Teaching and learning algebra word problems

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ABSTRACT

This study reports on a classroom design experiment into the teaching and learning of algebra word problems. The study was set in the mathematics department of a co-educational secondary school, and involved two teachers and 30 Year 12 students. The teachers and the researcher worked collaboratively to design and implement an intervention that focused explicitly on translation between word problems and algebra.

Two issues were considered: the impact of the intervention on students, and the impact of the study on teachers. Students' responses to classroom activities, supported by individual student interviews, were used to examine their approaches to solving algebra word problems. Video-stimulated focus group interviews explored students' responses to classroom activities, and informed the ongoing planning and implementation of classroom activities. Data about the impact on teachers' understandings, beliefs and practices was gathered through individual interviews and classroom observations as well as the ongoing dialogue of the research team.

The most significant impact on students related to their understandings of algebra as a tool. Some students were able to combine their new-found translation skills with algebraic manipulation skills to solve word problems algebraically. However, other students had difficulties at various stages of the translation process. Factors identified as supporting student learning included explicit objectives and clarity around what was to be learnt, the opportunity for students to engage in conversations about their thinking and to practise translating between verbal and symbolic forms, structured progression of learning tasks, time to consolidate understandings, and, a heuristic for problem solving.

Participation in the project impacted on teachers in two ways: firstly, with regards to the immediate intervention of teaching algebra; and secondly, with regards to teaching strategies for mathematics in general. Translation activities provided a tool for teachers to engage students in mathematical discussion, enabling them to elicit and build on student thinking. As teachers developed new understandings about how their students approached word problems they gained insight into the importance of selecting problems for which students needed to use algebra. However, teachers experienced difficulty designing quality instructional activities, including algebra word problems, that pressed for algebraic thinking. The focus on translation within the study encouraged a shift in teacher practice away from a skills-focus toward a problem-focus.

Whilst it was apparent that instructional focus on translation shifted teachers and students away from an emphasis on procedure, it was equally clear that translation alone is insufficient as an intervention. Students need both procedural and relational understandings to develop an understanding of the use of algebra as a tool to solve word problems. Students also need to develop fluency with a range of strategies, including algebra, in order to be able to select appropriate strategies to solve particular problems. This study affirmed for teachers that teaching with a focus on understanding can provide an effective and efficient method for increasing students' motivation, interest and success.

PREFACE AND ACKNOWLEDGEMENTS

This thesis was precipitated by teachers' expressed need to improve students' use of algebra to solve word problems. The study began to form in my mind when I became involved with school-wide professional development associated with literacy. I became aware that teachers in mathematics departments had particular needs associated with literacy. Teachers were concerned about the literacy demands of word problems within mathematics. Initial exploration around this topic led me to realise that this was a wide-spread concern for secondary teachers, which was in part fuelled by changes associated with the introduction of NCEA. Given the emphasis on word problems within high stakes assessment, it surprised me that there was little research information available on the impact of word problems on secondary students. The introduction of NCEA had also served to highlight national concerns about algebra, and so I decided to take the opportunity, in the final stages of my degree, to explore a topic that appeared important and timely.

I would like to acknowledge and thank the many people who made this study possible. Associate Professor Glenda Anthony, my main supervisor, who provided continuing interest and invaluable professional support. My thanks are extended to Brenda Bicknell, my second supervisor, who gave positive and encouraging feedback throughout the study.

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CHAPTER 1

INTRODUCTION

Mathematical literacy is the ability to formulate and solve mathematical problems in real life situations. This type of literacy is a foundation for participation as a reflective citizen in democracy and in occupational life. (Comparative Education Research Unit, December 2004)

1.1 Background

This study seeks to improve classroom practice by informing teachers' beliefs and knowledge about the teaching and learning of algebra word problems. In seeking answers to the question about classroom experiences that will enable students to solve algebra word problems more effectively, the teachers in this study collaborated with myself as researcher to engage in exploration of their own classroom practice. They were keen to participate in this project because they wanted to improve their classroom practices and student outcomes.

Motivation for this project arose directly out of my advisory work with mathematics teachers in New Zealand secondary schools. Specifically, questions about changing teacher practice arose from my engagement with teachers in professional development programmes. Questions about students' solving of algebra word problems came from classroom practitioners who identified an increasing emphasis on word problems in external assessments. Teachers were motivated by the requirement in national assessments for students to solve word problems by writing and solving algebraic equations. Concerned about students' difficulties, teachers wanted to know how they could improve the way they taught students to use algebra as a tool to solve algebra word problems. Given the significance of word problems in high stakes assessment in New Zealand there is a need for research on specific methods for teaching students to solve word problems. Although word problems are important within other domains of mathematics, it is particularly in the algebra strand that word problems form a barrier to student progress in secondary school.

The importance of algebra is stressed by Moses and Cobb (2001) who argue that algebra is the "key to the future of disenfranchised communities" (p. 5) because it is not only the gatekeeper to higher mathematics as well as "the gatekeeper for citizenship; and people who don't have it are like the people who couldn't read and write in the industrial age" (p. 14). However, despite the importance of algebra, it has proven to be a serious stumbling block for many students. Difficulties experienced by students at secondary school have contributed to the recent research and curricula emphasis on algebraic thinking and reasoning in the elementary years, with experiences in Pre-algebra and Early Algebra seen as critical for building the understandings and skills of formal algebra (Kieran, 2006; Stephens, 2006). This emphasis, however, does not abrogate responsibility for improving the teaching of formal algebra in the later years, which is the focus of this study.

Internationally, there has been a significant change in emphasis in school mathematics over the last three decades. Mathematics reform documents support an inquiry approach to teaching; students working in inquiry-based classrooms engage in mathematical discourse, sharing and refining their mathematical understandings by participating in learning communities. Inquiry classrooms involve a shift away from students' acquisition of procedural proficiencies to the development of their abilities to solve problems in meaningful contexts. Aligned with the focus on mathematical discourse and contextual problems is a growing awareness of the importance of language factors in the teaching and learning of mathematics (Curcio, 2004; Dowling, 2001; Ellerton & Clarkson, 1996; MacGregor & Price, 2002; Meaney, 2006).

Within New Zealand, word or story problems are emphasised in the mathematics curriculum (Ministry of Education, 1992) and feature prominently in high-stakes assessment. Although there is debate about the merits of assessment as a driving force for teaching, it is clear that what is measured in high-stakes assessments does influence what is taught in classrooms (Clarke, 2005). In New Zealand, the national assessment system has undergone significant changes since 2003 when the norm-referenced system was replaced by a standards-based system. Students now work towards a National Certificate of Educational Achievement (NCEA) which is assessed by performance against criteria defined by Achievement Standards. There are three levels of Achievement Standards,

and four categories of performance: not achieved, achieved, achieved with merit or achieved with excellence.

There are two achievement standards that focus on the use of algebra, one at Level One and one at Level Two (see Appendix A). Although schools can set their own course entry requirements, Level One Algebra is a common pre-requisite for Year 12 mathematics and Level Two Algebra is a common pre-requisite for Year 13 calculus. Both the algebra standards include the solving of algebraic word problems. Explanatory notes from the New Zealand Qualifications Authority (NZQA, 2005d) detailing the requirements for achieving the standards specify that students who achieve the standard are able to "use algebraic strategies to investigate and solve problems...Problems will involve modelling by forming and solving appropriate equations, and interpretation in context" (NZQA, 2005b, p. 2). Contextual problems are emphasised by the exam specifications which state: "Questions providing candidates with opportunities for achievement with merit and achievement with excellence will be set in real-life contexts" (NZQA, 2005d, p. 1). For the questions involving the solving of word problems, the assessment schedule states that a student "must form equations...at least one equation" (NZQA, 2005c, p. 2). According to this schedule non-algebraic methods are not recognised as valid solution methods.

National assessment results reflect poor achievement rates on the algebra standards. The most recent examiner's report highlighted students' difficulties with these achievement standards (NZQA, 2006b, 2006c). As an adviser, I facilitated NCEA professional development and facilitated workshops for teachers. Teachers suggested that student difficulties were exacerbated by three aspects of the algebra achievement standards: the emphasis on contextual problems; the writing of algebraic equations; and the need to solve these equations algebraically. They sought support to address these aspects.

The literature (e.g., Bennett, 2002; Koedinger & Nathan, 2004) suggests that writing and solving of equations is likely to be a significant cause of difficulty. International research indicates that secondary students tend to use informal methods even when they have been taught more formal algebraic methods. This is a concern beyond that of the achievement standards assessments, as reliance on informal methods hinders progress in higher mathematics. "There are important ideas that can best be communicated by using the symbols of algebra" (Foreman, 1997, p. 161).

The process of solving algebraic word problems can be viewed in terms of a comprehension stage and a solution stage. Writing equations involves translating from words into algebra as a part of the comprehension stage. This translation stage was a key focus for the project. The literature proposes a range of teaching practices to address the process of translation between algebraic and verbal representations, but research is needed to trial methods within the New Zealand secondary school context.

Alongside the focus on teaching practice, teacher learning was an important focus of this research. Recent studies have highlighted the importance of the teacher's role for student learning. Alton-Lee's (2003) *Quality teaching for diverse students in schooling: Best evidence synthesis* argued that "quality teaching is optimised when teachers have a good understanding of and are responsive to, the student learning processes involved" (p. 45). "Teachers' beliefs and knowledge… have a profound effect on the decisions they make regarding instruction" (Fennema, Sowder, & Carpenter, 1999, p. 10). However, Timperley, Fung, Wilson, and Barrar (2006) argue that understanding the learning processes involved in changing teacher practice is a neglected area of research.

1.2 Research Objectives

This study has a dual focus—teacher learning and student learning. It aims to create new insights and knowledge about effective teacher practice in relation to the teaching of algebra. It trials specific teaching strategies with the aim of enhancing students' use of algebra as a tool that replaces informal strategies in solving word problems. This research also aims to address the building of teachers' pedagogical content knowledge through the exploration of effective teaching strategies. The intervention involved the development and implementation of instructional activities that explicitly focus on translating between verbal and symbolic representations of algebra word problems.

Data is generated to address the following research questions:
1. In what ways does the introduction of explicit teaching activities that support the translation processes used to solve algebra word problems impact on student learning processes and outcomes?
2. How does participation in the classroom experiment impact on teachers' pedagogical practices, knowledge and beliefs?

1.3 Thesis Overview

Chapter 2 provides a review of the literature in the field and provides a background from which this project can be viewed. It summarises relevant and essential findings on the issues of mathematical pedagogical content knowledge, formal school algebra and student difficulties with algebra word problems and translation, the various pathways followed by students in solving algebra word problems, and the implications of these issues for instruction.

Chapter 3 presents a discussion of the methodology for the study with reference to effective approaches for teacher change. This chapter also includes the data generation methods and an outline of the project schedule.

In Chapter 4, the teaching activities used in the project are described. The results are reported and discussed in Chapters 5 and 6. Chapter 5 reports on the processes students used to solve algebra word problems. Chapter 6 presents and discusses teachers' views and responses to their involvement in the project.

The final chapter addresses the research questions, summarises key themes emerging from the project, discusses limitations of the project, and makes suggestions for further research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter presents an overview of the literature about algebra word problems and the implications of this for teaching. In order to situate the teaching and learning of algebra word problems, the first section of the chapter provides an overview of school algebra. Specifically, different interpretations of algebra are considered followed by a review of approaches to teaching algebra including the need for students to make the transition from arithmetic to algebra if they are to be successful with secondary school algebra. The remainder of the chapter focuses on word problems in algebra—the central focus of the research study.

Section 2.4 explores what an algebra word problem is. This is followed by a discussion of the processes involved in solving algebra word problems, focusing on comprehension, translation and solution phases, and student difficulties with these processes. Discussion of the solution phase leads into an exploration of solution paths with distinctions made between algebraic and non-algebraic strategies. Two particularly pertinent studies are discussed in some detail: Koedinger and Nathan (2004) and Stacey and MacGregor (2000). Both of these studies were situated in the secondary school context and explored the strategies students used to solve algebra word problems.

The final sections focus on the teaching of algebra word problems. Section 2.7 focuses on pedagogical content knowledge about algebra word problems, expert blind spot and instructional tasks. Section 2.8 outlines key findings from the literature says about the teaching of algebra. The final section summarises the issues discussed in this chapter.

2.2 School algebra

2.2.1 The many faces of algebra

The literature contains a multitude of definitions of algebra. A common thread running through these various interpretations is that algebra includes both content and processes.

The list of content that can be the focus of algebra is long and includes such things as symbols, functions, graphs, matrices and quaternions. The list of processes that can be the focus of algebra is also subject to a mix of terminology. Van Amerom (2003) provides a useful overview with her description of four perspectives: algebra as generalised arithmetic, algebra as a problem-solving tool, algebra as the study of relationships, and algebra as the study of structures.

Much of the research into the thinking of students as they develop algebraic skills and understandings has explored the difficulties they experience in developing algebraic ideas at elementary school (Booth, 1984; Cooper, Baturo, & Williams, 1999; Drouhard & Teppo, 2004; Filloy & Rojano, 1989; Herscovics, 1989; Kaput, 1999; Kieran, 1992; Koedinger & MacLaren, 2002; Kuchemann, 1981; van Amerom, 2003). Difficulties with signs and symbols, as well as process-object discontinuity, are associated with *Early Algebra*. Students progress from non-symbolic algebraic thinking of the *Pre-algebra* stage towards the expression of generalisations in increasingly formal algebraic ways appropriate to their developmental stage (Kaput, 1999; Kieran, 2006). Stephens (2006) emphasised the importance of *relational thinking*, an awareness of relations among numbers and the fundamental properties of number operations, which he views as a precursor to formal algebraic thinking. Jacobs and colleagues (2007) concur: "Relational thinking represents a fundamental shift from an arithmetic focus (calculating answers) to an algebraic focus (examining relations)" (p. 260). These authors suggest that focusing on relational thinking facilitates the transition to formal algebra.

Established views of formal algebra are dominated by symbol use (Asquith, Stephens, Grandau, Knuth, & Alibali, 2005). Traditionally, secondary school algebra has involved manipulation of algebraic expressions (Warren & Pierce, 2003); the use of symbols has distinguished an algebraic activity from other mathematical activity (Bell, 1996a). The move from generalised arithmetic to algebraic symbols captures the power of algebraic symbolism; it enables one to "squeeze the operationally conceived ideas into compact chunks and thus to make the information easier to comprehend and manipulate" (Sfard & Linchevski, 1994, p. 198). Vergnaud (1997) emphasises that algebra "requires symbolic calculations in a sense and to an extent never met before by students" (p. 25). Kieran's (1997) analysis of a range of approaches to algebra found that all "use algebra at least as a notation, a tool whereby we not only represent numbers and quantities with literal

symbols but also calculate with these symbols....The quintessential element distinguishing algebra from arithmetic is the presence of letters" (p. 137).

When considering approaches to algebra, a useful distinction can be made between students' solution strategies as *arithmetical* or *algebraic.* Arithmetical strategies involve operations on known numbers; their meaning remains connected to the original problem context (Kieran, 1992; Van Dooren, Verschaffel, & Onghena, 2002). In contrast, algebraic thinking and problem solving involves operating on unknowns; "the concrete meaning of these manipulations in relation to the problem context is temporarily suspended" (Van Dooren et al., p. 320).

Regardless of varied definitions of algebra and algebraic reasoning, the introduction to algebraic problem solving creates a barrier for many students (Kieran, 1992). This raises serious questions about the way algebra is taught.

2.2.2 Approaches to teaching algebra

"School algebra has traditionally been taught and learned as a set of procedures disconnected both from other mathematical knowledge and from students' real worlds" (Kaput, 1999, p. 133). This model of teaching, termed *learning by rote,* emphasises mastery of facts and procedures, as opposed to *learning by understanding* which emphasises understanding of concepts (Hiebert et al., 1997). The terms *instrumental understanding* and *relational understanding* were introduced by Skemp (1976) to distinguish between skill proficiency and conceptual understanding. Pesek and Kirshner (2002) highlight the tension between teaching for instrumental understanding and teaching for relational understanding. The first view leads to an emphasis on skills and the second commonly leans toward problem-solving approaches, with a focus on understanding what to do and why. There is general agreement that both aspects are necessary for meaningful mathematical learning. Sfard (2003) emphasises the circular nature of mathematical understanding: the need to understand something in order to operate on it, and the need to operate on something in order to understand it. "Understanding and doing are two sides of the same thing" (p. 366). However there is less agreement about the balance between the two or how to design programmes that develop both types of knowledge (Pesek & Kirshner, 2002; Siegler, 2003).

"Traditional approaches to teaching algebra are characterised by a lot of time being spent on learning skills before attempting to apply them to problems" (French, 2002, p. 6). There is debate about whether it is best for students to practise skills first before trying to understand them or understand procedures first before practising them. Hiebert and colleagues (1996) support a problem-solving focus: "Rather than mastering skills and applying them, students should be engaged in resolving problems" (p. 12). Research shows that students can master algebraic skills through problem solving: "Generic problems can provide authentic algebraic experiences that not only cover the strategies for problem solving by forming and solving equations, but also develop the key algebraic abilities of writing, reading, and manipulating symbolic expressions" (Bell, 1996b, p. 184). Hiebert (2003) concurs: "Problems to be solved can be used effectively as a construct for students to learn new concepts *and* skills, not just as applications of previously learned skills" (p. 17). However, many textbooks follow a common order starting with mathematical theory, followed by examples, then purely mathematical practice exercises, and finally the mathematics just covered embedded in contextualised problems (Lesh, 2002).

Current mathematics education reform documents recommend an inquiry-based approach as an effective way of helping students develop mathematical understanding (Anthony & Walshaw, 2007; National Council of Teachers of Mathematics, 2000). This approach is based on sociocultural theories about learning. From this perspective, mathematics teaching and learning are seen as social and communicative activities with a "clear shift away from viewing mathematics learning as *acquisition* towards understanding mathematics learning as *participation* in the discursive and cultural practices of a community" (Goos, 2004, p. 261). The focus is on establishing "*communities of mathematical inquiry* [italics in original]" (Goos, 2004, p. 259) where mathematical discourse is recognised as an important component of knowledge construction (Cobb, Boufi, McClain, & Whitenack, 1997; Irwin & Woodward, 2005; Khisty & Chval, 2002; Lo, Marton, Pang, & Pong, 2004; Marton, Runesson, & Tsui, 2004). In talking about mathematics, students explain their thinking; this helps them to clarify and refine their understandings as well as developing their awareness of alternative approaches and strategies (Sfard, 2001). Furthermore, classroom discourse is essential in mediating the tension that exists between acquiring knowledge and applying it (Watson & Mason, 2005).

However, the emphasis by experts and curricula on inquiry-based models tends to be at odds with common teacher understandings of what is important in learning and teaching mathematics (Timperley et al., 2006). Goos, Galbraith, and Renshaw (2004) noted several barriers to reform within secondary school classrooms including teacher beliefs, school structures and student resistance. The traditional view of mathematics teacher as a transmitter of knowledge is still prevalent and research suggests that factors involved in changing this view are complex (Cady, Meier, & Lubinski, 2006). Many secondary school mathematics lessons overseas follow a traditional expository or elicitation model typically involving a review/ introduction/ model example/ seatwork/ summary structure (Hiebert et al., 2003; Hollingsworth, Lokan, & McCrae, 2003). Many New Zealand mathematics classrooms are similar and most textbooks emphasise algebraic manipulation rather than problem-solving (Bennett, 2002). A report by the Education Review Office (2000) suggested teaching tends to be teacher-centred and questions the extent to which New Zealand mathematics teachers understand and use a problem-solving approach.

Recent studies suggest that traditional approaches to teaching algebra are strongly embedded. Although many algebra teachers reported that they value conceptual understanding, in practice they emphasised skills rather than understanding (Menzel & Clarke, 1999). Herscovics (1989) suggested that students taught the language of algebra without meaning develop Skemp's instrumental understanding—an understanding that does not transfer effectively to further learning. Chinnappan (2002) highlighted the importance of teaching developing *both* procedural and conceptual aspects of algebra. However, Thomas and Tall (2001) suggested that, in many classrooms, the need to develop procedural mastery in algebra appears to take precedence over any need for relational understanding. Clement (1982) suggested that the errors made by tertiary students with formulating and reading equations are to do with the way that schools have been "more successful in teaching students to manipulate equations than they have in teaching students to formulate them in a meaningful way" (p. 29). Further evidence for this focus on manipulation is provided by Vaiyautjamai and Clement's (2006) finding that many secondary school students who could solve quadratic equations had limited understanding of what they were doing.

2.2.3 The shift from arithmetic to algebra

The transition from arithmetic to algebra has proven difficult for many students. There has been a plethora of studies about students' abilities to perform algebraic tasks and to understand algebraic concepts (see for example Horne, 1999; Raymond & Leinenbach, 2000). The multiple meanings of signs and symbols are, in part, what gives algebra its power but also contribute to difficulties for students (Vergnaud, 1997). The way in which symbols are used is pivotal to progress in algebra. It is not enough to consider letters and numbers as unknown and known quantities; they are also variables. Many students experience persistent difficulty in making the transition from working with unknowns to understanding the concept of a variable (Cooper et al., 1999; Warren & Pierce, 2003).

A number of frameworks have been proposed to explain the stages of formation of abstract algebraic concepts. For many years, the pre-eminent model of learners' interpretations of letters in algebra was the hierarchy of four stages proposed by Kuchemann (1981). Kuchemann found that the majority of students treated letters as objects with few able to consider them as specific unknowns and fewer still as generalised numbers or variables. This was true even of older students; although the interpretation used depended in part on the question; a significant proportion of 13- to 15-year-old students in Kuchemann's study treated letters as concrete objects or ignored them. Kuchemann argued that "for any real understanding of even the beginnings of algebra [students] need to be able to cope with items that require the used of a letter as a specific unknown" (p. 105).

A similar framework, developed by MacGregor and Stacey (1997), involved six categories: "letter ignored, numerical value, abbreviated word, alphabetical value, use different letter for each unknown, unknown quantity" (p. 7). Their research into 11- to 15-year-old students learning to use algebraic notation found that difficulties were caused by "intuitive assumptions and pragmatic reasoning about a new notations, analogies with familiar symbol systems, interference from new learning in mathematics, and the effects of misleading teaching materials" (p. 1). They found slight improvement in success rate of 15-year-olds compared to 11-year-olds. Interestingly, they found that the older students' errors tended to reflect different understandings than shown by the younger students. Older students showed strong evidence of interference from new learning that

had been misunderstood: The misuse of exponential notation (e.g. x^3 instead of $3x$) increased with year level. They also found evidence that persistent difficulties with interpretation of algebraic letters is related to teaching approaches which introduce letters as abbreviated words rather than unknown numbers. For example, the use of letters as measurement labels interfered with students' understandings of the meaning of variable terms in an algebraic equation. Clement's (1982) "Students and Professors" problem highlighted the continued use of the label interpretation for literal terms even by mature algebra students.

In addition to understanding the notion of a letter as an unknown, understanding of the arithmetic operations and of equivalence is needed for understanding algebraic systems (Cooper et al., 1999; Kieran, 1992; MacGregor & Stacey, 1995). Students often carry a procedural view of the equals sign into their learning of algebra inhibiting the development of their understanding of equivalence (Filloy & Sutherland, 1996; Herscovics & Linchevski, 1994; Kieran, 1997; Reed, 1999; Sfard & Linchevski, 1994). Students who understand the notion of equivalence and who can discern dimensions of variation "in which some elements in mathematical sentences change while other elements remain unchanged" (Stephens, 2006, p. 481) are able to think relationally. Relational thinkers are able to see relationships without the need to close operations. This *acceptance of lack of closure* is another critical understanding along the path to formal algebra (Stephens, 2006).

Another significant discontinuity in the transition from arithmetic to algebraic thinking has been identified in the distinction between two ways of thinking about an algebraic expression or equation: one described as "procedural" (Kieran, 1992) or "operational" (Sfard, 1991), and the other described as "structural" (Sfard, 1991). These alternatives are also referred to as "process and object" (Kieran, 1992; Sfard, 1991). Process-object frameworks identify the need for learners to think both operationally, focusing on processes, and structurally, focusing on concepts. In shifting from a process conception to structural conception many students experience difficulty in going beyond the procedural part of the procedural-structural cycle. The operational way of thinking spells out the actions needed while the structural approach condenses the information and enables flexible problem-solving. Flexible thinking requires an understanding of both process and product. Without a structural understanding of algebraic concepts, a learner

must learn and recall a host of procedures each of which needs to be remembered as a separate device.

Operating structurally leads to compression of mathematics. Compressed representation is a feature of process-object development that is often used without experts being aware of the compression. Nunes (1997) found that 10-year-old students in his study "prefer an extended form of representation to a compressed one both in producing and interpreting problem-solving formulae containing letters" (p. 36). For example, "*a* x *a*" can be compressed into "a^2" which represents both the operation and the answer simultaneously as well as a move from multiplication to exponentiation. Kieran (1992) concluded from her analysis of research on algebra learning that "the majority of students do not acquire any real sense of the structural aspects of algebra… most students never reach the structural part of the procedural-structural cycle" (p. 412).

The importance of understanding structure was highlighted in MacGregor and Price's (1999) findings about the way 11- to15-year-olds learn to deal with symbols from a linguistic perspective. The researchers adopted a term used in research concerned with literacy development, *metalinguistic awareness,* to refer to "the linguistic ability that enables a language user to reflect on and analyse spoken or written language" (p. 451). They discuss the importance of

> awareness of potential ambiguity…the recognition that an expression may have more than one interpretation, depending on how structural relationships or referential terms are interpreted (e.g., knowing when brackets are required for ordering operations and being aware of the potential for mistranslating relational statements to equations). (p. 457)

This awareness is what enables students to analyse structure, choose methods of representation and manipulate expressions within algebra. Their investigation led them to suggest a link between understanding of symbol, syntax and ambiguity in algebra and understanding of symbol, syntax and ambiguity in ordinary language.

Stacey and MacGregor (2000) concluded that many student difficulties with algebra is due to the contrasting "ways in which problems are solved using arithmetic and algebra" (p. 151). Whereas arithmetic problems are solved by working with known numbers towards the answer, the algebraic method requires thinking that is the reverse of the

arithmetic process. Vergnaud (1997) expressed this idea: "The main difference between arithmetic and algebra is that algebra uses a formal detour where arithmetic would use a sequence of intuitive choices" (p. 25). For example, consider the word problem: *When 3 is added to 5 times a certain number, the sum is 50. Find the number.* The arithmetic solution involves subtracting 3 and dividing by 5 (using solving operations) but the algebraic form $5x + 3$ involves multiplication by 5 and addition of 3 (using forward operations). So, to set up the equation, students need to think in the opposite way to what they would set up using arithmetic. Furthermore, using algebra to solve the problem requires students to manipulate the equation with another set of simplifying operations.

The conceptual demand to "describe with 'forward operations' rather than with the solving operations" (Kieran, 1997, p. 145) is evidenced by the difficulties students have in translating and then solving an equation. According to Kieran, "one of the major obstacles in using algebra to solve problems is the translation of the verbal representation of the problem into an algebraic one" (p. 146). When translating word problems into symbols, Stacey and MacGregor (2000) suggest students' prior understandings that problems are solved by direct calculation makes it difficult for them to look for, select, and name the appropriate unknown or unknowns. Moreover, they suggest that this is what stops some students even attempting, and prevents others from making progress with, an algebraic approach.

Filloy and Rojano (1989) refer to the *didactic cut* or *cognitive gap* to differentiate between types of equations: "Arithmetic equations" of the type $ax + b = c$ can be solved by arithmetic methods, whereas "algebraic equations" of the type $ax + b = cx + d$ necessitate formal algebraic methods. Solving an equation which has the variable appearing only once involves only calculations with known numbers and so is not truly an *algebraic* process. Filloy and Rojano suggest that students are not required to operate algebraically until they work with equations which have the variable on both sides; understandably, students have greater difficulty with this task because of the greater complexity of the cognitive demands. This cognitive loading is further increased when students need to apply their equation-solving strategies to solve algebra word problems.

2.3 Algebra word problems

Word problems are seen as a way of learning and practising problem-solving without the need for direct contact with the real world situation. Word problems are special types of mathematical problems: "verbal descriptions of problem situations...[they] refer to an existent or imaginable meaningful context" (Verschaffel, Greer, & de Corte, 2000, pp. ix-x). They can be "algorithmic or non-algorithmic, closed or open" (Chapman, 2006, p. 211).

Mathematics curricula frequently advocate the use of contexts in order to make mathematics more meaningful and accessible for all learners (Ministry of Education, 1997, 2006; National Council of Teachers of Mathematics, 2000; Sullivan, Zevenbergen, & Mousley, 2002). "In this sense, 'context' refers to a real or imaginary setting for a mathematical problem, which illustrates the way that mathematics is used" (Anthony & Walshaw, 2007, p. 114). Advocates claim that solving real life problems, motivates and engages students as well as showing the usefulness of mathematics in real world situations and contributing to meaningful learning (Kouba, 1999; Webster, Young, & Fisher, 1999). Bicknell (1999), in a New Zealand study, found that both teachers and students recognised the solving of word problems as an important part of mathematics. A word problem, however, regardless of context, may fit into any particular strand of mathematics. The focus of this study is word problems within the algebra strand of mathematics.

The use of word problems has a long history within the early development of algebra (Charbonneau & Lefebvre, 1996). Even despite changes in emphasis of algebra over time, word problems have kept a prominent place in most algebra texts. They are used to contextualise algebra, to link algebra of the classroom to the real world. However, it is difficult to arrive at a consistent understanding of what constitutes an *algebra word problem*. Algebra is not inherent within a problem; "it is impossible to classify a word problem unequivocally as being arithmetical or algebraic" (Van Dooren et al., 2002, p. 325). One argument is that an algebraic problem is one where the solver perceives algebra to be the best method of finding a solution. Thus, just as what is a problem to one student may not be problematic to another student, what is an algebraic problem to one

student may not an algebraic problem for another if they more readily solve it non-algebraically.

One solution is suggested by Van Dooren and colleagues (2002). They classified word problems as arithmetic or algebraic depending on "the most efficient solution strategy for tackling a particular word problem" (p. 326). In their study, problems that can be solved by undoing operations were classified as arithmetic. To create algebraic problems with an identical semantic structure, they removed one value thereby creating an additional unknown value in a critical position so that the new values could not be generated by calculating with known values and an algebraic solution strategy is more efficient (see Figure 2.1). They argue that their classification indicates which strategy would most likely be the choice of an expert problem solver, but acknowledge that not all experts would agree on the most efficient strategy.

A similar scheme was introduced by Bednarz and Janvier (1996) to classify problems as arithmetic or *connected*, and algebraic or *disconnected*. With connected problems, bridges can be built between known information so students can work from the known to the unknown. With disconnected problems, no direct relation between known and unknown can be established. Acknowledging the subjective nature of all such definitions, in this project an *algebra word problem* will be taken to be a verbal presentation of a contextual situation which can be *efficiently* solved using algebra.

Studies about students' difficulties with solving algebra word problems have identified numerous factors that influence the processes students use, as well as the sources of difficulties at each stage. Verschaffel, Greer and de Corte (2000) identify four structural components of a word problem—the context, the semantic structure, the format, and the mathematical structure—any one of which can influence the difficulty level of problems and the strategies students use to solve problems. Models of algebra word problems commonly distinguish a comprehension phase and a solution phase (Ellerton & Clarkson, 1996; Koedinger & Nathan, 2004; MacGregor & Stacey, 1993; Mayer, 1992). The comprehension phase involves processing the text of the problem and translating it into internal or external representations corresponding to the relationships expressed in the text. The solution phase involves the use or transformation of the relationships represented to arrive at a solution. This model is helpful in explaining two kinds of

process explanations for students' difficulties with algebra word problems corresponding to the two phases.

Semantic category and mathematical structure	Schematization[a]
1. Unequal partition Arithmetic structure: A primary school with 345 pupils has a sports day. The pupils can choose between in-line skating, swimming and a bicycle ride. Twice as many pupils choose in-line skating as bicycling, and there are 30 pupils less for swimming than for in-line skating. 120 pupils want to go swimming. How many choose in-line skating and bicycling?	
Algebraic structure: In a large company, 372 people are working. There are 4 times as many workmen as clerks and 18 more clerks than managers. How many workmen, clerks, and managers are there in the company?	
2. Transformation Arithmetic structure: In 15 years, Jeroen will be twice as old as Stijn will be then. If Jeroen is 37 years old now, how old is Stijn?	
Algebraic structure: Last year, Farmer A had an area of land that was 9 hectares smaller than the land of Farmer B. This year, Farmer B bought 10 hectares of extra land, while Farmer A doubled his area of land. Consequently, the land of Farmer A is only 7 hectares smaller than that of Farmer B. How many hectares of land did they each have last year?	
3. Relation between quantities Arithmetic structure: The cashier of a cinema received 8220 francs in one evening. That evening, 30 tickets for adults were sold, at 210 francs per ticket. If you know that a child's ticket is 50 francs cheaper, how many tickets for children were sold that evening?	
Algebraic structure: A furniture factory uses large and small trucks to transport 632 beds from England to Germany. A large truck can carry 26 beds. A small truck can carry 20 beds. In the truck convoy that transports the beds, there were 4 more small trucks than large trucks. How many trucks of each type were in the convoy?	

Note. The test included two word problems of each type.
[a] The black boxes indicate the solutions for the word problem.

Figure 2.1 Examples of word problems by semantic category, mathematical structure, and schematization (Van Dooren et al., 2002, p. 327).

2.4 Solving algebra word problems

A variety of approaches have been used in describing the processes involved in solving word problems. Mayer (1992) described two distinct stages of translation and solution. Reed (1999) emphasised the need for students to read and understand, to comprehend the problem, before they can translate it. Difficulties can occur at any of these stages.

2.4.1 The comprehension phase

There is evidence that errors in the comprehension phase of word problems account for many student difficulties. Reading and understanding the problem are necessary first steps. Both these processes are influenced by linguistic factors including contextual, structural, and semantic aspects. Cooper and Dunne (2004) assert that a significant proportion of students experience difficulty in reading realistic mathematics items in a way which enables them to show their mathematical skills and understanding. They argue that contextualising mathematics creates another layer of difficulty for students, describing the difficulty of focusing on the mathematical problem when it is embedded in the "noise of the everyday context" (p. 88).

The context can make a problem easier or harder depending on an individual student's prior experience and familiarity with the context (Cooper & Dunne, 2004; Kouba, 1999; Lave, 1988; Lubienski, 1998; Perry, Howard, & Miller, 1999; Wiest, 2002). Sullivan, Zevenbergen, and Mousley (2002) assert that contextual tasks have the potential for "alienating, excluding or exacerbating disadvantage" (p. 656). Contexts that teachers think are accessible and meaningful to students may not be so. Bicknell's (1999) study of New Zealand Year 11 students noted that: "The context of some problems isolated students from being able to interpret the problem as intended by the writer" (p. 81).

Another difficulty relates to an apparent tendency for students to disregard the context, the reality of situations described in word problems, causing them to give senseless answers and find solutions even to problems lacking essential contextual information about the situation. Verschaffel et al. (2000) use the term *suspension of sense making* and suggest it is in part a result of classroom conditioning.

Difficulties created by the context are compounded by factors associated with the wording of a problem. Studies involving Year 8 to 10 students show that performance is affected by structural features including complex surface structures, sentence length, question length and the order in which information is presented (Ellerton & Clements, 1996; MacGregor & Stacey, 1993). The linguistic conventions of mathematics create difficulties particularly with the differences created by subtle changes in the use of conjunctions as exemplified by the differences between the phrases *divide in half* and *divide by half*. Technical words in problems also cause confusion. In particular, studies (e.g. Nesher, Hershkovitz, & Novotna, 2003) confirm the difficulties students have with the words *more* and *less*. Technical words include familiar words which have different meanings from their everyday meanings (as with *mean*) as well as mathematical terms (B. Barton, 1995). Although most of the studies cited above involve primary school students, indications are that contextual and grammatical features are also important influences on secondary student performance (see Lawrence & Patterson, 2005).

2.4.2 The translation phase

Algebra word problems "require a translation for the problems as stated in natural language to a new problem that can be solved by algebraic manipulation" (Forehand, 1975, p. 366). This transition between comprehending a problem and representing it algebraically is identified as a major site of difficulty even for more experienced students (Horne, 1999; Stacey & MacGregor, 2000). Translation between verbal and algebraic representations requires sound understanding of formal algebra which requires students to be at the formal stage of algebraic reasoning. "Before students can successfully translate word problems into expressions or equations they first need to have a thorough grasp of the meaning of algebraic symbols" (Hubbard, 2004, p. 310).

The ability to generate symbolic representations is essential for translation; such generation is at a higher level of difficulty than interpreting symbols or using symbols (Curcio, 2004). In order to create symbolic representations, students need the ability to use a letter to represent an unknown. Translation also requires students to generate equations. Research identifies this generation of equations from word problems as a key difficulty for students (e.g., Filloy & Rojano, 1989; Kieran, 1992; Sfard, 1991; van Amerom, 2003). Capraro and Joffrion (2006) investigated middle school students facility with translating between words and algebra; they found that the majority of students in

their study were not procedurally or conceptually ready to translate from the written word to mathematical equations. Heffernan and Koedinger (1997; 1998) suggest that writing algebraic expressions and equations is hard for students because of their difficulty recognising similarity between word problems and specifying the relations among variables. Moreover, Hubbard (2004) found that first year university students have difficulty with translating word problems into algebra. Similarly, many of the tertiary undergraduates in the study by Trigueros and Ursini (2003) had difficulty writing an equation. Although they could solve problems arithmetically, many were not able to use algebraic symbolism and had difficulty shifting to an algebraic way of tackling a problem

A number of errors with generating symbolic representations have been documented: reversal; use of algebraic letters as abbreviated names; interpretation of numerals as adjectives; and, letters as names of objects (Clement, 1982; Kaput, 1999; MacGregor & Stacey, 1993). Herscovics (1989) explored two lines of student thinking in his examination of translation errors: semantic and syntactic. Semantic translation, also called static comparison, involves interpreting the meaning behind the word problem as the relative size between groupings. Syntactic translation, in which the word order of a problem is use to form an equation, is often cited as the main source of students' errors and difficulties in translating from words to algebraic syntax (Clement, Lochhead, & Monk, 1981; Herscovics, 1989). Syntactic translation is cited as the reason for reversing the variables as illustrated by the well known "Students and Professors" problem:

> Write an equation using the variables S and P to represent the following statement: 'there are six times as many students as professors at this university.' Write S for the number of students and P for the number of professors. (Clement et al., 1981)

The answer, $6S = P$ provides an example of syntactic translation with the *six* preceding the word *students*. Semantic translation would produce the result $6P = S$ because there are more students as expressed by the number 6. Erbas and Ersoy (2002) attribute wide use of syntactic translation to teaching approaches and textbooks that emphasise procedures rather than meaning, with students often taught to solve word problems by searching for key words and word order matching techniques.

However, MacGregor and Stacey (1993) question whether syntactic translation was the cause of reversal; they found that students reverse variables even when syntactic translation would produce the correct equation. Hubbard (2004) suggested confusion between the "writing of the equation with the solution process" (p. 311) as a cause of reversal order in the writing of equations. Stacey and MacGregor (2000) identified two common ways in which arithmetic thinking interfered with the formulation of equations. One problem is that the basic logic of algebraic problem solving, by representing *forward operations* using the unknown, presents a major obstacle for many students. A second problem arises out of students' difficulties with knowing which quantity or quantities in a problem should be symbolised. They found that many students use x to stand for the quantity currently being calculated rather than for a specific unknown quantity.

2.4.3 The solution phase

Difficulties in solving word problems also occur in the solution phase, particularly with the strategies that students use to process the problem. Kieran (2006) highlighted the discontinuity faced by students in the introduction of formal representations and methods to solve problems that they had previously handled intuitively. Heffernan and Koedinger (1997; 1998) analysed student strategies and errors in solving matched word and algebra problems and found that students have informal algebra problem-solving knowledge prior to acquisition of symbolic equation solving skills. A key difficulty identified by Dickson (1989) in her study of 14-year-olds was that students' informal methods of solution did not match formal methods. Furthermore, Lawrence and Patterson (2005) found that senior students could understand problems but they lacked strategies for dealing with algebra word problems.

Numerous researchers have documented students' reluctance to reason algebraically (Bennett, 2002; Capraro & Joffrion, 2006; Koedinger & Nathan, 2004; Lawrence & Patterson, 2005; Stacey & MacGregor, 2000). This reluctance is evidenced by all age groups. Hubbard (2004) found that when first year university students were given a question requiring a numerical solution, their focus on getting the answer led them to abandon algebraic symbols. Booth (1984) found that both primary and secondary students often use informal methods and are very resistant to change as they fail to assimilate the formal taught procedures. Koedinger and colleagues (1999) found that middle school students tended to continue to use informal strategies even when

attempting more complex problems. In subsequent research, Koedinger and Nathan (2004) found that secondary students tackled word problems using informal methods rather than the formal algebraic approach that they had been taught. The research by Koedinger and Nathan is particularly pertinent to this study so is discussed in some detail.

Koedinger and Nathan (2004) reported on two studies exploring difficulty factors of problems. The first study involved 76 students taught by four different teachers from an urban high school; the second study involved 171 students taught by 12 teachers from three urban high schools. All but 18 of the students were enrolled in their first year of an algebra course; 18 students from the first study had completed their first year of algebra in the previous year. The researchers investigated differences in student performance on problems with different representational formats. In addition to the main contrast between word problems (which they termed *story problems*) and equations, they included an intermediate problem presentation, termed *word equations*, to isolate effects of situational knowledge from language comprehension demands between verbal and symbolic forms.

Problems with four different cover stories were created that systematically varied different factors: three levels of problem presentation (word problem, word equation, and symbol equation); two levels of unknown positions (result versus start); two number types (whole versus decimal); and, two operation types (multiplication and addition versus division and subtraction). They found evidence for effects of three difficulty factors: problem representation, unknown position, and number type. Students performed better on word problems and word equations than on number equations, better on start-unknown than result-unknown problems, and better on whole number problems than decimal number problems. There were statistically significant differences between performance on word problems and equations, and word equations and equations, but not word problems and equations. These results lend strong support for the *verbal facilitation hypothesis* that students tend to cope better with words than symbols; algebra equations are not readily understood and the equation comprehension skills of students may lag behind their English language comprehension skills.

Koedinger and Nathan (2004) identified three different solution strategies that students used and included these in their model of algebra word problems as summarised in Figure 2.2. Of the three methods, equation solving, guess-and-test, and unwinding, they classified the last two methods as "informal" meaning that "students do not rely on the use of mathematical (symbolic) formalisms" (p. 117); they also suggested that these strategies are not usually acquired through formal classroom instruction.

Different problem representations tended to elicit different strategies with word problems eliciting the unwind strategy half the time, and seldom eliciting the equation strategy. Situation-less word equations tended to elicit the informal strategies of guess-and-test or unwind. Even on equations, students frequently used these informal strategies. Interestingly, regardless of the problem, informal strategies tended to be more successful than use of symbol manipulation. Evidence supporting the verbal facilitation hypothesis led the researchers to suggest that students have difficulties with understanding algebraic equation because it is cognitively demanding, and that word problems are easier if the text elicits more effective solution strategies than those elicited by algebra problems.
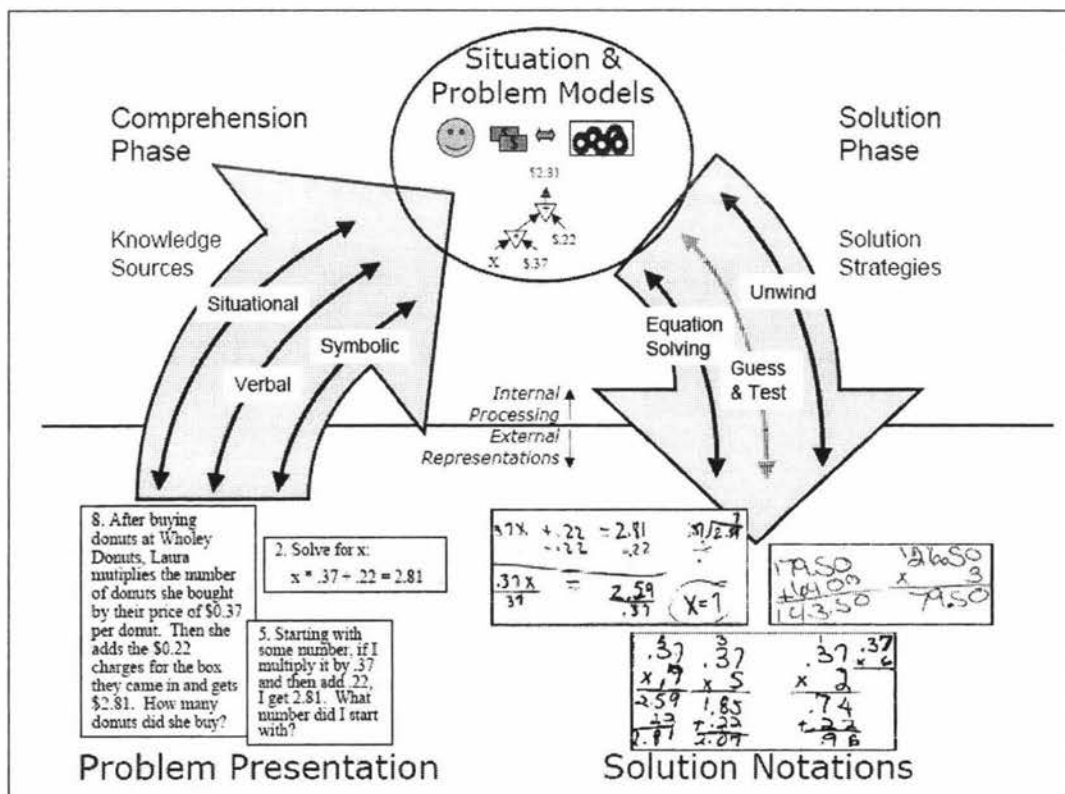

Figure 2.2 Processing algebra word problems (Koedinger & Nathan, 2004, p. 133)

Various explanations have been suggested as to why these informal methods, whether taught or informally derived, have become the preferred or dominant approach. Firstly, students frequently use them for *easy* problems; problems that they can solve easily by arithmetic reasoning. Secondly, students may use informal methods because they do not view algebra as a tool for solving problems. They may not lack algebraic skill, but typically view algebra as a rote response to an algebraic problem. Van Amerom (2003) found that algebraic symbolising and algebraic equation solving do not necessarily develop together. Her research showed that students who can solve equations may not have developed an understanding of equation structure, nor an understanding of algebraic manipulation. An instance of this is the student who translates a problem into a system of equations but then applies an informal strategy to solve the problem. Although the equations may have helped the student structure the problem, they are not used as part of the solution process.

The difficulty of making the transition to formal methods is another reason for students' use of informal methods. Students are unlikely to switch from an arithmetic approach unless they are explicitly taught and see the need to use algebra (Filloy & Rojano, 1989). It is only as problem complexity increases that the advantages of symbolic representations outweigh the more concrete solution method. However, students frequently revert to arithmetic when problems are simple (Nathan, Stephens, Masarik, Alibali, & Koedinger, 2002). Bell (1996b) describes the critical factor as students being '*willing to operate with symbolic (algebraic) expression* [italics in original]' (p. 174). Stacey and MacGregor (2000) assert that this lack of willingness arises out of the cognitive discontinuities between arithmetic and algebraic reasoning. Analysis of solution methods needs to consider the range of pathways or routes that students may choose.

2.5 Routes to the solution

Various classifications of solution pathways have been offered by researchers. Koedinger and Nathan's (2004) system classifies solution paths as algebraic or arithmetic (see Figure 2.2). The algebraic pathway involves writing and solving an equation and the arithmetic methods termed as *unwinding* and *guess-and-test* (p. 132). Van Dooren, Verschaffel, and Onghena (2002) classified solutions to word problems as arithmetical or

algebraic in a similar way. Arithmetical solutions are described as *manipulating the structure of the word problem* and *generating numbers* (p. 329). Figure 2.3 shows an example of each of these strategies.

Algebraic word problem	Arithmetical word problem
In a large company, 372 people are working. There are 4 times as many workmen as clerks, and 18 more clerks than managers. How many workmen, clerks, and managers are there in the company?	A primary school with 345 pupils has a sports day. The pupils can choose between in-line skating, swimming, and a bicycle ride. Twice as many pupils choose in-line skating as bicycling, and there are 30 pupils less for swimming than for in-line skating. 120 pupils want to go swimming. How many choose in-line skating and bicycling?

Algebraic solution	
Let x equal the number of managers. $x + (x + 18) + 4 (x + 18) = 372$ $6x + 5 \times 18 = 372$ $6x = 372 - 90$ $6x = 282$ $x = 47$	Let i equal the number of pupils chosing in-line skating. $345 = i + i/2 + i - 30$ $690 = 2i + i + 2i - 60$ $750 = 5i$ $i = 150$
There are 47 managers, 65 clerks $(47 + 18)$, and 260 workmen (4×65).	150 pupils went in-line skating, 120 went swimming, and 75 (150/2) chose the bicycle ride.

Arithmetical solution ("Manipulating the structure")	
Let us suppose for a moment that there are 18 more managers, thus as many managers as clerks. Then the total number of people = 390.	Let us assume that the number of pupils who swim equals the number of in-line skaters. The total augments with 30 → 375 pupils.

This total consists of 6 equal parts:
    4 parts of workmen
    1 part of clerks
    1 part of managers
	This total is divided in 5 groups:
	    2 groups of in-line skaters
	    2 groups of swimmers (in fact 30 less)
	    1 group of bicycle riders

Each part consists of 390/6 = 65 people. So there are 65 clerks and 260 workmen in the company, and the number of managers is 65 − 18 = 47.	Each group consists of 75 pupils. Thus 75 pupils choose the bicycle ride, 150 choose in-line skating, and 120 want to go swimming

Arithmetical solution	
"Guess-and-check"	"Generating numbers"
Suppose there were 80 clerks. Then there were 62 managers and 320 workmen for a total = 462. But this is too much. Then suppose	120 pupils who want to go swimming + 30 = 150 pupils who choose in-line skating divided by 2 gives 75 pupils who choose the bicycle ride.

Clerks	Managers	Workmen	Total	
60	42	240	342	= too few
70	62	280	412	= too much
65	47	260	372	= correct !

There are 65 clerks, 47 managers, and 260 workmen.

Figure 2.3 Classification scheme for solutions to the word problem test questions, including exemplary solutions (Van Dooren et al., 2002, p. 328).

Stacey and MacGregor (2000) identified a wide range of pathways that students follow in solving word problems based on an examination of written solutions from 900 students across 12 secondary schools and 30 student interviews. All the students, aged 13 to16 years, were in their third or fourth year of algebra learning. The students were given problems designed to eliminate as far as possible the features that increase problem difficulty in terms of language and vocabulary, context, and conceptual complexity. The authors reported on students' responses to four sample problems (see Figure 2.4). The first three problems lead to "arithmetic equations" (Filloy & Rojano, 1989) with the unknown on only one side while the fourth problem, NUMBER, involves an equation with the unknown on both sides. One-third of the students were able to write correct equations for all four problems. However, many of the students did not use equations to get the answers. NUMBER proved much more difficult with only 17% of students producing a correct solution compared to over 60% for each of the other three problems. They found no evidence of students choosing the most appropriate method for particular problems. "Almost no students... did the first three problems by logical reasoning but used algebra on harder problems such as NUMBER where it has definite advantages" (p. 153).

1. TRIANGLE

The perimeter of this triangle is 44 cm. Write an algebraic equation and work out x.

$x = \ldots\ldots$
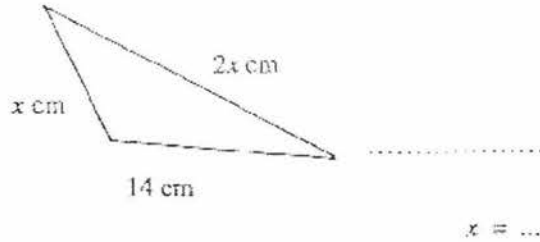
2. MARK

Some money is shared between Mark and Jan so that Mark gets $5 more than Jan gets. Jan gets x. Use algebra to write Mark's amount. The money to be shared is $47. Use algebra to work out how much Jan and Mark would get.

3. BUS

A bus took students on a 3-day tour. The distance traveled on Day 2 was 85 Km farther than on Day 1. The distance traveled on Day 3 was 125 Km farther than on Day 1. The total distance was 1410 Km. Let x stand for the number of Km traveled on Day 1.
Use algebra to work out the distance traveled each day.

4. NUMBER

I think of a number, multiply it by 8, subtract 3, and then divide by 3. The result is twice the number I first thought of. What was the number?

[Hint: Write an equation and solve it]

Figure 2.4 Four problems (Stacey & MacGregor, 2000)

Stacey and MacGregor noted that "at every stage of the process of solving problems by algebra, students were deflected from the algebraic path by reverting to thinking grounded in arithmetic problem-solving methods" (p. 149). A major impediment in secondary students shifting to use algebraic methods was their poor understanding of the logic of solving a problem by algebra. Although many students were able to solve word problems, "many of them had not learned how algebra can be used to solve problems" (p. 159). They noted the "great variety of methods that students used" (2000, p. 154) and drew up a classification system for algebraic and non-algebraic routes based on the

various pathways that students used to solve word problems (see Figure 2.5). The various paths are discussed in some detail below.
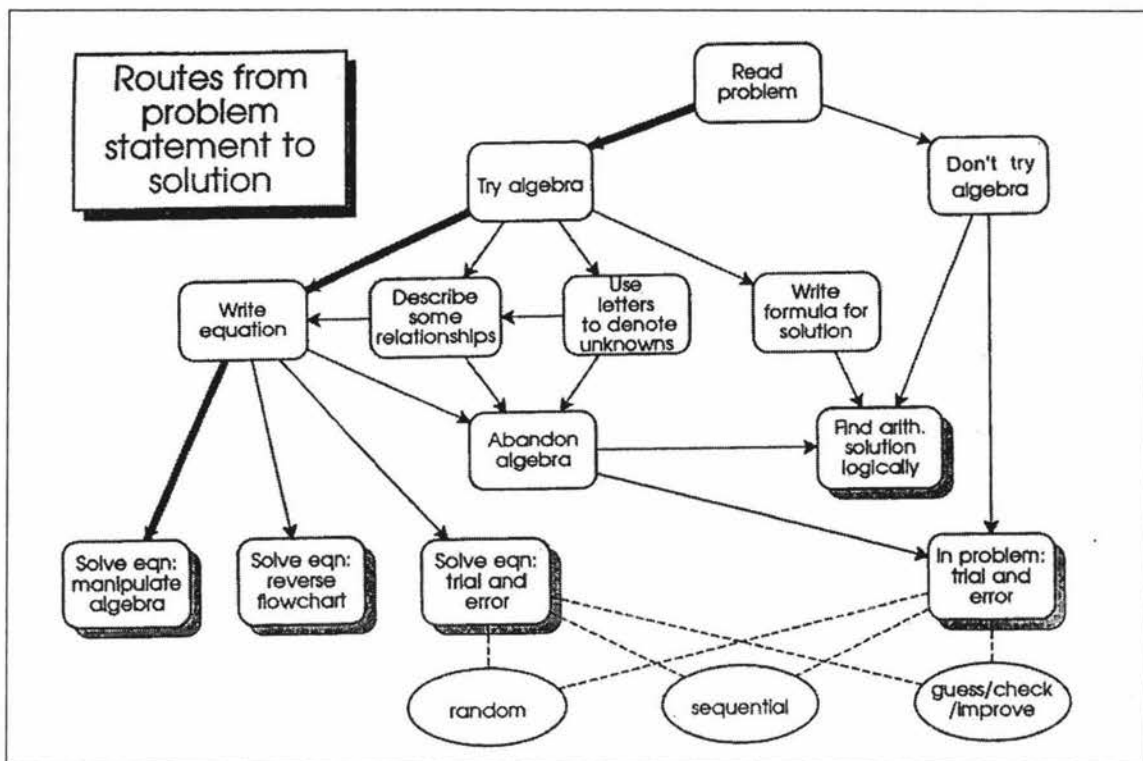
Figure 2.5 Routes from a problem statement to solution (Stacey & MacGregor, 2000, p. 155).

2.5.1 Non-algebraic routes

Informal methods involve some combination of arithmetic methods. These informal solution methods are effective with problems which have *easy* numbers. Although these informal solution strategies may include some form of algebraic thinking they are procedural and have limited generalisability.

- **Arithmetic reasoning**

In Stacey and MacGregor's (2000) study, many students solved all easy problems by arithmetic reasoning. This typically involves a logical analysis of the situation. They know what the *forward operations* (Kieran, 1992) are and they unwind, using *backward operations* to move directly to the solution, at every stage working with known numbers. This method of finding the arithmetic solution logically is called unwinding by Koedinger and Nathan (2004). Because of its reliance on backward operations, this method cannot be used in a situation with unknowns on both sides of the equation. Thus

the benefits of translating to equations may emerge only as problems get more complex (Koedinger et al., 1999).

- **Trial and error**

Trial and error methods are known by a number of terms including *guess and test* and *guess, check and improve*. Stacey and MacGregor (2000) use the term *in problem trial and error* to emphasise that this method relies on testing numbers in the problem statement (as opposed to the equation). In contrast to arithmetic reasoning, students using trial and error work with forward operations inherent in the problem. Students carry out trial and error in different ways; they may guess numbers randomly, or sequentially, or by using some sort of guess-check-improve method. Students using trial and error typically fail to find non-whole number solutions unless they use the guess-check-improve method.

- **Writing formulae**

Students who write formulae use a letter to label the unknown quantity to be worked out. Formulae describe a sequence of calculations; they show how to work out the answer from known information. A formula defines a procedure for computing; students "can see how to use it" (Stacey & MacGregor, 2000, p. 158). Although formulae use algebraic symbols, Stacey and MacGregor argue that their use is only superficially algebraic because formulae show how to work out the answer from *known* information and so represent the same reasoning as that shown in arithmetic reasoning.

2.5.2 Using algebra

Stacey and MacGregor (2000) suggest "the first evidence of a student's decision to use algebra is the use of a letter" (p. 158). Some students use letters to denote results of calculations. Other students use letters to denote unknowns or to express relationships but may go no further along the algebraic route than this.

- **Writing the equation**

Writing a complete equation is a key step in the algebraic pathway. An equation specifies the structure of an equality among *unknowns*. However, many students do not use the equation they have written, instead returning to the original problem to solve by an informal method; This suggests that "these students did not know that writing the equation was useful for solving the problem" (Stacey & MacGregor, 2000, p. 158).

- **Solving the equation**

Some students use their equation to generate a solution either by trial and error applied to an equation or using a reverse flow chart for solving their equation. Stacey and MacGregor (2000) categorise both these methods as informal or non-algebraic because they both involve operating with known values and forwards operations. "Equation solving by these two methods is 'doing algebra' only in the sense that the immediate problem to solve is represented with algebraic symbolism" (p. 158). They argue that manipulation is the only one method that is truly algebraic. They found that "very few students used the complete algebraic route even when alternative routes were very difficult or time consuming" (p. 154). Their description of the full algebraic route, formulating an equation and solving it algebraically, is in accord with the system used by Van Dooren and colleagues (2002) for scoring a solution as algebraic: "to be scored as an algebraic solution, the protocol should contain at least one equation in which known and unknown values are related to each other, and the answer is found through operating on the unknown" (p. 328).

2.6 Teaching algebra word problems

Despite the commonly reported difficulties, there is relatively little research literature about the teaching of word problems. However teachers' classroom practices are bound in complex ways with their knowledge, beliefs and understandings, all of which contribute to teachers' pedagogical content knowledge.

2.6.1 Pedagogical content knowledge

Research highlights the impact of teachers' subject-matter knowledge and pedagogical content knowledge on students' learning of mathematics (Fennema et al., 1999; Shulman, 1986; Thompson, 2004; Verschaffel, Greer, & de Corte, 2002). The term pedagogical content knowledge was coined by Shulman (1986) to reflect the specialised form of knowledge that is needed for effective teaching. Pedagogical content knowledge, "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9), encapsulates the idea that teachers need mathematics curriculum knowledge as well as knowledge about how to teach the curriculum. Pedagogical content knowledge includes an understanding of how specific

topics are (or should be) learned and taught, what makes the topics hard or otherwise, and what conceptions students bring with them to the learning of these topics.

Teachers' pedagogical content knowledge in the field of algebra word problems includes an understanding of the processes involved in learning algebra. Key to this are the differences between algebraic and arithmetical strategies as well as the cognitive leaps required in moving between the two ways of thinking. Teachers also need to understand the nature of algebra word problems, the difficulties they present to students, the ways students process them, and the various routes students may take in solving word problems. Each of these issues has implications for the sorts of experiences students need to participate in to develop their understanding, another key part of pedagogical content knowledge. In addition to their understanding of students' learning processes, teachers also

> need to be able to demonstrate to their students the validity and the pertinence or necessity of the new algebraic way of thinking as a powerful mathematical tool… to develop in their students a disposition to apply arithmetical or algebraic strategies in a flexible way, taking into account the characteristics of the problem to be solved. (Van Dooren et al., 2002, p. 322)

Both Bennett's (2002) and Lawrence's (2005) studies of New Zealand teachers' beliefs reveal that some teachers lack effective strategies for teaching word problems in mathematics. Chapman (2006) uses the term "paradigmatic" to describe a perspective on solving word problems common amongst teachers. She identified three approaches within this perspective. One approach is shown by teachers who use questioning, prompts and discussion to help students unpack the context so they can create their own mathematical interpretation. This approach reflects the greatest depth as it calls for reasoning, interpretation and connections, thereby developing Skemp's (1976) relational understanding. Another approach which can also develop relational understanding is to help students see how the mathematics structure can be independent of the context by highlighting similarities between parallel problems. The third approach is the key word approach in which teachers show students how to fragment and translate the context into mathematical representations by identifying key words and phrases and their corresponding symbolic representations. "Break it down first and translate the words into the mathematical sentence" (Chapman, 2006, p. 220). Anthony's (1996) study of a

Year 12 class reported frequent use of keyword as an instructional strategy. Chapman's suggestion that the key word approach develops Skemp's instrumental understanding is confirmed by Anthony and Walshaw (2007): "Keywords do little to help students construct meaningful mathematical knowledge...keywords enable students to complete problems without necessarily understanding the situation, without modeling it mathematically, and without acquiring the intended procedural knowledge" (p. 118).

2.6.2 Expert blind spot

A core aspect of teachers' pedagogical content knowledge is in comprehending the difficulties that need to be overcome by students in developing understandings. However, research by Koedinger and Nathan (2004) suggests that teachers often lack an understanding of student approaches to word problems. "Much of current educational decision making may be incorrectly biased by explicit knowledge and beliefs that are at odds with the reality of student thinking and learning" (p. 35). Moreover, teachers' misunderstandings are reinforced by textbooks which commonly portray methods that do not align with typical students' algebraic reasonings (Nathan & Koedinger, 2000b). The finding that teachers' expectations were quite contrary to student performance led to the suggestion that teachers suffer from "expert blind spot": the tendency "on one hand, to overestimate the ease of acquiring formal symbolic re languages, and on the other hand, to underestimate students' informal understandings and strategies" (Koedinger & Nathan, 2004, p. 163).

Teachers' tendency to a *symbol precedence view* of mathematical development was highlighted in research by Van Dooren, Verschaffel, and Onghena (2002) who found that secondary pre-service teachers prefer to used an algebraic method regardless of the nature of any given word problems. In Van Dooren et al.'s study of pre-service teachers, a large majority of the secondary pre-service group tended to use formal methods regardless of the problem and viewed the algebraic method as "'the one and only 'truly mathematical' solution method for such application problems" (p. 343). In contrast, primary pre-service teachers responded more flexibly, adapting their response according to the problem, often using strategies that matched the problem type. Researchers (Koedinger & Anderson, 1998; Van Dooren et al., 2002) noted that teachers tend to solve problems "without thinking", using the algebraic method regardless of the problem.

Many teachers expect students to find word problems more difficult than matched algebra problems because they expect students to translate the problem into formal algebra and then solve the algebra problem (Nathan & Koedinger, 2000a). This process would clearly make word problems harder than matched symbolic problems because of the intermediate step of writing the symbolic problem. "In this strategy equation solving is a sub-problem of word problem solving, and thus word problems will be harder to the extent that students have difficulty translating stories to equations" (Koedinger et al., 1999, p. 8). Lawrence and Patterson (2005) summarise this view of problem solving as a cycle (see Figure 2.6). The first stage of solving a word problem is taken to be the translation of the word problem into algebra. However, the research literature suggests that students acquire skills for solving simple problems expressed in words before symbols (Koedinger et al., 1999).
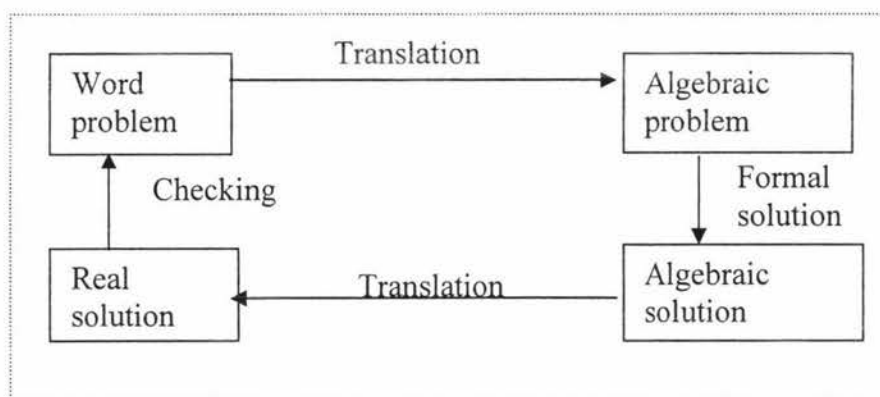
Figure 2.6 The algebraic problem-solving cycle (Lawrence & Patterson, 2005)

Stacey and MacGregor (2000) suggest that the mismatch with teachers' views and students' informal methods is one of the reasons students avoid algebra. They highlight student difficulty with identifying the unknown(s) in a problem and suggest the "standard instruction given by teachers to let a letter stand for 'the unknown' seems particularly inappropriate and highlights the technical knowledge underlying how teachers automatically classify problems as having a certain number of unknowns" (p. 163).

2.6.3 Instructional tasks

What students do in the mathematics classroom is crucial to their learning. However, the creation or selection of relevant tasks is complex. Hiebert and colleagues (1997) discuss

the difficulty of ensuring that: "what is problematic about the task should be the mathematics rather than other aspects of the situation" (p. 18). The relatively open nature of word problems affects students differently. Some students, particularly those with high socioeconomic status (SES), are able to explore open problems without becoming overly frustrated. In contrast, lower SES students, especially females, complain of "feeling completely confused about what to do with the problems, and asked (often passionately) for more teacher direction and a return to typical drill and practice problems" (Lubienski, 1998, p. 28).

Contextual tasks can be limited with the context often forming nothing more than a border around the mathematics. There is criticism of the use of "absurd" (Verschaffel et al., 2000, p. 4) problems with the argument that many contextual problems do not foster links with reality because they are meant to be solved by ignoring everyday knowledge. Cooper (2004) highlights difficulties raised for students by a lack of consistency in the weighting of two factors: the extent and/or depth of the contextual reference; and whether the question-setter wants factors implicit in the contexts to be attended to or ignored.

The fact that "engaging in high-level reasoning and problem solving involves more ambiguity and higher levels of personal risk for students" (Henningsen & Stein, 1997, p. 526), can lead to students trying to pressure teachers to reduce task complexity in order to reduce student anxiety. This tendency to subvert the cognitive demands of a task can minimise challenge and anxiety but may lead to tasks that are not sufficiently challenging. This, in turn, reduces the potential for learning of mathematics and presents teachers with the dilemma of "how to assist students in experiencing and acquiring mathematically powerful ideas but refrain from assisting so much that students abandon their own sense-making skills in favour of following the teacher's directions" (Hiebert et al., 1997, p. 29).

Ensuring the appropriate level of task difficulty is a complex issue. Dickinson and Butt (1989) found that when the difficulty level of mathematics problems ensured a minimum of 70% success rate, on-task behaviour of low-achieving students and some high-achieving students increased significantly. As well as open-ended, problematic tasks, students need to be presented with meaningful practice activities; these are important for achieving understanding with fluency (Watson & Mason, 2005).

Problems should sometimes be easy and straightforward so that students come to feel powerful and confident. But sometimes problems should lead to impasse, evoking puzzlement, bewilderment, and frustration, yet offer the possibility of proceeding with renewed determination and achieving the elation of sudden insights or the satisfaction of performing a difficult feat. (Goldin, 2003, p. 282)

In addition to the challenge of finding the right levels of difficulty, Sullivan and colleagues (2002) noted that teachers have difficulty developing tasks or activities to develop conceptual understanding, especially with the development of open-ended questions. Lesh (2002) also discussed these issues, emphasising the time needed to develop sound mathematical tasks.

2.7 Key findings for instruction

Teaching can make a difference. Stacey and MacGregor (2000) found that students in particular classes across the 12 different schools in their study were significantly more successful at dealing algebraically with word problems and concluded that this was the result of teaching. They suggest that teachers need to promote algebraic methods and be conscious not to reduce task complexity. Koedinger and colleagues (1999) discuss the importance of designing instructional programmes to "overcome the 'expert blind spot' that may occur when experts' ideas about what may be difficult for students are different from the reality" (p. 19). A review of the literature offers the following strategies for effective teaching of algebra word problems.

- **Explicit purpose of solving word problems**

Bennett (2002) highlights the need for teachers to be explicit about the purpose of word problems with students. Initial goals may focus on developing algebraic strategies, but longer term goals are to build algebraic reasoning and equip students with a wide range of strategies so that they can select an appropriate strategy for any particular problem. Swan (2000) found learning outcomes were significantly greater for students whose teacher was explicit about the purpose of tasks and emphasised the importance of learning with understanding. Teachers need to be explicit about algebra being more than manipulation of symbols. Bell (1996b) suggests that, to develop this understanding, students need to "experience the full activity of beginning with a problem, forming the equations, then solving it, and interpreting the result" (p. 181). Instructional activities need to provide students with opportunities for this experience.

- **Connections between informal and formal strategies**

Teachers need to encourage and build on students' informal methods (Stacey & MacGregor, 2000). Koedinger and colleagues (1999) suggest that "instruction that bridges from students' informal or grounded knowledge may be more effective than instruction that focuses directly on abstract representations" (p. 19). Nathan and colleagues (2002) found that bridging instruction that explicitly built on students' invented strategies and representations improved the performance of upper primary students to solve word problems, as well as their ability to translate between verbal, symbolic and graphical representations. They suggest that bridging provides conceptual grounding for the various representations by "explicitly connecting them to students' informal reasoning and their intuitions"(Nathan et al., 2002, p. 470) . At the same time, this encourages students, and teachers, to accept informal strategies as valid.

- **Problems that press for algebra**

The development of algebra as a method for solving problems is inhibited by focusing on the arithmetic aspects of solving numerically and writing formulae (Stacey & MacGregor, 2000). Teachers need to present students with problems to which algebra is the preferred option as opposed to the common practice of giving students problems that condone and encourage informal strategies (Bennett, 2002; Stacey & MacGregor, 2000). "To appreciate the value of algebra as a problem-solving tool, students must work on problems that they can't easily solve without algebra" (Angier & Povey, 1999, p. 153). Koedinger, Alibali, and Nathan (1999) stress that presenting students with more complex problems presses them to use algebra: "Students must translate to a formal symbolic representation and solve using algebraic manipulation" (p. 18).

- **Progression from words to algebra**

Instructional activities need take into account the importance of representational format (words versus equations) and problem complexity (single versus multiple unknowns). Koedinger and Nathan (2004) suggest

> starting with instructional activities involving story problems, which are easier for students to solve, and moving later to more abstract word-equation problems and then symbolic equations. In this view, decomposing instruction to focus on difficult equation solving skills is fine. However, such instruction should come *after* students have learned the meaning of algebraic sentences, in other words, after they have learned to translate back and forth between English and algebra. (p. 160)

Analysis supports the hypothesis that developing students' understandings of relationships in the verbal representation should facilitate learning the symbolic representation of relationships (Koedinger & MacLaren, 2002).

- **Building comprehension**

Curcio (2004) reports on an action research project investigating the systematic intervention of providing students with strategies for building comprehension in reading mathematics and in approaching and solving word problems. Secondary students involved in the interventions showed significant gains in dealing with word problems leading him to conclude that "mathematics teachers must view themselves as teachers of reading by designing instruction so that students can develop their ability to read, interpret and analyse problems" (p. 169). A range of strategies were used but the effect of individual strategies was not analysed. Strategies used included "cloze" reading tasks, development and discussion of concept maps and word problem analysis. Students were also presented with a task described as "Here's the equation, write the problem" which entailed giving students an equation and asking them to write a matching word problem.

- **Focusing on translation**

Another suggestion made by Koedinger and Nathan (2004) was for teachers to use students' understanding of verbal constraints as a bridge for understanding and manipulating symbolic constraints. They found that students' ability to use alternative methods to solve algebraic problems was influenced by explicit teaching of different representations and they recommended activities and exercises focus on translating back and forth between verbal and symbolic representations. According to this view, instruction should help students to make connections between their existing verbal knowledge and the new symbolic knowledge they are learning.

Relevant grounding activities might include:

1) matching equations and equivalent word equations;

2) translating equations to story problems and solving both;

3) solving story problems and summarising both the story and the solution in equations; and,

4) making students aware of the algorithmic nature of their intuitive verbal strategies, and generalising these procedures to include symbolic formalisms. (Koedinger & Nathan, 2004, p. 161)

Nickson (2000) also highlights the importance of "the alternation of the *arithmetic-to-algebra* and the *algebra-to-arithmetic* pathways" (p. 124).

2.8 Summary

The complex topic of algebra word problems presents difficulties for both teachers and students. Indications are that most teachers' current practices reflect limited understanding of students' strategies and difficulties with algebra and word problems. Resources tend to support a focus on rule mastery rather than conceptual understanding. Although students' difficulties with word problems are compounded by a variety of factors including contextual, structural and linguistic factors, a major difficulty for secondary students is their reliance on informal methods even with complex problems.

Secondary teachers tend to favour algebra as a solution strategy regardless of the problem. Expert blind spot explains the difficulty teachers have with appreciating students' informal approaches and understanding their difficulties with using algebra to solve word problems. Traditional classroom experiences tend to emphasise procedural understanding and research findings suggest that teachers need to provide more structured opportunities for students to develop confidence and competence with using algebra so that they can select it as a tool for solving appropriate word problems.

In order to use algebra to solve word problems students need to be fluent with the language of algebra; they need to be comfortable with the rules and conventions of algebraic notation. They need to be able to use the language of algebra to express relationships (to read and write the notation correctly) and to work with this representation (to manipulate symbols correctly and fluently). However, students who can do this may still not use algebra in solving problems. Koedinger and Nathan (2004) suggest that a cause of difficulty is students' lack of experience with algebraic descriptions compared to verbal descriptions. Students' lack familiarity with the language of algebra, and their ease with informal methods enable them to avoid tackling the conventions and symbols of the unfamiliar language. Many students view using algebra to solve a problem as an extra difficulty imposed by teachers for no good reason. This attitude is reinforced by textbook problems that can be readily solved without algebra.

Before they can use algebraic solution strategies, students need first to translate a word problem into algebra. Research indicates this phase is a key area for intervention. However, although the literature suggests strategies that may be helpful, little research has been done on their impact. A specific focus on translation may contribute to students' confidence and competence with the language of algebra. The suggestion is that this aspect of algebra word problems typically does not receive enough attention and contributes to students' reluctance to use algebra in responding to word problems. The teaching and learning of translation between word problems and algebra is a focus of this current study.

CHAPTER 3
RESEARCH DESIGN

Research is about the development of shared knowledge (Lesh, 2002, p. 31).

3.1 Introduction

There is a multiplicity of ways of conducting research, many of which have been used in mathematics education. Burton (2002) argues that this leads to legitimate questions about "which methods are best to do which job" (p. 9). As this study aims to investigate learning from the learner's perspective, a qualitative methodology was considered suitable. Qualitative research seeks to make sense of the world through individuals' experiences. In this study the relevant individuals are teachers and their students, and their classroom-based experiences. This requires a methodology that deals with the complexities of the classroom and the interdependencies of teachers and students. For this reason, a design experiment was used to examine the impact of introducing explicit teaching activities from both teacher and student perspectives.

3.2 Design experiment

Design experiment as a research methodology has its roots in Russian teaching experiments, Piagetian psychology, and radical and social constructivism (Steffe & Thompson, 2000). Design experiments fit the complex systemic nature of the teaching situation (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Lesh, 2002). They attempt to meet the need for conducting research within a context that contributes to both research and practice in that context (Burkhardt & Schoenfeld, 2003). Design experiments are participatory and collaborative, conducted by people directly concerned with the situation being researched. Cobb and associates highlight the potential of design experiments for impacting on practice because they "speak directly to the types of problems that practitioners address in the course of their work" (Cobb et al., 2003, p. 11).

Classroom design experiments, also called classroom teaching experiments (McClain, 2002), are a form of design experiment in which the researcher(s) collaborate(s) with teacher(s) as member(s) of the research team to assume responsibility for instruction

(Cobb, 2000). The focus of classroom design experiments is on designing and exploring interventions with a typical goal being the development of instructional activities for students. These experiments usually focus on the learning of students but teachers' learning is also a legitimate focus of investigation in classroom design experiments. "Teachers reorganise their beliefs and instructional practices as they attempt to make sense of classroom events and incidents" (Cobb, 2000, p. 312). This is in accord with the growing awareness of the importance of teacher inquiry for professional development (Jaworski, 2004; Timperley et al., 2006).

> Teacher knowledge in collaboration with university partnerships that are engaged in the practice of intentional change coupled with reflective conversations that serve as an enhancement to ongoing, long-term professional developments is a powerful tool for understanding the nature of teaching and learning. (Passman, 2002, p. 1)

Classroom design experiment is a research methodology that suited the time frame and situated nature of this project. The focus of the project was three fold: instructional activities, student learning, and teacher learning.

Teacher learning and changing teacher practice are complex issues. Teachers, like other learners, need a powerful reason to engage in sufficient depth with new information if they are to change their practice. There is wide-spread agreement that professional development should be "targeted and directly related to teachers' practice" (Willis, 2002, p. 6), as well as curriculum-based and "ongoing – part of a teacher's workweek not something that's tacked on" (Willis, p. 6). In their review of literature, Timperley et al. (2006) identify other key features of effective professional development. They highlight the need for integration of theory and practice and assistance for teachers to translate theory into classroom practice. They also suggest that teacher beliefs need to be challenged and their theories engaged for professional development to impact on student outcomes. Fullan (2001) agrees that changes in beliefs and understandings are the foundation of lasting change. "We are talking not about surface meaning, but rather deep meaning about new approaches to teaching and learning" (pp. 37, 38).

Evidence-based skills of inquiry into the impact of teaching on learning are critical for teacher change. "There is no question that engaging in inquiry about teaching and learning mathematics is an important (and often an extremely powerful) form of

professional development that enhances teachers' knowledge, skill and understandings" (Cochran-Smith, 2006, p. xvii) . "Classroom-based research, carried out by teachers, provokes teachers to analyse their classrooms, their practice, and their students' learning to depths that are difficult to reach with other types of professional-development activities" (D'Ambrosio, 1998, p. 147). Teachers are involved in three types of reflection: reflection-in-action which involves thinking "on your feet" and responding "on the spot", reflection-on-action which involves examining what happened, and reflection-for-action which involves developing new theories, consolidating, adapting or changing practice (Schon, 1995).

Acknowledging the importance of "a balance between teacher-led reflection and inquiry, and expert support" (Annan, Lai, & Robinson, 2003, p. 34; Schon, 1995), the project included an outside researcher (the author) working collaboratively with the teachers who were involved at all stages of the research process. This kept teacher involvement at the heart of the project providing opportunities for tapping into the wisdom of those responsible for classroom practice, as well as enabling the knowledge gained from the research to directly inform teacher practice (Breen, 2003).

The project was designed around the hypothesised learning process that activities with an explicit focus on translation between English and algebra would enhance students' solving of algebra word problems. Specific teaching interventions were designed, implemented and evaluated to see how they impacted on students' responses to word problems. Alongside this, the project investigated the impact on teachers of their involvement in the project.

3.3 The Project

3.3.1 The setting and sample

This project was conducted in the Mathematics Department of a large provincial co-educational secondary school. The two teachers involved in the study were members of the department who had expressed an interest in being involved in the research project. Two participating teachers were considered to provide the opportunities for learning talk (Annan et al., 2003) while being manageable in terms of time, management and

resources. The teachers were both experienced classroom practitioners who were accustomed to using information from the classroom to inform their practice.

The students who participated in the study came from the teachers' Year 12 mathematics classes. The assumption was made that students in these classes would have prior experience with algebra. The focus on the translation stage of solving algebra word problems required students who were familiar with algebraic conventions (Koedinger et al., 1999).

3.3.2 The schedule

The project consisted of five phases conducted over a six month period from March to August.

- **Phase One**

The first phase included a preliminary literature review and consultation with mathematics teachers. During this phase, information sheets and consent forms were given out to teachers and students (see Appendix D1 and D2 for sample letters). Interviews were held with teachers to gather information about their initial understandings of teaching algebra word problems. Students completed sample tasks on algebraic word problems. Their responses were analysed and six students whose work was identified as being of particular interest were selected from each class. This purposive sampling (Miles & Huberman, 1994) was designed to bring a diversity of perspectives to the data collection process. Interviews were held with the selected students to explore their responses to the tasks in more depth. Instructional goals were clarified.

- **Phase Two**

Phase Two, the development phase, involved the research team collaborating to develop tools and activities to support learning with an explicit focus on translation. The first step was for the research team to formulate the hypothetical learning trajectory (Simon, 1995). This consisted of identifying instructional starting points, student learning goals, instructional activities and conjectures about student responses to the activities. This process was informed by the body of knowledge on student difficulties with algebra

word problems as well as teachers' classroom experiences and knowledge of their students.

As the teachers and researchers planned the hypothetical learning trajectory for each lesson, they hypothesised about how students would approach problems, mistakes they might make in their reasoning, and difficulties they might need to overcome. These hypotheses formed the conjectured learning processes. Beginning with the learning goals, activities were developed for each lesson using knowledge gained from the previous lesson. The instructional activities and the main concepts they addressed are summarised in Table 3.1. Specific activities were developed only a day or two in advance so as to be informed by reflections on students' responses. In planning the activities, the complex nature of learning was stressed and teachers were aware of the need to be open to other contingencies and the possibility of building on these as they eventuated.

• **Phase Three**

Phase Three was the implementation phase. Teachers implemented the planned sequence of instructional activities from the hypothetical learning trajectory over a series of lessons. Each teacher targeted agreed objectives, implemented the teaching activities and modified them according to the context and demands of their classroom. Simon (1995) describes this continual modification as an essential part of the act of teaching and highlights the delicate balance needed between the sense of purpose and the flexibility to adapt to students.

The researcher observed all intervention lessons, video-recorded and made field notes. Video-stimulated recall interviews were held with a focus group of four students from each class to explore their responses to each lesson. The researcher met with each teacher as soon as possible following every lesson observation to clarify and discuss understandings of the events. This reflective process led to fine tuning of the learning activities on a daily basis; these adjustments are what McClain (2002) calls the *microlevel decisions* about appropriate tasks for the following lesson. Decisions made during these daily debriefing sessions were documented as field notes. Additionally, both teachers met with the researcher every two to three days to further develop and refine the planned learning trajectory.

Further samples of student class work on algebraic word problems were coded and the selected students were interviewed individually to explore their responses in more depth.

Table 3.1 Planned activities of the hypothetical learning trajectory

Objective	Title	Instructional activity
Students make links between informal solution methods and algebraic methods.	Own method	Solve problems (starting with the consecutive numbers problem) by their own informal methods.
		Compare different approaches and investigate connections between informal methods and algebraic methods.
Students understand that algebra can be used as a tool to solve word problems.	The cycle	Introduce and discuss the problem solving cycle
Students develop competence with translating between algebra and English.	Set 1	Match words and symbols Writing English for algebraic expressions
	Set 2	Write algebraic expressions from English
	Set 3	Cloze activity involving filling gaps in English expression from matched algebraic equation
	Set 4	Write English from algebraic equations
	Set 5	Close activity involving filling gaps for contextual English from algebraic equations
	Set 6	Match contextual English with algebraic equations in variety of forms
	Pass the paper	Write English from algebraic equations and back again
	Green cards	Write expressions from simple English and equating expressions to write and solve equations
Students develop competence in solving a word problem by translating from words to equations and then solving the equations.	Four-step method	Heuristic supported through group work

- **Phase Four**

Phase Four involved data gathering two to three months after the intervention. Teachers were interviewed individually to explore the project's impact on their understandings about, and teaching of, algebra word problems. Each teacher identified a class to which they were teaching algebra word problems at this time. One Year 10 class and one Year 11 class were selected and the researcher completed observations of three lessons selected at random for each teacher.

- **Phase Five**

This last phase involved retrospective analysis. In contrast to the ongoing analysis that informed the intervention on a daily basis, the retrospective analysis involved a re-visiting and analysis of all the data collected during the study.

3.4 Data collection

Three different tools were used to generate qualitative data: interviews, lesson observations, and artefacts. Vaughn, Schumm, and Sinagub (1996) emphasise that truth depends on context and perspective and assert that in qualitative research "the goal is to describe findings within a particular situation" (p. 16). The variety of tools used for data collection addressed the project's multiple aims of documenting shifts in teachers' understandings and practices as well as shifts in students' reasonings and the means by which these shifts were supported in the classroom.

Teachers and the researcher needed to treat many aspects of classroom interactions as background (Cobb et al., 2003). To avoid being swamped in data, they agreed what were target factors and ignored all non-target factors.

> The study of phenomena as complex as learning ecologies precludes complete specification of everything that happens. It is therefore all the more important to distinguish between elements that are the target of investigation and those that may be ancillary, accidental, or assumed as background conditions. (Cobb et al., 2003, p. 10)

In this project, the key aspects focused on were teacher and student processes around algebra word problems. Other aspects of the classroom including classroom norms and the role of the teacher were accepted as background to the study.

3.4.1 Tasks of sample word problems

Two tasks of sample word problems were completed by all Year 12 students in the study. Students completed Task A before, and Task B after, the intervention (see Appendices B1 and B2). The original intention was to use matched tasks. However, constraints imposed by the collaborative nature of the research combined with task construction issues, resulted in few parallel questions (see Section 6.3.2). The selection of questions was constrained in part by the types of questions that were common in formal assessment tasks. Teachers chose not to use questions developed to distinguish arithmetic and algebraic problems (Van Dooren et al., 2002) as they felt these were not what students would face in assessments. Teachers were keen to use questions with which they were familiar.

Students' responses to Task A highlighted the arithmetic nature of many of the questions. In recognition of this, teachers wanted to modify the questions for Task B so as to encourage students to use algebraic strategies. This resulted in most of the questions in the two tasks being unmatched with only the one question being meaningfully paralleled in Task A and Task B.

Both tasks contained five questions. Question 1 was included to provide information about students' inclusion of variables in answers. Questions 2, 3, 4 and 5 were all word problems. Question 2 of both tasks was a ratio problem modelled on an NCEA assessment question (see Appendix C1) but unfortunately this question in Task B contained a typing error which made the question contextually unsound. Question 3 was adapted from an in-school assessment and paralleled in the two tasks. Question 4, which was also adapted from an in-school assessment task and paralleled between Task A and B, required students to make assumptions in order to solve the problem. Question 5 was adapted from an NCEA assessment question (see Appendix C2) to create a word problem that required students to write and solve a quadratic equation. When teachers realised how readily informal methods solved this question in Task A, they re-wrote the question for Task B with numbers that made the answer difficult to arrive at without algebra.

3.4.2 Interviews

Interviews lie on a continuum, depending on their degree of flexibility, from the structured interview which uses set questions from which there is no deviation to the unstructured interview which flows like natural conversation with questions influenced by participant responses (Gillham, 2000b; Minichiello, Aroni, Timewell, & Alexander, 1990). For this research, the semi-structured or *semistandardised* (Berg, 2004) interview provided an appealing compromise. It allows respondents the freedom to develop their ideas but has some structure to ensure that information on crucial topics is gathered efficiently.

- **Individual student interviews**

Interviews were conducted with individual students to provide an in-depth look into student thinking about solving algebra word problems. During lesson observations the researcher identified students who were achieving varying success in solving algebra word problems. Semi-structured interviews were conducted with six consenting students from each class (see Appendix E2 for student interview schedule). The purpose of the interviews was to gain insight into the processes students used in solving algebra word problems. Information gained contributed to the planning of the intervention. The semi-structured approach enabled the interviewer to probe students' responses. Interviews were audio-taped. The students were interviewed before the intervention, and again shortly after the intervention was completed, providing a total of 24 interviews.

- **Individual teacher interviews**

Each teacher was interviewed at the start of the research, at the end of the implementation phase and again two to three months after the intervention to gauge changes over time. The semi-structured interviews examined each teacher's thinking about, and experience with, teaching and learning algebra, with a particular focus on classroom processes, planning and intentions (see Appendix E1 for teacher interview schedule).. Some questions were framed from a phenomenological perspective and included probes asking for examples so that the teachers could describe their teaching behaviours as stories of actual events.

- **Student focus group interviews**

A -stimulated focus group interview was conducted following each intervention lesson (see Appendix E3 for student focus group interview schedule). The underlying idea of

stimulated recall is that presenting subjects with a large number of cues enables them to revisit the experience with vividness and accuracy. Focus group interviews have the advantage of the "synergistic group effect" (Stewart & Shamdasani, 1990, p. 15). This effect means that the resulting synergy generates a larger number of issues and ideas than may be obtained through individual conversations. Although the freer atmosphere of the group creates challenges for the interviewer in controlling and accurately recording the interview, group interviews are particularly useful with adolescents (Keats, 2000).

Focus groups of four students were self-selected from each class. Students involved in focus group interviews were not involved in the individual interviews. Shortly after each lesson, using the video record as stimulus, the researcher interviewed the focus groups. The video record of the lesson was paused for discussion as required in response to the identification of a particular episode by any student or the researcher. Priority was given to those events to which students attached significance. The researcher used probing questions to assist students to identify and articulate aspects of the teachers' practices that engaged their learning, and conversely, aspects that served as barriers to their learning. Transcripts and commentary of student interviews were re-presented to students to ensure that authenticity of students' meanings was maintained. These were then shared with teachers, and thus contributed to the review and refinement of the intervention.

A significant potential limitation on the video-stimulated interviews arises from the self-selection process for members of the focus group. The possibility of bias in the selection process must be acknowledged. It was anticipated that the students wanting to be involved in the focus groups were likely to belong to a particular group of students. However, this did not eventuate. Teacher ratings of students identified the focus groups as being of mixed ability. The group included students with a high interest in algebra word problems as well as those who may have viewed the focus group interview as an opportunity to avoid class.

3.4.3 Classroom observation

Lesson observations provided another source of data. Direct observation can provide the researcher with data that remains hidden in other data collection methods (Altrichter, Posch, & Somekh, 1993). The researcher was present in the classroom and videoed every lesson during the intervention phase. The researcher also observed and videoed three randomly selected lessons two to three months after the intervention. Videos were

supported by detailed field notes. The main purpose of the videotapes was to stimulate discussion in student focus groups. The videotapes and field notes also served as a source of data about teachers' instructional behaviours and what teachers and students said and did during instruction.

3.4.4 Artefacts

Teaching artefacts comprised teaching notes, lesson plans, as well as instructional activities and exercises, all of which provided supporting evidence about teacher practice. Students' written work comprised the learning artefacts and provided supporting evidence about students' responses to tasks.

3.5 Analysis

"Analysis involves working with data, organising them, breaking them into manageable units, synthesising them, searching for patterns, discovering what is important and what is to be learned, and deciding what you will tell others" (Bogdan & Biklen, 1992, p. 157). There are a variety of systematic techniques for data analysis all of which follow an interactive, spiral process. The first step involves coding all data sources. The researcher uses themes identified from the data to form coding categories, sort the data into manageable units and code it into the categories (Berg, 2004).

Data analysis in this project was both ongoing and retrospective. The researcher used open-ended coding focusing on significant statements and actions that reflected students' and teachers' understandings, values, beliefs and intentions regarding algebra word problems. The researcher first worked through all the data chronologically, coding it, referencing the categories to the original data, authenticating interpretations and decisions with participating teachers. Each aspect of data gathered was scrutinised in order to create meaning.

Coding for student work was developed from Stacey and MacGregor's (2000). "Routes from a problem statement to the solution" (p. 155). The researcher coded the data on two occasions. The first coding was completed as soon as possible after collecting the work. Repeat coding two months later verified the consistency of codes.

In a second phase of the analysis, categories and themes were identified and information sorted according to these categories. Conjectures and refutations were made, and these became *meta-data* to be analysed so that themes were refined further. The teachers checked that decisions regarding inclusions and omissions of data were unbiased and indicative of other findings.

The results are discussed in subsequent chapters: Chapter 4 on instructional activities, Chapter 5 on students' responses, and Chapter 6 on teachers' perspectives. Students' responses are discussed in terms of the strategies used to solve algebraic word problems, and their reactions to the activities used in the intervention. Discussion of teachers' perspectives centres on the impact of the intervention on their beliefs and practices.

3.6 Validity and reliability

This research project is situated within a particular context and practice–two teachers working in their classrooms with their students. This introduces difficulties of generalising beyond the specific situation and the challenge of making the research replicable. While results are directly applicable only to the particular context and practice, they may be useful for similar students, in similar situations; teachers may be able to adapt the activities and approaches and translate the findings to their own particular context.

Issues arising out of the contextual nature of this project are addressed as far as possible by the systematic nature of the processes in this teaching experiment that differentiate this project as robust research rather than good practice which many teachers already do on a day-to-day basis. The robustness of any research is influenced by issues of validity and reliability. In research methodologies which are collaborative and participatory, questions about the value of practitioner research, academic legitimisation and the status of popular knowledge can raise particular concerns about reliability and validity (McTaggert, 1996).

Validity is concerned with the question of veracity or authenticity of research (Berg, 2004). It is the extent to which an instrument measures what it claims to measure. Internal validity is whether the researcher actually measures or observes what they think they are measuring or observing. External validity is concerned with the extent to which

the findings can be generalised or applied beyond the particular setting. However, Cobb (2000) takes a different view of generalisability as "the theoretical analysis developed when coming to understand one case is deemed to be relevant when interpreting other cases" (p. 327). A teaching experiment is unlikely to have generalisable findings as it is likely that any teaching intervention will work well for some students in some situations. However, the principles of the intervention and the theory behind it should be generalisable.

Reliability is an issue in qualitative research. It is related to replicability and is the extent to which a procedure produces similar results under consistent conditions on all occasions. Replication of any qualitative study can not be expected to produce the same result so achieving reliability in qualitative research in this sense is not possible (Merriam, 1998). Furthermore, in a teaching experiment "the conception of teachers as professionals who adjust their plans continually on the basis of ongoing assessments of individual and collective activity in the classroom would suggest that complete replicability is neither desirable, nor perhaps, possible" (Cobb, 2000, p. 329).

Lincoln and Guba (1985) suggest an alternative measure of reliability, that of dependability or consistency of results. Rather than asking whether findings will be found again, the question should be whether the results are consistent with the data collected. To assist with reliability, information about the context of the inquiry and the results are provided. This process should enable the reader to make an informed judgement as to the transferability of the results. This approach is facilitated by defining the researcher's position with respect to the subjects, triangulation and an audit trail explaining how the results were arrived at (Lincoln & Guba, 1985).

Triangulation is a commonly used method to improve validity. It is achieved by collecting different kinds of evidence from different sources and/or different tools (Gillham, 2000a). In this project, data were collected from three different viewpoints: teacher, student and observer (researcher). Data collection involved a range of tools including observation, interviewer and written artefacts. Data analysis involved ongoing checking of findings from different tools and different sources ensuring that the evaluation reflected the realities as experienced by both teachers and students.

This study aims to describe the processes used in data collection and evaluation in order to increase the transparency of the assumptions and decisions made by the researcher throughout the research.

3.6.1 The role of the researcher

The researcher is the primary instrument in qualitative research (Merriam, 1998). The researcher's experiences, expectations and values play a part in the decisions made throughout the research process so they need to be explicitly stated. In this project, the researcher undertakes a variety of roles. During the data gathering process the roles depended on the way evidence was collected, be it as an observer, interviewer, reader or interpreter (McNiff & Whitehead, 2002). In this project, my role as researcher was not an outsider but an insider. I worked collaboratively with participants who were significant partners in the research team. I became a member of the study population, who was accepted into the community on an equal footing.

I am an experienced secondary mathematics teacher currently working as an adviser to secondary school teachers. This project reflects my interest in working with mathematics teachers to improve outcomes for students. Having spent time in the project school as a visitor, rather than as part of the school hierarchy, meant that I was accustomed to being an unobtrusive visitor in the classroom. I had recently worked as a member of the school staff and had an established insider relationship with the teachers.

At all times within this project, in my role as researcher I aimed to make explicit my conjectures, suppositions, and assumptions that ground my interpretations so that they are open to scrutiny. I view mathematics learning from an emergent perspective in which learning is seen as both a psychological process of the individual and a social process of the group (Cobb, Jaworski, & Presmeg, 1996). Thus "accounts of [students'] mathematical development might involve the coordination of psychological analyses of their individual activities with social analyses of the norms and practices established by the classroom community" (Cobb, 2000, p. 309).

3.7 Limitations

Limitations of this project arise out of its situated, contextual nature and the complexities of the teaching/learning situation. Collaboration with teachers is likely to involve "realistic constraints as they explore what might be possible in students' mathematics education" (Cobb, 2000, p. 330). This study acknowledged the possibility of the researcher's actions at times being constrained by the collaborative decision making nature of the research design, thereby limiting her ability to act in accordance with what she judged to be best practice. The fact that, as researcher, I could only present the information and work within the decisions of the group meant that sometimes practices and activities were not closely aligned with the original intent of the project.

One limitation arose from constraints associated with maintaining the students' usual teaching/learning programmes. Students involved in the project could not be taken out of class for parallel pre and post tests and so data needed to be gathered in situ.

Another limitation arose from time constraints around gathering information about teachers' beliefs, understandings and practices. Teachers participated in interviews and meetings, but did not agree to complete questionnaires and surveys. Limitations such as these arise out of the contextual nature of the teaching experiment. In this project they limited what was measured, but they also served to authenticate the situated nature of the research.

Veracity of data can be a concern, particularly in research involving teachers in a collaborative role, the researcher faces the issue of knowing directly whether a communication is genuine. In this project, each teacher's equal status on the team was emphasised. This addressed the potential issue of teachers giving the researcher expert status and giving less honest information in an effort to please the expert. Two criteria for genuineness were used in this project:

> If the communication by the participant could be motivated by a desire to appear competent, these data may not be used as primary data. They may only be used as secondary data to corroborate other information. If the communication by the participant varies from what would be expected in order to demonstrate competence, then these data

may be considered genuine and used as primary data sources. (Simon, 2000, pp. 354-355)

The active involvement of the research team created another potential limitation in this project. The research team was an integral part of the system they were hoping to understand and explain. This dilemma of generating knowledge about a system at the same time as trying to change it is intrinsic to design experiment methodology. However, this is also a strength of the methodology since involving practitioners in identifying problems to be addressed and formulating solutions increases the relevance of research to practice.

3.8 Ethical considerations

Any research involving people must be scrutinised for potential ethical issues to ensure the safety and wellbeing of all participants are preserved. Central ethical questions focus on informed consent, privacy, confidentiality and anonymity, ownership of data and use of results (Miles & Huberman, 1994). Participants, both teachers and students, have the right to be fully informed concerning consent and time expectations. Other issues to be considered are those of honesty and trust, reciprocity, intervention and advocacy (Punch, 2000).

The following steps were taken to ensure that these ethical principles were applied in this project:

- Approval was given by the Massey University Human Ethics Committee.
- Approval was given by the school principal.
- Informed consent was gained in writing from the participants after they had been given the opportunity to discuss the implications of their participation. Care was taken to ensure that information was provided to students in a manner and form which they understood. For participating students, permission was gained from both students and a parent/caregiver.
- Confidentiality of information was a priority and safe custody of data was maintained. Student anonymity was challenging as the student focus groups were known to the teacher. However, the information was handled

so as to protect confidentiality as far as possible. The tapes and videotapes were transcribed by the researcher and the transcripts do not identify any individuals. The videotapes were viewed solely by participating students and the research team to inform the research process.

- Safety of students and staff was protected. Participating teachers made a commitment to valuing and respecting the contributions of students. Students were entitled to check and withdraw their comments at any stage of the project.

- The project was managed so that student learning was not adversely affected by their participation or their non-participation in the research. The project allowed for all the requirements of the normal school programme.

CHAPTER 4
RESULTS AND DISCUSSION: INSTRUCTIONAL ACTIVITIES

In the mathematics classroom, it is through tasks, more than any other way, that opportunities to learn are made available to students. (Anthony & Walshaw, 2007, p. 96)

4.1 Introduction

This chapter describes instructional activities that were used during the intervention. Data from lesson observations and field notes are supported by data gathered during student and teacher interviews, and teacher meetings. The activities were planned collaboratively (as summarised in Section 3.3.2) although implementation varied as teachers adapted, and at times omitted, activities in response to the contextual needs of each class. The following discussion focuses on activities as implemented rather than as planned.

The impact of any instructional sequence is influenced by a complex web of interacting factors. Hard conclusions cannot be drawn about the effectiveness of any one specific activity for all classroom situations. However, combining data about teaching activities from both student and teacher perspectives paints a picture of the impact of particular teaching activities for these particular students in their classrooms. From these results, suggestions can be made about approaches that may be effective in teaching students to use algebra to solve word problems.

The two teachers are referred to as TX and TY. The classes are referred to by year level and a letter according to which teacher taught them mathematics. Thus, 12X is the Year 12 class that was taught by TX. All students on each class list were randomly assigned a letter of the alphabet so their anonymity was retained. In the following discussions, students are referred to by their class and their assigned letter; for example, 12XA refers to the student in 12X who was assigned the letter A. Where a particular student can not be identified, they are referred to by year level and teacher and the letter S followed by a number so that 12XS1 is the first unidentified student from the Year 12 class taught by TX.

4.2 Instructional activities

Activities were designed for the intervention with the aim of supporting students to establish and strengthen connections between words and symbols. The intent was to link the instructional activities with the literature so they became a tool for putting theory into practice. The overall focus on increasing students' confidence with using symbols, aimed to address MacGregor and Stacey's (1997) assertion that "in a typical curriculum students do not get enough experience at using algebraic notation" (p. 19).

Students expressed a range of views about the intervention's instructional activities. Most comments were positive and there were no aspects of the interventions that received a significant number of negative comments. Significance was not determined statistically, but was decided by the number of students who commented on any aspect combined with teacher feedback. Assigning import was straightforward when students and teachers feedback was in accord but more difficult to interpret when responses were mixed. Aspects that received mixed responses are discussed separately so that individual voices are heard. The two key activities were "Pass-the-paper" and the "Four-step method": they were rated highly by both teachers and were specifically mentioned by all but one student in interviews at the end of the intervention.

4.2.1 Explicit expectations: the problem-solving cycle

A prominent aim for both teachers was for their students to view algebra as a tool for solving word problems. They believed this aligned with Bennett's (2002) suggestion that students be made explicitly aware of the purpose of word problems. In accord with Swan's (2000) emphasis on the importance of the teacher being explicit about the purpose of tasks, the teachers used the algebraic problem solving cycle (Lawrence & Patterson, 2005) to support them in making expectations about the use of algebra explicit. The problem-solving cycle (see Figure 2.6) formed the basis for class discussions. In the first lesson, both teachers showed students the cycle and discussed each step. In subsequent lessons both teachers used the cycle as part of the advance organiser to explain where each instructional activity fitted into the cycle and the links between activities.

Student responses following the interventions consistently indicated that they had a clear understanding about teacher expectations, the process of translation, and the use of algebra to solve word problems. They felt more confident because they knew what was expected, and knew how to do it and they attributed this mostly to the clear focus of each lesson.

> *12XF:* *You can see what you are trying to achieve—like where you are going.*

The problem-solving steps seemed to fit students' understandings of the problem solving process.

> *12YK:* *I think I already kind of carry out the steps but I don't think of it like that.*

Teachers also attributed students' understandings of expectations in part to the clear focus of each lesson. In addition, teachers felt that the cycle was helpful for clarifying their thinking about the process of solving algebra word problems.

> *TX:* *I haven't ever had it spelt out to me like this... definitely helped me...where you want to go with the students—what you want them to understand and do.*

4.2.2 Translation activities

The focus on translation arose from Koedinger and Nathan's (2004) suggestion that students need to learn to translate back and forth between English and algebra. The aim was to create a series of teaching activities and tasks that progressed from expressions to equations, and from literal translations to contextual translations. Recommendations by Koedinger and Nathan (2004) informed the planning of the matching equations and equivalent word equations activities. Two other types of activities were developed based on Curcio's (2004) research: "cloze reading tasks" (p. 165) and "here's the equation, write the problem" (p. 166).

Table 4.1 details the nine activities that were rated as particularly effective by both teachers and some students.

Table 4.1 Instructional activities rated as particularly effective

Action	Operating with	Activity	Instructional activity
Match contextual English and algebra	Expressions	Matching	Matching activity in Appendix F1
Creating literal English from algebra	Expressions	Write the problem	Set 1 in Appendix F2
Creating algebra from literal English	Expressions	Write the equation	Set 2 in Appendix F3
Completing literal English from algebra	Equations	Cloze activity	Set 3 in Appendix F4
Creating literal English from algebra	Equations	Write the problem	Set 4 in Appendix F5
Completing contextual English from algebra	Equations	Cloze activity	Set 5 in Appendix F6
Writing English from algebraic equations and back again	Equations	Writing	"Pass-the-paper" (see Figure 4.1)
Match contextual English with algebra	Equations	Matching	Set 6 in Appendix F7
Writing algebra from English	Equations	Writing and solving	The "Four-step method" (see Figure 4.2)

All students commented positively on the Pass-the-paper activity and all but one on the Four-step method. Apart from these two activities, students mostly commented on the translation activities as a whole. Features that were identified as significant included: translating both ways, the structured progression of the activities, and the structured opportunity for learning talk.

> 12XF: I liked how we learnt from both views—putting it into word problems and taking a word problem and putting it into algebraic. I understand it a bit better—and starting with simple ones then more complicated to see how they work rather than doing the complicated ones straight away.

12XL: *I always thought algebra was useless until we did that white sheet – the formula*
          stuff, Fahrenheit and stuff, used variables in the degrees and the formula so you
          could see it had a point... I thought algebra was pointless until it was put into
          word equations we really use.

12XB: *...because different people see it in different ways...different people's*
          interpretations of how to do it— it can be quite enlightening—different
          perspectives.

4.2.3 Pass-the-paper

All student comments were positive about the Pass-the-paper activity. This activity
focused on translating both ways between verbal and symbolic representations and
comparing outcomes. It was designed to be used by students working in pairs. Students
in each pair were given a paper with a set of equations on it. They each wrote literal
translations for the algebraic equations. They folded the paper so that the equations were
not visible and passed the paper to their partner who then used the literal translation to
write an algebraic equation. The pairs of students then compared the final algebraic
equations with the starting ones and discussed any differences. The aim of the activity
was to provide practice in translation between algebraic equations and literal translation,
as well as facilitating discussion between students about their thinking and understanding
about the steps in the translation process. Each teacher designed their own sheets for this
activity. (See Figure 4.1 for an example of one of TY's sheets).

Algebra	Literal translation	Algebra
$5n-2=26$		
$3(n+4)=18$		
$n^2 + 3n = 40$		
$(n+6)(n-2)=65$		
$3n+11=5n+1$		

Figure 4.1 Pass-the-paper activity

The Pass-the-paper activity was seen as particularly successful by the teachers because of
the high student engagement and the amount of focused student talk.

TY: *It's exciting when an activity works so well – the class was humming... they were*
        just so involved discussing what they had written and why.

Students in both classes commented on what they learnt from the activity.

> *12YF:* *It was good — the fact that you had to write every detail out cos you weren't going to be doing it yourself... you got to see other people's [thinking] cos sometimes what makes sense to you doesn't make sense to other people and it was good to see that.*

> *12XB:* *Because different people see it in different ways so like with the sheets — interpret it and write it down in words and pass it on. Different people's interpretations of how to do it can be quite enlightening— different perspectives. Some [questions] were ambiguous and so it was interesting to see what words you could use and have it totally not ambiguous.*

4.2.3 Four-step method

The Four-step method (Kushnir, 2001) was included in the intervention following on from activities aimed at strengthening students' connections between words and symbols. Bell (1996b) emphasised that for students to develop an understanding of algebra as more than manipulation of symbols, they need to "experience the full activity of beginning with a problem, forming the equations, then solving it, and interpreting the result" (p. 181). However, both teachers found it difficult to unpack their approach to word problems and they wanted a heuristic or rubric for solving word problems that they could share with students.

The Four-step method aligned with the problem-solving cycle and the translation emphasis of the intervention. The sequential steps were explicitly supported by a cooperative activity designed for groups of four students, with each student performing one of the steps as set out in Figure 4.2.

Student 1 performs Step 1 of assigning a variable;
Student 2 performs Step 2 of translating into algebra;
Student 3 performs Step 3 of solving the equation; and
Student 4 carries out Step 4 which is to check the solution.

Figure 4.2 The Four-step method (Kushnir, 2001)

Focusing on difficult equation solving skills was postponed in accord with Koedinger and Nathan's (2004) suggestion that

> decomposing instruction to focus on difficult equation solving skills is fine. However, such instruction should come *after* students have learned the meaning of algebraic sentences, in other words, after they have learned to translate back and forth between English and algebra. (p. 160)

Both teachers found the Four-step method valuable.

> *TX:* *When I looked at that Four-step process (assigning variables, writing an equation and solving it then checking), I have probably been through that process before but not with the lead-up of translating the words, not with specific focus of look at the variables, what do they represent?...The breakdown using that cooperative activity: identify the variable and the whole concept of variable and then write the equation—I think that is an important step that we've missed out [in the past].*

> *TY:* *I liked the way we kept going back to translation—the lead up of translating words with the specific focus on Four-steps of assign a variable, write the equation, solve, then check. This worked really well...It could be used over and over for a variety of different problems. The students just got into it and did it—easily adapted too.*

The usefulness of the Four-step method was stressed by all but one of the students in their final interviews. Students commented on its impact on their understanding and on the level of challenge.

> *12YO:* *Knowing what to let the variable be is critical and initially it seemed like it didn't matter...Some of them were quite challenging like you had to bring in other parts of maths so that was really good.*

Some students explained that they used the method when solving problems working individually. Success with this appeared to impact on students' feelings about the whole topic.

> *12XL:* *The Four-steps was really good. Working on the sheets when you had the first person did the first step and the next person did the next, and doing it by myself, doing the steps myself, that was good... You had to say what they were so you*

had to say "Let x", then next you had to do the equation, then the next step was to solve it, the last step was you had to put it back into the first. Those steps were really helpful. I hadn't done anything like that before. Before you'd think "Oh maths no", but this was stuff that I get so I felt lots better about it. I feel a bit more confident with writing algebra now.

TY suggested that the use of the heuristic had helped her identify that a key difficulty for students was deciding what to assign a variable to, when solving a contextual problem. This difficulty, identified in the literature (Stacey & MacGregor, 2000), was highlighted for TY when she introduced the Four-step method and asked the class to use it to solve the "John and Amy" problem (see Figure 4.3). During the 15 minutes that the class spent on the task, none of the students were able to assign a variable satisfactorily.

Amy is eight years older than John.
In six years, Amy will be twice as old as John will be then.
How old are they now?

Figure 4.3 The John and Amy problem

In the following lesson, TY revisited the problem and demonstrated the use of a table (as shown in the vignette below) to help establish the values of different quantities within the problem. Although Anthony and Walshaw (2007) suggest that teacher-supplied helps such as tables may inhibit rather than facilitate students' learning, in this situation, it appeared to act as a useful support without overly reducing the demands of the task. The table was presented as an option to students who had already attempted to tackle the problem and it became a tool that some groups adapted for use in subsequent problems.

TY was discussing the John and Amy problem.

> *TY: These kinds of problems can be quite confusing...we have two people, how old they are now and how old they will be in six years' time...let's use x to represent John's age now...if I draw up a little table...I find this helpful if I've got this kind of problem...*

TY talked through the setting up of a table as she wrote it on the board.

	John	Amy
Now		
Six years		

She then asked questions and used the volunteered responses from students to complete the table.

> *TY: Who can tell me what Amy's age is now in terms of John's age?*
>
> *12YS1: x plus 8*
>
> *TY: And John in six years?*
>
> *12YS1: x plus 6*
>
> *TY: And Amy in six years?*
>
> *12YS2: x plus...*
>
> *12YS3: 2 times x plus 6 in brackets*
>
> *TY: Yes but if we go six years on from what she is now?*
>
> *12YS2: x plus 14*
>
> *TY: And we know that is twice as much as John's age so that's how we can set up the equation can't we?*

The table on the board was now completed.

	John	Amy
Now	x	$x + 8$
Six years	$x + 6$	$x + 14$

TY wrote on the board $2(x + 6) = x + 14$

The groups then continued with their use of the Four-step method to solve the problem.

Most student groups used a table with at least one of the subsequent problems that they tackled that lesson.

> *12YO: We're doing that let thing. The table helps... helps me see what we've got.*

Interestingly, TY did not refer to the table in subsequent lessons and in final interviews only one student referred to the table.

*12YF: Our group used the table for one of the questions and it did help with the cricket
          question.*

Reflecting on difficulties after the first lesson with the John and Amy problem, TY
initially felt she should have started with an easier problem to avoid student frustration.
However, during the second lesson, she concluded that the difficulty of the problem had
not led to any loss of student motivation and may have provided a useful challenge for
many students. She suggested that assigning variables was an aspect that needed further
development and she planned to create an activity the next time she taught this topic that
focused on assigning variables before introducing the heuristic.

Students commented on the benefits of focusing on translation before solving problems
as suggested by the literature (Bell, 1996b; Curcio, 2004; Koedinger & Nathan, 2004).

*12YS1: Translating into words was really helpful before we had to solve the equations. It
        made it easier to solve them and it made it make more sense.*

12YS2: I understood what I was doing because I had translated it into words first.

4.3 Creating quality activities

The nature of the instructional activities impacts on students. Nardi and Steward (2003)
found that students perceive enjoyment as central to learning. Secondary students in their
study cited relevance, excitement and variety as factors making lessons enjoyable.
Furthermore, the positive emotion that accompanies moments of illumination (the 'aha!'
experience) has been shown to have a transformative effect on students' affective
domains, creating positive beliefs and attitudes not only about mathematics, but also
about an individuals' abilities to do mathematics (Barnes, 2002; Liljedahl, 2005).

When developing activities, teachers initially tended to rely on textbooks.

*TX: To actually find problems, exercises, that we can use is a huge difficulty. The
          textbooks just haven't focused on it.*

Teachers' difficulties locating suitable activities in textbooks supports Bennett's (2002)
findings about mathematics text books. He found that algebra texts commonly take the
approach of theory followed by examples, purely exercise and then the skills just covered

embedded into contextualized problems. Quinn (2004) supports this commenting on "the problematic trend in textbook writing that emphasises mere symbolic technique" (2004, p. 41).

Teachers recognised that the activities they developed were not perfect. They suggested that time was a constraint.

> *TX:* *Given more time, I could have done a better job, but the reality is we only have so much time, and there is only so much we can do.*

The challenge of writing quality instructional activities is reported by Sullivan, Siemon, Virgona, and Lasso (2002). They found that even after establishing criteria for quality tasks, teachers had difficulty creating tasks that fitted the criteria.

4.3.1 Errors in activities

Errors in activities highlight the difficulties of task writing. The first matching activity involved pairs of algebraic expressions and literal English translations (Appendix F1). This activity was an adaptation of an exercise in a Year 10 textbook (D. Barton, 1998, exercise 22.2) and was seen by both teachers as straightforward. Each group of students was given cards with a set of contextual expressions and a set of algebraic expressions to form into matching pairs. When TX was discussing the answers with the class, one question provoked class debate (see Figure 4.4) because some students identified that the algebraic expression provided was incorrect. There was much debate about whether the answer should be $n - 25$ or $25 - n$.

Profit made when an item is bought for $25, then sold for $n	**25 – *n***

Figure 4.4 Example from the first matching exercise

Initially, TX did not see that the answer of $25 - n$ was incorrect and tried to justify this answer. Students interrupted her with a number example.

> *12XS1: For example, if I buy for $25 and want to make a profit. So, say I sell for $30 then my profit is...*

TX: Oh, I see what you mean it has to be n - 25.

Although TX initially expressed frustration at finding an error in the activity, she decided that the student talk it provoked had been a rich source of student learning. She suggested that the activity would be improved if it was rewritten to include a number of errors.

4.3.2 Creating Cloze activities

Both teachers were familiar with Cloze activities through professional development in literacy. "Cloze refers to the 'reading closure' practice required when readers must fill blanks left in text" (Hornsby, 1992, p. 10). There are several cloze procedures, all of which follow some variation of a cycle of predicting, justifying, comparing and discussing until the piece is completed. Set 3 and Set 5 were written as cloze activities to support translation between algebra and English (see Appendices F4 and F6). Set 3 provided a scaffolded set of cloze activities involving an equation with a matching word problem. The word problem was a literal translation of the equation but had missing words that students were asked to fill. The activity provided decreasing amounts of scaffolding by increasing the number of gaps in a problem. Set 5 followed a similar structure but required students to complete the gaps in contextual word problems. Although teachers were aware that the aim was "not to make the procedure a test or to make it as difficult as possible" (Hornsby, 1992, p. 11), the cloze tasks proved difficult for students to complete.

> *12YF: It was dumb—you pretty much needed to guess what was in Mrs Y's head when she wrote it.*

4.3.3 Pressing for algebra

As the intervention progressed, teachers became increasingly aware of student tendency to avoid doing algebra. Stein, Schwan-Smith, Henningsen, and Silver (2000) describe task design as one of the factors that influences the commonplace tendency of students to subvert task demands. The teachers found that some of the activities they had written gave students clues that enabled them to complete the activities without doing the thinking that was the objective of a particular activity (Section 6.3.1). For example, students were able to discern a pattern in the Pass-the-paper activities without using algebraic tools.

> *12XJ: I found it easy cos they were all in sequence so I sort of knew what was coming. On my sheet it was in the same format, layout as the other one—my partner's—so I knew that's going to be brackets in my one. Made it easy—I didn't need to use algebra, I knew it had brackets...I just cheated. You could sort of tell by just looking at the numbers without looking at the words so didn't have to think too hard.*

However, once teachers were aware of the need to create activities that pressed for algebra, they still had difficulty producing activities that did not give unintended clues.

> *TX: You do have to be careful with the activities that they are not looking at the numbers and avoiding doing the deeper thinking.*

Although teachers had tried to include similar numbers within different problems in Set 6, many students discussed finding clues to avoid having to think so hard in this activity. Some students recognised the importance of doing tasks "properly".

> *12XS2: We short-cutted at first—like got ones with the same numbers that went with the questions but then we looked at them to see and worked it out properly... it was tricky but it was good...helpful in your mind, if you translated into the word problem so you looked at it to see what was actually happening to it and you could see if it was different or not—rather than just saying it's got the same numbers. People need to know it's helpful if you translate the algebra expressions into the written ones. It's best to do that.*

The common tendency to subvert task demands in Set 6 may have been in part due to the complexity of the task. In addition to the clues that students found, the activity involved too many items for most students to cope with comfortably.

> *12XS3: I had a bit of trouble—like I figured out what the word problem was saying then I went through all the other bits of paper equations and by then I kind of forgot what the word problem was saying- too hard trying to hold it in your head.*

TX amended this by reducing the number of items when developing the activity for her Year 10 class (See pink-blue-yellow matching activity in Appendix F8). Her use of similar numbers in each problem was designed to press students to use algebra to complete the activity. She also increased the level of challenge by leaving some cards blank so that students needed to create their own appropriate equations.

4.4 Discussion

The construction of the instructional activities for the intervention aimed to put theory into practice. Although variable student responses to activities make generalising about a particular activity problematic, aspects of the instructional activities can be identified as significant.

Key aspects were the structured progression of the activities, the explicit focus on translation and the amount of learning talk in which students engaged. The teachers used the problem-solving cycle to make expectations about translation into algebra, and the use of algebra to solve problems, explicit. Focusing on translation before asking students to solve problems helped them make sense of the problem solving.

Matching activities and cloze activities were effective for learning to go back and forth between verbal and symbolic representations but were difficult to design. The Pass-the-paper activity was an example of a translation activity that was associated with high levels of student engagement in learning talk. The heuristics of the Four-step method provided a useful tool for students to link the work on translation with an approach to solving a word problem. Teachers identified the assignation of variables as an aspect needing further development.

CHAPTER 5

RESULTS AND DISCUSSION: STUDENTS' RESPONSES

For all students in the mathematics classroom, the 'what' that they do is crucial to their learning.. (Anthony & Walshaw, 2007, p. 140)

5.1 Introduction

This chapter discusses Year 12 students' responses to algebra word problems, highlights common difficulties and explores evidence for changes in students' understandings. Data are from lesson observations, individual student interviews, and student focus group interviews, as well as students' written work. Written work included work completed by students during the intervention as well as student responses to two sample tasks, Task A before and Task B after the intervention.

Six students were purposefully selected from each class and interviewed individually at the start and end of the intervention. The students represented a range of achievement levels and strategy use. The interviews focused on students' responses to the sample tasks in order to explore their strategies and gain insight into their thinking. Focus group interviews with four students from each class were held following each intervention lesson. Video recordings of lessons were used to stimulate recall and prompt discussion of aspects of the lessons that were significant for student learning.

5.2 Strategies used to solve word problems

5.2.1 Coding of strategies

Students' responses to Tasks A and B were analysed according to the strategies used. However, the coding of solutions as either algebraic or arithmetic used by Koedinger and others (Koedinger & Nathan, 2004; Van Dooren et al., 2002) did not provide information about the range of responses; students who followed the algebraic pathway did not always follow it through to a solution. Stacey and MacGregor's (2000) routes provided a more complete description of the various pathways followed by students in responding to

word problem tasks. Codes were developed for of each of the strategies. Codes are shown as italicised capitals in Figure 5.1.
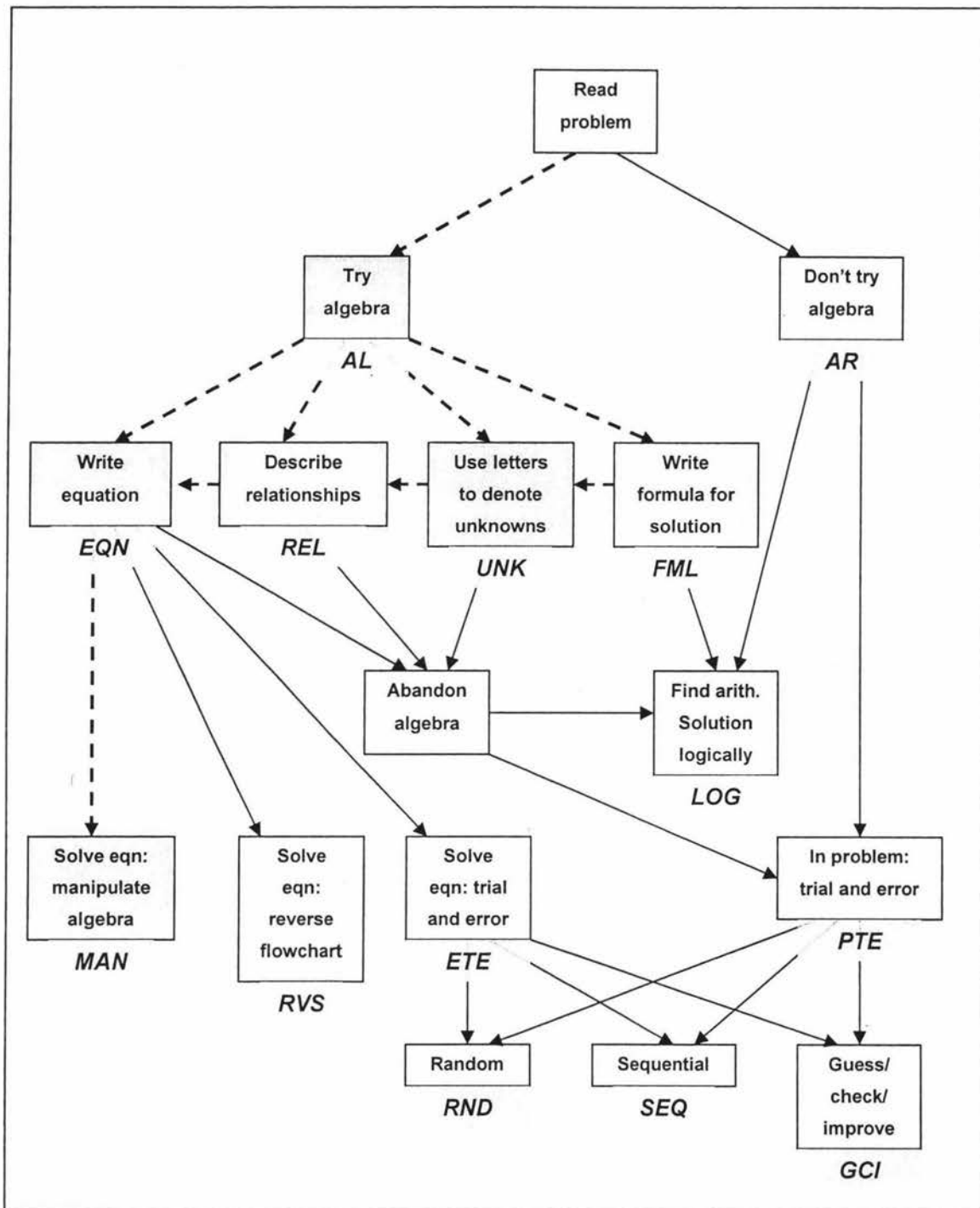

Figure 5.1 Routes from problem statement to solution (adapted from Stacey & MacGregor, 2000, p. 155).

Coding was based on written evidence (available for all students in classes 12X and 12Y), supplemented by verbal explanations for interviewed students (six from each class). A summary of the student results for Questions 2 to 5 of Tasks A and B is provided in Appendix G1. Students' responses were first coded as algebraic (AL) or non-algebraic (AR) depending on whether a student included letters in their response to a problem or not. Subsequent codes were then allocated according to the various strategies identified. A student may abandon algebra at any stage and either stop or use an arithmetic method to work toward a solution. Stacey and MacGregor argue that solving equations by reverse flowchart (RVS) and by trial and error (ETE) are not algebraic methods as they do not involve operating with the unknown. Other arithmetic methods include finding the solution logically (LOG), and trial and error in the problem (PTE). Trial and error methods, both ETE and PTE are then classified as random (RND), sequential (SEQ), or guess, check and improve (GCI).

Table 5.1 provides a summary of codes with the terminology as used by Stacey and MacGregor (2000) and other common terminology used in the literature referenced. In addition to the codes shown in Figure 5.1, five other codes were developed. Two codes were introduced to cover non-responses and unclear responses: NIL was used where a student gave no response, and NW where there was no evidence as to the strategy a student had used. Teachers were also interested in whether a student had obtained a correct answer or not, so three more codes were added: RA for the right answer; WA for the wrong answer; and NA for no answer. These last three codes were not used in the final analysis of student responses.

The researcher coded all student work. The coding system provided codes for all the variations seen in students' work. Year 12 students' responses included examples of all the strategies described in Stacey and MacGregor's (2000) routes apart from writing a formula for a solution so that FML was the only code not used.

Table 5.1 Coding for students' responses.

Code	Strategy (Stacey & MacGregor, 2000)	Other terminology
AR	Don't try algebra;	Informal (Koedinger & Nathan, 2004)
AL	Try algebra	
EQN	Write equation	
REL	Describe some relationships	
UNK	Use letters to denote unknowns	
FML	Write formula for solution	
MAN	Solve equation by manipulating algebra	
RVS	Solve equation by reverse flowchart	Backtracking
ETE	Solve equation by trial and error	Guess-and-check (Van Dooren et al., 2002)
LOG	Find arithmetic solution logically	Unwinding (Koedinger & Nathan, 2004); Generating numbers or Manipulating the structure (Van Dooren et al., 2002)
PTE	In problem trial and error	Guess and test (Koedinger & Nathan, 2004)
RND	Random trial and error	
SEQ	Sequential trial and error	
GCI	Guess/check/improve	
Code	Other coding used	
NIL	No response	
NW	No evidence of strategy	
RA	Right answer	
WA	Wrong answer	
NA	No answer	

5.2.2 Algebraic strategies

Strategies classified as algebraic are in shaded boxes in Figure 5.1 and dashed arrows have been used to indicate what Stacey and MacGregor (2000) consider the truly algebraic route. The full algebraic route is displayed by a student who solves a problem by writing an equation and solving it by manipulation. The minimum coding for this is AL, EQN, MAN; more codes are used if the student follows a less direct route to the solution.

Students' responses to Task B show increased use of algebraic strategies when compared with their responses to Task A. However, comparison of responses between the two tasks is problematic; constraints imposed by the collaborative nature of the research and task construction issues (as discussed in Sections 3. 4.1 and 6.3.2) meant that the tasks contained questions for which algebra was unlikely to be students' preferred strategy. Furthermore, despite original intentions of using matched tasks, there was only the one matched question, Question 3, in the two tasks.

Comparison of responses to Question 3 in the two tasks shows greater use of algebra in Task B than in Task A (See Appendix G2). Just over one third of the students did not show any evidence of use of algebra in response to this question in Task A but every student showed some use of algebra in responding to this question in Task B. 43% of the students followed the full algebraic pathway in their response to this problem in both Task A and 93% did so in Task B. This suggested a shift for 50% of the students who had not followed the full algebraic pathway in Task A but did so in Task B. Although performance on one question does not enable robust conclusions to be drawn, indications of increased recognition of the use of algebra as a strategy in response to the word problems were confirmed by student comments made in class and interviews.

5.2.3 Informal strategies

The most common informal approach used by students solving word problems was to find the arithmetic solution logically (coded LOG). This strategy, also referred to as unwinding or generating numbers the literature (Koedinger & MacLaren, 2002; Nathan & Koedinger, 2000a, 2000b; Van Dooren et al., 2002), was commonly described by

students as "just working it out" or "using logic". The response of 12XF to Question 2 of Task A was typical of many arithmetic responses at the start of the intervention.

12XF: I just did it logically.

2. In another class, there are twice as many boys as girls. If there are 33 students in the class and x of them are girls, how many boys are in the class?

$$\frac{33}{3} = 11 \qquad 11 \times 2 = 22$$

$$boys = 22$$

12XF Task A Question 2 (coded AR, LOG)

Trial and error applied to the problem was another common informal strategy. All three approaches to trial and error (random, sequential and guess/check/improve) were in evidence but no one approach stood out as more common. Written responses often did not provide enough evidence to classify the type of trial and error a student had used. This is exemplified by 12XJ who wrote nothing in response to Task A Question 3. His comments clarified his strategy which was coded AR, PTE, RND.

3. Sally gets paid x dollars per week for pocket money. Her older brother gets twice as much. Sally's younger brother gets $5 less than Sally. If their combined weekly total is $42, how much does Sally get?

Task A Question 3

12XJ: I tried to work it out in my head but I couldn't. I just tried to work out how much Sally had like a process of elimination; try a number and work out what it would be and then try another...not sure how to start it...

In responding to the same question in Task A, 12YB tried to use ratio as well as trial and error. From her written response it was unclear what type of trial and error was involved but her interview clarified that she had used a random approach. Although she has included an x in her response, this has not been classified as AL. In this case, her use of the letter appears to have been merely as repetition of the information from the question rather than something that she has used in her working.

12YB: I just tried lots of numbers but I couldn't do it.

3. Sally gets paid x dollars per week for pocket money. Her older brother gets twice as much. Sally's younger brother gets $5 less than Sally. If their combined weekly total is $42, how much does Sally get?

12YB Task A Question 3 (coded AR, PTE, RND)

A different method of trial and error seems to have been used by 12XJ. He was the only student to provide a response devoid of algebra to Question 5 in Task B. His use of the words "narrow in on the answer" suggested that he had used either guess, check and improve or a sequential approach. However, guess, check and improve seems most likely to have produced the answer correct to two significant figures.

12XJ: I couldn't remember how to do it so I just went back to my logical way... I did a bit of trial and error for that one – it took a while to narrow in on the answer.

5. A rectangle is 8 cm longer than it is wide. If its area is 135 cm^2, what is the width of the rectangle?
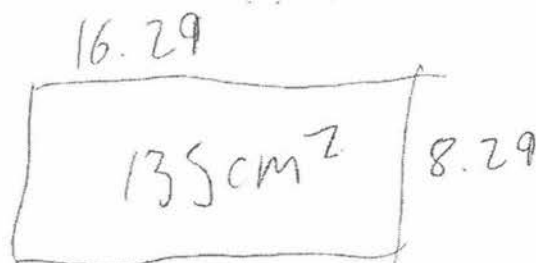
12XJ Task B Question 5 (coded AR, PTE, GCI).

5.2.4 Progressing along the algebraic route

Some students progressed no further along the algebraic route than using letters to denote unknowns or express relationships. Such responses were initially not recognised as algebraic by either teacher or most students. This is exemplified by 12XJ's response; his use of letters to denote unknowns in combination with his arithmetic solution strategy results in the coding of AL, UNK, LOG. This student's comments about his method and his use of pro-numerals suggest some awareness of expectations about algebra. However, his use of "G" and "b" may reflect an understanding of letters as labels, an interpretation which has been identified as a problem even for mature algebra students by Clement (1982).

> *12XJ: I didn't use algebra – I just worked it out.*

2. In another class, there are twice as many boys as girls. If there are 33 students in the class and *x* of them are girls, how many boys are in the class?

12XJ Task A Question 2 (coded AL, UNK, LOG)

The response of 12YB to this question involved the use of ratio as well as letters as labels. The use of ratios was not common, and none of the students used the term *ratio* to describe their responses. The use of ratios was coded as LOG.

> *12YB: I did that, 2 to 1. I think I divided that by 3 and I times that by 3 and then times the 2 by 11 cos that makes 22 and then that makes 33.*

2. In another class, there are twice as many boys as girls. If there are 33 students in the class and x of them are girls, how many boys are in the class?

$$2:1$$

$$B = \quad G$$

$$\frac{2}{3} \quad \frac{1}{3}$$

$$33 \div 3 = 11$$

$$11 \times 2 = 22 \text{ boys}.$$

$$11 \times 1 = 11 \text{ Girls}.$$

12YB Task A Question 2 (coded AL, UNK, LOG)

This student's use of letters (B = and G =) earned the coding of AL. However, his use of pro-numerals may have been as abbreviations for boys and girls, rather than standing in for unknowns (MacGregor & Stacey, 1997).

Some students progressed along the algebra pathway as far as writing relationships. This was seldom seen initially, but became more common during the intervention. 12YP used informal strategies and almost no algebra at the start of the project. By the end of the intervention he was using algebra to describe a relationship but commonly got stuck at the relationship stage and was unable to make useful progress beyond this, as illustrated in his response to Question 5 of Task B. He recognised that he had made progress, albeit limited progress.

> 12YP: *Question 2 I did without algebra in Task A but, in Task B, I wrote the equation. But I didn't get the solution ...and in Question 3, I did the same...wrote the equation...but that's the thing I didn't get to the answer.*

5. A rectangle is 8 cm longer than it is wide. If its area is 135 cm², what is the width of the rectangle?
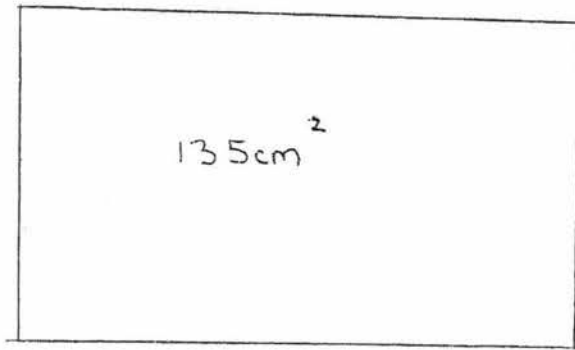
$135 cm^2$

~~Let x represent the area of the rectangle.~~

~~if the area~~

Let x represent the width of the rectangle.

Total area $= 135 cm^2$

$16.875 + 8 = 24.875.$

$x = 8 + y = 135$

$x = 135 + y = 8.$

12YP Task B Question 5 (coded AL, REL, EQN, NIL)

He used symbols to express length (*x*) in terms of width (*y*) but the equation he produced from these reflected a lack of understanding. It is not clear how he arrived at the number which he substituted into his expression for *x*. (Section 5.5.3 includes a discussion of this student's response to Task B Question 3).

5.2.5 Solving equations

The most common method by which students solved equations was algebraic manipulation. Some students responded to word problems by translating them into equations, and manipulating the equations to solve the problems regardless of the nature of the problem. Students who wrote equations did not necessarily use their equations to solve particular word problems. Some students were unable to solve their equations. Other students chose not to manipulate equations because equation solving was not the

most efficient strategy. This was highlighted by 12YK's explanation of her response to Question 5 in Task A. She wrote an equation but did not use this to solve the problem (coded AL, EQN, LOG). The use of the word *formula* to mean *equation* was not unusual for students in either class.

> 5. A rectangle is four cm longer than it is wide.
>
>   If its area is 21 cm^2, what is the width of the rectangle?

Task A Question 5

> *12YK:* *I didn't use any algebraic formulas really cos I knew that 3 times 7 is 21 and 7 is 4 longer than 3... it worked. I started off wanting to do it by formula and I had the rectangle and the width times the width plus 4 equals 21 and then I had the width squared plus the width and 4w is 21. I could [have] remembered and made into a quadratic. For the questions that are really obvious I don't usually bother with all the formula because I know that it is the right answer.*

5.2.6 Using algebra

At the start of the project, most students recognised when their methods did not involve algebra, but some were not clear what methods were preferred by the teacher. The lack of clarity about what sort of response was expected is highlighted by the conversation with 12YP. In his response to Question 3 of Task A, 12YP used an arithmetic unwinding strategy as well as labelling with a pro-numeral.

> *Interviewer:* *Did you do any algebra?*
>
> *12YP:* *Well probably not. Well, yes and no. I basically did a number orientated – division, multiplication*
>
> *Interviewer:* *Do you think that you did was what the teacher wanted you to?*
>
> *12YP:* *Well, I must say the teacher doesn't usually go for this sort of thing. It's very much numerically based*

3. Sally gets paid *x* dollars per week for pocket money. Her older brother gets twice as much. Sally's younger brother gets $5 less than Sally. If their combined weekly total is $42, how much does Sally get?

x $ per week.

42 $ total.
_____

42 - 5 = 37

37 - 2 = 18.5

4.25

18.50 + 9.25 =

27.75

40.00
- 27.75 = 12.25

43.00
- 13.50

Older brother gets: $18.50.

Sally gets: $12.25

Younger brother: $9.25

12YP Task A Question 3 (coded AL, UNK, LOG)

By the end of the project, students appeared to have a clear understanding of teacher expectations about the use of algebra. Their comments reflected increased awareness of the possibility of using algebra as a response to word problems.

> *12YP: I definitely got better at the transformation of the basic word problem into algebra equation and solve the algebra equation and so forth.*

Although some students knew, at the start of the project, that algebra was an expected response, they were also aware that some word problems could be readily be solved arithmetically. For many, arithmetic solutions were preferred in terms of efficiency, but for some this approach was seen as a way to "avoid" algebra. The conversation with 12XJ at the start of the project highlights this.

> *12XJ: I hate algebra because it's too hard. I don't get it cos the letters confuse me – I can do things with numbers but the letters confuse me. I think it's like I see some letters with numbers and it doesn't look right—I just don't get it. Just in general I just don't like it. I hate doing equations and stuff but the words are ok. I didn't do very well on Task A— I just went oh and couldn't do much of it…I just avoid algebra as much as possible.*

In the early stages of the intervention, this student only used informal strategies to solve word problems. By the end of the intervention, he was able to accurately translate word problems into algebraic equations but was seldom able to progress beyond the equation

writing as illustrated in his response to Question 3 of Task B. Although he was unable to solve the problem, his translation into an equation contrasts to his response to the parallel question in Task A where he wrote nothing.

> *12XJ: Well I knew she [the teacher] wanted me to do algebra and I tried but I think I got confused again...I didn't know how to write equations and now I do. It's really good but it's like "What do I do next?" With an equation, like, I don't even know the steps—what do you do after that and what do you do after that?*

3. Simon earns *x* dollars per week for his job at the pizza shop. One of his friends gets twice as much. Another friend gets $15 more than Simon. If their combined weekly total is $420, how much does Simon get?

12XJ Task B Question 3 (coded AL, EQN)

His interview as the end of the intervention highlighted the shift he had made in terms of his approach to algebra.

> *12XJ: I feel a lot better about algebra now... I can write equations but I still don't know what to do with them... I really needed teaching for solving cos then I would have been done!*

At the end of the intervention most students' comments reflected increased confidence with writing equations. Most students interviewed the end of the intervention, also commented positively on the usefulness of algebra.

> *12XF: Putting into word problems- things that have something to do with what we do in real life—doing relevant things like those questions—you can see why you use it.*

However, although more students recognised algebra as a good tool to use with appropriate problems, other students continued to avoid using algebra.

12XK: But if I had a real life situation I would not use algebra, I would just work it out.
Algebra still isn't all that relevant to real life.

5.3 Common Difficulties

Many of the difficulties associated with learning to use algebra that have been documented for students up to about age 15 (MacGregor & Stacey, 1997; Nunes, 1997; Thomas, 1995) were experienced by a number of the students in this study despite their age and prior experience with algebra. Their difficulties included confusion with notation, use of brackets, differences between multiplication and exponents, as well as difficulties with algebraic manipulation.

5.3.1 Algebraic notation

Difficulties with compression of representation as described by Nunes (1997) became evident during a lesson with class 12X early in the intervention. Some students did not recognise that "3 x n" could also be written "$3n$". This difficulty was addressed during the lesson by TX and did not arise again. However, other areas of confusion hinted at deeper conceptual issues as shown in the following conversations.

The class,12X, had been working on a translation activity (Set 2 in Appendix F3). TX was discussing the answers with the class. The following vignette has been selected to exemplify a range of issues related to symbolic notation.

The class were discussing translation of **Double the number**. Some students had written "n^2". After discussion the class settled on $2n$ as the answer.

>*12XS1: I did 2x instead of 2n. Is that ok?*

They then discussed the translation of **The product of a number and 6**.

>*12XS2: I wrote equals: "6n equals" because of the word product.*

The class moved on to discuss the translation of **Four times the sum of a number and seven.** One student's answer was written on the board: 4 ($n + 7$)

>*12XS3: Can the 4 be on the other side?*

This led to students debating whether order of multiplication mattered, and revealed some confusion about the need to include the multiplication sign.

> *12XS4: Second one's not right. You need the times sign in that one, but not if it's in front of the brackets.*

Students appeared to lack sound understanding of addition and subtraction concepts and these were complicated by difficulties with semantics. The following two vignettes highlight some of the difficulties.

> The students had been asked to write a word translation of the equation $7 - \dfrac{n}{2} = 11$. The teacher was moving around the class. This conversation was between her and two students who had been working together.
>
> > *12YD: 7 less than a number divided by 2*
> >
> > *12YN: Less than means takeaway*
> >
> > *12YD: Are we taking away from the number?*
> >
> > *12YN: 7 take-away the sum of the number divided by 2 ?*
> >
> > *TY: I'd leave out the words "the sum"... what does "the sum" mean?*
> >
> > *12YN: It means, like, do the sum, get the answer... 7 take-away the number divided by 2 ...*
> >
> > *12YD: That's just what I said!*
>
> 12YD continued for the next 4 minutes telling his neighbour he had said the answer right in the first place.

Students' attempts to write algebraic expressions from English expressions highlighted a number of difficulties; some of these were complicated with semantics as shown in this vignette from a lesson involving a translation activity.

> > *12XS1: Does product mean multiply? Then I got it wrong. I had x + x + 2 and it should be $x^2 + 2x$...*
> >
> > *12XS2: I was confused—when it said product I didn't know what it meant*
>
> The class then discussed **Double the number less the square of 12.** Two answers were volunteered and written on the board: $2n - n^2$ and $n^2 - 2n$.

> TX: *Which is less than?*
>
> 12XF: *Ah, I get it. 2n is less than n^2*
>
> 12XJ indicated he was confused. The conversation continued.
>
> TX: *If I've got 4 less than 30, how much have I got?*
>
> 12XJ: *4 minus 30 ... 30 minus 4 gives 26. So you've got 26.*
>
> TX: *40 less 60, and 40 less than 60. Do they say the same thing?*
>
> There was general debate about this. The teacher used more examples with numbers and the class then agreed that the two expressions were not the same.

5.3.2 Language

The intervention focus on translating between verbal and symbolic representations highlighted difficulties students experienced with developing clear understandings of some words—words which teachers described as "basic" (section 6.2.1). Some students confused area and perimeter as exemplified by 12YJ's response to Question 5 of Task A.

5. A rectangle is 4 cm longer than it is wide. If its area is 21 cm², what is the width of the rectangle?

$$4x = 21$$
$$x = 5 \cdot 25 \text{ cm}$$

12YJ Task A Question 5 (AL, EQN, NW)

Her response to Question 5 in Task B suggested some shifting of her understandings. Failing to solve the equation associated with area, she used an equation to solve the problem as a perimeter problem, but recognised this was incorrect and returned to the area equation.

5. A rectangle is 8 cm longer than it is wide. If its area is 135 cm², what is the width of the rectangle?

x

$x+8$ | A: 135cm² | $x+8$

x
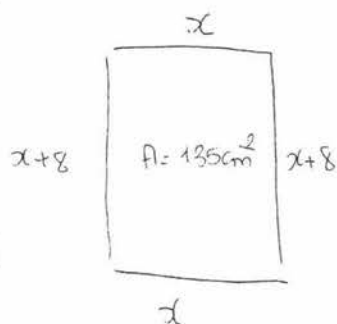
$A/$

$x(x+8) = 135$

$x^2 + 8x = 135$

$x^2 + 8x - 135 = 0$

$(x \quad)(x \quad) = 0$

$x + x + 8 + x + 8 + x = 135$

$4x + 8 = 135$

$4x = 127$

$x = 31.75$ ⟩ ✗

$x \times (x+8) = 135$

$x^2 + 8x = 135$

$x^2 + 8x - 135 = 0$

$(x \quad)(x \quad) = 0$

$x = \quad or \quad x =$

12YJ Task B Question 5 (coded AL, EQN, MAN)

5.3.3 Brackets

Developing awareness of potential ambiguity is identified as an important aspect of algebraic linguistic competence by MacGregor and Price (1999). They discuss the need for students to develop

> the recognition that an expression may have more than one interpretation, depending on how structural relationships or referential terms are interpreted (e.g., knowing when brackets are required for ordering operations and being aware of the potential for mistranslating relational statements to equations). (p. 457)

Writing verbal translations for expressions containing brackets caused problems for many of the Year 12 students. Some students simply ignored the order of operations and so didn't see any need for brackets, a few used the word *brackets* for a very literal translation, some expanded so as to avoid the brackets, and a few used words that indicated the result of the bracketed operation. Students developing confidence with brackets is illustrated in the following conversations between the teacher and students in 12X.

> The class had been translating *3 (n + 4)*. The teacher had agreed with a student answer of *Three times the result of a number added to 4*.

12XL: *I wrote "3 times a number added to 3 times 4".*

TX: *You've expanded the bracket.*

12XL: *Would that be right?*

TX: *Both are correct*

12XL: *Which is wrong?*

TX: *Neither. How could you explain it with the bracket? You might need to use the sum of.*

12XL: *We should just put brackets in as words.*

This discussion continued to focus on brackets.

The teacher then asked students for an algebraic expression for ***The difference between the number and eight and the result multiplied by four more than the number.***

Students suggested a range of expressions which the teacher wrote on the board:

$$n - 8 \quad (4 + n)$$

$$4 + n \left(\frac{n}{8} \right)$$

$$x \ + \ 4 \ (\ x \ - \ 8 \ )$$

Then the first student said: *Change mine to brackets...*

The teacher changed the first expression on the board to $(n - 8)(4 + n)$ and the first student nodded. The conversation continued with two other students.

12XN: *Will it work out the same if you have $(n - 8)(n + 4)$?*

TX: *Are they the same?*

12XF: *No – don't you have to do foil?*

TX: *Can you expand it to see?*

The teacher wrote on the board as 12XF talks through the expansion and simplification of $(n - 8)(n + 4)$

TX: *All right. Now expand the other one*

12XF: *What?*

TX: *Well, don't you want to expand the other one to see if they're the same?*

12XF: *I'm not sure. I don't know how to expand $(n - 8)(4 + n)$ ...*

> This was followed by a 90 second pause with the student staring at the board.
>
> > *12XF:* *The FOIL process—is that the same as leaving them in brackets and working it out? So if n is, like, 12 and you go (12-8)x(4+12), won't you do it a different way?*
> >
> > *TX:* *No.*
> >
> > *12XF:* *So you get exactly the same result with FOIL?*

Up until now 12XF had apparently seen *FOIL* as a fixed procedure that he used inflexibly. Reflecting on the lesson afterwards, TX suggested that this was a significant piece of learning for this student, and one that he may not have gained without the focus on language.

As the intervention progressed students showed an increasing awareness of the importance of brackets.

> *12XF:* *... say, for example, 7 and 5 times a number is 6 could be 5 m + 7 = 6 could be (7 + 5) m. It makes a big difference.*

Progress in understandings about brackets was illustrated by the difference between 12XA's responses in Task A and B. In her response to Task A, the omission of brackets was significant.

5. A rectangle is 4 cm longer than it is wide. If its area is 21 cm², what is the width of the rectangle?
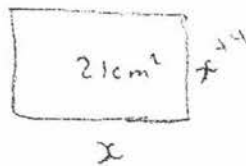
$$x \times x+4 = 21 cm^2$$
$$x^2 = 19 cm^2$$
$$x = 4.4 cm$$

12XA Task A Question 5 (coded AL, EQN, MAN)

In responding to Question 5 in Task B she does not make the same mistake. She gave two responses and used brackets in both. In the crossed out working, a response in terms of perimeter not area, she used brackets although they make no difference in the equation.

5. A rectangle is 8 cm longer than it is wide. If its area is 135 cm^2, what is the width of the rectangle?
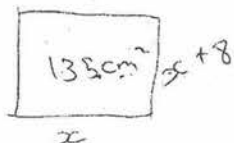
$$x \times (x+8) = 135 cm^2$$

$$x^2 + 8x = 135 cm^2$$

$$x^2 + 16.875x$$

$$x = 16.9$$

$$x + x + (x+8) + (x+8) = 135$$

$$2x + x + x = 119$$

$$4x = 29.75$$

12XA Task B Question 5 (coded AL, EQN, NIL)

5.3.4 Multiplication and exponents

Some of the Year 12 students displayed confusion between multiplication and exponents similar to that found in MacGregor and Stacey's (1997) study of 11-15 year olds. The confusion is highlighted in 12XA's response to Question 3 in Task A.

3. Sally gets paid x dollars per week for pocket money. Her older brother gets twice as much. Sally's younger brother gets $5 less than Sally. If their combined weekly total is $42, how much does Sally get?

$$x^2 + 5 = 42$$
$$x^2 \cdot 37$$
$$\$\ x : 6.08$$

Sally gets
$\$6.08$

12XA Task A Question 3 (coded AL, EQN, MAN)

12XA: Her older brother gets twice and her younger brother gets 5 less. And so x to the 2 equals 37 so that is 6.0. I square rooted it, so Sally gets $6.08.

Her use of square root suggests confusion with concepts rather than with the notation. In contrast, in responding to the parallel question of Task B, her translation showed no confusion between multiplication and exponents. Although her subsequent algebraic manipulation is confused, and she arrives at her answer by another method, there is evidence of a shift in her ability to translate into algebra between Tasks A and B.

3. Simon earns x dollars per week for his job at the pizza shop. One of his friends gets twice as much. Another friend gets $15 more than Simon. If their combined weekly total is $420, how much does Simon get?

$$f = 2x$$
$$x + (15+x) + 2x = 420$$
$$y = 15 + x$$
$$x + (15 + x = \frac{420}{2x}$$
$$\frac{x + 15 + x}{x} = \frac{210x}{x}$$
$$x = 210 - 15$$
$$x = 95 \text{ dollars}$$

Simon gets $\$101.30$

12XA Task B Question 3 (coded AL, EQN, MAN, NW)

However, some students continued to be confused by multiplication and exponents at the end of the intervention. 12YP's confusion was not apparent initially as this student avoided using algebra at the start of the intervention. His responses at the end of the intervention reflect a significant shift towards using symbols but a lack of fluency with them. His response to Question 3 on Task B shows a variety of strategies including the use of arithmetic and some confused use of symbols. One of his equations included an x^2 term which may have triggered his attempt to factorise.

12YP Task B Question 3 (coded AL, EQN, MAN, LOG)

12YN's confusion with exponents was also not apparent early in the intervention as she avoided using algebraic notation. However, her difficulties appeared to be related to weak understanding of notation rather than conceptual confusion as shown by her response to Question 2 of Task B. Her translation used exponents instead of coefficients but her manipulation of the equation suggests that she understood the question, and had translated with understanding.

12YN Task B Question 2 (coded AL, EQN, MAN)

5.3.5 Manipulation

Teachers were concerned that the project's focus on translation meant that algebraic manipulation was receiving less emphasis in class than usual. They expected students' algebraic manipulation skills to be a barrier and were surprised to observe the manipulation skills of some students improving as the intervention progressed.

12YK displayed limited algebraic manipulation skills at the start of the intervention as evidenced in her responses to Task A. In Questions 3 and 5, she made some progress along the algebraic path but was unable to manipulate her equations algebraically to arrive at the solution Her solution of the equation in Question 3 is coded RVS as it illustrates an unwinding approach, although no flowchart is shown. In contrast, in her response to this question in Task B, she solved the equation by manipulation.

3. Sally gets paid x dollars per week for pocket money. Her older brother gets twice as much. Sally's younger brother gets $5 less than Sally. If their combined weekly total is $42, how much does Sally get?

$$2x + x - 5 = 42$$

$$\text{(crossed out)} \quad 3x - 5 = 42$$

$$42 + 5 = 47$$

$$\frac{47}{3} = 15.67 \ (2 \ d.p)$$

Sally gets $15.67 per week.

12YK Task A Question 3 (AL, EQN, RVS)

3. Simon earns x dollars per week for his job at the pizza shop. One of his friends gets twice as much. Another friend gets $15 more than Simon. If their combined weekly total is $420, how much does Simon get?

$$x + 2x + x + 15 = 420$$

$$= 4x + 15 = 420$$

$$4x = 405$$

$$x = \$101.25$$

12YK Task B Question 3 (AL, EQN, MAN)

5. A rectangle is 4 cm longer than it is wide. If its area is 21 cm², what is the width of the rectangle?
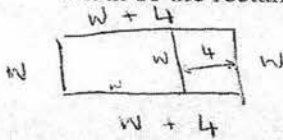
$$w \times (w + 4) = 21$$

$$w^2 + 4w = 21$$

$$21 - 4 = 17 \quad w(w + 4) \frac{1}{2} w$$

$$\sqrt{17} = 4.12 \ (2 \, d.p)$$

$$\boxed{w = 3}$$

$$21 - w^2 = 4w$$

$$21 - 9 = 12$$

12YK Task A Question 5 (AL, EQN, NW)

5. A rectangle is 8 cm longer than it is wide. If its area is 135 cm², what is the width of the rectangle?

$$x \qquad \boxed{135 \, cm^3} \quad x + 8$$
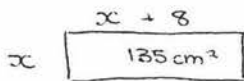
$$x(x + 8) = 135$$

$$x^2 + 8x = 135$$

$$x^2 + 8x - 135 = 0$$

$$\text{from} \quad x = \frac{-8 \pm \sqrt{8^2 - (4 \times -135 \times 1)}}{2}$$

$$x = \frac{-8 \pm \sqrt{604}}{2}$$

$$x = \frac{-8 + 24.58}{2} \qquad x = \frac{-8 - 24.58}{2}$$

$$x = 8.29 \ (2d.p) \ or \ -16.29 \ (2d.p)$$

$$8.29 \, cm$$

12YK Task B Question 5 (AL, EQN, MAN)

The differences between 12YK's responses to Question 5 in Task A and Task B also suggest an increase in her skills of algebraic manipulation.

However, algebraic manipulation remained an obstacle for some students; some students got "stuck" and were unable to follow the algebraic pathway all the way to a solution. Getting stuck was not common at the start of the intervention, but as the intervention progressed, equation writing became more common, and more students got stuck at the equation stage.

> *12YI: I didn't do as good as I would like to have done but I got better—got more understanding so I felt good about that. I could do the algebra equation thing but I couldn't finish it off cos my algebra isn't—I need more work on like what to do, to work it out—more practice putting it into algebra from words and then how to work it out from there.*

A few students expressed the need for need for more time and/or practice on particular aspects of algebraic processes. Practice and consolidation are recognised as necessary for mathematics learning (Anthony & Knight, 1999). Some students wanted practice with equation solving, while others expressed a desire for more practice with translation into words or equations.

> *12YP: It's all very well to write the equation but to find the solution...I just didn't understand it so more time to do those bits would have helped.*

> *12YG: More practice would have helped...writing the words for the equation...word problem is a bit hard but the algebra part is pretty easy for me.*

> *12YI: More of writing the equation would have been good for me.*

5.4 Discussion

At the outset of the intervention, some students did not understand what was expected of them when they were asked to solve algebra word problems. In class observations, interviews and student responses to Task A showed that initially students tended to use informal strategies to solve word problems. This tendency was reinforced when many of the problems could be solved readily without algebra. By the end of the intervention, in class observations, interviews and student responses to Task B indicated that students knew that algebra was an expected response and they knew what this meant. Students showed a greater confidence at translating word problems into symbols with the full algebraic pathway being followed more commonly. They showed an understanding about the process of translating word problems into algebra and were better able to do this.

Common difficulties in solving algebra word problems reflected students' poor understandings of language and algebraic notation. Confusion between multiplication and exponents was associated with both conceptual confusion and confusion about algebraic notation. Particular difficulty was caused by brackets and shallow understanding of algebraic procedures. There were mixed gains in appropriate use of brackets and algebraic syntax, but by the end of the intervention most students were able to write correct algebraic equations for word problems, and many students were able to manipulate their equations to solve the problems.

By the end of the project, students were able to describe the progress they had made and commonly described improvements in their ability to translate word problems into symbols. Although at the end of the intervention all students interviewed showed an understanding of the steps in the algebra problem solving cycle, equation solving remained a concern for some students. However, there was evidence that students felt more positive about algebra; they commented favourably on knowing what was expected. Students were more likely to be able to get started on a word problem, but some students felt that they needed more time to become competent with using algebra to solve equations.

CHAPTER 6

RESULTS AND DISCUSSION: TEACHERS' PERSPECTIVES

There is no question that engaging in inquiry about teaching and learning mathematics is an important (and often an extremely powerful) form of professional development that enhances teachers' knowledge, skill and understandings. (Cochran-Smith, 2006, p. xvii)

6.1 Introduction

This chapter presents and discusses key aspects of teachers' responses to involvement in the project, and explores evidence for change in teachers' understandings, attitudes and practices. Data are taken from teacher interviews, teaching documents and lesson observations. Semi-structured teacher interviews provided snapshots of teacher understandings and attitudes. These snapshots were triangulated by lesson observations. Teaching documents including lesson plans, mathematical tasks and student worksheets, provided additional evidence of teacher practice.

Language is a difficult medium for communication, producing meanings that are subjective. Listening to the recordings of each interview enabled the researcher to relive each conversation so that events and meaning were re-constructed with the benefit of hindsight. Although transcripts and interpretations were checked with teachers for accuracy, it must be acknowledged that the account presented below is viewed through the researcher's lens and reflects the researcher's interpretations of conversations and events.

6.2 Teacher Change

What teachers do is inherently complex and, in this project, teacher practice was influenced by the intervention planned. Indications of change have been taken account of only if they were corroborated by repeat instances over time and with alternative sources of evidence.

At the start of the research, both teachers said that they thought the topic of word problems was important, with TX describing students' ability to deal with word problems as "critical for their success in mathematics". They both expressed constraints about pressure of time and content coverage. They felt real pressure to complete the algebra topic and teach specific skills and were apprehensive about the time required for the intervention. However, these apprehensions vanished as the research progressed. At her second interview, TY recommended spending more time on everything in the intervention, particularly the introductory activities.

> TY: *Doing this next time, I would allocate more time- and much more time into the simpler building blocks. I might even allocate an extra week to the word problems. I would start with the translation- simple phrases -practising in both directions.*

Initially, both teachers had difficulty explaining their personal strategies or approaches to teaching algebra word problems. TX was concerned to realize that she was unable to elaborate how to teach students to tackle word problems.

> TX: *I've never been taught a way to do it. It's never been an expectation in my learning that we should explain the English, to break it down. "You can read— you should be able to do it".*

She wasn't sure that she knew *how* she solved word problems herself but acknowledged that she expected students to already know what to do.

> TX: *I do very little. I just expect them to read it. I guess I just expect them to know what I want them to do and then I am surprised when they don't do it. I haven't thought about it before but it's really no wonder they don't.*

Both teachers reported that during the intervention they introduced significant changes into their classrooms in terms of activities and approaches. What is less clear is the extent to which these changes impacted on their practice in the longer term. However, in discussing the impact of the interventions, it was clear that students' responses were a strong motivating factor influencing them to change their practice. Both teachers noticed an increase in student motivation as well as learning and said that this made them keen to continue with the new approaches.

> TX: *It is glaringly obvious that it has worked.*

> TY: Yes, it has. The whole idea of starting with the word problems and working on how to translate it and then develop the skills from that -I think that whole way of the understanding the use of algebra made them connect much better with the topic.

> TX: They understood the point of algebra...I had students answering in class with confidence at times who normally don't and seemingly enjoying what they were doing.

> TY: Engagement of students was fairly high—even the strugglers—they were able to make progress and feel good about what they had achieved. That was the buzz for me: the less able students seeing where- where and how they could use algebra...volunteers from a range of students who don't normally give answers and getting it right too.

6.2.1 Explicit focus on translation

A key focus of the intervention was on translation: students developing understanding of algebraic symbols and connecting them with English. The emphasis was on translating in both directions: from English to symbols and from symbols to English, as well as linking simple English with contextual English.

Prior to the research, neither teacher had focused on translation as a significant part of teaching algebra word problems. The teachers explained that, in the past, their focus had tended to be "on the maths not the words". They acknowledged that the literacy focus within the school had increased their awareness of vocabulary but had not changed the way they taught word problems.

> TY: Vocabulary—we have to be focusing a lot more on vocabulary.

> TX: Yes—product, difference, sum—those types of things—but not really breaking down the sentence structure.

In discussing their former practice, both teachers described key words as something they thought it was important to talk about with students. Neither teacher was aware of the potential pitfalls of reliance on key words rather than meaning (Anthony & Walshaw, 2007). Both reported difficulty with getting students to identify key words in anything other than simple word problems.

TX's normal practice included an activity for students to practise generating algebraic equations from contextual word problems but she acknowledged that she had never taught students how to do this.

> TX: *I just expect them to know how to do write them, and they really struggle.*

TY described her customary practice of modelling her own way of solving word problems, a practice which, on reflection, she saw as ineffective. Although she regularly modelled her way, student work seldom reflected use of the teacher's model. In discussing this, the teacher realised that she had never been explicit with students about what she was doing, or why.

> TY: *I model on the board, set out information given and work through solving it. Probably not very explicit about this—expect them to see what I have done as a model and use it but of course they don't and when I think about it, I have never told them.*

As the intervention progressed, both teachers increasingly structured the translation activities to allow students to discuss and share interpretations. TX came to recognise that her focus on vocabulary in previous years had not given students the opportunity to use the language.

> TX: *I wouldn't normally have spent as long on the simple translation at Year 12. Like n plus four converting it to words and vice versa—I wouldn't expect to have to do [this] at year 12 level but they found it useful. Cos even when we were doing problems they said things like "product means times" going back to those, what I would consider, quite basic words.*

At the end of the intervention, both teachers acknowledged the importance of the translation activities. They said they had not appreciated that connecting symbols and words was so difficult for students.

> TX: *I have done the vocab before but not to use it in expressions and get them writing expressions. This has made a real difference.*

When discussing how they would approach teaching algebra in the future, both teachers emphasized translation as a key focus.

> *TX:* *Building on what we had been looking at because they had been looking at variables and equations and translating it to English and the reverse—Going from equations to words and words to equations, looking at the translation process.*

> *TY:* *I would start with the translation—simple phrases—just practising, both directions.*

Evidence that teacher change had been embedded into practice to some extent was provided in lesson observations and interviews later in the year. After completing the intervention with their Year 12 classes, both teachers adapted the translation activities from the intervention and created more resources focusing on the translation process to meet the needs of students in other classes. TX recognised a difference in her approach to teaching algebra right from the start of the algebra topic with her Year 10 class.

> *TX:* *The translation focus is very different to what I normally would do, and very useful. I guess that's why I started with it—it linked with the measurement they had just done. There was a practical context they could see they used rules and shortcuts in the practical situation.*

In two of the three lessons of TX's Year 10 class randomly chosen for observation by the researcher later in the year, there was a strong emphasis on translation.

TX:	*With 5n ... what operation are we using in this expression?*
10XP:	*Times?*
TX:	*You sure?*
10XP:	*No*
TX:	*Who would agree with times? Hands high if you agree with her...5n...so how would we say this?*
10XD:	*5 times n*
10XG:	*A number multiplied by 5*
10XN:	*The product of 5 and n*
10XB:	*Result of 5 times n* [10X Lesson One]

> TX: What about 3(n - 7)... how would you read this? What operations have we got
>   there in this expression?
>
> 10XG :   A number minus 7 times three
>
> TX: I wouldn't quite agree with you
>
> 10XC: Is it 3 times n minus 3 times 7?
>
> TX: I would agree with that... what else? How else might we read that?
>
> 10XB: The result of a number minus 7 times 3.
>
> TX: Was that the same as Stdt G's? Do they mean the same thing? What do you
>   think? ...Are they the same or different? Let's put a number in there... if I put 10
>   in there... are they the same?   [10X Lesson Two]

TX also reported including translation in the assessment task that was common for all Year 10 students in the school.

6.2.2 Starting with problems

The focus on translation meant that skills of algebraic manipulation and equation solving were downplayed during the intervention. An unexpected consequence of this was that teacher practice shifted to teach skills as the need for them arose. The teachers identified this as a major change in their practice.

> TY: I would normally do the skills first like in the book.
>
> TX: If I taught it in my usual way, I would have focused on let's do this kind of
>   equations, then this kind, then this equation kind and [then] let's do some
>   applications.

In their first interview, both teachers acknowledged the importance of contextual problems within mathematics, but admitted that they did not focus on these in their usual practice. TX suggested that there was too much content to get through and too little time to explore applications with any of their classes.

> TX: What restricts me is how much time I've got to teach what I am teaching, but I
>   think we should be doing it. I think they are missing out really.

Midway through the intervention, TY commented on her enjoyment of the focus on word problems. Her comments reflected a real sense of ownership of this approach.

> TY: *I really like how we are starting with the applications and building up the skills as we go along—rather than teaching the skills and then going to do some applications. I will add in simultaneous equations to linear tomorrow and hopefully it will be a natural progression from one equation.*

After the first two lessons the *problems-first* approach became a feature of every one of the intervention lessons. The following vignette is provided as an illustration.

Students in 12X were using the four step method to solve problems some of which involved simultaneous equations. Students are working in groups. This is the conversation of one group:

> *12XB: Is it simultaneous?*

> *12XC: I've never done one though*

> *12XM: I used to be able to...*

12XC watches 12XB solve the equations using elimination with. 12XM solves them using substitution.

> *12XB: So how did you get that? Oh you just rearranged it, eh?*

> *12XM: Use x and y*

> *12XB: Are they just common numbers you use? Like it could be a and b?*

They move on to the next problem, again arriving at a pair of simultaneous equations.

> *12XB: It's simultaneous again.*

> *12XC: I think I might know what to do.*

> *12XM: Yeah, it's good when you remember how.*

12XB and 12XM solve using substitution and 12XC uses elimination. All solve the problem successfully, mostly on their own but occasionally talking to each other to check or share what they are doing.

TX expressed surprise that most groups of students were able to solve these problems without teacher input. She explained how simultaneous problems were an area of

difficulty for students and that normally she would have started by teaching the whole class how to solve simultaneous equations.

> *TX:*   *...then they would have spent the period practising them. Some of them would have got onto word problems by the end of the period, but most would be stuck on the algebra and not get that far.*

She spoke positively about the way algebraic skills developed out of the problem during the intervention. TY had a similar experience with 12Y.

> *TY:*   *Simultaneous equations followed rather seamlessly. "Now we've done this type of equation, we need to do these". They just followed on from that and I quite like that order of teaching. It felt like it was working.*

Interviews at the end of the intervention highlighted that the problems-first approach had developed into a key feature of the intervention. When students were focused on solving a problem, algebraic skills became subsidiary and could be incorporated into the lesson.

> *TY:*   *When I got to simultaneous and with quadratic formula, starting on problems and then working on skills relating to it- this was a very different approach. Occasionally we came across some things that we hadn't covered so we had to teach that skill like the quadratic formula and they were just slotted in, like "OK, we have to learn that".*

Following the intervention, both teachers were keen to try this approach in other topics with their Year 12 class.

> *TX:*   *I am going to do the next topic in the same way—focusing on the applications... and then getting the skills as we need them. I think it's an effective way of getting through the material. I think they'll use it with coordinate geometry—I think they will be able to look at the problem and use what they have learnt in algebra. Trig is another topic where they will be able to use it- another word problem one.*

> *TY:*   *I quite liked the order of teaching. It felt right with the focus on applications and then seamlessly bringing in the skills. In the next topic on coordinate geometry I am going to do the same thing—starting with the word problem and then ask how would you do this: "Here's a problem what information do you need to be able to solve it?" Like finding midpoints and so on—"What do you need to be able to do?" And distances that they don't know about—they can try and work out the formula... so starting from the word problem.*

Both teachers described the problems-first approach as a key feature of their learning throughout the research process. Both teachers intended to carry this shift in practice beyond the research and integrate it into their normal classroom practice. There were indications that this was happening to some extent. Changes were evidenced in interviews and observations later in the year. TX had approached the algebra topic with her Year 10 class in a different way to usual.

> *TX:* *I particularly liked giving them a problem, starting with a problem and working on the skill so they see a reason to use the skill…I intended to start with doing it in the usual way but I realized I had automatically included work on translation and I had given them a problems-first approach—getting them to develop their ideas rather than me telling them "This is how you do it"… so it was quite different to my normal approach.*

TY explained that she had also taught her Year 11 class by starting from word problems in contrast to her usual practice.

> *TY:* *…start with some more contextual problems rather than the skills so that they can see the point of factorising, expanding and so on. So instead of doing just normal old quadratics we started from word problems. It's given me some more tools and thoughts about how I would tackle that now. To be able to give them a problem… and say "How will you be able to solve these?"*

6.3 Issues

Surprisingly, teachers' expert status made it difficult to select and create good problems. The textbooks provided little support for this, and left them feeling unprepared and under-resourced.

6.3.1 Expert Blind Spot

The research highlighted for teachers the extent of their own expert blind spot. Both teachers described how attending to students' responses during the research gave them greater understanding about, and insight into, students' thinking. At the outset of the research TX was surprised by the way that students avoided algebra. She knew that students did not view algebra as useful but she did not really understand why. She recognised that her own enjoyment as a student of algebra for its own sake was not typical of students, and realised that students needed to see a reason for learning algebra.

TX: *I loved manipulating letters. I didn't need a context really but I think we need to bring in that sense of why we are doing this—some way we need students to see the sense of using algebra. They need to see why they are even bothering to learn algebra.*

In contrast, TY was not surprised to find that students commonly did problems without using algebra. Her recognition that students commonly avoid algebra fitted with her own memories as a student of algebra. Despite this, she was surprised by the difficulties students had with translation.

TY: *Before, I think I might have assumed too much about what the students knew or could do... I might have just assumed too much about how easily they could do the translating into algebra.*

During the intervention, both teachers became more aware of the way students managed to complete some tasks in the intervention without using algebra. In particular, teachers were surprised to learn that one of the intervention activities they had designed could be completed by focusing on numbers and ignoring the algebra and the words.

TX: *I was surprised about the way students discern patterns – they look for clues, for patterns we don't even see are there, sometimes the wrong clue, but they avoid doing the algebra.*

The element of surprise was also created by student difficulty with some of the initial tasks which teachers thought "simple".

TX: *I didn't expect them to have such difficulty. I thought this would be just straightforward for them. If I had known I would have tackled it differently.*

The vignette below is taken from the second lesson with 12Y. The teacher was surprised not only by the student's lack of understanding, but by the fact that she had not known he did not understand until she read the transcript of the interview.

The activity involved students working in pairs on a card matching activity. There were a range of algebraic expressions to match to each word expression. Not all the algebraic expressions were correct.

> *TY:* *I agree with what you've got but there are more...Take some paper and prove that they are the same.*

> *12YH: Don't know what you want me to do.*

> *TY:* *How are we going? Question 3 looks good. Question 4—are those equal? You need to check they say the same thing. Check by rearranging.*

> *12YH: Huh?*

> *TY:* *If you have two answers you need to check they are equivalent.*

12YH looks puzzled.

> *TY:* *We need to rearrange to see if they are the same.*

TY moves on to another pair of students.

At a subsequent interview, the student explained his response.

> *12YH: It was like she was talking a foreign language— like, what? I just switched off.*

The teacher found the transcript of the interview illuminating.

> *TY:* *You just assume they are on the same wave length— and sometimes you are just so wrong! I guess I though—I assumed—they would understand they need to rearrange to show expressions are equivalent. But I guess it wasn't that obvious—I mean we hadn't done anything like that for a while.*

6.3.2 Writing and selecting problems

When teaching equation solving, the usual practice for both teachers was to start with problems that were easy for students to do in their head in order to provide simple demonstrations of the "rules of algebra". However, both acknowledged that this made it difficult to persuade students why they should use algebra. Teachers' awareness of this problem was raised by readings from the literature including Hubbard (2004), Koedinger and Nathan (2004), and Nathan and Koedinger (2000a). Teachers realised that their

former instruction had involved presenting students with problems that could be done informally thereby encouraging informal strategies.

Whilst both teachers agreed on the importance of presenting students with problems to which algebra was the preferred option, they experienced difficulty with the creation and/or selection of suitable problems. Initially the teachers had real difficulty developing appropriate word problems. They tended to use the text book as a resource.

> *TX: What I find in textbooks and resources, there is not enough word problems as resources. We don't have access to the resources so I think that's why we don't do it.*

Many of the problems that the teachers automatically used algebra to solve were problems that most students solved informally. An example of such a question is the cake question which the teachers created early on in the intervention (see Figure 6.1)

> Jane baked a number of cakes. When she baked a second batch, she increased the recipe and got twice as many plus three more.
> The total number of cakes from both batches was 51.
> How many cakes were in the first batch?

Figure 6.1 The cake question

When tackling this problem, both teachers wrote and solved the equation, $x + 2x + 3 = 51$. However, the majority of students solved this problem informally. A typical student response involved no written working as described by one student in 12X.

> *12XS1: You just subtract 3 from 51 and then divide by 3.*

This informal strategy is variously described as *finding the solution logically* (Stacey & MacGregor, 2000) and *unwinding* (Koedinger et al., 1999; Nathan & Koedinger, 2000a).

Difficulty with selecting and/or creating *good* problems was highlighted by the problems used in Tasks A and Task B as discussed in Section 3.4.1. Students' responses to Task A highlighted their tendency to use informal responses. As the intervention progressed, teachers became increasingly aware of the importance of presenting students with tasks and questions that prompted them to use algebra. Task B was created part-way through

the intervention. Differences between the questions in Task A and Task B highlight shifts in teachers' understandings about problems that encourage use of informal strategies.

Teachers had difficulty creating problems involving quadratic equations. The rectangle problem adapted from an NCEA assessment to use in question 5 of Task A is an example of a typical textbook problem (see Figure 6.2).

A rectangle is 4 cm longer than it is wide.
If its area is 21 cm^2, what is the width of the rectangle?

Figure 6.2 Task A Question 5

Both teachers "automatically" used algebra to solve this problem. Neither teacher realised how easily this problem is solved by inspection. In contrast, most students who solved this problem did so using informal strategies. 12YA's response highlighted the fact that this question did not tend to elicit formal strategies.

> *12YA: This one is not hard. You know that 21 is 7 times 3 so it's got to be 3.*

Both teachers expressed surprise at how obvious the solution was to students.

> *TX: Once you see it, it's obvious. Why would a student use algebra? But algebra is what I would always do first. At least now I know I will have to be so careful with the problems I use.*

When teachers realised how readily informal methods solved this question in Task A, they re-wrote the question for Task B to include numbers that made the answer difficult to arrive at without using algebraic strategies. Teachers had gained a new awareness of the *cognitive gap* (Herscovics & Linchevski, 1994) or *didactic cut* (Filloy & Rojano, 1989).

Teachers' difficulties finding and creating problems that required algebraic solutions and were within their students' capability is in accord with other findings (Bednarz & Dufour-Janvier, 1994; Panizza, Sadovsky, & Sessa, 1996). In this project teachers used two types of problems to help shift students from operating with numbers to operating with or on unknowns. They used problems that involved equations with the unknown on both sides as suggested by Filloy and Rojano (1989) as well as problems that had non-integer solutions.

6.4 Discussion

The difficulty of changing teacher practice in substantive ways is highlighted in many studies (Askew, 2001; Borko, 2004; Cady et al., 2006; Franke, Carpenter, Levi, & Fennema, 2001; Haggarty, 2002; Timperley et al., 2006). The teachers' active involvement in the research intervention challenged their beliefs about algebra teaching and learning. Changes included both changes in teacher knowledge and beliefs and changes in pedagogical practices. Both of the teachers recognised that during the course of the research they had learnt more about how students interpret algebra word problems and how that affects their learning.

A particular strength of the intervention was that it provided alternative models of teaching mathematics which often drew positive responses from students. The increases in student motivation, interest, and understanding encouraged teachers to change their practice (Timperley et al., 2006). At the start of the project, teachers' concerns focused on their own lack of knowledge about teaching algebra word problems and the pressure to cover content, constraining the amount of time they could afford for the topic. By the end of the project, teachers were focused on those aspects students had difficulties with, what students had learnt, and the need to put more time into the topic in the future. This shift in focus signifies a change in what they both recognise as important within their teaching of mathematics, indicating a change in their underlying beliefs.

Understandings, beliefs and practices are not fixed. Once the research started, discussions, readings, and experiences informed a number of understandings and practices that teachers identified as new to them. The adoption of a problems-first approach was the change that teachers talked about the most and they increasingly adapted this into much of their classroom practice. The surprise both teachers expressed at the success of this approach indicates that a major barrier to implementing such an approach was a lack of knowledge about how to successfully implement it. Although Guskey (2000) suggests changes in practice are a precursor to changes in beliefs, these findings highlight the complex web of factors that govern teachers' learning and practices.

Both teachers were convinced that translation "worked" and they expressed their determination to include this within any future teaching of the topic of algebra word problems. They included it within assessment materials and disseminated the practice with other colleagues in the department. In addition, the teachers were prepared to remove other things from the teaching programme in order to spend the time that was needed for students to make connections between words and symbols. They recognised their own expert blind spots and were aware of their tendency to use algebra on problems that students would solve informally. Knowing more about how students avoid algebra, and under what conditions, raised teachers' awareness of the complexities of task design and problem selection.

CHAPTER 7
CONCLUDING REMARKS

*Teaching is a complex activity. Quality teaching is not simply the fact of 'knowing your subject' or the condition of 'being born a teacher'.... We cannot claim that teaching causes student outcomes.... But if student outcomes are not **caused** by teaching practices, they can at least be **occasioned** by those practices. (Anthony & Walshaw, 2007, p. 205)*

7.1 Effectiveness of the intervention

This project was precipitated by teachers' expressed need to improve students' use of algebra to solve word problems. It sought to address concerns that there has been "little attention given explicitly to the teaching of word problems" (Chapman, 2006, p. 211) through the implementation of a collaborative classroom design experiment focused on translation between word problems and algebra.

In analysing the impact of the study, two issues were considered: the impact of the intervention on students, and the impact of the study on teachers. Responses to these two concerns are interconnected and overlapping. However, in order to address each fully, responses framed around two questions are discussed separately.

1.   **In what ways does the introduction of explicit teaching activities that support the translation processes used to solve algebra word problems impact on student learning processes and outcomes?**

The most significant impact was related to students' understandings of algebra as a tool. Initially, as has been noted in other studies (e.g., Koedinger, Alibali, & Nathan, 2001; Koedinger & Nathan, 2004; Stacey & MacGregor, 2000), the students tended to use informal arithmetic strategies when solving problems; this was true even for problems expressly presented to them for the purpose of practising or assessing algebraic knowledge. The explicit review of the translation process appeared to be successful in convincing these adolescent students "why they should abandon methods which have served them well in the past" (Watson, 2007, p. 88).

As the study progressed, changes in students' attitudes reflected a growing awareness of algebra as something that they could get better at. Such beliefs are important in terms of students' theories of learning. Dweck (1999) highlighted the importance of supporting students to develop an understanding that mathematics is something that can be learnt. He suggested that for many students, progress is hindered by their belief that the ability to do mathematics is innate. It appears that the intervention helped develop or reinforce students' views of algebra as something that can be learnt, although it is not clear which specific aspects of the intervention were associated with this.

Activities were developed with an explicit focus on translation between verbal and symbolic representations. These were effective in enhancing students' facility at translating word problems into algebra. The activities served to provide practice opportunities for students in moving forwards and backwards across the bridge connecting verbal and symbolic representations. Once students had practised translation both ways, they were introduced to a heuristic that built on the earlier translation activities. The heuristic provided a structured approach that organised problem-solving into explicit stages that students could use to scaffold the solution process.

Some students were able to combine their new-found translation skills with algebraic manipulation skills to solve word problems algebraically. However, other students experienced difficulties and "got stuck" at various stages of the translation process. This raises questions about how teachers might support such students to continue to use algebra so that they can develop algebraic fluency. Students' comments highlighted the value of completing the full problem solving cycle, from translation through to answer, a process suggested by Bell (1996b) as important for developing an understanding of algebra as more than symbol manipulation.

Students identified a number of factors associated with the intervention as valuable. These included: the explicit objectives and clarity around what was to be learnt; the opportunity to engage in conversations about their thinking and to practise translating between verbal and symbolic forms; structured progression of learning tasks; time to consolidate understandings; and the heuristic for problem solving. Overall, students made little mention of algebraic manipulation; instead, they discussed their understanding of

the process of algebraic problem solving. This suggests that the intervention supported the development of relational understanding (Skemp, 1976).

2. How did participation in the classroom experiment impact on teachers' pedagogical practices, knowledge and beliefs?

There were indications that participation in the project impacted on teachers in two ways: firstly, with regards to the immediate intervention of teaching algebra; and secondly, with regards to teaching strategies for mathematics in general.

Translation activities provided a tool for teachers to engage students in mathematical discussion, enabling them to elicit and build on student thinking. Both the problem-solving cycle and the heuristic providing a structured approach to solving word problems were new to the teachers. As teachers developed new understandings about how their students approached word problems they gained insight into their own expert blind spot, and the importance of using problems for which students needed to use algebra.

An unanticipated consequence of collaborating with teachers in this study was the apparent difficulty that the teachers experienced in developing word problems. Teachers' initial attempts to develop activities that took account of student understandings highlighted their previous uncritical reliance on textbook examples. In particular, the teachers experienced difficulty designing quality instructional activities, including algebra word problems, that pressed for algebraic thinking. For word problems, getting the difficulty level right involves a delicate balance between problems that are too hard for students to solve and problems that are too easy and can thus be solved without algebra.

Within the study, the requirement that teacher instruction focus on translation encouraged a shift in teacher practice away from a skills-focus toward a problems-focus. This was associated with a shift toward a more student-centred approach to teaching as highlighted by a teacher comment:

So it's not what I say that's important, it's what the students do.

The resulting *problems-first* approach led both teachers to question their long-standing approach to teaching the topic of algebra. Teacher talk reflected changes in their classroom practice toward adopting more of an inquiry-based model of teaching. The focus on problems rather than skills supports Watson's (2007) suggestion that teachers introducing a new technique, such as algebra, focus on tool selection: "Rather than it being a *rule*, it becomes a *tool* to be used when appropriate... 'supermethods' need to be rehearsed so that they are ready to use when necessary, and have the status of tools, rather than rules" (p. 88).

Teachers' initial concerns about coverage were challenged by the experience of sustained attention to a small number of ideas consistent with suggestions from the literature: "Emphasis needs to shift beyond superficial coverage of a large number of small tasks to the comprehensive treatment of a small number of big ideas" (Lesh & Clarke, 2000, p. 141). Such a change in focus is likely to have helped develop students' relational understanding rather than the instrumental understanding that would have developed with a focus on skills (Skemp, 1976).

Although the scope of the research prohibited in-depth examination of longer term change, teacher talk about plans for future lessons suggested generative change (Franke, Carpenter, Fennema, Ansell, & Behrend, 1998) was already occurring, and was likely to continue. Teachers reacted positively to being involved as research collaborators, and appeared to gain confidence in their ability to reflect on their teaching processes, and the value of this reflective process. They valued working together. Their ongoing conversations with one another, particularly during the implementation phase, provided important support for developing ideas, planning activities, and reflecting:

> *It is really good having another teacher involved. You are not on your own... We talk about it [the intervention] lots... I like the way we each bring our own perspective and go from there, sometimes ending up in quite unexpected places.*

The development of "shared histories of learning" (Wenger, 1998, p. 87) is characteristic of a community of practice. As noted by Jacobs, Franke, Carpenter, Levi, and Battey (2007) "teachers learn through participation in practice" (p. 27) and participation in communities of practice is a key factor in effective professional development for teachers.

The link to student learning was critical for enabling reflection to be enacted in changes in teacher practices. The research provided teachers with structured opportunities to learn from close observation of their students. By focusing on students' thinking and what students could do (as opposed to could not), teachers were able to build on student thinking and thus create more appropriate opportunities for student learning. Successful outcomes for students were a key influence for teachers adopting new practices as suggested by the literature (e.g., Hattie, 2002; Timperley et al., 2006).

7.2 Implications

7.2.1 Extending the translation activities

Instructional activities that focus on translation appear to be a valuable, yet commonly overlooked, task in helping students develop understanding of algebra word problems. In this project, the introduced instructional activities included students writing literal word translations of algebraic equations. However, this study did not include any activities involving students writing contextual problems from algebra. Teachers' identification of this as an activity worth investigating further is supported by Watson and Mason's (Timperley et al., 2006; 2005) contention that student-generated examples can extend students' mathematical thinking, shifting the responsibility for learning to the students. Likewise, Bernado's (2001) research found that secondary students who constructed their own problems analogous to ones they had previously solved, developed deeper levels of understanding of the problems and were better at transferring their skills. It may also be useful to extend the focus on translation between verbal and symbol to include diagrams and graphs as a way to enhance students' representational fluency (Nathan et al., 2002).

7.2.2 Flexible approaches to problem-solving

Teachers' tendency to use algebra to solve a problem regardless of the nature of a problem reflects the current emphasis of the NCEA algebra achievement standards and associated assessment tasks. However, privileging algebra as the preferred solution strategy is at odds with developing flexible approaches to solving problems. A more adaptive approach would involve the capacity and willingness to adjust strategies to the

characteristics of the problem. "Successful and flexible problem solvers... are able to be adaptive in applying the most appropriate strategy" (Van Dooren et al., 2002, p. 347).

The recently released draft of suggested changes to NCEA level one achievement standards supports a more flexible approach to problem-solving.

> Most of the proposede [sic] standards now contain the leader "Demonstrate an understanding" and then define the type of problems being solved rather than specifying the method of solution. A broader range of strategies will be acceptable in the solution of a problem. Students should be able to use graphical, algebraic, numerical or tabular representation to solve problems within any of the standards. (NZQA, 2007)

It these proposals are mandated, teachers will have a real incentive to focus on developing flexible approaches to problem solving, both for themselves and their students. This does, however, have significant implications for professional development in terms of broadening teachers' own problem-solving approaches, as well as their teaching of problem solving.

7.2.3 Designing tasks

The difficulty of creating instructional tasks and assessment tasks, particularly those involving algebraic word problems, is an issue. Teachers need professional development, time and support if they are to develop quality tasks. Hodge, Visnovska, Zhao, and Cobb (2007) acknowledge the complex issues surrounding task design and highlight the importance of considering the potential of a task for "developing students' pragmatic and mathematical interests" (p. 400). They emphasise the need for analysis of how instructional tasks serve as resources for teachers and how the classroom situation mediates this process.

Furthermore, the complexities associated with designing word problems to assess algebraic strategies have implications for those involved in writing assessment tasks. High-stakes assessments should provide models of good practice. However, currently many NCEA assessment tasks do not address the issues associated with algebra word problems.

7.3 Limitations

As with any classroom study conducted in a naturalistic setting there are often real constraints and unanticipated factors that impact on the study implementation. With a specific focus on the translation phase of word problems, this study was able to make limited links to the broader aspects of students' understandings of algebra as they were evidenced in classroom observations and student interviews. It is clear that many factors impact on students' solving of word problems. The research literature highlights the complexity of factors and misconceptions related to the concept of variables and solving equations that also impact on student learning outcomes.

Moreover, the focus of the intervention mainly involved the introduction of specific instructional activities. Other significant aspects, including classroom climate/culture, the nature of discourse and the role played by the teacher, were treated as background. "Altering a single influencing factor, such as the mathematical structure of a task, does not necessarily lead to changes in student strategy use; the task, students' cognitive structures, and social influences all contribute to student strategy selection" (Lannin, Barker, & Townsend, 2006, p. 26).

It is also recognised that some data was based on teachers' perceptions. Teachers' perceptions formed the basis of discussion about teachers' levels of understanding, ways of thinking, beliefs and attitudes. Although there was some triangulation by way of researcher classroom observations, definitive claims cannot be made about teachers' actual practices.

7.4 Further Research

The cognitive models that students use to write mathematical equations are complex and there is a need for more research on "how they can be helped to develop the thought processes required to translate word problems into equations correctly" (Hubbard, 2004, p. 304). This study suggests some ways forward.

- **Translation focus with younger students**

This research focused on translation with senior secondary students. This gives rise to questions about the effectiveness of a translation focus with younger students. It would

be worthwhile investigating the impact of a focus on translation on younger students' understandings before they are exposed to other aspects of algebra.

- **Alternative routes to a solution**

Stacey and MacGregor's (2000) "routes to solution" proved useful in this study for coding student work as well as for deepening teachers' understandings of students' strategies. This leads to questions about the impact of sharing knowledge about these routes with students. Perhaps sharing the range of routes with students would help them to develop a more flexible approach to solving problems.

- **Language acquisition models**

Meaney (2006) highlights the importance of students acquiring the mathematics register. Her model suggests students progress through four stages in acquiring the mathematics register: noticing, intake, integration and output along the lines of models explaining second language acquisition. Meaney's model deals with the spoken and written text of mathematics but does not currently address the language of symbols which is a key aspect of the language of mathematics. If developing fluency with symbols is a language acquisition process, Meaney's model may help to explain the process by which students acquire symbolic language.

- **Teacher change and learning**

Many of the recent suggestions for changes in teacher practice are broad and ill-defined in terms of how they are to be interpreted in classroom practice. However, a key feature of this project was its focus on one specific aspect of teaching practice which was identified by teachers as an issue. Despite this narrow focus, there are indications of wide-reaching changes in teachers' understandings and practices. While Timperley et al. (2006) provide a comprehensive review of evidence-based professional development practices, mathematics educators such as Beswick (2007) argue that we need to continue to learn more about the kinds of professional learning experiences that are effective for mathematics teachers.

7.5 Concluding thoughts

Teaching mathematics is about building understanding, and understanding mathematics involves both procedural understanding, knowing how, and relational understanding, knowing why. Traditionally, procedural understanding has tended to dominate the mathematics classroom, particularly within the domain of school algebra. Even so, traditional views of algebra and word problems can be changed. Time and coverage pressures need not constrain teachers to focus on procedural understanding. Whilst it was apparent that instructional focus on translation shifted teachers and students away from an emphasis on procedure, it was equally clear that translation alone is insufficient as an intervention. Students need both procedural and relational understandings to develop an understanding of the use of algebra as a tool to solve word problems Students also need to develop fluency with a range of strategies, including algebra, in order to be able to select appropriate strategies to solve particular problems. This study affirmed for teachers that teaching with a focus on understanding can provide an effective and efficient method for increasing students' motivation, interest and success.

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APPENDICES

Appendix A: Algebra Achievement Standard (AS90284)

Subject Reference		Mathematics 2.1			
Title		Manipulate algebraic expressions and solve equations			
Level	2	**Credits**	4	**Assessment**	External
Subfield			Mathematics		
Domain			Algebra		
Registration date	20 October 2004		**Date version published**		20 October 2004

This achievement standard requires the manipulation of algebraic expressions and the solution of equations.

Achievement Criteria

	Achievement Criteria	Explanatory Notes
Achievement	• Manipulate algebraic expressions.	• Assessment of manipulation will be based on a selection from: – expanding brackets up to 3 factors – factorising expressions including quadratics – using fractional and negative indices – using elementary properties of logarithms – simplifying rational expressions.
	• Solve equations.	• Assessment of solving equations will be based on a selection from: – multi-step linear equations or inequations eg $3(2x-5) = 5x+7$ – quadratics that can be factorised eg $2x^2 - 11x = 21$ – simple logarithmic equations eg $\log_x 25 = 2, \ 3^x = 25$ – forming and solving linear/linear simultaneous equations.

	Achievement Criteria	Explanatory Notes

Achievement with Merit	• Solve problems involving equations.	• Assessment will be based on a selection from:
		   – quadratics requiring the use of the quadratic formula
		   – linear/non-linear simultaneous equations
		   – exponential eg $13^{4x-5} = 6$.
		 • Non-linear equations may be given as appropriate to the complexity of the problem.
		 • Students will be expected to solve problems in context.

Achievement with Excellence	• Choose algebraic techniques and strategies to solve problem(s).	• When solving a problem the student may be required to:
		   – interpret the solution
		   – explore the nature of the roots of a quadratic
		   – complete a multi-step algebraic manipulation
		   – complete an algebraic proof.

General Explanatory Notes

1   This achievement standard is derived from *Mathematics in the New Zealand Curriculum*, Learning Media, Ministry of Education, 1992:
- achievement objectives p. 158
- suggested learning experiences p. 159
- sample assessment activities pp. 160-161
- mathematical processes p. 26.

2   The use of the Factor/Remainder Theorem will not be assessed.

3   An algebraic proof will involve a multi-step manipulation of a given algebraic statement to generate another given expression.

4   For this standard the problems may be set in a mathematical context.

Quality Assurance

1   Providers and Industry Training Organisations must be accredited by the Qualifications Authority before they can register credits from assessment against achievement standards.

2   Accredited providers and Industry Training Organisations assessing against achievement standards must engage with the moderation system that applies to those achievement standards.

Appendix B: Common Tasks

Appendix B1: Task A

Note: when the task was printed out for students, space was provided between questions for students to write

Please show all your working

1. In a test, the number of students who pass, is three times the number of students who fail. If the number of students who fail is f, how many students are there in this class?

    (Adapted from Version 2, Question 2 Hubbard, 2004)

2. In another class, there are twice as many boys as girls. If there are 33 students in the class and x of them are girls, how many boys are in the class?

    (Adapted from Version 1, Question 3 Hubbard, 2004; also Question 6 NZQA, 2006a)

3. Sally gets paid x dollars per week for pocket money. Her older brother gets twice as much. Sally's younger brother gets $5 less than Sally. If their combined weekly total is $42, how much does Sally get?

    (Adapted from in-school assessment task)

4. To rent a car from Tiger Motors costs $100 per day and 20 cents per km. To rent a car from Kiwi Motors costs $120 per day and 15 cents per km. For what distance is each company the same price?

    (Adapted from in-school assessment task)

5. A rectangle is four cm longer than it is wide. If its area is 21 cm^2, what is the width of the rectangle?

    (Adapted from question 6 NZQA, 2005a)

Appendix B2: Task B

Note: when the task was printed out for students, space was provided between questions for students to write

Please show all your working

1. Records show that, the number of people who pass their driving licence on their first attempt is three times the number of people who fail in the month of May. If the number of people who pass is p, how many people sat their licence in May?
   (Adapted from Version 2, Question 2 Hubbard, 2004)

2. In the same month, there were twice as many females as males who sat their licence. If there were 143 people and x of them are female, how many males were there?
   (Adapted from Version 1, Question 3 Hubbard, 2004; also Question 6 NZQA, 2006a)

3. Simon earns x dollars per week for his job at the pizza shop. One of his friends gets twice as much. Another friend gets $15 more than Simon. If their combined weekly total is $420, how much does Simon get?
   (Adapted from Question 3 of Task A)

4. To rent a car from Tiger Motors costs $50 per day and 40 cents per km. To rent a car from Kiwi Motors costs $60 per day and 35 cents per km. For what distance is each company the same price?
   (Adapted from Question 4 of Task A)

5. A rectangle is eight cm longer than it is wide. If its area is 135 cm^2, what is the width of the rectangle?
   (Adapted from Question 5 of Task A)

Appendix C: NCEA questions

Appendix C1: Question six, 2004 NCEA level 1 assessment for achievement standard 90147

Merit level question used as a basis for Question 2 of Task A and B

Peter has more than twice as many CDs as Mary.

Altogether they have 97 CDs.

Write a relevant equation, and use it to find the **least number** of CDs that Peter could have.

NZQA. (2004). AS90147. Retrieved 20 April, 2007, from

http://www.nzqa.govt.nz/nqfdocs/ncea-resource/achievements/2007/as90147.pdf
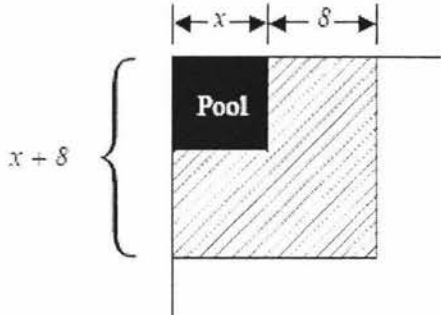
Appendix C2: Question six, 2005 NCEA level 1 assessment for achievement standard 90147

Merit level question used as a basis of Question 5 of Task A

QUESTION SIX

The diagram shows a square courtyard with a square pool in one corner.

The area of the courtyard is $225\,\text{m}^2$, and the courtyard extends 8 m beyond the pool.

Solve the equation $225 = (x + 8)^2$, to find x, the **length of the side of the pool**.

NZQA. (2005). AS90147. Retrieved 20 April, 2007, from

http://www.nzqa.govt.nz/nqfdocs/ncea-resource/achievements/2007/as90147.pdf

Appendix D: Information sheets for participants

Appendix D1: Parent information sheet

Dear Parent/Guardian
Mathematics Research Project
PARENT INFORMATION SHEET

My name is Anne Lawrence. I am a registered teacher currently working for Massey University as an Adviser to teachers of Mathematics. As part of my continuing education, I am completing a thesis for a Master of Educational Studies (Mathematics) through Massey University. My thesis is a collaborative project that involves me working in a partnership with two teachers from *** High School. Teachers have selected two of their classes as a focus for this research. .

The aim of my research project is to improve the teaching of algebra. In particular I am interested in finding out how effective particular teaching strategies are at helping students to use algebra to solve word problems. I believe that students will benefit from participation through the opportunity to reflect on the way they learn algebra, and the impact of specific teaching strategies on their learning.

Students have been invited to take part in this research if they are currently in a mathematics class selected as a focus class by a teacher participating in this project. Students' perspectives will provide important information for the evaluation of teaching strategies. Their participation is voluntary and is independent of any assessment procedures associated with their course of study.

There are two levels that participating students will be involved:
1. Test and follow-up interviews.
    *   As part of their normal classroom programme, students will sit a test at the start and end of the unit of work on algebra. I would like your permission to copy your son/daughter's test responses for this study on the understanding that their name cannot be identified. Their name will not be published in any reports or other publications, and the school will remain anonymous.
    *   I would like your permission for your son/daughter to participate in two individual interviews about how he/she solved word problems during the algebra tests. The first interview will be at the start of the unit of work on algebra, and the second will be at the end of this unit. Both interviews will be audio-taped.
2. Focus Group Stimulated Recall interviews.
    *   Some of your son/daughter's classes will be recorded onto DVD. Participating students will participate in a maximum of two focus group interviews with 4 other students from their mathematics class. The group will watch excerpts from the DVD recording of the lesson and talk to the researcher about teaching strategies that helped or restricted their learning. Interviews will be audio-taped. Teachers will not be present in interviews, but will be provided with a record of the group feedback. No individual student comments will be identified. Focus group interviews will last a maximum of 40 minutes each.

Your son/daughter is under no obligation to accept this invitation.   If he/she decides to participate, he/she has the right:
    *   to refuse to answer any particular questions;
    *   to withdraw from the study at any time;
    *   to ask any questions about the study at any time during participation;

- to provide information on the understanding that his/her name will not be used unless he/she gives permission to the researcher;
- to be given access to a summary of the findings of the study when it is concluded;
- to ask for the audio tape to be turned off at any time during any interview.

I will do my best to maintain the confidentiality of participants throughout the research, e.g. by using pseudonyms. It should be noted, however, that there is a clear expectation that all participants, including the researcher, will respect any information shared through the research process and will treat it with confidentiality. Neither the school nor any individuals will be identified either directly or indirectly in verbal or written form. Where direct quotes from the interview tapes or written correspondence are used in subsequent publications pseudonyms will be assigned to maintain anonymity.

If after reading this information sheet you are willing for your son/daughter to be involved in the project, can you please complete the consent form and return it in a sealed envelope to your mathematics teacher. If more than five students from the class consent to be part of the case study I will select a representative sample and you will be informed in writing whether your son/daughter has been selected to take part or not.

I would like to thank you for your careful consideration of this opportunity. If you have any questions about the project please contact me (the researcher) or my chief supervisor here at Massey University.

Researcher:	**Supervisor:**
Anne Lawrence	Associate Professor Glenda Anthony
Phone 06 350 9303	Phone 06 350 9600
Email: a.lawrence@massey.ac.nz	Email: G.J.Anthony@massey.ac.nz

This project has been reviewed and approved by the Massey University Human Ethics Committee, PN Application 05/122. If you have any concerns about the conduct of this research, please contact Dr John O'Neill, Chair, Massey University Campus Human Ethics Committee: Palmerston North, telephone 06 350 5799 x 8635, email humanethicspn@massey.ac.nz.

Appendix D2: Student information sheet

Mathematics Research Project
STUDENT INFORMATION SHEET

My name is Anne Lawrence. I am a registered teacher currently working for Massey University as an Adviser to teachers of Mathematics. As part of my continuing education, I am completing a thesis for a Master of Educational Studies (Mathematics) through Massey University. My thesis is a collaborative project that involves me working in a partnership with two teachers from *** High School.

The project aims to improve the teaching of algebra. In particular I am interested in finding out how effective particular teaching strategies are at helping students to use algebra to solve word problems. I believe that students will benefit from participation through the opportunity to reflect on the way they learn algebra, and the impact of specific teaching strategies on their learning.

Why me?
You have been invited to take part in this research because you are currently in a mathematics class taught by a teacher who has agreed to take part in this project. Your participation is voluntary and is independent of any assessment procedures associated with your course of study.

What will I be asked to do?
If you agree to participate, there are two levels that you would be involved:
1. Test and follow-up interviews.
   - As part of your normal classroom programme, your class will sit a test at the start and end of the unit of work on algebra. I would like your permission to copy your test responses for this study on the understanding that your name cannot be identified. Your name will not be published in any reports or other publications, and the school will remain anonymous.
   - I would like you to participate in two individual interviews about how you solved word problems during the algebra tests. The first interview will be at the start of the unit of work on algebra, and the second will be at the end of this unit. Both interviews will be audio-taped.
2. Focus Group Stimulated Recall interviews.
   - Some of your classes will be recorded onto DVD. You may be asked to participate in one or two focus group interviews with 4 other students from your mathematics class. The group will watch excerpts from the DVD recording of the lesson and talk to the researcher about teaching strategies that helped or restricted your learning. Interviews will be audio-taped. Teachers will not be present in interviews, but will be provided with a record of the group feedback. Each focus group interview will last no more than 40 minutes.

What rights do I have?
You are under no obligation to accept this invitation. If you decide to participate, you have the right:
   - to decline to participate;
   - to refuse to answer any particular questions;
   - to withdraw from the study at any time;
   - to ask any questions about the study at any time during participation;
   - to provide information on the understanding that your name will not be used unless you give permission to the researcher;

- to be given access to a summary of the findings of the study when it is concluded;
- to ask for the audio tape to be turned off at any time during any interview.

When any research is conducted it must be recognized that there is always a risk of a breach of confidentiality and that I can only give an assurance of confidentiality and anonymity to the extent allowed by law. It should be noted, however, that there is a clear expectation that all participants, including the researcher, will respect any information shared through the research process and will treat it with confidentiality. Neither the school nor any individuals will be identified either directly or indirectly in verbal or written form. Where direct quotes from the interview tapes or written correspondence are used in subsequent publications pseudonyms will be assigned to maintain anonymity.

What do I do now?

If after reading this information sheet you are willing to be involved in the project, can you please complete the consent form and return it in a sealed envelope to your mathematics teacher. If more than five students from your class consent to be part of the case study I will select a representative sample and you will be informed in writing whether you have been selected to take part or not.

I would like to thank you for your careful consideration of this opportunity. If you have any questions about the project please contact me (the researcher) or my chief supervisor here at Massey University.

Researcher:
Anne Lawrence
Phone 06 350 9303
Email: a.lawrence@massey.ac.nz

Supervisor:
Associate Professor Glenda Anthony
Phone 06 350 9600
Email: G.J.Anthony@massey.ac.nz

This project has been reviewed and approved by the Massey University Human Ethics Committee, PN Application 05/122. If you have any concerns about the conduct of this research, please contact Dr John O'Neill, Chair, Massey University Campus Human Ethics Committee: Palmerston North, telephone 06 350 5799 x 8635, email humanethicspn@massey.ac.nz.

Appendix E: Interview schedules

Appendix E1: Initial Interview Schedule with Teachers

Remind about confidentiality and that teacher can ask to have the tape recorder switched off at any time. Ensure teacher is comfortable. Start interview only when teacher is ready.

Tell me about yourself as a teacher:
- How long have you been teaching?
- What subjects?
- What levels of student?

I am interested in why you chose to participate in this study:
- What motivates you to be involved?
- What do you hope to achieve?

This study focuses on students' use of algebra to solve word problems. I am interested in what you know about this from classroom experience:
- What are some of the things you think are critical for student success with learning to use algebra to solve word problems?
- Are there any particular ways you specifically seek to support student learning about word problems in algebra?
- What are your challenges, concerns, unanswered puzzles about the teaching of word problems in algebra?

You have selected two classes as the focus of this research. Tell me about these classes:
- Why did you choose these classes to work with in this research?
- Tell me a little about the classes, how you feel about them, what are the positives and negatives of being their mathematics teacher.
- Are there particular characteristics of these classes that will make the research particularly challenging?

Are there any other comments you would like to make about teaching and learning of word problems in algebra?

Thank you for your time.

Appendix E2: Interview Schedule about Student Test Responses

Remind about confidentiality and that the student can ask to have the tape recorder switched off at any time. Ensure student is comfortable. Start interview only when student is ready.

Tell me about your work in the algebra test:
- How did you feel about how you went in it?
- How difficult did you find it?
- Any specific parts that you found particularly difficult or easy?

This study focuses on how students tackle word problems. I am interested in how you went about tackling the word problems in the test. Here is your test responses.
- Tell me what you were doing when you tackled this question.
- What was the question asking you to do?
- What do you think your teacher would have wanted you to do?
- Are there any particular things that make this question particularly easy or difficult for you to do?
- Are there any things you can think of that might make the question easier for you to do?

Are there any other comments you would like to make about the way you tackle word problems in algebra?

Thank you for your time.

Appendix E3: Stimulated Recall Interview Schedule with Student Focus Groups

Reflection on the lesson

Before we watch a recording of your mathematics class, I want you to think about the last lesson and tell me how you feel the lesson went for you.
- What were the main things you learned in that lesson?
- What parts of the lesson did you enjoy?
- What things about the lesson did you find annoying?
- What do you think your teacher's main goal was?

Stimulated Recall

I am going to play the recording of the class on the television. I am really interested in you telling me what was happening in this class for you.

I would like you to stop the recording at any point where you want to comment on things like:
- How the class was going for you
- Important points where the class really interested you
- Important points where you were confused, bored, unable to understand what was going on.
- Can you identify times when you were actively learning?
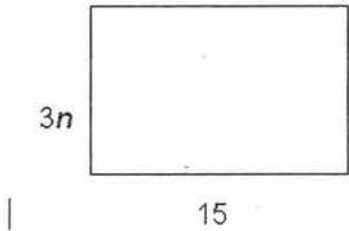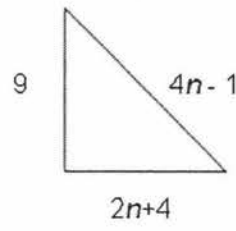
I will also sometimes stop the recording and ask you to talk to me about what was going on for you during the class at different times.

Appendix F: Instructional Activities

Appendix F1: Matching activity

Match each yellow card with the appropriate blue card

Total cost of n books at $12 each	$12n$	The even number just before n if n is also even	$n - 2$
Amount of each share when $45n$ is evenly shared among 5 people	$9n$	Monica's age n years ago if she is 12 today	$12 - n$
Profit made when an item is bought for $25, then sold for n	$25 - n$	Change from $15 after buying three books at n each	$15 - 3n$
Average of n and $n+2$	$n + 1$	The odd number just after n if n is also odd	$n + 2$
Perimeter of this rectangle 3n, 15	$6n + 30$	Area of this Triangle 9, 4n - 1, 2n+4	$9n + 18$

Appendix F2: Set 1

Write an English phrase for each of these expressions in Algebra:

1. $n + 4$
2. $10 - n$
3. $n - 7$
4. $5n$
5. n^2
6. $4n - 6$
7. $n^2 + 2n$
8. $3(n + 7)$
9. $15 - 3n$
10. $\dfrac{n + 8}{2}$

Appendix F3: Set 2

Write algebraic expressions for the following:

1. Eleven more than a number.
2. A number decreased by five.
3. Twelve divided by a number.
4. Three less than four times a number.
5. The product of a number and six.
6. The sum of a number and seven and the result multiplied by four.
7. The product of a number and two more than the number.
8. Double the number less than the square of the number.
9. Triple the number divided by four less than the number.
10. The difference between the number and eight and the result multiplied by four more than the number.

Appendix F4: Set 3

Choose the correct words to fill in the gaps:

$$3 (n + 5) = 2n$$

I _____ five to a number then _____ my answer by _____ . This gives the

same result as _____ my number.

$$n^2 + 3n = 4$$

If I multiply my number by _____ then add _____ times my number the

_____ is four.

$$\frac{n+4}{n-5} = 15$$

If I _____ four _____ than ____ _____ by _____ less than ____

_____ the _____ is _____ .

$$(n + 7) (n - 3) = 24$$

If I _____ _____ more than ____ _____ by _____

_____ than ____ _____ the _____ is _____ .

$$n (n + 5) = 4n - 3$$

The _____ of ____ _____ and _____ _____ than ____

_____ is the _____ as three _____ _____ my number

_____ ____ _____ .

Appendix F5: Set 4

Example $2n - 12 = 5$ could be written

Twelve less than twice a number is five. What is the number?

Or When I double a number and take away twelve, the answer is five. What is the number?

Write the following equations in words:

1. $5n + 7 = 6$

2. $n^2 - 3 = 5$

3. $7 - \dfrac{n}{2} = 11$

4. $\dfrac{n+4}{3} = 6$

5. $2n - 7 = 3n + 8$

Appendix F6: Set 5

Choose the correct words to fill in the gaps:

Ferry Charges

$$15x + 16 = 44.50$$

The charge for a vehicle on the fast ferry depends on the length of the vehicle, given by x.

If it costs $15 _____ _____ and a handling fee of _____ is _____ , then the fare for a motorcycle is _____ .

Matthew's Allowance

$$2x + 20 = 154$$

where Matthew's allowance is given by x.

Matthew's brother Samuel gets _____ the allowance that Matthew does. After Samuel is _____ _____ for his birthday, Samuel now has _____ in _____ . How much is Matthew's allowance?

Converting Temperature

$$\frac{5F - 160}{9} = C$$

Where the temperature in degrees Fahrenheit is given by F and the temperature in degrees Celsius is given by C.

To convert _____ Fahrenheit to _____ _____ you _____ _____ from _____ _____ the _____ in degrees Fahrenheit and then _____ the result by _____ . This gives the temperature in _____ _____ .

Drinking a Toast $C = \dfrac{n(n-1)}{2}$

Where C is the number of clinks and n is the number of people.

To calculate the number of _____ you _____ the number you get when you _____ the number of _____ by _____ _____ than the _____ ____ _____ .

Interior Angles in Polygons

$$S = (n - 2) \times 180$$

Where S is the sum of the interior angles and n is the number of sides.

To _____ the _____ of the _____ _____ in a polygon, _____ _____ from the _____ ____ _____ and _____ the _____ by _____ .

Appendix F7: Set 6

Match each word problem with the appropriate equation(s).

Note that some of the equations may not be correct for any of the problems.

Question 1

Mary goes to the movies with $x. Anne has three times as much as Mary, Joanne has $6 more than Mary. Altogether they have $41.

How much money do they each go to the movies with?

$x + 3x + x + 6 = 41$

$41 - 3x - x + 6 = x$

$x + 3x + x - 6 = 41$

Question 2

I can cycle an average of 18 kilometres per hour. On one trip I cycle for x hours but I have also walked 7 kilometres up steep hills. If I travelled 106 kilometres in all, find the value of x.

$18x + 7 = 106$

$18/x + 7/x = 106$

$18x = 106 - 7x$

Question 3

A man purchases a number of tools at $5 each. Unfortunately, five do not work but by selling those that do at $12 each, he is able to make $45 profit.

How many tools did he buy originally?

$12(x - 5) - 5x = 45$

$(60 - 25) + 7(x - 5) = 45$

$5(0 - 5) + 7x = 45$

Question 4

Two buckets hold 12L and 4L of liquid respectively. To each bucket is now added another x litres, so that the first one holds twice as much as the second one. Find the value of x.

$12 + x = 2(4 + x)$

$2(12 + x) = x + 4$

$x + 12 = 2x + 4$

Question 5

Two men are digging in the gardens. One prepares $60m^2$ for planting vegetables and the other prepares $18m^2$. If both men now prepare an extra $x m^2$ each, then the first man has prepared exactly 3 times the area of the second man.

Find the value of x.

$60 + x = 3(18 + x)$

$1/3 (x + 60) = x + 18$

$3 (x + 60) = x + 18$

$x + 60 = 3x + 18$

Question 6

One girl has four times as much money as her friend. She gives her friend $12 and as a result, they now have the same amount of money.

How much did each have originally?

$4y - 12 = y + 12$

$x - 12 = x/4 + 12$

$4x - 12 = x + 12$

Appendix F8: Pink-blue-yellow matching activity

Pink cards are for algebraic equations.
Blue cards are for literal word equations.
Yellow cards are for contextual word equations.
You need to form matching set of pink, blue and yellow cards.
You will need to write your own appropriate equation for any blank cards.

$X + 3.5 = 12.3$		John had 3.5kg of apples. He was given some more and he now has 12.3kg of apples. How many was he given?
	Twelve point three divided by an amount gives three point five	If we have 12.3m of ribbon, how many 3.5m lengths can be cut from this length?
$12.3 \times 3.5 = X$	The result of three and a half lots of twelve point three.	
$2X - 3.5 = 12.3$		I cycled 12.3km to work. This is the same as twice the distance to the local school less 3.5km. How far is it to the local school?

Appendix G: Coding of Students' Responses to Sample Tasks

Appendix G1: Responses to Questions 2 to 5 on Task A and Task B

Student		Task A Q2	Task A Q3	Task A Q4	Task A Q5	Task B Q2	Task B Q3	Task B Q4	Task B Q5
12X	A	AL, EQN, NW, WA	AL, EQN, MAN, WA	AR, T/E RND, NA	AL, EQN, MAN, WA	AL, EQN, NA	AL, EQN, MAN, NW, WA	AR, NA	AL, EQN, NIL, WA
12X	B	AR, LOG, RA	AL, EQN, MAN, WA	AR, T/E, LOG, RA	AL, REL, LOG, RA	AL, UNK, LOG, RA	AL, EQN, MAN, RA	AL, REL, LOG, WA	AL, EQN, MAN, NA
12X	C	AL, EQN, RVS, RA,	AL, EQN, NW, RA	NIL	NIL	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AR, T/E, SEQ, RA	AL, EQN, NW, RA
12X	D	NW, RA	AL, EQN, MAN, RA	NW, WA	AL, EQN, LOG, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, T/E, GC, WA	AL, EQN, MAN, RA
12X	E	AL, EQN, MAN, RA	AL, EQN, MAN, RA	NIL	AL, EQN, LOG, RA	AL, REL, NA	AL, EQN, MAN, RA	NIL	AL, EQN, MAN, NA
12X	F	AR, LOG, RA	NIL	NIL	NIL	AL, REL, NA	AL, EQN, MAN, RA	AL, EQN, MAN, WA	AL, EQN, MAN, NA
12X	G	AR, LOG, RA	AR, LOG, WA	NIL	AL, REL, LOG, WA	AL, REL, AR, LOG, RA	AL, EQN, MAN, RA	NIL	AL, UNK, AR, T/E, SEQ, WA
12X	H	AL, REL, LOG, RA	AL, UNK, LOG, RA	AL, EQN, T/E, SEQ, NA	AL, EQN, LOG, WA	AL, EQN, MAN, NA	AL, UNK, LOG, RA	AR, T/E, SEQ, NA	AL, UNK, LOG, WA
12X	I	AL, EQN, MAN, RA	AL, EQN, MAN, WA	AL, EQN, NA	AL, EQN, NA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, NA
12X	J	AL, UNK, LOG, RA	AR, PTE, RND, NA	AR, LOG, RA	NW, RA	AL, EQN, NA	AL, EQN, NA	NW, RA	AR, PTE, GCI, RA
12X	K	AR, LOG, RA	AR. LOG. WA	AR, NA	AR, LOG, WA	AL, EQN, MAN, RA	AL, EQN, MAN, WA	AL, EQN, NA	AL, EQN, MAN, ETE, GCI, WA
12X	L	AL, REL, LOG, RA	AL, REL, NA	NIL	AL, EQN, MAN, WA	AL, REL, NA	AL, EQN, MAN, WA	NIL	AL, EQN, MAN, WA
12X	M	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, NA	AL, EQN, NA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, NA	AL, EQN, NA
12Y	A	AR, LOG, RA	AR, PTE, SEQ, WA	AR, T/E, RND, NA	AR, LOG, RA	AL, REL, LOG, WA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA
12Y	B	AL, UNK, LOG, RA	AR, PTE, GCI, WA	NIL	NIL	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, WA
12Y	C	AR, LOG, RA	AL, EQN, MAN, RA	AL, EQN, MAN, WA	AL, EQN, LOG, RA	AL, EQN, NW, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA
12Y	D	AL, UNK, NA	NIL	NIL	NIL	AL, UNK, NA	AL, REL, NA	AR, PTE, SEQ, NA	NIL
12Y	E	NW, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA

Cont'd.		Task A	Task A	Task A	Task A	Task B	Task B	Task B	Task B
Student		Q2	Q3	Q4	Q5	Q2	Q3	Q4	Q5
12Y	F	AR, LOG, RA	AL, EQN, MAN, WA	NIL	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA
12Y	G	AL, REL, NA	AL, EQN, MAN, RA	AR, T/E, SEQ, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	NIL	AL. EQN, MAN, RA
12Y	H	AR, LOG, RA	AR, LOG, WA	NIL	NIL	AL, REL, NA	AL, EQN, MAN, RA replaced with NIL, WA	AR, NA	AL, REL, NA
12Y	I	AL, UNK, NA	AL, REL, NA	NIL	NIL	AL, UNK, NA	AL, EQN, MAN, WA	AR, NA	AL, UNK, NA
12Y	J	AL, EQN, NA	AL, EQN, MAN, WA	NIL	AL, EQN, NW, WA	AL, EQN, MAN, WA	AL, EQN, MAN, RA	NIL	AL, EQN, MAN, NA
12Y	K	AL, EQN, MAN, RA	AL, EQN, RVS, WA	AL, REL, LOG, RA	AL, EQN, LOG, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA
12Y	L	AL, REL, NA	AR, GCI, RND, WA	AR, NA	AR, GC, RAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA
12Y	M	NW, WA	NW, WA	NIL	NIL .	AL, EQN, MAN, WA	AL, EQN, MAN, RA	NIL	AL, EQN, NA
12Y	N	AL, UNK, NA	NIL	NIL	NIL	AL, EQN, MAN, RA	AL, EQN, MAN, RA	NIL	NIL
12Y	O	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA
12Y	P	AR, LOG, RA	AL, UNK, LOG, WA	NIL	AR, NW, RA	AL, REL, NA	AL, EQN, MAN, LOG, WA	AR, NA	AL, REL, EQN, NIL, NA
12Y	Q	AL, EQN, NW, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, NA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA	AL, EQN, MAN, RA

Appendix G2: Responses to Questions 3 on Task A and B

12X	Task A	Task B	12Y	Task A	Task B
Student	Q3	Q3	Student	Q3	Q3
12X A	AL, EQN, MAN, WA	AL, EQN, MAN, NW, WA	12Y A	AR, PTE, SEQ, WA	AL, EQN, MAN, RA
12X B	AL, EQN, MAN, WA	AL, EQN, MAN, RA	12Y B	AR, PTE, GCI, WA	AL, EQN, MAN, RA
12X C	AL, EQN, NW, RA	AL, EQN, MAN, RA	12Y C	AL, EQN, MAN, RA	AL, EQN, MAN, RA
12X D	AL, EQN, MAN, RA	AL, EQN, MAN, RA	12Y D	NIL	AL, REL, NA
12X E	AL, EQN, MAN, RA	AL, EQN, MAN, RA	12Y E	AL, EQN, MAN, RA	AL, EQN, MAN, RA
12X F	NIL	AL, EQN, MAN, RA	12Y F	AL, EQN, MAN, WA	AL, EQN, MAN, RA
12X G	AR, LOG, WA	AL, EQN, MAN, RA	12Y G	AL, EQN, MAN, RA	AL, EQN, MAN, RA
12X H	AL, UNK, LOG, RA	AL, UNK, LOG, RA	12Y H	AR, LOG, WA	AL, EQN, MAN, RA replaced with NIL, WA
12X I	AL, EQN, MAN, WA	AL, EQN, MAN, RA	12Y I	AL, REL, NA	AL, EQN, MAN, WA
12X K	AR. LOG. WA	AL, EQN, MAN, WA	12Y J	AL, EQN, MAN, WA	AL, EQN, MAN, RA
12X L	AL, REL, NA	AL, EQN, MAN, WA	12Y K	AL, EQN, RVS, WA	AL, EQN, MAN, RA
12X J	AR, PTE, RND, NA	AL, EQN, NA	12Y L	AR, GCI, RND, WA	AL, EQN, MAN, RA
12X M	AL, EQN, MAN, RA	AL, EQN, MAN, RA	12Y M	NW, WA	AL, EQN, MAN, RA
			12Y N	NIL	AL, EQN, MAN, RA
			12Y O	AL, EQN, MAN, RA	AL, EQN, MAN, RA
			12Y P	AL, UNK, LOG, WA	AL, EQN, MAN, LOG, WA
			12Y Q	AL, EQN, MAN, RA	AL, EQN, MAN, RA