A graph theoretic proof that Wada's type seven link invariant is determined by the double branched cover

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Abstract

The fundamental group of a link L is a group-valued link invariant that can be defined by assigning a generator to each arc of a link diagram of L, and introducing a relation between them at each crossing. Wada studied what he called shift representations to look for other crossing relations that might define group-valued link invariants. He found seven shift representations, two of which he noted do not define group-valued link invariants. One of the seven defines an infinite family G_m of invariants that includes the fundamental group as G_1, and these have since been shown to distinguish knots up to reflection for $m \geq 2$. Wada showed that three of the remaining four give no new information, leaving just his type seven invariant, which we call W_7. Sakuma showed that the seventh of Wada's shift representations is isomorphic to the free product of $\mathbb{Z}$ and the fundamental group of the double branched cover of L, $\pi_1(\tilde{L}_2)$, that is $W_7(L) \cong \pi_1(\tilde{L}_2) * \mathbb{Z}$. We will use graph theoretic methods to give a new proof of Sakuma's result.

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Chapter 1

Introduction

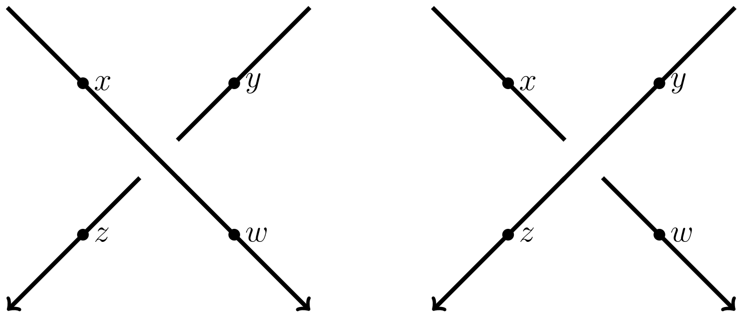
The *fundamental group* is a well-known and important group-valued invariant of links and knots. It can be defined using a diagram of the link by introducing a generator for each strand of the link, and a relation at each crossing, which is discussed more in Section 2. However, although the fundamental group carries a lot of information about the link, it is unable to distinguish between composite knots that differ by replacing a summand with its reflection. For instance, the *granny knot* GK and the *square knot* SK are both connected sums of two trefoil knots. The granny knot GK is the connected sum of a trefoil knot and its reflection, while the square knot SK is the connected sum of two identical trefoil knots, and their fundamental groups are isomorphic. This provides motivation to look for new group-valued link invariants that might carry more information about the link.

With this aim in mind, Wada [11] studied *shift representations* to look for *group-valued link invariants*. He started by introducing a generator for each arc (rather than strand) of the link diagram. Then he looked for words u, v in the free group $F_2 = \langle a, b \rangle$ such that $a \mapsto u, b \mapsto v$, defines an automorphism of F_2, and the crossing relations shown in Figure 1.1 define a group-valued link invariant. Wada used a computer to test all pairs of words (u, v) with $|u + v| < 10$. Up to two natural symmetries, Wada found seven types of shift representations, five of which define link invariants, which we will call W_n for $3 \leq n \leq 7$. It was later shown by Ito [4] that Wada's list of shift representations is complete.

First we will briefly look at each of the shift presentations.

Wada showed that the third type of shift representation, $W_3(L)$ is isomorphic to the free group F_r, where r is the number of components of the link L, and thus it is not a new invariant.

The fourth type of shift representation, W_4, defines an infinite family of invariants with crossing relations $z = x^m y x^{-m}$ for the left crossing, and



(a) $z = u(x, y)$, $w = v(x, y)$. (b) $x = u(z, w)$, $y = v(z, w)$.

Figure 1.1: Left and right crossings with Wirtinger generators.

$z = x^{-m}yx^m$ for the right crossing, as in Figure 1.2. The value $m = 1$ gives the *fundamental group*. For a link L the values $m > 1$ define new invariants called the *generalised link groups* $G_m(L)$, which were introduced independently by Kelly [5]. Tuffley [10] showed that for $m > 1$ $G_m(GK)$ and $G_m(SK)$ are not isomorphic by counting homomorphisms into a suitably chosen finite group, confirming a conjecture of Lin and Nelson [6]. Nelson and Neumann [8] subsequently showed that the generalised knot groups in fact distinguish knots up to reflection. Al Fran and Tuffley [1] extended [8] to show that the difference between the generalised knot groups of certain square and granny knot analogues can be detected by counting homomorphisms into a finite group.

Wada [11] also showed that for a link L

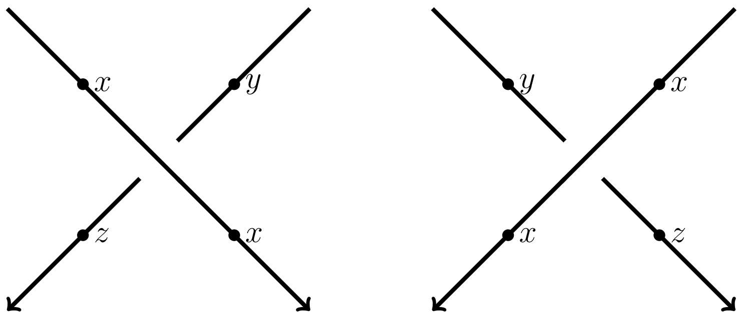
$$W_5(L) \cong \pi_1(\tilde{L}_2) * \mathbb{Z}, \tag{1.1}$$

where $\pi_1(\tilde{L}_2)$ is the fundamental group of the *double branched cover of L*. Hence $W_5(L)$ provides no new invariant. Wada also showed that W_6 reduces to W_5 by conjugation in F_n, and is therefore not a new invariant either. Wada showed that W_6 and W_7 have the same abelianisation, which led Wada to wonder if the above relation is true for $W_7(L)$ as well. However, he reports having done some calculations that indicated they were different. Sakuma [9] showed that in fact the relation does hold for $W_7(L)$ too:

Theorem 1.1 (Sakuma [9]). *Wada's type 7 link invariant is determined by the double branched cover:*

$$W_7(L) \cong \pi_1(\tilde{L}_2) * \mathbb{Z}. \tag{1.2}$$

Sakuma proved the theorem by working directly with the shift representation, and used the π-orbifold [2] of the link with an additional unlinked and unknotted component added. We will work with a Wirtinger presentation for $W_7(L)$ and use graph theoretic methods to give a new proof of the result.



(a) $z = x^m y x^{-m}$. (b) $z = x^{-m} y x^m$.

Figure 1.2: Wirtinger generators and crossing relations for $W_4(L)$.

Chapter 2

Background

We will start with some basic definitions concerning links and knots. For further reading see [7].

Definition 2.1. (Knot) A *knot* K is a finite simple closed polygonal curve in $\mathbb{R}^3$.

Definition 2.2. (Link) A *link* L is a union of one or more disjoint finite simple closed polygonal curves in $\mathbb{R}^3$.

The disjoint curves of a link L are called *components*. Thus a knot is a link with one component. Two links are considered equivalent if space can be continuously deformed in such a way as to carry one link to the other. This is known as *ambient isotopy*. The components of the link being finite polygonal curves restricts us to a class of links called *tame links*, and excludes the class of links called *wild links*, which can have pathological behaviour. For the remainder of this work tame links will simply be called links, and components of links will be assumed to be tame.

Links can also have an orientation, where there is a direction of travel specified around each component of the link. A link with an orientation specified on each component or a knot with orientation specified is called *oriented*.

A way to study links and to differentiate between links is to project them onto a plane. A projection of a link onto a plane gives a *link diagram* as a representation of the link. In order to not lose information when projecting onto the plane, we will use regular projections.

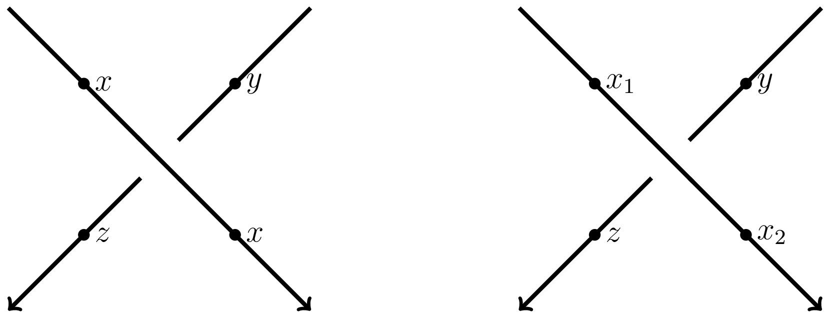
Definition 2.3. (Regular projection) A link projection is *regular* if no three points on the link project to the same point, and no vertex projects to the same point as any other point on the link.

A *link crossing* is a point of the regular projection where the projection of the link crosses itself. Every crossing of a link diagram is the intersection of two edges, and which edge of the link is on top and which is underneath is indicated in the link diagram, as in Figure 2.2. The edge that passes over is shown normally in the link diagram, while the edge that passes under is broken at the crossing, to signify it passes under the other edge. The part of the link that passes underneath is the *undercrossing*, while the part of the link passing over the top is the *overcrossing*.

Definition 2.4. (Strand) A *strand* of a link projection is a section of the link from one undercrossing to another.

Definition 2.5. (Arc) An *arc* of a link projection is a section of the link from one crossing to another.

With these definitions each strand of the link can be split at a crossing into two arcs, as in Figure 2.1, where the strand labelled x is split into the two arcs x_1 and x_2.



(a) Strand labelled left crossing.

(b) Arc labelled left crossing.

Figure 2.1: Left crossing with strands and arcs labelled.

With an oriented link the crossings also have an orientation. Consider Figure 2.2, a figure-eight knot with two left and two right crossings. Crossings are labelled as left or right depending on which strand of the crossing is the over-strand. When oriented with both strands travelling downward, for the *left crossing* the strand that comes from the left is the overcrossing, while for the *right crossing* the strand that comes from the right is the overcrossing.

Next we will introduce the concept of faces of a link diagram.

Definition 2.6. (Faces of a link diagram.) Let L be a link with diagram D. Consider the regular projection of D onto the surface of a sphere. The faces of D are the areas of the surface of the sphere bounded by D.

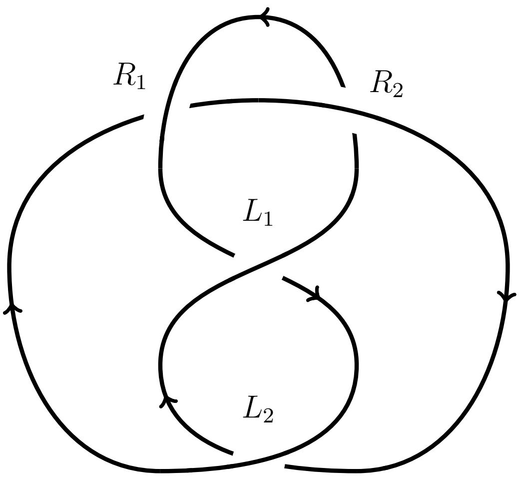


Figure 2.2: A directed figure-eight knot with labelled left and right crossings.

With this definition the outer area of the link diagram is also a face. Figure 2.3 shows the faces of a trefoil knot.

We will also make use of a type of colouring of a link diagram called the *checkerboard colouring*, which we will now define.

Definition 2.7. (Checkerboard colouring.) The checkerboard colouring is a two colouring of a link diagram where faces of the link diagram that meet opposite each other at a crossing are the same colour, and faces that are adjacent at a crossing are opposite colours.

The existence of the checkerboard colouring can be proved by considering the parity of the sum of the winding numbers of the components around a point in the face. Then faces adjacent at a crossing have opposite parities and faces opposite at crossings have the same parities. Figure 2.4 shows how the colours of faces that meet at a crossing relate. For the faces of Figure 2.3, faces two and five would be one colour, and faces one, three and four the other colour.

Now we will introduce the concept of link invariants. A link invariaint is a mathematical object such as a number, polynomial or group that is associated with the link in such a way that the value depends only on the equivalence class of the link. Many link invariants are defined using the link diagram, and for such definitions we need to ensure that the value of the invariant doesn't depend on the choice of diagram used to calculate it. This can be done using the Reidemeister moves. These are three types of deformations that can be done to a diagram that do not change the associated link, shown in Figure 2.5 on page 9. These moves affect an isolated part of the link, and

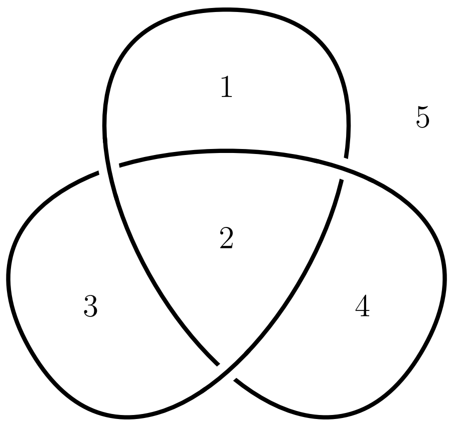


Figure 2.3: A trefoil knot with numbered faces.

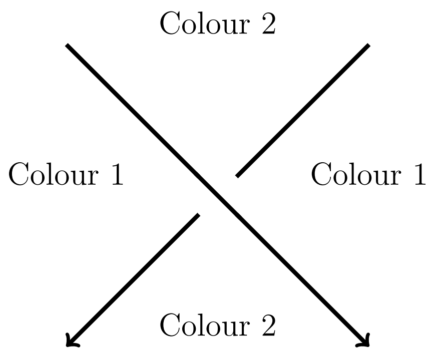


Figure 2.4: The checkerboard colouring of faces at a crossing.

leave the rest of the link unchanged. By Theorem 2.8 these Reidemeister moves are sufficient to change any diagram of a link into any other diagram of the same link, so they be used to check that a given definition of a link invariant doesn't depend on the diagram used.

Theorem 2.8. *Any two link diagrams D_1 and D_2 for a link L are related by a sequence of Reidemeister moves.*

Next we will define the fundamental group.

Definition 2.9. (Fundamental group) The *fundamental group* of a link L is the fundamental group of the link complement $\mathbb{R}^3 - L$, written as $\pi_1(\mathbb{R}^3 - L)$. The elements are homotopy classes of paths in the complement, where paths are considered up to deformation fixing the basepoint, where the operation for the group is concatenation.

The fundamental group as defined in Definition 2.9 is clearly a link invariant. However, for practical purposes we need to be able to calculate it from the link diagram. This is done using the Wirtinger presentation. We associate a *Wirtinger generator* to each strand of the link L, and associate a movement with the generator, as shown in Figure 2.6 on page 10. Each generator represents a path starting from the observer, passing through the plane and around behind a strand before returning out through the plane back towards the observer. The direction of the movement behind the strand is determined by the right hand rule, where the thumb of the right hand is in the direction of the strand and the fingers of the right hand are in the direction of the movement behind the strand, with the palm facing out of the plane.

The relations will come from the crossings, as shown in Figure 2.7 on page 10. The two paths shown at each crossing can be deformed into each other, and thus a relation is formed between their associated Wirtinger generators. The relation for the left crossing with the crossings labelled as in Figure 2.7 is $y_i = y_k y_j y_k^{-1}$, and the relation for the right crossing is $y_i = y_k^{-1} y_j y_k$. These are the only relations between the Wirtinger generators.

Theorem 2.10. *The Wirtinger presentation of a link complement is a group presentation of the fundamental group of the link complement.*

Thus, the fundamental group has a presentation consisting of a generator for each strand of the link, and a relation for each crossing. The Wirtinger presentation depends on the link diagram, and the Reidemeister moves can be used to give an independent proof that the group defined by the Wirtinger presentation is a link invariant.

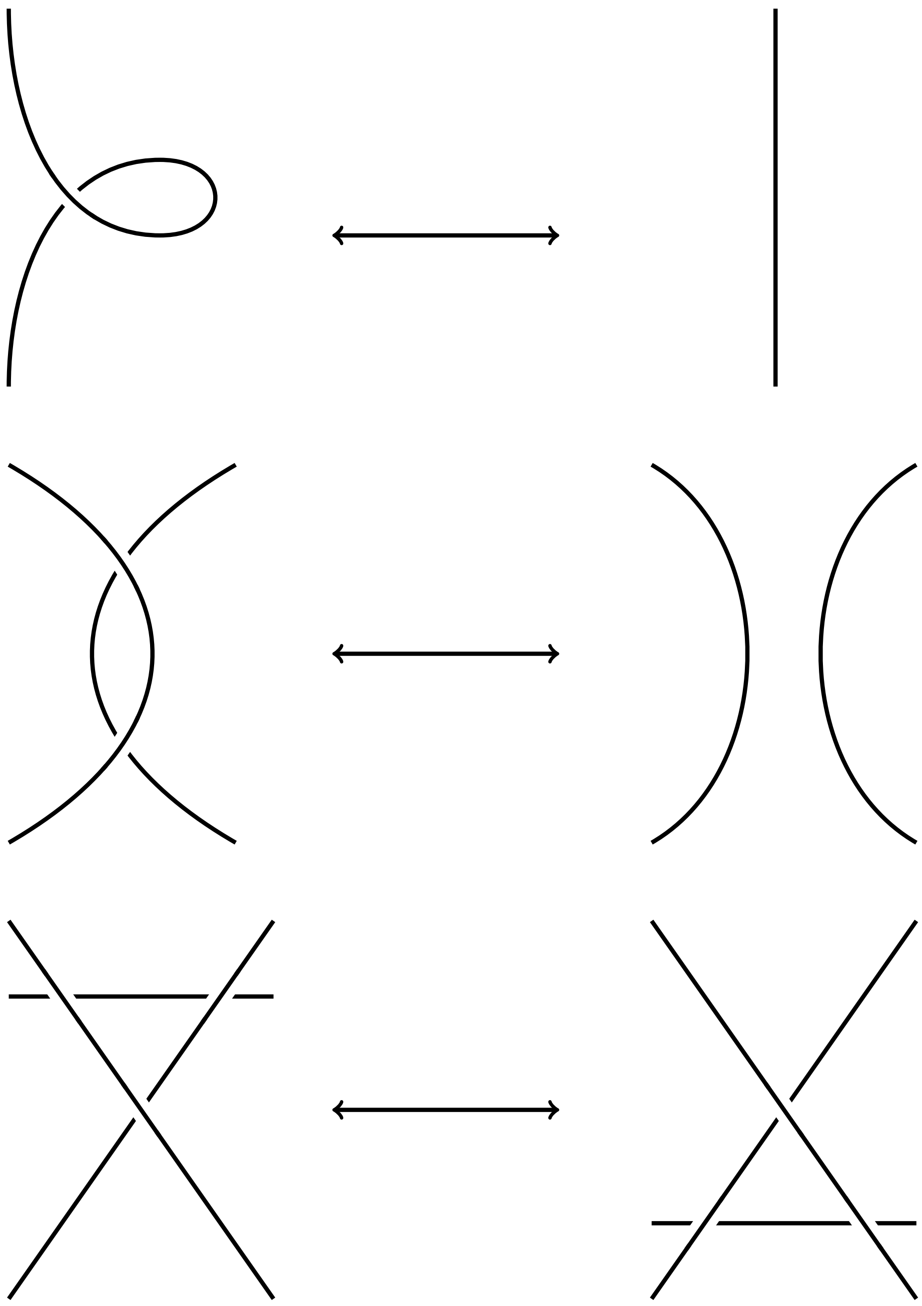
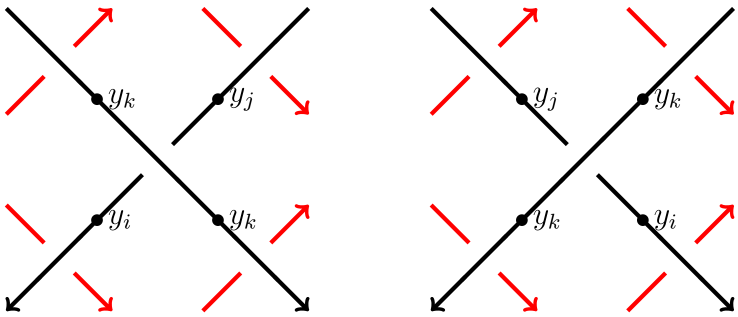
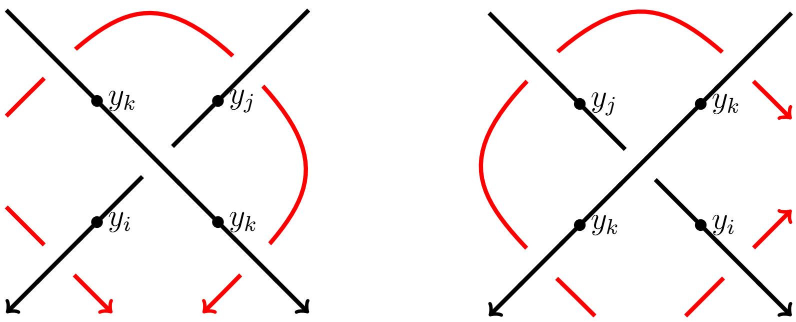


Figure 2.5: Type I - III Reidemeister moves.



(a) Left crossing. (b) Right crossing.

Figure 2.6: Crossings with Wirtinger generators and movements.



(a) Left crossing. (b) Right crossing.

Figure 2.7: Crossings with Wirtinger generators and equivalent movements.

Chapter 3

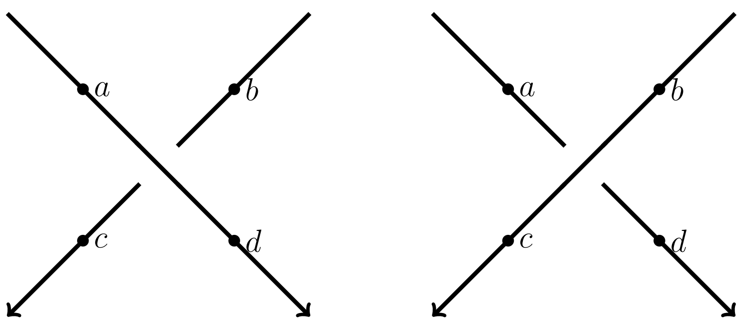
Definition and presentation of $W_7(L)$

3.1 Introduction

We will begin by defining $W_7(L)$. Recall that an arc of a link diagram is a section of the link from one crossing to another

Definition 3.1. ($W_7(L)$) Let L be an oriented link with diagram D. Introduce a Wirtinger generator a_i for each arc of D. Then $W_7(L)$ is the group generated by the a_i, with crossing relations $c = a^2b$, $d = (ab)^{-1}b$ for the left crossing and $a = c^2d$, $b = (cd)^{-1}d$ for the right crossing, with the crossings labelled as in Figure 3.1.

In contrast to the fundamental group, $W_7(L)$ has a generator for each arc of the diagram, rather than each strand. It can be shown using the Reidemeister moves that $W_7(L)$ is well defined.



(a) Left crossing. (b) Right crossing.

Figure 3.1: Crossings with Wirtinger generators.

Note that for the left crossing we have

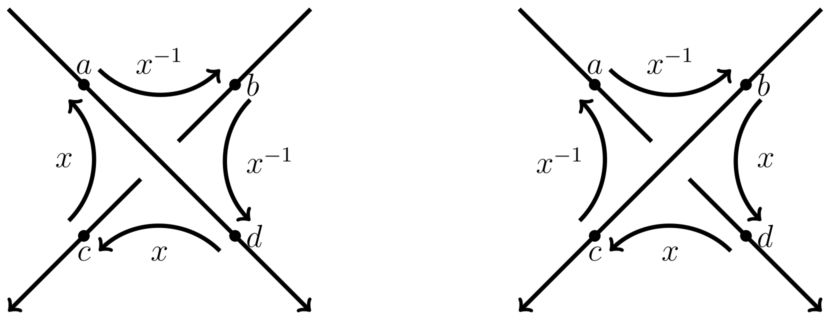
$$cd = a^2b(ab)^{-1}b = ab, \tag{3.1}$$

and for the right crossing we have

$$ab = c^2d(cd)^{-1}d = cd. \tag{3.2}$$

At each crossing we introduce the *crossing generator* $x = ab = cd$. This allows the relations to be rewritten as $c = ax, d = x^{-1}b$ for the left crossing and $a = cx, b = x^{-1}d$ for the right crossing.

The relations at a crossing can be considered as equivalent to movements along the crossing from a Wirtinger generator on one of the arcs to a Wirtinger generator on the other arc. For example, with the labellings as in Figure 3.2, the relation $d = x^{-1}b$ can be viewed as moving along the crossing from b to d, where b and d are Wirtinger generators on two different arcs connected at a crossing with generator x.



(a) Left crossing with clockwise movement.

(b) Right crossing with clockwise movement.

Figure 3.2: Crossings with crossing generator signs for the indicated movements.

For consistency, expressions will be written so the right hand side contains the crossing generator and the Wirtinger generator started at, while the left hand side contains the Wirtinger generator ended at. The exponent of each Wirtinger generator is determined by the direction travelled around the crossing, relative to the direction of the arc containing that generator. If the movement is in the same direction as an arc, the exponent of the Wirtinger generator for that arc is $+1$. If the movement is opposite the direction of an arc, the exponent of the Wirtinger generator for that arc is -1. For the left crossing relation $d = x^{-1}b$, moving from b to d is moving in the same direction as the arc containing b and then moving in the same direction as the arc containing d, hence their exponents are both $+1$.

With the manner of writing the expressions as stated above, and the centre of the clock in the centre of the crossing, travelling clockwise puts the crossing generator on the left of the Wirtinger generator in the right hand side expression. Travelling anticlockwise puts the crossing generator on the right of the Wirtinger generator in the right hand side expression. With the labellings as in Figure 3.2, the relations between the Wirtinger generators at the crossing are:

Left crossing:

Clockwise:

$$d = x^{-1}b, \qquad b^{-1} = x^{-1}a, \qquad a^{-1} = xc^{-1}, \qquad c = xd^{-1}.$$

Anticlockwise:

$$b^{-1} = d^{-1}x^{-1}, \qquad a^{-1} = bx^{-1}, \qquad c = ax, \qquad d = c^{-1}x.$$

Right crossing:

Clockwise:

$$d = xb, \qquad b^{-1} = x^{-1}a, \qquad a^{-1} = x^{-1}c^{-1}, \qquad c = xd^{-1}.$$

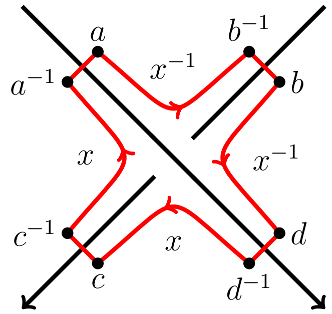
Anticlockwise:

$$b^{-1} = d^{-1}x, \qquad a^{-1} = bx^{-1}, \qquad c = ax^{-1}, \qquad d = c^{-1}x.$$

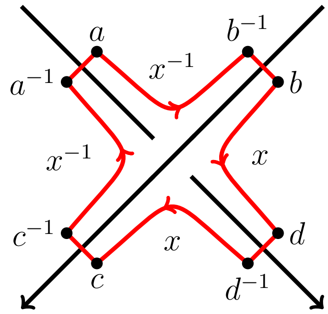
3.2 Graphical representation of the presentation

The idea of movement around a crossing and movement along the link diagram motivates building a properly defined graphical representation for the link to understand the relations coming from the link. In fact we can define a planar graph representation of the link directly from a link diagram, and apply graph theory knowledge to derive some results about relations coming from the link.

Definition 3.2. (Graph of $W_7(L)$) Let L be an oriented link with a given link diagram D. We can then associate an edge labelled directed graph with D, called $\mathcal{G}(D)$, in the following way. For each arc of D we place a vertex on each side of the arc, such that if a_i is the Wirtinger generator for the arc, a vertex labelled a_i is on the left of the arc relative to the direction of the arc in the link diagram and a vertex labelled a_i^{-1} is on the right of the arc. These will be the vertices of the graph, with two types of edges:



(a) Left crossing with graph vertices, edges and crossing generators.



(b) Right crossing with graph vertices, edges and crossing generators.

Figure 3.3: Graph expression of a left and right crossing.

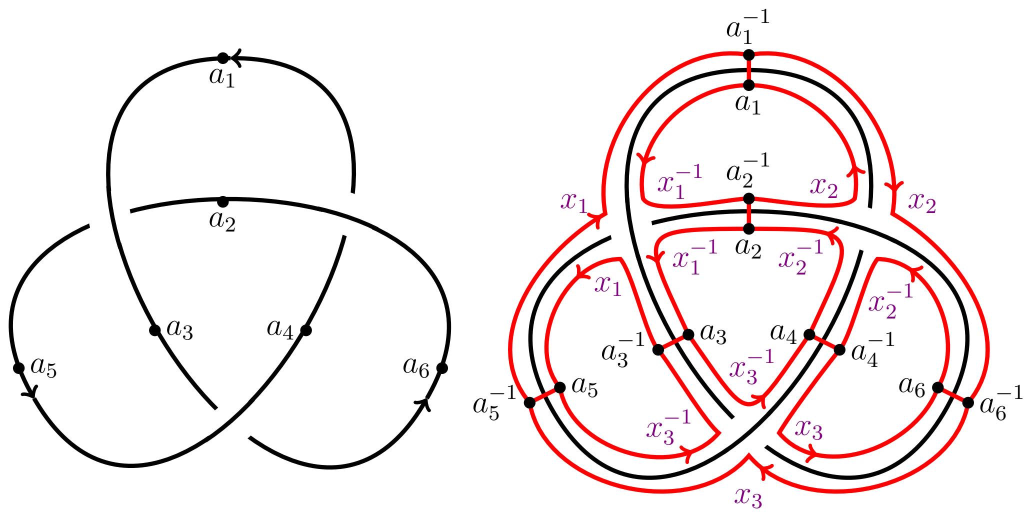
1. an edge from a_i^{-1} to a_i for all i,
2. a directed edge joining each pair of vertices in the same quadrant of each crossing, for all crossings in D. The direction of these edges is clockwise, where the centre of the clock is the centre of the crossing.

The two types of edges, with $\alpha_i, \alpha_j, \varepsilon \in \pm 1$, encode relations in the following way:

1. Edges from $a_i^{\alpha_i}$ to $a_i^{-\alpha_i}$ have the trivial relation $a_i^{-\alpha_i} = (a_i^{\alpha_i})^{-1}$ for each i.
2. Edges from $a_i^{\alpha_i}$ to $a_j^{\alpha_j}$ for $i \neq j$ encode a relation of the form $a_j^{\alpha_i} = x^\varepsilon a_i^{\alpha_i}$, where x is the crossing generator of the crossing where the arc containing a_i and the arc containing a_j meet.

Figure 3.3 shows the graph representation of the left and right crossings from Figure 3.2, with labelled vertices and the signed crossing generator associated with each edge.

Figure 3.4 shows the graph representation for the trefoil knot, with vertices labelled in black and crossing generators labelled in purple.



(a) Trefoil knot with Wirtinger generators.

(b) Graph representation of trefoil knot.

Figure 3.4: Graphical expression of trefoil knot.

3.3 Words and relations from walks in $\mathcal{G}(D)$

Now we have defined a graphical representation of a link diagram for $W_7(L)$, we will consider the relations that arise from walks in $\mathcal{G}(D)$. Let L be a link with connected diagram D. Then walks in $\mathcal{G}(D)$ will form relations involving the Wirtinger generators associated with the beginning and ending vertices, and the signed crossing generators associated with each of the edges in the walks.

By construction of $\mathcal{G}(D)$, a_i and a_i^{-1} are on opposite coloured faces of the checkerboard colouring of D for all i, so moving from one coloured face to another will invert the current relation, and the crossing generators that were on the left of the Wirtinger generator in the relation will move to the right of it. This means the crossing generators associated with the edges of a walk in $\mathcal{G}(D)$ can appear on both the right and left of the Wirtinger generator in the right hand side expression. If the signed crossing generator associated with the directed edge from vertex a_i to vertex a_j is x_k, the relation associated with the edge walk from a_i to a_j is $a_j = x_k a_i$. Then the relation associated with travelling in the opposite direction along this edge from a_j to a_i is $a_i = x_k^{-1}a_j$. This means the sequence of signed crossing generators associated with a walk in $\mathcal{G}(D)$ accounts for the direction travelled along each edge, as well as the order of crossing generators encountered.

Lemma 3.3 shows the relation formed from a walk in $\mathcal{G}(D)$.

Lemma 3.3. *Let L be a link, and let W be a walk in $\mathcal{G}(D)$ from $a_i^{\alpha_i}$ to $a_j^{\alpha_j}$ where $\alpha_i, \alpha_j \in \{1, -1\}$. Let $x_{l_1}^{\varepsilon_1}, x_{l_2}^{\varepsilon_2}, \ldots, x_{l_s}^{\varepsilon_s}$ be the sequence of signed crossing generators encountered along W on faces of the same checkerboard colouring colour as $a_i^{\alpha_i}$, and $x_{r_1}^{\eta_1}, x_{r_2}^{\eta_2}, \ldots, x_{r_t}^{\eta_t}$ the sequence of signed crossing generators encountered along W on faces of the opposite colour to $a_i^{\alpha_i}$, where $\varepsilon_k, \eta_k \in \{1, -1\}$ for each k. Then the relation formed from the walk W is*

$$a_j^{\alpha_j} = [x_{l_s}^{\varepsilon_s} \cdots x_{l_2}^{\varepsilon_2} x_{l_1}^{\varepsilon_1} a_i^{\alpha_i} x_{r_1}^{-\eta_1} x_{r_2}^{-\eta_2} \cdots x_{r_t}^{-\eta_t}]^{\beta}, \tag{3.3}$$

where $\beta = +1$ if $a_j^{\alpha_j}$ is on a face of the same colour as $a_i^{\alpha_i}$, and $\beta = -1$ if $a_j^{\alpha_j}$ is on a face of the opposite colour.

Proof. Consider a walk W of length n from $a_i^{\alpha_i}$ to $a_j^{\alpha_j}$. For $n = 0$, the walk is trivial, $j = i$, no crossing generators are encountered and the relation is $a_i^{\alpha_i} = a_i^{\alpha_i}$. Assume the hypothesis is true for walks of length $n = 0, 1, 2, \ldots, k$. Consider a walk W of length $n = k + 1$ from $a_i^{\alpha_i}$ to $a_j^{\alpha_j}$. Let W' be the walk of length k obtained from W by excluding the last edge. Let $a_u^{\alpha_u}$ be the last vertex of W'. Let $x_{l_1}^{\varepsilon_1}, x_{l_2}^{\varepsilon_2}, \ldots, x_{l_{t'}}^{\varepsilon_{t'}}$ be the sequence of signed crossing generators encountered on faces of the same colour as $a_i^{\alpha_i}$ and $x_{r_1}^{\eta_1}, x_{r_2}^{\eta_2}, \ldots, x_{r_{s'}}^{\eta_{s'}}$ the sequence of signed crossing generators encountered on faces of the opposite colour to $a_i^{\alpha_i}$, where $s' = s$ or $s - 1$, $t' = t$ or $t - 1$. By the inductive hypothesis:

$$a_u^{\alpha_u} = [x_{l_{s'}}^{\varepsilon_{s'}} \cdots x_{l_2}^{\varepsilon_2} x_{l_1}^{\varepsilon_1} a_i^{\alpha_i} x_{r_1}^{-\eta_1} x_{r_2}^{-\eta_2} \cdots x_{r_{t'}}^{-\eta_{t'}}]^{\gamma}, \tag{3.4}$$

where $\gamma = +1$ if $a_u^{\alpha_u}$ is on a face of the same colour as $a_i^{\alpha_i}$, and $\gamma = -1$ if $a_u^{\alpha_u}$ is on a face of the opposite colour to $a_i^{\alpha_i}$. There are two cases for the last edge.

1. The last edge of the walk is from $a_u^{\alpha_u}$ to $a_u^{-\alpha_u}$, where $a_u^{-\alpha_u}$ is on an opposite coloured face to $a_u^{\alpha_u}$ by construction of the graph of the link. Then $a_j^{\alpha_j} = a_u^{-\alpha_u}$, $\beta = -\gamma$, $s = s'$, $t = t'$ and

$$\begin{aligned} a_j^{\alpha_j} = a_u^{-\alpha_u} &= [x_{l_{s'}}^{\varepsilon_{s'}} \cdots x_{l_2}^{\varepsilon_2} x_{l_1}^{\varepsilon_1} a_i^{\alpha_i} x_{r_1}^{-\eta_1} x_{r_2}^{-\eta_2} \cdots x_{r_{t'}}^{-\eta_{t'}}]^{-\gamma} \\ &= [x_{l_s}^{\varepsilon_s} \cdots x_{l_2}^{\varepsilon_2} x_{l_1}^{\varepsilon_1} a_i^{\alpha_i} x_{r_1}^{-\eta_1} x_{r_2}^{-\eta_2} \cdots x_{r_t}^{-\eta_t}]^{\beta}, \end{aligned} \tag{3.5}$$

as required.

2. The last edge is not from $a_u^{\alpha_u}$ to $a_u^{-\alpha_u}$, so the expression for the last edge is of the form $a_j^{\alpha_j} = x_m^{\delta} a_u^{\alpha_u}$ and $a_j^{\alpha_j}$ is on the same face as $a_u^{\alpha_u}$. It follows that $\beta = \gamma$, and x_m^{δ} is the next crossing generator in the sequence of signed crossing generators encountered on faces of the same colour as $a_j^{\alpha_j}$. The expression for W is

$$a_j^{\alpha_j} = x_m^{\delta}[x_{l_{s'}}^{\varepsilon_{s'}} \cdots x_{l_2}^{\varepsilon_2} x_{l_1}^{\varepsilon_1} a_i^{\alpha_i} x_{r_1}^{-\eta_1} x_{r_2}^{-\eta_2} \cdots x_{r_{t'}}^{-\eta_{t'}}]^{\gamma}. \tag{3.6}$$

If $\gamma = +1$ then $a_u^{\alpha_u}$ and consequently $a_j^{\alpha_j}$ are on a face of the same colour as $a_i^{\alpha_i}$. Then $\beta = +1$, $t = t'$, $s = s' + 1$, $l_s = m$ and $\varepsilon_s = \delta$. The expression for W is

$$\begin{aligned} a_j^{\alpha_j} &= [x_m^{\delta} x_{l_{s'}}^{\varepsilon_{s'}} \cdots x_{l_2}^{\varepsilon_2} x_{l_1}^{\varepsilon_1} a_i^{\alpha_i} x_{r_1}^{-\eta_1} x_{r_2}^{-\eta_2} \cdots x_{r_{t'}}^{-\eta_{t'}}]^1 \\ &= [x_{l_s}^{\varepsilon_s} \cdots x_{l_2}^{\varepsilon_2} x_{l_1}^{\varepsilon_1} a_i^{\alpha_i} x_{r_1}^{-\eta_1} x_{r_2}^{-\eta_2} \cdots x_{r_t}^{-\eta_t}]^{\beta}. \end{aligned} \tag{3.7}$$

If $\gamma = -1$ then $a_u^{\alpha_u}$ and consequently $a_j^{\alpha_j}$ are on a face of the opposite colour to $a_i^{\alpha_i}$. Then x_m^{δ} is the next crossing generator in the sequence of signed crossing generators encountered on faces of the opposite colour to $a_i^{\alpha_i}$, $\beta = -1$, $s = s'$, $t = t' + 1$, $r_t = m$ and $\eta_t = \delta$. The expression for W is

$$\begin{aligned} a_j^{\alpha_j} &= x_m^{\delta}[x_{l_{s'}}^{\varepsilon_{s'}} \cdots x_{l_2}^{\varepsilon_2} x_{l_1}^{\varepsilon_1} a_i^{\alpha_i} x_{r_1}^{-\eta_1} x_{r_2}^{-\eta_2} \cdots x_{r_{t'}}^{-\eta_{t'}}]^{-1} \\ &= [x_{l_{s'}}^{\varepsilon_{s'}} \cdots x_{l_2}^{\varepsilon_2} x_{l_1}^{\varepsilon_1} a_i^{\alpha_i} x_{r_1}^{-\eta_1} x_{r_2}^{-\eta_2} \cdots x_{r_{t'}}^{-\eta_{t'}} x_m^{-\delta}]^{-1} \\ &= [x_{l_s}^{\varepsilon_s} \cdots x_{l_2}^{\varepsilon_2} x_{l_1}^{\varepsilon_1} a_i^{\alpha_i} x_{r_1}^{-\eta_1} x_{r_2}^{-\eta_2} \cdots x_{r_t}^{-\eta_t}]^{\beta}. \end{aligned} \tag{3.8}$$

□

Since we can form a walk from a single Wirtinger generator in $\mathcal{G}(D)$ to any other Wirtinger generator on the same component of D, Lemma 3.3 tells us we can express the Wirtinger generators of a component in terms of a single Wirtinger generator in that component, and the crossing generators that appear in that component. Then if the diagram D for L is connected we can reduce to a single Wirtinger generator. Lemma 3.4 shows us we can always form a connected diagram for a link L.

Lemma 3.4. *Every link L has a connected diagram D.*

Proof. Let L be link, and D a diagram for L with as few connected components as possible. If D is connected we are done. If D is not connected, it has at least two components, D_1 and D_2. We then perform a type 2 Reidemeister on an arc of each of D_1 andD_2 to connect them, as in Figure 3.5. Now D_1 and D_2 are a single component, and this new diagram D' has fewer components then D. This contradicts the choice of D, so D must in fact be connected. □

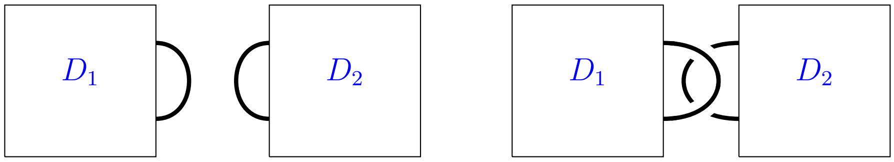


Figure 3.5: Two components joined by a type 2 Reidemeister move.

Since we can form a connected diagram D for a given link L, we can express every Wirtinger generator in $W_7(L)$ in terms of the crossing generators and a single Wirtinger generator $a_i^{\alpha_i}$. This gives the following corollary:

Corollary 3.5. *Let L be a link and let D be a connected diagram for L. Then $W_7(L)$ has a presentation that is generated by any single Wirtinger generator and the crossing generators.*

3.4 Faces and their relations in $\mathcal{G}(D)$

We will begin by introducing the notion of a face walk:

Definition 3.6. (Face walk) Let L be a link with diagram D. A *face walk* is a non-trivial simple closed walk in $\mathcal{G}(D)$ that involves no edge from $a_i^{\alpha_i}$ to $a_i^{-\alpha_i}$ for any i. Such a walk necessarily travels around a face F of the link diagram.

Figure 3.6 shows the five face walks of the trefoil knot in red.

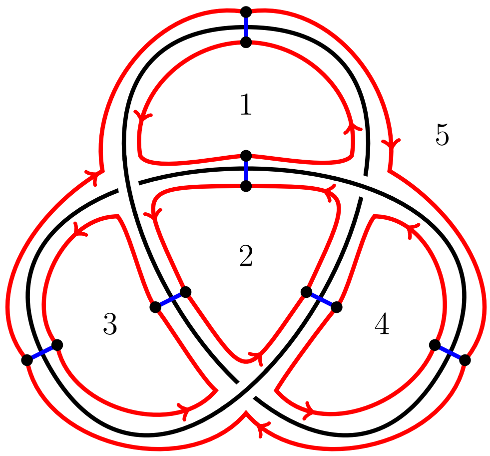


Figure 3.6: Trefoil knot with numbered faces.

Every face walk is a walk from some $a_{j_1}^{\alpha_1}$ to $a_{j_1}^{\alpha_1}$ which by Lemma 3.3 will give a relation of the form

$$a_{j_1}^{\alpha_1} = x_{i_s}^{\varepsilon_s} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} a_{j_1}^{\alpha_1}, \tag{3.9}$$

where the crossing generators all appear on the left of the Wirtinger generator since all edges in the walk are on the same face. This relation can be simplified to

$$x_{i_s}^{\varepsilon_s} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} = 1, \tag{3.10}$$

or $f_j = 1$, where the word f_j is a product of crossing generators,

$$f_j = x_{i_s}^{\varepsilon_s} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1}. \tag{3.11}$$

We call the relation $f_j = 1$, the *face relation* associated with the face F_j of the link diagram.

What vertex is chosen for the start and hence end of a face walk does not provide a different face relation for any face walk. Instead, it produces a cyclic permutation of the crossing generators and thus a cyclic permutation of $f_j = 1$.

Lemma 3.7. *Let W be a simple walk around a face F, in a given direction, starting and ending at $a_{j_1}^{\alpha_1}$, with relation*

$$a_{j_1}^{\alpha_1} = x_{i_s}^{\varepsilon_s} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} a_{j_1}^{\alpha_1}. \tag{3.12}$$

Then

$$f = x_{i_s}^{\varepsilon_s} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} = 1 \tag{3.13}$$

is the face relation for F. Let W' be a walk on F, in the same direction around F as W, with face relation f', starting at $a_{j_2}^{\alpha_2}$, where $a_{j_2}^{\alpha_2}$ is r steps around W from $a_{j_1}^{\alpha_1}$. Then f' is a cyclic permutation of f, so $f' = 1$ follows from $f = 1$.

Proof. As $a_{j_2}^{\alpha_2}$ is r edges from $a_{j_1}^{\alpha_1}$ around W, we have

$$\begin{aligned} a_{j_2}^{\alpha_2} &= x_{i_r}^{\varepsilon_r} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} a_{j_1}^{\alpha_1} \\ &= v a_{j_1}^{\alpha_1}, \end{aligned} \tag{3.14}$$

where $v = x_{i_r}^{\varepsilon_r} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1}$. Then W' encounters the sequence of signed crossing generators

$$x_{i_{r+1}}^{\varepsilon_{r+1}}, \ldots, x_{i_s}^{\varepsilon_s}, x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}, \ldots, x_{i_r}^{\varepsilon_r}, \tag{3.15}$$

so the relation for W' is

$$a_{j_2}^{\alpha_2} = x_{i_r}^{\varepsilon_r} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} x_{i_s}^{\varepsilon_s} \cdots x_{i_{r+2}}^{\varepsilon_{r+2}} x_{i_{r+1}}^{\varepsilon_{r+1}} a_{j_2}^{\alpha_2}. \tag{3.16}$$

And the word related to f' is

$$f' = x_{i_r}^{\varepsilon_r} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} x_{i_s}^{\varepsilon_s} \cdots x_{i_{r+2}}^{\varepsilon_{r+2}} x_{i_{r+1}}^{\varepsilon_{r+1}}. \tag{3.17}$$

Since f' is a cyclic permutation of f, f' is conjugate to f. We see

$$\begin{aligned} vfv^{-1} &= x_{i_r}^{\varepsilon_r} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} (x_{i_s}^{\varepsilon_s} \cdots x_{i_{r+1}}^{\varepsilon_{r+1}} x_{i_r}^{\varepsilon_r} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1})(x_{i_r}^{\varepsilon_r} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1})^{-1} \\ &= x_{i_r}^{\varepsilon_r} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} x_{i_s}^{\varepsilon_s} \cdots x_{i_{r+2}}^{\varepsilon_{r+2}} x_{i_{r+1}}^{\varepsilon_{r+1}} = f'. \end{aligned} \tag{3.18}$$

Then since $f = 1$, it follows that $f' = 1$. □

If the face walk is done in the opposite direction the relation formed will be $a_{j_1}^{\alpha_1} = f^{-1} a_{j_1}^{\alpha_1}$ which reduces to $f^{-1} = 1$. Thus, changing the direction of the face walk does not provide a new relation.

Lemma 3.8. *Let W be a simple walk around a face F, in a given direction, starting and ending at $a_{j_1}^{\alpha_1}$ with relation*

$$a_{j_1}^{\alpha_1} = x_{i_s}^{\varepsilon_s} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} a_{j_1}^{\alpha_1}. \tag{3.19}$$

Let

$$f = x_{i_s}^{\varepsilon_s} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} = 1. \tag{3.20}$$

Let W' be the simple walk around F starting and ending at $a_{j_1}^{\alpha_1}$, in the opposite direction to W. Then the relation for W' is

$$a_{j_1}^{\alpha_1} = x_{i_1}^{-\varepsilon_1} x_{i_2}^{-\varepsilon_2} \cdots x_{s+1}^{-\varepsilon_{s+1}} x_s^{-\varepsilon_s} a_i^{\alpha_i}. \tag{3.21}$$

Let

$$f' = x_{i_1}^{-\varepsilon_1} x_{i_2}^{-\varepsilon_2} \cdots x_{s+1}^{-\varepsilon_{s+1}} x_s^{-\varepsilon_s}. \tag{3.22}$$

Then $f' = f^{-1}$, and since $f = 1$, $f' = 1$ follows from $f = 1$ and does not provide a new relation.

Proof. Let W' be the simple walk around F starting and ending at $a_i^{\alpha_i}$, in the opposite direction to W. The walk W' encounters the same crossing generators as W but in the opposite order, and since W' crosses the edges in the opposite direction the signs of the crossing generators will be reversed. Therefore the relation for W' is

$$a_{j_1}^{\alpha_1} = x_{i_1}^{-\varepsilon_1} x_{i_2}^{-\varepsilon_2} \cdots x_{i_{s+1}}^{-\varepsilon_{s+1}} x_{i_s}^{-\varepsilon_s} a_{j_1}^{\alpha_1} = f' a_{j_1}^{\alpha_1}, \tag{3.23}$$

so

$$\begin{aligned} f' &= x_{i_1}^{-\varepsilon_1} x_{i_2}^{-\varepsilon_2} \cdots x_{i_{s+1}}^{-\varepsilon_{s+1}} x_s^{-\varepsilon_s} \\ &= (x_{i_s}^{\varepsilon_s} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1})^{-1} = f^{-1}. \end{aligned} \tag{3.24}$$

Then since $f = 1$, $f' = 1$ does not provide any new information. □

There are faces of $\mathcal{G}(D)$ that are not faces of D. These are faces that surround a crossing of the link diagram, as in Figure 3.7. These faces have associated simple closed walks that are not face walks, but do not provide new relations, as shown in Lemma 3.9 below:

Lemma 3.9. *(Cross face walk) The relation from a walk around a crossing, a cross face walk relation, is trivial, and so provides no new information.*

Proof. Consider the crossings as in Figure 3.7. There are two cases, according to whether the crossing is a left or right crossing.

1. The walk $W = a, b^{-1}, b, d, d^{-1}, c, c^{-1}, a^{-1}, a$ around the left crossing is a simple closed walk around a face of $\mathcal{G}(D)$ that is not a face walk. The relations for each edge of this walk are:

$$b^{-1} = x^{-1}a, \qquad d = x^{-1}b, \qquad c = xd^{-1}, \qquad a^{-1} = xc^{-1}.$$

Substitution gives:

$$b^{-1} = x^{-1}(cx^{-1}) = x^{-1}(xd^{-1})x^{-1} = x^{-1}x(b^{-1}x)x^{-1} = (x^{-1}x)b^{-1}(xx^{-1}) = b^{-1},$$

and the relation from W provides no new informationl.

2. The walk $W' = a, b^{-1}, b, d, d^{-1}, c, c^{-1}, a^{-1}, a$ around the right crossing is a simple closed walk around a face of $\mathcal{G}(D)$ that is not a face walk. The relations for each edge of this walk are:

$$b^{-1} = x^{-1}a, \qquad d = xb, \qquad c = xd^{-1}, \qquad a^{-1} = x^{-1}c^{-1}.$$

Substitution gives

$$b^{-1} = x^{-1}(cx) = x^{-1}(xd^{-1})x = x^{-1}x(b^{-1}x^{-1})x = (x^{-1}x)b^{-1}(xx^{-1}) = b^{-1},$$

and the relation from W' provides no new information.

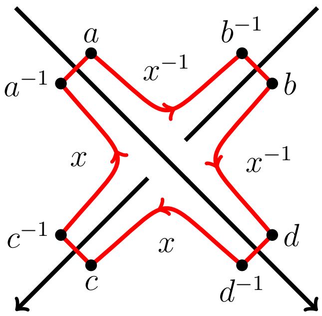
□

Corollary 3.5 tells us that we only need one Wirtinger generator and the crossing generators for a presentation of $W_7(L)$. The next step will be to prove that we can reduce the relations of the presentation to the face relations, $f_i = 1$ for all i. This will give us Theorem 3.10:

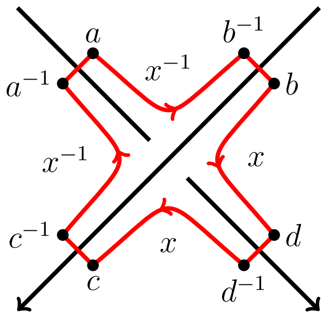
Theorem 3.10. *Let L be a link with connected diagram D. Then*

$$W_7(L) \cong \langle a_p, x_1, x_2, \ldots, x_k | f_1, f_2, \ldots f_t \rangle, \tag{3.25}$$

where a_p is the single Wirtinger generator, $x_1, x_2, \ldots, x_k$ are the crossing generators and $f_1, f_2, \ldots f_t$ the face relations.



(a) Left crossing with graph vertices, edges and crossing generators.



(b) Right crossing with graph vertices, edges and crossing generators.

Figure 3.7: Graph expression of a left and right crossing.

Chapter 4

Graph theory preliminaries

We need a way to decompose more complex walks to show that their relations can be derived from the face relations. We will start by defining a type of walk that we will use in our decomposition.

Definition 4.1. (Lollipop) Let G be a graph drawn on a plane. A *lollipop* is a walk on G that consists of a simple walk, called the *stick*, followed by a simple closed walk, called the *head*, followed by the reverse of the initial simple walk.

A simple lollipop is a lollipop where the head is a walk around a face of the graph. Figure 4.1 shows an example of a non-simple lollipop and its simple lollipop decomposition.

Figure 4.2 shows an example of a lollipop and a simple lollipop on the trefoil knot. Lollipops will provide a way to decompose large closed walks into simpler walks to understand their relations.

We will decompose walks into simpler ones using the notion of *homotopy*, which we define as follows:

Definition 4.2. (Homotopic) Two walks W and W' on a graph G are *homotopic* if one can be obtained from the other by a series of insertions and/or deletions of cancelling edge pairs.

Definition 4.3. (Homotopically decomposed) A closed walk on a graph G can be *homotopically decomposed* if it is homotopic to a product of lollipops.

Definition 4.4. (Reduced) A walk on a planar graph G is *reduced* if it contains no cancelling edge pairs.

Then from the definition of homotopic, any non-reduced walk is homotopic to a reduced walk.

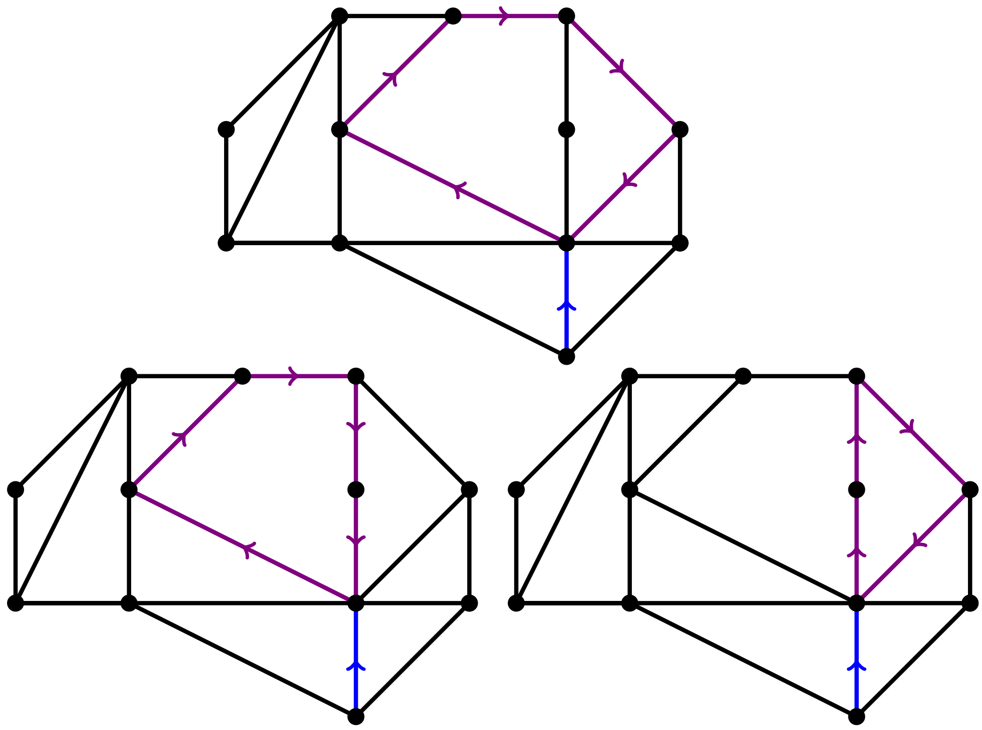
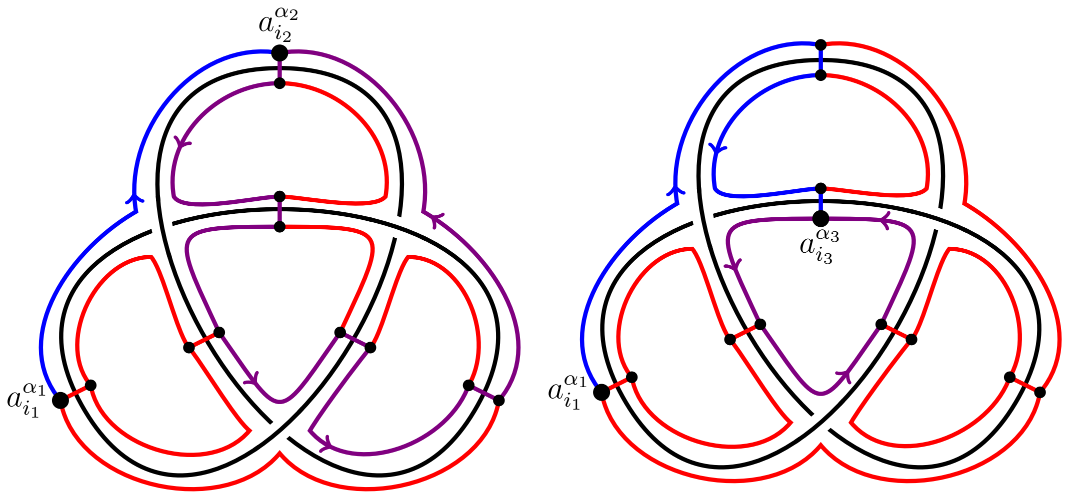


Figure 4.1: Non-simple lollipop and its simple lollipop decomposition, with the sticks of the lollipops shown in blue and the heads of the lollipops shown in purple.

Lemma 4.5. *Every closed walk in a planar graph G can be decomposed homotopically into a product of lollipops.*

Proof. Let T be a spanning tree for a planar graph G. Let W be a closed walk on G, and let n be the number of edges in W not in T. For $n = 1$, we can write W as UeV, where U is the walk in T from the start of W to the first vertex of e, and V is the walk in T from the second vertex of e back to the start of W. U and V can be split up as $U = gU'$, $V = V'g^{-1}$, where g is the walk in T from the start of W to the first point at which V meets U. It's possible that g might be empty. Then g will be the stick of the lollipop, and $U'eV'$ is the head.

Assume the hypothesis is true for $n = 1, 2, 3, \ldots, k$. Let $n = k + 1$. Then we can decompose W into lollipops in the following way: walk W until we reach the first edge e of W not in T. Walk e, then follow the spanning tree back to the beginning of W. Then retrace back to the second vertex of e and resume W. This inserts a cancelling path in W, so the new walk is homotopic



(a) A non-simple lollipop with stick from $a_{i_1}^{\alpha_1}$ to $a_{i_2}^{\alpha_2}$.

(b) A simple lollipop with stick from $a_{i_1}^{\alpha_1}$ to $a_{i_3}^{\alpha_3}$.

Figure 4.2: A simple and non-simple lollipop walk on the trefoil knot, where the sticks are shown in blue and the heads are shown in purple.

to W, and it can be split up as w_1w_2, where w_1 has one edge not in T, and w_2 has k edges not in T. By the inductive hypothesis, w_1, w_2 and consequently w_1w_2 can be homotopically decomposed into a product of lollipops, and thus so can W, since it is homotopic to w_1w_2. □

Lemma 4.6. *Any lollipop on a planar graph G can be homotopically decomposed into the product of simple lollipops.*

Proof. Let G be a planar graph. Consider a lollipop P on G whose head is made up of n faces of G. If $n = 1$, P is a simple lollipop and is decomposed. Assume the hypothsis is true for $n = 1, 2, 3, \dots, k$. Let P have $n = k + 1$ faces of G. Then there is a walk W that connects two vertices on the head of P, and divides the region enclosed by the head of P into two regions, H_1 and H_2. Let g be the walk for the stick of P, and let W be from $a_j^{\alpha_j}$ to $a_l^{\alpha_l}$. Let $a_i^{\alpha_i}$ be the vertex where the stick of P meets the head of P. Let P_1 be the walk on P from $a_i^{\alpha_i}$ to $a_j^{\alpha_j}$, P_2 the walk from $a_j^{\alpha_j}$ to $a_l^{\alpha_l}$ on P, and P_3 the walk on P from $a_l^{\alpha_l}$ to $a_i^{\alpha_i}$, as shown in Figure 4.3. P is homotopic to the walk $gP_1WP_3g^{-1}gP_3^{-1}W^{-1}P_2P_3g^{-1}$. This walk is the product of two lollipops, $gP_1WP_3g^{-1}$ and $gP_3^{-1}W^{-1}P_2P_3g^{-1}$, so P can be homotopically decomposed into a product of two lollipops.

Let d_1 be the number of faces of G in H_1, and d_2 the number of faces of G in H_2. Then $d_1 + d_2 = k + 1$, and $d_1, d_2 \geq 1$, so $d_1, d_2 < k + 1$ and by the inductive hypothesis the two new lollipops can be decomposed into simple lollipops, and hence the lollipop with $k + 1$ faces of G can be homotopically decomposed into simple lollipops. □

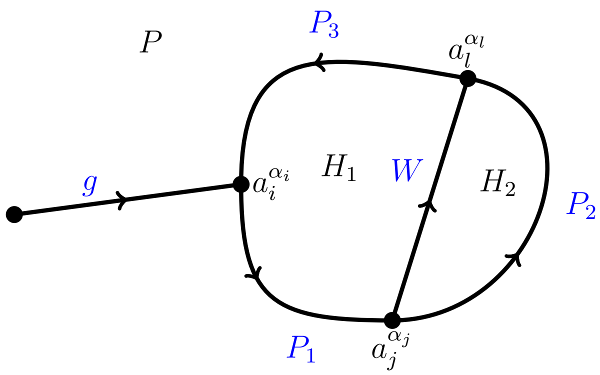


Figure 4.3: Lollipop decomposition, see proof of Lemma 4.6

Chapter 5

Proof of Theorem 3.10

We will begin with a lemma that relates the relations from homotopic walks in $\mathcal{G}(D)$.

Lemma 5.1. *Let L be a link with link diagram D, and let W and W' be walks in $\mathcal{G}(D)$ such that W' is obtained from W by inserting a cancelling edge pair. Then W and W' have the same relation.*

Proof. Let W be a walk in $\mathcal{G}(D)$ such that $W = w_1 w_2$, and W' the walk in $\mathcal{G}(D)$ such that $W' = w_1 e e^{-1} w_2$, where e is an edge in $\mathcal{G}(D)$. Let W be from $a_i^{\alpha_i}$ to $a_j^{\alpha_j}$, and let w_1 be from $a_i^{\alpha_i}$ to $a_m^{\alpha_m}$. Then by Lemma 3.3 w_1 has a relation of the form

$$a_m^{\alpha_m} = (l_1 a_i^{\alpha_i} r_1)^{\beta_1}, \tag{5.1}$$

where l_1 is the product of signed crossing generators from faces of the same colour as $a_i^{\alpha_i}$, and r_1 is the product of signed crossing generators from faces of the opposite colour to $a_i^{\alpha_i}$. By Lemma 3.3 w_2 has a relation of the form

$$a_j^{\alpha_j} = (l_2 a_m^{\alpha_m} r_2)^{\beta_2}, \tag{5.2}$$

where l_2 is the product of signed crossing generators from faces of the same colour as $a_m^{\alpha_m}$, and r_2 is the product of signed crossing generators from faces of the opposite colour to $a_m^{\alpha_m}$.

The edge e can be one of two types.

1. The edge e is from $a_m^{\alpha_m}$ to $a_m^{-\alpha_m}$. Then the walk $w_1 e e^{-1}$ has relation

$$a_m^{\alpha_m} = (l_1 a_i^{\alpha_i} r_1)^{(-1)^2 \beta_1} = (l_1 a_i^{\alpha_i} r_1)^{\beta_1}, \tag{5.3}$$

the same relation as the walk w_1.

2. The edge e is from $a_m^{\alpha_m}$ to $a_n^{\alpha_n}$ with relation $a_n^{\alpha_n} = x_e^{\delta} a_m^{\alpha_m}$. Then the relation for $w_1 e e^{-1}$ is

$$a_m^{\alpha_m} = x_e^{-\delta} x_e^{\delta} (l_1 a_i^{\alpha_i} r_1)^{\beta_1}, \tag{5.4}$$

which reduces to the relation for w_1.

Then in both cases since $w_1 e e^{-1}$ has the same relation as w_1, W and W' have the same relation. □

This gives us the following corollary:

Corollary 5.2. *Let L be a link with diagram D. Then homotopic walks in $\mathcal{G}(D)$ have the same relation.*

Next we will look at the relation for a simple lollipop, and how it can be reduced to its associated face relation or the trivial relation $1 = 1$.

Lemma 5.3. *Let L be a link with link diagram D. The relation for a simple lollipop P in $\mathcal{G}(D)$ reduces to the face relation $f_j = 1$ when the head of P is the face walk f_j, and is trivial when the head of P is a cross face walk.*

Proof. Let $a_i^{\alpha_i}$ be the starting vertex for the lollipop. Let g_l be the product of crossing generators associated with edges of the stick on faces of the same colour to $a_i^{\alpha_i}$, and let g_r be the product of crossing generators associated with edges of the stick on faces of the opposite colour to $a_i^{\alpha_i}$.

Face walk head: By Lemma 3.3 the relation for a simple lollipop with head f_j can have two forms:

1. The face walk f_j associated with the head of the lollipop is on a face of the same colour to $a_i^{\alpha_i}$. Then the relation for P is

$$a_i^{\alpha_i} = [g_l^{-1} f_j g_l a_i^{\alpha_i} g_r^{-1} g_r]^{\beta}, \tag{5.5}$$

where $\beta = +1$ since the starting and ending vertex is the same, and thus they are on the same face. This relation reduces to $f_j = 1$.

2. The face walk f_j associated with the head of the lollipop is on a face of the opposite colour to $a_i^{\alpha_i}$. Then the relation for P is

$$a_i^{\alpha_i} = [g_l^{-1} g_l a_i^{\alpha_i} g_r^{-1} f_j^{-1} g_r]^{\beta}, \tag{5.6}$$

where $\beta = +1$ since the starting and ending vertex is the same, and thus they are on the same face. This relation also reduces to $f_j = 1$.

Cross face walk head: If the head of P is a cross face walk, by Lemmas 3.3 and 3.9 the relation for P becomes:

$$a_i^{\alpha_i} = [g_l^{-1} g_l a_i^{\alpha_i} g_r^{-1} g_r]^{\beta}, \tag{5.7}$$

where $\beta = +1$ since the starting and ending vertex is the same, and thus they are on the same face. This relation reduces to $a_i^{\alpha_i} = a_i^{\alpha_i}$.

□

Now since any closed walk in $\mathcal{G}(D)$ is homotopic to a product of simple lollipops, and the relation for a simple lollipop reduces to the face relation $f_j = 1$ associated with the face of the lollipop, the relations from closed walks in $\mathcal{G}(D)$ are consequences of the face relations. This establishes Theorem 3.10

Chapter 6

Examples of presentations for $W_7(L)$

We will now find presentations for $W_7(L)$ in three examples: the knot 6_3 and two infinite families, the $(2, k)$ torus knots and the k-twist knots with odd numbers of twists.

6.1 The knot 6_3

We will start with finding the presentation for $W_7(6_3)$. Figure 6.1 on page 35 shows the graph representation for the 6_3 knot with labelled vertices, crossings and faces.

Travelling anticlockwise around each face, with the center of the clock being in the middle of the face, the next crossing generator will appear on the left of the current product. The relations for each of the eight faces are as follows:

1. Starting at a_2, the relation for F_1 is:

$$f_1 = x_1^{-1} x_3^{-1} x_2^{-1} = 1. \tag{6.1}$$

2. Starting at a_3, the relation for F_2 is:

$$f_2 = x_1 x_2^{-1} = 1. \tag{6.2}$$

3. Starting at a_5^{-1}, the relation for F_3 is:

$$f_3 = x_1 x_2 x_4^{-1} x_5 = 1. \tag{6.3}$$

4. Starting at a_7, the relation for F_4 is:

$$f_4 = x_2 x_3^{-1} x_4 = 1. \tag{6.4}$$

5. Starting at a_9^{-1}, the relation for F_5 is:

$$f_5 = x_5^{-1} x_4^{-1} x_6^{-1} = 1. \tag{6.5}$$

6. Starting at a_{11}^{-1}, the relation for F_6 is:

$$f_6 = x_6^{-1} x_4 x_3 = 1. \tag{6.6}$$

7. Starting at a_{12}^{-1}, the relation for F_7 is:

$$f_7 = x_5^{-1} x_6 = 1. \tag{6.7}$$

8. Starting at a_{11}, the relation for F_8 is:

$$f_8 = x_3 x_1^{-1} x_5 x_6 = 1. \tag{6.8}$$

Each of the face relations equates a product of crossing generators to one. We can form a group presentation of the knot from these face relations. From Equation (6.2) we get the relation $x_1 = x_2$. Substituting this into Equation (6.1) gives:

$$x_1^{-1} x_3^{-1} x_2^{-1} = x_1^{-1} x_3^{-1} x_1^{-1} = 1, \qquad x_3^{-1} = x_1^2, \qquad x_3 = x_1^{-2}.$$

Next from Equation (6.4) we get:

$$x_2 x_3^{-1} x_4 = x_1 x_1^2 x_4 = 1, \qquad x_4 = x_1^{-3}.$$

Then Equation (6.3) becomes:

$$x_1 x_2 x_4^{-1} x_5 = x_1 x_1 x_1^3 x_5, \qquad x_5 = x_1^{-5}.$$

Then from Equation (6.7), $x_6 = x_5 = x_1^{-5}$. Now from Equation (6.8):

$$x_3 x_1^{-1} x_5 x_6 = x_1^{-2} x_1^{-1} x_1^{-5} x_1^{-5} = 1, \qquad x_1^{13} = 1,$$

Now substituting into Equation (6.6):

$$\begin{gathered} x_6^{-1} x_4 x_3 = 1 \\ x_1^5 x_1^{-3} x_1^{-2} = 1, \end{gathered}$$

which reduces to $1 = 1$. Lastly we check Equation (6.5):

$$\begin{aligned} x_5^{-1}x_4^{-1}x_6^{-1} &= 1 \\ x_1^5x_1^3x_1^5 = x_1^{13} &= 1, \end{aligned}$$

which provides no new information. We see that x_1 is a cyclic generator for the set of crossing generators, and the relations reduce to the relation x_1^{13}=1. Then we can express every Wirtinger generator in terms of a_1 and x_1, and we conclude:

$$W_7(6_3) \cong \langle a_1, x_1 | x_1^{13} = 1\rangle \cong \mathbb{Z} * \mathbb{Z}_{13}. \tag{6.9}$$

6.2 The $(2, k)$ torus knot

Figure 6.2 on page 36 shows the graph representation for the $(2, k)$ torus knot.

We consider what happens at a section of the torus knot to form a presentation for the torus knot with k crossings. Consider the section of the torus knot as in Figure 6.3 on page 37.

Each of the faces F_1 through F_{k-1} have relations of the form

$$\begin{aligned} x_i^{-1}x_{i+1} &= 1 \\ x_i &= x_{i+1}. \end{aligned} \tag{6.10}$$

The face F_k has the relation:

$$\begin{aligned} x_k^{-1}x_1 &= 1 \\ x_1 &= x_k. \end{aligned} \tag{6.11}$$

Then the middle face F_{k+1} of the knot has the relation

$$x_1^{-1}x_2^{-1} \cdots x_k^{-1} = 1. \tag{6.12}$$

Then with the relations F_1 through F_{k-1} we have

$$\begin{aligned} x_1^{-1}x_1^{-1} \cdots x_1^{-1} &= 1 \\ x_1^k &= 1. \end{aligned} \tag{6.13}$$

The outer face has the relation

$$x_1x_2 \cdots x_k = 1. \tag{6.14}$$

Then with the relations F_1 through F_{k-1} we have

$$\begin{aligned} x_1x_1 \cdots x_1 &= 1 \\ x_1^k &= 1, \end{aligned} \tag{6.15}$$

so the outer face provides no new information.

We conclude that:

$$W_7((2,k) \text{ torus}) \cong \langle a_1, x_1 | x_1^k = 1\rangle \cong \mathbb{Z} * \mathbb{Z}_k. \tag{6.16}$$

6.3 The k-twist knot

Next we look at the k-twist knot with an odd number of twists, as in Figure 6.4 on page 38. It can be shown that the k-twist with an even number of twists has the same group presentation.

To form a presentation we will look at a section and the clasp of the k-twist knot as in Figure 6.5 on page 39. Note that the strands of the twist section travel in opposite directions for the k-twist knot, while they travelled in the same direction for the $(2,k)$ torus knot.

The section of the twist shows the relations for the faces F_1 through F_{k-1} alternate

$$x_i^{-1}x_{i+1}^{-1} = 1, \qquad x_{i+1}x_{i+2} = 1,$$

which give

$$x_i^{-1} = x_{i+1} \qquad x_{i+1} = x_{i+2}^{-1},$$

for $i = 1, 2, \ldots, k-2$. The relation from the face F_{k+1} is

$$x_{k+2}x_{k+1}^{-1} = 1, \tag{6.17}$$

which rearranges to

$$x_{k+2} = x_{k+1}. \tag{6.18}$$

Then the face F_k has relation

$$x_k x_{k+1} x_{k+2} = 1. \tag{6.19}$$

In terms of x_{k+1} we have

$$x_k = x_{k+1}^{-2}. \tag{6.20}$$

The large central face F_{k+3} has relation

$$x_1^{-1}x_2x_3^{-1}\cdots x_{k-1}x_k^{-1}x_{k+1} = 1, \tag{6.21}$$

and with the other face relations we get

$$\begin{aligned} x_{k+1}^2 x_{k+1}^2 x_{k+1}^2 \cdots x_{k+1}^2 x_{k+1}^2 x_{k+1} &= 1 \\ x_{k+1}^{2k+1} &= 1. \end{aligned} \tag{6.22}$$

Next we check the remaining two faces. From F_{k+2} we get:

$$x_1 x_{k+1} x_{k+2} = 1. \tag{6.23}$$

In terms of x_{k+1} we have

$$x_1 = x_{k+1}^{-2} \tag{6.24}$$

which does not provide new information. From the outer face F_{k+4} we get:

$$x_1 x_2^{-1} x_3 \cdots x_{k-1}^{-1} x_k x_{k+2}^{-1} = 1. \tag{6.25}$$

Then with the other face relations we get

$$\begin{aligned} x_{k+1}^{-2} x_{k+1}^{-2} x_{k+1}^{-2} \cdots x_{k+1}^{-2} x_{k+1}^{-2} x_{k+1}^{-1} &= 1 \\ x_{k+1}^{-(2k+1)} &= 1 \\ x_{k+1}^{2k+1} &= 1, \end{aligned} \tag{6.26}$$

and the outer face provides no new information.

We conclude that:

$$W_7(k\text{-twist}) \cong \langle a_1, x_1 | x_1^{2k+1} = 1 \rangle \cong \mathbb{Z} * \mathbb{Z}_{2k+1}. \tag{6.27}$$

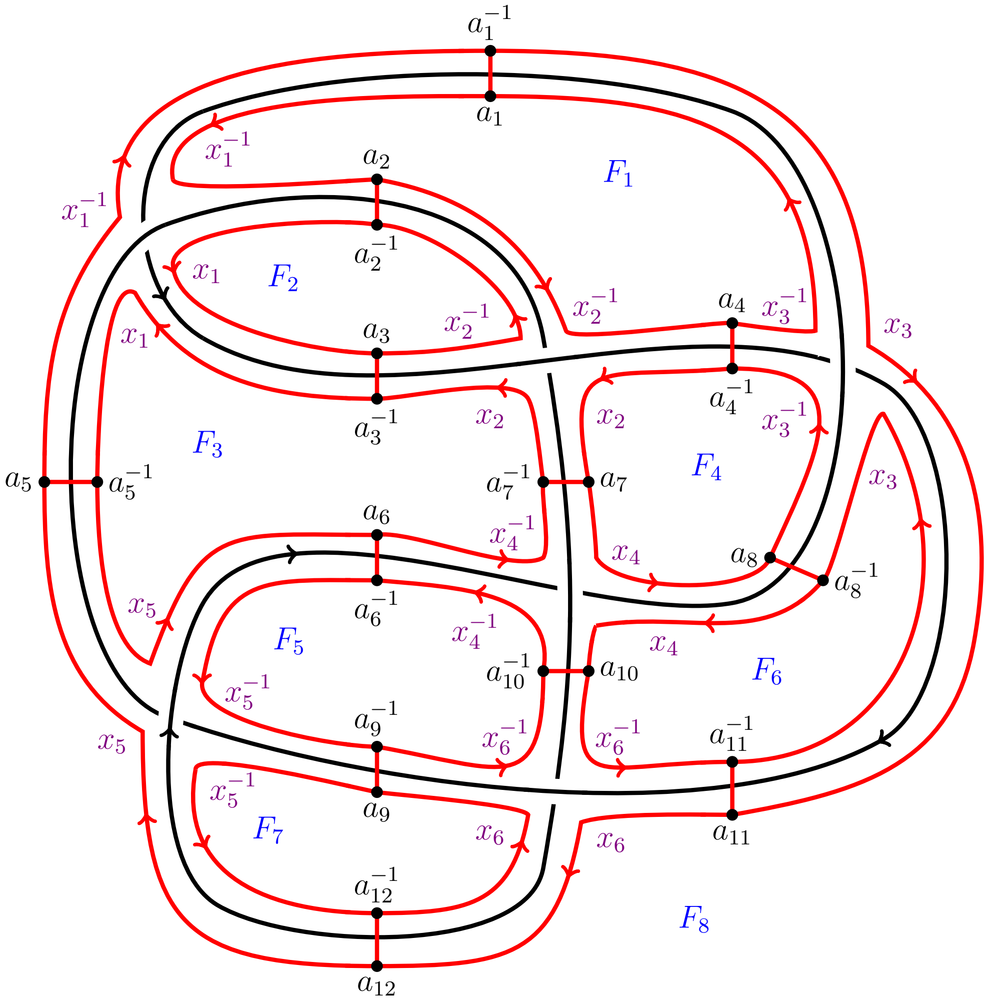


Figure 6.1: Graph representation of the 6_3 knot.

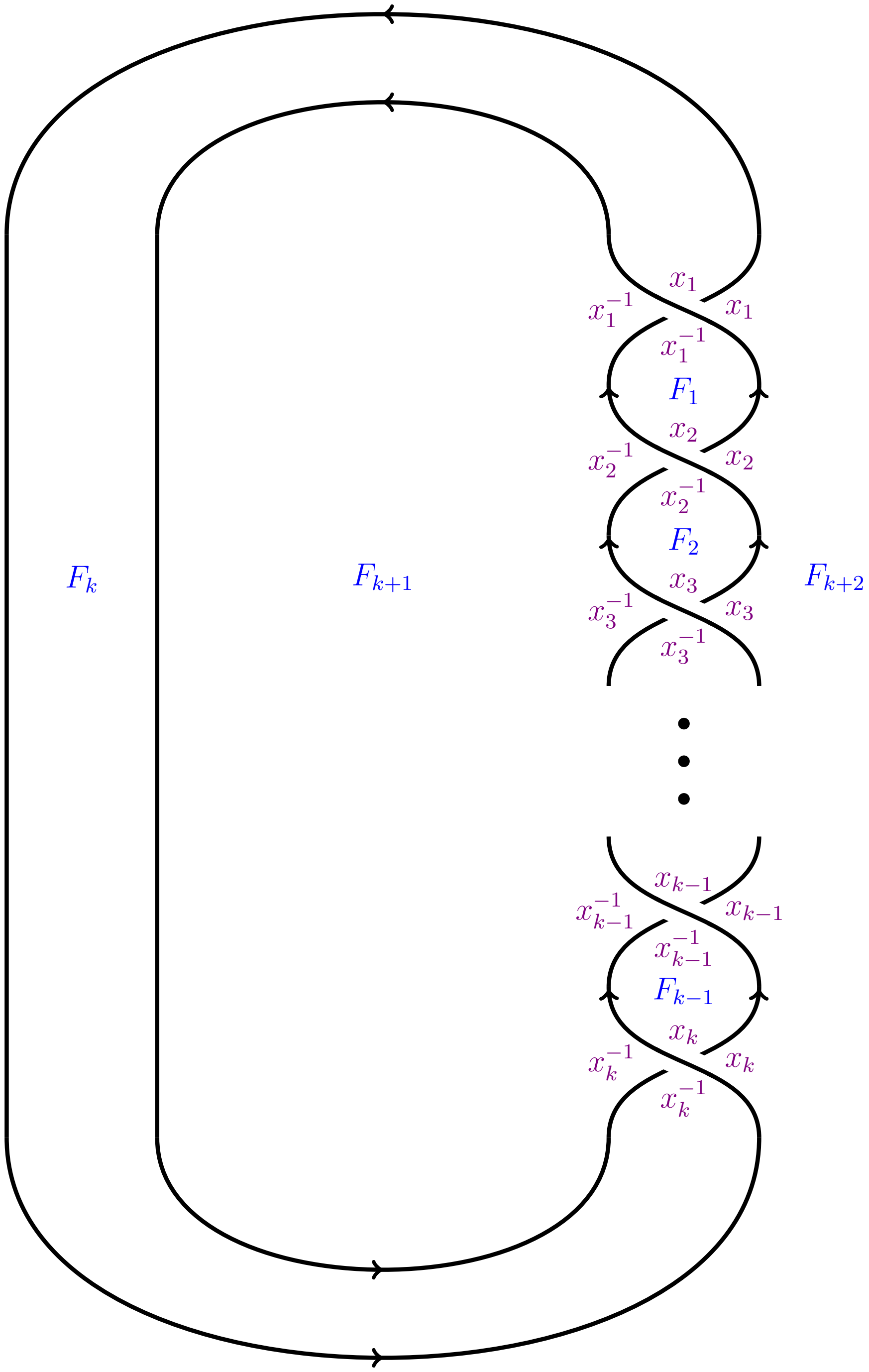


Figure 6.2: $(2, k)$ torus knot.

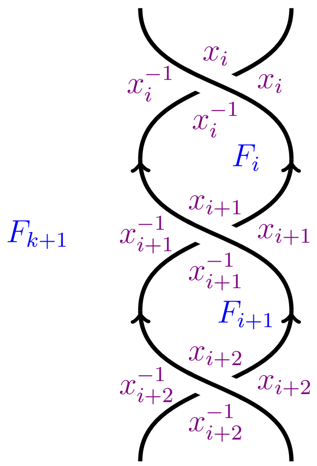


Figure 6.3: A section of the torus knot from Figure 6.2.

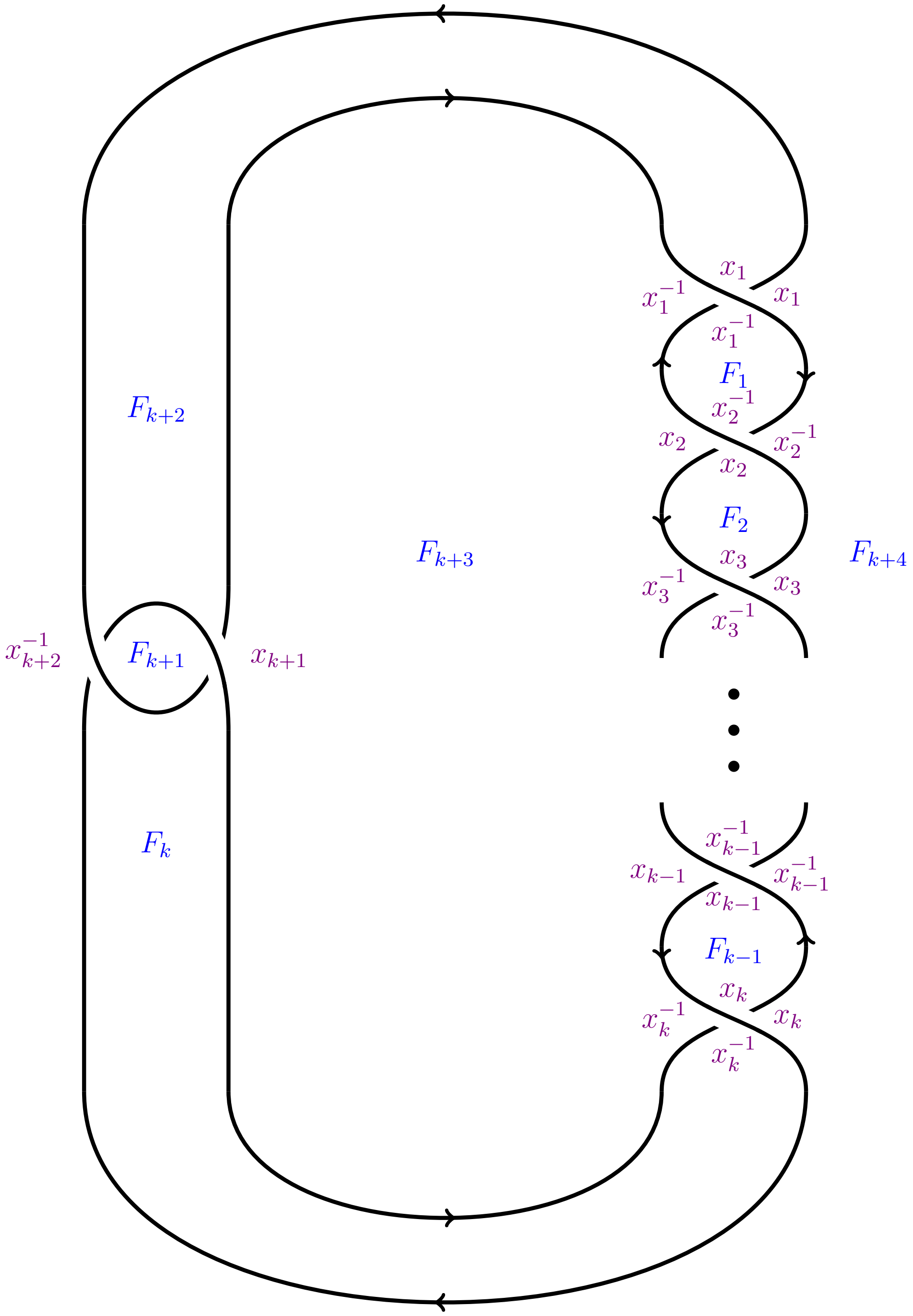
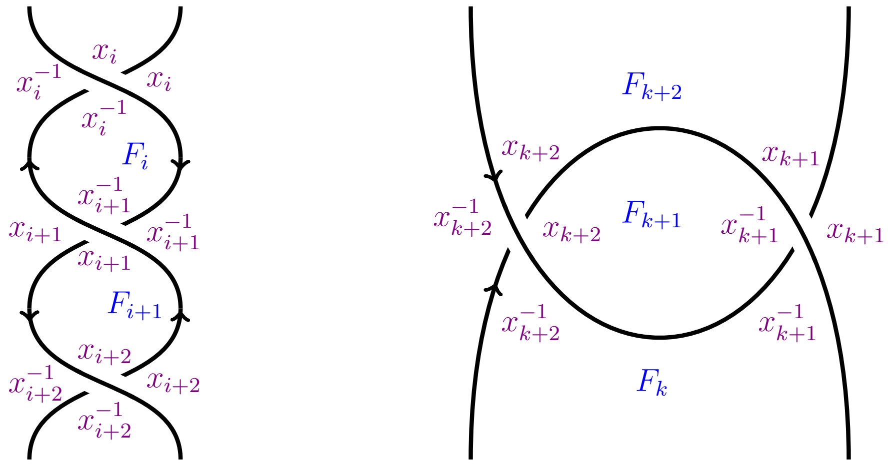


Figure 6.4: k-twist knot.



(a) A section of the twist knot. (b) The clasp of the twist knot.

Figure 6.5: A section of the twist knot and the clasp of the twist knot from Figure 6.4.

Chapter 7

Presentation for the fundamental group of the double branched cover

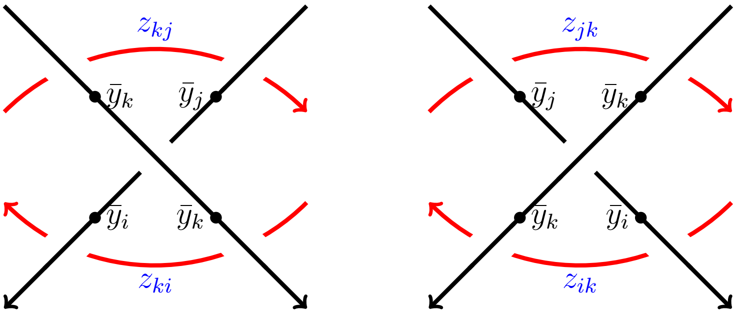
7.1 Introduction

We will be starting with the fundamental group of an oriented link L, $\pi_1(\mathbb{R}^3 - L)$. There is a homomorphism from $\pi_1(\mathbb{R}^3 - L)$ to $\mathbb{Z}$, where every Wirtinger generator $y_i \in \pi_1(\mathbb{R}^3 - L)$ maps to 1 for all i. There is also a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_2$. Then let ϕ be the composition of these homomorphisms, mapping $\pi_1(\mathbb{R}^3 - L)$ to $\mathbb{Z}_2$. We are interested in a quotient of $\ker \phi$. Let y_i, for $i = 1, 2, \ldots, l$ be the generators for $\pi_1(\mathbb{R}^3 - L)$, then let N be the normal subgroup of $\pi_1(\mathbb{R}^3 - L)$ generated by y_i^2 for all i; then the group we want is $(\ker \phi)/N$. This group is the fundamental group of a space associated with L, its *double branched cover* $\tilde{L}_2$. We wish to find a presentation for this group.

Let the Wirtinger generators for $\pi_1(\mathbb{R}^3 - L)$ be $y_1, y_2, \ldots, y_l$. Then $\ker \phi$ is generated by products of an even number of the $y_i^{\pm 1}$. So elements of $\ker \phi$ have the form $[y_{i_1}^{\eta_1} y_{i_2}^{\eta_2}][y_{i_3}^{\eta_3} y_{i_4}^{\eta_4}] \cdots [y_{i_{2n-1}}^{\eta_{2n-1}} y_{i_{2n}}^{\eta_{2n}}]$, where $\eta_i \in \pm 1$ for all i. Now we introduce the relation $y_i = y_i^{-1}$, or $y_i^2 = 1$ for all i, and we will use $\bar{y}_i$ to express we are working with the added relation $y_i^2 = 1$ for all i. Then we can define $z_{ij} = \bar{y}_i \bar{y}_j$ for all i, j, so that the z_{ij} generate $\pi_1(\tilde{L}_2)$.

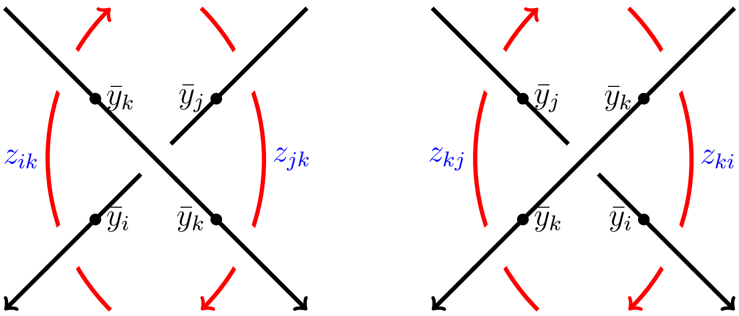
Some of the z_{ij} will be associated with a crossing. We will call them crossing generators. Consider Figures 7.1 and 7.2. Then define a *crossing generator* to be a generator z_{ab}, such that $z_{ab} = \bar{y}_a \bar{y}_b$ for $a, b \in \{i, j, k\}$ with $a \neq b$ and $\{a, b\} \neq \{i, j\}$, as in Figures 7.1 and 7.2.

Our initial presentation will have the Wirtinger generators and the relations from the crossings, $\bar{y}_i = \bar{y}_k \bar{y}_j \bar{y}_k$, as well as the relations $\bar{y}_i^2 = 1$. Then



(a) Left crossing. (b) Right crossing.

Figure 7.1: Crossings with crossing generators and horizontal movements.



(a) Left crossing. (b) Right crossing.

Figure 7.2: Crossings with crossing generators and vertical movements.

we replace the Wirtinger generators with the z_{ij}. Then we have four families of relations among the z_{ij}. Each relation either comes from our inital crossing relations or the relation $\bar{y}_i^2 = 1$; or from these relations and Wirtinger generators associated with arcs that meet at a crossing. For the following relations we will consider the left crossing and right crossing in Figures 7.1 and 7.2. With these labellings the relations for the left and right crossings are the same.

1. Since $\bar{y}_j^2 = 1$ we see $\bar{y}_i\bar{y}_j\bar{y}_j\bar{y}_k = \bar{y}_i\bar{y}_k$, so
$$z_{ij}z_{jk} = z_{ik}, \tag{7.1}$$
for all i, j, k.

2. From the relation $\bar{y}_i^2 = 1$ for all i we see
$$z_{ii} = 1, \tag{7.2}$$
for all i.

3. At both a left and right crossing we see $\bar{y}_i = \bar{y}_k\bar{y}_j\bar{y}_k$, or $\bar{y}_i\bar{y}_k = \bar{y}_k\bar{y}_j$. Then since $z_{ik} = \bar{y}_i\bar{y}_k$ and $z_{kj} = \bar{y}_k\bar{y}_j$ we have

$$z_{ik} = z_{kj}, \tag{7.3}$$

whenever strands i, j and k meet at a crossing as in Figures 7.1 and 7.2.

4. At both a left and right crossing $\bar{y}_i\bar{y}_k = \bar{y}_k\bar{y}_j$, and

$$\begin{aligned} \bar{y}_a\bar{y}_i\bar{y}_k\bar{y}_b &= \bar{y}_a\bar{y}_k\bar{y}_j\bar{y}_b \\ z_{ai}z_{kb} &= z_{ak}z_{jb}, \end{aligned} \tag{7.4}$$

whenever strands i, j and k meet at a crossing as in Figures 7.1 and 7.2.

Note that from relations 1 and 2 we get $z_{ij}z_{ji} = z_{ii} = 1$, and we see $z_{ij} = z_{ji}^{-1}$.

These four families of relations along with the generators z_{ij} will form our initial presentation for $\pi_1(\tilde{L}_2)$.

7.2 Definition of the graph for $\pi_1(\tilde{L}_2)$

First we will define a graph for $\pi_1(\tilde{L}_2)$, as we did for $W_7(L)$. Let L be a link with diagram D. We will now define a graph for $\pi_1(\tilde{L}_2)$, by modifying $\mathcal{G}(D)$ in the following way. Contract the edges from a_i to a_i^{-1} for all i, and label the new vertex $\bar{y}_i$. Each edge of this new graph will be from some $\bar{y}_i$ to some $\bar{y}_j$, and we will associate a crossing generator to each edge, such that the directed edge from $\bar{y}_i$ to $\bar{y}_j$ will have crossing generator z_{ij}, as in Figure 7.3. Call this new graph $\mathcal{G}_\pi(D)$.

Figure 7.3 below shows an example of a left and right crossing of $\mathcal{G}_\pi(D)$, where the directed red edges are the edges of $\mathcal{G}_\pi(D)$.

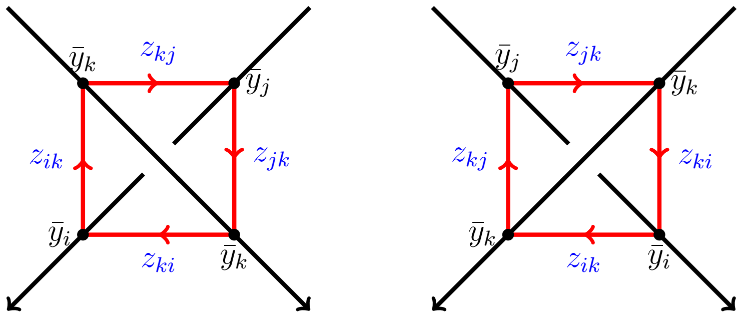
With this construction any walk in $\mathcal{G}_\pi(D)$ has an associated product of crossing generators of the form

$$z_{i_1i_2}z_{i_2i_3}\cdots z_{i_mi_n}, \tag{7.5}$$

where the sequence of crossing generators encountered is

$$z_{i_1i_2}, z_{i_2i_3}, \ldots, z_{i_mi_n}, \tag{7.6}$$

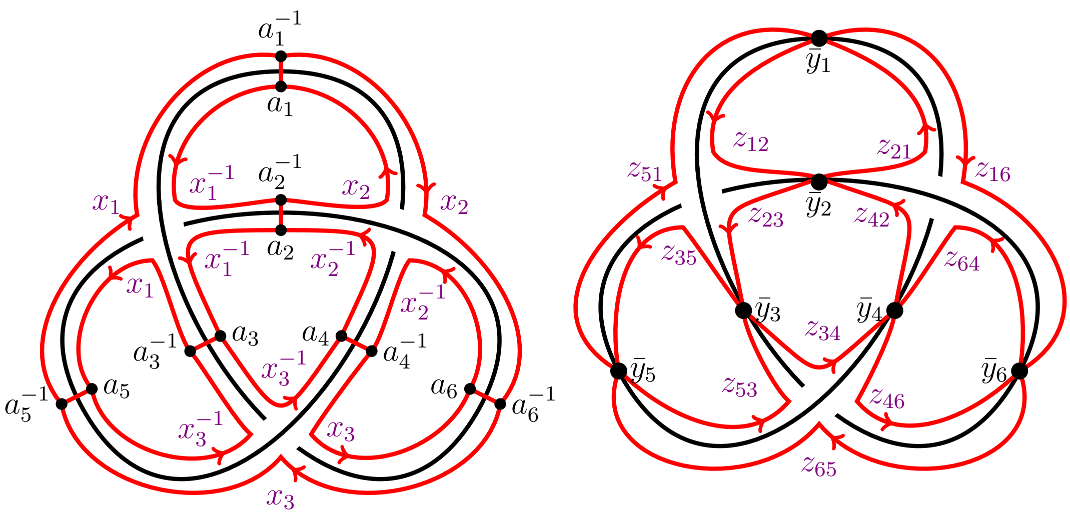
and the next crossing generator encountered appears on the right of the product.



(a) Left crossing. (b) Right crossing.

Figure 7.3: Left and right crossing of $\mathcal{G}_\pi(D)$ with crossing generators.

The directed edge from $\bar{y}_i$ to $\bar{y}_k$ will have associated generator z_{ik}. Then traversing the edge in the opposite direction, from $\bar{y}_k$ to $\bar{y}_i$, will have associated crossing generator z_{ik}^{-1}. Earlier we showed that $z_{ik}^{-1} = z_{ki}$, so traversing the edge from $\bar{y}_k$ to $\bar{y}_i$ has associated generator z_{ki}, regardless of whether we are travelling with or against the direction of the edge in $\mathcal{G}_\pi(D)$. Figure 7.4 shows both graphs for the trefoil knot.



(a) $\mathcal{G}(D)$ for the trefoil knot. (b) $\mathcal{G}_\pi(D)$ for the trefoil knot.

Figure 7.4: $\mathcal{G}(D)$ and $\mathcal{G}_\pi(D)$ for the trefoil knot.

With our earlier graph theory work and the following lemma and corollary, we see that the product associated with any closed walk in $\mathcal{G}_\pi(D)$ is equal to the product associated with a product of lollipops, as we did with the $\mathcal{G}(D)$.

Lemma 7.1. *Let L be a link with a connected diagram D, and let W and W' be walks in $\mathcal{G}_\pi(D)$ such that W' is obtained from W by inserting a cancelling edge pair. Then the products of crossing generators associated with W and W' are equal in the free group on the crossing generators.*

Proof. Let W be a walk in $\mathcal{G}_\pi(D)$ such that $W = W_1W_2$, and W' a walk in $\mathcal{G}_\pi(D)$ such that $W' = W_1ee^{-1}W_2$, where e is an edge in $\mathcal{G}_\pi(D)$. Let the product associated with W_1 be w_1, let the product associated with W_2 be w_2, and let the crossing generator associated with e be z_e. Then W has associated product w_1w_2, and W' has associated product $w_1z_ez_e^{-1}w_2$, and we see the products of crossing generators associated with W and W' are equal in the free group on the crossing generators. □

Corollary 7.2. *Let L be a link with a connected diagram D. Then homotopic walks in $\mathcal{G}_\pi(D)$ have equal associated products as elements of the free group on the crossing generators.*

7.3 The faces and their relations in $\mathcal{G}_\pi(D)$

We will start with introducing the concept of face relations in $\mathcal{G}_\pi(D)$, for a given link L with diagram D. As with $\mathcal{G}(D)$ we define a face walk in $\mathcal{G}_\pi(D)$ as a non-trivial closed walk that travels around a face F_j of the link diagram. From Lemma 7.3 below we see that the product associated with F_j, which we will call h_j, the face relation for F_j in $\mathcal{G}_\pi(D)$, is equal to 1.

Lemma 7.3. *Let L be a link with diagram D. The product of crossing generators associated with any closed walk in $\mathcal{G}_\pi(D)$ is equal to 1 in $\pi_1(\tilde{L}_2)$.*

Proof. Let W be a closed walk of length n in $\mathcal{G}_\pi(D)$. Let

$$z_{i_1i_2}z_{i_2i_3}z_{i_3i_4}\cdots z_{i_{n-1}i_n}z_{i_ni_1} \tag{7.7}$$

be the product of crossing generators associated with W. Then using relation 1 from page 41 we can reduce this product in the following way. If the expression for the closed walk has more than two terms left, we use the relation $z_{i_1i_2}z_{i_2i_3} = z_{i_1i_3}$ from relation 1 to combine the first and second term into a single term. Note that doing this will leave the last crossing generator and first subscript of the first crossing generator unchanged:

$$z_{i_1i_2}z_{i_2i_3}z_{i_3i_4}\cdots z_{i_{n-1}i_n}z_{i_ni_1} \tag{7.8}$$

becomes

$$z_{i_1i_3}z_{i_3i_4}\cdots z_{i_{n-1}i_n}z_{i_ni_1}. \tag{7.9}$$

Continuing in this way we reduce the relation to

$$z_{i_1 i_n} z_{i_n i_1} = z_{i_1 i_1} = 1, \tag{7.10}$$

by relations 1 and 2 from page 41. □

Corollary 7.4. *Let F be a face in $\mathcal{G}_\pi(D)$ with associated product h. Then $h = 1$ in $\pi_1(\tilde{L}_2)$.*

We will prove the following lemmas in the same way we did with W_7 in Lemmas 3.7 and 3.8 on pages 19 and 20.

We first show that what vertex is chosen for the start and hence end of a face walk does not provide a different face relation for any face walk on $\mathcal{G}_\pi(D)$. Instead, it produces a cyclic permutation of the crossing generators and thus a cyclic permutation of $h = 1$, where h is the product of crossing generators associated with F in $\mathcal{G}_\pi(D)$.

Lemma 7.5. *Let L be a link with diagram D. Let F be a face of $\mathcal{G}_\pi(D)$, and let W be a simple walk around F starting at $\bar{y}_{i_1}$, with associated product*

$$h = z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_{a-1} i_a} z_{i_a i_1} = 1. \tag{7.11}$$

Let W' be a walk around F, in the same direction as W, starting at $\bar{y}_{i_r}$, where $\bar{y}_{i_r}$ is $r-1$ steps around W from $\bar{y}_{i_1}$. Let the product associated with W' be h'. Then h' is a cyclic permutation of h, so $h' = 1$ follows from $h = 1$.

Proof. As $\bar{y}_{i_r}$ is $r-1$ edges from $\bar{y}_{i_1}$ around W, let the product of crossing generators associated with the walk from $\bar{y}_{i_r}$ to $\bar{y}_{i_1}$ be

$$v = z_{i_r i_{r+1}} \cdots z_{i_{a-1} i_a} z_{i_a i_1}. \tag{7.12}$$

Then the product associated with W' is

$$h' = z_{i_r i_{r+1}} \cdots z_{i_{a-1} i_a} z_{i_a i_1} z_{i_1 i_2} \cdots z_{i_{r-1} i_r}, \tag{7.13}$$

which is a cyclic permutation of h. Since h' is a cyclic permutation of h, h' is conjugate to h. We see

$$\begin{aligned} vhv^{-1} &= (z_{i_r i_{r+1}} \cdots z_{i_{a-1} i_a} z_{i_a i_1}) z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_{a-1} i_a} z_{i_a i_1} (z_{i_r i_{r+1}} \cdots z_{i_{a-1} i_a} z_{i_a i_1})^{-1} \\ &= z_{i_r i_{r+1}} \cdots z_{i_{a-1} i_a} z_{i_a i_1} z_{i_1 i_2} \cdots z_{i_{r-1} i_r} = h'. \end{aligned} \tag{7.14}$$

Then since $h = 1$, it follows that $h' = 1$. □

If the face walk with relation $h = 1$ is done in the opposite direction the relation formed will be $h^{-1} = 1$. Thus, changing the direction of the face walk does not provide a new relation.

Lemma 7.6. *Let W be a simple walk of length n around a face F, in a given direction, starting and ending at $\bar{y}_i$, with associated product*

$$z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_{n-1} i_n} z_{i_n i_1}. \tag{7.15}$$

Let W' be the simple walk around F starting and ending at $\bar{y}_i$, in the opposite direction to W. Then W' has associated product

$$z_{i_n i_1}^{-1} z_{i_{n-1} i_n}^{-1} \cdots z_{i_3 i_2}^{-1} z_{i_2 i_1}^{-1} = h'. \tag{7.16}$$

Then $h' = h^{-1}$, so $h' = 1$ follows from $h = 1$ and does not provide a new relation.

Proof. W' encounters the same crossing generators as W but in the opposite order, and since W' crosses the edges in the opposite direction the crossing generators will be inverted. Then the product associated with W' is

$$\begin{aligned} h' &= z_{i_n i_1}^{-1} z_{i_{n-1} i_n}^{-1} \cdots z_{i_3 i_2}^{-1} z_{i_2 i_1}^{-1} \\ &= (z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_{n-1} i_n} z_{i_n i_1})^{-1} = h^{-1}. \end{aligned} \tag{7.17}$$

Then since $h = 1$, $h' = 1$ does not provide a new relation. □

The purpose of the next section will be to prove the following theorem.

Theorem 7.7. *Let L be a link with connected diagram D. Then*

$$\pi_1(\tilde{L}_2) \cong \langle z_1, z_2, \ldots, z_n | h_1, h_2, \ldots, h_s \rangle, \tag{7.18}$$

where $\{z_1, z_2, \ldots z_n\}$ is a set consisting of a single crossing generator from each crossing, and $h_1, h_2, \ldots h_s$ are the face relations expressed in terms of $z_1, z_2, \ldots z_n$.

7.4 Proof of Theorem 7.7

First we will show that $\pi_1(\tilde{L}_2)$ can be generated by only the crossing generators.

Lemma 7.8. *Let D be a connected diagram for a link L. Let P be any path from $\bar{y}_{i_1}$ to $\bar{y}_{i_n}$ in $\mathcal{G}_\pi(D)$. Then $z_{i_1 i_n}$ is equal to the product of crossing generators associated with P.*

Proof. If P is a path of length 1 in $\mathcal{G}_\pi(D)$, then there is an edge connecting $\bar{y}_{i_1}$ and $\bar{y}_{i_n}$, and the edge has associated crossing generator $z_{i_1 i_n}$. Thus the lemma holds for paths of length 1. If P is a path of length greater than 1 in $\mathcal{G}_\pi(D)$ let the product of crossing generators associated with P be

$$z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_{n-1} i_n}. \tag{7.19}$$

Then by relation 2 from page 41 we can reduce this product as we did in Lemma 7.3,

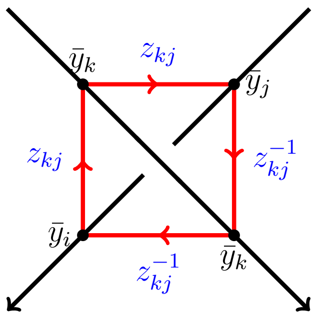
$$z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_{n-1} i_n} = z_{i_1 i_n}, \tag{7.20}$$

and we have an expression for $z_{i_1 i_n}$ in terms of a product of crossing generators. □

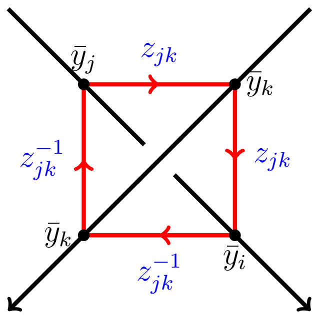
Now that we can express any product of an even number of the $\bar{y}_{i_m}$ in terms of crossing generators, we can reduce our generating set to only the crossing generators.

Corollary 7.9. *The generating set for the presentation of $\pi_1(\tilde{L}_2)$ can be reduced to the crossing generators.*

Then with relation 3 from page 42 we can reduce the set of crossing generators to a single crossing generator at each crossing. Let the crossing be oriented downward, so that both strands are travelling downward, as in Figure 7.3. Then we choose the generator related to the top edge as our single generator for the crossing, for both left and right crossings. With the labellings as in Figure 7.3, we would use z_{kj} for the left crossing and z_{jk} for the right crossing. This will produce crossings as shown in Figure 7.5.



(a) Left crossing.



(b) Right crossing.

Figure 7.5: Left and right crossings of $\mathcal{G}_\pi(D)$ with a single crossing generator.

Now we have shown that relations 1–4 from Section 7.1 can be used to reduce our generating set to only one generator arising from each crossing, and to derive the face relations. Next we will show we can do the reverse of this process, and derive relations 1–4 from the face relations and a single crossing generator from each crossing.

First we show that the face relations imply Lemma 7.3.

Lemma 7.10. *Let L be a link with connected diagram D. Then the face relations $h_k = 1$ imply that the product associated with any closed walk in $\mathcal{G}_\pi(D)$ is equal to 1 in $\pi_1(\tilde{L}_2)$.*

Proof. Let W be a closed walk in $\mathcal{G}_\pi(D)$. From Corollary 7.2 we know that homotopic walks in $\mathcal{G}_\pi(D)$ have equal associated products in the free group on the crossing generators. Then since every closed walk in $\mathcal{G}_\pi(D)$ is homotopic to a product of lollipops, we can write $W \simeq P_1 P_2 \cdots P_n$, where $P_1, P_2, \ldots, P_n$ are lollipops. Let the associated product for P_i be $g_i h_{j_i} g_i^{-1}$. Then the associated product for $P_1 P_2 \cdots P_n$ in $\mathcal{G}_\pi(D)$ is

$$\begin{aligned} g_1 h_{j_1} g_1^{-1} g_2 h_{j_2} g_2^{-1} \cdots g_n h_{j_n} g_n^{-1} &= g_1 g_1^{-1} g_2 g_2^{-1} \cdots g_n g_n^{-1} \\ &= 1, \end{aligned} \tag{7.21}$$

since $h_{j_i} = 1$ in $\mathcal{G}_\pi(D)$ for all j, i. From Corollary 7.2 we know that homotopic walks in $\mathcal{G}_\pi(D)$ have equal associated products in the free group on the crossing generators, so the product associated with W is equal to 1. □

We can now derive Lemma 7.8 from the face relations in the following way. Let $\bar{y}_a$ and $\bar{y}_b$ be two vertices in $\mathcal{G}_\pi(D)$ not connected by an edge. Then we define z_{ab} as a product of crossing generators associated with a walk in $\mathcal{G}_\pi(D)$ from $\bar{y}_a$ to $\bar{y}_b$.

Lemma 7.11. *With this definition the product z_{ab} is well defined.*

Proof. Consider two different walks from $\bar{y}_a$ to $\bar{y}_b$, W_1 and W_2. Let the products associated with W_1 and W_2 be w_1 and w_2 respectively. Then $W_1 {W_2}^{-1}$ is closed so by Lemma 7.3 $w_1 {w_2}^{-1} = 1$ and $w_1 = w_2$ in $\pi_1(\tilde{L}_2)$. □

Note that the walk $z_{ij} z_{ji}$ is closed, so we have the relation $z_{ij} z_{ji} = 1$ by Lemma 7.3, and so $z_{ij} = z_{ji}^{-1}$.

Now we are ready to prove that the four relations can be derived from the face relations.

Lemma 7.12. *The four relations from Section 7.1 can be derived from the face relations, $h_i = 1$ for all i.*

Proof. 1. We will start with deriving relation 1. Consider the product associated with the closed walk $W_{ij}W_{jk}W_{ki}$, where W_{ab} is a walk from $\bar{y}_a$ to $\bar{y}_b$. By Lemma 7.10 this product is equal to 1 in $\pi_1(\tilde{L}_2)$. By Lemma 7.11 this product is equal to $z_{ij}z_{jk}z_{ki} = 1$ in $\pi_1(\tilde{L}_2)$. Then $z_{ij}z_{jk} = z_{ki}^{-1} = z_{ik}$ in $\pi_1(\tilde{L}_2)$.

2. For relation 2 we have shown already that $z_{ij}z_{ji} = 1$, and then from relation 1 we know $z_{ij}z_{ji} = z_{ii} = 1$.

3. For relation 3 we introduce an extra crossing generator at each crossing, and let it be equal to the generator we have for the crossing. With both strands travelling downward, we let the new crossing generator be associated with the left edge of the graph at the crossing, and label it z_{ab}, where the left edge travels from $\bar{y}_a$ to $\bar{y}_b$. Consider Figure 7.5. Our new crossing generator would be z_{ik}, and we set z_{ik}=z_{kj}.

4. To derive relation 4 we will use Lemma 7.11 and relation 3 as follows.

   From Lemma 7.11 the product $z_{ai}z_{kb}$ can be calculated using any path from $\bar{y}_a$ to $\bar{y}_i$ and any path from $\bar{y}_k$ to $\bar{y}_b$. Let P_{ak} be a path from $\bar{y}_a$ to $\bar{y}_k$, and P_{ib} a path from $\bar{y}_i$ to $\bar{y}_b$. Then $P_{ak}z_{ki}$ is a path in $\mathcal{G}_\pi(D)$ from $\bar{y}_a$ to $\bar{y}_i$, and $z_{kj}P_{jb}$ is a path from $\bar{y}_k$ to $\bar{y}_b$. Then $z_{ai}z_{kb} = (P_{ak}z_{ki})(z_{kj}P_{jb})$, and since $z_{kj} = z_{ik} = z_{ki}^{-1}$ by relation 3, we have

$$\begin{aligned} z_{ai}z_{kb} &= (P_{ak}z_{ki})(z_{kj}P_{jb}) \\ &= (P_{ak}z_{ki})(z_{ki}^{-1}P_{jb}) \\ &= P_{ak}P_{jb} \\ &= z_{ak}z_{jb}. \end{aligned} \tag{7.22}$$

□

Now that we have one crossing generator per crossing for our presentation of $\pi_1(\tilde{L}_2)$, we will relabel our crossing generators to better match our crossing generators for $W_7(L)$. Given a link L with connected diagram D, if the crossing generator for a given crossing of $\mathcal{G}(D)$ is x_i, we let the crossing generator for the same crossing of $\mathcal{G}_\pi(D)$ be z_i. Thus our presentation for $\pi_1(\tilde{L}_2)$ for a link with k crossings and t faces is:

Lemma 7.13.

$$\pi_1(\tilde{L}_2) \cong \langle z_1, z_2, \ldots, z_k | h_1, h_2, \ldots, h_t \rangle, \tag{7.23}$$

where $z_1, z_2, \ldots, z_k$ are the crossing generators and $h_1, h_2, \ldots, h_t$ are the face relations, expressed in terms of $z_1, z_2, \ldots, z_k$.

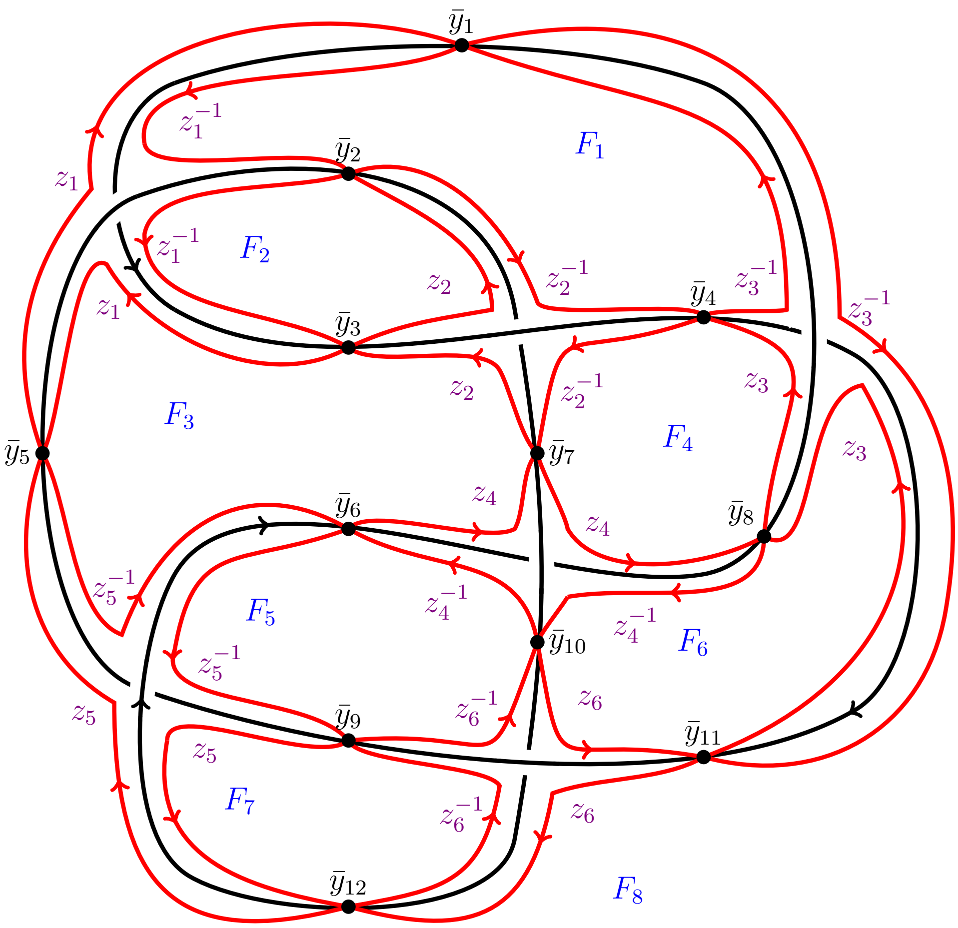


Figure 7.6: $\mathcal{G}_\pi(6_3)$.

Now as an example, we will work out the presentation for $\pi_1(\tilde{L}_2)$ for the 6_3 knot with labellings as in Figure 7.6, as we did with $W_7(6_3)$.

Now we will find the face relations for each of the eight faces. We travel anticlockwise around each face, with the center of the clock being in the middle of the face. Then the next crossing generator appears on the right of the current product. The eight face relations are:

1. Starting at $\bar{y}_2$, the relation for F_1 is:
$$h_1 = z_2^{-1} z_3^{-1} z_1^{-1} = 1. \tag{7.24}$$

2. Starting at $\bar{y}_3$, the relation for F_2 is:
$$h_2 = z_2 z_1^{-1} = 1. \tag{7.25}$$

3. Starting at $\bar{y}_5$, the relation for F_3 is:

$$h_3 = z_5^{-1} z_4 z_2 z_1 = 1. \tag{7.26}$$

4. Starting at $\bar{y}_7$, the relation for F_4 is:

$$h_4 = z_4 z_3 z_2^{-1} = 1. \tag{7.27}$$

5. Starting at $\bar{y}_9$, the relation for F_5 is:

$$h_5 = z_6^{-1} z_4^{-1} z_5^{-1} = 1. \tag{7.28}$$

6. Starting at $\bar{y}_{11}$, the relation for F_6 is:

$$h_6 = z_3 z_4^{-1} z_6 = 1. \tag{7.29}$$

7. Starting at $\bar{y}_{12}$, the relation for F_7 is:

$$h_7 = z_6^{-1} z_5 = 1. \tag{7.30}$$

8. Starting at $\bar{y}_{11}$, the relation for F_8 is:

$$h_8 = z_6 z_5 z_1 z_3^{-1} = 1. \tag{7.31}$$

Next we will reduce our face relations via substitution.

1. From h_2 we get

$$z_2 = z_1. \tag{7.32}$$

2. From h_1 we get

$$h_1 = z_1^{-1} z_3^{-1} z_1^{-1} = 1, \qquad z_3 = z_1^{-2}.$$

3. From h_4

$$h_4 = z_4 z_1^{-2} z_1^{-1} = 1, \qquad z_4 = z_1^3.$$

4. From h_3

$$h_3 = z_5^{-1} z_1^3 z_1 z_1 = 1, \qquad z_5 = z_1^5.$$

5. From h_7

$$z_6 = z_5, \qquad z_6 = z_1^5.$$

6. Then substituting into h_8

$$h_8 = z_1^5 z_1^5 z_1 z_1^2 = z_1^{13} = 1. \tag{7.33}$$

Now checking the remaining two face relations:

$$h_6 = z_1^{-2} z_1^{-3} z_1^5 = 1, \tag{7.34}$$

and

$$h_5 = z_1^{-5} z_1^{-3} z_1^{-5} = z_1^{-13} = 1, \tag{7.35}$$

which provide no new information. Thus our presentation for $\pi_1(\tilde{L}_2)$ with $L = 6_3$ is:

$$\langle z_1 | z_1^{13} = 1 \rangle \cong \mathbb{Z}_{13}. \tag{7.36}$$

Comparing (7.36) with the result (6.9) of Section 6.1 we see that for $L = 6_3$ we have

$$W_7(L) \cong \pi_1(\tilde{L}_2) * \mathbb{Z}. \tag{7.37}$$

We will prove that this is always the case in Theorem 8.3 in Chapter 8.

Chapter 8

Proof of the main theorem, Theorem 1.1

The purpose of this chapter is to complete our proof of the main theorem, Theorem 1.1.

First we will restate our presentations for $W_7(L)$ and $\pi_1(\tilde{L}_2)$ for an oriented link L with connected diagram D.

1. Our presentation for $W_7(L)$ for a link L with k crossings and t faces is

$$W_7(L) \cong \langle a_p, x_1, x_2, \ldots, x_k | f_1, f_2, \ldots, f_t \rangle, \tag{8.1}$$

   where a_p is a Wirtinger generator, $x_1, x_2, \ldots, x_k$ are the crossing generators and $f_1, f_2, \ldots, f_t$ the face relations. A face relation f_n is of the form

$$f_n = x_{i_l}^{\varepsilon_l} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1}, \tag{8.2}$$

   where the $x_{i_p}^{\varepsilon_p}$ are the signed crossing generators encountered by travelling the face F_n in $\mathcal{G}(D)$ anticlockwise, by Lemma 3.3.

2. Our presentation for $\pi_1(\tilde{L}_2)$ for a link L with k crossings and t faces is

$$\pi_1(\tilde{L}_2) \cong \langle z_1, z_2, \ldots, z_k | h_1, h_2, \ldots, h_t \rangle, \tag{8.3}$$

   where $z_1, z_2, \ldots, z_k$ are the crossing generators and $h_1, h_2, \ldots, h_t$ the face relations. A face relation h_b is of the form

$$h_b = z_{j_1}^{\alpha_1} z_{j_2}^{\alpha_2} \cdots z_{j_r}^{\alpha_r}, \tag{8.4}$$

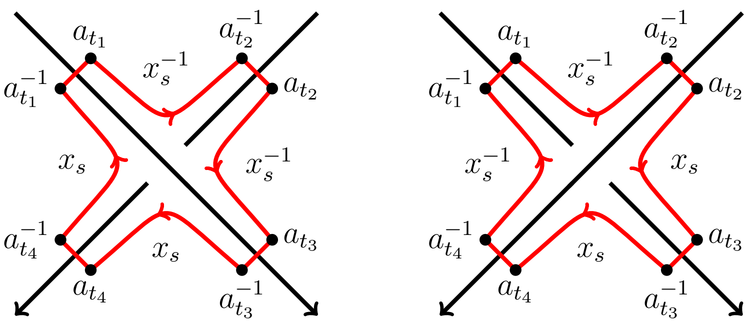
   where

$$z_{j_1}^{\alpha_1}, z_{j_2}^{\alpha_2}, \ldots, z_{j_r}^{\alpha_r} \tag{8.5}$$

   is the sequence of signed crossing generators encountered by travelling the face F_b in $\mathcal{G}_\pi(D)$ anticlockwise, by Lemma 7.13.

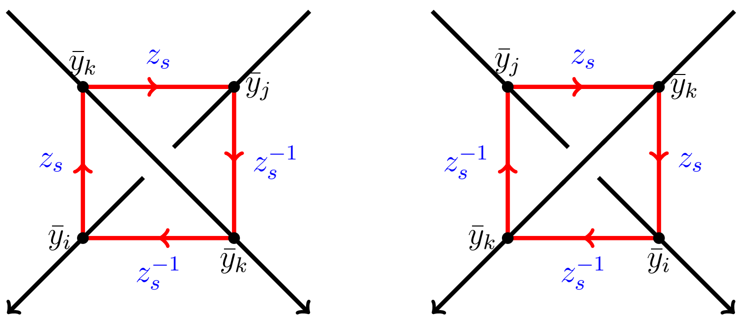
We can see that the relations for both presentations are the corresponding face relations, so if we can determine how the crossing generators relate we can find an isomorphism between the presentations.

In Figures 8.1 and 8.2 we show our chosen crossing generators for $W_7(L)$ and $\pi_1(\tilde{L}_2)$ at a crossing of a link L with connected diagram D.



(a) Left crossing. (b) Right crossing.

Figure 8.1: Left and right crossing of $\mathcal{G}(D)$ with a single crossing generator.



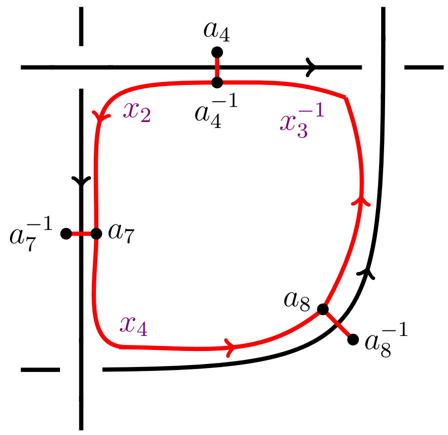
(a) Left crossing. (b) Right crossing.

Figure 8.2: Left and right crossing of $\mathcal{G}_\pi(D)$ with a single crossing generator.

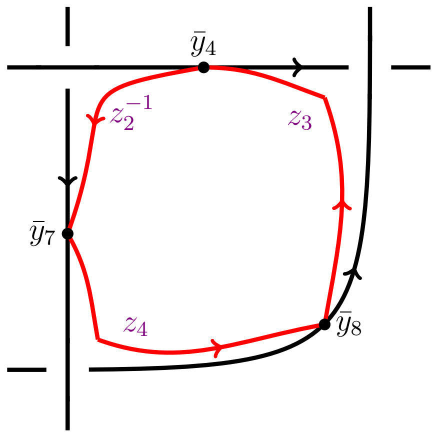
We want an isomorphism between the two groups, so we will look at how the crossing generator at a given crossing for $W_7(L)$ compares to the crossing generator for $\pi_1(\tilde{L}_2)$ for the same crossing.

First we note that by Lemma 3.3 for a face relation for $W_7(L)$, the next crossing generator appears on the left, while as discussed in Section 7.2 for a face relation for $\pi_1(\tilde{L}_2)$ the next crossing generator appears on the right. To compensate for this we will be inverting our relation for $\pi_1(\tilde{L}_2)$, so that the orders of crossing generators for the relations match. Then if we want the face relations to be identical once inverted we need the sign of the crossing generator for a crossing in $\pi_1(\tilde{L}_2)$ to be the negative of the sign of the crossing generator for the same crossing in $W_7(L)$.

As an example, consider the relation for the fourth face of the 6_3 knot, with labellings as in Figures 6.1 and 7.6. $\mathcal{G}(6_3)$ and $\mathcal{G}_\pi(6_3)$ for this face are shown in Figure 8.3 below.



(a) Face 4 of $\mathcal{G}(6_3)$.



(b) Face 4 of $\mathcal{G}_\pi(6_3)$.

Figure 8.3: Face 4 of the 6_3 knot from Figures 6.1 and 7.6.

Starting and ending at a_7 and travelling anticlockwise in $\mathcal{G}(6_3)$, the sequence of signed crossing generators encountered is x_4, x_3^{-1}, x_2. The face relation for $W_7(6_3)$ is

$$x_2x_3^{-1}x_4 = 1. \tag{8.6}$$

Then starting and ending at $\bar{y}_7$ and travelling anticlockwise for $\mathcal{G}_\pi(6_3)$, the sequence of signed crossing generators encountered is z_4, z_3, z_2^{-1}. Then the face relation for $\pi_1(\widetilde{(6_3)}_2)$ is

$$z_4z_3z_2^{-1} = 1, \tag{8.7}$$

or equivalently

$$z_2z_3^{-1}z_4^{-1} = 1. \tag{8.8}$$

We take the inverse of the relation so that the order of the crossing generators in the products match. Since we will be inverting our relation for

$\pi_1(\tilde{L}_2)$, we want the sign of the crossing generator in each quadrant of a crossing for $\pi_1(\tilde{L}_2)$ to be the negative of the sign of the crossing generator for each quadrant of the corresponding crossing for $W_7(L)$.

From Figures 8.1 and 8.2 we can see that the exponent of the crossing generator in the bottom quadrant of the left crossing of $\mathcal{G}_\pi(D)$ is the negative of the exponent in the same quadrant of the left crossing of $\mathcal{G}(D)$, as required. However, the sign of the exponent of the crossing generator for the right quadrant of the left crossing of $\mathcal{G}_\pi(D)$ is not the negative of the exponent of the crossing generator of the same quadrant for the left crossing of $\mathcal{G}(D)$. Thus the face relations will not hold under a map between them. However, we can see that the exponent of the crossing generator at each of the quadrants of the left crossing for $\mathcal{G}_\pi(D)$ is the negative of the exponent of the crossing generator at each of the quadrants of the right crossing of $\mathcal{G}(D)$. Likewise, the exponent of the crossing generator at each of the quadrants of the right crossing for $\mathcal{G}_\pi(D)$ is the negative of the exponent of the crossing generator at each of the quadrants of the left crossing for $\mathcal{G}_\pi(L)$. Thus, if we switch all the left crossings to right crossings and the right crossings to left crossings for $\mathcal{G}_\pi(D)$, the face relations will hold under our map, and we will have an isomorphism, once we account for the extra Wirtinger generator in the presentation for $W_7(L)$.

We can switch the left and right crossings in the following way. Given a link L, we can place a mirror behind the link L, and the reflection of L in the mirror will have its crossings switched, so that left crossings will become right crossings and right crossings will become left crossings.

We will introduce a theorem to show how we can switch the left and right crossings without changing the associated fundamental group for a link L. For further reading see [3].

Theorem 8.1. *Homotopy equivalent spaces have the same fundamental group.*

Since a reflection is a homeomorphism, and homeomorphic spaces are homotopy equivalent, the fundamental group of a link L and the fundamental group of its reflection L' are isomorphic. The fundamental group of the double branched cover, $\pi_1(\tilde{L}_2)$, depends only on the fundamental group of the link, $\pi_1(L)$. Then since the fundamental group of the double branched cover constructed from the link, $\pi_1(\tilde{L}_2)$, is isomorphic to the fundamental group of the double branched cover constructed from the reflection, $\pi_1(\tilde{L'}_2)$, we have the following corollary:

Corollary 8.2. *Let L be a link, and let L' be the reflection of L found by placing a mirror behind L. Then $\pi_1(\tilde{L}_2) \cong \pi_1(\tilde{L'}_2)$.*

Now we need to account for the extra Wirtinger generator present in the presentation for $W_7(L)$ that does not appear in $\pi_1(\tilde{L}_2)$. We do this by taking the free product of $\pi_1(\tilde{L}_2)$ and $\mathbb{Z}$, and our presentation for this product is

$$\langle b, z_1, z_2, \ldots, z_k | h_1, h_2, \ldots, h_t \rangle, \tag{8.9}$$

and so our isomorphism is:

$$W_7(L) \cong \pi_1(\tilde{L}_2) * \mathbb{Z}. \tag{8.10}$$

Now we will restate our conclusion as a theorem and recap the discussion as the proof.

Theorem 8.3. *Let L be a link, then*

$$W_7(L) \cong \pi_1(\tilde{L}_2) * \mathbb{Z}. \tag{8.11}$$

Proof. Let D be a connected diagram for L. By Theorem 3.10 our presentation for $W_7(L)$ is

$$\langle a_p, x_1, x_2, \ldots, x_k | f_1, f_2, \ldots, f_t \rangle. \tag{8.12}$$

Let L' be the reflection of L made by placing a mirror behind L. By Lemma 7.13 our presentation for $\pi_1(\tilde{L'}_2) * \mathbb{Z}$ is

$$\langle b, z_1, z_2, \ldots, z_k | h_1, h_2, \ldots, h_t \rangle. \tag{8.13}$$

Consider the face F_s of $\mathcal{G}(D)$, and let the sequence of signed crossing generators associated with F_s in $\mathcal{G}(D)$ be

$$x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}, \ldots, x_{i_r}^{\varepsilon_r}. \tag{8.14}$$

Then by Lemma 3.3,

$$f_s = x_{i_r}^{\varepsilon_r} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1}. \tag{8.15}$$

Then since the sign of the exponent of the crossing generator in each quadrant of the left and right crossings of $\mathcal{G}_\pi(D)$ is the negative of those of $\mathcal{G}(D)$, and z_i is at the same crossing as x_i, the sequence of signed crossing generators associated with F_s in $\mathcal{G}_\pi(D)$ is

$$z_{i_1}^{-\varepsilon_1}, z_{i_2}^{-\varepsilon_2}, \ldots, z_{i_r}^{-\varepsilon_r}, \tag{8.16}$$

and the relation for h_s is

$$h_s = z_{i_1}^{-\varepsilon_1} z_{i_2}^{-\varepsilon_2} \cdots z_{i_r}^{-\varepsilon_r}. \tag{8.17}$$

Equivalently, by Corollary 7.4, $h_s = 1$ and

$$h_s = z_{i_r}^{\varepsilon_r} \cdots z_{i_2}^{\varepsilon_2} z_{i_1}^{\varepsilon_1}. \tag{8.18}$$

Then we can show that with a relabeling of the z_j the face relations are identical. We relabel z_j to x_j, then

$$h_s = x_{i_r}^{\varepsilon_r} \cdots x_{i_2}^{\varepsilon_2} x_{i_1}^{\varepsilon_1} = f_s, \tag{8.19}$$

and we conclude that

$$\langle b, z_1, z_2, \ldots, z_k | h_1, h_2, \ldots, h_t \rangle = \langle a_p, x_1, x_2, \ldots, x_k | f_1, f_2, \ldots, f_t \rangle. \tag{8.20}$$

Thus

$$W_7(L) \cong \pi_1(\tilde{L'}_2) * \mathbb{Z}. \tag{8.21}$$

Then by Corollary 8.2,

$$W_7(L) \cong \pi_1(\tilde{L}_2) * \mathbb{Z}. \tag{8.22}$$

□

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