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Design and Parametric Control of Co-Axes Driven Two-Wheeled Balancing Robot

A thesis presented in partial fulfilment of the requirements for the degree of

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In
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To my dearest mother, father and Lisa for their encouragement and support
Abstract

Nowadays robots can be seen in our daily life. Recently, robotic applications and their wide range of functionalities have drawn many engineers’ attentions. Two-wheeled balancing robots are typical example of unstable dynamic system. Understanding the classical theory of inverted pendulum and its dynamic system are initial steps for developing a two-wheeled balancing robot. A balancing robot’s structure has two different sections. The first section contains the moving parts or wheels and the second section contains the rigid parts or chassis. An initial physical structure was designed and built and robot’s specifications were measured for developing the mathematical model of two-wheeled balancing robot. Existing energies of dynamic model were observed separately and substituted into Lagrangian equation to generate the mathematical model of balancing robot. Mathematical model was generated to observe the behaviour of the model. State-space model of robot was developed and a controller was designed according to state-space model. Tilt sensor and gyroscope provide the feedbacks of closed-loop system. Two-wheeled balancing robot has some key parameters that are directly engaged with system’s performance and responses. Parametric studies were done and system responses were observed by variation of key parameters. Observed results from parametric studies were applied into physical model to improve the robot performance. Kalman filter was implemented for fusing the gyroscope angular rate and raw tilt angle. A proportional-integral-derivative (PID) controller was designed to generate the required input for motor controllers to control the rotation of wheels based on the Kalman filter’s output.
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Chapter 1

Overview

MASSEY UNIVERSITY
School of Engineering & Advanced Technology
1.1. Introduction

Nowadays robots can be seen in our daily life. Robots are employed on production lines in factories, appeared as intelligent machines to do a specific task or used as commercial products. Recently, robotic applications and their wide range of functionalities have drawn many engineers’ attentions. Two-wheeled balancing robots are typical example of unstable dynamic system.

Two-wheeled balancing robots can be used in several applications with different perspectives such as an intelligent gardener in agricultural fields (e.g. vineyards), an autonomous trolley in hospitals, shopping malls, offices, airports, healthcare applications or an intelligent robot to guide blind or disable people.
1.2. **How Does Balancing Robot Balance Itself?**

Two-wheeled balancing robot is an unstable dynamic system. It means that robot is free to fall forward or backward without any applied forces. Robot is balance when its centre of gravity and wheels are located on an identical imaginary vertical line. Otherwise, wheels should follow the robot’s falls until robot balance itself. Using two wheels only for two-wheeled balancing robot provide lighter weight and smoother manoeuvre. Figure 1.1 illustrates the basic functions of balancing robot.

![Figure 1.1 - Three different modes of balancing robot](image)

Figure 1.1 - Three different modes of balancing robot

Figure 1.2 shows the basic flow chart of balancing robot.
Figure 1.2 - Balancing robot flow chart
1.3. Literature Review

There are several successful commercial products and many other research prototypes around the world. Following paragraphs review some of products, researches and prototypes that built based on two-wheeled balancing robot basics.

Segway is a well-known self-balancing personal transporter, which was produced by Segway Inc. of New Hampshire, USA [22]. It was invented by Dean Kamen in 2001. Segway is driven by servo motors and it can speed up to 20 km/h. Several tilt sensor and gyroscopes are employed in Segway to detect the angular motion of robot. Segway rider is able to accelerate and decelerate by leaning forward or backward. In 2006 iBot was another product that developed by Kamen [10]. iBot is a powered mobile wheelchair. It climbs the stairs and balances itself on two wheels. iBot increases its height when it balances on two wheels. Increasing the height provides an eye level vision for a disable person to communicate with other people. Figure A1.1 (Appendix 1.1) illustrates iBot functionalities.

Figure 1.3 - Segway i180 series (left) and iBot 4000 (right)
EMIEW is a humanoid balancing robot developed by Hitachi research group [8]. EMIEW stands for Excellent Mobility and Interactive Existence as Workmate. This robot is a guide or surveillance robot in a real world environment such as offices, factories or hospitals. Hitachi has developed two models of EMIEW. These two models can avoid obstacles and have a top speed of 6 km/h. Emiew2 was given an enhanced mobility by deploying a suspension system for each leg and voice recognition feature. Figure A1.2 (Appendix 1.2) illustrates more pictures of EMIEWs.

![Figure 1.4 - EMIEW (left) and EMIEW2 (right)](image)

Anybots is a robotic company founded by Trevor Blackwell in 2001 [2]. QA and QB are two mobile telepresence robots developed by Anybots that balance on two wheels and manoeuvre smoothly. These robots provide an easy communication via Internet for a user, who cannot attend at the second place. QB is the newest version of Anybot’s telepresence robot. It can rotate around its vertical axis and drive at 5.6 km/h. Figures 1.5 illustrates QA and QB respectively.
David P. Anderson has developed a prototype of two-wheeled balancing robot in his workshop [1]. Anderson was chosen four essential measurements that define the motion and position of nBot. These measurements are:

- Position of robot
- Velocity of robot
- Angular position of wheels
- Angular velocity of wheels
These measurements were summed as a feedback to microcontroller. Then microcontroller generates the required signal as a motor voltage, which is proportional to motor’s torque.

Figure 1.6 - nBot

The Industrial Electronics Laboratory at the Swiss Federal Institute of Technology in Switzerland has built a truly mobile and autonomous two-wheeled balancing robot (JOE) [6]. Researchers have developed two state-space controllers. One controller controls rotation around lateral axis (pitch) and the second one control the dynamics around its vertical axis (yaw). Each controller produce required torque for right and left motors.
All the mentioned projects and products (figures 1.3 - 1.7) use back and forth motion of wheels to balance the robot. There are some projects that use other methods to balance the structure. Department of mechanical engineering at university of Auckland in New Zealand used a reaction wheel on top of the robot to generate an external force to balance the robot [12]. The reaction wheel provides rotation inertia to the opposite direction of robot’s fall and stabilise the system. Figure 1.8 illustrates the two wheeled platform with reaction wheel actuator.

Figure 1.8 - Two wheeled platform with reaction wheel actuator
1.4 Conclusion

Two-wheeled robots provide higher level of mobility and manoeuvrability. The application of two-wheeled balancing robot varies with different environment and requirements. Understanding the classical theory of inverted pendulum and its dynamic system are initial steps for developing a two-wheeled balancing robot. Safety and robustness of two-wheeled products are so important and these aspects will be achieved by employing accurate sensors and designing a precise control unit.
Chapter 2

System Configuration
2.1. Introduction

Two-wheeled balancing robot is a clear example of a natural unstable dynamic system. It means that robot falls forward or backward freely without having a control unit. Basically, it follows the inverted pendulum theory. A balancing robot’s structure has two different sections. The first section contains the moving parts or wheels and the second section contains the rigid parts or chassis. An initial physical structure was designed and built. Then, robot’s specifications were measured for developing the mathematical model of two-wheeled balancing robot. This chapter explains the system configuration it shows the robot’s specifications.
2.2. Components

2.2.1. Microcontroller

An Arduino Duemilanove USB module was used as balancing robot’s processor. This module is based on ATmega328 IC. It has 14 digital input/output pins, 6 analogue inputs and a 16MHz crystal oscillator. The module can be powered via the USB connection or an external power supply. The Arduino module selects the power source automatically and can be programmed with the Arduino software. Figure 2.1 illustrates the Arduino module.

![Arduino Duemilanove USB module](image)

*Figure 2.1 - Arduino Duemilanove USB module*

Table A2.1 (Appendix 2.1) illustrates the specifications of Arduino module [3].

2.2.2. Gyroscope

CRS03, single axis angular rate sensor was used to detect the angular velocity of robot. This sensor is manufactured by Silicon Sensing System Ltd [23]. It provides an analogue output, which is proportional to the angular rate about its sensing axis. The analogue output is between 0 to 5 volts. The output voltage is 2.5V (plus offset) at zero angular rate and 4.5V or 0.5V at full scale angular rate depending on direction of rotation. CRS03
sensor utilizes silicon micromachining and other solid state fabrication techniques and employs the Coriolis effect to measure angular rate. Figure 2.2 illustrates the CRS03 sensor.

![CRS03, single axis angular rate sensor](image)

**Figure 2.2 -** CRS03, single axis angular rate sensor

CRS03 has a ratiometric output. It means that the sensor’s output is a function of angular rate and supply voltage. Equation 2.1 shows the typical rate output of the sensor.

\[
V_O = \frac{1}{2} \times V_{dd} + \left( R_a \times SF \times \frac{V_{dd}}{5} \right)
\]  

(2.1)

Where, \( V_O \) is rate output (V); \( V_{dd} \) is supply voltage (V); \( R_a \) is applied rate (°/s) and \( SF \) is scale factor (V/°/s). Table A2.2 (Appendix 2.2) illustrates the sensor’s pin definitions.

### 2.2.3. Accelerometer

Memsic 2125, a dual axis thermal accelerometer was used to measure the tilt angle. This sensor is manufactured by Parallax, Inc and it is able to measure dynamic acceleration (vibration) and static acceleration (gravity) with a range of ±2 g [18]. Memsic 2125 has
two outputs that show the pulse output of g-force for X and Y axis. There is a small heater inside a chamber of gas that warms a bubble of air within the device (fig. 2.3). This bubble moves, when gravitational forces act on it. This movement is detected by four temperature sensors that are located around the chamber and onboard electronics convert the position of bubble into pulse output for the X and Y axis. The pulse outputs from the Memsic 2125 are set to a 50% duty cycle at 0 g. Duty cycle of pulse changes in proportion to acceleration of robot. Figure A2.2 and A2.3 (Appendix 2.3) illustrates the Memsic 2125 and its pin definitions.

Figure 2.3 - Accelerometer heated gas chamber

Figure 2.4 illustrates the pulse output of Memsic 2125 and equation 2.2 illustrates the required formula for observing the g force [23].

Figure 2.4 - Memsic 2125 pulse output
\[ A(g) = \left( \frac{1}{T1} - \frac{1}{T2} \right) - 0.5 \right) / 12.5\% \] (2.2)

2.2.4. Motor Controller

Two HB-25 motor controllers were used to control the speed and direction of DC motors’ rotation. This type of motor controller was controlled by a PWM signal. Figure 2.5 illustrates a typical form of input signal for HB-25 [19].

Figure 2.5 - HB-25 input signal

Figure 2.5 illustrates that there is a hold-off time of 5 ms and HB-25 ignores incoming pulses during this period. Therefore, the unit should not be refreshed more frequently than about 5.25 ms plus pulse time. Pulse time can be anywhere between 0.8 ms to 2.2 ms. HB-25 and its features are illustrated in appendix 2.4.

2.2.5. Gear-Head DC Motor

Two EMG30 gear-head DC motors were used for initial design. EMG30 is a 12V motor with gearbox 30:1. It produces the rate torque of 1.5 Kg/cm with 170 rpm. Appendix 2.5 illustrates the specification of EMG30 DC motor.
2.2.6. Batteries

A 12V-2.3AH battery was used for motor controllers and four AA batteries were used to power up the microcontroller.
2.3. Robot’s Structure

Inverted pendulum theorem was followed to design and develop the two-wheeled balancing robot. In this project chassis was developed by four hard plastic levels (Figure A2.7) that were separated by M6 bolts and nuts (Figure A2.8) and heaviest parts (batteries) were located at the highest level of robot. Two co-axis DC motors were mounted under the chassis and other parts were distributed on levels. Figure 2.6 illustrates the initial design of balancing robot.

![Developed two-wheeled balancing robot](image_url)

*Figure 2.6 - Developed two-wheeled balancing robot*
Figure 2.7 illustrates the schematic diagram of two-wheeled balancing robot and it shows how the different components are related together.

![Figure 2.7 - Schematic diagram of two-wheeled balancing robot](image-url)
2.4. Calibration

Tilt sensor, gyroscope and DC motor were calibrated before integrating with physical model. Calibration of these three parts provided better understanding of their outputs. Calibration results were used in programming the microcontroller.

2.4.1. Tilt Sensor Calibration

A home-made calibration jig was developed for calibrating the tilt sensor. Tilt sensor was located at different angles and sensor’s output was observed as its pulse output of X axis g-force. Figure 2.8 and 2.9 illustrate the developed calibration jig and tilt sensor calibration’s graph respectively.

Figure 2.8 - Tilt sensor calibration jig
Equation 2.3 shows the equation of linear fitted line on tilt sensor’s calibration graph.

\[ y = -0.29x + 200 \]  \hspace{1cm} (2.3)

### 2.4.2. Gyroscope Calibration

Full-scale output graph of gyroscope and its linear equation were observed. Figure 2.10 and equation 2.4 illustrates the full-scale output graph and its related equation respectively.
2.4.3. DC Motor and Motor Controller Calibration

An experiment was done to determine the corresponding duty cycle of PWM signal for desired rotation per minute (RPM) of DC motors. Figure 2.11 illustrates the DC motor calibration graph.

\[ y = 0.02x + 2.5 \]  

(2.4)
Figure 2.11 - DC motor calibration graph
2.5. Conclusion

Initial structure of two-wheeled balancing robot was developed. Symmetrical weight distribution on robot’s chassis was an important step for designing the structure. Robot’s weight could be reduced by using the lighter materials. Best location of centre of gravity and sensors were determined after mathematical simulation.
Chapter 3

Mathematical Modelling

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3.1. Introduction

Mathematical model of two-wheeled balancing robot was generated to observe the behaviour of model such as maximum overshoot, settling time and etc. Robot’s specifications were observed and substituted into mathematical model. In this chapter existing energies of dynamic model were observed separately and substituted into Lagrangian equation to generate the mathematical model. Then, state-space model of system was design after developing the differential equations of system.
3.2. **Mathematical Modelling of Two-wheeled Balancing Robot and Its Dynamic System**

It is necessary to understand the dynamics of two-wheeled balancing robot for modelling the system. Figure 3.1 illustrates the dynamic model of two-wheeled balancing robot.

Two-wheeled balancing robot has three degrees of freedom (3DOF). Degrees of freedom are described by three types of robot’s rotations around X, Y and Z axes, which are called roll, pitch and yaw respectively. In this project the main focus was on the rotation around Y axes (pitch). It assumed that centre of gravity of robot is located at point \( P \) and \( \theta_p \) represents the pitch angle. The coordinate of point \( P \) will change if robot moves away from its initial location along \( X_0 \) axis. Therefore, equations 3.1 and 3.2 illustrate the displacement of point \( P \) along \( X_0 \) and \( Z_0 \) axes [11].

\[
X_p = x + l \sin \theta_p \tag{3.1}
\]

\[
Z_p = l \cos \theta_p \tag{3.2}
\]
3.3. **Lagrangian Dynamic Analyses**

Lagrangian approach was used to generate the ordinary differential equations (ODE) of system. Equation 3.3 shows the general form of Lagrangian equation [13].

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = Q_r \quad (3.3)
\]

Where,

- $L= T-V$, is called the Lagrange function where, $T$ is total Kinetic energy of the system from all inertia elements and $V$ is total potential energy of the system from all stiffness elements and inertia elements due to gravitational effect.
- $q_r (r = 1, 2, 3, ..., n)$ are generalised coordinate/variable or displacement variable such as $x_1, x_2, \theta_1, \theta_2$ and etc.
- $Q_r$ is sum of non-conservative forces projected on direction of generalised coordinate such as applied forces.

The first step for developing the Lagrangian equation of balancing robot was to observe the existing energies in the system. There are three types of energy for chassis and wheels. These energies are:

- Kinetic energy
  - Translational kinetic energy of chassis
  - Rotational kinetic energy of wheel
- Potential energy
- Dissipation energy
Equation 3.4 illustrates the Lagrangian equation of two-wheeled balancing robot where, $E$ is kinetic energy, $U$ is potential energy, $F$ is dissipation energy and $\tau_r$ is required torque for left and right wheels.

$$\frac{d}{dt} \left( \frac{\partial E}{\partial \dot{q}_r} \right) - \frac{\partial E}{\partial q_r} + \frac{\partial F}{\partial \dot{q}_r} + \frac{\partial U}{\partial q_r} = \tau_r$$  \hspace{1cm} (3.4)

### 3.3.1. Total Kinetic Energy of System

Translational kinetic energy of chassis ($E_{TKC}$) with constant mass and centre of gravity located at point $P$ was observed by classical equation of kinetic energy shown on equation 3.5 [16].

$$E_{TKC} = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{X}_p^2 + \frac{1}{2} m \dot{Z}_p^2$$  \hspace{1cm} (3.5)

Where, $m$ is mass of robot’s chassis and $v$ is travelling speed of chassis. Equation 3.6 was developed by substituting equations 3.1 and 3.2 into equation 3.5.

$$E_{TKC} = \frac{1}{2} m \left( v + l \omega_p \cos \theta_p \right)^2 + \left( -l \omega_p \sin \theta_p \right)^2$$  \hspace{1cm} (3.6)

Equation 3.6 was simplified by following algebraic equations.
Therefore, translational kinetic energy of chassis was defined by equation 3.7.

\[ E_{TKC} = \frac{1}{2} m \left( v^2 + 2vl\omega_p \cos \theta_p + l^2 \omega_p^2 \cos^2 \theta_p + l^2 \omega_p^2 \sin^2 \theta_p \right) \]

\[ E_{TKC} = \frac{1}{2} m \left( v^2 + l^2 \omega_p^2 + 2vl\omega_p \cos \theta_p \right) \]

Rotational kinetic energy of chassis \((E_{RKC})\) was observed by equation 3.8.

\[ E_{RKC} = \frac{1}{2} J_C \omega_p^2 \]  
(3.8)

Where, \(J_C\) is rotation inertia of chassis and \(\omega_p\) is angular velocity of point \(P\). Total kinetic energy of chassis \((E_{KC})\) was observed by adding the translational and rotational kinetic energy of chassis.

\[ E_{KC} = E_{TKC} + E_{RKC} \]

\[ E_{KC} = \frac{1}{2} m \left( v^2 + l^2 \omega_p^2 \right) + mvl\omega_p \cos \theta_p + \frac{1}{2} J_C \omega_p^2 \]  
(3.9)
Translational and rotational kinetic energy of two wheels were observed by equations 3.10 and 3.11 respectively.

\[ E_{TKW} = 2 \times \frac{1}{2} M v^2 = Mv^2 \]  \hspace{1cm} (3.10)

\[ E_{RKW} = 2 \times \frac{1}{2} J_w \omega_w^2 = J_w \frac{v^2}{R^2} \]  \hspace{1cm} (3.11)

Where, \( M \) and \( J_w \) are mass and rotation inertia of one wheel respectively. Total kinetic energy of wheels (\( E_{KW} \)) was observed by adding the translational and rotational kinetic energy of wheels.

\[ E_{KW} = E_{TKW} + E_{RKW} \]

\[ E_{KW} = Mv^2 + J_w \frac{v^2}{R^2} \]  \hspace{1cm} (3.12)

Total kinetic energy of two-wheeled balancing robot was observed by adding equations 3.9 and 3.12.

\[ E = E_{KC} + E_{KW} \]

\[ E = \frac{1}{2} m \left( v^2 + l^2 \omega_p^2 \right) + mvl \cos \theta_p \omega_p + \frac{1}{2} J_c \omega_p^2 + Mv^2 + J_w \frac{v^2}{R^2} \]
\[
E = \frac{1}{2}mv^2 + \frac{1}{2}ml^2\omega_p^2 + mvl\cos\theta_p\omega_p + \frac{1}{2}J_c\omega_p^2 + Mv^2 + J_w\frac{V^2}{R^2}
\]  
(3.13)

3.3.2. Total Potential Energy of System

There are no changes in potential energy of wheels while robot moves. Therefore, total potential energy of system \((U)\) was defined by equation 3.14 [16].

\[
U = mgl\cos\theta_p
\]  
(3.14)

Where, \(m\) is mass of chassis, \(g\) is gravity and \(l\) is distance between centre of gravity and rotational axis of wheels.

3.3.3. Total Dissipation Energy of System

Total dissipation energy of system \((F)\) was defined by equation 3.15 [16].

\[
F = \mu_0v^2 + \mu_1\omega_p^2
\]  
(3.15)

Where, \(\mu_0\) is coefficient of friction between wheel and ground and \(\mu_1\) is coefficient of friction between chassis and rotational axis of wheels. It was assumed that \(\mu_0=0.1\) and \(\mu_1=0.1\).

Displacement of balancing robot was selected as the first generalised variable \((q_1)\) in Lagrangian equation of system.
Following equations show the expansion of Lagrangian equation of system When \(q_1 = x\).

\[
\frac{\partial E}{\partial q_1} = \frac{\partial E}{\partial \dot{x}} = \frac{\partial E}{\partial v} = mv + ml \cos \theta_p \omega_p + 2Mv + 2J_w \frac{v}{R^2}
\]

\[
\frac{d}{dt} \left( \frac{\partial E}{\partial v} \right) = m\ddot{x} + ml(\cos \theta_p)\dot{\theta}_p + 2M\ddot{x} + 2J_w \frac{\ddot{x}}{R^2}
\]

\[
\frac{d}{dt} \left( \frac{\partial E}{\partial v} \right) = \left( m + 2M + \frac{2J_w}{R^2} \right)\ddot{x} + ml(\cos \theta_p)\dot{\theta}_p
\]

\[
\frac{\partial E}{\partial q_1} = \frac{\partial E}{\partial \dot{x}} = 0
\]

\[
\frac{\partial F}{\partial q_1} = \frac{\partial E}{\partial \dot{x}} = \frac{\partial F}{\partial \dot{v}} = 2\mu_0 v
\]

\[
\frac{\partial U}{\partial q_1} = \frac{\partial E}{\partial \dot{x}} = 0
\]

The first differential equation of system (eq. 3.21) was observed by adding the equations 3.17 to 3.20.

\[
\left( m + 2M + \frac{2J_w}{R^2} \right)\ddot{x} + ml(\cos \theta_p)\dot{\theta}_p + 2\mu_0 v = 0
\]
Pitch angle of balancing robot was selected as the second generalised variable \( q_2 \) in Lagrangian equation of system.

\[
q_2 = \theta_p \quad \text{(3.22)}
\]

Following equations show the expansion of Lagrangian equation of system when \( q_2 = \theta \).

\[
\frac{\partial E}{\partial q_2} = \frac{\partial E}{\partial \theta_p} = \frac{\partial E}{\partial \omega_p} = ml^2 \omega_p + mvl(\cos \theta_p) + J_c \omega_p
\]

\[
\frac{d}{dt} \left( \frac{\partial E}{\partial \omega_p} \right) = ml^2 \ddot{\theta}_p + ml(\cos \theta_p) \dddot{x} + J_c \dddot{\theta}_p
\]

\[
\frac{d}{dt} \left( \frac{\partial E}{\partial \omega_p} \right) = \left( ml^2 + J_c \right) \ddot{\theta}_p + ml(\cos \theta_p) \dddot{x} \quad \text{(3.23)}
\]

\[
\frac{\partial E}{\partial q_2} = \frac{\partial E}{\partial \theta_p} = -mvl(\sin \theta_p) \omega_p \quad \text{(3.24)}
\]

\[
\frac{\partial F}{\partial q_2} = \frac{\partial F}{\partial \omega_p} = 2 \mu_i \omega_p \quad \text{(3.25)}
\]

\[
\frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial \theta_p} = -mgl(\sin \theta_p) \quad \text{(3.26)}
\]

The second differential equation of system (eq. 3.27) was observed by adding the equations 3.23 to 3.26.
Equations 3.21 and 3.27 are two non-linear equations of two-wheeled balancing robot. Before any further analysis, differential equations of system were linearised. There is only one equilibrium point for balancing robot and it is when robot is balanced or up-right. Therefore, it was assumed that $\theta_p = 0$. So,

$$\sin \theta_p \approx \theta_p \quad (3.28)$$

$$\cos \theta_p = 1 \quad (3.29)$$

Linear equations of two-wheeled balancing robot were observed by substituting equations 3.28 and 3.29 into non-linear equations. Equations 3.30 and 3.31 show the linear equations of system.

$$\left( m + 2M + \frac{2J_w}{R^2} \right) \ddot{x} + ml \dddot{\theta}_p + 2 \mu_0 \dot{v} = 0 \quad (3.30)$$

$$ml \dddot{x} + (ml^2 + J_c) \dddot{\theta}_p - mg \dot{\theta}_p + 2 \mu_1 \omega_p = \tau_L + \tau_R \quad (3.31)$$
3.4.  **Robot’s Specifications**

Linear equations of two-wheeled balancing robot contain different constants such as mass of a wheel \( (M) \), mass of chassis \( (m) \), radius of a wheel \( (R) \) and etc. Each constant was measured separately. Then, observed specifications were used for mathematical modelling and simulating the system.

3.4.1.  **Weights**

Weights of chassis and wheels were observed by SJ-HS bench scale (Figure A3.1).

3.4.2.  **Location of Centre of Gravity**

Location of centre of gravity was observed by hanging the robot horizontally. Then, distance between the location of centre of gravity and rotational axis (motor’s shafts) was measured. Figure 3.2 illustrates the way of observing the location of centre of gravity.

![Figure 3.2 - Observing the location of centre of gravity](image)
3.4.3. Rotation Inertia of Chassis

The chassis was assumed as a rectangular cube in order to simplify the calculation. Equation 3.32 was used to calculate the rotation inertia of chassis [16].

\[ I_C = \frac{1}{12}m\left(h^2 + d^2\right) \]  \hspace{1cm} (3.32)

Where, \(h\) and \(d\) are height and width of the chassis respectively (Appendix 3.2).

3.4.4. Rotation Inertia of Wheel

It was assumed the robot’s wheel as a cylinder. Equation 3.33 was used for calculating the rotation inertia of wheel [16].

\[ I_w = \frac{1}{2}MR^2 \]  \hspace{1cm} (3.33)

Where, \(M\) and \(R\) are mass and radius of the wheel respectively (Appendix 3.3).

Therefore, the following are physical parameters of two-wheeled balancing robot.

\[ g = 9.81 \quad [m/sec^2] : \quad \text{Gravity acceleration} \]
\[ m = 2.63 \quad [kg] : \quad \text{Chassis weight} \]
\[ J_C = 0.018 \quad [kgm^2] : \quad \text{Rotation inertia of chassis} \]
$R = 0.05 \ [m] : \text{Wheel radius}$

$J_W = 1.75 \times 10^{-4} \ [kgm^2] : \text{Rotation inertia of a wheel}$

$M = 0.14 \ [kg] : \text{Wheel weight}$

$W = 0.18 \ [m] : \text{Chassis width}$

$D = 0.12 \ [m] : \text{Chassis depth}$

$H = 0.24 \ [m] : \text{Chassis height}$

$l = 0.18 \ [m] : \text{Distance of the centre of mass from the wheel axle}$

$\mu_0 = 0.1 : \text{Coefficient of friction between wheel and ground}$

$\mu_1 = 0 : \text{Coefficient of friction between chassis and wheel}$
3.5. State-Space Modelling

State-space model of a dynamic system is a mathematic model that includes a set of input, output and state variables of system. The state of a dynamic system is described by a set of state variables \( [x_1(t), x_2(t), ..., x_n(t)] \). The state variables determine the future behaviour of system when the present state of the system and input signals are known [4]. The minimum number of state variables is equal to order of system’s differential equations. State-space model of a linear system contains four matrices, which are:

- \( A \) matrix is known as state matrix
- \( B \) matrix is known as input matrix
- \( C \) matrix is known as output matrix
- \( D \) matrix is known as direct transmission matrix

Equations 3.34 and 3.35 are two equations of general state-space model for a dynamic system, which are called state and output equations respectively.

\[
\dot{X} = AX + Bu \\
Y = CX + Du
\]  
(3.34)  
(3.35)

Figure 3.3 illustrates the related block diagram of the state-space equations.
Four state variables were chosen for dynamic system of two-wheeled balancing robot. Equation 3.36 shows the state vector of dynamic system \( X \). Four elements of state vectors are:

- Chassis displacement, \( x \)
- Chassis pitch angle, \( \theta_p \)
- Chassis velocity, \( v \)
- Chassis angular velocity, \( \omega_p \)

\[
X = \begin{bmatrix} x & \theta_p & v & \omega_p \end{bmatrix}^T \tag{3.36}
\]

State variables of velocity and angular velocity are derivative of displacement and pitch angle respectively. Therefore,

\[
X_1 = x \tag{3.37}
\]

\[
X_2 = \theta_p \tag{3.38}
\]
According to state equation (3.34), state variables were written in first derivative format. Therefore, equivalent of first derivative of each state variable were observed. Target is to find an equivalent for $\dot{X}_3$ and $\dot{X}_4$.

$$X_3 = v = \dot{X}_1$$

$$X_4 = \dot{\omega}_p = \dot{X}_2$$

Appendix 3.4 shows all the algebraic calculations for observing $\dot{X}_3$ and $\dot{X}_4$. Therefore, $\dot{X}_3$ and $\dot{X}_4$ are:

$$\dot{X}_1 = X_3$$

$$\dot{X}_2 = X_4$$

$$\dot{X}_3 = ?$$

$$\dot{X}_4 = ?$$

$$\dot{X}_3 = \left[ -\frac{(ml)^2 g}{\text{den}} \right] X_2 + \left[ -\frac{2\mu_0 (ml^2 + J_C)}{\text{den}} \right] X_3 + \left[ \frac{2m\mu_1}{\text{den}} \right] X_4 + \left[ -\frac{ml}{\text{den}} \right](\tau_L + \tau_R)$$

$$\dot{X}_4 = \left[ \frac{(m + 2M + 2J_w/R^2)mgL}{\text{den}} \right] X_2 + \left( \frac{2\mu_0 ml}{\text{den}} \right) X_3 + \left[ -\frac{2\mu_0 (m + 2M + 2J_w/R^2)}{\text{den}} \right] X_4 + \left[ \frac{m + 2M + 2J_w/R^2}{\text{den}} \right](\tau_L + \tau_R)$$
Where,

\[ \text{den} = \left( m + 2M + 2 \frac{J_m}{R^2} \right) (ml^2 + J_c) - (ml)^2 \]

and \( \tau_L \) and \( \tau_R \) are required torques for left and right wheels. Equation 3.41 shows the expanded version of state equation.

\[
\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\dot{X}_3 \\
\dot{X}_4
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4
\end{bmatrix}
\times u
\]

(3.41)

Each elements of \( A \) and \( B \) matrices were observed after developing the derivative form of state variables.

\[
\begin{align*}
\dot{X}_1 &= A_{11}X_1 + A_{12}X_2 + A_{13}X_3 + A_{14}X_4 \\
\dot{X}_2 &= A_{21}X_1 + A_{22}X_2 + A_{23}X_3 + A_{24}X_4 \\
\dot{X}_3 &= A_{31}X_1 + A_{32}X_2 + A_{33}X_3 + A_{34}X_4 + B_3(\tau_L + \tau_R) \\
\dot{X}_4 &= A_{41}X_1 + A_{42}X_2 + A_{43}X_3 + A_{44}X_4 + B_4(\tau_L + \tau_R)
\end{align*}
\]
Therefore, $A$, $B$, $C$ and $D$ matrices are shown below.

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
-\frac{(ml)^2 g}{\text{den}} & 2\mu_0 (ml^2 + J_c) & \frac{1}{2ml\mu_i} & 0 \\
\frac{(m+2M+2\frac{J_w}{R^2})mgl}{\text{den}} & \frac{2\mu_0 ml}{\text{den}} & -\frac{2\mu_i (m+2M+2\frac{J_w}{R^2})}{\text{den}}
\end{bmatrix}
\] (3.42)

\[
B = \begin{bmatrix}
0 \\
-\frac{ml}{\text{den}} \\
\frac{m+2M+2\frac{J_w}{R^2}}{\text{den}}
\end{bmatrix}
\] (3.43)

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\] (3.44)

\[
D = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\] (3.45)

Numeric state-space model of two-wheeled balancing robot was observed by substituting the constant values (shown in table 3.1) into equations 3.42 and 3.43. Equations 3.46 and 3.47 show the state and output equations of state-space model.

\[
\dot{X} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -24.2421 & -0.2276 & 0 \\
0 & 156.1861 & 1.044 & 0
\end{bmatrix} \times \begin{bmatrix}
x \\
\theta_p \\
v \\
\omega_p
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
-5.22 \\
33.63
\end{bmatrix}u
\] (3.46)
State-space matrices \((A, B, C\) and \(D)\) were used to observe the further analysis and simulation with MATLAB and SIMULINK. Figure 3.4 illustrates the designed block diagram of state-space model.
Figure 3.4 - State-space model block diagram
3.6. Conclusion

In this chapter differential equations of system were observed and existing constants of equations were measured. State-space model of system were designed for further simulation and analyses.
Chapter 4

Controller Design
(MATLAB & SIMULINK Implementations)
4.1. Introduction

MATLAB and SIMULINK were used in this chapter to observe the system responses and its performance. Designing a full-state feedback controller was required to stabilise the unstable balancing robot. Set point or reference point of the system is when robot stands upright.
4.2. Open-Loop Control System

Those systems in which the output has no effect on the control action are called open-loop control systems [17]. Figure 4.1 illustrates the block diagram of open-loop system.

![Block diagram of open-loop system](image)

**Figure 4.1** - Block diagram of open-loop system

Transfer function of a system is a mathematical relation between input and output of a linear system. This function can be observed by taking the Laplace transforms of differential equations of system. In this project MATLAB was used to generate the transfer functions of two-wheeled balancing robot (Appendix 4.1).

There are two proofs that show the two-wheeled balancing robot is an unstable dynamic system. These proofs are:

4.2.1. Open-Loop Impulse Response

Figure 4.2 illustrates the open-loop impulse response of displacement ($x$), pitch angle ($\theta_P$), velocity ($v$) and angular velocity ($\omega_P$) of balancing robot.
Open-loop impulse response of balancing robot was proved that open-loop system is an unstable system. Figure 4.2 shows that there is no settling point for impulse responses.

### 4.2.2. Open-Loop System Poles

The system poles were found by observing the eigenvalues of matrix $A$, defined as the solutions $\lambda$ in equation 4.1[21].

$$|A - \lambda I| = 0$$  \hspace{1cm} (4.1)

Where, $I$ is the identity matrix and the symbol ‘$| |$’ refers to the matrix determinant. Appendix 4.2 shows the algebraic calculations for observing the eigenvalues of matrix $A$. Equation 4.2 illustrates the eigenvalues of matrix $A$, which are open-loop system poles.
As equation 4.2 shows, there is one positive pole located on right-hand plane of pole-zero plot and it means open-loop system is unstable. Figure 4.3 illustrates the location of open-loop system poles.

\[
Poles = \begin{bmatrix} 0 & -12.5792 & 12.4171 & -0.0656 \end{bmatrix}
\]  

(4.2)

Figure 4.3 - Open-loop system poles

Therefore, full-state feedback control was designed to stabilise the unstable dynamic system.
4.3. Controllability and Observability

Full-state feedback design commonly relies on pole-placement techniques [4]. It was necessary to note that the dynamic system is completely controllable and observable. The observed state-space matrices in section 3 were used to check the controllability and observability of system.

- A system is completely controllable if $C_M$ matrix is of rank $n$. Where $n$ is number of state variables.

$$C_M = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$ (4.3)

Appendix 4.3 illustrates more details of generating the $C_M$ matrix. Rank of matrix $C_M$ was observed and it is equal to number of state variables. Therefore, dynamic system of balancing robot is completely controllable.

$$\text{Rank} \left( C_M \right) = 4$$

- A system is completely observable if $O_M$ matrix is of rank $n$. Where $n$ is number of state variables.

$$O_M = \begin{bmatrix} C^T & A^T C^T & \left( A^T \right)^2 C^T & \left( A^T \right)^3 C^T \end{bmatrix}$$ (4.4)

Appendix 4.4 illustrates more details of generating the $O_M$ matrix. Rank of matrix $O_M$ was observed and it is equal to number of state variables. Therefore, dynamic system of balancing robot is completely observable.

$$\text{Rank} \left( O_M \right) = 4$$
4.4. Closed-loop Control System

Figure 4.4 illustrates general block diagram of closed-loop system. Tilt sensor and gyroscope measurements were used as controller feedbacks.

![Figure 4.4 - Block diagram of closed-loop system](image)

A full-state feedback controller was designed to locate all the closed-loop system poles on left-hand plane of pole-zero plot. Figure 4.5 illustrates the block diagram of full-state feedback control.

![Figure 4.5 - Block diagram of full-state feedback control](image)
Where, $X_{ref}$ is reference of system and $K$ matrix contains negative feedback gains that were used to make the input of system proportional to the given state.

$$U = -KX = -\left( k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 \right)$$

Equation 4.5 describes the full-state feedback control shown on figure 4.5.

$$u(t) = -K \left( x(t) - x_{ref} \right) \quad (4.5)$$

State matrix of closed-loop control system were defined as $A_c = A - BK$. Therefore state equation of balancing robot system was described as equation 4.6.

$$\dot{x}(t) = (A - BK) x(t) + BK x_{ref} \quad (4.6)$$

Linear-quadratic regulator (LQR) method was used to generate the $K$ matrix. Appendix 4.1 shows required MATLAB commands to generate the $K$ matrix. Equation 4.7 illustrates the closed-loop system gains in $K$ matrix.

$$K = \begin{bmatrix} -0.006 & 82.2429 & 75.6809 & 13.9637 \end{bmatrix} \quad (4.7)$$

Equation 4.8 shows the eigenvalues of matrix $A_c$, which are the controlled closed-loop system poles.
\[ \text{Poles} = \begin{bmatrix} -0.0649 & -37.3627 + 35.2102i & +37.3627 + 35.2102i & 0 \end{bmatrix} \] (4.8)

**Figure 4.6 - Closed-loop system poles**

Figure 4.7 and 4.8 illustrates the closed-loop impulse response of displacement and pitch angle of system respectively.
**Figure 4.7** - Impulse response of robot displacement

**Figure 4.8** – Impulse response of robot pitch angle
A closed-loop SIMULINK model was designed for dynamic system of two-wheeled balancing robot. In this model, initial pitch angle of robot was set at 4.5 degree and behaviour of robot was observed. Following graphs show robot moved forward and balanced itself after 0.2 seconds. Figure 4.9 illustrates the closed-loop SIMULINK model and following plots show the robot impulse responses respectively.

**Figure 4.9 - Closed-loop SIMULINK model of balancing robot**
Figure 4.10 - Closed-loop impulse response of robot displacement

Figure 4.11 - Closed-loop impulse response of robot pitch angle
Figure 4.12 - Closed-loop impulse response of robot velocity

Figure 4.13 - Closed-loop impulse response of robot angular velocity
Table 4.1 illustrates the specifications of pitch angle closed-loop impulse response.

**Table 4.1 - Specifications of closed-loop impulse response**

<table>
<thead>
<tr>
<th>Closed-loop impulse response of Pitch angle ($\theta_p$)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Rise time ($t_r$)</td>
<td>0.05 s</td>
</tr>
<tr>
<td>Delay time ($t_d$)</td>
<td>0.03 s</td>
</tr>
<tr>
<td>Peak time ($t_p$)</td>
<td>0.09 s</td>
</tr>
<tr>
<td>Settling time ($t_s$)</td>
<td>0.15 s</td>
</tr>
</tbody>
</table>
4.5. Conclusion

Full-state feedback control was designed successfully to control the unstable two-wheeled balancing robot. Tilt sensor and gyroscope provide the closed-loop system feedback. Reference point was compared with generated feedback and microcontroller was sent the right command to motor controllers.
Chapter 5

Parametric Studies

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School of Engineering & Advanced Technology
5.1. Introduction

Two-wheeled balancing robot has some key parameters that are directly engaged with system’s performance and responses. The robot’s movement was optimised by choosing the suitable parameters and specifications for physical model. These parameters are:

1. Coefficient of friction (CoF) between wheels and ground ($\mu_0$).
2. Location of centre of gravity ($l$).
3. Radius of the wheel ($R$).

In this chapter parametric studies were done and system responses were observed by variation of key parameters based on designed closed-loop SIMULINK model in chapter 4 (Figure 4.8).
5.2. Variation of $\mu_0$

Variation of $\mu_0$ affects on some of controller gains and closed-loop impulse response of displacement. Figure 5.1 illustrates the changes of $K_1$, $K_2$, $K_3$ and $K_4$ that are the controller gains of $x$, $\theta_P$, $v$ and $\omega_P$ respectively.

**Figure 5.1 - Controller gains vs. coefficient of friction ($\mu_0$)**

Figure 5.1 illustrates that variation of $\mu_0$ has no affect on controller gains of displacement ($K_1$) and pitch angle ($K_2$). As figure 5.1 shows, controller gain of velocity ($K_3$) is more sensitive to variation of $\mu_0$. Therefore robot velocity is the most important factor that was controlled precisely for balancing the robot.

Variation of $\mu_0$ affects on impulse response of robot displacement too. Figure 5.2 illustrates that displacement overshoot decreases slightly by increasing $\mu_0$. 
Figure 5.2 - Displacement impulse response vs. coefficient of friction ($\mu_0$)

Figure 5.3 - Pitch angle impulse response vs. coefficient of friction
Figure 5.3 illustrates that variation of $\mu_0$ has minor affect on impulse response of robot’s pitch angle.
5.3. Variation of $l$

Location of centre of gravity is another important factor that affects the performance of two-wheeled balancing robot. Figure 5.3 illustrates the sensitivity of impulse response overshoot of displacement with variation of $l$.

![Displacement impulse response](image)

**Figure 5.4** - Displacement impulse response vs. $l$

Figure 5.4 illustrates that displacement overshoot increases with increasing the location of centre of gravity. Also this fact was observed by several experiments that robot was tended to fall faster when centre of gravity located at higher point of robot because of existence the higher rotation inertia. There are some singular points for $l$ that are shown by crosses on figure 5.4. These points were not chosen as location of centre of gravity of physical model. Figure 5.5 illustrates that variation of $l$ has a minor affect on impulse response of robot’s pitch angle.
Figure 5.5 - Pitch angle impulse response vs. $l$
5.4. Variation of $R$

Size of the wheel is another important factor that affects the performance of balancing robot. Bigger wheels increase the stability of robot, because it takes shorter time to travel from point A to point B. Figure 5.6 illustrates that displacement overshoot increases by increasing the radius of wheel.

![Displacement overshoot vs. R](image)

**Figure 5.6 - Displacement overshoot vs. $R$**

There were some singular points in simulation that were not chosen as size of robot’s wheel in physical model.
5.5. Conclusion

According to observed results in this chapter, ideal constants were chosen for location of centre of gravity \( (l) \) and radius of robot’s wheel \( (R) \) to improve the robot’s performance. The observed plots in this chapter show that variations of key parameters were caused small changes on robot’s performance. But several experiments were proved that deploying the observed constants on physical model had a visible improvement in robot’s motion.
Chapter 6

Results and Implementation

MASSEY UNIVERSITY
School of Engineering & Advanced Technology
6.1. Introduction

Observed results from parametric studies were applied into physical model to improve the robot performance. Figure 6.1 illustrates the final physical model of balancing robot.

Figure 6.1 - Final design of two-wheeled balancing robot
Few changes were done in final design of balancing robot to improve the performance of robot. These changes are:

- Location of centre of gravity ($l$) was pulled down and located on lower level of chassis. This change provided less rotation inertia and increased the stability of robot.
- Size of the wheels ($R$) was increased.
- Tilt sensor was located parallel to rotation axis of wheels. Therefore sensor performs more accurate and detects small changes of pitch angle.
- Initial DC motors were replaced with higher torque DC motor to increase the stability of robot.

Therefore, new physical parameters of two-wheeled balancing robot after few changes in physical model are:

\[
\begin{align*}
    m &= 3.18 \ [kg] \quad : \quad \text{Chassis Weight} \\
    J_C &= 0.022 \ [kgm^2] \quad : \quad \text{Rotation inertia of chassis} \\
    R &= 0.06 \ [m] \quad : \quad \text{Wheel radius} \\
    J_W &= 5.44 \times 10^{-4} \ [kgm^2] \quad : \quad \text{Rotation inertia of a wheel} \\
    M &= 0.25 \ [kg] \quad : \quad \text{Wheel weight} \\
    l &= 0.183 \ [m] \quad : \quad \text{Distance of the centre of mass from the wheel axle}
\end{align*}
\]
6.2. Optimised Closed-loop Impulse Responses

Figure 6.2 and 6.3 illustrates the optimised closed-loop impulse response of displacement and pitch angle of two-wheeled balancing robot. The overshoot of optimised closed-loop impulse responses were decreased compare with initial impulse responses (figures 4.7 and 4.8).

![Closed-loop impulse response of "displacement"](image1)

**Figure 6.2** - Displacement impulse response

![Closed-loop impulse response of "pitch angle"](image2)

**Figure 6.3** – Pitch angle impulse response
6.3. Kalman Filter

Kalman filter was implemented for gyroscope rate and raw tilt angle fusion. The Kalman filter is the minimum variance state estimator for linear dynamic systems with Gaussian noise [20]. Kalman filter is an iterative filter that requires two inputs or states. In this project tilt sensor and gyroscope provide two inputs of Kalman filter. A linear model is required for Kalman filter. Equation 6.1 and 6.2 illustrate the general form of a linear model using the gyroscope data respectively. Including the time (\( dt \)) between two measurements are essential.

\[
\begin{align*}
    x_{k+1} &= A \cdot x_k + B \cdot u_k \\
    \begin{bmatrix}
        \theta_p \\
        \text{bias}
    \end{bmatrix}_{k+1} &= 
    \begin{bmatrix}
        1 & -dt \\
        0 & 1
    \end{bmatrix} \begin{bmatrix}
        \theta_p \\
        \text{bias}
    \end{bmatrix}_k + 
    \begin{bmatrix}
        dt \\
        0
    \end{bmatrix} u_k
\end{align*}
\]  

(6.1) (6.2)

The output of tilt sensor and gyroscope were fused together by Kalman filter. Following lines show the simple pseudo code of Kalman filter [9].

- Read the value of the last measurement
- Update the state \( X \) of model every \( dt \) with a biased gyro measurement
- Read the angle from tilt sensor
- Calculate the difference between the measured value and predicted value
- Calculate the covariance matrix
- Calculate the Kalman gain
- Update the covariance matrix
- Calculate the error

Appendix 5 illustrates the implemented Kalman codes. For observing the performance of developed Kalman filter’s algorithm, robot was tilted back and forth randomly and figure 6.4 was generated and this figure shows that Kalman filter was implemented correctly and estimated angle followed the raw angle smoothly. Figure 6.5 illustrates the Kalman filter’s output while robot was balanced.
Figure 6.4 - Estimated angle by Kalman filter

Figure 6.5 - Kalman filter’s output in balance mode
Table 6.1 shows the sample of numeric values of raw angle and estimated angle shown on figure 6.5, when robot was balance for 2 seconds.

**Table 6.1 - Raw and estimated angle**

<table>
<thead>
<tr>
<th>Raw angle</th>
<th>Estimated angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.2</td>
<td>-0.73</td>
</tr>
<tr>
<td>-2.8</td>
<td>-0.84</td>
</tr>
<tr>
<td>-0.22</td>
<td>-0.75</td>
</tr>
<tr>
<td>-0.22</td>
<td>-0.64</td>
</tr>
<tr>
<td>-1.21</td>
<td>-0.51</td>
</tr>
<tr>
<td>-1.21</td>
<td>-0.35</td>
</tr>
<tr>
<td>-2.4</td>
<td>-0.26</td>
</tr>
<tr>
<td>-3.99</td>
<td>-0.35</td>
</tr>
<tr>
<td>-2.6</td>
<td>-0.44</td>
</tr>
<tr>
<td>-1.41</td>
<td>-0.57</td>
</tr>
<tr>
<td>1.16</td>
<td>-0.52</td>
</tr>
<tr>
<td>-0.22</td>
<td>-0.38</td>
</tr>
<tr>
<td>-1.21</td>
<td>-0.52</td>
</tr>
<tr>
<td>0.17</td>
<td>-0.33</td>
</tr>
<tr>
<td>-1.81</td>
<td>-0.24</td>
</tr>
<tr>
<td>-0.81</td>
<td>-0.24</td>
</tr>
<tr>
<td>-0.02</td>
<td>-0.1</td>
</tr>
<tr>
<td>0.37</td>
<td>0.05</td>
</tr>
<tr>
<td>0.57</td>
<td>0.25</td>
</tr>
<tr>
<td>-3.79</td>
<td>0.15</td>
</tr>
<tr>
<td>-1.21</td>
<td>0.06</td>
</tr>
<tr>
<td>2.34</td>
<td>0.08</td>
</tr>
<tr>
<td>0.37</td>
<td>-0.13</td>
</tr>
<tr>
<td>2.34</td>
<td>-0.13</td>
</tr>
<tr>
<td>1.55</td>
<td>-0.12</td>
</tr>
<tr>
<td>1.55</td>
<td>-0.04</td>
</tr>
<tr>
<td>0.37</td>
<td>0</td>
</tr>
<tr>
<td>-0.02</td>
<td>0.05</td>
</tr>
<tr>
<td>-0.62</td>
<td>0.01</td>
</tr>
<tr>
<td>-1.21</td>
<td>-0.14</td>
</tr>
<tr>
<td>1.55</td>
<td>-0.18</td>
</tr>
<tr>
<td>1.36</td>
<td>-0.22</td>
</tr>
<tr>
<td>-0.62</td>
<td>-0.36</td>
</tr>
<tr>
<td>-0.42</td>
<td>-0.46</td>
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<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>-1.01</td>
<td>-0.62</td>
</tr>
<tr>
<td>-0.81</td>
<td>-0.8</td>
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<tr>
<td>5.86</td>
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<td>-2.6</td>
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<td>-4.39</td>
<td>-0.78</td>
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<td>-0.22</td>
<td>-0.71</td>
</tr>
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<td>-0.22</td>
<td>-0.65</td>
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<td>-2.4</td>
<td>-0.69</td>
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<tr>
<td>-0.62</td>
<td>-0.57</td>
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<tr>
<td>-1.01</td>
<td>-0.5</td>
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</tr>
<tr>
<td>-1.61</td>
<td>-0.47</td>
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<tr>
<td>-2.8</td>
<td>-0.58</td>
</tr>
<tr>
<td>1.95</td>
<td>-0.4</td>
</tr>
<tr>
<td>-2.2</td>
<td>-0.48</td>
</tr>
<tr>
<td>-0.62</td>
<td>-0.44</td>
</tr>
<tr>
<td>-0.02</td>
<td>-0.35</td>
</tr>
<tr>
<td>-0.02</td>
<td>-0.24</td>
</tr>
<tr>
<td>-0.22</td>
<td>-0.15</td>
</tr>
<tr>
<td>0.96</td>
<td>0.05</td>
</tr>
<tr>
<td>-0.02</td>
<td>0.2</td>
</tr>
<tr>
<td>-1.21</td>
<td>0.26</td>
</tr>
<tr>
<td>-3.2</td>
<td>0.08</td>
</tr>
<tr>
<td>1.36</td>
<td>0.08</td>
</tr>
<tr>
<td>1.95</td>
<td>0.02</td>
</tr>
<tr>
<td>0.57</td>
<td>-0.1</td>
</tr>
<tr>
<td>1.75</td>
<td>-0.1</td>
</tr>
</tbody>
</table>
6.4. PID Controller

A proportional-integral-derivative (PID) controller was designed to generate the required input for motor controllers to control the rotation of wheels based on the Kalman filter’s output. Figure 6.6 illustrates the general block diagram of PID controller [17].

![Figure 6.6 - PID controller’s block diagram](image)

A PID controller consists of three types of controllers, which are proportional, integral and derivative controller. Each of these controllers generates the corresponding P, I and D values. Basically error \( e \) in PID controller is the difference between set point and the measured value. There are three types of error, which are used to generate the PID values:

1. Current error = set point - Kalman output
   
   \[ P = K_p \times \text{current error} \]

2. Accumulator error = accumulator error - current error
   
   \[ I = K_i \times \text{accumulator error} \]
Results and Implementations

3. Delta error = current error – previous error
   \[ D = K_d \times \text{delta error} \]

The sum of P, I and D generate the required input for HB-25 motor controller.
6.5. Robot Performance

\[ \theta_p = -2^\circ \]

\[ \theta_p = -3^\circ \]

\[ \theta_p = +2^\circ \]
Results and Implementations

$\theta_P = -1^\circ$

$\theta_P = 0^\circ$

$\theta_P = -2.5^\circ$
$\theta_P = -3^\circ$

$\theta_P = 0^\circ$

$\theta_P = -2^\circ$
Figure 6.7 - Balancing robot in action

\[
\theta_p = -2.5^\circ
\]

\[
\theta_p = 0^\circ
\]

\[
\theta_p = -1.5^\circ
\]
Figure 6.7 illustrates the performance of balancing robot. The sequential pictures in figure 6.7 show that motors turned the wheels to direction of chassis fall and kept the robot upright. Figure 6.8 and 6.9 show the measured pitch angle and displacement of balancing robot respectively while robot was balanced. Figure 6.8 shows that linear fitted line of all collected pitch angles is located at zero degree, where robot stands vertically.

![Pitch angle vs. time](image_url)

**Figure 6.8 - Pitch angle vs. time**
Figure 6.9 - Displacement vs. time

Figure 6.9 illustrates that robot’s travelling distance is longer when pitch angle is higher (as shown on first peak) for keeping the robot balance. Figure 6.10 shows the Kalman filter’s output in balance mode when a disturbance force disturbed the robot’s balancing.
Figure 6.10 - Estimated angle vs. disturbance force
6.6 Conclusion

A closed-loop control unit was implemented and desired performance of robot was observed. Fusion of tilt sensor’s output and gyroscope rate by Kalman filter was an essential step for developing a two-wheeled balancing robot that increase the smoothness of robot’s manoeuvre. Increasing the number of gyroscope and tilt sensor will improve the performance of balancing robot as well as deploying two encoders and better DC motors with higher torque.
Chapter 7

Overall Conclusion

MASSEY UNIVERSITY
School of Engineering & Advanced Technology
Overall Conclusion

Two-wheeled robots provide higher level of mobility and manoeuvrability. The application of two-wheeled balancing robot varies with different environment and requirements. Understanding the classical theory of inverted pendulum and its dynamic system are initial steps for developing a two-wheeled balancing robot. Safety and robustness of two-wheeled products are so important and these aspects will be achieved by employing accurate sensors and designing a precise control unit. Initial structure of two-wheeled balancing robot was developed. Symmetrical weight distribution on robot’s chassis was an important step for designing the structure. Best location of centre of gravity and sensors were determined after mathematical simulation. Differential equations of system were observed and existing constants of equations were measured. State-space model of system were designed for further simulation and analyses. Full-state feedback control was designed successfully to control the unstable two-wheeled balancing robot. Tilt sensor and gyroscope provide the closed-loop system feedback. Reference point was compared with generated feedback and microcontroller was sent the right command to motor controllers. According to observed results of parametric studies, ideal constants were chosen for location of centre of gravity (l) and radius of robot’s wheel (R) to improve the robot’s performance. A closed-loop control unit was implemented and desired performance of robot was observed. Fusion of tilt sensor’s output and gyroscope rate by Kalman filter was an essential step for developing a two-wheeled balancing robot that increase the smoothness of robot’s manoeuvre. Increasing the number of gyroscope and tilt sensor will improve the performance of balancing robot as well as deploying two encoders and better DC motors with higher torque.
References


Appendices
Appendix 1

Appendix 1.1 - iBot Functionalities

Figure A1.1a illustrates that iBot provides eyes contact for disable person and figure A1.1b illustrates that iBot provides higher level seat for disable person to reach an item located at higher level.

(a)  (b)

Figure A1.1 - iBot functionalities
Appendix 1.2 - Hitachi Balancing Robot

Figure A1.2 shows the EMIEWs at office. EMIEW is 70 kg with 130 cm height and EMIEW2 is 13 kg with 80 cm height.

![EMIEW and EMIEW2](image)

(a) EMIEW  
(b) EMIEW2

**Figure A1.2 - Hitachi balancing robots**
Appendices

Appendix 2

Appendix 2.1 - Arduino Duemilanove Specifications

Table A2.1 - Arduino duemilanove USB module specifications

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Microcontroller</td>
<td>ATmega328</td>
</tr>
<tr>
<td>Operating Voltage</td>
<td>5V</td>
</tr>
<tr>
<td>Input Voltage (recommended)</td>
<td>7-12V</td>
</tr>
<tr>
<td>Input Voltage (limits)</td>
<td>6-20V</td>
</tr>
<tr>
<td>Digital I/O Pins</td>
<td>14 (of which 6 provide PWM output)</td>
</tr>
<tr>
<td>Analogue Input Pins</td>
<td>6</td>
</tr>
<tr>
<td>DC Current per I/O Pin</td>
<td>40mA</td>
</tr>
<tr>
<td>DC Current for 3.3V Pin</td>
<td>50mA</td>
</tr>
<tr>
<td>Flash Memory</td>
<td>32KB (of which 2KB used by boot loader)</td>
</tr>
<tr>
<td>SRAM</td>
<td>2KB</td>
</tr>
<tr>
<td>EEPROM</td>
<td>1KB</td>
</tr>
<tr>
<td>Clock Speed</td>
<td>16 MHz</td>
</tr>
</tbody>
</table>
Appendix 2.2 - Gyroscope CRS03

Table A2.2 - CRS03 pin definition

<table>
<thead>
<tr>
<th>Pin Number</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+VDC</td>
</tr>
<tr>
<td>2</td>
<td>Ground (0V)</td>
</tr>
<tr>
<td>3</td>
<td>Rate Output</td>
</tr>
</tbody>
</table>

Figure A2.1 - CRS03 pin definition
Appendix 2.3 - Memsic 2125 Dual Axis Accelerometer

Figure A2.2 - Memsic 2125 dual axis thermal accelerometer

Figure A2.3 - Memsic 2125 pin definitions
Appendix 2.4 - HB-25 Motor Controller

![HB-25 motor controller](image)

**Figure A2.4 - HB-25 motor controller**

<table>
<thead>
<tr>
<th>Pin</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vin</td>
<td>Power input terminal, +6 to +16 VDC</td>
</tr>
<tr>
<td>GND</td>
<td>Negative side of battery terminal</td>
</tr>
<tr>
<td>M1 &amp; M2</td>
<td>Motor connectors (Polarity reverses direction)</td>
</tr>
<tr>
<td>W</td>
<td>Servo pulse input</td>
</tr>
<tr>
<td>R</td>
<td>Not connected</td>
</tr>
<tr>
<td>B</td>
<td>Servo ground</td>
</tr>
<tr>
<td>J</td>
<td>Mode jumper pins</td>
</tr>
</tbody>
</table>

**Table A2.3 - HB-25 pin definitions**
Figure A2.5 - HB-25 pin definitions

Table A2.4 - HB-25 specifications

<table>
<thead>
<tr>
<th>Specification</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Motor Supply</td>
<td>6.0 VDC - 16.0 VDC</td>
</tr>
<tr>
<td>Load Current</td>
<td>25A Continuous 35A Surge (13.8 V)</td>
</tr>
<tr>
<td>Standby Current</td>
<td>50 mA @ 6 V, 80 mA @ 13.8 V (fan on)</td>
</tr>
<tr>
<td>PWM Frequency</td>
<td>9.2 KHz</td>
</tr>
<tr>
<td>Pulse Input</td>
<td>1.0 ms Full Reverse</td>
</tr>
<tr>
<td></td>
<td>1.5 ms Neutral (off)</td>
</tr>
<tr>
<td></td>
<td>2.0 ms Full Forward</td>
</tr>
<tr>
<td>Modes</td>
<td>Single/Dual Motor Control</td>
</tr>
<tr>
<td>Weight</td>
<td>71 grams</td>
</tr>
<tr>
<td>Size</td>
<td>40.6 mm × 40.6 mm × 50.0 mm</td>
</tr>
</tbody>
</table>
Appendix 2.5 - EMG30 DC Motor

Table A2.5 - EMG30 gear-head DC motor specifications

<table>
<thead>
<tr>
<th>Specification</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rated Voltage</td>
<td>12 V</td>
</tr>
<tr>
<td>Rated Torque</td>
<td>1.5 Kg/cm</td>
</tr>
<tr>
<td>Rated Speed</td>
<td>170 rpm</td>
</tr>
<tr>
<td>Rated Current</td>
<td>530 mA</td>
</tr>
<tr>
<td>No Load Speed</td>
<td>216</td>
</tr>
<tr>
<td>No Load Current</td>
<td>150 mA</td>
</tr>
<tr>
<td>Stall Current</td>
<td>2.5 A</td>
</tr>
<tr>
<td>Rated Output</td>
<td>4.22 W</td>
</tr>
</tbody>
</table>

Figure A2.6 - EMG30 gear-head DC motor
Appendix 2.6 - Physical Model

Figure A2.7 - Dimension of plastic levels in millimetres

Figure A2.8 - Developed structure for two-wheeled balancing robot
Appendices

Appendix 3

Appendix 3.1 - Bench Scale

Figure A3.1 - SJ-HS bench scale
Appendix 3.2 - Rotation Inertia of Robot’s Chassis

\[ I_C = \frac{1}{12} m \left( h^2 + d^2 \right) \]

Where,

\( h = 0.26 \) m
\( d = 0.12 \) m

Then,

\[ I_C = \frac{1}{12} \times (2.63 \text{ Kg}) \times \left( (0.26 \text{ m})^2 + (0.12 \text{ m})^2 \right) \]

\[ I_C = 0.018 \text{ Kgm}^2 \]

Figure A3.2 - Robot’s chassis
Appendix 3.3 - Rotation Inertia of Robot’s Wheel

\[ I_W = \frac{1}{2} MR^2 \]

Where,

\[ M = 0.14 \text{ Kg} \]
\[ R = 0.05 \text{ m} \]

Then,

\[ I_W = \frac{1}{2} \times (0.14 \text{ Kg}) \times (0.05 \text{ m})^2 \]

\[ I_W = 1.75 \times 10^{-4} \text{ Kgm}^2 \]

Figure A3.3 – Robot’s wheel
Appendix 3.4 - Algebraic calculations for observing $\dot{X}_3$ and $\dot{X}_4$

First step is to cancel the $\dot{X}_4$ term and observe the $\dot{X}_3$.

\[
\left( m + 2M + 2\frac{J_w}{R^2} \right)\dot{X}_3 + ml\dot{X}_4 + 2\mu_0 X_3 = 0
\]

\[
- \frac{ml}{ml^2 + J_C} \left[ ml\dot{X}_3 + (ml^2 + J_C)\dot{X}_4 - mgl X_2 + 2\mu_1 X_4 = \tau_L + \tau_R \right]
\]

\[
\left( m + 2M + 2\frac{J_w}{R^2} \right)\dot{X}_3 + ml\ddot{X}_4 + 2\mu_0 X_3 = 0
\]

\[
- \frac{m^2 l^2}{ml^2 + J_C} \dot{X}_3 - ml\dot{X}_4 + \frac{m^2 l^2 g}{ml^2 + J_C} X_2 - \frac{2ml\mu_1}{ml^2 + J_C} X_4 = - \frac{ml}{ml^2 + J_C} (\tau_L + \tau_R)
\]

Equivalent terms of above equations were added together. Therefore,

\[
\left( m + 2M + 2\frac{J_w}{R^2} \right) - \frac{m^2 l^2}{ml^2 + J_C} \left[ \dot{X}_3 + \frac{m^2 l^2 g}{ml^2 + J_C} X_2 + 2\mu_0 X_3 - \frac{2ml\mu_1}{ml^2 + J_C} X_4 \right] = - \frac{ml}{ml^2 + J_C} (\tau_L + \tau_R)
\]
The long term in equation A3.1 was replaced with den to shorten the equation.

\[
\text{den} = \left( m + 2M + \frac{J_w}{R^2} \right) \left( ml^2 + J_c \right) - \left( ml \right)^2
\]

Therefore,

\[
\dot{X}_3 = -\left( \frac{ml^2 + J_c}{\text{den}} \right)\dot{X}_3 = -\left( \frac{ml^2}{\text{den}} + J_c \right) - 2\mu_0 X_3 + \frac{2m\mu_1}{m^2 + J_c} X_4 - \frac{ml}{m^2 + J_c} \left( \tau_L + \tau_R \right)
\]
Equation A3.2 shows the equivalent expression of $\dot{X}_3$.

$$\dot{X}_3 = \left[ -\frac{(ml)^2 g}{\text{den}} \right] X_2 + \left[ -\frac{2\mu_0 (ml^2 + J_C)}{\text{den}} \right] \dot{X}_3 + \left[ \frac{2ml\mu_i}{\text{den}} \right] X_4 + \left[ -\frac{ml}{\text{den}} \right] (\tau_L + \tau_R) \quad \text{(A3.2)}$$

Next step is to cancel the $\dot{X}_3$ term and observe $\dot{X}_4$.

$$\left( m + 2M + 2\frac{J_w}{R^2} \right) \dot{X}_3 + ml\dot{X}_4 + 2\mu_0 X_3 = 0$$

$$-\left( m + 2M + 2\frac{J_w}{R^2} \right) \dot{X}_3 + \left[ \frac{m + 2M + 2\frac{J_w}{R^2}}{ml} \right] \dot{X}_4 + \left[ \frac{m + 2M + 2\frac{J_w}{R^2}}{ml} \right] X_2 - \left[ \frac{2\mu_i \left( m + 2M + 2\frac{J_w}{R^2} \right)}{ml} \right] X_4 = \left( m + 2M + 2\frac{J_w}{R^2} \right) (\tau_L + \tau_R)$$
\[
- \frac{\text{den}}{ml} \dot{X}_4 = \left[ - \left( m + 2M + 2 \frac{J_w}{R^2} \right) g \right] X_2 - 2\mu_0 X_3 + \frac{2\mu_1 \left( m + 2M + 2 \frac{J_w}{R^2} \right)}{ml} X_4 + \left[ \frac{m + 2M + 2 \frac{J_w}{R^2}}{ml} \right] \left( \tau_L + \tau_R \right)
\]

Equation A3.3 shows the equivalent expression of \( \dot{X}_4 \).

\[
\dot{X}_4 = \left[ \frac{m + 2M + 2 \frac{J_w}{R^2}}{\text{den}} \right] mgl X_2 + \left( \frac{2\mu_0 ml}{\text{den}} \right) X_3 + \left[ \frac{2\mu_1 \left( m + 2M + 2 \frac{J_w}{R^2} \right)}{\text{den}} \right] X_4 + \left[ \frac{m + 2M + 2 \frac{J_w}{R^2}}{\text{den}} \right] \left( \tau_L + \tau_R \right) \quad (A3.3)
\]
Appendix 4

Appendix 4.1 - Following m file was written to generate the open- and closed-loop transfer functions of displacement and pitch angle. Also the relevant impulse responses were plotted by MATLAB.

% close all the open boxes and clear the workspace

clear all;
close all;

% defining the constants of linear differential equations of system

g=9.81; % gravity acceleration (m/s^2)
rad=0.05; % wheel radius (m)
M=0.14; % wheel mass (kg)
m=2.63; % chassis mass (kg)
Jw=1.75e-4; % rotation inertia of wheel (kg*m^2)
Jc=0.018; % rotation inertia of body (kg*m^2)
l=0.18; % distance of the center of mass from the wheel axle (m)
u0=0.1; % coefficient of friction between wheel and ground
u1=0; % coefficient of friction between chassis and wheel
% denominator

\[ d = \left( (m+(2*M)+(2*(Jw/(\text{rad}^2))))*(m*(l^2))+Jc \right) - (m*l)^2; \]

% defining the state-space matrices

\[
\begin{align*}
A &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{g*(m*l^2)}{d} & -\frac{2*(m*l^2)+Jc}{d} & \frac{2*m*l*u1}{d} \\ 0 & \frac{(m*g*l)*(m+(2*M)+(2*(Jw/(\text{rad}^2))))}{d} & -\frac{2*(m+(2*M)+(2*(Jw/(\text{rad}^2))))}{d} & \frac{(m+(2*M)+(2*(Jw/(\text{rad}^2))))*u1}{d} \end{bmatrix}; \\
B &= \begin{bmatrix} 0 \\ 0 \\ -\frac{m*l}{d} \\ \frac{m+2*M+2*Jw/\text{rad}^2}{d} \end{bmatrix}; \\
C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \\
D &= \begin{bmatrix} 0 \\ 0 \end{bmatrix};
\]
% defining the time interval

\[ t=0:0.001:0.5; \]

% developing transfer functions of displacement and pitch angle in open-loop System

\[ [\text{num},\text{den}]=\text{ss2tf}(A,B,C,D,1); \]
\[ \text{sys1}=\text{tf}(\text{num}(1,:),\text{den}); \quad \% \text{transfer function of pitch angle} \]
\[ \text{sys2}=\text{tf}(\text{num}(2,:),\text{den}); \quad \% \text{transfer function of displacement} \]

% Linear Quadratic Regulator method was used to develop the closed-loop system

\[ x=5000; \]
\[ y=6000; \]
\[ Q=[x \ 0 \ 0 \ 0; 0 \ y \ 0 \ 0; 0 \ 0 \ 0 \ 0; 0 \ 0 \ 0 \ 0]; \]
\[ R=1; \]
\[ K=\text{lqr}(A,B,Q,R); \]
\[ A_c=A-B*K; \]
\[ B_c=B; \]
\[ C_c=C; \]
\[ D_c=D; \]
% developing transfer functions of displacement and pitch angle in closed-loop System

[numc,denc]=ss2tf(Ac,Bc,Cc,Dc,1);

sysc1=tf(numc(1,:),denc);       % transfer function of pitch angle
sysc2=tf(numc(2,:),denc);       % transfer function of displacement

% plotting the open-loop impulse responses of displacement and pitch angle

impulse(sys1,t);

title('Open-loop impulse response of "displacement"', 'fontsize', 9, 'fontweight', 'b');
ylabel('Displacement (m)');

grid

figure;

impulse(sys2,t);

title('Open-loop impulse response of "pitch angle"', 'fontsize', 9, 'fontweight', 'b');
ylabel('Pitch Angle (rad.)');

grid

figure;
% plotting the closed-loop impulse responses of displacement and pitch angle

impulse(sysc1,t);

title('Closed-loop impulse response of "displacement"','fontsize',9,'fontweight','b');
ylabel('Displacement (m)');

grid

figure;

impulse(sysc2,t);

title('Closed-loop impulse response of "pitch angle"','fontsize',9,'fontweight','b');
ylabel('Pitch Angle (rad.)');

grid
Appendix 4.2 - Eigenvalues of matrix $A$

\[ |A - \lambda I| = 0 \]  \hspace{1cm} (A4.1)

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -24.2421 & -0.2276 & 0 \\
0 & 156.1861 & 1.044 & 0
\end{bmatrix}
- \lambda
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -24.2421 & -0.2276 & 0 \\
0 & 156.1861 & 1.044 & 0
\end{bmatrix}
- \lambda
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & \lambda
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
-\lambda & 0 & 1 & 0 \\
0 & -\lambda & 0 & 1 \\
0 & -24.2421 & -0.2276 & -\lambda \\
0 & 156.1861 & 1.044 & -\lambda
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
-\lambda & 0 & 1 \\
0 & -\lambda & 1 \\
0 & -24.2421 & -\lambda \\
0 & 156.1861 & -\lambda
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
-\lambda & -0.2276 - \lambda & 0 \\
0 & 1.044 & -\lambda \\
-24.2421 & -0.2276 - \lambda & 156.1861 & 1.044 \\
-\lambda & 156.1861 & 1.044 & -\lambda
\end{bmatrix}
+ \lambda
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -\lambda & 0 & 1 \\
0 & 156.1861 & 0 & -24.2421
\end{bmatrix}
= 0
\]
The roots of equation A4.2 are the open-loop system poles.
Appendix 4.3 - Observing the $C_M$ matrix

$$C_M = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

$$C_M = \begin{bmatrix}
0 & -5.22 & 0.0012 & -0.8156 \\
0 & 33.6314 & -0.0054 & 5.254 \\
-5.22 & 1.1882 & -0.8156 & 0.3178 \\
33.6314 & -5.4498 & 5.254 & -1.7026 \\
\end{bmatrix}$$

$\text{Rank } (C_M) = 4$
Appendix 4.4 - Observing the $O_M$ matrix

$$O_M = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & (A^T)^3 C^T \end{bmatrix}$$

$$O_M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -24.2421 & 156.1861 & 5.5175 & -25.3088 \\ 0 & 0 & 1 & 0 & -0.2276 & 1.044 & 0.0518 & -0.2376 \\ 0 & 0 & 0 & 1 & 0 & 0 & -24.2421 & 156.1861 \end{bmatrix}$$

$Rank\ (O_M) = 4$
Appendix 5

#include <math.h>

/*******************************

//Defining the Constants

*******************************/
#define NumReadings 5 // number of reading from gyroscope
#define GyroPin 0 // assign the gyroscope pin
#define xPin 2 // tilt sensor output pin for X axis
#define yPin 3 // tilt sensor output pin for Y axis
#define HB25R 9 // assign a pin for right motor controller
#define HB25L 10 // assign a pin for left motor controller
#define SF 0.02 // a constant used for rounding the gyro’s
    // output
const int Current=0;
const int Accumulator=1;
const int Previous=2;
const int Delta=3;
//Defining the variables

//*******************************************
//Defining the variables
//*******************************************

float error[4]; // define the error matrix with 4 elements
float oldAngle=0; // assign the oldAngle as zero when robot is
// balance
float P=0; // initialise the P, I and D value for PID
  // controller
float I=0;
float D=0;
float Kp=20; // assign the Kp, Ki and Kd values of PID
  // controller
float Ki=0.2;
float Kd=1;
float motors=0; // initialise the command value for motor
  // controller
float dt=0.6;
float angle=0; // assign the current angle
float q_bias=0; // initialise the gyro bias
float rate=0; // assign the unbiased angular rate
float q_m=0;
float R_angle=100; // assign the covariance noise matrix, which
    // is a 1x1 matrix

int readings[NumReadings]; // define the gyroscope output as a matrix

int index=0;

int total=0; // initialise the total and average of angle
    // value

int average=0;

int mydt=30;

int pulseX,pulseY; // define the required variables for tilt sensor

int accelerationX,accelerationY;

int ax_m=0;

int ay_m=0;

int cnt=0;

unsigned long lastread=0;

unsigned long startmillis=0;

static float p[2][2]= {{1,0},{0,1},}; // assign the covariance matrix of Kalman
    // filter

static const float Q_angle=0.001;

static const float Q_gyro=0.003;
```cpp
void setup()
{
    Serial.begin(9600);
    pinMode(xPin, INPUT);  // define the tilt sensor pins as input pins
    pinMode(yPin, INPUT);
    for(int i=0; i<NumReadings; i++)
        readings[i]=0;     // initialise all the gyroscope readings to zero
    startmillis=millis();
    P=0;
    I=0;
    D=0;
}

//*******************************************

void loop()
{
    int delta=millis()-lastread;
    if(delta>=mydt)  // sample gyroscope output every dt ms
        {
```
lastread=millis();

total-=readings[index];     // subtract the last gyro reading from total value

readings[index]=analogRead(GyroPin);      // read gyro’s output

readings[index]=((readings[index]/204.8)/SF)-125;       // centering the gyro’s output

// output

total+=readings[index];       // add every gyro reading to total

index=index+1;               // increase the index of reading

if (index>=NumReadings)       // reset the index if index is greater than 5
index=0;

average=total/NumReadings;      // averaging the gyro’s output

q_m=((float)average)*(1500.0/1024.0)*PI/180;

const  float  q=q_m-q_bias;

cost  float  pdot[2*2] = {
    Q_angle-p[0][1]-p[1][0],
    -p[1][1],
    -p[1][1],
    Q_gyro
};
rate=q;                           // store the unbiased gyro estimate
angle+=q*dt;                      // update the angle estimate

p[0][0] += pdot[0] * dt;         // update the covariance
                               // matrix
p[0][1] += pdot[1] * dt;
p[1][0] += pdot[2] * dt;
pulseX = pulseIn(xPin,HIGH);     // read the tilt sensor’s output
ax_m = (((pulseX / 10) - 500) * 8)-7.125;  // X acceleration
float  angle_m=atan2(-4000,ax_m);
angle_m = (angle_m*100)+146.10;    // centring the tilt sensor’s
                               // output

const  float angle_err=angle_m-angle;  // observing error
const  float c_0=1;
const  float PCt_0 = c_0 * p[0][0];
const  float PCt_1 = c_0 * p[1][0];
const  float E =R_angle+ c_0 * PCt_0;
const  float K_0 = PCt_0 / E;
const  float K_1 = PCt_1 / E;
const float t_0 = PCt_0;
const float t_1 = c_0 * p[0][1];

p[0][0] -= K_0 * t_0;
p[0][1] -= K_0 * t_1;
p[1][0] -= K_1 * t_0;
p[1][1] -= K_1 * t_1;
angle += K_0 * angle_err;

q_bias += K_1 * angle_err;
error[Current] = angle-myAngle;

// another method to generate the PID values.
/*P=Kp*error[Current];
error[Accumulator] = error[Accumulator] + error[Current];
I =Ki*error[Accumulator];
error[Delta] = error[Current]-error[Previous];
D = Kd * error[Delta];
error[Previous]=error[Current];*/
float   Index=0;

float   myAngle=angle+Index;

P=Kp*myAngle;   // observing the PID values according to  
// Kalman filter’s output

I =Ki*myAngle;

D = Kd * myAngle;

if ((angle<-20) || (angle>20))  // stop the motors when robot falls over
{
    motors=185;
} else {
    motors=P+I+D+194;  // generating the required command for
// motor controllers, based on its calibration
// graph
}

if (motors>=250) motors = 250.0; // defined boundaries for motor’s value
if (motors<=110) motors = 110.0;
analogWrite(HB25L,motors-1.2); // send signal to motor controllers
analogWrite(HB25R,motors);
delay(1.5);

analogWrite(HB25L,0);
analogWrite(HB25R,0);

delay(6.0);

oldAngle=myAngle;
Serial.print(angle_m);
Serial.print("\t");
Serial.print(angle);
Serial.print("\t");
Serial.println(ax_m);

}