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# The Quotient Between Length and Multiplicity

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## Abstract

This dissertation examines the finiteness of the algebraic invariants  $n_A(M)$  and  $\theta_A(M)$ . These invariants, based on the ratio of length and multiplicity and the ratio of Loewy length and multiplicity respectively, are studied in general and under certain conditions. The finiteness of  $\theta_A(M)$  is established for a large class of algebraic structures.  $n_A(M)$  is shown to be finite in the low dimensional case as well as when we restrict our attention to special sets of ideals. Also considered in this dissertation are equivalent conditions for the local case to be bounded by the graded case when evaluating  $n_A(M)$ .

## 0.1 Dedication

This dissertation is dedicated to the memory of Professor Wolfgang Vogel. Professor Vogel inspired my leap into the field of mathematics and his constant enthusiasm and encouragement kept me going. During my three and a half years working with him, Professor Vogel was always positive and optimistic about mathematics and life in general. He was always available to talk with me and he introduced me to people from all over the world. I feel truly lucky to have had the opportunity to know Professor Vogel and to work with him. His contribution to mathematics and his ability to inspire his students are just two of the legacies he has left behind. Wolfgang Vogel was a great mathematician and a truly great man, he will be deeply missed.

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