Minimising Weighted Mean Distortion

A thesis presented in partial fulfilment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

at Massey University, Albany, New Zealand

Maarten Nicolaas M^{c}Kubre-Jordens

2009

ABSTRACT

There has been considerable recent interest in geometric function theory, nonlinear partial differential equations, harmonic mappings, and the connection of these to minimal energy phenomena. This work explores Nitsche's 1962 conjecture concerning the nonexistence of harmonic mappings between planar annuli, cast in terms of distortion functionals. The connection between the Nitsche problem and the famous Grötzsch problem is established by means of a weight function. Traditionally, these kinds of problems are investigated in the class of quasiconformal mappings, and the assumption is usually made *a priori* that solutions preserve various symmetries. Here the conjecture is solved in the much wider class of mappings of finite distortion, symmetry-preservation is proved, and ellipticity of the variational equations concerning these sorts of general problems is established. Furthermore, various alternative interpretations of the weight function introduced herein lead to an interesting analysis of a much wider variety of critical phenomena — when the weight function is interpreted as a thickness, density or metric, the results lead to a possible model for tearing or breaking phenomena in material science. These physically relevant critical phenomena arise, surprisingly, out of purely theoretical considerations.

ACKNOWLEDGEMENTS

First and foremost, my thanks to my supervisor Gaven Martin, for his valuable and sage guidance and support. This work is an ode to his skill as supervisor.

To the people who have helped me throughout my study by contributing their time and knowledge — my co-supervisor Carlo Laing, Graeme Wake, Winston Sweatman, Shaun Cooper, Chris Tuffley, and many others — thank you.

I am grateful also to the various institutes who have hosted me, funded me, or otherwise contributed. These are: the New Zealand Institute for Mathematics and its Applications, who have generously funded my study; Massey University's Institute of Information and Mathematical Sciences, who have hosted me, providing space, computer equipment, access to Massey University's resources, and support for conferences; and the New Zealand Institute for Advanced Study, which has provided support in many ways.

I am especially grateful and indebted to the love of my life, my wife Alexandra, for her tireless dedication and support, and for the mammoth task of proofreading at the end. Special thanks to my fellow postgraduate students, who had the tenacity both to put up with me and to let me put up with them.

The invaluable support I have received from a much longer list of people, academic and otherwise, should not go unmentioned. Those who I have acknowledged here by name are but the tip of the proverbial iceberg. A thesis takes much more than talent or hard work: it takes supporters of all kinds.

Dedicated to the memory of Jan Vermeulen, my Opa.
24 September 1932 – 21 September 2009

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INTRODUCTION

In 1962, J. C. C. Nitsche [34] made a conjecture concerning harmonic mappings of annuli in the plane. This conjecture has recently been solved in [21]. Nitsche's conjecture is that the target annulus could not be too thin relative to a given fixed domain annulus, or there would be no harmonic mappings. This research shows that the inverses to these mappings are the mappings of minimal mean distortion between doubly connected regions (perhaps composed with some conformal mappings). Within this work, the conjecture is cast in terms of the inverse of these mappings, and it is confirmed to hold in specific cases. Furthermore, the thesis extends it and generalises the results to a much wider variety of problems.

The thesis solves the problem of minimising various convex functions of mean distortion of a domain under various kinds of mapping. Conformal and quasiconformal mappings have found extensive use in kinetic and elasticity theory—in particular, in geometric function theory and nonlinear partial differential equations (PDEs) — concerning how materials deform when physical processes occur (for example, heating, stressing or straining a material). Mappings of finite distortion are a natural generalisation of the quasiconformal class, and it is in this class of mappings that we shall look for solutions.

Consider deforming a doubly connected region in the plane with some given conformal metric. A question one is urged to ask, and which this thesis will investigate, is whether it is possible to attain a particular final deformation while minimising some weighted average of the local conformal distortion. The distortion function, and the conformal metric, may be taken to represent certain physical properties of the material; for example, the true shape of a physical structure and its local anisotropic stretching.

Sharp results for classical distortion were revealed in the project preceding this thesis [24], but only a limited case was studied, and under a number of assumptions. In the research presented here, sharp results are obtained in the wider class of mappings of finite distortion.

Furthermore, it is possible to interpret the weight function on the distortion in a number of ways; one way leads to drawing an equivalence between Nitsche phenomena and another famous problem, due to Grötzsch. Other interpretations lead to models for tearing and similar phenomena in material science, or to finding whether such critical phenomena occur in spaces of various different curvatures.

This thesis shows that, perhaps surprisingly, very specific and physically relevant predictions about critical phenomena (tearing, breaking etc.) can be made by merely considering purely theoretical, analytic points of view. Moreover, the main theorem embodied in this work explains how and why these types of phenomena arise, and admits a qualitative analysis of where such phenomena occur. It is possible to interpret the main theorem (Theorem 3.3) as showing how critical phenomena may occur in various geometric settings, allowing these results to become widely used in analysis of manifolds and their behaviour.

A brief overview

Chapter 1 lays down most of the pertinent analytical foundations of the thesis. In particular, a short survey of the Riemann Mapping Theorem and its classical proof is presented and the key elements of the proof are outlined. Proofs of this type are standard and powerful in the more general setting of quasiconformal analysis. Some important techniques from the calculus of variations are set out here. These techniques are often used in finding extremals of functionals that reflect physical situations: the famous brachistochrone; problems in the theory of potentials; and many others. The chapter also observes that harmonic mappings, which arise naturally in many situations, are solutions to Dirichlet type problems (when solutions exist) and satisfy variational equations.

Motivation for the particular extremal problems found herein are furthered in Chapter 2, where various kinds of minimisation problems are discussed, and some known results are outlined. Nitsche's 1962 conjecture, as it is applicable to the situation considered here, is expounded; the equivalence of Nitsche-type problems and those of Grötzsch is also shown.

The main theorem of this thesis is proved in Chapter 3. After first surveying a proof for a similar, special case; it transpires that, to generalise, a different technique involving the use of a key inequality needs to be employed. The inequality is chosen in such a way as to make the last part of an estimate on an integral vanish, equality holding only in the special case of the minimiser of mean distortion. It is shown that natural generalisations of Nitsche's conjecture are true for the L^1 norm, but false for the L^p norms with $p > 1$. This motivates analysing the theorem in particular cases, demonstrating how interesting critical phenomena may arise and showing cases in which they do not. The end of this chapter contains preliminary calculations and attempts to generalise the theory further that may prove fruitful for future projects.

Chapter 4 then exposes the ellipticity condition on the variational equations. Perhaps surprisingly, in the weighted case, the conditions for ellipticity are the same as in the unweighted case — irrespective of the particular (positive, finite) weight function. Some conjectures are made at further conditions that, when placed on the weight function, may yield the presence or absence of critical phenomena.

The penultimate chapter, Chapter 5, contains calculations of particular examples of the weight function as well as some illustrative ideas on how to analyse when and where critical phenomena arise. A possible connection to the Frank-Kamenetskii theory of thermal ignition is noted, thereby linking the behaviour of deformations within metrics to critical phenomena that occur in other areas of physical science.

In Chapter 6 the thesis draws to a close with some remarks about behaviour of minimisers near the minimum of the weight function. The last section suggests directions for future research.

1. PRELIMINARIES

This chapter outlines the necessary analytical background for the rest of the thesis and reviews briefly some of the work that has been done in this area. More specific literature relevant directly to later chapters will be reviewed within those chapters. Most of the theorems in this chapter will be laid out without proof; the appendix and the references provide more details. There is also a basic list of notation in the appendix.

The concepts of the various arithmetic operations, exponents, absolute value, conjugation, elementary aspects of the theory of real functions of real variables and notions of limits and continuity will be taken as primitives and assumed. Let us begin with differentiation of complex functions, advancing quickly into deeper study.

Definition 1.1. *Let* $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ *denote the complex plane. The* ***derivative of*** f ***at*** z_0 *is*

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

or, equivalently,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

if this limit exists, and otherwise is undefined.

As is well-known, the derivative of a real function of complex variables is either zero or does not exist. The case of a complex function of a real variable yields simply $z'(t) = x'(t) + iy'(t)$ if $z(t) = x(t) + iy(t)$. But the existence of the derivative of a complex function of a complex variable has many structural consequences.

Firstly, $f(z)$ must be continuous wherever it has a derivative; the numerator of the definition is, essentially, a paraphrase of this condition. If further we write $f(z) = u(z) + iv(z)$ where u and v are real-valued, it follows that necessarily u and v are continuous if $f'(z)$ exists.

It is easily shown (e.g. [2]) that the usual rules for derivatives (the Product Rule, the Chain Rule, etc.) for real-valued functions carry over to complex functions of complex variables without significant modification.

1.1 Holomorphic functions

The class of *holomorphic functions* (also sometimes known as *analytic*, *complex differentiable* or simply *differentiable functions*) is formed simply by the collection of all complex functions of a complex variable that have a derivative wherever they are defined:

Definition 1.2. *A function $f(z)$ is **holomorphic on** $\Omega \subset \mathbb{C}$ if and only if it is differentiable at each $z \in \Omega$; $f(z)$ is **entire** if and only if it is analytic on $\mathbb{C}$.*

Notable in the definition of the derivative is that it does not matter how z_0 is approached; this leads to the formulation of the *Cauchy-Riemann* equations; a proof of the following theorem may be found in [2, 38].

Theorem 1.3. *Let $f(z) = f(x, y) = u(x, y) + iv(x, y)$ where $z = x + iy$, with x, y, u and v real. Then f is holomorphic if and only if it, together with u and v, satisfies*

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

or, equivalently,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{1.1}$$

the Cauchy-Riemann equations.

The existence of these partial derivatives is implied by the existence of $f'(z)$; the continuity of u and v must be assumed to prove the implication in the other direction. Notice that using this theorem, we can write the derivative of f in several ways; the simplest is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Now, by (1.1),

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

This shows that for holomorphic f, $|f'(z)|^2$ is the Jacobian $J(z, f)$ of u and v with respect to x and y for holomorphic f.

The derivative of a holomorphic function is itself holomorphic. A proof of this is more difficult and uses Cauchy's integral formula; see for example [3, pp. 120–122]. Using this fact, it is easy to see that u and v will have continuous partial derivatives of all orders, and their mixed derivatives will be equal. Thus

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Definition 1.4. *A function u is said to be **harmonic** if it satisfies Laplace's equation*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

If two harmonic functions u and v satisfy the Cauchy-Riemann equations (1.1)*, then they are said to be **conjugate harmonic functions**.*

The real and imaginary parts of a holomorphic function are therefore conjugate harmonic functions.

One may consider a complex function $f(x, y)$ of two real variables instead as a function of the complex variable $z = x + iy$ and its conjugate $\bar{z} = x - iy$. Observe that $x = \frac{1}{2}(z + \bar{z})$ and $y = -\frac{1}{2}i(z - \bar{z})$. Treating z and $\bar{z}$ as independent variables, we obtain

Definition 1.5. *The z- and $\bar{z}$-derivatives of f are given by*

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Note the introduced subscript notation for the derivatives. By comparison with the earlier formulation of holomorphic functions, it is immediate that a C^1 function is holomorphic if and only if its $\bar{z}$-derivative is zero; hence one may say that a holomorphic function is a function of z alone, independent of $\bar{z}$.

With a little work, it can be shown that a harmonic function u satisfies the formal differential equation

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

1.2 Conformal mappings & Riemann's Mapping Theorem

Loosely speaking, a conformal mapping is a mapping that preserves angles between tangents, both in size and orientation. Formally, we have

Definition 1.6. *Let f be a mapping of a region Ω into the plane and let $z_0 \in \Omega$. Suppose that z_0 has a deleted neighborhood $D'(z_0, r) \subset \Omega$ in which $f(z) \neq f(z_0)$. We say that f* ***preserves angles at*** *z_0 if*

$$\lim_{r \to 0} e^{-i\theta} \frac{f\left(z_0 + re^{i\theta}\right) - f(z_0)}{\left|f\left(z_0 + re^{i\theta}\right) - f(z_0)\right|}$$

for $r > 0$ exists and is independent of θ. For holomorphic functions, this condition is equivalent to having a nonvanishing derivative in Ω, and we call these functions ***conformal****.*

Properties of conformal mappings are many and their use in complex analysis is widespread. The use of conformal mappings in the present project will be limited to simplifying distortion problems. As will become evident in Section 2.7, an equivalence between two types of minimal-distortion problem can be found so that there are multiple approaches possible.

Often in the study of mappings of the complex plane (or, in fact, in mappings in $\mathbb{R}^n$) it is important to consider what quantities are conformally invariant – that is, invariant under a conformal mapping. One such quantity is the modulus of a curve family (see Section 1.4), an invariant widely used in the study of quasiconformal mappings.

1.2.1 Normal Families

Let $\mathscr{F}$ be a family of functions defined on a region Ω of the complex plane, and let d be the usual metric on the complex plane.

Definition 1.7. *The functions in a family $\mathscr{F}$ are* ***equicontinuous on a set*** *$S \subset \Omega$ if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $f \in \mathscr{F}$, $d(f(z), f(z_0)) < \varepsilon$ whenever $|z - z_0| < \delta$ with $z, z_0 \in S$.*

Note that this immediately implies that if $\mathscr{F}$ is equicontinuous on S, then each member of $\mathscr{F}$ is itself uniformly continuous on S.

Definition 1.8. *A family of functions $\mathscr{F}$ is* ***normal in*** *Ω if every sequence $(f_n)_{n \geqslant 0}$ of functions $f_n \in \mathscr{F}$ contains a subsequence that converges uniformly on every compact subset of Ω.*

The property of being normal is in a sense a compactness property; sometimes the term *relatively compact* is used to describe a normal family [3, p. 221]. It follows from this definition that should the space containing $\mathscr{F}$ be complete, then $\mathscr{F}$ is normal if and only if it is totally bounded, for the closure of $\mathscr{F}$ is compact if and only if it is complete and totally bounded. For more in-depth discussion, pick up most any textbook on complex analysis (there is a good selection in the bibliography).

1.2.2 Riemann's Mapping Theorem

This important theorem was the subject of Riemann's 1851 dissertation. His proof was incomplete, but later completed by other notable mathematicians such as Hilbert [32, p. 404]. It is now a standard component of almost any graduate course in complex analysis.

Theorem 1.9. ***Riemann's Mapping Theorem.*** *Every simply connected region Ω in the plane (other than the plane itself or the extended complex plane) is conformally equivalent to the open unit disc.*

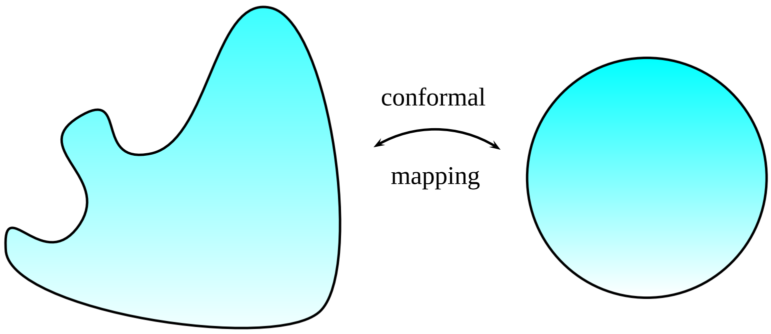

Fig. 1.1: The Riemann Mapping Theorem.

The exceptions to this theorem are: the complex plane, because a bounded analytic function that is entire on the whole complex plane is constant (by Liouville's Theorem, B.6), and thus the image of the complex plane cannot be the unit disc; and the extended plane, $\widehat{\mathbb{C}}$, because it is topologically distinct.

Proofs of this theorem generally involve the solution of a particular kind of extremal problem; that of holomorphic mappings of the region Ω into the unit disc (See e.g. [38, pp. 282–284]). The argument comprising the proof is lengthy, but can be briefly outlined as follows:

1. Show that the family $\mathscr{F}$ of (one-to-one) holomorphic functions mapping Ω *into* the unit disc is nonempty.

2. Prove that the family $\mathscr{F}$ is normal, by establishing equicontinuity on compact subsets and invoking the Arzelà-Ascoli Theorem (B.10) .

3. For a fixed $z \in \Omega$, take a function $f_0 \in \mathscr{F}$ and establish that if f_0 does not cover the whole unit disc, then there exists $f_1 \in \mathscr{F}$ with the property $|f_1'(z)| > |f_0'(z)|$.

4. Take s to be the supremum of the set $\{f'(z_0)|f \in \mathscr{F}\}$ for some $z_0 \in \Omega$.

5. The limit of a sequence of functions in $\mathscr{F}$ for which the derivative at z_0 tends to s is the function required; that is, a conformal mapping taking Ω onto the unit disc – seen to be *onto* since otherwise it would contradict the third step in the construction.

For a more detailed analysis, see [3, 10, 29, 35, 38].

It is important to note this argument as a tool used in this classic proof: a normal families type argument to establish the existence of some extremising function. This method is used to great effect in many other extremal problems and is a useful technique in analysis of conformal or quasiconformal problems.

The handiness of a modulus of continuity type argument is that those functions which share the same modulus of continuity are exactly equicontinuous families. In general the rôle of a modulus of continuity ω is to fix some explicit functional dependence of δ to ε in the ε-δ definition of uniform continuity. For example, functions for which $\omega(t) = kt$ are k-Lipschitz, and those functions that are Hölder continuous satisfy $\omega(t) = kt^\alpha$ — and in fact quasiconformal mappings are Hölder continuous functions.

1.3 Sobolev spaces

The kinds of functionals dealt with in this thesis result in differential equations of some finite degree; the solutions to these equations have traditionally been investigated in some appropriate class of continuous functions. However, solutions to differential equations are examined more naturally in *Sobolev spaces*; modern literature reflects this (for example, [33]).

Loosely speaking, a Sobolev space is a space of functions that are weakly differentiable to a certain degree. More precisely,

Definition 1.10. *The* ***Sobolev space*** $W^{k,p}_{loc}(\Omega, \Omega')$ *is the set of equivalence classes of all functions* $f \in L^p(\Omega, \Omega')$ *such that* f*, together with its weak derivatives up to order* k*, have locally finite* L^p *norm.*

The equivalence classes here are understood to be those functions (and derivatives) that are equal almost everywhere (i.e. their L^p norm is the same).

It is important that the derivative in this definition is understood in a suitably weak sense:

Definition 1.11. *The i^{th} weak derivative of $f \in L^1_{loc}$ is understood as $f^{(i)} \in L^1_{loc}$ with*

$$\int f(t)\, \varphi^{(i)}(t)\, dt = (-1)^i \int f^{(i)}\, \varphi(t)\, dt,$$

for all i times continuously differentiable compactly supported φ.

This sense of weak derivative is recognisably motivated by the integration by parts technique. Since a function's weak derivative is found in a Lebesgue class, it is unique up to a set of measure zero. As with the L^p spaces, the Sobolev space with $p = 2$ forms a Hilbert space, $W^{k,2}$.

The natural norm on the space $W^{k,p}$ (making $W^{k,p}$ a Banach space) is the norm

$$||f||_{k,p} = \begin{cases} \left(\sum_{i=0}^{k} \int |f^{(i)}(t)|^p dt \right)^{\frac{1}{p}} & \text{if} \quad 1 \leqslant p < \infty; \\ \max_{0 \leqslant i \leqslant k} ||f^{(i)}||_\infty & \text{if} \quad p = \infty. \end{cases}$$

but it is actually sufficient to take only the first and last in the sequence; i.e. this norm is equivalent to the norm $||f||_{k,p} = ||f||_p + ||f^{(k)}||_p$; see for instance [33].

1.4 Quasiconformal mappings & moduli of curve families

Suppose that Γ is a curve family in $\mathbb{R}^n$ (i.e. elements of Γ are curves in $\mathbb{R}^n$).We recount a little of the theory of quasiconformal mappings before moving on to their more general counterparts.

Definition 1.12. *A non-negative Borel function* $\rho : \mathbb{R}^n \to \mathbb{R}$ *is* ***admissible*** *if*

$$\int_\gamma \rho \, ds \geqslant 1$$

for every locally rectifiable curve $\gamma \in \Gamma$. *Denote the collection of all admissible functions on* Ω *by* $\mathrm{adm}(\Omega)$.

Definition 1.13. *For each* $p \geqslant 1$, *set the* p***-modulus of*** Ω

$$M_p(\Omega) = \inf_{\rho \in \mathrm{adm}(\Omega)} \int_{\mathbb{R}^n} \rho^p \, dm.$$

When $p = n$, *we simply write* $M(\Omega)$ *as the* ***modulus*** *of* Ω.

Note that $0 \leqslant M_p(\Omega) \leqslant \infty$. If it so happens that $\mathrm{adm}(\Omega) = \emptyset$, take $M_p(\Omega) = \infty$ – though this case will not occur in this thesis. Often, in the literature, the related concept of *extremal length* ($\lambda(\Gamma)$) is used; but simply $\lambda(\Gamma) = 1/M(\Gamma)$ so its use and discussion are practically interchangeable. This thesis uses whichever is appropriate within context. Also, note that M_p is an outer measure in the space of all curves in $\mathbb{R}^n$ (for a proof, see [42]).

Theorem 1.14. *The modulus of a curve family is conformally invariant.*

Proof. (see e.g. [42, p. 25].) Consider a conformal homeomorphism $f : \Omega \to \Omega'$, from a region Ω to another region Ω' in $\mathbb{R}^n$. Let Γ be the family of curves contained inside the region Ω, and likewise set $\Gamma' = f\Gamma$ (contained inside Ω'). The differential matrix Df gives rise to the Jacobian determinant of f:

$$|Df|^n = \det(Df) = J(x, f). \tag{1.2}$$

Since $|Df|$ is the largest eigenvalue of the differential matrix Df, observe that because $|\lambda_{\max}|^n = |\lambda_1 \lambda_2 \ldots \lambda_n|$, $|Df|$ must be (a multiple of) an orthogonal matrix. Choose $\rho \in \mathrm{adm}(f\Gamma)$. Then $\rho(f(x)) : \Omega \to \mathbb{R}^+ \cup \{0\}$ (ρ is extended by 0 to $\mathbb{R}^n$, but this contributes nothing to the integral, so we may consider ρ to be defined on Ω only). Pick $\gamma \in \Gamma$. Then $\gamma \circ f = \gamma' \in \Gamma'$. Now

$$1 \leqslant \int_{\gamma'} \rho \, ds = \int_{\gamma \circ f} \rho \, ds \leqslant \int_\gamma \rho(f)|Df| \, ds$$

so that $\rho(f)|Df| \in \text{adm}(\Gamma)$. Hence, using (1.2),

$$M(\Gamma) \leqslant \int_{\Omega} \rho^n(f)|Df|^n \, dx = \int_{\Omega} \rho^n(f) J(x,f) \, dx = \int_{\Omega'} \rho^n \, dm(x)$$

by change of variables. Taking the infimum of this over all admissible ρ, we get that $M(\Gamma) \leqslant M(\Gamma')$. Now, we may repeat this process of obtaining an upper bound on the modulus $M(\Gamma')$, using f^{-1} instead. Since f is conformal, so is f^{-1}, and we obtain the other inequality, $M(\Gamma') \leqslant M(\Gamma)$. Hence, in fact, $M(\Gamma) = M(\Gamma')$. □

Computing $M(\Gamma)$ can often be difficult, but considerations of the geometry of the situation often greatly simplify the task. Furthermore, an upper bound is often considerably easier to find. In particular, note that for $\rho \in \text{adm}(\Gamma)$,

$$M_p(\Gamma) \leqslant \int \rho^p \, dm.$$

Note that if $\ell(\gamma) \geqslant r > 0$ for each $\gamma \in \Gamma$ (where the γ all lie in some Borel set G), then

$$M_p(\Gamma) \leqslant \frac{m(G)}{r^p}. \tag{1.3}$$

For, defining $\rho : \mathbb{R}^n \to \mathbb{R}$ by $\rho(x) = \frac{1}{r}$ for $x \in G$ and $\rho(x) = 0$ otherwise, then $\rho \in \text{adm}(\Gamma)$, and the required inequality follows.

1.4.1 Modulus of an annulus

Here we calculate the modulus of a spherical ring and, in particular, for an annulus in the complex plane. The annulus plays a crucial rôle in later considerations.

Let A be a spherical ring, with inner radius a and outer radius b. Take the family of curves Γ to be the set of all curves joining the sphere of radius a to the sphere of radius b in A (see Fig. 1.2). Note that we could also consider the curves that join these two spheres that are not contained entirely in A, but the size of $\int \rho$ on such curves is always at least as big as the size of $\int \rho$ on curves that are contained in A. Also, the other curve family for the planar

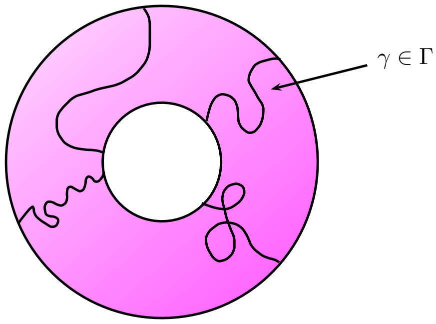

Fig. 1.2: A curve family for the planar annulus.

annulus is that of closed paths in A that circumnavigate the central circle; however, as we shall see later, the symmetry of the situation requires us to consider only the curve family joining the two boundaries.

What follows is a calculation from Väisälä [42, pp. 22–23].

Let $\rho \in \operatorname{adm}(\Gamma)$. Consider the lines $\gamma_u : [a, b] \to \mathbb{R}^n$, defined by $\gamma_u = tu$, where $t \in [a, b]$ and $u \in S^{n-1}$. We get

$$\int_{\gamma_u} \rho \, ds = \int_a^b \rho(tu) t^{\frac{n-1}{n}} t^{-\frac{n-1}{n}} \, dt$$

and therefore, by Hölder's inequality,

$$1 \leqslant \left(\int_{\gamma_u} \rho \, ds \right)^n \leqslant \int_a^b \rho^n(tu) t^{n-1} \, dt \left(\int_a^b t^{-1} \, dt \right)^{n-1}$$

$$= \left(\log \frac{b}{a} \right)^{n-1} \int_a^b \rho^n(tu) t^{n-1} \, dt.$$

Hence, as $b > a$,

$$\int_{S^{n-1}} \frac{1}{\left(\log \frac{b}{a}\right)^{n-1}} \, du \leqslant \int_{S^{n-1}} \int_a^b \rho^n(tu) t^{n-1} \, dt \, du = \int_A \rho^n \, dm$$

by change of variables (t^{n-1} is the Jacobian). Noting that

$$\int_{S^{n-1}} \frac{1}{\left(\log \frac{b}{a}\right)^{n-1}}\, du = \frac{\omega_{n-1}}{\left(\log \frac{b}{a}\right)^{n-1}},$$

where ω_{n-1} is the area of S^{n-1}, and taking the infimum over all ρ, we see that

$$M(\Gamma) \geqslant \frac{\omega_{n-1}}{\left(\log \frac{b}{a}\right)^{n-1}}.$$

On the other hand, define ρ by

$$\rho(x) = \begin{cases} \frac{1}{\left(|x| \log \frac{b}{a}\right)^{n-1}} & \text{if } x \in A, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\rho \in \mathrm{adm}(\Gamma)$, and so in fact

$$M(\Gamma) = \frac{\omega_{n-1}}{\left(\log \frac{b}{a}\right)^{n-1}}.$$

In the case of the annulus in the complex plane, we have $n = 2$ so that in particular, we have the following theorem:

Theorem 1.15. *If Γ is the curve family in the complex plane joining the circle of radius $a > 0$ to the circle of radius $b > a$ then*

$$M(\Gamma) = \frac{2\pi}{\log \frac{b}{a}}.$$

Remember that modulus and extremal length are reciprocal, and either will be used as necessary.

The modulus of a curve family is important to our discussion for two reasons. The first and foremost is that it is a conformal invariant (i.e. it is a quantity that has the same value for any conformal deformation), and so it gives us geometric information about how close to conformal a mapping is. The second reason is that the ratio of a to b is relevant in an important theorem from Schottky [9], which will help us to simplify an important class of mapping problems: that of conformal mappings between doubly connected regions. The relevance of this will become clear in Section 2.1.

1.4.2 Quasiconformality, geometrically and analytically.

Early work on quasiconformal mappings largely uses a definition of quasiconformality that can be generalised to the context of arbitrary metric spaces. The rationale of this is to give a bound on the eccentricity of the infinitesimal ellipsoids that result when infinitesimal balls are mapped (illustrated below); that is, a bound on local distortion.

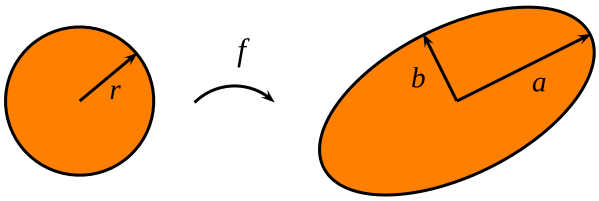

Fig. 1.3: Local action of a quasiconformal map.

Definition 1.16. *Let $f : X \to Y$ be a homeomorphism between two metric spaces $(X, d_X), (Y, d_Y)$. For $x \in X$ and $r > 0$, set*

$$L(x, r, f) = \sup \{d_Y(f(x), f(y)) : d_X(x, y) \leqslant r\}$$

and

$$l(x, r, f) = \inf \{d_Y(f(x), f(y)) : d_X(x, y) \leqslant r\}.$$

The ratio

$$H(x, r, f) = \frac{L(x, r, f)}{l(x, r, f)}$$

*is a measure of the eccentricity of the image of ball $B(x, r)$ under f. Then f is K**-quasiconformal at** x, $K \geqslant 1$, if*

$$\limsup_{r \to 0} H(x, r, f) = H(x, f) \leqslant K. \tag{1.4}$$

*We say f is K**-quasiconformal** if it is K-quasiconformal at every $x \in X$.*

Naturally we are normally interested in the smallest number K that will satisfy (1.4); and when the term 'K-quasiconformal' is used it will normally imply that K is the smallest number that will do so. Homeomorphisms that are 1-quasiconformal are precisely the conformal mappings, reflecting the fact that conformal mappings preserve angles and "roundness"; as such, the quasiconformal mappings are a natural generalisation of conformal mappings. Figure 1.3 illustrates this idea; the quasiconformal coefficient K for map f in this figure will be (a function of) the ratio a/b.

This geometric definition, while elegant, is unfortunately difficult to work with. In the literature an analytic definition equivalent to the following is often used:

Definition 1.17. *Let Ω, Ω' be domains in $\mathbb{R}^n$ and $f \in W^{1,n}_{loc}(\Omega, \Omega')$. Then f is K-**quasiconformal** if there exists a constant K such that*

$$|Df(x)|^n \leqslant KJ(x, f) \tag{1.5}$$

almost everywhere in Ω.

Note that no more generality is obtained by assuming $f \in W^{1,1}_{loc}(\Omega, \mathbb{C})$ since a homeomorphism of this class has a locally integrable Jacobian, so the distortion inequality (1.5) implies that $f \in W^{1,n}_{loc}(\Omega, \mathbb{C})$.

The Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z, f)\frac{\partial f}{\partial z} \tag{1.6}$$

where μ is a measurable function with $||\mu||_\infty \leqslant k < 1$, is satisfied by quasiconformal mappings. This observation will become important later, when we investigate properties of variational equations (see Section 1.6).

The modulus of a curve family naturally leads to another (equivalent) definition for a quasiconformal mapping; a mapping $f : \Omega \to \Omega'$ is K-quasiconformal if for every curve family Γ in Ω,

$$\frac{1}{K}M(\Gamma) \leqslant M(f\Gamma) \leqslant KM(\Gamma).$$

For a proof that this is an equivalent definition, see [42, pp. 46–48]. Given the previous reasoning concerning modulus as a conformal invariant, it is natural that this is how a quasiconformal map may also be defined.

1.5 Mappings of finite distortion & distortion functions

In the previous sections, the quantities providing a bound on a mapping's distortion are constant. In the case of conformal mappings, there was no distortion; with quasiconformal mappings, a kind of "finite distortion" was seen and an *a priori* uniform bound was assumed as part of the definition. However there is a natural generalisation of this idea, extending beyond the class of quasiconformal mappings. If the bounding value is allowed to vary between different inputs, we obtain a much broader idea of what it is to have *finite distortion*:

Definition 1.18. *(See [8, pp. 657–659].) A mapping $f : \Omega \to \Omega'$ between subdomains of $\mathbb{R}^n$ and belonging to the Sobolev class $W^{1,1}_{loc}(\Omega, \Omega')$ is said to have* ***finite distortion*** *if there is a measurable* ***distortion function*** *$K(x)$ such that*

$$|Df(x)|^n \leqslant K(x)J(x,f) \tag{1.7}$$

for almost every $x \in \Omega$. Here $1 \leqslant K(x) < \infty$, $|Df(x)|$ is the usual operator norm of the linear map $Df(x)$ and $J(z,f) \in L^1_{loc}(\Omega)$ is the Jacobian determinant of f at x. The smallest function $K(x)$ for which the distortion inequality (1.7) holds is called the ***outer distortion of*** *f and is defined by*

$$K(x,f) = \begin{cases} \dfrac{|Df(x)|^n}{J(x,f)} & \textit{whenever } Df(x) \textit{ exists and is nonsingular;} \\ 1 & \textit{otherwise.} \end{cases}$$

With reference to Fig. 1.3, in this case we allow the ratio a/b to vary, though we require it to be finite and nonzero almost everywhere.

For Sobolev homeomorphisms the condition $J(z,f) \in L^1_{loc}(\Omega)$ is redundant. Also, the distortion function $K(x)$ need not be restricted to specific use in the above definition; in general, a distortion function is simply a function which measures how far a mapping deviates from conformality — in the same sense that the constant of quasiconformality does. There is a counterpart to outer distortion called *inner distortion*, but this will not be addressed within this thesis; throughout, outer distortion is shortened to simply *distortion*.

The function $K(x, f)$ corresponds directly to the constant K in the definition of a quasiconformal map; a mapping f is K-quasiconformal at x if $K(x, f) \leqslant K$ in a neighbourhood of x. In fact, it can be seen from the definitions (1.5) and (1.7) that quasiconformal mappings are exactly those mappings with *uniformly bounded* distortion. In allowing the distortion function $K(x, f)$ to vary with x (and to be finite *almost everywhere*) the class of functions we consider is now much wider. This is problematic if we wish to pursue a normal families type argument as is so useful in quasiconformal (and conformal) cases. For the quasiconformal case, the argument typically runs

$$\text{quasiconformal} \implies \text{Hölder continuous} \implies \text{equicontinuous} \implies \text{normal}$$

but in the class of finite distortion mappings, there are in fact no *a priori* estimates for a modulus of continuity, ruling out this kind of argument at the very first step.

There is another obstacle here: the operator norm used in this definition is insufficiently regular for use in the variational equations (see Section 1.6). A pertinent example may be found in [8, pp. 663–665], in which the minimisation problem

$$\iint_{\mathbb{Q}} K(z, f)|dz|^2, \tag{1.8}$$

where f maps $\mathbb{Q} = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ to $\mathbb{Q}' = [0, 2] \times [0, 1]$ preserving the order of boundary components, is shown to have infinitely many extremals.

Given these drawbacks, we use the distortion function

$$\mathbb{K}(x, f) = \frac{||Df(x)||^2}{J(x, f)} \tag{1.9}$$

at points where $Df(x)$ exists and is non-singular instead. Here $||A||^2 =$

$\frac{1}{2}\mathrm{tr}(A^tA)$ (the mean Hilbert-Schmidt norm). The minimisation problem at (1.8), with K replaced by $\mathbb{K}$, in contrast to the case for $K(z,f)$, is shown to have a unique extremal.

In two dimensions, take the example of a linear mapping A (a 2×2 matrix). If A^tA has eigenvalues λ_1, λ_2, then

$$K(z,A) = \max\left\{\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}\right\},$$

and

$$\mathbb{K}(z,A) = \frac{1}{2}\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right).$$

If A varies from point to point, then so do K and $\mathbb{K}$. Note that as the eigenvalues cross K fails to be differentiable whereas $\mathbb{K}$ is. In two dimensions we observe that

$$\mathbb{K}(z,f) = \frac{1}{2}\left(K(z,f) + \frac{1}{K(z,f)}\right)$$

which is a convex function of $K(x,f)$, and therefore the L^∞ minimisers are the same.

In general (as is evident above), distortion functions give a measure of the deviation from conformality of a given mapping f by considering its differential (Jacobian) matrix Df. A general linear transformation can be given as

$$Az = az + b\bar{z}$$

with $a, b \in \mathbb{C}$. The Jacobian determinant and mean Hilbert-Schmidt norm are then

$$J(z,f) = |a|^2 - |b|^2, \qquad ||A||^2 = |a|^2 + |b|^2.$$

In the case of holomorphic mappings, these relationships reduce to the familiar Cauchy-Riemann equations (i.e. $b = 0$). Comparing these observations with (1.9) shows immediately that

$$\mathbb{K}(x,f) = \frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2},$$

a form of the distortion relation that will be of utility in this study.

The mapping

$$f(x+iy) = u(x) + iy$$

will also become important in our study. For this mapping,

$$Df = \begin{bmatrix} u_x & 0 \\ 0 & 1 \end{bmatrix}$$

whence

$$||Df||^2 = \frac{1}{2}(u_x^2 + 1), \quad J(z,f) = u_x$$

and therefore

$$\mathbb{K}(z,f) = \frac{1}{2}\left(u_x + \frac{1}{u_x}\right).$$

Since we require $\mathbb{K} \geqslant 1$, we restrict ourselves to the study of the orientation-preserving mappings, where $u_x > 0$.

1.6 The calculus of variations

The analysis of the *brachistochrone problem*, by John Bernoulli, contributed significantly to the formulation of the calculus of variations, which has seen many successful applications in a wide variety of problems in nonlinear elasticity, physics and engineering [17, pp. 36–43]. In this thesis, the calculus of variations will be used both to solve mean distortion problems and to confirm solutions derived by other means.

To find a function that minimises a certain functional (e.g. mean distortion), a variational process will be employed. The (first-)variational process generally results in a second-order differential equation to solve; the generality of this process is best started in the context of Hilbert spaces. The definition of a local maximum or minimum is almost analogous to that of a function of one real variable (*cf.* [43, p. 28]):

Definition 1.19. *Let $J : X \to \mathbb{R}$ be a functional defined on the function space $(X, ||\cdot||)$, and take a subset $S \subseteq X$. The functional J is said to have a* ***local maximum (local minimum)*** *in S at $f \in S$ if there exists an $\varepsilon > 0$ such that $J(\tilde{f}) - J(f) \leqslant 0$ ($J(\tilde{f}) - J(f) \geqslant 0$) for all $\tilde{f} \in S$ such that*

$||\tilde{f} - f|| < \varepsilon$.

The set S here is a set of functions satisfying a given set of boundary conditions on a domain of interest. The variational process then uses the idea that any $\tilde{f} \in S$ within an ε-neighborhood of f can be represented by a perturbation of f:

$$\tilde{f} = f + \varepsilon\eta$$

for some suitable $\eta \in X$, normally assumed to vanish on the boundary. One can view η as a *test function*; this approach is usually taken in a more general setting where the smoothness condition on f is relaxed.

Here the theory will be specialized to fixed boundary variational problems; the theory is, however, easily generalised (for example [12] or [43]). What follows is a short version of the calculation found in [12, pp. 24–30] and is demonstrative of the variational process in general. Some liberties have been taken with assumptions and specifics to illustrate the point better — for a rigorous version see [12] or [43].

Suppose we want to extremise the functional defined on $W^{1,2}(\mathbb{R}, \mathbb{R})$,

$$J = \int_{x_1}^{x_2} F(x, f(x), f'(x)) \, dx,$$

subject to some boundary conditions $f(x_1) = y_1, f(x_2) = y_2$. Suppose also that f is an extremum of J. Now take $\tilde{f}$ in an ε-neighborhood of f and substitute into J to get J as a function of ε:

$$J = \int_{x_1}^{x_2} F(x, \tilde{f}(x), \tilde{f}'(x)) \, dx = \int_{x_1}^{x_2} F(x, f + \varepsilon\eta, f' + \varepsilon\eta') \, dx$$

and taking its derivative with respect to ε via the chain rule,

$$J_\varepsilon = \int_{x_1}^{x_2} (F_f \, \eta + F_{f'} \, \eta') \, dx,$$

where subscripts denote appropriate derivatives (regarding F as a function with x, f and f' as independent variables) and primes indicate derivatives with respect to x. Integrating the second term by parts yields

$$J_\varepsilon = F_{f'}\ \eta|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[F_f\ \eta - \left(\frac{d}{dx} F_{f'} \right)\ \eta \right]\ dx.$$

Recall η here is a test function which vanishes on the boundary, and also recall the necessary condition from the theory of functions for real variables to have a smooth extremum. Thus

$$\int_{x_1}^{x_2} \left[F_f - \left(\frac{d}{dx} F_{f'} \right) \right]\ \eta\ dx = 0. \tag{1.10}$$

In the general setting of a Hilbert space, then, this corresponds to the condition

$$\langle E, \eta \rangle = 0$$

where

$$E = F_f - \left(\frac{d}{dx} F_{f'} \right).$$

To simplify condition (1.10) further, appeal is made to the *fundamental lemma of the calculus of variations* (see [43, p. 32]):

Lemma 1.20. *Suppose that $\langle E, \eta \rangle = 0$ for all $\eta \in H$, and that $E : [x_1, x_2] \to \mathbb{R}$ is continuous. Then $E = 0$ for all $x \in [x_1, x_2]$.*

Here H is the set of all test functions η such that $f + \varepsilon\eta$ is in an ε-neighborhood of f:

$$H = \{\eta \in X : f + \varepsilon\eta \in S\}.$$

Recall that S is the set of functions in the function space $(X, ||\cdot||)$ over which we are looking for an extremum. The proof of this fundamental idea is omitted; it can be found in most texts on the calculus of variations (for example, [12, pp. 27–29]).

Applying this lemma to the condition (1.10) gives us (*cf.* [43, p. 33]):

Theorem 1.21. *Let $\Omega = [x_1, x_2] \subset \mathbb{R}$ where $x_1 < x_2$, and let $\Omega' \subset \mathbb{R}$. Let*

$J : W^{1,2}(\Omega, \Omega') \to \mathbb{R}$ be a functional of the form

$$J(f) = \int_{x_1}^{x_2} F(x, f, f')\, dx$$

where F has continuous partial derivatives of order 2 with respect to x, f and f'. Let

$$S = \left\{ f \in W^{1,2}[x_1, x_2] : f(x_1) = y_1 \text{ and } f(x_2) = y_2 \right\},$$

where y_1 and y_2 are given real numbers. If f is an extremal for J in S, then

$$\frac{\partial F}{\partial f} - \frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right) = 0. \tag{1.11}$$

Equation (1.11) is a second-order differential equation that any smooth extremal f must satisfy, called the *Euler-Lagrange equation*, after the mathematicians Leonhard Euler and Joseph Lagrange who discovered it in the 18^{th} century [18, p. 680].

1.6.1 Multiple independent variables

The idea of a small perturbation may be extended to functions of multiple variables. In the setting discussed here, the two-variable case will be the most important — proof of the more general case is omitted; see [12, pp. 47–54].

Suppose the aim is to extremise a functional of the form

$$J = \iint_\Omega F\left(x, y, f, f_x, f_y\right)\, dx\, dy \tag{1.12}$$

with the values of f on the boundary of the domain $\Omega \subset \mathbb{R}^2$ prescribed and where subscripts denote derivatives. Suppose also that f is an extremum of J. Now take $\tilde{f}$ in an ε-neighborhood of f

$$\tilde{f} = f + \varepsilon\eta$$

where η is a test function on Ω of the two independent variables which vanishes on the boundary. The functional J, considered as a function of one variable ε (as before), should have an extremum at $\varepsilon = 0$. By the chain rule,

$$J_\varepsilon = \iint_\Omega \left(F_f \eta + F_{f_x}\eta_x + F_{f_y}\eta_y\right) dx\,dy.$$

Observe that

$$\frac{\partial}{\partial x}\left(F_{f_x}\eta\right) = \frac{\partial}{\partial x}\left(F_{f_x}\right)\eta + F_{f_x}\eta_x$$

and similarly for the total partial derivative with respect to y; hence the last two terms of J_ε are, by integrating by parts,

$$\begin{aligned}\iint_\Omega \left(F_{f_x}\eta_x + F_{f_y}\eta_y\right) dx\,dy \;=\; & \iint_\Omega \left[\frac{\partial}{\partial x}\left(F_{f_x}\eta\right) + \frac{\partial}{\partial y}\left(F_{f_y}\eta\right)\right] dx\,dy \\ & - \iint_\Omega \left[\frac{\partial}{\partial x}\left(F_{f_x}\right)\eta + \frac{\partial}{\partial y}\left(F_{f_y}\right)\eta\right] dx\,dy.\end{aligned}$$

Green's formula now shows that

$$\iint_\Omega \left[\frac{\partial}{\partial x}\left(F_{f_x}\eta\right) + \frac{\partial}{\partial y}\left(F_{f_y}\eta\right)\right] dx\,dy = \int_{\partial\Omega} \left(F_{f_x}\eta\,dy - F_{f_y}\eta\,dx\right) = 0,$$

since the test function $\eta \equiv 0$ on the boundary. Thus, the necessary condition for an extremum assumes the form

$$\iint_\Omega \left(F_f - \frac{\partial}{\partial x}\left(F_{f_x}\right) - \frac{\partial}{\partial y}\left(F_{f_y}\right)\right)\eta\,dx\,dy = 0.$$

Invoking the fundamental lemma (Lemma 1.20) yields

Theorem 1.22. *Let Ω, Ω' be regions in $\mathbb{R}^2$. Let $J : W^{1,2}(\Omega, \Omega') \to \mathbb{R}$ be a functional of the form*

$$J(f) = \iint_\Omega F(f, x, y, f_x, f_y)\,dx\,dy$$

where F has continuous partial derivatives of order 2 with respect to each of its arguments, considered independently. Let

$$S = \left\{f \in W^{1,2}(\Omega, \Omega') : f(\partial\Omega) \textit{ is prescribed}\right\}.$$

If f is an extremal for J in S, then

$$F_f - \frac{\partial}{\partial x}(F_{f_x}) - \frac{\partial}{\partial y}(F_{f_y}) = 0. \tag{1.13}$$

Equation (1.13) is the Euler-Lagrange equation for functions of two variables. It is possible (using vector calculus) to generalise this result to n dimensions; however this thesis only requires the two-dimensional result.

It is important to remember that the partial derivative

$$\frac{\partial}{\partial x}(F_{f_x})$$

in equation (1.13) is *total*, in the sense that computing it via the chain rule gives

$$\frac{\partial}{\partial x}(F_{f_x}) = F_{f_x x} + F_{f_x f}\frac{\partial f}{\partial x} + F_{f_x f_x}\frac{\partial f_x}{\partial x} + F_{f_x f_y}\frac{\partial f_y}{\partial x}. \tag{1.14}$$

1.7 Harmonic mappings & Dirichlet's principle

The study of harmonic mappings is extremely important in mathematical physics; solutions to many extremal problems take a harmonic form. The kinds of functionals this thesis discusses are energy functionals; solutions to these will sometimes be inverse to harmonic mappings. Many problems in electropotential theory, heat conduction and fluid flow are of this kind — for some elementary examples and background, see [19, Chs. 3,7], [37, Ch. 3] or [29, §5.3].

Classically, to solve a Dirichlet problem one must find a harmonic extension of a function, continuous on the boundary of the unit disk, to the closed disk. This problem has found many applications; it is an example of finding a minimal energy transformation (see, for example, [15]). Here a simpler definition is given, to illustrate that the functionals considered in this thesis are of essentially the same form as finite energy functionals.

Theorem 1.23. ***Dirichlet's Principle.*** *Let $\Omega \subset \mathbb{R}^n$ be a domain. Given* *the boundary condition* $u = 0$ *on* $\partial\Omega$*, if* u *satisfies Laplace's equation*

$$\Delta u = 0$$

then u *is the minimiser of the Dirichlet energy functional*

$$E(v) = \int_\Omega |\nabla v|^2 \, dx$$

among twice-differentiable functions v *with* $v = 0$ *on* $\partial\Omega$*.*

Note that this principle is often stated for somewhat more general boundary conditions; however the connection between harmonic mappings and energy functionals is clear.

We now have sufficient analytic background to meaninfully discuss minimisation problems of various kinds. These will be presented in the next chapter, motivating the exact problem kind that this thesis solves.

2. MINIMISATION PROBLEMS

Consider deforming an annular region in the plane with a given conformal metric (possibly viewed as some material property of the region) so as to minimize some weighted L^p-average of the local conformal distortion — a measure of the local anisotropic stretching of the material. This is illustrated in Fig 2.1, below, with two different metrics, namely the usual planar metric and the flat metric on $\mathbb{C} \setminus \{0\}$.

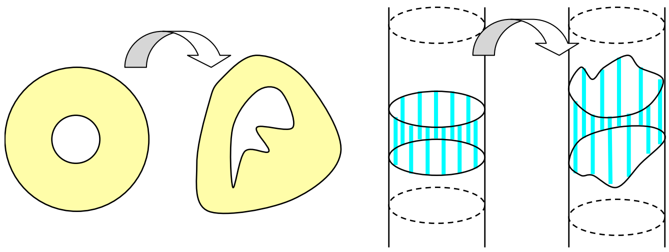

Fig. 2.1: Deformations in the plane (left) and the flat metric (right).

Recall the distortion function $\mathbb{K}$ at (1.9) from the previous chapter. The main focus of this thesis will be to show the existence, uniqueness and form of minimisers for certain kinds of functionals. In particular we seek to extremise the functional

$$I(f) = \int_\Omega \Phi\left(\mathbb{K}(z,f)\right)\ \eta(z)\ |dz|^2 \tag{2.1}$$

where $\Phi : [1,\infty) \to [0,\infty)$ is convex (the L^p norms with $p \geqslant 1$ essentially take this form: $\Phi(t) = t^p$), Ω is a region in the complex plane, and $\eta(z)$ is some positive C^1 weight function, interpreted as a density function, thickness, metric, or some other physical property, and f is of finite distortion.

The special case of this problem, $\Phi(t) = t$ and $\eta \equiv 1$, has solutions that are the inverse to those of a physical problem of finding a minimal energy deformation among harmonic mappings; see, for example, [8]

2.1 Conformal mapping problems

The theory of conformal mapping problems for simply connected regions is well-known, culminating in Riemann's Mapping Theorem, which ensures that there exists a mapping with no local distortion (i.e. a conformal mapping) for simply connected regions. Of interest, then, is what occurs where the region is not simply but multiply connected. Doubly connected regions will form the basis of a theory for multiply connected regions. A theorem from Schottky plays a crucial rôle:

Theorem 2.1. *(as stated in [9]; originally from [39]) An annulus* $\mathbb{A} = \{z : r < |z| < R\}$ *can be mapped conformally onto the annulus* $\mathbb{A}' = \{z' : r' < |z'| < R'\}$ *if and only if* $\frac{R}{r} = \frac{R'}{r'}$*. Moreover, every conformal mapping* $f : \mathbb{A} \to \mathbb{A}'$ *takes the form* $f(z) = \lambda z^{\pm 1}$*, where* $|\lambda| = \frac{r'}{r}$ *or* $|\lambda| = r'R$ *as the case may be.*

In essence, this theorem says that doubly connected regions are conformally equivalent if and only if their moduli (extremal length) are the same. This theorem provides one step towards simplifying our mapping problem.

Another relevant result in classical complex analysis is:

Theorem 2.2. *Every doubly connected region* Ω *in the complex plane is conformally equivalent to a round annulus* $\mathbb{A} = \{z \in \mathbb{C} : r \leqslant |z| \leqslant R\}$*,* $r \neq R$*, with modulus* $M(\mathbb{A}) = \log \frac{R}{r} = M(\Omega)$*.*

The definition of modulus included here is the interpretation of "extremal length" from Theorem 1.15, discarding multiplicative constants. This is a special case of a result found in Ahlfors' book [3, pp. 255–256]. Together with Schottky's theorem, this simplifies the search for mappings between doubly connected regions: we need only consider mappings between annuli whose moduli are not equal.

However, there are physical considerations which do not allow relatively nice solutions like conformal mappings. In fact, when one considers how

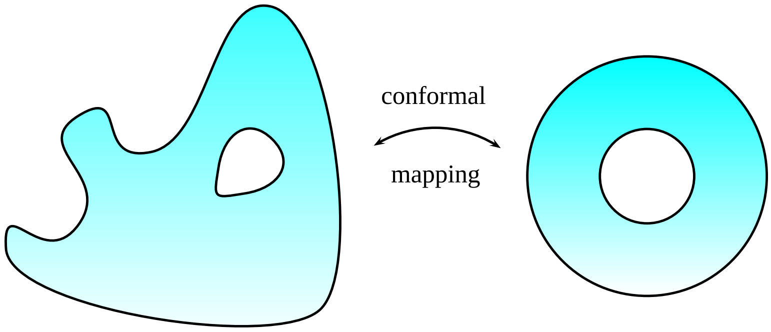

Fig. 2.2: A doubly connected region is conformally equivalent to a circular annulus.

unlikely it is that one doubly connected region encountered has exactly the same extremal length as another, it becomes obvious that the class of functions being considered must be widened. Quasiconformal mappings are a natural generalisation of conformal mappings.

2.2 Quasiconformal mapping problems

Where the region under consideration is simply connected, the Riemann Mapping Theorem reduces the problem to a conformal mapping problem. However, this does not guarantee that given boundary conditions are satisfied — to account for this we must enlarge the class of functions to (at least) quasiconformal mappings.

It is possible to view the distortion inequality (1.5) for quasiconformal mappings in a number of ways when considering mapping problems. Firstly, one may wish to minimise the *maximal* distortion on a domain Ω. In this case,

Theorem 2.3. *Between doubly connected regions Ω, Ω' in the complex plane, subject to a given set of boundary conditions, there is a quasiconformal mapping, unique up to a conformal transformation of the target, which minimises maximal distortion if and only if the ratio of the moduli of Ω and Ω' is bounded.*

A proof of this (in all dimensions) may be found by combining Theorems 34.3 and 39.1 in Väisälä's book [42, pp. 114,131].

However, in the case of *mean* distortion, it has been shown [6] that

Theorem 2.4. *The minimisation problem at (2.1) with $\eta \equiv 1$ throughout Ω and Φ the identity map has a unique solution only within particular ranges of the ratio of the moduli of Ω and Ω'.*

As an illustrative example, the L^1-Grötszch problem (see [8, p. 663]) seeks to minimise distortion among functions mapping the unit square $\square$ to some non-square rectangle $\square'$, *with the boundary condition* that edges map to edges. Consider, however, the following calculation (see [2, p. 12] for a similar calculation).

Let Ω be the closed rectangle $[0, b] \times [0, a]$ in the complex plane. Let Γ be the set of all curves in Ω joining the side $[0, a]$ to its opposite side. Consider the lines $\gamma_y : [0, b] \to \mathbb{C}$ defined by $\gamma_y = x + iy$ where $x \in [0, b]$. Clearly $\gamma_y \in \Gamma$. We get by Hölder's inequality

$$1 \leqslant \left(\int_{\gamma_y} \rho \, ds \right)^2 \leqslant b \int_0^b \rho^2(\gamma_y) \, dx.$$

Thus integrating over $y \in [0, a]$ gives, by Fubini's Theorem,

$$a \leqslant b \int_0^a \int_0^b \rho^2(\gamma_y) \, dx \, dy = b \int_\Omega \rho^2 \, dm \leqslant b \int \rho^2 \, dm.$$

Since this holds for every $\rho \in \mathrm{adm}(\Gamma)$, we have $M(\Gamma) \geqslant a/b$. By the earlier result (1.3), we also know that $M(\Omega) \leqslant ab/b^2 = a/b$. Hence in fact

$$M(\Gamma) = \frac{a}{b},$$

that is, the modulus of a rectangle is the ratio of the length of its sides.

In this case, while the Riemann Mapping Theorem guarantees the existence of a conformal mapping from $\square$ to $\square'$, certainly edges cannot match up exactly under this deformation. The preceding calculation shows that the extremal length of the curve family *joining edges* for $\square$ is 1, while that of $\square'$

is the ratio of larger to smaller edge length — and since extremal length is a conformal invariant, the Riemann mapping ought to preserve it, which it cannot achieve by mapping edges to corresponding edges.

The solution in this case is, however, quasiconformal — in fact minimisers can be shown to be linear stretchings. For example, in [8, pp.663–665] it is shown that stretching a square to a rectangle of the same height (in the y coordinate) but twice the width (x coordinate) results in the 2-quasiconformal mapping

$$f(x + iy) = 2x + iy.$$

It is worth noting that in this case, as in the conformal case, the argument for the existence of a minimiser uses a modulus of continuity type argument. Such an argument usually runs similarly to the proof of the Riemann Mapping Theorem: find a candidate mapping (i.e. any candidate within the class of mappings under consideration); use Hölder continuity to establish equicontinuity of a family of functions within the considered class that contains the candidate, which implies this family is normal; then choose a sequence that minimises the necessary quantity.

In general, the tools used to establish the existence of minimisers are exactly such a normal family type argument, using (consequences of) some conformal invariant — for example in the case where the invariant is the modulus of a curve family, first Hölder continuity is established, from which the rest of the argument follows.

2.3 Mappings of finite distortion

Mappings of finite distortion lend themselves very well to extremal problems where solutions may degenerate on sets of zero measure [7]. Unfortunately, this means that their distortion is not uniformly bounded, unlike the quasiconformal mappings.

When we widen consideration of mappings to this class, we lose some powerful tools found in classical quasiconformal analysis. The typical argument as mentioned in the previous section fails, as there are no *a priori* modulus of continuity estimates available for these mappings. We may, however, look at sequences that degenerate, in the sense that their limit is not of finite distortion — by such a means it may be shown that minimisers do not exist outside certain ranges (*cf.* the technique from [6, pp. 16–21]).

2.4 The Teichmüller problem

The classical Teichmüller problem asks one to identify deformations of the unit disk with minimal distortion, under the condition that the boundary remains fixed and the origin is moved to a given (distinct) point; see Fig. 2.3. Teichmüller identified the minimiser for maximal distortion (a quasiconformal mapping) [30, p. 233].

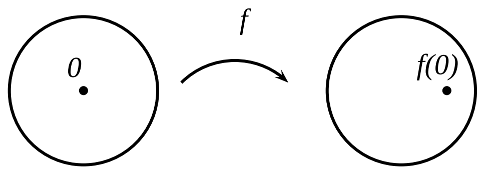

Fig. 2.3: The Teichmüller problem.

In [30], it is shown that while there are minimisers for the classical *maximal* distortion term among the set of quasiconformal mappings, the extreme value for the *mean* distortion is not attained by any mapping in the wider class of finite distortion, unless $f(0) = 0$ whereupon $f(z) = z$ is the obvious minimiser.

A point is raised by Martin in [30] is that the boundary hypotheses are important here:

Theorem 2.5. *Not every homeomorphism of finite distortion with*

$$\iint_{\mathbb{D}} \mathbb{K}(z, f)\ |dz|^2 < \infty$$

has a continuous extension to the closed disk $\overline{\mathbb{D}}$.

That is to say, a kind of Dirichlet-like principle may not be satisfiable in this setting. The counterexample Martin provides is one in which the image under a certain mapping of finite distortion of a single boundary point is a continuous arc. Perhaps surprising here is that the inverse mapping *does* have a continuous extension to the boundary and is harmonic [30, pp. 234–235].

Asymptotically sharp bounds on the mean distortion are also proven in [30]. Further, Martin emphasizes that, while in the classical case of maximal distortion there is a minimiser that can be exhibited, in the case of mean distortion the infimum is never attained (provided the solution is not the identity map) but the minimum value tends to ∞ as $|f(z)| \to 1$. Under the observation that harmonic mappings are often the result of minimal energy calculations in physics, he proposes that the solutions to these problems may represent critical phenomena of materials — a possible model for tearing or similar.

The fact that critical phenomena already occur in this situation (let alone doubly connected regions) suggests the importance of studying mappings of finite distortion in a more general setting — in fact, even for classical distortion, widening the class of functions to those of finite distortion still does not guarantee the existence of minimisers in general; this will be proven in Chapter 3.

2.5 Nitsche-type problems & Nitsche's conjecture

Define annuli

$$\mathbb{A}_1 = \{1 \leqslant |z| \leqslant R\}, \qquad \mathbb{A}_2 = \{1 \leqslant |z| \leqslant S\}$$

with moduli $\sigma_1 = \log(R)$ and $\sigma_2 = \log(S)$. In 1962, Nitsche published a note [34] in which he proved that $\mathbb{A}_2$ can be mapped harmonically onto $\mathbb{A}_1$ only if $\mathbb{A}_1$ is not too thin relative to $\mathbb{A}_2$; further, he conjectured about the critical value beyond which no harmonic mappings are possible. In the present minimisation problem, it is the inverse mapping $f : \mathbb{A}_1 \to \mathbb{A}_2$ that is of interest. When posed in terms of this scenario, Nitsche's conjecture is

Conjecture 2.6. *Let $\Omega = \mathbb{A}_1$ and $\Omega' = \mathbb{A}_2$. For the minimisation problem (2.1), with $\eta(z) = 1$ and $\Phi(t) = t$, minimisers exist if and only if*

$$S + \frac{1}{S} \leqslant 2R. \tag{2.2}$$

This has recently been proven [21].

We consider homeomorphisms of finite distortion $f : \mathbb{A}_1 \to \mathbb{A}_2$ mapping the boundary components to each other,

$$f(\{|z| = 1\}) = \{|z| = 1\}, \qquad \text{and} \qquad f(\{|z| = R\}) = \{|z| = S\}.$$

On the annulus $\mathbb{A}_1$ place a positive weight $\eta : \mathbb{A}_1 \to \mathbb{R}_+$ (we view $\eta(z)|dz|^2$ as a conformal measure on $\mathbb{A}_1$ or a material property of $\mathbb{A}_1$). In polar coordinates

$$f_z = \frac{1}{2}\, e^{-i\theta} \left(f_\rho - \frac{i}{\rho}\, f_\theta \right), \qquad f_{\bar{z}} = \frac{1}{2}\, e^{i\theta} \left(f_\rho + \frac{i}{\rho}\, f_\theta \right)$$

and

$$\begin{aligned} |f_z|^2 + |f_{\bar{z}}|^2 &= \frac{1}{2}(|f_\rho|^2 + \rho^{-2}|f_\theta|^2), \\ J(z,f) = |f_z|^2 - |f_{\bar{z}}|^2 &= \frac{1}{\rho}\, \Im m(f_\theta \overline{f_\rho}), \end{aligned}$$

which together yield

$$\mathbb{K}(z,f) = \frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2} = \frac{\rho|f_\rho|^2 + \rho^{-1}|f_\theta|^2}{2\Im m(f_\theta \overline{f_\rho})}.$$

Definition 2.7. *Given a convex function $\Phi : [1,\infty) \to [0,\infty)$ a* ***Nitsche-type problem*** *asks one to establish the existence or nonexistence of a minimiser (or perhaps a stationary point) of the functional*

$$f \mapsto \iint_{\mathbb{A}_1} \Phi(\mathbb{K}(z,f))\, \eta(z)\, |dz|^2,$$

with Φ, η, f as in (2.1).

It is proven in [6] that if (2.2) is satisfied, then there is a unique minimiser whose inverse is harmonic. It is further shown that outside this range there are no minimisers, and the way a minimising sequence degenerated is explained — thus proving Conjecture 2.6 in the affirmative. This will be addressed further in Section 5.2. Given the symmetry here one expects the minimiser to be a radial mapping; one of the form

$$z = re^{i\theta} \mapsto \rho(r)e^{i\theta}, \qquad \rho(1) = 1, \quad \rho(R) = S \tag{2.3}$$

and indeed the minimiser is

$$z \mapsto \frac{|z| + \sqrt{|z|^2 + \omega}}{1 + \sqrt{1 + \omega}} \frac{z}{|z|}, \qquad \omega = \frac{2 - SR}{R^2 - 1}.$$

We refer to the resulting bound on (a function of) R, S from the inequality at (2.2) as the *Nitsche bound* for this problem. In general,

Definition 2.8. *A* ***Nitsche bound*** *for an extremal problem between two regions* Ω *and* Ω' *is a bound on (some function of) given conformal invariants* σ *and* σ'*, associated with* Ω *and* Ω' *respectively, such that an extremal exists if and only if this bound is respected.*

2.6 Grötszch-type problems

The classical Grötzsch problem is to identify the linear mapping as the homeomorphism of least *maximal* distortion between two rectangles (assuming edges go to edges). Thus we set

$$\mathbb{Q}_1 = [0, \ell] \times [0, 1], \qquad \mathbb{Q}_2 = [0, L] \times [0, 1]$$

and suppose we have a deformation of finite distortion $f : \mathbb{Q}_1 \to \mathbb{Q}_2$ with

$$\Re e f(0, y) = 0, \quad \Re e f(\ell, y) = L, \quad \Im m f(x, 0) = 0, \quad \Im m f(x, 1) = 1 \tag{2.4}$$

(so f is orientation-preserving and maps edges to edges). This Sobolev map is absolutely continuous on almost all lines and so $\int_0^\ell \Re e(f_x)\, dx = L$ and

$\int_0^1 \Im m(f_y)\,dy = 1$ for almost all y and x respectively, and hence

$$\Re e \iint_{\mathbb{Q}_1} f_x(z)\,|dz|^2 = L, \quad \Im m \iint_{\mathbb{Q}_1} f_y(z)\,|dz|^2 = \ell. \tag{2.5}$$

The distortion function is

$$\mathbb{K}(z,f) = \frac{|f_x|^2 + |f_y|^2}{J(z,f)} \geqslant 1.$$

Definition 2.9. *A* ***Grötzsch-type problem*** *seeks a minimiser, satisfying the boundary conditions (2.4), to the functional*

$$f \mapsto \iint_{\mathbb{Q}_1} \Phi(\mathbb{K}(z,f))\,\lambda(z)\,|dz|^2 \tag{2.6}$$

for some positive weight function λ.

Observe that this definition considers mean distortion, rather than the traditional maximal distortion. As will shortly be seen, solving this type of problem has an intimate connection with Nitsche-type problems, and any Nitsche bound therefore carries over (via an appropriate weight function).

2.7 Equivalence of Nitsche- and Grötszch-type problems

The universal cover of an annulus is effected by the exponential map, so $z \mapsto \exp(2\pi z)$ takes $z = x+iy \in [0,\ell]\times[0,1]$ to $\mathbb{A}_1$ if $\sigma_2 = \log(S) = 2\pi L$. A branch of the logarithm must be chosen to define the inverse map $\mathbb{A}_2 \to [0,\ell]\times[0,L]$. If $f : \mathbb{A}_1 \to \mathbb{A}_2$ is given, then we can define $\tilde{f}(z) = \frac{1}{2\pi}\log(f(\exp 2\pi z))$. A particular point here is that log is conformal (in fact we only really need log to define a univalent conformal mapping from $\mathbb{A}_2$ to $\mathbb{Q}_2$ with edges matching up) so

$$\mathbb{K}(z,\tilde{f}) = \mathbb{K}\left(z, \frac{1}{2\pi}\log\big(f(e^{2\pi z})\big)\right) = \mathbb{K}\big(z, f(e^{2\pi z})\big),$$

and hence the change of variables $w = \exp 2\pi z$ yields

$$\begin{aligned}\iint_{\mathbb{Q}_1} \Phi\big(\mathbb{K}(z,\tilde{f})\big)\lambda(z)\,|dz|^2 &= \iint_{\mathbb{Q}_1} \Phi\big(\mathbb{K}(z, f(e^{2\pi z}))\big)\lambda(z)\,|dz|^2 \\ &= \frac{1}{4\pi^2}\iint_{\mathbb{A}_1} \Phi\big(\mathbb{K}(w,f)\big)\,\lambda(z)e^{-4\pi\Re e(z)}\,|dw|^2.\end{aligned}$$

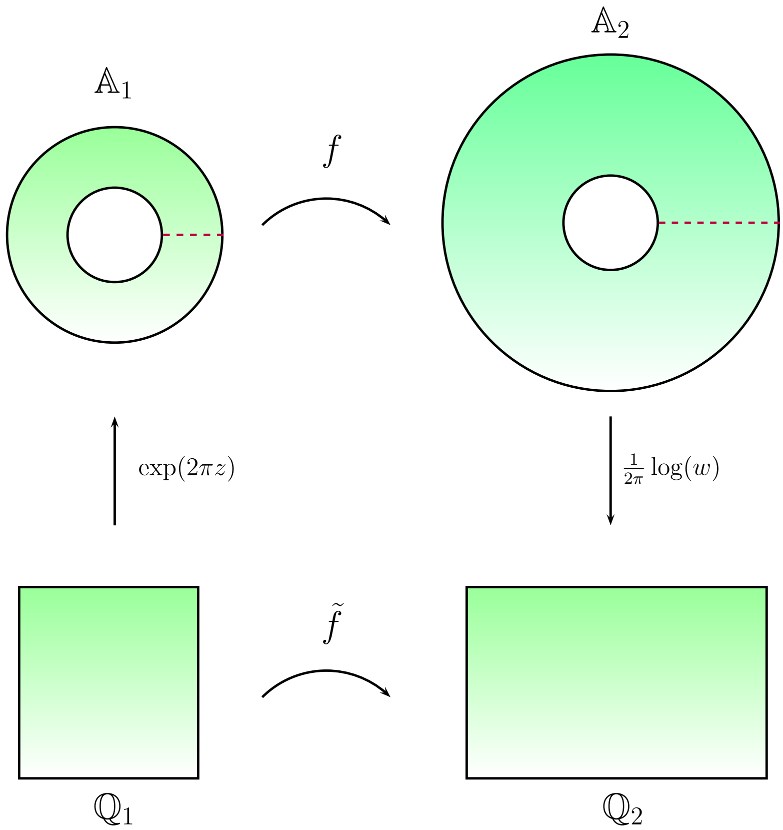

Fig. 2.4: Equivalence of Nitsche- and Grötzsch-type problems.

With the choice

$$\eta(w) = \frac{1}{4\pi^2}\,\lambda(z)e^{-4\pi\Re e(z)}, \qquad e^{2\pi z} = w, \tag{2.7}$$

the equivalence between the two problems (with related weights) is seen. Figure 2.4 depicts this equivalence.

It is of course possible to choose a different branch of the logarithm to accomplish this mapping. Doing this will simply result in a different rectangle (say $\mathbb{Q}_3$), which can be translated to any rectangle obtained by another branch of the logarithm, in particular to $\mathbb{Q}_2$ mentioned above, so this amounts to using the original fixed branch.

3. MINIMISERS OF DISTORTION FUNCTIONALS

Previous chapters have observed a variety of mapping problems, and techniques used for studying them, motivating us to consider the mapping problem (2.1) as important and worthy of deeper study. This chapter proves under which conditions minimisers for the mean distortion problem (2.1) exist and are unique.

3.1 The symmetry assumption for the Grötzsch problem

The aim of this section is to outline a proof that minimisers of the distortion functional at (2.1) are radially symmetric in some cases — full generality will be established later in this chapter. Initially, take $\eta(z) = 1/|z|^2$, so that via the equivalence from Section 2.7 this idea reduces, by restricting ourselves to the L^p norms only, to finding minimisers to

$$\iint_{\mathbb{Q}} \mathbb{K}^p(z, f)\ |dz|^2,$$

in the Grötzsch problem (2.6). To simplify this section, note that it is possible to find a conformal scaling map to take a rectangle to another of the same modulus while respecting the order of boundary components (see Section 2.2). Consider $\mathscr{F}$, consisting of orientation-preserving homeomorphisms $f : \mathbb{Q} \to \mathbb{Q}'$ in the Sobolev class $W^{1,1}(\mathbb{Q}, \mathbb{C})$ of finite distortion mapping vertical edges to vertical edges and horizontal edges to horizontal edges. Note that in this case, $W^{1,1}_{loc}(\mathbb{Q}, \mathbb{C}) = W^{1,1}(\mathbb{Q}, \mathbb{C})$ since $\mathbb{Q}$ is compact. Thus the aim is to prove

Theorem 3.1. *Let* $\mathbb{Q} = [0, a] \times [0, 1] \subset \mathbb{C}$ *and* $\mathbb{Q}' = [0, b] \times [0, 1] \subset \mathbb{C}$. *For* *the Grötzsch type distortion problem*

$$\min_{f\in\mathscr{F}} \iint_{\mathbb{Q}} \mathbb{K}^p(z,f)\,|dz|^2, \tag{3.1}$$

minimisers take the form

$$f(z) = \frac{b}{a}\Re e(z) + i\Im m(z), \tag{3.2}$$

and the minimal mean distortion is

$$\iint_{\mathbb{Q}} \mathbb{K}^p |dz|^2 = a\left(\frac{a^2+b^2}{2ab}\right)^p. \tag{3.3}$$

This is proven for the special case $a = 1, b = 2, p = 1$ in [8, pp.663–665]; the inspiration for this proof is drawn from there.

Proof. Let

$$f(x+iy) = u(x+iy) + iv(x+iy) \tag{3.4}$$

for homeomorphisms $f \in W^{1,1}_{loc}(\mathbb{Q},\mathbb{Q}')$. Note that for almost every $y \in \mathbb{Q}$,

$$\int_0^a u_x(t+iy)\,dt = u(a+iy) - u(iy) = b$$

and for almost every $x \in [0,a]$,

$$\int_0^1 v_y(x+is)\,ds = v(x+i) - v(x) = 1.$$

Hence, upon further integration,

$$\iint_{\mathbb{Q}} u_x\,dx\,dy = b \quad \text{and} \quad \iint_{\mathbb{Q}} v_y\,dx\,dy = a.$$

Next, take $\alpha := \frac{2p}{p+1} \geqslant 1$ (with $p > 0$) and use the measure-theoretic version of Jensen's Inequality to observe

$$\left(\int_0^1\int_0^a [bu_x + av_y]\,dx\,dy\right)^\alpha \leqslant \int_0^1\int_0^a a^{\alpha-1}\left(bu_x+av_y\right)^\alpha\,dx\,dy, \tag{3.5}$$

noting that equality holds when u_x and v_y are constant almost everywhere. Evaluating the left hand side gives

$$\left(a^2+b^2\right)^\alpha \leqslant \iint_{\mathbb{Q}} a^{\alpha-1}\left(bu_x+av_y\right)^\alpha |dz|^2. \tag{3.6}$$

Now

$$0 \leqslant (au_x-bv_y)^2 = a^2u_x^2 - 2abu_xv_y + b^2v_y^2 \tag{3.7}$$

and add $b^2u_x^2+2abu_xv_y+a^2v_y^2$ to both sides, obtaining

$$b^2u_x^2+2abu_xv_y+a^2v_y^2 \leqslant a^2u_x^2+b^2u_x^2+a^2v_y^2+b^2v_y^2.$$

Factorising gives

$$(bu_x+av_y)^2 \leqslant (a^2+b^2)(u_x^2+v_y^2)$$

and, since both sides are nonnegative,

$$(bu_x+av_y)^\alpha \leqslant \left(\sqrt{a^2+b^2}\right)^\alpha \left(\sqrt{u_x^2+v_y^2}\right)^\alpha$$

for $\alpha \geqslant 1$. Substituting into (3.6) and observing that the integrands are nonnegative throughout yields

$$\left(a^2+b^2\right)^\alpha \leqslant \iint_{\mathbb{Q}} a^{\alpha-1}\left(\sqrt{a^2+b^2}\right)^\alpha \left(\sqrt{u_x^2+v_y^2}\right)^\alpha |dz|^2. \tag{3.8}$$

Recall the distortion function

$$\mathbb{K}(z,f) = \frac{||Df(z)||^2}{J(z,f)}$$

and observe that for a function of the form (3.4)

$$2||Df||^2 = \operatorname{tr}(Df^tDf) = u_x^2+u_y^2+v_x^2+v_y^2.$$

Since a, b and α are constant, rearranging (3.8) shows that

$$a^{1-\alpha}\left(\sqrt{a^2+b^2}\right)^\alpha \quad \leqslant \quad \iint_{\mathbb{Q}} \left(\sqrt{u_x^2+v_y^2}\right)^\alpha |dz|^2$$

$$\begin{aligned}
&\leqslant \iint_{\mathbb{Q}} \left(\sqrt{u_x^2 + u_y^2 + v_x^2 + v_y^2}\right)^{\alpha} \, dx \, dy \qquad (3.9)\\
&= \iint_{\mathbb{Q}} 2^{\frac{\alpha}{2}} ||Df||^{\alpha} |dz|^2 \\
&= 2^{\frac{\alpha}{2}} \iint_{\mathbb{Q}} \mathbb{K}^{\frac{\alpha}{2}} J^{\frac{\alpha}{2}} |dz|^2 .
\end{aligned}$$

From Hölder's inequality we find that for r and q Hölder conjugates,

$$\iint_{\mathbb{Q}} \mathbb{K}^{\frac{\alpha}{2}} J^{\frac{\alpha}{2}} |dz|^2 \leqslant \left(\iint_{\mathbb{Q}} \mathbb{K}^{\frac{\alpha}{2} r} |dz|^2 \right)^{\frac{1}{r}} \left(\iint_{\mathbb{Q}} J^{\frac{\alpha}{2} q} \, |dz|^2 \right)^{\frac{1}{q}} .$$

Choose $q = \frac{2}{\alpha}$ (note that $q \geqslant 1$ since $\alpha \leqslant 2$) and $r = \frac{2}{2-\alpha}$ and square both sides to obtain

$$2^{-\alpha} a^{2-2\alpha} \left(a^2 + b^2\right)^{\alpha} \leqslant \left(\iint_{\mathbb{Q}} \mathbb{K}^{\frac{\alpha}{2-\alpha}} |dz|^2 \right)^{2-\alpha} \left(\iint_{\mathbb{Q}} J \, |dz|^2 \right)^{\alpha} . \qquad (3.10)$$

For an arbitrary Sobolev map $f : \mathbb{Q} \to \mathbb{Q}'$ we know that

$$\iint_{\mathbb{Q}} J \, |dz|^2 \leqslant |\mathbb{Q}'| = b$$

and so, substituting $\alpha = \frac{2p}{1+p}$ into (3.10), simplifying and rearranging,

$$\iint_{\mathbb{Q}} \mathbb{K}^p |dz|^2 \geqslant a \left(\frac{a^2 + b^2}{2ab} \right)^p .$$

This is a lower bound on the mean L^p distortion.

It remains to check when equality holds. First, equality holds in (3.9) if and only if $u_y = v_x = 0$ everywhere, whence u is independent of y and v is independent of x. Next, (3.5) attains equality if and only if u_x and v_y are constant; and (3.7) forces $bv_y = au_x$ almost everywhere in $\mathbb{Q}$. It is now obvious that equality holds throughout the proof if and only if

$$f(x + iy) = \frac{b}{a} x + iy,$$

as required. □

At this juncture it must be pointed out that, contrary to intuition, radial symmetry for minimisers of mean distortion need not hold in higher dimensions; in particular, there is a recent theorem from Iwaniec and Onninen which points out that radial symmetry of minimisers is lost in dimension 4 and higher; the proof is a classical *reductio ad absurdum* [23, p. 936, pp. 977–979].

However, in this thesis the more general case for dimension 2 will be proved — a somewhat more difficult calculation than the special case mentioned at Theorem 3.1. Before launching into this proof, we derive a key inequality.

3.2 A key inequality

Take complex numbers X and X_0, and positive real J and J_0. Rearranging the elementary inequality

$$\left| \frac{X}{X_0} - \frac{J}{J_0} \right|^2 \geqslant 0,$$

with equality holding if and only if $X/X_0 = J/J_0$, a positive real number, gives

$$\left| \frac{XJ_0 - X_0 J}{X_0 J_0} \right|^2 \geqslant 0.$$

Observe that $|X_0 J_0|^2 > 0$. On expansion (recall $|z|^2 = z\overline{z}$ for any $z \in \mathbb{C}$), the numerator yields

$$\overline{X} X J_0^2 - \overline{X} X_0 J J_0 - X \overline{X_0} J J_0 + \overline{X_0} X_0 J^2 \geqslant 0.$$

Dividing by JJ_0^2 gives

$$\frac{\overline{X} X}{J} - \frac{\overline{X} X_0}{J_0} - \frac{X \overline{X_0}}{J_0} + \frac{\overline{X_0} X_0 J}{J_0^2} \geqslant 0.$$

and so (by inserting 0 in the form $\overline{X_0}X_0/J_0 - \overline{X_0}X_0/J_0$ twice)

$$\frac{\overline{X}X}{J} - \frac{\overline{X_0}X_0}{J_0} - \frac{\overline{X_0}X}{J_0} + \frac{\overline{X_0}X_0}{J_0} - \frac{X_0\overline{X}}{J_0} + \frac{\overline{X_0}X_0}{J_0} + \frac{\overline{X_0}X_0 J}{J_0^2} - \frac{\overline{X_0}X_0}{J_0} \geqslant 0.$$

Rearranging and multiplying the last term by J_0/J_0 gives

$$\frac{\overline{X}X}{J} - \frac{\overline{X_0}X_0}{J_0} \geqslant \left(\frac{\overline{X_0}X}{J_0} - \frac{\overline{X_0}X_0}{J_0} + \frac{X_0\overline{X}}{J_0} - \frac{\overline{X_0}X_0}{J_0}\right) - \left(\frac{\overline{X_0}X_0 J}{J_0^2} + \frac{\overline{X_0}X_0 J_0}{J_0^2}\right).$$

Pulling out common factors and further simplifying yields

$$\frac{\overline{X}X}{J} - \frac{\overline{X_0}X_0}{J_0} \geqslant \left(\frac{\overline{X_0}}{J_0}(X - X_0) + \overline{\frac{\overline{X_0}}{J_0}(X - X_0)}\right) - \frac{\overline{X_0}X_0}{J_0^2}(J - J_0),$$

and this proves

Proposition 3.2. *For any complex numbers X and X_0, and positive real J and J_0,*

$$\frac{|X|^2}{J} - \frac{|X_0|^2}{J_0} \geqslant 2\Re e\left(\frac{\overline{X_0}}{J_0}(X - X_0)\right) - \frac{|X_0|^2}{J_0^2}(J - J_0). \tag{3.11}$$

This inequality may be used to identify minima for the mean distortion, by studying the function

$$(X, Y, J) \mapsto \frac{|X|^2 + |Y|^2}{J},$$

convex on $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ (meaning its graph lies above its tangent plane). Recalling the distortion function

$$\mathbb{K}(z, f) = \frac{|f_z|^2 + |f_{\overline{z}}|^2}{|f_z|^2 - |f_{\overline{z}}|^2}$$

the connection is clear; when X and Y are partial derivatives, these functions are the same. In this thesis, this technique will be used to establish a much more general result.

3.3 The Main Theorem

This section proves

Theorem 3.3. *Let*

(i) ℓ, L *be positive real numbers, and* $\mathbb{Q} = [0,\ell] \times [0,1]$, $\mathbb{Q}' = [0,L] \times [0,1]$,

(ii) $\lambda(x) : [0,\ell] \to (0,\infty)$ *be a continuous weight function,*

(iii) $\Phi : [1,\infty) \to [1,\infty)$ *be an increasing convex* $\mathcal{C}^\infty$ *function,*

(iv) $f_0(z) = u_0(x) + iy$ *where* u_0 *is a solution to the boundary value problem*

$$\lambda(x)\left(1 - \frac{1}{u_x^2(x)}\right)\Phi'\left(u_x(x) + \frac{1}{u_x(x)}\right) = \alpha, \tag{3.12}$$

$$u(0) = 0, \qquad u(\ell) = L \tag{3.13}$$

where α *is a real constant.*

(v) $\mathscr{F}$ *be all homeomorphisms of finite distortion* $f : \mathbb{Q} \to \mathbb{Q}'$ *with*

$$\lim_{x\searrow 0} \Re e\, f(x,y) = 0, \qquad \lim_{x\nearrow \ell} \Re e\, f(x,y) = L,$$
$$\lim_{y\searrow 0} \Im m\, f(x,y) = 0, \qquad \lim_{y\nearrow 1} \Im m\, f(x,y) = 1.$$

Then for any $f \in \mathscr{F}$,

$$\iint_{\mathbb{Q}} \Phi\left(\mathbb{K}(z,f)\right)\lambda(x)|dz|^2 \geqslant \iint_{\mathbb{Q}} \Phi\left(\mathbb{K}(z,f_0)\right)\lambda(x)|dz|^2$$

holds, with equality attained uniquely by the minimiser f_0.

Proof. Consider a mapping of the form

$$f_0(z) = u(x) + iy, \tag{3.14}$$

which, when considered on a rectangle, is analogous to the radial stretchings (2.3). We have $(f_0)_x = u_x \neq 0$ (since $\ell, L > 0$) and $(f_0)_y = i$. Therefore, if we set $\omega(x) = 1/u_x(x)$, a real valued function, then for any $f \in \mathscr{F}$ we have the identity

$$|\omega(x) f_x + i f_y|^2 \geqslant 0. \tag{3.15}$$

Equality here clearly holds if $f = f_0$. If $f \neq f_0$, then observe that:

1. f cannot be constant since $f \in \mathscr{F}$ must map $\mathbb{Q}$ to $\mathbb{Q}'$; and

2. since equality in (3.15) for nonconstant f demands $\omega(x) f_x = -i f_y$, the Beltrami coefficient

$$\mu(z, f) = \frac{f_{\bar{z}}}{f_z} = \frac{f_x + i f_y}{f_x - i f_y} = \frac{f_x - \omega(x) f_x}{f_x + \omega(x) f_x} = \frac{u_x - 1}{u_x + 1} = \mu(z, f_0)$$

and hence equality holds in (3.15) only if f and f_0 are conformally equivalent. The boundary conditions (3.13) force this conformal equivalence to be the identity; hence equality holds in (3.15) if and only if $f = f_0$.

Now (3.15) gives

$$\begin{aligned} 0 &\leqslant |\omega(x) f_x + i f_y|^2 = (\omega(x) f_x + i f_y)(\omega(x)\overline{f_x} - i\overline{f_y}) \\ &= \omega^2(x)|f_x|^2 + |f_y|^2 - 2\Im m(\omega(x) f_y \overline{f_x}) \end{aligned}$$

which yields

$$\omega^2(x)|f_x|^2 + |f_y|^2 \geqslant 2\omega(x)\,\Im m(f_y \overline{f_x}). \tag{3.16}$$

Notice that (letting $f = U + iV$)

$$\Im m(f_y \overline{f_x}) = \Im m(U_x(z) - iV_x(z))(U_y(z) + iV_y(z)) = J(z, f),$$

so (3.16) gives

$$\omega^2(x)|f_x|^2 + |f_y|^2 \geqslant 2\omega(x) J(z, f) \tag{3.17}$$

with equality if and only if $f = f_0$. It is possible to rewrite (3.17) in two different ways, namely

$$\begin{aligned} |f_x|^2 + |f_y|^2 &\geqslant (1 - \omega^{-2}(x))|f_y|^2 + 2\omega^{-1}(x) J(z, f), \\ |f_x|^2 + |f_y|^2 &\geqslant (1 - \omega^2(x))|f_x|^2 + 2\omega(x) J(z, f). \end{aligned}$$

This gives two estimates on the distortion function (writing $J = J(z,f)$),

$$2\mathbb{K}(z,f) \geqslant (1-\omega^{-2}(x))\frac{|f_y|^2}{J} + 2\omega^{-1}(x),$$

$$2\mathbb{K}(z,f) \geqslant (1-\omega^{2}(x))\frac{|f_x|^2}{J} + 2\omega(x),$$

and therefore, for a general mapping f, either

$$2(\mathbb{K}(z,f) - \mathbb{K}(z,f_0)) \geqslant (1-\omega^{-2}(x))\left[\frac{|f_y|^2}{J} - \frac{|(f_0)_y|^2}{J_0}\right],$$

or

$$2(\mathbb{K}(z,f) - \mathbb{K}(z,f_0)) \geqslant (1-\omega^{2}(x))\left[\frac{|f_x|^2}{J} - \frac{|(f_0)_x|^2}{J_0}\right].$$

To apply the inequality (3.11) from Proposition 3.2 we require positivity of the first term; that is, in the first case we want $(1-\omega^{-2}(x)) > 0$ and in the second case, $(1-\omega^2(x)) > 0$. This depends pointwise on u_x for the candidate extremal mapping. Next note that if $\Phi : \mathbb{R} \to \mathbb{R}$ is convex, then its graph lies above any tangent line:

$$\Phi(\mathbb{K}) - \Phi(\mathbb{K}_0) \geqslant \Phi'(\mathbb{K}_0)\big(\mathbb{K} - \mathbb{K}_0\big)$$

where we have adopted the shorthand $\mathbb{K} = \mathbb{K}(z,f), \mathbb{K}_0 = \mathbb{K}(z,f_0)$. Equality holds here if and only if $\mathbb{K} = \mathbb{K}_0$. This therefore yields the inequalities:

$$\begin{aligned} 2(\Phi(\mathbb{K}) - \Phi(\mathbb{K}_0)) &\geqslant (1-\omega^{-2}(x))\Phi'(\mathbb{K}_0) \\ &\left[2\,\Re e\Big(\frac{\overline{(f_0)_y}}{J_0}\,\big(f_y - (f_0)_y\big)\Big) - \frac{|(f_0)_y|^2}{J_0^2}\big(J - J_0\big)\right], \\ 2(\Phi(\mathbb{K}) - \Phi(\mathbb{K}_0)) &\geqslant (1-\omega^{2}(x))\Phi'(\mathbb{K}_0) \\ &\left[2\,\Re e\Big(\frac{\overline{(f_0)_x}}{J_0}\,\big(f_x - (f_0)_x\big)\Big) - \frac{|(f_0)_x|^2}{J_0^2}\big(J - J_0\big)\right]. \end{aligned}$$

Now $(f_0)_y = i$ and $(f_0)_x = 1/\omega(x) = J_0$ so, after substitution,

$$\begin{aligned}2(\Phi(\mathbb{K}) - \Phi(\mathbb{K}_0)) &\geqslant \big(1 - \tfrac{1}{\omega^2(x)}\big)\Phi'(\mathbb{K}_0)\left[\tfrac{2}{J_0}\,\Im m\big(f_y - 1\big) - \tfrac{J-J_0}{J_0^2}\right]\\ &= 2\left(\omega(x) - \tfrac{1}{\omega(x)}\right)\Phi'(\mathbb{K}_0)\Im m\big(f_y - 1\big) + \big(\omega^2(x) - 1\big)\Phi'(\mathbb{K}_0)(J_0 - J)\,,\end{aligned}$$

$$2(\Phi(\mathbb{K}) - \Phi(\mathbb{K}_0)) \;\geqslant\; (1 - \omega^2(x))\Phi'(\mathbb{K}_0)\left[2\,\Re e\big(f_x - (f_0)_x\big) - \big(J - J_0\big)\right].$$

The next step is to multiply by a weight function $\lambda(x)$ and integrate. Assume that f_0 (that is $\omega(x)$) is chosen so that

$$\lambda(x)(1 - \omega^2(x))\Phi'(\mathbb{K}_0) = \alpha \tag{3.18}$$

for a real constant α. This is already recognisable as

$$\lambda(x)\Big(1 - \frac{1}{u_x^2}\Big)\Phi'\Big(u_x + \frac{1}{u_x}\Big) = \alpha,$$

which is the Euler-Lagrange equation for the variational problem of finding the extremal for

$$\iint_{\mathbb{Q}} \Phi(\mathbb{K}(z,f))\lambda(x)\;|dz|^2$$

among functions of the form (3.14) — this is easy to verify using the results from Section 1.6. This equation is therefore necessarily satisfied by the minimiser. Observe that if $\alpha = 0$ then $u_x = 1$, whence f is the identity map, meaning $\Omega = \Omega'$; $\alpha \neq 0$ is the nontrivial case.

We now have

$$\begin{aligned}2\iint_{\mathbb{Q}} \Phi(\mathbb{K})\lambda(x)\;|dz|^2 &\geqslant 2\iint_{\mathbb{Q}} \Phi(\mathbb{K}_0)\lambda(x)\;|dz|^2 - \alpha\iint_{\mathbb{Q}}(J_0 - J)\;|dz|^2\\ &\quad +2\iint_{\mathbb{Q}} \lambda(x)\left(\omega(x) - \frac{1}{\omega(x)}\right)\Phi'(\mathbb{K}_0)\Im m\big(f_y - 1\big)\;|dz|^2,\end{aligned}$$

$$\begin{aligned}2\iint_{\mathbb{Q}} \Phi(\mathbb{K})\lambda(x)\ |dz|^2 &\geqslant 2\iint_{\mathbb{Q}} \Phi(\mathbb{K}_0))\lambda(x)\ |dz|^2 + \alpha\iint_{\mathbb{Q}} (J_0 - J)\ |dz|^2 \\ &+ 2\alpha\iint_{\mathbb{Q}} \Re e\big(f_x - (f_0)_x\big)\ |dz|^2.\end{aligned}$$

From the boundary conditions at (2.4) and (2.5) we see that

$$\begin{aligned}&\iint_{\mathbb{Q}} \lambda(x)\left(\omega(x) - \frac{1}{\omega(x)}\right)\Phi'(\mathbb{K}_0)\Im m\big(f_y - 1\big)\ |dz|^2 \\ &= \int_0^\ell \lambda(x)\left(\omega(x) - \frac{1}{\omega(x)}\right)\Phi'(\mathbb{K}_0)\Big[\int_0^1 \Im m\big(f_y - 1\big)dy\Big]\ dx = 0\end{aligned}$$

and

$$\iint_{\mathbb{Q}} \Re e\big(f_x - (f_0)_x\big)\ |dz|^2 = \int_0^1 \Big[\int_0^\ell \Re e\big(f_x - (f_0)_x\big)dx\Big]\ dy = 0.$$

This yields, in both cases,

$$\iint_{\mathbb{Q}} \Phi(\mathbb{K})\lambda(x)\ |dz|^2 \geqslant \iint_{\mathbb{Q}} \Phi(\mathbb{K}_0)\lambda(x)\ |dz|^2 \pm \frac{\alpha}{2}\iint_{\mathbb{Q}} \big(J_0 - J\big)\ |dz|^2.$$

To look for a minimum here, the sign of α depends on whether $u_x(x) > 1$ or $u_x(x) < 1$ for all x (this must be the case for the minimiser since if u_x ever changes sign, there is a better candidate for the minimum; namely a u_x which does *not* change sign). Note that this already is an interesting property of the minimiser. For an arbitrary Sobolev mapping we know [31]

$$\iint_{\mathbb{Q}} J\ |dz|^2 \leqslant |\mathbb{Q}'| = \iint_{\mathbb{Q}} J_0\ |dz|^2.$$

This establishes the desired result. □

Theorem 3.3 proves the existence and uniqueness of minimisers for general minimisation problems of the Grötzsch type, and, by extension, to Nitsche-type problems, provided we can solve (3.12). In [6, pp. 16–21] minimising sequences are identified outside the Nitsche range (see the bound at (2.2)) and from this, minimisers can be shown not to exist outside this range in, specifically, the unweighted case for the L^1 norm.

Similar arguments may show the nonexistence of minimisers outside various ranges in our more general setting as well. However, precise results are quite challenging and we shall not address this matter here.

3.3.1 The theorem for the annulus

Recall the equivalence between Grötzsch-type and Nitsche-type problems seen in Section 2.7. With the choice

$$\eta(w) = \frac{1}{4\pi^2}\,\lambda(z)e^{-4\pi\Re e(z)}, \qquad e^{2\pi z} = w,$$

(z is the complex coordinate in the rectangle, w in the annulus) the equivalence between the Grötzsch and Nitsche problems is seen. By means of this change of variables, it is possible to convert Theorem 3.3 to the annulus:

Theorem 3.4. *Let*

(i) R, S *be positive real numbers, and* $\mathbb{A} = \{z \in \mathbb{C} : 1 \leqslant |z| \leqslant R\}$, $\mathbb{A}' = \{z \in \mathbb{C} : 1 \leqslant |z| \leqslant S\}$,

(ii) $\eta(z) : \mathbb{A} \to (0, \infty)$ *be a radially symmetric continuous weight function,*

(iii) $\Phi : [1, \infty) \to [1, \infty)$ *be an increasing convex* $\mathcal{C}^\infty$ *function,*

(iv) $f_0(z) = \rho_0(|z|)\dfrac{z}{|z|}$ *where* ρ_0 *is a solution to the boundary value problem*

$$r^2\eta(r)\left(1 - \frac{\rho^2(r)}{r^2\rho_r^2(r)}\right)\Phi'\left(\frac{r\rho_r(r)}{\rho(r)} + \frac{\rho(r)}{r\rho_r(r)}\right) = \alpha$$

$$\rho(1) = 1, \qquad \rho(R) = S$$

where α *is a real constant.*

(v) $\mathscr{F}$ *be all homeomorphisms of finite distortion* $f : \mathbb{A} \to \mathbb{A}'$ *with*

$$\lim_{r \searrow 1} f(re^{i\theta}) = e^{i\theta}, \quad \lim_{r \nearrow R} f(re^{i\theta}) = Se^{i\theta}.$$

Then for any $f \in \mathscr{F}$,

$$\iint_{\mathbb{A}} \Phi\left(\mathbb{K}(z,f)\right)\eta(z)|dz|^2 \geqslant \iint_{\mathbb{A}} \Phi\left(\mathbb{K}(z,f_0)\right)\eta(z)|dz|^2$$

holds, with equality attained uniquely by the minimiser f_0.

It is easily verified that, under the transformation between rectangle and annulus, this gives the same minimisers.

For a given minimisation problem (with suitable parameters), these two theorems (3.3 and 3.4) show that the minimiser is required to satisfy a particular differential equation. This is the Euler-Lagrange equation for extremal problems of this kind — and it is in fact a *first*-order differential equation. Here is where the Nitsche phenomenon arises: there are two boundary conditions (3.13) to satisfy, and this may not be possible by solving a first-order equation.

3.4 Critical Nitsche-type phenomena

Theorem 3.3 strongly motivates us to study the ordinary differential equation (3.12) for solutions that will identify minimisers of Nitsche and Grötzsch-type problems. Note also that the transformation from the Nitsche-type problem to the Grötzsch problem yields a significantly simpler equation to study.

3.4.1 The Nitsche phenomenon

Let us first observe how the Nitsche phenomenon arises for the special case $\Phi(t) = t$. Here we have $\Phi' \equiv 1$, and $\lambda(x) = 4\pi^2 e^{4\pi x}$ as $\eta(w) = 1$. We have

$$1 - \frac{1}{u_x^2(x)} = \frac{\alpha}{4\pi^2} e^{-4\pi x}, \qquad u_x(x) = \frac{2\pi}{\sqrt{4\pi^2 - \alpha e^{-4\pi x}}},$$

$$u(x) = \int \frac{2\pi e^{2\pi x}\, dx}{\sqrt{4\pi^2 e^{4\pi x} - \alpha}} = \frac{1}{2\pi} \int \frac{dt}{\sqrt{t^2 - \alpha}}, \qquad t = 2\pi e^{2\pi x}.$$

So

$$u(x) \quad = \quad \frac{1}{2\pi}\log\left(\frac{e^{2\pi x}+\sqrt{e^{4\pi x}-\beta}}{1+\sqrt{1-\beta}}\right), \qquad \beta = \frac{\alpha}{4\pi^2}$$

noting $u(0)=0$. Recall $u:[0,\ell]\to[0,L]$ and we must solve $u(\ell)=L$, that is

$$L = \frac{1}{2\pi}\log\left(\frac{e^{2\pi\ell}+\sqrt{e^{4\pi\ell}-\beta}}{1+\sqrt{1-\beta}}\right) \tag{3.19}$$

by choice of our free parameter α. As λ is continuous, the extreme value theorem informs us that it attains both an absolute maximum and an absolute minimum on $[0,\ell]$. Set

$$\lambda_0 = \min_{x\in[0,\ell]} \lambda(x).$$

Notice that α is not bounded from below, and as $\alpha\to-\infty$, $\beta\to-\infty$ we can make the right hand side of (3.19) arbitrarily small. Thus there is always a minimiser if $0<L\leqslant\ell$. Now if $\alpha>0$ we see that (3.12) requires $\alpha<\lambda_0$ so that $\beta<1$. Observe that (after simplification)

$$\frac{dL}{d\beta} = \frac{e^{2\pi\ell}\sqrt{e^{4\pi\ell}-\beta}+e^{4\pi\ell}-\sqrt{1-\beta}-1}{4\pi\left(e^{2\pi\ell}\sqrt{e^{4\pi\ell}-\beta}+e^{4\pi\ell}-\beta\right)\left(\sqrt{1-\beta}+1-\beta\right)},$$

which is nonnegative whenever $\ell\geqslant 0$ and $\beta\leqslant 1$. Therefore L is bounded by the choice $\beta=1$:

$$L < \frac{1}{2\pi}\log\left(e^{2\pi\ell}+\sqrt{e^{4\pi\ell}-1}\right). \tag{3.20}$$

With some manipulation, it can be shown that (3.20) is precisely the Nitsche bound (2.2).

3.4.2 More generality for $\Phi'=0$

For more general weights $\lambda(x)$,

$$1-\frac{1}{u_x^2(x)} = \frac{\alpha}{\lambda(x)}, \qquad u_x(x) = \sqrt{\frac{\lambda(x)}{\lambda(x)-\alpha,}}$$

and we must study the behaviour of

$$u(x) = \int \sqrt{\frac{\lambda(x)}{\lambda(x) - \alpha}}\, dx.$$

Again, as $\alpha \to -\infty$ we can make this integral arbitrarily small. Notice that $\alpha/\lambda(x) < 1$, so if we put $\lambda_0 = \min_{x\in[0,\ell]} \lambda(x)$, then this integral is dominated by that with the choice $\alpha = \lambda_0$ and one must decide whether

$$\int_0^\ell \sqrt{\frac{\lambda(x)}{\lambda(x) - \lambda_0}}\, dx < \infty.$$

As λ is taken as a positive weight, the numerator is a bounded function and so the issue is around the convergence or otherwise of the integral

$$\int_0^\ell \frac{dx}{\sqrt{\lambda(x) - \lambda_0}} < \infty$$

A comparison test with $\int dt/t$ shows that convergence will require

$$\lambda(t) \approx \lambda_0 + t^{2s}, \qquad s < 1$$

(with $t = x - x_0$) near the minimum x_0.

Divergence of this integral will indicate that no critical phenomena occur. For, if the integral diverges, a suitable choice of α will ensure that the boundary conditions $u(0) = 0$, $u(\ell) = L$ can be satisfied.

There are two cases for the integral to diverge

- $\lambda(x)$ has a smooth interior minimum on $0 < x < \ell$; or
- $\lambda(x)$ has a minimum on one of the boundary values, $x = 0$ or $x = \ell$. Divergence here requires the minimum also to be smooth, in the sense that the appropriate directional derivative is zero.

When viewed in the negative, the condition for a critical phenomenon to occur is straightforwardly that the weight λ has a non-smooth minimum in the domain. This leads to the postulation of

Conjecture 3.5. *Suppose that Φ is convex and has bounded derivative. Then in the minimisation problem at (2.1), critical Nitsche-type phenomena will occur if and only if $|z|^2\eta(z)$ is not smooth at the minimum value.*

The reasoning above verifies this for $\Phi(t) = t$. This conjecture is phrased in terms of annular domains; the following reasoning supports this conjecture.

In the case of the annulus, solutions are radially symmetric on the annulus (independent of $\arg w$), and we can rewrite (2.7) as

$$\eta(r) = \frac{1}{4\pi^2}\,\lambda(x)e^{-4\pi x}, \qquad e^{2\pi x} = r,$$

or, rearranging,

$$\lambda(x) = 4\pi^2\, r^2\, \eta(r).$$

Since we require $\lambda'(x) = 0$ at $x = x_0$, and (by the chain rule) $\frac{d}{dx} = \frac{dr}{dx}\frac{d}{dr}$, we have

$$\frac{d}{dx}\lambda(x)\bigg|_{x=x_0} = 0 = 4\pi^2\frac{dr}{dx}\cdot\frac{d}{dr}\left(r^2\,\eta(r)\right)\bigg|_{r=r_0}$$

where $r_0 = r(x_0)$ is a minimum of $r^2\eta(r)$. Note that $r = e^{2\pi x}$ (which justifies the use of the chain rule) so that $\frac{dr}{dx} = 2\pi e^{2\pi x} = 2\pi r$ and hence

$$8\pi^3 r\cdot\frac{d}{dr}\left(r^2\eta(r)\right) = 0$$

at the minimum. As $r > 0$, division by $8\pi^3 r$ and integration gives the condition

$$\eta(r) \approx \frac{c}{r^2}$$

for some constant c near the minimum; in other words, $r^2\eta(r)$ must be flat near the minimum.

In order for a deformation to be able to stretch the domain infinitely far, $r^2\eta(r)$ must have a smooth minimum inside the domain, in the sense that if the minimum occurs on the boundary, then its directional derivative (orthogonal to and towards the boundary) is zero. The Nitsche range is simply that range within which the minimum of $r^2\eta(r)$, if it occurs in the domain (or on its boundary), is smooth.

An explanation from [6, pp. 16–21] shows what happens in situations outside the Nitsche range. Here, mappings of finite distortion map the domain annulus to a subannulus of the image; the remainder of the target annulus is a region of finite, nonzero measure which is the image of a boundary (zero measure) — thus of infinite distortion; Fig. 3.1 illustrates this.

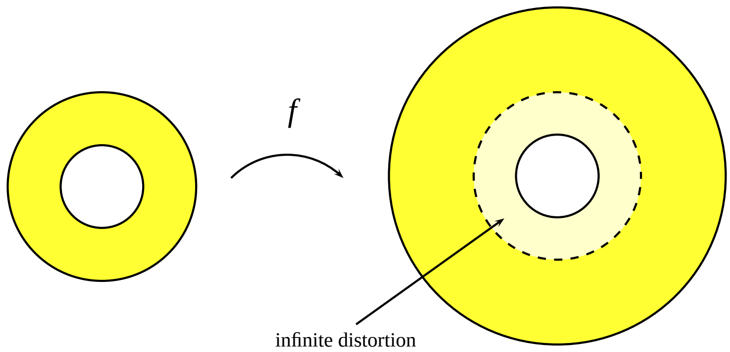

Fig. 3.1: Beyond the Nitsche bound.

In this picture, the minimum value of $r^2\eta(r)$ occurs on the inner boundary of the domain annulus. Despite being beyond the Nitsche bound, the mapping of minimal mean distortion is the limit of a sequence of mappings of finite distortion; however, this limit map has *infinite distortion* on the inner boundary, which is mapped to a region of nonzero measure.

These conditions for critical phenomena will be revisited when various interpretations of the weight function are discussed in Chapter 5.

3.4.3 Failure of Nitsche phenomenon: Φ' unbounded

Should the convex function Φ have unbounded derivative, then there is always a minimiser. In particular we do not see any critical Nitsche-type phenomena for the L^p-norms of mean distortion; that is, with $\Phi(t) = t^p$. Let us deal with the case $\lambda(x) \equiv 1$; the case where λ is bounded is quite similar here.

From Theorem 3.3, recall that

$$\left(1 - \frac{1}{u_x^2(x)}\right) \Phi'\left(u_x(x) + \frac{1}{u_x(x)}\right) = \alpha.$$

The relevant concern is whether u_x is able to satisfy this differential equation. If $\alpha \searrow 0$, then $u_x \searrow 1$ and when $\alpha \nearrow \infty$ we obtain $u_x \nearrow \infty$, so for $\alpha > 0$ the intermediate value theorem provides us with a solution. The remaining case is the question of what happens when $\alpha \to -\infty$, and this is analysed similarly.

3.4.4 Borderline case: Φ' bounded

In this case there are some subtleties. We may view the condition

$$\lambda(x)\left(1 - \frac{1}{u_x^2(x)}\right)\Phi'\left(u_x(x) + \frac{1}{u_x(x)}\right) = \alpha$$

as imposing a condition on the integrability of $\psi(\lambda_0/\lambda(x))$ where ψ is the inverse of the bounded function $t \mapsto \Phi'(t+t^{-1})(1-t^{-2})$. Whether or not there are critical Nitsche-type phenomena will be determined by this integrability condition (*cf.* Conjecture 3.5).

Let us give some illustrative examples in the standard critical Nitsche case with $\ell = 1$, $\lambda(x) = e^{4\pi x}$ (ignoring multiplicative constants). We may assume that $\Phi'(t) \nearrow 1$ and the limiting case $\alpha = e^{4\pi}$:

Case: $\Phi(t) = t - \log(t)$, $\Phi'(t) = 1 - \frac{1}{t}$, $a = a(x) = \alpha/\lambda(x) = e^{-4\pi x} \leqslant 1$.

Here u_x is the largest real root of the polynomial:

$$\begin{aligned} \left(1 - \frac{1}{t+t^{-1}}\right)(1-t^{-2}) &= a \\ P(t) = -1 + t - at^2 - t^3 + (1-a)t^4 &= 0. \end{aligned}$$

Since

$$P\left(\frac{1}{1-a}\right) = -\frac{a^2}{(1-a)^2} < 0,$$

the largest real root

$$u_x(x) > \frac{1}{1-a(x)},$$

and

$$\int_x^1 u_s(s)\, ds > \int_x^1 \frac{ds}{1-e^{-4\pi s}}. \tag{3.21}$$

Now

$$\int_x^1 \frac{ds}{1-e^{-4\pi s}} = \int_x^1 \frac{e^{4\pi s}ds}{e^{4\pi s}-1} = \frac{1}{4\pi}\int_{s=x}^{s=1} \frac{d\zeta}{\zeta-1} = \frac{1}{4\pi}\log|\zeta-1|\Big|_{s=x}^{s=1}$$

where $\zeta = e^{4\pi s}$; hence the left hand side of (3.21) diverges as $x \searrow 0$. Therefore with appropriate choice of α we can always solve $u(0) = 0$ and $u(1) = L$. Hence we find no critical Nitsche phenomena here.

The above case encourages investigation of functions of the form

$$\Phi(t) = t + \frac{1}{(p-1)t^{p-1}} \tag{3.22}$$

for $p \neq 1$. Note that the first case above corresponds essentially to the case $p = 1$, though in the general formula (3.22) Φ is undefined when $p = 1$. This is suggestive of interesting occurrences around $p = 1$.

We have $\Phi'(t) = 1 - \frac{1}{t^p}$, $0 < a = a(x) = e^{-4\pi x} < 1$ for $0 < x < 1$, and hence u_x is the largest real root of the polynomial

$$P(t) = \left(1 - \frac{1}{(t+t^{-1})^p}\right)\left(1 - \frac{1}{t^2}\right) - a = 0. \tag{3.23}$$

Note that when $t > 0$, $P(t)$ is continuous, and

$$P'(t) = \frac{2}{t^3}\left(1 - \frac{1}{(t+t^{-1})^p}\right) + \left(1 - \frac{1}{t^2}\right)^2 \frac{p}{(t+t^{-1})^{p+1}} > 0,$$

meaning P increases monotonically. Also note that $P(1) = -a < 0$, and

$$\lim_{t\to\infty} P(t) = 1 - a > 0,$$

so that P has exactly one real positive root $u_x > 1$. Furthermore, if $p < 0$ then Φ' is unbounded, and when $p = 0$, $\Phi' = 0$. Thus we may assume $p > 0$.

Case: $0 < p < 1$.

Firstly, observe that

$$\begin{aligned} 0 &> -a^2(1-a)^2 \\ &= -a^2+2a^3-a^4 \\ &= 2a-a^2-2a^2+a^3+2a^3-a^4-2a+2a^2-a^3 \\ &= (2a-a^2)(1-a+a^2)-a-a+2a^2-a^3 \\ &= \left(1-(1-a)^2\right)\left(\left(1+(1-a)^2\right)-(1-a)\right)-a\left(1+(1-a)^2\right). \end{aligned}$$

Division by $(1+(1-a)^2)$ gives

$$\left(1-(1-a)^2\right)\left(1-\frac{1-a}{1+(1-a)^2}\right)-a \;<\; 0$$

or, rewriting slightly,

$$\left(1-\frac{1}{\left(\frac{1}{1-a}\right)^2}\right)\left(1-\frac{1}{\frac{1}{1-a}+\frac{1-a}{1}}\right)-a \;<\; 0 \tag{3.24}$$

Now for $0<p<1$,

$$\left(\frac{1}{1-a}+\frac{1-a}{1}\right)^p < \frac{1}{1-a}+\frac{1-a}{1}.$$

Together with (3.24), this shows

$$P\left(\frac{1}{1-a}\right) = \left(1-\frac{1}{\left(\frac{1}{1-a}\right)^2}\right)\left(1-\frac{1}{\left(\frac{1}{1-a}+\frac{1-a}{1}\right)^p}\right)-a<0$$

and hence the largest real root

$$u_x > \frac{1}{1-a}.$$

The integral of the right hand side diverges (see the reasoning for the case $\Phi' = 1-t^{-1}$). Thus with appropriate choice for α we can always solve $u(0)=0, u(1)=L$ and we see no Nitsche phenomena.

Case: $p \geqslant 2$.

Recall equation (3.23). Note that

$$\left(t+\frac{1}{t}\right)^p > \left(t+\frac{1}{t}\right)^2 > t^2.$$

Thus

$$1-\frac{1}{(t+t^{-1})^p} > 1-\frac{1}{t^2}$$

and therefore

$$\left(1-\frac{1}{(t+t^{-1})^p}\right)\left(1-\frac{1}{t^2}\right)-a > \left(1-\frac{1}{t^2}\right)^2-a.$$

The largest real root of $P(t)$ is therefore dominated by the largest real root of

$$Q(t) = \left(1-\frac{1}{t^2}\right)^2-a.$$

Solving $Q(t)=0$ gives the largest real root of $P(t)$

$$u_x < \frac{1}{\sqrt{1-\sqrt{a(x)}}} = \frac{1}{\sqrt{1-e^{-2\pi x}}}.$$

Now

$$\int_0^1 \frac{1}{\sqrt{1-e^{-2\pi x}}}\,dx = \frac{1}{\pi}\int_{x=0}^{x=1}\frac{d\zeta}{\sqrt{\zeta^2-1}}$$

with $\zeta = e^{\pi x}$. Observe that

$$\frac{1}{\pi}\int_{x=0}^{x=1}\frac{d\zeta}{\sqrt{\zeta^2-1}} = \log\left(\zeta+\sqrt{\zeta^2-1}\right)\Big|_{x=0}^{x=1} = \log\left(e^{\pi}+\sqrt{e^{2\pi}-1}\right),$$

a finite number. Therefore $u_x(x)$ is dominated by an integrable function and we must see the Nitsche phenomenon. It is no coincidence that the value of the integral here is strongly reminiscent of that for the "standard" Nitsche case outlined in Section 3.4.1 — the function dominating the root of $P(t)$ here is essentially the same.

It remains to cover

Case: $1 < p < 2$.

For $p > 1$,

$$\left(t + \frac{1}{t}\right)^p > t^p$$

whence

$$1 - \frac{1}{(t+t^{-1})^p} > 1 - \frac{1}{t^p}.$$

Also, for $p < 2$, $t^2 > t^p$ for $t > 1$ and therefore we also have

$$1 - \frac{1}{t^2} > 1 - \frac{1}{t^p}.$$

Therefore the polynomial

$$P(t) = \left(1 - \frac{1}{(t+t^{-1})^p}\right)\left(1 - \frac{1}{t^2}\right) - a > \left(1 - \frac{1}{t^p}\right)^2 - a = Q(t),$$

and the largest real root of $P(t)$ is again dominated by the largest real root of $Q(t)$. Solving $Q(t) = 0$ yields

$$u_x < \frac{1}{\left(1 - \sqrt{a(x)}\right)^{1/p}}.$$

Near $x = 0$, $\sqrt{a(x)} = e^{-2\pi x} \approx 1 - 2\pi x$ and so

$$\int_0^1 \frac{1}{\left(1 - \sqrt{a(x)}\right)^{1/p}}\, dx \approx \left(\frac{1}{2\pi}\right)^{1/p} \int_0^1 \frac{1}{x^{1/p}}\, dx,$$

which converges (to $(2\pi)^{-1/p}$) if and only if $p > 1$. Therefore in this case, too, u_x is dominated by an integrable function and we must see a critical Nitsche-type phenomenon.

The conditions surrounding critical phenomena will be further discussed in Chapter 5. Also, while the Euler-Lagrange equations may be difficult (or, at worst, impossible) to solve analytically, we may still learn some things about them. The next chapter will exhibit the ellipticity of these equations.

4. ELLIPTICITY OF THE EULER-LAGRANGE VARIATIONAL EQUATIONS

Elliptic differential equations arise naturally in the study of materials, where static solutions are found for steady-state phenomena [27, pp. 44–48]. In this chapter, the ellipticity of the variational equations for the minimisation problem is exposed, from which it may be said that the theory expounded here applies, generally, to the material sciences.

4.1 Variational equations

What follows is a long formal calculation, with little or no assumptions about the symmetry or topology of the domain for a given extremal problem. On the basis of examples in the calculus of variations [8, p. 683], however, we do assume the extremal function f for the minimisation problem (2.1) has enough regularity to guarantee that

$$f_{z\bar{z}} = f_{\bar{z}z},$$

for example $f \in W^{2,2}_{loc}(\Omega, \Omega')$.

In [8], extremal problems for mappings of finite distortion are studied, particularly in an elliptic setting, for the minimisation problem

$$\iint_\Omega \mathbb{K}_\Phi(z,f)|dz|^2 = \iint_\Omega \Phi\left(\mathbb{K}(z,f)\right)|dz|^2 \tag{4.1}$$

with $\Phi \in \mathcal{C}^\infty[1,\infty)$ a strictly increasing convex function ($\Phi(1) = 1$). Astala *et al.* [8, pp. 684–687] carry out the following calculation.

Begin with the complex Beltrami coefficient from (1.6)

$$\mu(z,f) := \frac{f_{\bar z}}{f_z}$$

for $f : \overline{\Omega} \to \overline{\Omega'}$ an orientation-preserving diffeomorphism. Vary by a complex-valued test function $\eta \in \mathcal{C}_0^\infty(\Omega)$. For all sufficiently small complex parameters ε, $J(z, f + \varepsilon\eta) > 0$. Note also that boundary values are the same for f and $f + \varepsilon\eta$. Now the complex differential of μ, denoted $\dot\mu$, is a complex linear operator on $\mathcal{C}_0^\infty(\Omega)$, and acts on a test function η as follows:

$$\dot\mu[\eta] = \left.\frac{\partial\mu(z, f+\varepsilon\eta)}{\partial\varepsilon}\right|_{\varepsilon=0} = \frac{f_z\,\eta_{\bar z} - f_{\bar z}\,\eta_z}{(f_z)^2}.$$

The complex differential of

$$\kappa(z,f) = |\mu(z,f)|^2 = \frac{|f_{\bar z}|^2}{|f_z|^2}$$

is calculated by the chain rule:

$$\dot\kappa = \overline{\mu}\dot\mu,$$

so for each test function η,

$$\dot\kappa[\eta] = \kappa\left(\frac{\eta_{\bar z}}{f_{\bar z}} - \frac{\eta_z}{f_z}\right).$$

The linear distortion function $\mathbb{K}$ is simply

$$\mathbb{K}(z,f) = \frac{|f_z|^2 + |f_{\bar z}|^2}{|f_z|^2 - |f_{\bar z}|^2} = \frac{||Df(z)||^2}{J(z,f)} = \frac{1+\kappa(z,f)}{1-\kappa(z,f)},$$

and hence

$$\dot{\mathbb{K}} = \dot{\mathbb{K}}(z,f) = \frac{2\dot\kappa(z,f)}{(1-\kappa(z,f))^2}$$

or

$$\dot{\mathbb{K}}[\eta] = \frac{2\kappa}{(1-\kappa)^2}\left(\frac{\eta_{\bar z}}{f_{\bar z}} - \frac{\eta_z}{f_z}\right).$$

Therefore for $\Phi : [1, \infty) \to [1, \infty)$ convex,

$$\mathbb{K}_\Phi = \Phi\left(\frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2}\right)$$

and

$$\begin{aligned}\dot{\mathbb{K}}_\Phi[\eta] &= \Phi'\left(\frac{1+\kappa}{1-\kappa}\right)\frac{2\kappa}{(1-\kappa)^2}\left(\frac{\eta_{\bar{z}}}{f_{\bar{z}}} - \frac{\eta_z}{f_z}\right) \\ &= \frac{2}{(1-\kappa)^2}\Phi'\left(\frac{1+\kappa}{1-\kappa}\right)\left(\frac{\overline{f_{\bar{z}}}\,\eta_{\bar{z}} - \kappa\,\overline{f_z}\,\eta_z}{|f_z|^2}\right).\end{aligned}$$

The preceding calculation done, the authors of [8] discuss minimisers of the variational integrals as seen in (4.1). We now extend this discussion to include weight functions.

Suppose the aim is to find the variational equations for

$$\iint_\Omega \mathbb{K}_\Phi(z, f)\; w(z)\; |dz|^2, \tag{4.2}$$

where $w(z)$ is some complex-valued nonzero weight function (we take $w(z)$ in favour of our earlier notation of η and λ to emphasize more generality here). Our aim here will be to derive some conditions on the ellipticity of the Euler-Lagrange equations, taking into account this weight function.

If f is in $\mathcal{C}^1(\Omega)$ then

$$\frac{\partial}{\partial \varepsilon}\iint_\Omega \mathbb{K}_\Phi(z, f + \varepsilon\eta)\; w(z)\; |dz|^2\bigg|_{\varepsilon=0} = 0$$

for every test function $\eta \in \mathcal{C}_0^\infty(\Omega)$. Now

$$\frac{\partial}{\partial \varepsilon}\iint_\Omega \mathbb{K}_\Phi(z, f + \varepsilon\eta)\; w(z)\; |dz|^2 = \iint_\Omega \dot{\mathbb{K}}_\Phi(z, f)\; w(z)\; |dz|^2$$

and hence

$$\iint_\Omega \frac{2}{(1-\kappa)^2}\Phi'\left(\frac{1+\kappa}{1-\kappa}\right)\left(\frac{\overline{f_{\bar{z}}}\,\eta_{\bar{z}} - \kappa\,\overline{f_z}\,\eta_z}{|f_z|^2}\right)\; w(z)\; |dz|^2 = 0.$$

That is,

$$\iint_\Omega \frac{2}{(1-\kappa)^2}\Phi'\left(\frac{1+\kappa}{1-\kappa}\right)\left(\frac{\overline{f_{\bar z}}\,\eta_{\bar z}}{|f_z|^2}\right)\ w(z)\ |dz|^2 \\ = \iint_\Omega \frac{2}{(1-\kappa)^2}\Phi'\left(\frac{1+\kappa}{1-\kappa}\right)\left(\frac{\kappa\,\eta_z}{f_z}\right)\ w(z)\ |dz|^2$$

Now set

$$A(\kappa) = \frac{2\kappa}{(1-\kappa)^2}\Phi'\left(\frac{1+\kappa}{1-\kappa}\right) = \Phi'\left(\frac{|f_z|^2+|f_{\bar z}|^2}{|f_z|^2-|f_{\bar z}|^2}\right)\frac{2|f_z|^2|f_{\bar z}|^2}{(|f_z|^2-|f_{\bar z}|^2)^2} \tag{4.3}$$

so we have

$$\iint_\Omega \left(\frac{A(\kappa)w(z)}{f_{\bar z}}\right)\eta_{\bar z}|dz|^2 = \iint_\Omega \left(\frac{A(\kappa)w(z)}{f_z}\right)\eta_z|dz|^2.$$

Integrating by parts gives

$$\left(\frac{A(\kappa)w(z)}{f_{\bar z}}\right)\eta\Big|_{\partial\Omega} - \iint_\Omega \frac{\partial}{\partial\bar z}\left(\frac{A(\kappa)w(z)}{f_{\bar z}}\right)\eta|dz|^2 \\ = \left(\frac{A(\kappa)w(z)}{f_z}\right)\eta\Big|_{\partial\Omega} - \iint_\Omega \frac{\partial}{\partial z}\left(\frac{A(\kappa)w(z)}{f_z}\right)\eta|dz|^2$$

whence, as the test function η vanishes on the boundary,

$$\iint_\Omega \frac{\partial}{\partial\bar z}\left(\frac{A(\kappa)w(z)}{f_{\bar z}}\right)\eta|dz|^2 = \iint_\Omega \frac{\partial}{\partial z}\left(\frac{A(\kappa)w(z)}{f_z}\right)\eta|dz|^2.$$

Rearranging,

$$\iint_\Omega \left[\frac{\partial}{\partial\bar z}\left(\frac{A(\kappa)w(z)}{f_{\bar z}}\right) - \frac{\partial}{\partial z}\left(\frac{A(\kappa)w(z)}{f_z}\right)\right]\eta|dz|^2 = 0.$$

The fundamental lemma of the calculus of variations (Lemma 1.20) now informs us that

$$\frac{\partial}{\partial\bar z}\left(\frac{A(\kappa)w(z)}{f_{\bar z}}\right) - \frac{\partial}{\partial z}\left(\frac{A(\kappa)w(z)}{f_z}\right) = 0,$$

or

$$\frac{\partial}{\partial \bar{z}}\left(\frac{A(\kappa)w(z)}{f_{\bar{z}}}\right) = \frac{\partial}{\partial z}\left(\frac{A(\kappa)w(z)}{f_z}\right). \tag{4.4}$$

It is easy to verify that this corresponds directly to the result from (1.13).

4.2 *The initial result*

First let us proceed as in [8], by setting $w(z) \equiv 1$. The authors do not detail this formal calculation, and mention essentially nothing of the inner workings. However, in order to generalise to the case of weight functions, it is necessary to check how the weight function will behave during this computation.

If $\Phi(t) = t$, then we can use the rule $(\overline{h})_{\bar{z}} = \overline{h_z}$ to simplify (4.4), obtaining

$$\frac{\partial}{\partial z}\left(\frac{|f_z|^2 f_{\bar{z}}}{J(z,f)^2}\right) = \frac{\partial}{\partial \bar{z}}\left(\frac{|f_{\bar{z}}|^2 f_z}{J(z,f)^2}\right). \tag{4.5}$$

First observe that

$$\begin{aligned} J(z,f) &= |f_z|^2 - |f_{\bar{z}}|^2 \\ J(z,f)^2 &= |f_z|^4 - 2|f_z|^2|f_{\bar{z}}|^2 + |f_{\bar{z}}|^4 \end{aligned}$$

and

$$\frac{\partial}{\partial z}\left(J(z,f)^2\right) = 2J(z,f)\left[\left((\overline{f_z})_z f_z + f_{zz}\overline{f_z}\right) - \left((\overline{f_{\bar{z}}})_z f_{\bar{z}} + f_{\bar{z}z}\overline{f_{\bar{z}}}\right)\right] \tag{4.6}$$

$$\frac{\partial}{\partial \bar{z}}\left(J(z,f)^2\right) = 2J(z,f)\left[\left((\overline{f_z})_{\bar{z}} f_z + f_{z\bar{z}}\overline{f_z}\right) - \left((\overline{f_{\bar{z}}})_{\bar{z}} f_{\bar{z}} + f_{\bar{z}\bar{z}}\overline{f_{\bar{z}}}\right)\right] \tag{4.7}$$

$$\frac{\partial}{\partial z}\left(|f_z|^2 f_{\bar{z}}\right) = (\overline{f_z})_z f_z f_{\bar{z}} + \overline{f_z} f_{zz} f_{\bar{z}} + |f_z|^2 f_{\bar{z}z} \tag{4.8}$$

$$\frac{\partial}{\partial \bar{z}}\left(|f_{\bar{z}}|^2 f_z\right) = (\overline{f_{\bar{z}}})_{\bar{z}} f_{\bar{z}} f_z + \overline{f_{\bar{z}}} f_{\bar{z}\bar{z}} f_z + |f_{\bar{z}}|^2 f_{z\bar{z}} \tag{4.9}$$

so that by the quotient rule the numerator of the left hand side of (4.5) becomes

$$N\text{-}LHS(4.5) = J(z,f)^2 \frac{\partial}{\partial z}\left(|f_z|^2 f_{\bar{z}}\right) - |f_z|^2 f_{\bar{z}} \frac{\partial}{\partial z}\left(J(z,f)^2\right)$$

and the numerator of the right hand side

$$N\text{-}RHS(4.5) = J(z,f)^2 \frac{\partial}{\partial \bar{z}} \left(|f_{\bar{z}}|^2 f_z\right) - |f_{\bar{z}}|^2 f_z \frac{\partial}{\partial \bar{z}} \left(J(z,f)^2\right)$$

Note that the denominator is the same in each case — the fourth power of the Jacobian, which is nonzero (almost everywhere) — and cancels. Now set

$$\frac{N\text{-}LHS(4.5)}{J(z,f)} = \frac{N\text{-}RHS(4.5)}{J(z,f)}, \tag{4.10}$$

so that by substituting equations (4.6)–(4.9) into (4.10),

$$\begin{aligned} LHS(4.10) \;=\; & \left(|f_z|^2 - |f_{\bar{z}}|^2\right)\left((\overline{f_z})_z f_z f_{\bar{z}} + \overline{f_z} f_{zz} f_{\bar{z}} + |f_z|^2 f_{\bar{z}z}\right) \\ & -2\,|f_z|^2 f_{\bar{z}} \left[\left((\overline{f_z})_z f_z + f_{zz}\overline{f_z}\right) - \left((\overline{f_{\bar{z}}})_z f_{\bar{z}} + f_{\bar{z}z}\overline{f_{\bar{z}}}\right)\right], \end{aligned}$$

$$\begin{aligned} RHS(4.10) \;=\; & \left(|f_z|^2 - |f_{\bar{z}}|^2\right)\left((\overline{f_{\bar{z}}})_{\bar{z}} f_{\bar{z}} f_z + \overline{f_{\bar{z}}} f_{\bar{z}\bar{z}} f_z + |f_{\bar{z}}|^2 f_{z\bar{z}}\right) \\ & -2\,|f_{\bar{z}}|^2 f_z \left[\left((\overline{f_z})_{\bar{z}} f_z + f_{z\bar{z}}\overline{f_z}\right) - \left((\overline{f_{\bar{z}}})_{\bar{z}} f_{\bar{z}} + f_{\bar{z}\bar{z}}\overline{f_{\bar{z}}}\right)\right]. \end{aligned}$$

Expanding yields

$$\begin{aligned} LHS(4.10) \;=\; & |f_z|^2 (\overline{f_z})_z f_z f_{\bar{z}} + |f_z|^2 \overline{f_z} f_{zz} f_{\bar{z}} + |f_z|^4 f_{\bar{z}z} \\ & -|f_{\bar{z}}|^2 (\overline{f_z})_z f_z f_{\bar{z}} - |f_{\bar{z}}|^2 \overline{f_z} f_{zz} f_{\bar{z}} - |f_{\bar{z}}|^2 |f_z|^2 f_{\bar{z}z} \\ & -2\,|f_z|^2 f_{\bar{z}} (\overline{f_z})_z f_z - 2\,|f_z|^2 f_{\bar{z}} f_{zz} \overline{f_z} \\ & +2\,|f_z|^2 f_{\bar{z}} (\overline{f_{\bar{z}}})_z f_{\bar{z}} + 2\,|f_z|^2 f_{\bar{z}} f_{\bar{z}z} \overline{f_{\bar{z}}}, \end{aligned}$$

$$\begin{aligned} RHS(4.10) \;=\; & |f_z|^2 (\overline{f_{\bar{z}}})_{\bar{z}} f_{\bar{z}} f_z + |f_z|^2 \overline{f_{\bar{z}}} f_{\bar{z}\bar{z}} f_z + |f_z|^2 |f_{\bar{z}}|^2 f_{z\bar{z}} \\ & -|f_{\bar{z}}|^2 (\overline{f_{\bar{z}}})_{\bar{z}} f_{\bar{z}} f_z - |f_{\bar{z}}|^2 \overline{f_{\bar{z}}} f_{\bar{z}\bar{z}} f_z - |f_{\bar{z}}|^4 f_{z\bar{z}} \\ & -2\,|f_{\bar{z}}|^2 f_z (\overline{f_z})_{\bar{z}} f_z - 2\,|f_{\bar{z}}|^2 f_z f_{z\bar{z}} \overline{f_z} \\ & +2\,|f_{\bar{z}}|^2 f_z (\overline{f_{\bar{z}}})_{\bar{z}} f_{\bar{z}} + 2\,|f_{\bar{z}}|^2 f_z f_{\bar{z}\bar{z}} \overline{f_{\bar{z}}}. \end{aligned}$$

Collecting coefficients of the various second derivatives, and making use of the equivalence $(\overline{h})_{\bar z} = \overline{h_z}$, gives

$$\begin{aligned} LHS(4.10) &= f_{zz}\left(|f_z|^2\overline{f_z}f_{\bar z} - |f_{\bar z}|^2\overline{f_z}f_{\bar z} - 2\,|f_z|^2 f_{\bar z}\overline{f_z}\right) \\ &\quad + f_{\bar z z}\left(|f_z|^4 - |f_{\bar z}|^2|f_z|^2 + 2\,|f_z|^2 f_{\bar z}\overline{f_{\bar z}}\right) \\ &\quad + \overline{f_{z\bar z}}\left(|f_z|^2 f_z f_{\bar z} - |f_{\bar z}|^2 f_z f_{\bar z} - 2\,|f_z|^2 f_{\bar z} f_z\right) \\ &\quad + 2\,\overline{f_{\bar z\bar z}}\,|f_z|^2 f_{\bar z} f_{\bar z}, \end{aligned}$$

$$\begin{aligned} RHS(4.10) &= f_{\bar z\bar z}\left(|f_z|^2\overline{f_{\bar z}}f_z - |f_{\bar z}|^2\overline{f_{\bar z}}f_z + 2\,|f_{\bar z}|^2 f_z\overline{f_{\bar z}}\right) \\ &\quad + f_{z\bar z}\left(|f_z|^2|f_{\bar z}|^2 - |f_{\bar z}|^4 - 2\,|f_{\bar z}|^2 f_z\overline{f_z}\right) \\ &\quad + \overline{f_{\bar z z}}\left(|f_z|^2 f_{\bar z} f_z - |f_{\bar z}|^2 f_{\bar z} f_z + 2\,|f_{\bar z}|^2 f_z f_{\bar z}\right) \\ &\quad - 2\,\overline{f_{zz}}|f_{\bar z}|^2 f_z f_z. \end{aligned}$$

Further simplification yields

$$\begin{aligned} LHS(4.10) &= -f_{zz}\overline{f_z}f_{\bar z}\left(|f_z|^2 + |f_{\bar z}|^2\right) + f_{\bar z z}|f_z|^2\left(|f_z|^2 + |f_{\bar z}|^2\right) \\ &\quad - \overline{f_{z\bar z}}f_z f_{\bar z}\left(|f_z|^2 + |f_{\bar z}|^2\right) + 2\,\overline{f_{\bar z\bar z}}\,|f_z|^2 f_{\bar z}^2, \end{aligned}$$

$$\begin{aligned} RHS(4.10) &= f_{\bar z\bar z}f_z\overline{f_{\bar z}}\left(|f_z|^2 + |f_{\bar z}|^2\right) - f_{z\bar z}|f_{\bar z}|^2\left(|f_z|^2 + |f_{\bar z}|^2\right) \\ &\quad + \overline{f_{\bar z z}}f_z f_{\bar z}\left(|f_z|^2 + |f_{\bar z}|^2\right) - 2\,\overline{f_{zz}}|f_{\bar z}|^2 f_z^2. \end{aligned}$$

Equating $LHS(4.10) = RHS(4.10)$ and rearranging now gives

$$\begin{aligned} & f_{z\bar z}|f_{\bar z}|^2\left(|f_z|^2 + |f_{\bar z}|^2\right) + f_{\bar z z}|f_z|^2\left(|f_z|^2 + |f_{\bar z}|^2\right) - \overline{f_{z\bar z}}f_z f_{\bar z}\left(|f_z|^2 + |f_{\bar z}|^2\right) \\ &\quad - \overline{f_{\bar z z}}f_z f_{\bar z}\left(|f_z|^2 + |f_{\bar z}|^2\right) \\ = \; & f_{\bar z\bar z}f_z\overline{f_{\bar z}}\left(|f_z|^2 + |f_{\bar z}|^2\right) + f_{zz}\overline{f_z}f_{\bar z}\left(|f_z|^2 + |f_{\bar z}|^2\right) - 2\,\overline{f_{zz}}|f_{\bar z}|^2 f_z^2 \\ &\quad - 2\,\overline{f_{\bar z\bar z}}\,|f_z|^2 f_{\bar z}^2, \end{aligned}$$

and further simplification yields

$$\gamma^2 f_{z\bar z} - 2\,\gamma f_z f_{\bar z}\overline{f_{z\bar z}} = \gamma f_z\overline{f_{\bar z}}f_{\bar z\bar z} + \gamma\overline{f_z}f_{\bar z}f_{zz} - 2\,|f_{\bar z}|^2 f_z^2\overline{f_{zz}} - 2\,|f_z|^2 f_{\bar z}^2\overline{f_{\bar z\bar z}}, \quad (4.11)$$

with $\gamma = |f_z|^2 + |f_{\bar{z}}|^2$. Note that this is precisely the step at which the additional assumption, introduced at the start of this calculation, that $f_{z\bar{z}} = f_{\bar{z}z}$ (f is continuously differentiable) is required.

Taking equation (4.11) and its conjugate (bearing in mind that γ is real) gives a set of simultaneous equations:

$$\begin{aligned} \gamma^2 f_{z\bar{z}} - 2\,\gamma f_z f_{\bar{z}} \overline{f_{z\bar{z}}} &= \gamma f_z \overline{f_{\bar{z}}} f_{\bar{z}\bar{z}} + \gamma \overline{f_z} f_{\bar{z}} f_{zz} - 2\,|f_{\bar{z}}|^2 f_z^2 \overline{f_{zz}} \\ &\quad -2\,|f_z|^2 f_{\bar{z}}^2 \overline{f_{\bar{z}\bar{z}}}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \gamma^2 \overline{f_{z\bar{z}}} - 2\,\gamma \overline{f_z}\,\overline{f_{\bar{z}}} f_{z\bar{z}} &= \gamma \overline{f_z} f_{\bar{z}} \overline{f_{\bar{z}\bar{z}}} + \gamma f_z \overline{f_{\bar{z}}}\,\overline{f_{zz}} - 2\,|f_{\bar{z}}|^2 (\overline{f_z})^2 f_{zz} \\ &\quad -2\,|f_z|^2 (\overline{f_{\bar{z}}})^2 f_{\bar{z}\bar{z}}. \end{aligned} \tag{4.13}$$

Multiplying (4.12) through by γ, and (4.13) by $2 f_z f_{\bar{z}}$, and rearranging the latter yields

$$\begin{aligned} \gamma^3 f_{z\bar{z}} - 2\,\gamma^2 f_z f_{\bar{z}} \overline{f_{z\bar{z}}} &= \gamma^2 f_z \overline{f_{\bar{z}}} f_{\bar{z}\bar{z}} + \gamma^2 \overline{f_z} f_{\bar{z}} f_{zz} - 2\,\gamma |f_{\bar{z}}|^2 f_z^2 \overline{f_{zz}} \\ &\quad - 2\,\gamma |f_z|^2 f_{\bar{z}}^2 \overline{f_{\bar{z}\bar{z}}}, \\ 2\,\gamma^2 f_z f_{\bar{z}} \overline{f_{z\bar{z}}} - 4\,\gamma |f_z|^2 |f_{\bar{z}}|^2 f_{z\bar{z}} &= 2\,\gamma |f_z|^2 f_{\bar{z}}^2 \overline{f_{\bar{z}\bar{z}}} - 4\,\overline{f_z} f_{\bar{z}} |f_{\bar{z}}|^2 |f_z|^2 f_{zz} \\ &\quad -4\, f_z \overline{f_{\bar{z}}} |f_z|^2 |f_{\bar{z}}|^2 f_{\bar{z}\bar{z}} + 2\,\gamma f_z^2 |f_{\bar{z}}|^2\, \overline{f_{zz}}. \end{aligned}$$

Adding these two equations together to eliminate $\overline{f_{z\bar{z}}}$ and simplifying gives

$$\gamma \iota f_{z\bar{z}} = \iota \overline{f_z} f_{\bar{z}} f_{zz} + \iota f_z \overline{f_{\bar{z}}} f_{\bar{z}\bar{z}}$$

with $\iota = \gamma^2 - 4|f_z|^2|f_{\bar{z}}|^2 = J(z,f)^2 \neq 0$, and division by $\gamma = |f_z|^2 + |f_{\bar{z}}|^2$ and cancelling ι yields

$$f_{z\bar{z}} = \frac{\overline{f_z} f_{\bar{z}}}{|f_z|^2 + |f_{\bar{z}}|^2} f_{zz} + \frac{f_z \overline{f_{\bar{z}}}}{|f_z|^2 + |f_{\bar{z}}|^2} f_{\bar{z}\bar{z}}.$$

Thus (4.5) reduces to the equation

$$f_{z\bar{z}} = \alpha f_{zz} + \beta f_{\bar{z}\bar{z}}, \tag{4.14}$$

with

$$\alpha = \frac{\mu(z,f)}{1 + |\mu(z,f)|^2} = \frac{\overline{f_z} f_{\bar{z}}}{|f_z|^2 + |f_{\bar{z}}|^2} = \overline{\beta}.$$

Note that

$$|\alpha| + |\beta| = \frac{2|\mu|}{1+|\mu|^2} < 1$$

whence (4.14) is elliptic [7, pp. 231–232].

This recreates the proof from Astala *et al.* [8], with the lengthy missing stages included. The next section expands this result to the weighted case.

4.3 The weighted expansion

Again assuming $\Phi(t) = t$, in the case where a weight function $w = w(z)$ is introduced in the minimisation problem (4.2), equation (4.4) becomes

$$\frac{\partial}{\partial z}\left(\frac{\overline{w}|f_z|^2 f_{\bar{z}}}{J(z,f)^2}\right) = \frac{\partial}{\partial \bar{z}}\left(\frac{\overline{w}|f_{\bar{z}}|^2 f_z}{J(z,f)^2}\right), \tag{4.15}$$

observing again the useful property $(\bar{h})_{\bar{z}} = \overline{h_z}$. Note that now

$$\begin{aligned}
\frac{\partial}{\partial z}\left(\overline{w}|f_z|^2 f_{\bar{z}}\right) &= \overline{w}_z|f_z|^2 f_{\bar{z}} + \overline{w}\left((\overline{f_z})_z f_z f_{\bar{z}} + \overline{f_z} f_{zz} f_{\bar{z}} + |f_z|^2 f_{\bar{z}z}\right), \\
\frac{\partial}{\partial \bar{z}}\left(\overline{w}|f_{\bar{z}}|^2 f_z\right) &= \overline{w}_{\bar{z}}|f_{\bar{z}}|^2 f_z + \overline{w}\left((\overline{f_{\bar{z}}})_{\bar{z}} f_{\bar{z}} f_z + \overline{f_{\bar{z}}} f_{\bar{z}\bar{z}} f_z + |f_{\bar{z}}|^2 f_{z\bar{z}}\right).
\end{aligned}$$

Proceeding as before (and cancelling $J(z,f)$ whenever possible) gives

$$\begin{aligned}
LHS(4.15) &= \overline{w}\left[f_{\bar{z}z}|f_z|^2\left(|f_z|^2+|f_{\bar{z}}|^2\right) - f_{zz}\overline{f_z} f_{\bar{z}}\left(|f_z|^2+|f_{\bar{z}}|^2\right)\right. \\
&\qquad \left.+2\,\overline{f_{\bar{z}\bar{z}}}\,|f_z|^2 f_{\bar{z}}^2 - \overline{f_{z\bar{z}}} f_z f_{\bar{z}}\left(|f_z|^2+|f_{\bar{z}}|^2\right)\right] \\
&\qquad\quad +\overline{w}_z|f_z|^2 f_{\bar{z}} J(z,f), \\
RHS(4.15) &= \overline{w}\left[f_{\bar{z}\bar{z}} f_z \overline{f_{\bar{z}}}\left(|f_z|^2+|f_{\bar{z}}|^2\right) - f_{z\bar{z}}|f_{\bar{z}}|^2\left(|f_z|^2+|f_{\bar{z}}|^2\right)\right. \\
&\qquad \left.+\overline{f_{\bar{z}z}} f_z f_{\bar{z}}\left(|f_z|^2+|f_{\bar{z}}|^2\right) - 2\,\overline{f_{zz}}|f_{\bar{z}}|^2 f_z^2\right] \\
&\qquad\quad +\overline{w}_{\bar{z}}|f_{\bar{z}}|^2 f_z J(z,f).
\end{aligned}$$

Rearranging and simplifying now yields

$$\begin{aligned}
\overline{w}\left(\gamma^2 f_{z\bar{z}} - 2\,\gamma f_z f_{\bar{z}} \overline{f_{z\bar{z}}}\right) = \overline{w}\left(\gamma f_z \overline{f_{\bar{z}}} f_{\bar{z}\bar{z}} + \gamma \overline{f_z} f_{\bar{z}} f_{zz} - 2\,|f_{\bar{z}}|^2 f_z^2 \overline{f_{zz}} - 2\,|f_z|^2 f_{\bar{z}}^2 \overline{f_{\bar{z}\bar{z}}}\right) \\
-\overline{w}_z|f_z|^2 f_{\bar{z}} J(z,f) + \overline{w}_{\bar{z}}|f_{\bar{z}}|^2 f_z J(z,f)
\end{aligned}$$

with $\gamma = |f_z|^2 + |f_{\bar z}|^2$. Note again that the additional assumption that $f_{z\bar z} = f_{\bar z z}$ is required here. Taking this equation and its conjugate gives a set of simultaneous equations:

$$\begin{aligned}\overline{w}\left(\gamma^2 f_{z\bar z} - 2\,\gamma f_z f_{\bar z}\overline{f_{z\bar z}}\right) &= \overline{w}\left(\gamma f_z \overline{f_{\bar z}} f_{\bar z\bar z} + \gamma \overline{f_z} f_{\bar z} f_{zz} - 2\,|f_{\bar z}|^2 f_z^2 \overline{f_{zz}}\right.\\ &\quad \left. -2\,|f_z|^2 f_{\bar z}^2 \overline{f_{\bar z\bar z}}\right) - \overline{w}_z |f_z|^2 f_{\bar z} J(z,f)\\ &\quad +\overline{w}_{\bar z}|f_{\bar z}|^2 f_z J(z,f), \qquad (4.16)\end{aligned}$$

$$\begin{aligned}w\left(\gamma^2 \overline{f_{z\bar z}} - 2\,\gamma \overline{f_z}\,\overline{f_{\bar z}} f_{z\bar z}\right) &= w\left(\gamma \overline{f_z} f_{\bar z} \overline{f_{\bar z\bar z}} + \gamma f_z \overline{f_{\bar z}}\,\overline{f_{zz}} - 2\,|f_{\bar z}|^2 (\overline{f_z})^2 f_{zz}\right.\\ &\quad \left. -2\,|f_z|^2 (\overline{f_{\bar z}})^2 f_{\bar z\bar z}\right) - \overline{\overline{w}_z}|f_z|^2 \overline{f_{\bar z}} J(z,f)\\ &\quad +\overline{\overline{w}_{\bar z}}|f_{\bar z}|^2 \overline{f_z} J(z,f). \qquad (4.17)\end{aligned}$$

Multiplying (4.16) through by $w\gamma$, and (4.17) by $2\overline{w} f_z f_{\bar z}$, and rearranging yields

$$\begin{aligned}|w|^2\left(\gamma^3 f_{z\bar z} - 2\,\gamma^2 f_z f_{\bar z}\overline{f_{z\bar z}}\right) &= |w|^2\left(\gamma^2 f_z \overline{f_{\bar z}} f_{\bar z\bar z} + \gamma^2 \overline{f_z} f_{\bar z} f_{zz} - 2\,\gamma |f_{\bar z}|^2 f_z^2 \overline{f_{zz}}\right.\\ &\quad \left. -2\,\gamma |f_z|^2 f_{\bar z}^2 \overline{f_{\bar z\bar z}}\right)\\ &\quad + w f_z f_{\bar z}\gamma J(z,f)\left(\overline{w}_{\bar z}\overline{f_{\bar z}} - \overline{w}_z \overline{f_z}\right),\end{aligned}$$

$$\begin{aligned}|w|^2 \left(-4\,\gamma |f_z|^2 |f_{\bar z}|^2 f_{z\bar z} \right.\\ \left. +2\,\gamma^2 f_z f_{\bar z}\overline{f_{z\bar z}}\right) &= |w|^2\left(-4\,\overline{f_z} f_{\bar z}|f_{\bar z}|^2|f_z|^2 f_{zz} - 4\, f_z \overline{f_{\bar z}} |f_z|^2 |f_{\bar z}|^2 f_{\bar z\bar z}\right.\\ &\quad \left. +\,2\,\gamma |f_z|^2 f_{\bar z}^2 \overline{f_{\bar z\bar z}} + 2\,\gamma f_z^2 |f_{\bar z}|^2\,\overline{f_{zz}}\right)\\ &\quad + 2\overline{w}|f_z|^2|f_{\bar z}|^2 J(z,f)\left(f_{\bar z}\overline{\overline{w}_{\bar z}} - f_z \overline{\overline{w}_z}\right).\end{aligned}$$

Adding these two equations together to eliminate $\overline{f_{z\bar z}}$ gives

$$|w|^2\gamma\iota f_{z\bar z} = |w|^2 \iota \overline{f_z} f_{\bar z} f_{zz} + |w|^2 \iota f_z \overline{f_{\bar z}} f_{\bar z\bar z} + |w|^2 \iota\gamma F(z)$$

with $\iota = \gamma^2 - 4|f_z|^2|f_{\bar z}|^2 = J(z,f)^2 \neq 0$ and

$$F(z) = \frac{1}{|w|^2\gamma J(z,f)}\left[w f_z f_{\bar z}\gamma\left(\overline{w}_{\bar z}\overline{f_{\bar z}} - \overline{w}_z\overline{f_z}\right) + 2\overline{w}|f_z|^2|f_{\bar z}|^2\left(f_{\bar z} w_z - f_z w_{\bar z}\right)\right]$$

contains only first derivatives of f. Division by $|w|^2\iota\gamma$ yields

$$f_{z\bar{z}} = \frac{\overline{f_z} f_{\bar{z}}}{|f_z|^2 + |f_{\bar{z}}|^2} f_{zz} + \frac{f_z \overline{f_{\bar{z}}}}{|f_z|^2 + |f_{\bar{z}}|^2} f_{\bar{z}\bar{z}} + F(z),$$

and so (4.15) reduces to a partial differential equation of the form

$$f_{z\bar{z}} = \alpha f_{zz} + \beta f_{\bar{z}\bar{z}} + F(z),$$

with α and β as in (4.14). Note that here the ellipticity conditions are the same as those for (4.14).

Here we have introduced

$$F(z) \quad = \quad \frac{f_z f_{\bar{z}}}{J(z,f)} \left(\frac{\overline{w}_{\bar{z}} \overline{f_{\bar{z}}} - \overline{w}_z \overline{f_z}}{\overline{w}} \right) + \frac{2|f_z|^2 |f_{\bar{z}}|^2}{\gamma J(z,f)} \left(\frac{f_{\bar{z}} w_z - f_z w_{\bar{z}}}{w} \right). \qquad (4.18)$$

Note that

- $F(z)$ is homogeneous of degree 1 in f.
- $F(z)$ is homogeneous of degree 0 in w.
- If w is constant then $F(z) = 0$, confirming that this result agrees with the unweighted case.
- $F(z)$ does not involve f directly, only its first derivatives.

This chapter shows, perhaps surprisingly, that the ellipticity conditions of the Euler-Lagrange equations are independent of the weight function $w(z)$ — generalising the ellipticity results from [8]. The above list of properties of $F(z)$ may also provide opportunities for interesting future research.

5. INTERPRETING THE WEIGHT FUNCTION — SPECIFIC CASES & APPLICATIONS

Here will be discussed some interesting interpetations of the weight function; we put forth ideas on how the weight function might be interpreted to lend a physical (or other) application to the results of this thesis. Elliptic partial differential equations arise naturally in the study of materials; it is not surprising, therefore, that our results provide links to the physical sciences.

5.1 Thickness

With $\lambda(x) = e^{4\pi x}$ (i.e. $\eta = 1$ in (2.1)), the Nitsche phenomenon for annuli obtains. Observe also that if $\lambda(x)$ is constant, $u_x(x)$ is constant and therefore $u(x)$ in such cases is a linear mapping that can be stretched to any length, as determined by the constant α. The weight function $\lambda(x)$ can also be viewed as a thickness or density. In particular an object with a "cut" gives an interesting Nitsche-type phenomenon.

Consider the (continuous positive) weight function on $[0, \ell]$

$$\lambda(x) = \begin{cases} 1 - \frac{x}{\ell} & \text{if } \; 0 \leqslant x < \frac{\ell}{2}; \\ \frac{x}{\ell} & \text{if } \; \frac{\ell}{2} \leqslant x \leqslant \ell. \end{cases}$$

This is stretched to $[0, L]$ with $\Phi'(x) = 1$ (the case $\Phi'(x)$ constant is similar). First, note that for each x, $\alpha \leqslant \lambda(x)$ and hence $\alpha \leqslant \frac{1}{2}$. Recall that Theorem 3.3 gives (3.12), which yields

$$u = \int \sqrt{\frac{\lambda(x)}{\lambda(x) - \alpha}}\, dx.$$

Taking $\lambda_1 = 1 - \frac{x}{\ell}$ and $\lambda_2 = \frac{x}{\ell}$, and simplifying,

$$\begin{aligned} u_1 &= \int \sqrt{\frac{\ell - x}{\ell - x - \ell\alpha}}, \\ u_2 &= \int \sqrt{\frac{x}{x - \ell\alpha}}. \end{aligned}$$

The change of variables $s = \ell - x$ in u_1 shows that the analysis of u_1 is similar to u_2 with a change of sign (this should be no surprise as λ is symmetric). Further changing variables by $t = \sqrt{s - \ell\alpha}$, carrying out the integration and back-substituting,

$$u_1 = -\sqrt{\ell - x}\sqrt{\ell - x - \ell\alpha} - \ell\alpha \log(\sqrt{\ell - x - \ell\alpha} + \sqrt{\ell - x}) + C$$

for some constant C. The constant is found using the boundary condition $u(0) = 0$ (preserving the order of boundary components). Rearranging gives

$$u_1 = \left[\sqrt{1-\alpha} - \sqrt{\left(1 - \tfrac{x}{\ell}\right)^2 - \alpha\left(1 - \tfrac{x}{\ell}\right)} \right.$$
$$\left. - \alpha \log\left(\tfrac{\sqrt{1-\frac{x}{\ell}} + \sqrt{1 - \frac{x}{\ell} - \alpha}}{1 + \sqrt{1-\alpha}}\right)\right] \ell.$$

A similar calculation, keeping in mind that $u(\ell) = L$, shows that

$$u_2 = L - \left[\sqrt{1-\alpha} - \sqrt{\left(\frac{x}{\ell}\right)^2 - \alpha\left(\frac{x}{\ell}\right)} - \alpha \log\left(\frac{\sqrt{\frac{x}{\ell}} + \sqrt{\frac{x}{\ell} - \alpha}}{1 + \sqrt{1-\alpha}}\right)\right] \ell.$$

Now we require that $u_1(\ell/2) - u_2(\ell/2) = 0$ (i.e. that they meet in the middle). Thus

$$L = \left[2\sqrt{1-\alpha} - \sqrt{1-2\alpha} - 2\alpha \log\left(\frac{1 + \sqrt{1-2\alpha}}{\sqrt{2}\left(1 + \sqrt{1-\alpha}\right)}\right)\right] \ell,$$

and hence L can be made arbitrarily small by letting α tend to $-\infty$. However, there is an upper limit on L;

$$\alpha \leqslant \min_{x \in [0,\ell]} \lambda(x) = \frac{1}{2},$$

and thus

$$L_{\max} = \left[\sqrt{2} - \log\left(\sqrt{2} - 1\right)\right] \ell \approx 2.296\ell.$$

This is a Nitsche bound on the maximal stretch. It is the value α that determines how far the final stretch can be; the bound on α determines the bound on L; see Figure 5.1.

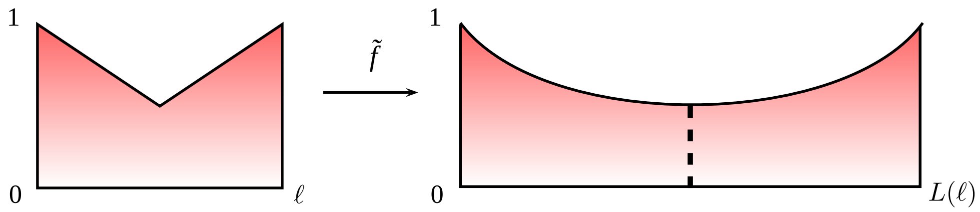

Fig. 5.1: Stretching of a cut of base length ℓ ($\alpha = \frac{1}{2}$).

As ℓ tends to 0, so does L. But now suppose the cut is embedded in a block of some fixed length (say 1), the result is not quite as clear.

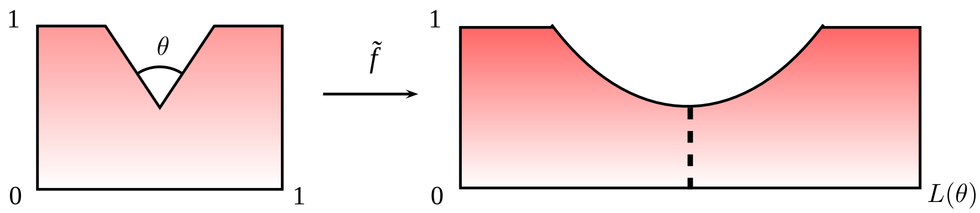

Fig. 5.2: Stretching of a block with an open cut ($\alpha = \frac{1}{2}$).

Figure 5.2 might suggest (initially at least) that as θ approaches 0, L approaches 1 (i.e. the block is unable to stretch). But this is in fact not the case. Simply calculating the limit reveals that L tends to $\sqrt{2}$; see Figure 5.3.

Once again we observe that α determines the length of the final stretch; it is in fact $\sqrt{\frac{1}{1-\alpha}}$. That is, the deeper the cut, the smaller the maximal stretch — which coheres with intuition when interpreting the weight function as a thickness of some elastic material.

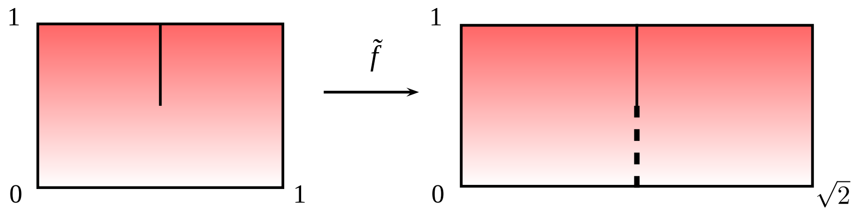

Fig. 5.3: Stretching of a block with a straight-line cut ($\alpha = \frac{1}{2}$).

A different sequence of weight functions with the same (qualitative) limiting case as that of the above is a sequence of cusps (Figure 5.4),

$$\lambda_1(x) = 1 + (1 - x^n)^{1/n}, \qquad \lambda_2(x) = 1 + \big(1 - (2 - x)^n\big)^{1/n}.$$

The minimum of the weight function λ determines the critical value $\alpha = 1$. Here the maximum stretch is limited (between $2\sqrt{2}$ and π). Carrying out

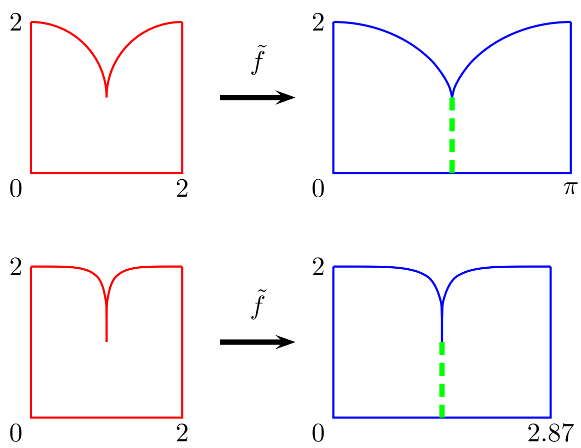

Fig. 5.4: Stretching of a cusp ($\alpha = 1$, $n = 2$ (top), $n = 5$ (bottom)).

the calculation for the sequence of weight functions by letting $n \to \infty$ shows that this sequence also decreases to a limit of $2\sqrt{2}$. This confirms the results of the cut block case.

It is possible to interpret the weight function as a kind of density function;

this case is essentially analogous to the thickness interpretation already presented — where density has a non-smooth minimum value inside the domain being stretched, critical phenomena will occur.

The functional that is being minimised is, as previously discussed, a kind of energy functional (see Chapter 1). As Theorem 3.3 shows, the critical value for the constant α is attained by setting α equal to (or in some nice relation to) the minimum value of the weight function. Thus the following interpretation may be made: during a stretching process, the energy that is applied focuses at the weakest point (the minimum value of the weight), and a critical phenomenon corresponds to an "infinite energy density" buildup at that point, meaning that there is too much energy for the substance being stretched to maintain integrity.

5.2 Metric interpretation

We may choose to interpret the quantity $\eta(z)\ |dz|^2$ (or, equivalently, $\sqrt{\eta(z)}\ |dz|$) as being some kind of surface metric on the annulus. Curvature metrics η that are conformally flat — meaning that locally they behave like a Euclidean manifold — are characterised by the fundamental equation

$$-\frac{\Delta \log \eta(r)}{\eta^2(r)} = k, \tag{5.1}$$

where k is the *(Gaussian) curvature* of the surface (see [20]). Note the radial symmetry — such symmetry will be assumed throughout this section. Of particular interest are those metrics with constant curvature. Where $k > 0$, the space is said to have *spherical* curvature; where $k = 0$ we have the *flat* (Euclidean) metric; and where $k < 0$ spaces exhibit *hyperbolic* curvature. Bearing in mind that, in polar coordinates, the Laplace operator

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and, restricting ourselves to the radially symmetric case, η is independent of θ, thus the $\partial^2/\partial\theta^2$ term vanishes. An observation key to the next few examples is that, as a consequence of Theorem 3.3, a Nitsche-type critical phenomenon occurs if and only if the minimum of $r^2\eta(r)$ on the domain annulus is not smooth (recall the end of Section 3.4.2).

5.2.1 Solving the metric equation

Substituting $\sigma(r) = \log \eta(r)$ in (5.1) gives

$$\sigma_{rr}(r) + \frac{1}{r}\sigma_r(r) = -ke^{2\sigma(r)},$$

where subscripts denote derivatives. Further substituting $\Theta(r) = 2\sigma(r)$ and rearranging gives

$$\Theta_{rr}(r) + \frac{1}{r}\Theta_r(r) + 2ke^{\Theta(r)} = 0.$$

Using substitution and a change of variables

$$\Upsilon(s) = 2\log r + \Theta(r), \qquad r = e^s$$

we obtain

$$\Upsilon_s = 2 + r\Theta_r, \qquad \Upsilon_{ss} = r\Theta_r + r^2\Theta_{rr},$$

whence

$$\Theta_r = \frac{1}{r}\left(\Upsilon_s - 2\right), \qquad \Theta_{rr} = \frac{1}{r^2}\left(\Upsilon_{ss} + 2 - \Upsilon_s\right).$$

Following these substitutions through transforms the curvature equation (5.1) into

$$\frac{1}{r^2}\left(\Upsilon_{ss} + 2 - \Upsilon_s\right) + \frac{1}{r^2}\left(\Upsilon_s - 2\right) + 2ke^{\Upsilon - 2\log r} = 0$$

which reduces to the elegant second-order equation

$$\Upsilon_{ss}(s) + 2ke^{\Upsilon(s)} = 0. \tag{5.2}$$

It is interesting to note that this is exactly the Frank-Kamenetskii equation with parameter $\delta = 2k$ (see [41] and [26, pp. 76–80]). The parameter δ in this model determines a critical point in reactions, specifying exactly where thermal runaway occurs. It is also interesting to note the direct connection between δ and k — this suggest there may be further critical phenomena depending on the size of the curvature. These results show potential for further research.

With the help of the further substitution

$$\Gamma = \Upsilon_s$$

and observing that

$$\Upsilon_{ss} = \Gamma_s = \Gamma \frac{d\Gamma}{d\Upsilon}$$

via the chain rule, equation (5.2) becomes separable:

$$\Gamma \frac{d\Gamma}{d\Upsilon} = -2ke^{\Upsilon}.$$

Solving yields

$$\Gamma^2 = -4ke^{\Upsilon} + C$$

whence

$$\Upsilon_s = \pm 2\sqrt{-ke^{\Upsilon} + C} \tag{5.3}$$

where C is an integration constant and will depend on boundary conditions.

5.2.2 Euclidean (flat) metrics

This is the situation in which the classical Nitsche phenomenon arises, the curvature $k = 0$. Substituting $\sigma(r) = \log \eta(r)$ into (5.1) and evaluating the Laplacian yields the differential equation

$$\sigma_{rr} + \frac{1}{r}\sigma_r = 0.$$

Solving this gives $\sigma(r) = \log \frac{C}{r^\alpha}$, or

$$\eta(r) = \frac{C}{r^\alpha} \tag{5.4}$$

where C is constant. Equivalently, we can take $k = 0$ in (5.3) to get

$$\Upsilon_s = constant.$$

Following the substitutions backwards confirms the solution (5.4).

The case $\alpha = 2$ corresponds to the choice $\eta(w) = C/|w|^2$ in a Nitsche type problem (Theorem 3.4). Notice that $ds^2 = |dw|^2/|w|^2$ is the flat metric of $\mathbb{C} \setminus \{0\}$. In particular all circles centered on the origin in this metric have the same length and $\mathbb{C} \setminus \{0\}$ with this metric is isometric to a (flat) cylinder (recall Figure 2.1).

If we translate this to the Grötzsch problem via (2.7), we obtain

$$\eta(w) = \frac{C}{|w|^2} = \frac{1}{4\pi}\lambda(z)e^{-4\pi\Re e(z)}.$$

Bearing in mind that $w = e^{2\pi z}$, this relation shows $\lambda(z)$ must be a (nonzero) constant.

Correspondingly, in the Grötzsch-type problem (Theorem 3.3) with $\lambda(z)$ constant there is always a linear minimiser $f : [0, \ell] \times [0, 1] \to [0, L] \times [0, 1]$,

$$f(z) = \frac{L}{\ell}x + iy$$

irrespective of the convex functional Φ. Hence within a flat metric space we see no Nitsche phenomena.

5.2.3 Spherical metrics

We may read the solution (5.3) as

$$\frac{d\Upsilon}{ds} = \pm 2\sqrt{k}\sqrt{c_0^2 - e^{\Upsilon}}$$

for some suitable constant c_0. Separating variables and integrating,

$$\int \frac{d\Upsilon}{\sqrt{c_0^2 - e^{\Upsilon}}} = \int \pm 2\sqrt{k}\, ds. \tag{5.5}$$

The right hand side is straightforward; the left hand side may be evaluated by multiplying by 1 in the form $e^{-\Upsilon/2}/e^{-\Upsilon/2}$ to get

$$LHS(5.5) = \int \frac{e^{-\Upsilon/2}\, d\Upsilon}{e^{-\Upsilon/2}\sqrt{c_0^2 - e^{\Upsilon}}} = \int \frac{e^{-\Upsilon/2}\, d\Upsilon}{\sqrt{c_0^2 e^{-\Upsilon} - 1}}.$$

The substitution $u = c_0 e^{-\Upsilon/2}$ with $-2c_0^{-1} du = e^{-\Upsilon/2}\, d\Upsilon$ transforms this integral into

$$-\frac{2}{c_0}\int \frac{du}{\sqrt{u^2 - 1}}.$$

Evaluating both sides of (5.5) and backsubstituting for u therefore yields

$$-\frac{2}{c_0}\cosh^{-1}\left(c_0 e^{-\Upsilon/2}\right) = \pm 2\sqrt{k}\, s + \alpha$$

for a constant of integration α. Rearranging and using the relation $s = \log r$ gives

$$\Upsilon = \log\left[\frac{c_0^2}{\cosh^2\left(\mp c_0\sqrt{k}\log r - \frac{c_0\alpha}{2}\right)}\right],$$

and simplifying the cosh term using exponent and logarithm rules, and the definition of cosh in terms of exp, gives

$$\cosh\left(\mp c_0\sqrt{k}\log r - \frac{c_0\alpha}{2}\right) = \frac{1}{2}\left(c_1 r^{\mp c_0\sqrt{k}} + c_1^{-1} r^{\pm c_0\sqrt{k}}\right)$$

where the constant $c_1 = \exp\left(-\frac{c_0\alpha}{2}\right) > 0$ to simplify constants. Thus

$$\Upsilon = \log\left[\frac{4c_0^2}{\left(c_1 r^{\mp c_0\sqrt{k}} + c_1^{-1} r^{\pm c_0\sqrt{k}}\right)^2}\right].$$

Undoing the substitutions backward throughout the calculation and multiplying both denominator and numerator of the result by $c_1 r^{\pm c_0\sqrt{k}}$ shows that

$$\eta(r) = \frac{2\ c_0\ c_1\ r^{\pm c_0\sqrt{k}-1}}{c_1^2 + r^{\pm 2c_0\sqrt{k}}}. \tag{5.6}$$

Note that $c_1 > 0$ and since we require $\eta(r) > 0$ we have $c_0 > 0$ also.

Now, the criterion for the occurrence of a critical phenomenon is the smoothness at the minimum of $r^2\eta(r)$. Thus we wish to find the nature of the minimum of

$$F(r) = \frac{2\ c_0\ c_1\ r^{\pm c_0\sqrt{k}+1}}{c_1^2 + r^{\pm 2c_0\sqrt{k}}}.$$

With a little work, it can be shown that

$$F'(r) = \frac{2\ c_0\ c_1\ r^{\pm c_0\sqrt{k}}\left(c_1^2 + r^{\pm 2c_0\sqrt{k}} \pm c_0 c_1^2\sqrt{k} \mp c_0\sqrt{k}\ r^{\pm 2c_0\sqrt{k}}\right)}{\left(c_1^2 + r^{\pm 2c_0\sqrt{k}}\right)^2}.$$

Observe F' is only possibly undefined only when $r = 0$ but in this case F is also undefined (and we require $r > 0$ as $s = \log r$). Hence F exhibits no minimum here. Solving $F'(r) = 0$ shows

$$c_1^2 + r^{\pm 2c_0\sqrt{k}} \pm c_0 c_1^2\sqrt{k} \mp c_0\sqrt{k} r^{\pm 2c_0\sqrt{k}} = 0$$

or

$$r = \left(\frac{c_1^2(1 \pm c_0\sqrt{k})}{-1 \pm c_0\sqrt{k}}\right)^{\pm 1/(2c_0\sqrt{k})}. \tag{5.7}$$

Let us illustrate how this result may be generally analysed with a typical case: $\eta(r) = (1 + r^2)^{-1}$. This metric can be shown to have constant positive curvature $k = +4$ and is often used as a prototypical example of a constant positive curvature metric. This corresponds to the case $c_0 = \frac{1}{2}, c_1 = 1$ and $k = 4$ in (5.6).

Locating where $r^2\eta(r) = 0$ via (5.7) yields the only critical points $r = 0$ and $r = \infty$. Therefore, for any annulus of bounded modulus the extreme values of $r^2\eta(r)$ occur on the boundary and have nonzero derivative there — meaning we must see critical Nitsche-type phenomena.

5.2.4 Hyperbolic metrics

To solve equation (5.1) meaningfully for $k < 0$, restrict z to $|z| < 1$. Then the solution (5.3) becomes

$$\frac{d\Upsilon}{ds} = \pm 2\sqrt{-k}\sqrt{e^{\Upsilon} + c_0^2}$$

for some appropriate constant c_0. By reasoning analogous to the previous section (with a few sign changes, and observing the use of the hyperbolic sine function instead of the hyperbolic cosine as substitution to carry out the integration), it is easy to arrive at

$$\eta(r) = \frac{2\ c_0\ c_1\ r^{\pm c_0\sqrt{-k}-1}}{c_1^2 - r^{\pm 2c_0\sqrt{-k}}}. \tag{5.8}$$

Setting $F(r) = r^2\eta(r)$ we obtain

$$F'(r) = \frac{2\ c_0\ c_1\ r^{\pm c_0\sqrt{-k}}\left(c_1^2 - r^{\pm 2c_0\sqrt{-k}} \pm c_0 c_1^2\sqrt{-k} \pm c_0\sqrt{-k}\ r^{\pm 2c_0\sqrt{-k}}\right)}{\left(c_1^2 - r^{\pm 2c_0\sqrt{-k}}\right)^2}$$

and solving for critical points we find either $r = 0$ or

$$r = \left(\frac{c_1^2(1 \pm c_0\sqrt{-k})}{1 \mp c_0\sqrt{-k}}\right)^{1/\pm 2c_0\sqrt{-k}} \tag{5.9}$$

In the illustrative case $\eta(r) = (1 - r^2)^{-1}$, the hyperbolic metric on the unit disk with constant curvature -4, we have $c_0 = 1/2, c_1 = 1$ and $k = -4$ in (5.8). Solving (5.9) again shows that extrema for annuli with bounded modulus must occur only on their boundary; thus, there must always be critical Nitsche phenomena.

5.2.5 Constant nonzero curvature

In (5.7) and (5.9) we may experiment with different values for k and the positive constants c_0 and c_1. Take $k > 0$ for instance, and set $\beta = \pm c_0\sqrt{k} \neq 0$

for ease of notation. Then (5.7) becomes

$$r = \left(\frac{c_1^2(\beta+1)}{\beta-1}\right)^{1/2\beta}. \tag{5.10}$$

If $|\beta| < 1$ there are no real solutions, and $|\beta| \searrow 1$ leads to $r = 0$ or $r = \infty$ (as was the case for $\eta(r) = (1+r^2)^{-1}$). Yet for $|\beta| > 1$ there are valid solutions; and now the relevant question is whether the solutions to (5.10) locate a minimum or a maximum. Recall the derivative

$$F'(r) = \frac{2\, c_0\, c_1\, r^\beta \left(c_1^2 + r^{2\beta} + c_1^2\beta - \beta\, r^{2\beta}\right)}{\left(c_1^2 + r^{2\beta}\right)^2}.$$

After some computation, it can be shown that

$$F''(r) = \frac{2\, c_0\, c_1\, r^{\beta-1}}{\left(c_1^2 + r^{2\beta}\right)^3}\left[c_1^4\beta(\beta+1) - 6c_1^2\beta^2 r^{2\beta} + \beta(\beta-1)r^{4\beta}\right]$$

and we inspect the sign of this second derivative at the critical point (5.10). Note that as r, c_1 and c_2 are all positive, we need only determine the sign of

$$c_1^4\beta(\beta+1) - 6c_1^2\beta^2 r^{2\beta} + \beta(\beta-1)r^{4\beta}.$$

Substituting (5.10) yields

$$c_1^4\beta(\beta+1) - 6c_1^2\beta^2\left(\frac{c_1^2(\beta+1)}{\beta-1}\right) + \beta(\beta+1)\left(\frac{c_1^2(\beta+1)}{\beta-1}\right)^2$$

which simplifies to

$$-4c_1^4\beta^2\frac{\beta+1}{\beta-1} < 0$$

whenever $|\beta| > 1$. Thus the critical point is a maximum — and hence once again, for an annulus of bounded modulus the minimum of $r^2\eta(r)$ must occur on the boundary and has nonzero derivative, showing that critical Nitsche phenomena will occur.

Next, take metrics of constant negative curvature; setting $\beta = \pm c_0\sqrt{-k}$, we obtain

$$r = \left(\frac{c_1^2(1+\beta)}{1-\beta} \right)^{1/2\beta}. \tag{5.11}$$

Recall

$$F'(r) = \frac{2\ c_0\ c_1\ r^\beta \left(c_1^2 - r^{2\beta} + c_1^2\beta + \beta\ r^{2\beta}\right)}{\left(c_1^2 - r^{\pm 2c_0\sqrt{-k}}\right)^2}.$$

This time $|\beta| > 1$ yields no real solutions and $|\beta| \nearrow 1$ again gives the critical points at 0 and ∞. Hence we consider $|\beta| < 1$. Calculation of the second derivative of F shows

$$F''(r) = \frac{2\ c_0\ c_1\ r^{\beta-1}}{\left(c_1^2 - r^{2\beta}\right)^3} \left[c_1^4\beta(\beta+1) + 6c_1^2\beta^2 r^{2\beta} + \beta(\beta-1)r^{4\beta}\right]$$

where the sign is straightforward to analyse. The numerator is positive (being a product and sum of positive terms), and we are left to consider only the sign of

$$c_1^2 - r^{2\beta}.$$

Substituting (5.11) gives

$$c_1^2 - \left(\frac{c_1^2(1+\beta)}{1-\beta} \right) = c_1^2 \left(1 - \frac{1+\beta}{1-\beta} \right) = \frac{-2\beta c_1^2}{1-\beta} > 0$$

as $|\beta| < 1$. Thus $r^2\eta(r)$ is convex and has a smooth minimum at the critical value in (5.11). Therefore, whether or not there are critical phenomena will depend on where the minimum of $r^2\eta(r)$ falls. If it is in the interior of the domain, then the minimum is smooth and no Nitsche phenomena obtain. If it falls on the boundary, then again the minimum is smooth. But if it is outside the domain, then the minimum of $r^2\eta(r)$ *in* the domain is on the boundary and not smooth, and therefore we must see a critical Nitsche phenomenon.

5.2.6 Nonconstant curvature

This section contains some test cases, motivated by the previous sections on constant curvature, of cases where the curvature may be varied with r.

Take

$$\eta(r) = \frac{r^{-p}}{1 - r^p}$$

for $r < 1$. This is an *incomplete* metric on the unit disc (for some p) — a metric on the *punctured* unit disc. On substituting into the characteristic equation (5.1) to determine its curvature, we see (after some simplification) that its curvature is negative,

$$k = -p^2 r^{3p-2} < 0.$$

To determine whether critical phenomena occur, again look at where the minimum of $r^2\eta(r)$ occurs, and whether it is smooth. Thus, after some simplification,

$$\frac{d}{dr}(r^2\eta(r)) = \frac{2 - p + (2p-2)r^p}{r^{p-1}(1-r^p)^2}$$

which, on solving equal to zero for a smooth minimum, yields

$$r_0 = \left(\frac{p-2}{2p-2}\right)^{1/p}.$$

This requires $p > 2$ (or $p < 1$) to be real, as might be expected; the critical case is $p = 2$ which yields $r^2\eta(r) = (1 - r^2)^{-1}$. When $p < 2$, $r^2\eta(r)$ has a smooth minimum at $r = 0$, and hence there is no smooth interior minimum for annuli of bounded modulus and we must see critical phenomena. However, for $p > 2$, it is easily verified that $0 < r_0 < 1$ and that this value gives a minimum for $r^2\eta(r)$. Thus if this minimum value lies inside the domain annulus, there will be no critical phenomena; otherwise, there will be.

On the other hand, we may take

$$\eta(r) = \frac{r^{-p}}{1 + r^p}$$

and investigate it for critical phenomena. Again, observe that this metric is incomplete as it is undefined when $r = 0$. The characteristic equation (5.1) yields

$$k = p^2 r^{3p-2} > 0$$

whence this is a spherical metric. After some simplification, we find

$$\frac{d}{dr}(r^2\eta(r)) = \frac{2-p+(2-2p)r^p}{r^{p-1}(1-r^p)^2}$$

which yields

$$r_0 = \left(\frac{p-2}{2-2p}\right)^{1/p}.$$

For r_0 to be real, this equation requires $1 < p < 2$. For these values, however, a second derivative test reveals that $r^2\eta(r)$ has a maximum at r_0; whence the minimum must occur on a boundary and we must see critical phenomena.

Putting together everything from Section 5.2, we have shown the following:

Theorem 5.1. *In the minimisation problem (2.1),*

$$I(f) = \int_\Omega \Phi\left(\mathbb{K}(z,f)\right)\ \eta(z)\ |dz|^2,$$

if $\eta(z)$ is radially symmetric and satisfies the characteristic equation for Gaussian curvature (5.1)

$$-\frac{\Delta \log \eta(z)}{\eta^2(z)} = k,$$

with k constant, the following holds:

(i) If $k > 0$, critical phenomena always obtain.

(ii) If $k = 0$, no critical phenomena obtain.

(iii) If $k < 0$, critical phenomena obtain if and only if the minimum of $|z|^2\eta(z)$ occurs on the boundary and is not smooth, in the sense that its directional derivative is nonzero.

The last subsection (Section 5.2.6) further encourages the postulation of:

Conjecture 5.2. *The conclusions from Theorem 5.1 also hold in cases of nonconstant curvature.*

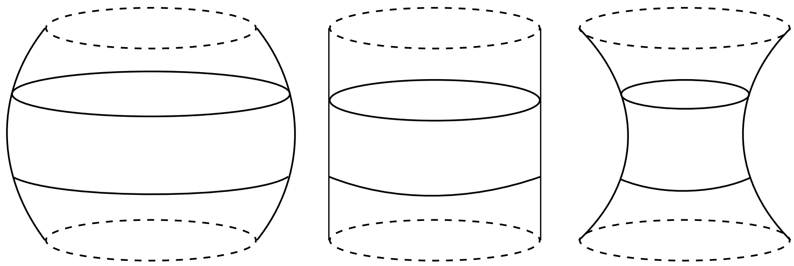

Fig. 5.5: Spherical (left), flat (center), and hyperbolic (right) cylinders.

Figure 5.5 shows these results qualitatively, the (topological) annulus in each case is the domain of the deformation f. In the case of the spherical metric, the minimum value of $r^2\eta(r)$ (qualitatively represented by the smallest circle in the diagram) always occurs at a boundary, and this minimum is not smooth. In the flat metric, the minimum of $r^2\eta(r)$ is everywhere (since it is constant) and smooth; whence no critical phenomena. In the hyperbolic case, however, the minimum of $r^2\eta(r)$ may occur on a boundary, or it may occur in the interior of the domain annulus. The latter is the case in Fig. 5.5 — the domain contains the "tightest" part of the hyperbolic surface and, since this is smooth, no critical phenomena occur.

6. SEQUITUR

The most significant result of this thesis is, essentially, Theorem 3.3. It examines energy functionals in a wide range of integrability classes as they arise from mean pointwise distortion, from a theoretical point of view. This is an idea central to many problems in the physical sciences. The theorem proves for the first time that symmetries of the functional are inherited by the solution — normally an *a priori* assumption. It also provides an alternative proof to part of Nitsche's 1962 conjecture, and expands on it significantly.

Initiating the study of functionals of this kind is motivated by various kinds of distortion problems as seen in Chapter 2. The celebrated Teichmüller problem, for example, may be seen as a problem of this kind. For this problem the *maximal* distortion may be solved within the class of quasiconformal mappings — however, *mean* distortion has been shown not to be minimisable in this class, subject to certain boundary conditions [30]. Closely related is the Dirichlet problem, first encountered in Chapter 1.

As a result of Theorem 3.3, it becomes possible to determine qualitatively when minimisers to energy functionals exist and when they do not. One result in particular is that the important L^p integrability classes for $p > 1$ admit no critical phenomena, which is perhaps surprising considering that the classical Nitsche critical case occurs in exactly the space L^1. Furthermore, this theorem is established within the class of mappings of finite distortion, a wide class of mappings including the classically important conformal and quasiconformal mappings; these are generally solutions to Dirichlet-type minimal energy problems.

In the search for a proof of this theorem, it becomes clear in Chapter 2 that some powerful tools typically used in the study of quasiconformal mappings are not available when considering mappings of finite distortion.

The usual normal families type argument does not apply as there are no obvious modulus of continuity estimates. Another more recent technique has been adapted to tackle this problem; the inequality found at Section 3.2. Additionally, the important connection between Nitsche-type and Grötzsch-type problems is elucidated in Section 2.7. In fact, Theorem 3.3 concerns Grötzsch-type problems; the theorem, when applied to the annulus, is a little more difficult to analyse; this is seen in Chapter 3.

Further, the results from Chapter 4 show that the variational equations arising from considering distortion functionals are generally elliptic, irrespective of the particular weight function, wherefore the wealth of the theory of elliptic partial differential equations may be applied broadly.

These findings are examined in more specific settings in Chapter 5, where it is discovered that, within a suitable interpretation, the nonexistence of minimisers corresponds to a range of critical phenomena. In the case of a thickness or density function, this corresponds to tearing of the material. When minimising mean distortion one must place the largest distortion where the weight is smallest, so that this contributes least to the total distortion. This may correspond to a material being stretched until there is “zero density” (or minimum density necessary for structural integrity) left at the point of minimum weight, where the structure will tear or break.

Chapter 5 also discusses weight functions as metrics. One result is the possible connection between interpreting the weight function as a metric and the well-known Frank-Kamenetskii thermal ignition theory in chemistry. Another result is that critical phenomena occur only in non-flat settings (of course, under the assumption there are no additional conditions on the convex function Φ) and, in particular, they always occur in spherical metrics but only sometimes in hyperbolic metrics.

One of the highlights of the theorem is that, due to the differential equation from Theorem 3.3,

$$\lambda(x) \left(1 - \frac{1}{u_x^2}\right) \Phi' \left(u_x + \frac{1}{u_x}\right) = \alpha,$$

the constant α determines (or is determined by) the final deformation. In cases where such a relationship cannot be satisfied, then for that particular λ, deformations may still exist — but there will be no deformations of *minimal* weighted mean distortion. In fact, it may even be possible to identify minimising sequences (as per [8]) that degenerate.

Thus, we may extend our idea of interpreting the weight as a thickness to discuss the case where there are multiple cuts. Observe that α is constrained by the minimum of λ over the interval of interest. Hence the bound on the maximal stretch is also constrained by the lowest dip in λ that gives the maximal α, and critical phenomena occur at this point — this is precisely what one might expect.

It is a profound result that the Euler-Lagrange equation for minimal weighted mean distortion is first order with free parameter α. The differential equation in the theorem is in fact the Euler-Lagrange variational equation; these equations are normally *second*-order, with two boundary conditions and one free parameter. Here we see a first order equation, with two boundary conditions. It is precisely this which explains how critical Nitsche-type phenomena arise, for a first-order equation with two boundary conditions does not have solutions in all cases, and varying the parameter may not lead to a desired solution.

6.1 Further research

The interpretations of the weight function seen in Chapter 5 are a start on modelling tearing type phenomena practically that can be expanded upon. The cut block (for example) incorporates no mechanism that would allow the region to become thinner under the mapping, whereas in reality it often is the case that thickness changes under deformations.

Recall the function $F(z)$ from (4.18) that arises from considering the ellipticity of the variational equations in the generalised setting. An important research question is: can we obtain some condition on the weight $w(z)$ in order for critical phenomena to occur (for example, perhaps its derivative must not be zero on the boundary)? Special cases of the weight function may yield useful results and insights — for example, what if the weight is a function of only one variable, or if the weight is a real-valued function?

In analysing the main theorem in terms of critical Nitsche-type phenomena, recall Conjecture 3.5:

> Critical Nitsche-type phenomena in the minimisation problem at (2.1), where Φ is convex and has bounded derivative, obtain if and only if the minimum value of $|z|^2\eta(z)$ is not smooth.

This is verified for the case $\Phi(t) = t$. It will be interesting to see whether this conjecture holds. Since Section 3.4.3 shows that there is always a minimiser (i.e. no critical phenomena) when Φ' is unbounded, the case Φ' bounded is the case worthy of investigation. In fact, Section 3.4.4 uses this idea to arrive at a critical case for Φ.

Furthermore, Theorem 3.3 speaks only to doubly connected regions with radial symmetry. What happens if the region is more generally multiply connected? Only a special case of a result from Ahlfors [3, pp. 255–256] is used to simplify the problem (Theorem 2.2). But the full result from [3, pp. 255–256] hints at how results may be generalised from here — to look at "sub-annuli" within an annulus, insofar as symmetry considerations allow.

The results from the end of Chapter 5 seem to indicate a close connection between the theory developed in this thesis and thermal ignition phenomena in physical chemistry; it may be worthwhile exploring this connection to see how deep it goes.

The author wishes to finish here, with a thought: *etiam eruditio*. This thesis has accomplished a lot, and there is a lot of potential for further research, both theoretical and applied.

APPENDIX

A. COMMON DEFINITIONS AND NOTATION

Throughout this thesis, familiarity with some topological notions and some aspects of analysis will be assumed, and certain notational conventions will be used:

- $\widehat{\mathbb{C}}$ denotes the extended complex plane, also known as the Riemann sphere. That is, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
- If $z = x + iy$ is a complex number, x denotes the real part of z, written as $\Re(z)$, and y denotes the imaginary part of z, written as $\Im(z)$.
- $D(z_0, r)$ denotes an *open disc*, centered at z_0, with radius $r > 0$. That is, $D(z_0, r) = \{z : |z - z_0| < r\}$. Similarly, a closed disc centered at z_0 with radius $r > 0$ is denoted by $\overline{D}(z_0, r)$.
- A set S containing z_0 is called a *neighborhood of* z_0 if there exists a real number $r > 0$ such that $D(z_0, r) \subset S$.
- A *deleted neighborhood of* z_0 is a set $S \setminus \{z_0\}$ such that S is a neighborhood of z_0.
- A set S is said to be *open* if for each $z \in S$ there exists $r > 0$ such that $D(z, r) \subset S$.
- For any set S, $\widetilde{S}$ denotes the complement of S. That is, $\widetilde{S} = \mathbb{C} \setminus S$.
- A set S is said to be *closed* if $\widetilde{S}$ is open.
- A function $f : X \to Y$ is called a *homeomorphism* from the space X to the space Y if it is a continuous bijection with a continuous inverse.

- A point z is in the *boundary of* S, ∂S, if every neighborhood of z intersects both S and its complement $\widetilde{S}$.

- The *closure of* S, $\overline{S}$, is given by $\overline{S} = S \cup \partial S$.

- S is called *bounded* if there exists some $r > 0$ such that $S \subset D(0, r)$.

- *Compact* sets are sets that are both closed and bounded.

- S is said to be *disconnected* if there exist open sets A and B such that:

  (i) $A \cap S \neq \emptyset$;

  (ii) $B \cap S \neq \emptyset$;

  (iii) $S \subset A \cup B$; and

  (iv) $(A \cap S) \cap (B \cap S) = \emptyset$.

  A set that is not disconnected is called *connected*.

- A *region* is a nonempty open connected subset of the complex plane.

- The function $f(z)$ is said to have *limit* L *as* z *tends to* a,

$$\lim_{z \to a} f(z) = L$$

  if and only if to each real number $\varepsilon > 0$ there corresponds a real number $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - a| < \delta$.

- The function $f(z)$ is said to be *continuous at* a if and only if $\lim_{z \to a} f(z) = f(a)$. A *continuous function* is one that is continuous at all points where it is defined.

- If $f(x, y) = u(x, y) + iv(x, y)$ is a function of a complex variable $z = x + iy$, then the matrix of partials

$$Df(x, y) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

is known as the *differential matrix* of f.

- The term *Jacobian* refers to the *Jacobian determinant*, the determinant of the differential matrix, $\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$.

- A *path* in $\mathbb{R}^n$ is a continuous mapping $\gamma : I \to \mathbb{R}^n$ where I is an interval in R. The path is said to be *open* or *closed* according to whether I is open or closed.

- The *locus* of a path is the point set $\gamma I \subset \mathbb{R}^n$. A *subpath* of a path $\gamma : I \to \mathbb{R}^n$ is the restriction of γ to a subinterval of I.

- Let $\gamma : [a, b] \to \mathbb{R}^n$ be a closed path, and let $a = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k = b$ be a subdivision of $[a, b]$. The supremum of the sums

$$\sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})|$$

over all subdivisions of $[a, b]$ is called the *length* of γ and denoted by $\ell(\gamma)$. Note that $0 \leqslant \ell(\gamma) \leqslant \infty$, where $\ell(\gamma) = 0$ if and only if γ is constant. Furthermore, if $\ell(\gamma) < \infty$, γ is *rectifiable*; otherwise γ is non-rectifiable.

- A path given by $\gamma(t) = x(t) + iy(t)$ for $a \leqslant t \leqslant b$ is called a *smooth path* if its derivative $\gamma'(t) = x'(t) + iy'(t)$ with respect to the real parameter t exists for each t in $[a, b]$ and if the function γ' is continuous on the interval $[a, b]$.

- A path $\gamma : [a, b] \to \mathbb{C}$ is said to be *piecewise smooth* if there exists a partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$ with the property that the restriction of γ to each $[t_{k-1}, t_k]$ for $1 \leqslant k \leqslant n$ is a smooth path.

- If $A \subset \mathbb{R}^n$, then $m_n^*(A)$ denotes the Lebesgue outer measure of A. If A is measurable the star may be omitted; the n is omitted when there is no danger of misunderstanding.

- By $C^n(\Omega)$ we denote the class of all complex-valued functions that are n times continuously differentiable on the region Ω, and $C(\Omega)$ denotes the class of all complex-valued functions that are continuous on the region Ω.

- A *diffeomorphism*, $f : \Omega \to \Omega'$ is a C^1-homeomorphism whose Jacobian $J(x, f)$ does not vanish.

- Let (X, μ) be a measure space. A function is *p-integrable* if its L^p norm, defined as

$$||f||_p = \left(\int |f|^p \right)^{\frac{1}{p}},$$

  is finite. The collection of all p-integrable functions is denoted by L^p.

- S^n is a unit n-sphere. That is to say, a unit sphere in $\mathbb{R}^{n+1}$.

- Two regions Ω, Ω' (in $\mathbb{R}^n$ or $\mathbb{C}$) are *conformally equivalent* if there exists a conformal mapping $f : \Omega \to \Omega'$.

- An integral over a domain is *locally finite* if, for every point x of the domain, there is an open neighbourhood N_x about x such that the integral over N_x is finite.

- A function $f : (X, d_X) \to (Y, d_Y)$ admits $\omega : [0, \infty) \to [0, \infty)$ as a *modulus of continuity* at x_0 if and only if

$$d(f(x_0)), f(x)) \leqslant \omega\left(d(x_0, x)\right)$$

  for all x in X. If f admits ω as a modulus of continuity at each x_0 in X, then f admits ω as a (global) modulus of continuity. We may equivalently say that f is ω-continuous (at x_0).

- Given two normed vector spaces V and W (over the same field, say $\mathbb{R}$ or $\mathbb{C}$), the *operator norm* $|A|$ of a (continuous) linear map $A : V \to W$ is given by

$$|A| = \inf\left\{c : ||Av|| \leqslant c||v|| \text{ for all } v \in V\right\}.$$

Equivalently,

$$|A| = \sup\{||Av|| : v \in V \text{ with } ||v|| \leqslant 1\}.$$

There are other, equivalent definitions often used in literature.

- The *Hilbert-Schmidt* norm of a linear operator A is

$$||A||^2 = \sum_i ||Ae_i||^2$$

  where the e_i form an orthonormal basis for the space on which A operates. The *mean Hilbert-Schmidt* norm of A is

$$||A||^2 = \frac{1}{n}\mathrm{tr}\left(A^t A\right)$$

- L^p is the space of all functions (on a measure space (X, Σ, μ)) that are Lebesgue p-integrable; i.e. the set of all functions for which

$$||f||_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} < \infty.$$

- A *test function* is a smooth function (infinitely differentiable) with compact support (identically zero outside of a bounded set). Within this thesis, we are interested in test functions that vanish on the boundary of a given domain; the author may at times neglect to state this explicitly, but will be assumed.

B. SELECTED BACKGROUND THEOREMS AND PROPOSITIONS

Theorem B.1. *(The **Intermediate Value Theorem**)*
If f is a continuous real-valued function on an interval $[a, b]$ and y is a real number satisfying $f(a) \leqslant y \leqslant f(b)$, or $f(b) \leqslant y \leqslant f(a)$. Then there exists $x \in [a, b]$ such that $f(x) = y$.

For a proof of this fundamental theorem from analysis, see e.g. [11, p. 51]. In short, this theorem says that the image of an interval is an interval.

Theorem B.2. *(The **Inverse Function Theorem**)*
Let $f : \Omega \to \mathbb{C}$ be analytic, with f' continuous and $f'(z_0) \neq 0$. Then there exists a neighborhood U of z_0 and a neighborhood V of $f(z_0)$ such that $f : U \to V$ is a bijection and the inverse function f^{-1} is analytic, with derivative given by

$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z_0)} \qquad \text{where} \qquad w = f(z_0).$$

For a proof, see [29, pp. 77–78].

Theorem B.3. *(The **Bolzano-Weierstrass Theorem**)*
Suppose that $(z_n)_{n \geqslant 1}$ is a bounded sequence in $\mathbb{C}$. Then (z_n) has at least one accumulation point. Moreover, this sequence has exactly one accumulation point if and only if it is a convergent sequence with the unique accumulation point as its limit.

For a proof, see [35, p. 53], or [11, p. 48].

Theorem B.4. *(The **Cauchy Criterion for Uniform Convergence**)*
Suppose that each function in a sequence (f_n) is defined on a set U. The sequence converges uniformly on U if and only if it is a uniform Cauchy sequence on U.

For a proof, see [35, p. 246].

Proposition B.5. *If Ω is a simply connected region of the complex plane, then for every function f that is both analytic and free of zeros in Ω there exists a branch of $\log f(z)$ in this region.*

For a proof, see [35, pp. 196–197].

Theorem B.6. *(**Liouville's Theorem**)*
The only bounded entire functions on $\mathbb{C}$ are constant.

For a proof, see [29, pp. 171–172].

Theorem B.7. *(**Cauchy's Integral Formula**)*
Let f be analytic in an open disk D and γ a closed, piecewise smooth path in D. Then

$$n(\gamma, z)f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

for every z in $D \setminus |\gamma|$.

For a proof, see [35, pp. 161–162]

Theorem B.8. *Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ a nonconstant analytic mapping. Then $f(\Omega)$ is also a domain (in particular, it is open).*

For a proof, see [29, pp. 435–436].

Theorem B.9. *(**Hurwitz's Theorem**)*
Suppose that each function in a sequence (f_n) is analytic and zero-free in a domain Ω and that $f_n \to f$ normally in Ω. Then either f is free of zeros in Ω or it is identically zero there.

For a proof, see [35, pp. 348–349].

Theorem B.10. *(The **Arzelà-Ascoli Theorem**)*
A family $\mathscr{F}$ of functions that are defined and continuous on some region Ω is a normal family if and only if it is both equicontinuous and pointwise bounded in Ω.

For a proof, see [35, pp. 282–284].

Theorem B.11. ***(Montel's Theorem)***
Let $\mathscr{F}$ be a family of functions that are analytic in an open set Ω. Suppose that $\mathscr{F}$ is locally bounded in Ω. Then $\mathscr{F}$ is a normal family in this set.

For a proof, see [35, p. 285].

Theorem B.12. *(**Green's Formula**, Flux-Divergence Form)*
Let $\partial\Omega$ be a positively oriented, piecewise smooth, simple closed curve in the plane $\mathbb{R}^2$, and let Ω be the region bounded by $\partial\Omega$. If M and N are functions of (x, y) defined on an open region containing Ω and have continuous partial derivatives there, then

$$\iint_\Omega \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy = \oint_{\partial\Omega} M\, dy - N\, dx.$$

For a proof, see e.g. [5, § 18.4] or [1, § 7.4].

Theorem B.13. ***(Fubini's Theorem)***
Suppose A and B are complete measure spaces. Let $f(x, y)$ be $A \times B$-measurable. If $\int_{A\times B} |f|\, d(x, y) < \infty$ where $d(x, y)$ is a product measure on the space $A \times B$, then

$$\int_A \left(\int_B f(x, y)\, dy \right) dx = \int_B \left(\int_A f(x, y)\, dx \right) dy = \int_{A\times B} f(x, y)\, d(x, y).$$

For a particularly neat proof and further discussion, see [25].

Theorem B.14. *(**Jensen's Inequality**, measure-theoretic version)*
Let a, b be real numbers with $a < b$, and $f : [a, b] \to \mathbb{R}$ a Lebesgue-integrable function, and let Φ be a convex real function. Then Jensen's inequality states

$$\Phi \left(\int_a^b f(x)\, dx \right) \leqslant \frac{1}{b-a} \int_a^b \Phi\left((b-a) f(x)\right)\, dx$$

Note that this is not the most general statement of Jensen's inequality in a measure-theoretic setting; this is just the version needed here. For a proof, see [38, p. 62]

Theorem B.15. *(**Hölder's Inequality** for L^p spaces)*
*Let $f \in L^p$ and $g \in L^q$ for **Hölder conjugates** p and q, meaning*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\int fg \leqslant \left(\int |f|^p \right)^{\frac{1}{p}} \left(\int |g|^q \right)^{\frac{1}{q}},$$

where the (Lebesgue) integrals are assumed with respect to the appropriate measure in each space.

For a proof, see almost any graduate text on analysis or measure theory; e.g. [38, pp. 63,66]

Theorem B.16. *(**Gauss' Divergence Theorem**, planar version)*
Let Ω be a compact subset of $\mathbb{R}^2$ with a piecewise smooth boundary. If $\vec{F}$ is a continuously differentiable vector field defined on a neighbourhood of Ω, then

$$\iint_\Omega (\nabla \cdot \vec{F})\, dx\, dy = \oint_{\partial\Omega} \vec{F} \cdot \hat{n}\, ds$$

where $\hat{n}$ is the unit normal vector.

For a proof, see [1, § 7.3].

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