irreducible signature of Q_n is connected.*

Proof. Let $\mathcal{I}$ be an irreducible signature of Q_n, and let $\mathcal{I}'$ be an ordered permutation of $\mathcal{I}$. By hypothesis $\mathcal{I}'$ has an ordered irreducible splitting signature $\mathcal{D}$ such that the set of upright spanning trees of Q_n with signature $\mathcal{I}'$ and splitting signature $\mathcal{D}$ forms a block. Then by Theorem 11.1 the edge slide graph of signature $\mathcal{I}'$ is connected, and then by Lemma 2.2.20 the edge slide graph of signature $\mathcal{I}$ is connected. □

Thus, to complete the proof of Conjecture 6.3.1 it suffices to prove under the inductive hypothesis that every irreducible signature $\mathcal{I}$ has a splitting signature $\mathcal{D}$ such that the upright spanning trees with signature $\mathcal{I}$ and splitting signature $\mathcal{D}$ lie in a single component. We have established this step for certain classes of signatures, including all irreducible signatures of Q_4. We therefore have:

Corollary 11.3. *The edge slide graph of every irreducible signature of Q_4 is connected.*

Proof. The inductive hypothesis of Theorem 11.1 holds for $n = 4$ by our discussion above. Let $\mathcal{I} = (a_1, a_2, a_3, a_4)$ be an ordered irreducible signature of Q_4. Then $\mathcal{I}$ has the splitting signature (2, 2, 3) and by either Theorem 7.3.1 or Theorem 7.3.2, according to whether (2, 2, 3) is a unidirectional splitting signature or has multiple directions in $\mathcal{F}_4^{4+}$, the set of upright spanning trees with signature $\mathcal{I}$ and the splitting signature (2, 2, 3) forms a block. Therefore the edge slide graph of every irreducible signature of Q_4 is connected, by Theorem 11.2. □

In the general case, we have shown in Corollary 5.6.14 that for $n \geq 5$ every irreducible signature $\mathcal{I}$ has a unidirectional splitting signature or a super rich splitting signature. The unidirectional case is covered by Theorem 7.3.2, and we have:

Corollary 11.4. *Under the inductive hypothesis of Theorem 11.1 suppose that $\mathcal{I}$ has a unidirectional splitting signature. Then the edge slide graph of signature $\mathcal{I}$ is connected.*

Combining Corollary 11.3 with Corollary 11.4 we get:

Theorem 11.5. *Let $\mathcal{I} = (a_1, a_2, a_3, a_4, a_5)$ be an irreducible signature of Q_5 such that $\mathcal{I}$ has a unidirectional splitting signature. Then $\mathcal{I}$ is connected.*

We have also proved Conjecture 6.3.1 holds for two infinite families of irreducible signatures, namely $\mathcal{I}_n^{(-1)}$ and $\mathcal{I}_{(3,n)}^{(-1,-1)}$. When $\mathcal{I}$ does not have a unidirectional splitting signature it has a super rich splitting signature, and under the inductive hypothesis our results in Chapter 7 establish several tools for rearranging the labels of a tree with a super rich splitting signature. In particular, Lemma 7.3.6 shows that if T and T' are settled and agree at all vertices of $\mathcal{F}_n^{n+}$ of level $\alpha + 1$ and above, then we can increase by one the number of vertices that agree at level α, without affecting the vertices of higher levels. This gives us grounds to believe the following conjecture:

Conjecture 7.3.3. *Let $n \geq 5$, and suppose that every irreducible signature of Q_{n-1} is connected. Let $\mathcal{I} = (a_1, \dots, a_n)$ be an irreducible signature of Q_n with $a_n \geq a_t$ for all $t \in [n-1]$. Let $\mathcal{D} = (d_1, \dots, d_{n-1})$ be a super rich splitting signature of $\mathcal{I}$ with respect to n. Let $\mathcal{B}$ be the set of upright spanning trees of Q_n with signature $\mathcal{I}$ and signature $\mathcal{D}$ in $\mathcal{F}_n^{n-}$. Then $\mathcal{B}$ forms a block.*

Establishing Conjecture 7.3.3 would complete the proof of the inductive step, completing the proof of Conjecture 6.3.1. The difficulty with our current approach to proving Conjecture 7.3.3 is that Lemma 7.3.6 requires the trees to be settled. At present we are unable to ensure the hypothesis that the tree is settled is still satisfied after applying Lemma 7.3.6, without potentially destroying the progress already made in bringing the trees into agreement. We have not been able to resolve this issue within the time frame of this research. We believe the approach we had in Chapter 7 could be used with some improvements to completely prove the conjecture.

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