Optimal Robust Control Systems Design and Analysis By State Space Approaches

A thesis presented in partial fulfilment of the requirements for the degree of Doctor of Philosophy Technology at Massey University.

Hao Wei

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To my mother for her love

仅以此论文献给我最亲爱的母亲

ABSTRACT

This thesis provides a fundamental investigation of robust control, both the issues of robust controller design and robustness analysis of control systems are addressed. The techniques presented evolve from time domain descriptions of linear systems and employ state space approaches. A comprehensive review of the field is given and several significant advances are presented. These include some new design and analysis techniques and some new perspectives on existing techniques. The thesis is fundamental in nature, systematically developing and criticising algorithms and methodologies. Some numerical examples are employed to illustrate the results.

Robust control addresses problems caused by discrepancies between nominal system models used for conventional controller design and analysis, and actual 'real' systems. Much of the classical work in the field assumed no knowledge of possible (or even probable) uncertainties and considered system tolerance to some general, imprecise classes of discrepancy. This tended to lead to conservative designs which degraded system performance to an unnecessary extent.

The modern trend is to provide a 'precise' prediction of possible (probable) uncertainties, described by an uncertainty model. This aims to avoid the consideration of unfeasible discrepancies which often caused the conservatism and will tend to minimise performance degradation. However, tolerance to further (hopefully small) unpredicted uncertainties should still be considered as such residual discrepancies will always exist. This modern trend is supported in this thesis and one of the main potential benefits of the new methodologies will be less conservative designs.

The principle contributions include: systematic methods for the design of cost-optimal robust controllers for both full state feedback and output feedback systems. These explicitly consider a nominal system model and an admissible domain of uncertainties and also provide some inherent robustness to residual uncertainties. Furthermore, a new method for the analysis of the robustness of given full state feedback controllers is presented. For an admissible domain of uncertainty of given structure, the maximal magnitude is determined such that stability and performance criteria are upheld.

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CHAPTER 1

INTRODUCTION

This thesis is concerned with uncertainties in system models which affect control system performance. The analysis is performed predominantly in the time domain using state space approaches. The ability of a controller to perform satisfactorily in the presence of uncertainty is called "Robustness", it is the motivation behind entire works and to which increasing attention is being paid. The current theories of designing a feedback controller such that the controlled system has good robustness (Robust Controller Design) and analysing the robustness for a given controlled system (Robustness Analysis) have been summarised in this thesis and have been further developed. It is believed that a robust feedback control system is necessary for many applications and hence that it will have a great effect on industrial control practice.

1.1 ROBUST CONTROL

Mathematical models are commonly used in the design and analysis of control systems. They will not give a totally accurate description of the real system and this discrepancy may lead to operational problems: performance degradation or even instability. Robust control aims to tackle the problems created by such discrepancies. The field may essentially be divided into two areas: robust controller design, this aims to produce controllers which can tolerate plant/model discrepancies and robustness analysis, for any given controller this enables us to evaluate how much discrepancy may be tolerated. To expand further, robustness has two facets: stability robustness and performance robustness. In the presence of plant/model discrepancy, stability robustness describes the ability to remain stable and performance robustness describes the ability to minimise performance degradation (as measured by some performance index). Robustness is a critical attribute for satisfactory operation of control systems and hence is very important for industrial applications. The slow acceptance of advanced modem control algorithms in industrial applications may be attributed to the insufficient attention paid to the robustness issue. It is now common for controller design and analysis methodologies to explicitly consider robustness.

To perform robust controller design or robustness analysis, a model of the discrepancy, often termed *model uncertainty*, between the nominal model and the real plant is needed. It is necessary to have a description, termed the *uncertainty model*, of the possible uncertainties which the controller should tolerate. Uncertainties may be described in two ways: *parametrically* or *nonparametrically*. Parametric uncertainty models describe possible parameter variations thus prescribing admissible changes in parameter values in the nominal model that should be tolerated. Typically such changes will occur due to variations in masses, stiffness or inertia's in a dynamic system. Nonparametric uncertainty models give a more general description of possible model/plant discrepancies, they are commonly expressed as additive constrained transfer functions. Nonparametric discrepancies will typically occur when high order dynamics are neglected during model reduction, linearisation of a nonlinear system model and due to the approximation of continuum by lumped parameter models. In addition to the predicted, likely discrepancies described above, a control system should tolerate some, preferably small amount, of unpredicted discrepancy as this will always arise. This is termed the need for inherent robustness to unknown residual uncertainty.

So, for controller design and analysis we are typically given a nominal model, an uncertainty model and some unknown residual uncertainty. Thus, design and analysis are performed with respect to both the nominal model and the uncertainty model and the system's ability to tolerate both modelled and residual unknown uncertainty is considered. However, if the uncertainty is believed to be negligible or no prediction of it may be made, there may be no uncertainty model available. In this case design and analysis is performed with respect to the nominal model alone and it's ability to tolerate some unknown residual uncertainty is considered. In such cases it is likely that a larger residual unknown uncertainty will have to be tolerated.

By definition, the discrepancy between the nominal system model and the 'real' system can never be exactly known thus some prediction of it is made. When the nominal model does not adequately describe the 'real' system then tolerance to such discrepancy will prevent undesirable performance degradation, this is the reason of robust controller design. However, when a control system is made robust, an opposite consequence must be taken, i.e., when the nominal model adequately describes the 'real' system the performance of a robust controller is often worse than that of a controller 'optimised' for the nominal model alone. There is an inherent trade off between robustness and performance when the system behaves as the nominal model. So robust controller design and analysis requires a precise prediction of possible uncertainties, if the prediction is too conservative or simply imprecise then unnecessary performance degradation will result, if it is too optimistic then the control system may not be able to tolerate discrepancies that actually arise.

Robustness to plant/model discrepancy is related to but not equivalent to robustness to process disturbance or measurement noise. The latter is often termed disturbance/noise rejection or sensitivity reduction. Techniques for creating robustness to disturbance/noise such as $H\infty$ design, have been applied to the plant/model discrepancy robustness problem with varying degrees of success. An alternative approach is to transfer the plant/model discrepancy problem to an equivalent one of disturbance/noise rejection and then to employ a sensitivity reduction technique.

This thesis generally addresses multivariable feedback controllers, however, for the purposes of robust controller design and analysis these are further delineated into full state feedback controllers and dynamic output feedback controllers. Other configurations such as partial state feedback or static output feedback controllers exist but their robustness is not explicitly considered.

1.2 Review of Previous Work and Related Literature

Over the past six decades there has been a great deal of interest in the problem of robust control and this can be divided into two areas: nominal model based and uncertainty model based. Nominal model based methods for design do not generally refer explicitly to an uncertainty model, the design is often performed iteratively with respect to some analysis procedure. Such analysis procedures include gain and phase margin or H_∞ bound criteria. The major drawback with nominal model based methods is that they tend to produce conservative designs and hence unnecessarily large performance degradation. This is essentially due to the lack of explicit consideration of existing information of uncertainty in the design, thus the development of uncertainty model based methods was motivated. Uncertainty model based methods typically refer to both a nominal system model and an uncertainty model thus the robustness can be tailored to tolerate predicted plant/model discrepancies.

1.2.1 Nominal model based robust design and analysis

The "robust control problem" appeared in the literature for the first time in the early thirties, it was firstly studied as a "sensitivity reduction problem" by Black (1927), Bode (1945) and Nyquist (1932), hence the period from 1927 to 1960 can be called the classical sensitivity design period. The focus during this period was on stability, sensitivity reduction and noise suppression in single-input single-output (SISO) systems. A good review of these works can be found in Horowitz (1963).

The next major period was between 1960 and 1975, this is called the state variable period. A number of key state-variable concepts, such as controllability, observability, optimal linear quadratic state feedback, optimal state estimation (Kalman filtering), etc., were introduced in the early 1960's by R. E. Kalman, and some major results associated with this period may be found in Anderson and Moore (1971). At the same time, there were also some attempts to extend SISO sensitivity results to MIMO systems (Cruz, 1973).

Unfortunately, the problem of robustness of plant uncertainty was largely ignored during this period.

The foundation of modern robust control was also laid in this period by two important papers. Zames (1963) introduced the concept of the "small gain" principle which plays a key role in robust stability criteria and Kalman (1964) demonstrated that for SISO systems optimal linear quadratic (LQ) state feedback control laws had some very strong robustness properties in terms of gain and phase margins. Safonov (1977) demonstrated that these gain and phase margins results extended to MIMO systems for gain and phase variations in each input channel to the plant. Unfortunately, when state-estimate feedback is used instead of state feedback, Doyle (1978) showed that these desirable robustness properties vanish. This caused a resurgence in the interest in robustness and started the modern robust control period which is still very active.

The first major result of the modern period was from Doyle and Stein (1979) who were able to show that the desirable robustness properties of the optimal LQ state feedback control law could be recovered by suitable design of the Kalman filter in the feedback loop. This idea motivated the development of the LQG/LTR (linear-quadratic-Gaussian loop transfer recovery) approach by Doyle and Stein (1981). Safonov (1980) presented a generalised stability criterion which was useful for the study of robustness in multivariable systems, this book also contains an excellent summary of LQG robustness and stability results and it is the first book on feedback systems to include the term robust.

One powerful method of the modern period is the H^∞ optimisation control technique. The investigation of H^∞ optimisation of control systems was begun by Zames (1979). He found that a possible way to reduce the sensitivity of control system is the minimisation of the H^∞ norm of the sensitivity function of a SISO linear feedback system. The H^∞ optimisation design method based on frequency domain methods and transfer function descriptions is comprehensively described by Frances (1987). Doyle et. al (1989) provided a solution of the H^∞ optimal control problem for regular systems by state space methods. This gives a useful controller synthesis methodology from a state space time domain problem description. When it was recognised that the H^∞ optimisation approach can deal with robustness far more directly than the current optimisation methods, it was soon extended to more general problems. Kwakernaak (1993) in his summary paper, provides a comprehensive review of how robust control systems may successfully be designed by H^∞ optimisation.

Mixed H_2/H_∞ optimal control design is introduced by Bernstein and Haddad (1989), Rotea and Khargonekar (1991), they attempt to design a controller such that a H_2 norm performance measure is minimised subject to an H_∞ norm constraint. However, direct analytic solutions to this design problem are not yet available. The problem of mixed H_2/H_∞ optimal control design is still not totally solved and is currently receiving much attention.

This section has reviewed previous work on controller design and analysis which is based on a nominal system model and assuming no knowledge of the uncertainties. However, it is acknowledged that, if some reasonably accurate prediction of the possible uncertainties may be made then less conservative design may be achieved. Thus uncertainty model based robust controller design and analysis has evolved.

1.2.2 Uncertain model based robust design and analysis

Over the past two decades there has been a great deal of interest in the problem of robust control to avoid the conservative description of uncertainty, some robust design and analysis methods based on uncertainty models have been developed in the modern robust control period. A notable recent paper (Douglas and Athans, 1994) promotes a design method which offers robustness against both modelled parametric uncertainties and residual unknown uncertainties.

Early work concentrated on the guarantee of stability for all admissible modelled uncertainties using Lyapunov stability theory (Barmish et al., 1983; Barmish and Galimidi, 1986 and Chen, 1988), and they lead to the concept of quadratic stability for uncertain linear systems. A Riccati equation approach to the design of such robust stabilising controllers was developed by Petersen and Hollot (1986), Petersen (1987) and Khargonekar et al. (1990). Furthermore Khargonekar et al. (1990) established a connection to H_∞ controller synthesis, this enables results on H_∞ control to be applied to the problem of robust stabilisation. However, none of these methods address the issue of cost performance.

Chang and Peng (1972) presented the Guaranteed Cost Control approach which provided an upper performance bound for all admissible uncertainty values. Uncertainty was present in the system matrix alone and the admissible domain was described by a specific structured format. Here the nominal system model is in state space form and the performance measure is taken to be the maximum of a quadratic cost function over all admissible parameter variations. This approach leads to a guaranteed level of performance (guaranteed-cost) for permissible parameter variations and is an early attempt to design for both robust performance and robust stability. Continuing research had been made by Kosmidou (1987), Bernstein and Haddad (1988), Liuo and Yang (1987) and Kosmidou (1990). Luo et al. (1993) presented a method to find optimal robust linear quadratic regulators (RLQR) based on an exhaustive numerical search, an analytic method was presented by Petersen and McFarlane (1992) and further developed by Jiang and Clements (1993) and Petersen (1994). The concepts of Guaranteed Cost Control approach are used by Khargonekar et al. (1990), Petersen (1991) and Xie and Souza (1990) to develop robust H^∞ state feedback controller design methods which provide $H\infty$ norm bound guarantees for classes of uncertain systems.

Since it is not usually possible to measure all state variables of the plant, it is not usually possible to implement the state feedback solution and often output feedback controllers should be used. In the area of robust dynamic output feedback control, Jabbari and Schmitendorf (1991) and Jabbari and Schmitendorf (1993) proposed a method which considers closed loop robust stability, it uses a full state feedback RLQR design method to provide a robust control law then a high gain observer is employed to estimate the system states. A drawback of these design methods is that they do not tend to the nominal observer design when the uncertainties tend to zero. Xie et al. (1992) proposed a method which designs a robust controller to stabilise an uncertain system with a prescribed level of disturbance attenuation for all admissible parameter uncertainties.

Robustness analysis based on uncertainty models may give a good measure of the robustness of a controlled system. A technique by Neto et al. (1992) derives robustness bounds with respect to a given state feedback controller and bounded parametric uncertainty, for uncertainties within these bounds stability is guaranteed. Luo *et al.* (1993), Chen and Dong (1989) and Sobel et. al (1989) analysed the robustness of a standard Linear Quadratic Gaussian (LQG) controller with respect to some given uncertainty model, however these methods tend to be conservative since they are based on the Bellman-Gronwall inequality.

1.3 AIMS AND STRUCTURE OF THE THESIS

In this section an overview of some of the issues in robust control is given, an attempt is made to describe the state-of-the-art for each. The contributions made in this thesis in each area are then described. The issues are divided in those relating to full state feedback and those to dynamic output feedback and further into those with a given nominal system model alone and those with both nominal system model and uncertainty model (Table 1.1). Methods of robust controller design and robustness analysis will be addressed for each problem in turn.

Robust Control Problems	Full State Feedback	Dynamic Output Feedback
Nominal model alone	Problem 1	Problem 2
Nominal and uncertainty models	Problem 3	Problem 4

Table 1.1. Overview of Issues in Robust Control

Problem 1

The design of full state feedback controllers given a nominal system model may be approached by either LQR or H_∞ methods. The LQR method is suitable when the control objective is to minimise a quadratic cost function and it is known to provide good robustness to unknown uncertainties (or inherent robustness). Anderson and Moore (1989) provided a comprehensive review of the method and demonstrate that it provides good gain and phase margins. H_∞ methods are suitable when the control objective is to minimise the sensitivity to process disturbance and measurement noise and they have also been shown to possess good robustness properties (Khargonekar et al., 1990).

A method is developed and presented in Chapters 2 and 3 which offers some criteria which, if satisfied, will guarantee the robust stability of any full state feedback control system to a given uncertainty. Thus, the inherent robustness to parametric uncertainties offered by both of these methods and in fact for any given full state feedback controller, may be determined.

Problem 2

There are several controller design methods available here: LQG, LQG/LTR, H_∞ and RLQG (minimum entropy H_∞). The LQG method is suitable when the control objective can be described as the minimisation of a quadratic cost function, however, it was shown to have poor robustness (Doyle, 1978) and thus the LQG/LTR method was developed (Doyle and Stein, 1981). This enables the robustness properties of the LQR method to be recovered by the deployment of a high gain observer or by adjusting the weighting matrices in the quadratic cost function, see Chapter 3. As for the full state feedback problem, H_∞ methods may be applied and will offer good robustness properties. The robustness of these techniques is reviewed in Chapter 3.

Minimum entropy H∞ control (Mustafa and Glover, 1989) uses the cost function weighting matrices to determine suitable definitions for the disturbance and performance vectors of the H∞ problem. Thus it attempts to combine the robustness properties of H∞ control with the cost optimal properties of LQG control. In Section 6.2 of this thesis the method is developed from the perspective of improving the robustness of the LQG solution (Marsh and Wei, 1995). The technique produced is identical to that of Mustafa and Glover (1989) however the new perspective is believed to offer valuable insight into the method.

For any given dynamic output feedback controller, the level of robustness to unknown uncertainties may be evaluated by calculating a suitable H∞ norm bound, this is discussed in Chapters 2 and 3. The analysis method is based on the 'small gain theorem' (Zames, 1963) and uses the state space solutions to H∞ problems presented by Doyle et al. (1989).

Problem 3

The Robust LQR (RLQR) method (Chapter 4) is suitable for the design of robust full state feedback controllers given a nominal model and an uncertainty model. It will provide a controller which guarantees stability for all admissible uncertainties and provides minimal performance degradation across the admissible domain. As for the LQR method, it is shown to posses good inherent robustness to unknown residual uncertainty. The method is an extension to that of Petersen (1994), a broader range of uncertainty descriptions is accommodated and a different solution technique and proof of optimal performance degradation are offered. A generalised version of this methodology, applied to partial state feedback systems was presented by Wei and Marsh (1994). An alternative technique is robust H∞ controller synthesis (Khargonekar et al., 1990) this extends the disturbance/noise rejection and robustness properties of the H∞ technique to classes of uncertain systems.

A new method of robustness analysis for full state feedback control systems with a given uncertainty model is presented in Chapter 5 (also Wei and Marsh, 1995). For a given controller and performance degradation requirement, a bound for the uncertainty can be found such that the controlled system remains stable and the performance degradation is within the requirement over the admissible domain. An alternative approach is permitted to design a controller such that a performance criterion may be specified and a robustness bound found which may then be used to specify the uncertainty magnitude for the design procedure. Thus a robust controller is designed using the performance criterion as a design variable and a (maximal) robustness bound is offered. It should be noted that though this approach may be taken iteratively using the standard RLQR approach, this method permits a direct one-step solution.

Problem 4

The RLQG method presented in Chapter 6 and by Marsh and Wei (1995) enables dynamic output feedback controllers to be designed for a class of uncertain systems and provides some inherent robustness to unknown uncertainty. The modelled uncertainty is assumed to lie within some measurable domain the magnitude of which is a design parameter. The H_∞ normal bound constraint is the other design parameter and the method extends the validity of this, and thus provides a stability guarantee, to all admissible parametric uncertainties. An alternative approach to dynamic output feedback controller design could be approached by extending H_∞ techniques to cover such classes of uncertain systems. The author is not aware of any literature on such approaches.

The analysis of the robustness of any given dynamic output feedback controller to a given class of modelled uncertainties and some unknown residual uncertainties is a very challenging problem. It is possible to give a general robustness analysis of an arbitrary dynamic output feedback controller to unknown residual uncertainties, this is discussed in Problem 2, but to extend that guarantee to all members of a class of uncertain systems is believed to be very difficult. However, since good robustness properties are guaranteed for the RLQG design method and general analysis of robustness is not believed to be an important requirement.

This outlines the state-of-the-art in the areas considered and aims to put the contributions of this thesis into context. The thesis is organised as follows: the fundamentals of robust control are described in Chapter 2; configurations, notation and terminology are introduced. Descriptions are given of nominal system and uncertainty models in both the frequency and time domains and of some common performance objectives of control system design. Finally robustness principles are introduced, this Chapter provides the core of the whole thesis.

The inherent stability robustness of some well-known modem control system design techniques is assessed in Chapter 3. These include LQR, H^∞ and H_2/H^∞ for full state feedback controllers and LQG, LQG/LTR, H^∞ and H_2/H^∞ for dynamic output feedback controllers. These techniques normally refer to nominal system models alone for controller design. This Chapter attempts to quantify each technique's robustness subject to model uncertainty.

In Chapter 4, an optimal full state feedback RLQR design methodology is presented for systems with bounded parametric uncertainties, it offers both good stability robustness and good performance robustness. Robustness analysis for full state feedback control systems is addressed in Chapter 5. Both stability robustness and performance robustness for classes of uncertain systems are analysed, this is a new approach and is one of the main contributions of this thesis. It is shown that the RLQR presented offers excellent stability and performance robustness.

A new RLQG design technique for dynamic output feedback controllers is presented in Chapter 6. It is believed to be less conservative than other approaches in many circumstances. For systems with bounded parametric uncertainties the method guarantees both robust stability to admissible uncertainties and some inherent robustness to unknown residual uncertainties. System performance is also explicitly considered with respect to a quadratic cost function. In essence the method enables the designer to trade off robustness to modelled uncertainty, inherent robustness to unknown residual uncertainty and cost performance. The thesis is concluded by Chapter 7 which provides a general discussion of the results and outlines areas of further work in robust control.

CHAPTER 2

FUNDAMENTALS OF ROBUST CONTROL

As noted in the previous Chapter, robust control may be summarised as the problem of analysing and designing a controller for a system that contains significant uncertainty. To define the problem more precisely, a number of elements will be carefully developed in this chapter which will be a fundamental part of this thesis.

Classes of plant models, uncertainty models, and performance measures will be introduced as a background, then some important results will be developed which will be used throughout the thesis. These include the description of uncertain systems and uncertain controlled systems, as well as conditions of robustness.

2.1 DESCRIPTION OF CONTROL SYSTEMS

For simplicity, the work in control system design and analysis in this thesis will be done under the assumption that the process to be controlled has a linear input-output behaviour and a model of the plant is available. The model is a mathematical description of the plant and, sometimes, the control objective or performance vector, will also be presented as part of it.

2.1.1 Description of systems

There are two common linear multivariable plant model descriptions used here; state-space models and transfer-function matrix models. Generally, the state-space model of a linear, time-invariant, finite dimensional system is given by:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} & \text{state equation} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} & \text{output equation} \end{cases} \tag{2.1.1}$$

Where $\mathbf{x} \in \Re^{n\times 1}$ the state vector

$\mathbf{A} \in \Re^{n\times n}$ system matrix whose elements are constant

$\mathbf{B} \in \Re^{n\times r}$ input matrix whose elements are constant

$\mathbf{u} \in \Re^{r\times 1}$ the input vector

$\mathbf{y} \in \Re^{m\times 1}$ the output vector

$\mathbf{C} \in \Re^{m\times n}$ output matrix whose elements are constant

$\mathbf{D} \in \Re^{m\times r}$ feedthrough matrix whose elements are constant

For plant subject to stochastic disturbance/noise inputs, the state-space variable model is extended to:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} + \mathbf{Ed} \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} + \mathbf{Fv} \end{cases} \tag{2.1.2}$$

where $\mathbf{d} \in \Re^{d1\times 1}$ vector of random Gaussian white noise disturbance process

$\mathbf{v} \in \Re^{d2\times 1}$ vector of random Gaussian white noise measurement noise process

$\mathbf{E} \in \Re^{n\times d1}$ disturbance weighting matrix

$\mathbf{F} \in \Re^{m\times d2}$ measurement noise weighting matrix

It is assumed $\mathbf{d}$ and $\mathbf{v}$ are uncorrelated, that is $\mathcal{E}[\mathbf{dv}^T] = \mathbf{0}$, their mean values are zero, i.e., $\mathcal{E}[\mathbf{d}] = \mathcal{E}[\mathbf{v}] = \mathbf{0}$, and their intensities matrices are $\mathbf{W}$ and $\mathbf{V}$ respectively:

$$\begin{aligned} \mathcal{E}[\mathrm{d}(t)\mathrm{d}^T(\tau)] &= \mathrm{W}\delta(t-\tau) \\ \mathcal{E}[\mathrm{v}(t)\mathrm{v}^T(\tau)] &= \mathrm{V}\delta(t-\tau) \end{aligned}$$

and for simplicity, it is also assumed that **D=0** for model (2.1.1) and (2.1.2).

The system outputs may be further delineated into measured outputs and a performance vector, this will produce a general state-space model which can be described in following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}_2\mathbf{u} + \mathbf{B}_1\boldsymbol{\omega} \\ \mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_1\mathbf{u} \\ \mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{D}_2\boldsymbol{\omega} \end{cases} \tag{2.1.3}$$

Where $\boldsymbol{\omega}$ is a vector of the union of disturbance and noise processes as: $\boldsymbol{\omega} = \begin{bmatrix} \mathbf{d} \\ \mathbf{v} \end{bmatrix}$, it may include reference inputs, disturbances and noise. The performance vector $\mathbf{z}$ may include errors, performance vectors, process outputs and control inputs. The internal compensation signals are represented by vectors $\mathbf{y}$ and $\mathbf{u}$, and correspond to the sensor signals and actuator demands, respectively.

A transfer-function matrix model may be also used to describe a system, this is denoted:

$$\mathbf{G}(s) = \frac{\mathcal{L}[\mathbf{y}(t)]}{\mathcal{L}[\mathbf{u}(t)]} = \frac{\mathbf{y}(s)}{\mathbf{u}(s)} \tag{2.1.4}$$

Where $\mathbf{u}(s)$ Laplace transform of input vector $\mathbf{u}(t)$

$\mathbf{y}(s)$ Laplace transform of output vector $\mathbf{y}(t)$

This can be related to the state space representation (2.1.1) by:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

The system description methods mentioned in this section will be used throughout the rest of the thesis.

2.1.2 Description of controlled systems

This thesis is concerned with design and analysis of feedback controllers, which can be divided into two distinct types: *state feedback* controllers and *output feedback* controllers. To enable the state-feedback controller to be used, all the states must be measurable, this is known as a full information system. Because of the considerable design experience available on full state feedback control systems, as well as some useful properties such systems can provide, state feedback controller design could be a very powerful tool for attaining control objectives and hence, it has been extensively studied in this thesis. However, it is not always possible to implement the state-feedback solution, the reason is that it is not always possible to measure all the state variables of the plant. So sometimes an output feedback controller should be used. In fact, an output feedback controller is more practicable than a state feedback controller.

Firstly the state feedback controller is considered for the system which can be described by the state space model (2.1.1). The closed loop system can be described as Fig. 2.1. The state feedback controller can be described by:

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) \tag{2.1.5}$$

Where matrix **K** is the vector of static gains.

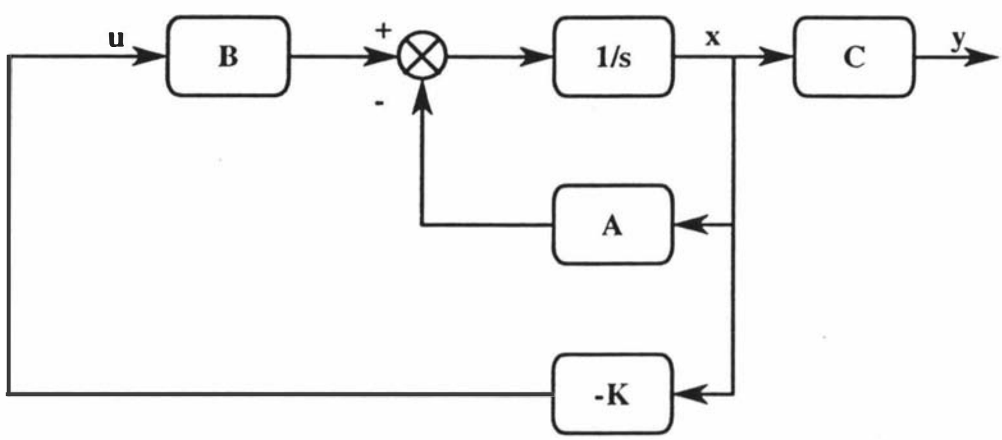

Fig. 2.1 Full state feedback controlled system

Secondly the dynamic output feedback controller, **K**(s), which is commonly described by a dynamic system representation, is considered for the system (2.1.1). The closed loop system can be described as Fig. 2.2.

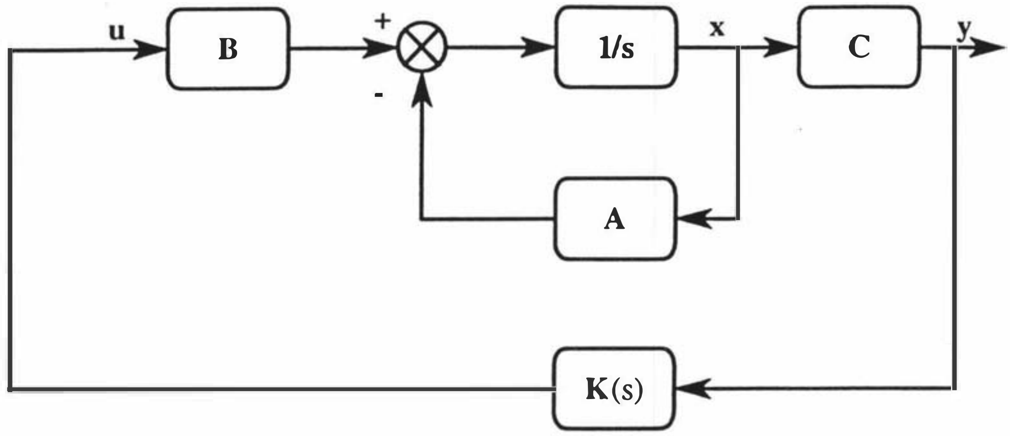

Fig. 2.2 Output feedback controlled system

In this thesis **K**(s) is only considered as a proper, dynamic output feedback controller with the same order (or less) as the plant and which may be described as:

$$\mathbf{K}(s) = \mathbf{C}_c(s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}_c$$

Alternatively, the control input may be related to the measured output by the following state space description:

$$\begin{cases} \dot{\varsigma} = \mathbf{A}_c\varsigma + \mathbf{B}_c\mathbf{y} \\ \mathbf{u} = \mathbf{C}_c\varsigma \end{cases} \tag{2.1.6}$$

Where ς is the state vector of the dynamic controller.

2.2 Description of Uncertain Systems

Linear time invariant models which are used in control system design can only approximately describe the actual dynamics of a plant. This means that some differences exist between the nominal model and actual plant, and this difference is called "model uncertainty". To permit robustness analysis and robust controller design, this model uncertainty may be described by an uncertainty model. Two possible types of uncertainty models, *parametric* and *nonparametric*, are presented and discussed in this section.

Parametric uncertainty models are motivated by an imprecise knowledge of the parameters of the system. The structure of the model equations can be determined by means of the basic laws of physics and engineering, but the numerical values of the parameters are only known within tolerances. If the model parameters are estimated experimentally, the remaining uncertainties depend on the level of disturbances that excited the plant during the experiment. Parameter variations and nonlinearities have to be omitted if the system is to be described by a linear model. Sensor or actuator failures which, from the controller point of view, yield changes or even restrictions of the input-output behaviour of the plant, parameter drifts, or parameter variations caused by moves of the operating point, also belong to this source of model uncertainties.

Nonparametric uncertainty models offer a general description of model uncertainty. Such an uncertainty model may be suitable if the nominal model has been reduced in order to simplify calculations or to avoid difficulties that arise from the high complexity of the complete model. For instance, parasitic dynamically elements in actuators, transmitters or measurement devices are often neglected. Besides that, if the controller has to be designed when the system to be controlled is still under construction, then there might be some estimates of the static or dynamic behaviour of the plant.

Furthermore, these uncertainty models may also be characterised as *structured* or *unstructured* models. If a suitable structure is known for the uncertainty model, then this may be used to give a more precise description of the uncertainty, else, the default is to use an unstructured uncertainty model. Descriptions of parametric and nonparametric uncertainty models are now given in more detail.

2.2.1 Models of parametric uncertainty

Parametric uncertainties are those that can be compensated for by correcting the model parameter values. Normally, parametric uncertainty is denoted in an uncertainty model by some uncertain elements, and these uncertain elements may be bounded by some kind of norm. A system with parametric uncertainties can be described as:

$$\begin{cases} \dot{\mathbf{x}} = (\mathbf{A} + \Delta\mathbf{A})\mathbf{x} + (\mathbf{B} + \Delta\mathbf{B})\mathbf{u} \\ \mathbf{y} = (\mathbf{C} + \Delta\mathbf{C})\mathbf{x} \end{cases}$$

or $$\mathbf{G}(s) = (\mathbf{C} + \Delta\mathbf{C})(s\mathbf{I} - (\mathbf{A} + \Delta\mathbf{A}))^{-1}(\mathbf{B} + \Delta\mathbf{B})$$

To illustrate the different types of parametric uncertainty models, we just consider the state variable model here, and for simplicity, a closed loop system $\dot{\mathbf{x}} = (\mathbf{A} + \Delta\mathbf{A})\mathbf{x}$ will be considered. The general form for parametric uncertainty model of the uncertain term $\Delta\mathbf{A}$ is a matrix norm bounded structured format:

$$\Delta\mathbf{A} = \mathbf{N}\mathbf{\Phi}(t)\mathbf{M}$$

Where $\mathbf{N}$ and $\mathbf{M}$ are constant matrices which imply the structure of uncertainty, the uncertain matrix, $\mathbf{\Phi}(t)$, is constrained by the maximal singular value, i.e., $\overline{\sigma}(\mathbf{\Phi}(t)) \leq \varepsilon$.

Two special cases of this format are considered:

(1). Scalar norm bounded structured parametric uncertainty:

$\Delta\mathbf{A} = \sum_i q_i \mathbf{A}_i$ $\quad i = 1,2,\ldots,r$, where $\mathbf{A}_i$ are constant structure matrices, q_i are bounded scalar parameters with $|q_i| \le \varepsilon$. This is a *structured parametric* uncertainty model.

(2). Unstructured norm bounded parametric uncertainty:

When $\mathbf{N}$ and $\mathbf{M}$ are identity matrices, $\Delta\mathbf{A}$ can only be described as an *unstructured parametric* model, which is constrained by the maximal singular value, i.e., $\bar{\sigma}(\Delta\mathbf{A}) \le \varepsilon$.

Hence, in this thesis, the parametric uncertainties $\Delta\mathbf{A}$ $\Delta\mathbf{B}$ and $\Delta\mathbf{C}$ are generally modelled using norm bounded, structured, parametric uncertainty models as:

$$\Pi = \left\{ \begin{array}{ll} \Delta\mathbf{A} = \mathbf{N}_a \Phi_a(t) \mathbf{M}_a & \bar{\sigma}(\Phi_a(t)) \le \varepsilon \\ \Delta\mathbf{B} = \mathbf{N}_b \Phi_b(t) \mathbf{M}_b : & \bar{\sigma}(\Phi_b(t)) \le \varepsilon \\ \Delta\mathbf{C} = \mathbf{N}_c \Phi_c(t) \mathbf{M}_c & \bar{\sigma}(\Phi_c(t)) \le \varepsilon \end{array} \right\}$$

2.2.2 Models of nonparametric uncertainty

Nonparametric uncertainty models represent the uncertainty which can not be represented in terms of the parameters of the nominal model, so it represents the general case of the system uncertainty. A system with nonparametric uncertainty may be described as:

$$\mathbf{G}(s) = \mathbf{G}_0(s) + \Delta\mathbf{G}(s)$$

where $\mathbf{G}_0(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ is the nominal model of the system.

The nonparametric uncertainty may be modelled generally in *additive nonparametric* format:

$$\frac{\mathbf{y}(s)}{\mathbf{u}(s)} = \mathbf{G}_0(s) + \Delta\mathbf{G}(s) = \mathbf{G}_0(s) + \mathbf{N}\Delta(s)\mathbf{M}$$

Where $\mathbf{N}$ and $\mathbf{M}$ are constant matrices which imply the structure of the uncertainty. $\Delta(s)$ represents a nonparametric uncertainty matrix which could typically be constrained by an H^∞ norm, i.e., $\|\Delta(s)\|_\infty \le \eta$.

Two special cases of this description are:

(1). $\dfrac{\mathbf{y}(s)}{\mathbf{u}(s)} = (\mathbf{I} + \mathbf{L}(s))\mathbf{G}_0(s)$

Where $\mathbf{L}(s)$ represents an *output multiplicative unstructured nonparametric* uncertainty which could be constrained by a norm, i.e., $\|\mathbf{L}(j\omega)\| \leq \ell_m(\omega)$.

(2). $\dfrac{\mathbf{y}(s)}{\mathbf{u}(s)} = \mathbf{G}_0(s)(\mathbf{I}+\mathbf{L}(s))$

Where $\mathbf{L}(s)$ represents an *input multiplicative unstructured nonparametric* uncertainty which could be constrained by a norm, i.e., $\|\mathbf{L}(j\omega)\| \leq \ell_m(\omega)$.

2.2.3 Modelling of uncertain systems

Uncertain systems are normally described in one of following two ways: firstly, we assume that the system has uncertainties, but no information about these uncertainties is available, (i.e., system with *unknown uncertainty)*. In this case, only the nominal model is used for robust controller design and robustness analysis, the closed loop system should have some inherent robustness which can guarantee the robust stability and performance. The unknown uncertainty could be described by parametric or nonparametric models, or even both, such as:

Real system = nominal model + *unknown uncertainty*

or $$\frac{\mathbf{y}(s)}{\mathbf{u}(s)} = (\mathbf{C}+\Delta\mathbf{C}_u)(s\mathbf{I}-\mathbf{A}-\Delta\mathbf{A}_u)^{-1}(\mathbf{B}+\Delta\mathbf{B}_u)+\Delta\mathbf{G}(s)$$

Where $\Delta\mathbf{G}(s)$ is the nonparametric part of the unknown uncertainty, and $\Delta\mathbf{A}_u$, $\Delta\mathbf{B}_u$ and $\Delta\mathbf{C}_u$ are the parametric part of the unknown uncertainty.

Secondly, we assumed that part of the uncertainty can be represented by some particular parametric uncertainty model. This representation of parametric uncertainty should be used in the design and analysis to give a precise description of model uncertainty and thus avoid conservatism. Normally, since it is impossible to describe all model uncertainty with a parametric uncertainty model, so to guarantee robustness, it is assumed that there also exists an *unknown residual uncertainty* for this uncertain system. This residual unknown uncertainty could have a nonparametric part $\Delta\mathbf{G}(s)$ and parametric parts $\Delta\mathbf{A}_u$, $\Delta\mathbf{B}_u$ and $\Delta\mathbf{C}_u$. Hence the closed loop system should be robust for modelled uncertainties and also has some inherent robustness included for the unknown uncertainty.

Real system = nominal model + modelled uncertainty + *unknown residual uncertainty*

or $$\frac{\mathbf{y}(s)}{\mathbf{u}(s)} = (\mathbf{C}+\Delta\mathbf{C}_p+\Delta\mathbf{C}_u)(s\mathbf{I}-\mathbf{A}-\Delta\mathbf{A}_p-\Delta\mathbf{A}_u)^{-1}(\mathbf{B}+\Delta\mathbf{B}_p+\Delta\mathbf{B}_u)+\Delta\mathbf{G}(s)$$

Where $\Delta\mathbf{A}_p$, $\Delta\mathbf{B}_p$ and $\Delta\mathbf{C}_p$ are modelled (parametric) uncertainties of the system. Due to the omnipresence of unknown uncertainty, a practical and reasonable control system should always be designed with some inherent robustness included for it.

2.2.4 Effect of uncertainties on the closed loop behaviour

To summarise the results from the previous section, for the system with a nominal model, modelled uncertainties and residual unknown uncertainty, it is necessary to study the effect of the unknown uncertainty for the robust design and robustness analysis. The general description of the system of concern can be found as:

$$\frac{\mathbf{y}(s)}{\mathbf{u}(s)} = (\mathbf{C}+\Delta\mathbf{C})(s\mathbf{I}-\mathbf{A}-\Delta\mathbf{A})^{-1}(\mathbf{B}+\Delta\mathbf{B})+\Delta\mathbf{G}(s)$$

Where $\Delta\mathbf{A}=\Delta\mathbf{A}_p+\Delta\mathbf{A}_u$, $\Delta\mathbf{B}=\Delta\mathbf{B}_p+\Delta\mathbf{B}_u$ and $\Delta\mathbf{C}=\Delta\mathbf{C}_p+\Delta\mathbf{C}_u$ are all possible parametric uncertainties of the system such as modelled uncertainties and the parametric part of the residual unknown uncertainty.

For full state feedback control design and analysis, uncertain controlled systems with the above uncertainty can be described by Fig. 2.3.

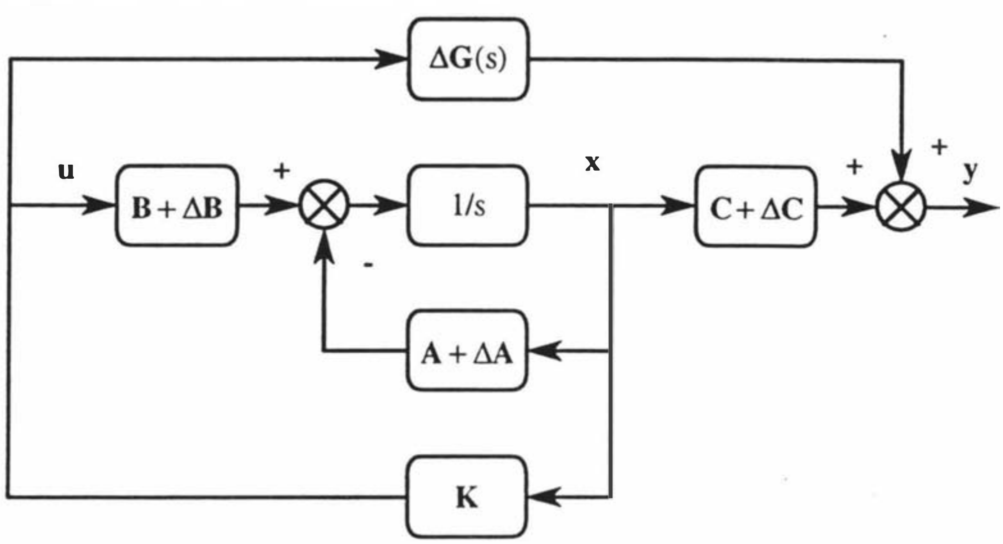

Fig. 2.3 Full state feedback control system with uncertainty

It is clear that the transfer function between input and output can be described as:

$$\frac{\mathbf{y}(s)}{\mathbf{u}(s)} = (\mathbf{C}+\Delta\mathbf{C})(s\mathbf{I}-\mathbf{A}-\Delta\mathbf{A}-(\mathbf{B}+\Delta\mathbf{B})\mathbf{K})^{-1}(\mathbf{B}+\Delta\mathbf{B})+\Delta\mathbf{G}(s) \tag{2.2.1}$$

For a stable nonparametric uncertainty $\Delta\mathbf{G}(s)$, the stability of the closed loop system only depends on the closed loop behaviour $(s\mathbf{I}-\mathbf{A}-\Delta\mathbf{A}-(\mathbf{B}+\Delta\mathbf{B})\mathbf{K})^{-1}$; it is clear that uncertainties $\Delta G(s)$ and ΔC do not affect this behaviour. Hence to design a control system with good robustness means finding a full state feedback controller, $\mathbf{K}$, such that the closed loop system has good robustness subject to uncertainties $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$, i.e., only uncertainties $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ will affect the robustness of controlled system in robust design and analysis.

For dynamic output controller design and analysis, uncertain controlled systems with modelled as well as unknown uncertainty can be described by Fig. 2.4.

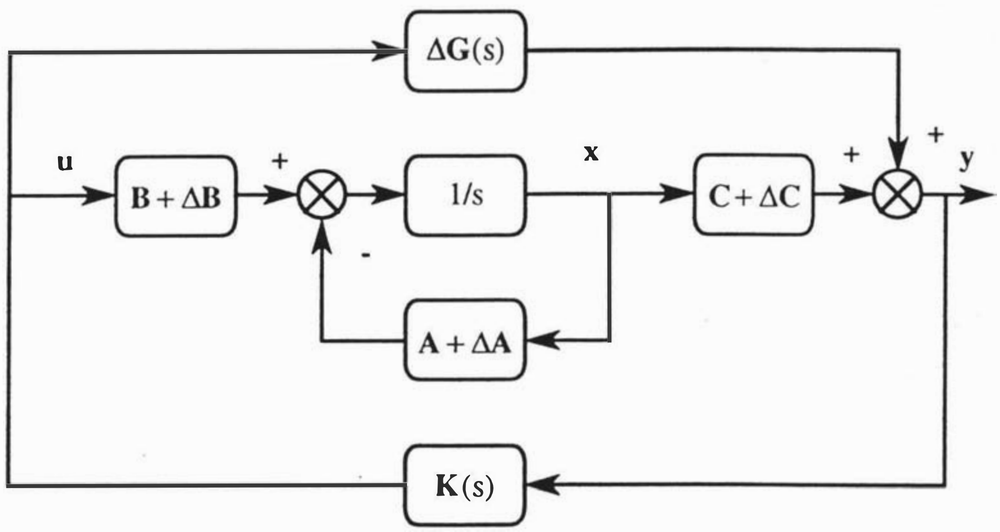

Fig. 2.4 Output feedback control system with uncertainty

It is evident that both parametric and nonparametric uncertainties will affect the closed loop behaviour, so both should be considered in the robust control design and analysis procedure. However, since (2.2.1) may be rewritten as:

$$\begin{aligned}\frac{\mathbf{y}(s)}{\mathbf{u}(s)} &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \Gamma(s,\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{K},\Delta\mathbf{A},\Delta\mathbf{B},\Delta\mathbf{C}) \\ &= \mathbf{G}_0(s) + \Delta\mathbf{G}(s)\end{aligned}$$

Where $\Gamma(s,\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{K},\Delta\mathbf{A},\Delta\mathbf{B},\Delta\mathbf{C})$ is a function whose formulation can be found by some algebraic manipulation. So the parametric uncertainty, $\Delta\mathbf{A}, \Delta\mathbf{B}, \Delta\mathbf{C}$, can also be represented as a special case of the nonparametric uncertainty $\Delta\mathbf{G}(s)$.

Hence, for the unknown uncertainty, it is reasonable to consider nonparametric uncertainty $\Delta G(s)$ only because good robustness for this will provide some inherent robustness for the parametric uncertainty parts, $\Delta\mathbf{A}_u$, $\Delta\mathbf{B}_u$ and $\Delta\mathbf{C}_u$. So for simplicity, we will only consider the additive nonparametric uncertainty part, $\Delta\mathbf{G}(s)$, of the unknown uncertainty for robust design and analysis of output feedback control systems. For less conservative design, modelled uncertainties $\Delta\mathbf{A}_p, \Delta\mathbf{B}_p, \Delta\mathbf{C}_p$ should be always considered in robust design and analysis.

2.3 Performance Measures for Controlled Systems

The ultimate objective of control system design is that the controller performs "well" when it is implemented on the real plant. To assess this objective it is necessary to establish performance measures for controlled (or closed-loop) system; the following general measures are used in this thesis:

- Stability
- Dynamic performance
- Robustness

In this section the first two of them; *stability* and *dynamic performance* will be introduced. Dynamic performance will be further broken down into integral quadratic cost, H2 and H∞ norm performances. Robustness is the focus of this thesis it will be discussed in detail in §2.4.

2.3.1 Stability

Stability, of course, is always a necessary performance requirement in control system design. Simply stated, a system is stable if for every bounded input, the output is also bounded. This is normally referred to as *bounded-input bounded-output (BIBO)* stability. Since it usually is not easy to measure every possible input and output of a system, BIBO stability is difficult to determine, we therefore consider further two stability definitions, *asymptotic stability* and *Lyapunov stability.*

Consider a closed loop system which is described in state space by:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} \\ y = C_0 x \end{cases} \tag{2.3.1}$$

1). **Asymptotic stability**

The system (2.3.1) is said to be asymptotically stable if the output, $\mathbf{y}(t)$, is, such that, for any initial condition:

$$\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$$

A necessary and sufficient condition (Ogata 1990) of asymptotic stability for above system is:

$$\mathrm{Re}\{\lambda_i(\mathbf{A}_0)\} < 0, \quad \forall i \quad \text{when } (C_0, A_0) \text{ is observable.} \tag{2.3.2}$$

Since for uncertain systems it is not easy to find the exact eigenvalues of the system matrix, $\mathbf{A}_0$, the test for asymptotic stability is difficult to employ. It is necessary to introduce the following stability measurement method which will play an important role in the robust stability analysis of control systems.

2). **Lyapunov stability**

A general sufficient condition for a system to be stable is that there exists a scalar function, $\mathbf{V}(\mathbf{x}, t)$, that has a continuous first partial derivative and satisfies the conditions:

- $V(\mathbf{x}, t)$ is positive definite.
- $\dot{V}(\mathbf{x}, t)$ is negative definite.

This function is known as a Lyapunov function and its existence is a sufficient condition of system stability. This sufficient condition is a very useful judgement of stability, particularly to nonlinear systems.

A special case of the Lyapunov stability applicable to *linear time-invariant systems*, which is generally referred to as *quadratic stability*, is given in the following Lemma:

Lemma 2.3.1 For any positive definite matrix $\mathbf{Q}$, the closed-loop system $\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x}$ is quadratic stable if and only if there exists a positive definite matrix $\mathbf{P}$ which satisfies the so-called *Lyapunov matrix equation*:

$$\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\mathbf{A}_0 + \mathbf{Q} = \mathbf{0}, \quad \text{or} \quad \mathbf{A}_0^T\mathbf{P} + \mathbf{P}\mathbf{A}_0 < \mathbf{0} \tag{2.3.3}$$

Furthermore, this makes $\mathbf{V} = \mathbf{x}^T\mathbf{P}\mathbf{x}$ a valid quadratic Lyapunov function for the system as $\mathbf{P} > \mathbf{0} \Rightarrow \mathbf{V} > \mathbf{0}$ and $\dot{\mathbf{V}} = \mathbf{x}^T(\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\mathbf{A}_0)\mathbf{x} < 0$.

Proof is given by Ogata (1990), pp. 733

To extend this result to *linear time-varying systems*, consider the following Corollary.

Corollary 2.3.1 The linear time-varying system $\dot{\mathbf{x}} = \mathbf{A}_0(t)\mathbf{x}$ is asymptotically stable if there exists a positive definite matrix $\mathbf{P}$ such that the following expression is valid for all time $t \in [0, \infty)$:

$$\mathbf{A}_0^T(t)\mathbf{P} + \mathbf{P}\mathbf{A}_0(t) < \mathbf{0} \tag{2.3.4}$$

Proof: If there exists a positive definite matrix $\mathbf{P}$ such that the expression (2.3.4) can be held for all time $t \in [0, \infty)$, then scalar function $\mathbf{V}(\mathbf{x}, t) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ is a valid quadratic Lyapunov function for this system, and the system is said to be quadratically stable. ◆

The Lyapunov stability condition therefore provides a useful stability criterion, the quadratic stability measure, which is important in the stability analysis of control systems described by state space equations. It is also particularly useful for robust stability analysis. This measure is fundamental to the Riccati equation approach of robust stabilising controller design.

There may exist many controllers which can stabilise the system. To compare them and find the most useful one, it is necessary to introduce some measures of dynamic performance; such as integral-quadratic cost performance, H_2 norm performance and H_∞ norm performance.

2.3.2 Integral-quadratic cost performance

Of the various performance measures of system input and output energies, the integral-quadratic cost performance is very popular in control design. Consider the plant which is modelled by the deterministic plant model (2.1.1), the common description of the integral-quadratic cost performance measure is:

$$J = \int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt \tag{2.3.5}$$

and the integral-quadratic cost performance measure of the closed loop system is:

$$J = \int_0^\infty \mathbf{x}^T\mathbf{Q}_0\mathbf{x}dt \tag{2.3.6}$$

with $\mathbf{Q}_0 = \mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}$

If the plant is subject to stochastic disturbance/noise inputs and described by the model (2.1.2), the common description of the integral-quadratic cost performance measure is:

$$J = \lim_{t_0 \to \infty} \frac{1}{t_0} \mathcal{E}\int_0^{t_0} [\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}]dt \tag{2.3.7}$$

and the integral-quadratic cost performance measure of the closed loop system is:

$$J = \lim_{t_0 \to \infty} \frac{1}{t_0} \mathcal{E}\int_0^{\infty} \mathbf{x}^T\mathbf{Q}_0\mathbf{x}dt \tag{2.3.8}$$

Where for all of these descriptions, $\mathbf{Q} \geq \mathbf{0}$ and $\mathbf{R}>\mathbf{0}$ are performance weighting matrices.

These cost values can be calculated from the following Lemma.

Lemma 2.3.2 For a stable closed-loop system, $\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x}$, with initial state vector, $\mathbf{x}(0) = \mathbf{x}_0$, and performance index (2.3.6), the cost value is given by:

$$J = \mathbf{x}_0^T\mathbf{P}\mathbf{x}_0.$$

Where $\mathbf{P}$ is the positive definite solution of the Lyapunov equation:

$$\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\mathbf{A}_0 + \mathbf{Q}_0 = \mathbf{0} \tag{2.3.9}$$

Proof: If $\mathbf{P}$ is the positive definite solution (2.3.9), then

$$\mathbf{Q}_0 = -(\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\mathbf{A}_0)$$

hence

$$\begin{aligned} J &= -\int_0^\infty \mathbf{x}^T(\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\mathbf{A}_0)\mathbf{x}dt \\ &= -\int_0^\infty \frac{d}{dt}(\mathbf{x}^T\mathbf{P}\mathbf{x})dt \\ &= \mathbf{x}(0)^T\mathbf{P}\mathbf{x}(0) - \mathbf{x}(\infty)^T\mathbf{P}\mathbf{x}(\infty) \end{aligned}$$

Since the closed loop system is stable, $\mathbf{x}(\infty) = \mathbf{0}$, so it follows that $J = \mathbf{x}_0^T\mathbf{P}\mathbf{x}_0$, thus proving Lemma 2.3.2. ◆

For the system subject to stochastic disturbance/noise inputs, we have that:

Lemma 2.3.3 For the closed-loop system, $\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} + \mathbf{E}\mathbf{d}$, where $\mathbf{d}$ is assumed to be the vector of Gaussian random disturbance process driving the plant whose covariance matrix is (2.1.2) the performance index as (2.3.8), the cost value is: $J = \mathrm{tr}(\mathbf{P}\mathbf{Q}_0)$. Where $\mathbf{P}$ is the positive definite solution of the Lyapunov equation:

$$\mathbf{A}_0\mathbf{P} + \mathbf{P}\mathbf{A}_0^T + \mathbf{E}^T\mathbf{W}\mathbf{E} = \mathbf{0} \tag{2.3.10}$$

Proof: The proof of this Lemma can be found in Kwakernaak & sivan (1972) ◆

A possible objective of the control system design here could be to find a stabilising controller, $\mathbf{u}(t)$, such that the above cost performance value is minimal. Some powerful and popular control design methods, such as LQR and LQG address to this objective, will be introduced in the next chapter.

2.3.3 H2-norm performance

The H2 norm is a measure of system input and output energies. Consider a closed loop system description of system (2.1.3), where a full state feedback controller, $\mathbf{u} = \mathbf{K}\mathbf{x}$, has been employed to relate state to control input.

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} + \mathbf{B}_1\omega \\ \mathbf{z} = \mathbf{C}_0\mathbf{x} \end{cases}$$

It's transfer function can be described as

$$\mathbf{G}(s) = \frac{\mathbf{z}(s)}{\omega(s)} = \mathbf{C}_0(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}_1$$

Then the H2 norm can be defined for the above transfer function as:

$$\|\mathbf{G}(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\{\mathbf{G}(j\omega)^*\mathbf{G}(j\omega)\}d\omega} \tag{2.3.11}$$

Where * means conjugate transpose, and (2.3.11) will be finite if the closed-loop system is stable. The H2 norm may be computed in the following way (Doyle, et. al 1989).

Lemma 2.3.4 Let $\mathbf{L}_c$ be the controllability Gramian of ($\mathbf{A}_0$, $\mathbf{B}_1$) and $\mathbf{L}_0$ the observability Gramian of ($\mathbf{A}_0$, $\mathbf{C}_0$). These can be found from

$$\mathbf{A}_0\mathbf{L}_c + \mathbf{L}_c\mathbf{A}_0^T + \mathbf{B}_1\mathbf{B}_1^T = 0 \tag{2.3.12}$$

$$\mathbf{A}_0^T\mathbf{L}_o + \mathbf{L}_o\mathbf{A}_0 + \mathbf{C}_0^T\mathbf{C}_0 = 0 \tag{2.3.13}$$

Then the H2 norm is given by

$$\|\mathbf{G}(s)\|_2^2 = \mathrm{Tr}(\mathbf{L}_c\mathbf{C}_0^T\mathbf{C}_0) = \mathrm{Tr}(\mathbf{L}_o\mathbf{B}_1\mathbf{B}_1^T) \tag{2.3.14}$$

The H2 norm performance measure is closely related to the integral-quadratic performance measure, to show this relationship, suppose that the H2 norm of the transfer function $\mathbf{G}(s) = \frac{\mathbf{z}(s)}{\omega(s)}$ is given by:

$$\|\mathbf{G}(s)\|_2 = \sqrt{\mathbf{Tr}(\mathbf{P}\mathbf{C}_0^T\mathbf{C}_0)}$$

Where the controllability Gramian **P** is the positive definite solution of the following Lyapunov equation:

$$\mathbf{A}_0\mathbf{P} + \mathbf{P}\mathbf{A}_0^T + \mathbf{B}_1\mathbf{B}_1^T = \mathbf{0} \tag{2.3.15}$$

If we choose $\mathbf{C}_0^T\mathbf{C}_0 = \mathbf{Q}_0$ and the disturbance is assumed to have identity covariance matrix, i.e., **W** = **I**, then by comparison with Lemma 2.3.3 it follows that:

$$\|\mathbf{G}(s)\|_2^2 = \mathrm{tr}(\mathbf{P}\mathbf{Q}_0) = \mathrm{J}$$

Hence, for a system with a particular performance weighting and uncorrelated Gaussian disturbances, the H2-norm performance will give the same measure as the integral-quadratic cost performance.

2.3.4 H∞-norm performance

The peak value of a frequency response is called the H∞ norm. This is a powerful measure of the disturbance rejection and noise suppression ability of systems. It can be interpreted as the maximum energy gain over the whole frequency range. In this section the H∞ norm performance of the control system will be described. The use of the H∞ norm measure to robustness will be discussed in next section.

Before continuing, an understanding of the concepts of *singular values* is necessary. The singular values of a rank r matrix, $\mathbf{M} \in \Re^{m \times n}$, denoted σ_i are the non-negative square-roots of the eigenvalues of $\mathbf{M}^T\mathbf{M}$ ordered such as that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$. If r<n then there are n-r zero singular values. Furthermore, there exist two unitary matrices $\mathbf{U} \in \Re^{m \times m}$ and $\mathbf{V} \in \Re^{n \times n}$, and a diagonal matrix, $\Sigma \in \Re^{m \times n}$, such that

$$\mathbf{M}^T\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^T = \mathbf{U}\begin{bmatrix} \Sigma_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{V}^T$$

Where $\Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$. This representation is called the singular-value decomposition(SVD) of matrix **M**. The greatest singular value, σ_1, is denoted $\bar{\sigma}(\mathbf{M}) = \sigma_1$; the n-th singular value (i.e., the least singular value) is denoted $\underline{\sigma}(\mathbf{M}) = \sigma_r$. Some useful properties of singular values are given here:

(p1). $\underline{\sigma}(\mathbf{M}) \leq |\lambda_i(\mathbf{M})| \leq \bar{\sigma}(\mathbf{M})$

(p2). If $\mathbf{M}^{-1}$ exists, $\bar{\sigma}(\mathbf{M}) = \dfrac{1}{\underline{\sigma}(\mathbf{M}^{-1})}$, and $\underline{\sigma}(\mathbf{M}) = \dfrac{1}{\bar{\sigma}(\mathbf{M}^{-1})}$

(p3). $\bar{\sigma}(\alpha\mathbf{M}) = |\alpha|\bar{\sigma}(\mathbf{M})$, where α is any scaled parameter.

(p4). $\bar{\sigma}(\mathbf{M}+\mathbf{N}) \leq \bar{\sigma}(\mathbf{M}) + \bar{\sigma}(\mathbf{N})$, and $\bar{\sigma}(\mathbf{MN}) \leq \bar{\sigma}(\mathbf{M})\bar{\sigma}(\mathbf{N})$

(p5). $\underline{\sigma}(\mathbf{M}) - \bar{\sigma}(\mathbf{N}) \leq \underline{\sigma}(\mathbf{M}+\mathbf{N}) \leq \underline{\sigma}(\mathbf{M}) + \bar{\sigma}(\mathbf{N})$

(p6). $\max\{\bar{\sigma}(\mathbf{M}), \bar{\sigma}(\mathbf{N})\} \leq \bar{\sigma}([\mathbf{M} \quad \mathbf{N}]) \leq \sqrt{2}\max\{\bar{\sigma}(\mathbf{M}), \bar{\sigma}(\mathbf{N})\}$

(p7). $\max\{\bar{\sigma}(\mathbf{M}), \bar{\sigma}(\mathbf{N})\} \leq \bar{\sigma}(\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix})$

(p8). $\max_{i,j}|m_{i,j}| \leq \bar{\sigma}(\mathbf{M}) \leq n\max_{i,j}|m_{i,j}|$

(p9). $\sum_{i=1}^{n}\sigma_i^2 = \mathrm{Tr}(\mathbf{M}^T\mathbf{M})$

To use these concepts and properties in an H∞ optimisation design method, let us study the H∞ norm measure of the control system. Consider a stable transfer function $\mathbf{G}(s) = \frac{\mathbf{y}(s)}{\mathbf{u}(s)}$. The H∞ norm can be defined, in the frequency domain, as

$$\|\mathbf{G}(s)\|_\infty = \operatorname*{Sup}_{\omega}\left(\overline{\sigma}[\mathbf{G}(j\omega)]\right) \tag{2.3.16}$$

To interpret the H∞ norm as the maximum energy gain, suppose $\mathbf{G}(s)$ describes a stable dynamic system with input vector $\mathbf{u}(t)$ and output vector $\mathbf{y}(t)$. Let $\mathbf{u}(t)$ be bounded in energy by which we mean that the total input energy is finite, i.e.,

$$\text{total input energy} = \int_0^\infty \mathbf{u}(t)^T\mathbf{u}(t)dt$$

Then the square root of the maximum energy gain from input to output over all non-zero $\mathbf{u}(t)$ is equal to the H∞ norm of $\mathbf{G}(s)$:

$$\operatorname*{Sup}_{\mathbf{u}(t)\neq 0}\left(\frac{\int_0^\infty \mathbf{y}(t)^T\mathbf{y}(t)dt}{\int_0^\infty \mathbf{u}(t)^T\mathbf{u}(t)dt}\right)^{\frac{1}{2}} = \operatorname*{Sup}_{\omega}\left(\overline{\sigma}[\mathbf{G}(j\omega)]\right) = \|\mathbf{G}(s)\|_\infty \tag{2.3.17}$$

Hence, the H∞ norm is a powerful measure of the disturbance rejection and noise suppression ability of system.

The following Lemma will present some important relationships between H∞ norm performance and quadratic stability through the use of Riccati equations.

Lemma 2.3.5 For a system $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

(1). $\|\mathbf{G}(s)\|_\infty < \gamma_0$ if and only if there exists a positive definite matrix **P** which satisfies

$$\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A}+\gamma_0^{-2}\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P}+\mathbf{C}^T\mathbf{C}<\mathbf{0} \tag{2.3.18}$$

(2). If [**A**, **C**] is observable, then this condition may be relaxed to:

$$\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A}+\gamma_0^{-2}\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P}+\mathbf{C}^T\mathbf{C}=\mathbf{0} \tag{2.3.19}$$

Proof: (1). (**Sufficient**): Assume that there exists a positive definite matrix, **P**, which satisfies (2.3.16). We can then define two constant matrices $\mathbf{Q}_1 > \mathbf{0}$ and $\mathbf{E}_0$ as follows:

$$\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A}+\gamma_0^{-2}\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P}+\mathbf{C}^T\mathbf{C}+\mathbf{Q}_1=\mathbf{0}$$

$$\mathbf{E}_0^T\mathbf{E}_0=\mathbf{C}^T\mathbf{C}+\mathbf{Q}_1$$

then we obtain

$$\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A}+\gamma_0^{-2}\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P}+\mathbf{E}_0^T\mathbf{E}_0=\mathbf{0}$$

and since $(j\omega\mathbf{P})^* + (j\omega\mathbf{P}) = \mathbf{0}$ it follows that

$$-(j\omega\mathbf{I}-\mathbf{A})^*\mathbf{P} - \mathbf{P}(j\omega\mathbf{I}-\mathbf{A}) + \gamma_0^{-2}\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P} + \mathbf{E}_0^T\mathbf{E}_0 = \mathbf{0} \tag{2.3.20}$$

Where * means take the conjugate transpose. Premultiply (2.3.20) by $\gamma_0^{-1}(-j\omega\mathbf{I}-\mathbf{A})^{-T}$ and postmultiply by $\gamma_0^{-1}(j\omega\mathbf{I}-\mathbf{A})^{-1}$ to obtain:

$$\begin{aligned}&-\mathbf{P}\gamma_0^{-2}(j\omega\mathbf{I}-\mathbf{A})^{-1} + \gamma_0^{-4}(-j\omega\mathbf{I}-\mathbf{A})^{-T}\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P}(j\omega\mathbf{I}-\mathbf{A})^{-1}\\ &\quad + \gamma_0^{-2}(-j\omega\mathbf{I}-\mathbf{A})^{-T}\mathbf{E}_0^T\mathbf{E}_0(j\omega\mathbf{I}-\mathbf{A})^{-1} - \gamma_0^{-2}(-j\omega\mathbf{I}-\mathbf{A})^{-T}\mathbf{P} = \mathbf{0}\end{aligned} \tag{2.3.21}$$

Premultiply (2.3.21) by $\mathbf{B}^T$ and postmultiply by $\mathbf{B}$, defining:

$$\mathbf{H}(j\omega) = \gamma_0^{-2}\mathbf{B}^T(-j\omega\mathbf{I}-\mathbf{A})^{-T}\mathbf{P}\mathbf{B}$$

and

$$\mathbf{G}_1(s) = \mathbf{E}_0(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$

to obtain:

$$-\mathbf{H}(j\omega) - \mathbf{H}^T(-j\omega) + \mathbf{H}(j\omega)\mathbf{H}^T(-j\omega) + \gamma_0^{-2}\mathbf{G}_1^T(-j\omega)\mathbf{G}_1(j\omega) = 0 \tag{2.3.22}$$

i.e., $\mathbf{I} - [\mathbf{I}-\mathbf{H}(-j\omega)]^T[\mathbf{I}-\mathbf{H}(j\omega)] - \gamma_0^{-2}\mathbf{G}_1^T(-j\omega)\mathbf{G}_1(j\omega) = 0$

$$\mathbf{I} - \gamma_0^{-2}\mathbf{G}_1^T(-j\omega)\mathbf{G}_1(j\omega) \geq 0$$

so $\mathbf{I} - \gamma_0^{-2}\mathbf{G}^T(-j\omega)\mathbf{G}(j\omega) - \gamma_0^{-2}\mathbf{B}^T(-j\omega\mathbf{I}-\mathbf{A})^{-T}\mathbf{Q}_1(j\omega\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} \geq 0$

Since $\mathbf{B}$ should be full column rank and $\mathbf{Q}_1$ is positive definite, it is clear that for finite ω,

$$\gamma_0^{-2}\mathbf{B}^T(-j\omega\mathbf{I}-\mathbf{A})^{-T}\mathbf{Q}_1(j\omega\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} > 0$$

So $\mathbf{I} - \gamma_0^{-2}\mathbf{G}^T(-j\omega)\mathbf{G}(j\omega) > 0$ (2.3.23)

Therefore

$$\|\mathbf{G}(s)\|_\infty < \gamma_0$$

(**Necessary**): If $\|\mathbf{G}(s)\|_\infty < \gamma_0$ then it follows that

$$\gamma_0^{-2}\mathbf{G}^T(-j\omega)\mathbf{G}(j\omega) < \mathbf{I}$$

Using this inequality and from the results of Brockett (1970) we find that there exist matrices, $\mathbf{P} > \mathbf{0}$ and $\mathbf{Q}_2 > \mathbf{0}$, such that:

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \gamma_0^{-2}\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P} + \mathbf{C}^T\mathbf{C} < \mathbf{0} \tag{2.3.24}$$

This implies that:

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \gamma_0^{-2}\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P} + \mathbf{C}^T\mathbf{C} < \mathbf{0}$$

The proof of result (2) could be obtained in (Lemma 4 of Doyle 1989). ◆

Corollary 2.3.5 It is also evident that, for any finite positive parameter γ_0, if the transfer function $\mathbf{G}(s)$ for given system satisfies $\|\mathbf{G}(s)\|_\infty < \gamma_0$, this implies that there exists a positive definite matrix, $\mathbf{P}$, which satisfies (2.3.18) so the system is stable by the use of Lemma 2.3.1.

A further useful result can be found in the following Lemma by the use of the properties of singular values.

Lemma 2.3.6 For any appropriately dimension matrices, $\mathbf{B}_1$ and $\mathbf{C}_1$, it follows that:

$$\left\| \begin{bmatrix} \mathbf{C} \\ \mathbf{C}_1 \end{bmatrix} (s\mathbf{I}-\mathbf{A}_0)^{-1} [\mathbf{B} \quad \mathbf{B}_1] \right\|_\infty \geq \left\| \mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B} \right\|_\infty$$

Proof: From the properties (P6) and (P7) of singular values, we can easily find that:

$$\left\| \begin{bmatrix} \mathbf{C} \\ \mathbf{C}_1 \end{bmatrix} (s\mathbf{I}-\mathbf{A}_0)^{-1} [\mathbf{B} \quad \mathbf{B}_1] \right\|_\infty = \left\| \begin{bmatrix} \mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}[\mathbf{B} \quad \mathbf{B}_1] \\ \mathbf{C}_1(s\mathbf{I}-\mathbf{A}_0)^{-1}[\mathbf{B} \quad \mathbf{B}_1] \end{bmatrix} \right\|_\infty$$

From the property (P6) of singular values, it follows that:

$$\left\| \begin{bmatrix} \mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}[\mathbf{B} \quad \mathbf{B}_1] \\ \mathbf{C}_1(s\mathbf{I}-\mathbf{A}_0)^{-1}[\mathbf{B} \quad \mathbf{B}_1] \end{bmatrix} \right\|_\infty \geq \left\| \mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}[\mathbf{B} \quad \mathbf{B}_1] \right\|_\infty$$

Since $\left\| \mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}[\mathbf{B} \quad \mathbf{B}_1] \right\|_\infty = \left\| [\mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B} \quad \mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B}_1] \right\|_\infty$

From the property (P7) of singular value, it follows that:

$$\left\| [\mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B} \quad \mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B}_1] \right\| \geq \left\| \mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B} \right\|_\infty$$

So

$$\left\| \begin{bmatrix} \mathbf{C} \\ \mathbf{C}_1 \end{bmatrix} (s\mathbf{I}-\mathbf{A}_0)^{-1} [\mathbf{B} \quad \mathbf{B}_1] \right\| \geq \left\| \mathbf{C}(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B} \right\|_\infty$$ ◆

In this section some different kinds of performance measures of control systems have been introduced, stability guarantee, integral quadratic cost and H∞ performance will be the main objectives of control design in this thesis.

2.4 ROBUSTNESS OF CONTROL SYSTEMS

In this section, measures of the robustness of uncertain systems described by *nominal model and unknown uncertainty* will be established. Hence, only a nominal model with some assumed uncertainties is available for robust design and robust analysis. From the discussions of §2.2.4, the unknown uncertainty should be described as parametric uncertainty for a state-feedback control system, but for a dynamic output feedback control system, unknown uncertainty should be described in general as an additive nonparametric uncertainty. In robustness analysis there are two principle concerns, namely, stability robustness and performance robustness.

As a preliminary, we introduce the following result from linear algebra which will be used to derive the subsequent results.

Lemma 2.4.1 For any matrices $\mathbf{X}$, $\mathbf{Y}$ and a full rank matrix $\mathbf{V}$ with appropriate dimensions:

(i). $\mathbf{X}^T\mathbf{Y}+\mathbf{Y}^T\mathbf{X} \le \alpha\mathbf{X}^T\mathbf{X}+\frac{1}{\alpha}\mathbf{Y}^T\mathbf{Y}$, for any scalar $\alpha > 0$

(ii). $\mathbf{X}^T\mathbf{Y}+\mathbf{Y}^T\mathbf{X} \le \alpha\mathbf{X}^T\mathbf{V}\mathbf{X}+\frac{1}{\alpha}\mathbf{Y}^T\mathbf{V}^{-1}\mathbf{Y}$, for any $\alpha > 0$ and $\mathbf{V} > 0$

Proof: Since

$$\alpha\mathbf{X}^T\mathbf{X}+\frac{1}{\alpha}\mathbf{Y}^T\mathbf{Y}-\mathbf{X}^T\mathbf{Y}-\mathbf{Y}^T\mathbf{X}=(\frac{1}{\sqrt{\alpha}}\mathbf{Y}-\sqrt{\alpha}\mathbf{X})^T(\frac{1}{\sqrt{\alpha}}\mathbf{Y}-\sqrt{\alpha}\mathbf{X}) \ge \mathbf{0}$$

So the result of (i) follows. Similarly, since

$$\alpha\mathbf{X}^T\mathbf{V}\mathbf{X}+\frac{1}{\alpha}\mathbf{Y}^T\mathbf{V}^{-1}\mathbf{Y}-\mathbf{X}^T\mathbf{Y}-\mathbf{Y}^T\mathbf{X}=(\frac{\mathbf{V}^{-1/2}}{\sqrt{\alpha}}\mathbf{Y}-\sqrt{\alpha}\mathbf{V}^{1/2}\mathbf{X})^T(\frac{\mathbf{V}^{-1/2}}{\sqrt{\alpha}}\mathbf{Y}-\sqrt{\alpha}\mathbf{V}^{1/2}\mathbf{X}) \ge \mathbf{0}$$

So the result of (ii) also follows. ◆

2.4.1 Stability robustness

Stability robustness concerns the problem of whether the system remains stable for all plant uncertainty within a specific class of uncertainties. A related problem involves determining the largest class of uncertainties under which stability is preserved.

As discussed in §2.2.4, the unknown uncertainty should be described as parametric uncertainty for state feedback control system, or for an output feedback control system, unknown uncertainty should be described as nonparametric uncertainty. Hence in this section two kinds of uncertain controlled system will be studied: uncertain systems with assumed parametric uncertainty and a state feedback controller, and uncertain systems with assumed nonparametric uncertainty and an output feedback controller.

I. Systems with assumed parametric uncertainty and a state feedback controller

For a system and given full state feedback controller, **K**, only uncertainties $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$, which are the parametric part of the unknown uncertainty, will affect the stability robustness of the closed loop system. The effects of uncertainties $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ can be united in $\Delta\mathbf{A}_0$ by: $\Delta\mathbf{A}_0 = \Delta\mathbf{A} + \Delta\mathbf{B}\mathbf{K}$, so for full state feedback controlled system, without loss of generality, we can only consider the closed loop system uncertainty: $\Delta\mathbf{A}_0 = \mathbf{N}\Phi(t)\mathbf{M}$, where **N** and **M** are constant matrices which imply the structure of uncertainty, the uncertain matrix is constrained by singular value. Hence consider an uncertain system described as:

$$\dot{\mathbf{x}} = (\mathbf{A}_0 + \mathbf{N}\Phi(t)\mathbf{M})\mathbf{x} \tag{2.4.1}$$

with $\bar{\sigma}(\Phi(t)) \le \varepsilon$. The problem studied here is to find a condition such that the uncertain system (2.4.1) can remain stable for all admissible uncertainty. Since the uncertain matrix is time-varying, from Corollary 2.3.1, uncertain system (2.4.1) will be robustly stable if,

$$(\mathbf{A}_0 + \mathbf{N}\Phi(t)\mathbf{M})^T\mathbf{P} + \mathbf{P}(\mathbf{A}_0 + \mathbf{N}\Phi(t)\mathbf{M}) < \mathbf{0} \tag{2.4.2}$$

has a positive definite solution, $\mathbf{P} > \mathbf{0}$. It is obvious that to check this for all admissible $\Phi(t)$ is impossible, the following result will give a basic theory of the robust control design and analysis techniques for the system with parametric uncertainties.

Lemma 2.4.2 Consider the uncertain system (2.4.1), if there exists a parameter $\alpha > 0$, such that

$$\mathbf{A}_0^T\mathbf{P}_0 + \mathbf{P}_0\mathbf{A}_0 + \alpha\varepsilon^2\mathbf{P}_0\mathbf{N}\mathbf{N}^T\mathbf{P}_0 + \frac{1}{\alpha}\mathbf{M}^T\mathbf{M} < \mathbf{0} \tag{2.4.3}$$

has a positive definite solution, $\mathbf{P}_0$, then this solution also satisfies expression (2.

Proof: Suppose there exists a parameter $\alpha > 0$ such that (2.4.3) has a positive definite solution, $\mathbf{P}_0$, from (i) of Lemma 2.4.1 and $\bar{\sigma}(\Phi(t)) \le \varepsilon$ it follows that for this positive definite matrix $\mathbf{P}_0$ we always have that:

$$(\mathbf{N}\Phi(t)\mathbf{M})^T\mathbf{P}_0 + \mathbf{P}_0(\mathbf{N}\Phi(t)\mathbf{M}) \le \alpha\varepsilon^2\mathbf{P}_0\mathbf{N}\mathbf{N}^T\mathbf{P}_0 + \frac{1}{\alpha}\mathbf{M}^T\mathbf{M}$$

then adding this to (2.4.3) it follows that

$$(\mathbf{A}_0 + \mathbf{N}\Phi(t)\mathbf{M})^T\mathbf{P}_0 + \mathbf{P}_0(\mathbf{A}_0 + \mathbf{N}\Phi(t)\mathbf{M}) < \mathbf{0} \tag{2.4.4}$$

This means that there exists a positive definite solution which satisfies (2.4.4). From the definition of the Corollary 2.3.1 we know that system (2.4.1), $\dot{\mathbf{x}} = (\mathbf{A}_0 + \mathbf{N}\Phi(t)\mathbf{M})\mathbf{x}$, is quadratically stable. ◆

So the existence of a positive definite solution for (2.4.3) is sufficient to guarantee the robust stability of uncertain system (2.4.1). Alternatively, a robust stability corollary may be established in the form of an H∞ norm.

Corollary 2.4.2 System $\dot{\mathbf{x}} = (\mathbf{A}_0 + \Delta\mathbf{A}_0)\mathbf{x}$ is stable for all $\Delta\mathbf{A}_0 = \mathbf{N}\Phi(t)\mathbf{M}$ with $\overline{\sigma}(\Phi(t)) \le \varepsilon$ if

$$\left\|\mathbf{M}(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{N}\right\|_\infty < 1/\varepsilon \tag{2.4.5}$$

or, if there exist any two matrices $\mathbf{N}_0$, $\mathbf{M}_0$ and any $\alpha > 0$ such that:

$$\left\|\mathbf{M}_0(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{N}_0\right\|_\infty < 1 \tag{2.4.6}$$

and $\alpha\varepsilon\mathbf{M}^T\mathbf{M} \le \mathbf{M}_0^T\mathbf{M}_0$; $\frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T \le \mathbf{N}_0\mathbf{N}_0^T$

Proof: It is clear that if

$$\alpha\varepsilon\mathbf{M}^T\mathbf{M} \le \mathbf{M}_0^T\mathbf{M}_0; \quad \frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T \le \mathbf{N}_0\mathbf{N}_0^T$$

then there exist two matrices $\mathbf{B}_1$ and $\mathbf{C}_1$ such that

$$\mathbf{M}_0^T\mathbf{M}_0 = \alpha\varepsilon\mathbf{M}^T\mathbf{M} + \mathbf{C}_1^T\mathbf{C}_1; \quad \mathbf{N}_0\mathbf{N}_0^T = \frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T + \mathbf{B}_1\mathbf{B}_1^T$$

i.e.

$$\mathbf{M}_0 = \begin{bmatrix} \sqrt{\alpha\varepsilon}\mathbf{M} \\ \mathbf{C}_1 \end{bmatrix} \text{ and } \mathbf{N}_0 = \begin{bmatrix} \sqrt{\frac{\varepsilon}{\alpha}}\mathbf{N} & \mathbf{B}_1 \end{bmatrix}$$

So from Lemma 2.3.6 we find that if the system satisfies (2.4.6), then it will satisfy (2.4.5). However, from Lemma 2.3.5 it follows that if (2.4.5) is satisfied, then there exists a positive solution for the following

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{P}\frac{\varepsilon^2}{\alpha}\mathbf{N}\mathbf{N}^{T^T}\mathbf{P} + \alpha\mathbf{M}^T\mathbf{M} < \mathbf{0}$$

From Lemma 2.4.1 we can find that

$$(\mathbf{A} + \mathbf{N}\Phi(t)\mathbf{M})^T\mathbf{P} + \mathbf{P}(\mathbf{A} + \mathbf{N}\Phi(t)\mathbf{M}) < \mathbf{0}$$

Hence $V = \mathbf{x}^T\mathbf{P}\mathbf{x}$ is a Lyapunov function and the system will be robustly stable. ◆

II. Systems with assumed nonparametric uncertainty and an output feedback controller

From §2.2.4 we know that parametric part $\Delta\mathbf{A}$, $\Delta\mathbf{B}$ and $\Delta\mathbf{C}$ of unknown uncertainty are some special cases of non-parametric part $\Delta\mathbf{G}(s)$, and it is reasonable to consider nonparametric $\Delta\mathbf{G}(s)$ only because good robustness for non-parametric uncertainty part $\Delta\mathbf{G}(s)$ will provide some inherent robustness properties for the parametric uncertainty parts $\Delta\mathbf{A}$, $\Delta\mathbf{B}$ and $\Delta\mathbf{C}$. Hence we consider an uncertain system described in §2.2.2 as:

$$\mathbf{G}(s) = \mathbf{G}_0(s) + \mathbf{N}\Delta(s)\mathbf{M} \tag{2.4.7}$$

Where $\Delta(s)$ is uncertain matrix and its size is bounded by $\|\Delta(s)\|_\infty \leq \eta$. Matrices **N** and **M** could describe the structure of non-parametric uncertainty, if the uncertainty is unstructured, then **N** and **M** can be chosen as identity matrices.

To establish a general condition for robust stability of the uncertain system, consider the closed loop system of uncertain system (2.4.7) with a dynamic output feedback controller **K**(s) could be presented by Fig. 2.5:

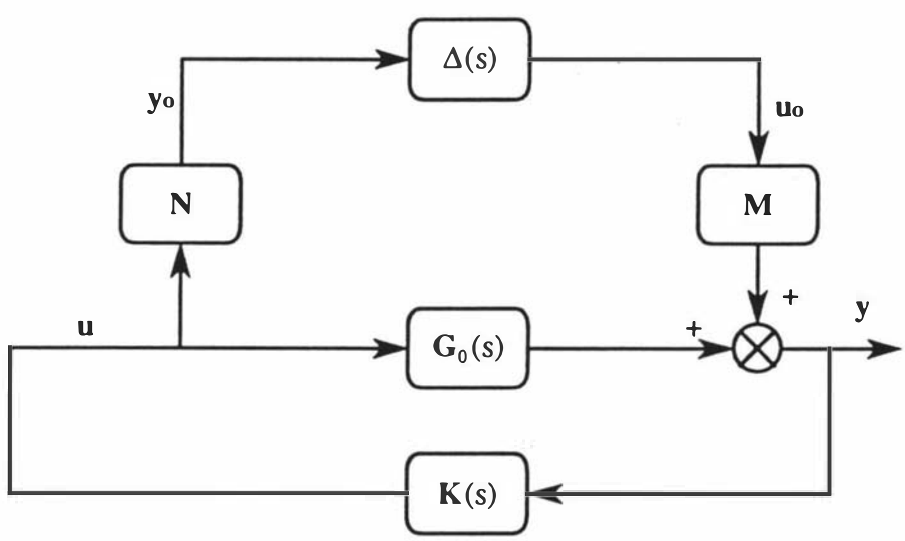

Fig. 2.5 The uncertain closed-loop system

Furthermore, diagram of Fig. 2.5 can be transferred to the following form:

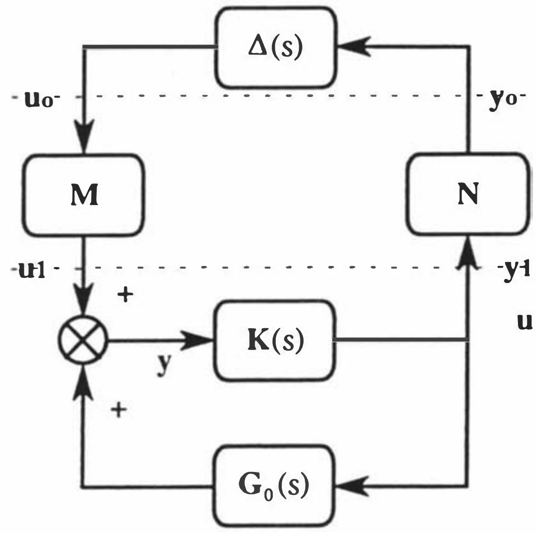

Fig. 2.6 The transferred uncertain closed-loop system

Then we can get the diagram for the Small Gain Theorem as:

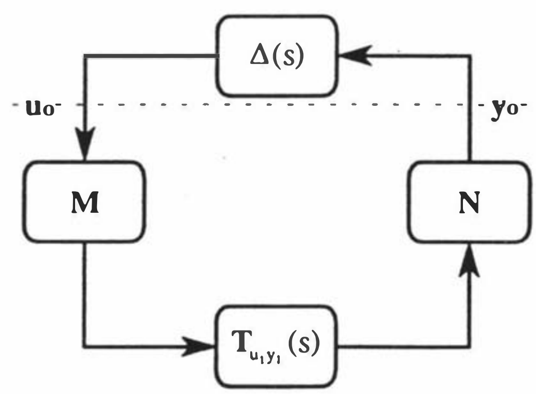

Fig. 2.7 The diagram for the Small Gain Theorem

The well-known Small Gain Theorem is introduced in the following (Balas, 1991):

Lemma 2.4.3 The closed loop system shown in Fig 2.7 is stable for all uncertainty $\Delta(s)$ with $\|\Delta(j\omega)\|_\infty \leq \eta$, if

$$\|\mathbf{T}_{u_o y_o}(j\omega)\|_\infty < \frac{1}{\eta} \tag{2.4.9}$$

However, $\mathbf{T}_{u_o y_o}(s)$ can be found from the Fig. 2.8.

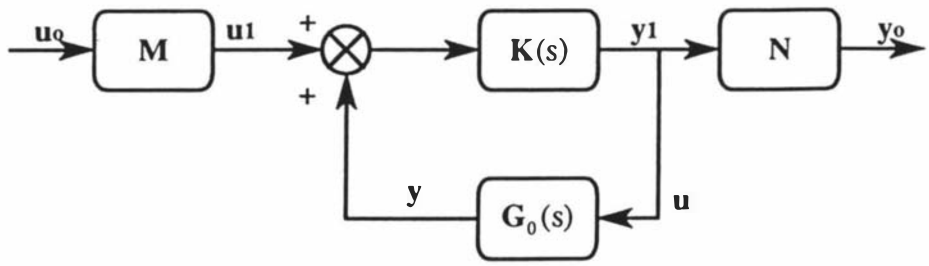

Fig. 2.8 The diagram for the certain system part

So $\quad \mathbf{T}_{u_o y_o}(s) = \mathbf{M}\mathbf{K}(s)[\mathbf{I}+\mathbf{G}_0(s)\mathbf{K}(s)]^{-1}\mathbf{N}$

General state space descriptions of such plant with transfer function $\mathbf{G}_0(s)$ and dynamic output feedback controller $\mathbf{K}(s)$ were given in §2.1.1 and §2.1.2 which can be represented as:

$$\mathbf{G}_0(s) \Rightarrow \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \qquad \mathbf{K}(s) \Rightarrow \begin{cases} \dot{\varsigma} = \mathbf{A}_c\varsigma + \mathbf{B}_c\mathbf{e} \\ \mathbf{y}_1 = \mathbf{C}_c\varsigma \end{cases}$$

It is clear from the Fig. 2.8 that

$$\mathbf{u} = \mathbf{y}_1, \qquad \mathbf{e} = \mathbf{y} + \mathbf{M}\mathbf{u}_o, \qquad \mathbf{y}_o = \mathbf{N}\mathbf{y}_1.$$

Then closed loop transfer function from $\mathbf{u}_o$ to $\mathbf{y}_o$ can be found as:

$$\mathbf{T}_{u_o y_o}(s) = \mathbf{C}_0(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}_0 \tag{2.4.10}$$

Where

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{C}_c \\ \mathbf{B}_c\mathbf{C} & \mathbf{A}_c \end{bmatrix}, \quad \mathbf{C}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{N}\mathbf{C}_c \end{bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}\mathbf{B}_c \end{bmatrix}$$

So from Lemma 2.4.3 an alternative interpretation of this robust stability criterion can be formulated as the corollary 2.4.3.

Hence, for the system with nominal model and nonparametric uncertainty, to improve the robustness of the closed loop system subject to this non-parametric uncertainty implies to design the controller such that $\left\|\mathbf{T}_{u_o y_o}(j\omega)\right\|_\infty$ can be reduced. The choice of $\mathbf{N}$ and $\mathbf{M}$ depends on the structure of uncertainty which the open loop system will have, if we don't know anything of uncertainty structure, then matrices $\mathbf{N}$ and $\mathbf{M}$ should be chosen as identity matrix.

Corollary 2.4.3 If the H∞ norm of the transfer function of the closed loop system (2.4.10) satisfies the condition of

$$\left\|\mathbf{T}_{u_o y_o}(j\omega)\right\|_\infty < \frac{1}{\eta} \tag{2.4.11}$$

then the system will be robustly stable to the non-parametric uncertainty $\Delta\mathbf{G}(s) = \mathbf{N}\Delta(s)\mathbf{M}$ with $\left\|\Delta(j\omega)\right\|_\infty \le \eta$.

2.4.2 Performance robustness

Robust stability is the minimum requirement for a control system with significant model uncertainties. However, robust stability alone is often not enough, once it has been satisfied, it is of interest to investigate quantitatively the performance degradation within a given uncertainty domain. In most cases, long before the onset of instability, the closed-loop performance will degrade to the point of unacceptability. Hence a "robust performance" measure is necessary for system analysis, such a measure can be indicated by the worst case performance associated for a given level of uncertainties.

Here we also consider the uncertain system described by a nominal model and unknown uncertainty, for simplicity, only robustness of integral-quadratic cost performance (or H_2 norm) for the full state-feedback controlled system is considered. The case of output feedback controlled systems is very complex, it is an interesting issue but beyond the scope of this thesis.

Consider the same uncertain closed loop system of (2.4.1) with initial state vector $\mathbf{x}(0) = \mathbf{x}_0$.

$$\dot{\mathbf{x}} = (\mathbf{A}_0 + \Delta\mathbf{A}_0(t))\mathbf{x} \quad \text{where } \Delta\mathbf{A}_0(t) = \mathbf{N}\Phi(t)\mathbf{M}, \ \bar{\sigma}(\Phi(t)) \le \varepsilon$$

If the performance index is given by (2.3.6) as

$$J = \int_0^\infty \mathbf{x}^T\mathbf{Q}_0\mathbf{x}dt$$

then we try to apply Lemma 2.3.2, the cost performance is $J = \mathbf{x}_0^T\mathbf{P}\mathbf{x}_0$, where $\mathbf{P}$ is the *constant positive definite solution* of the following Lyapunov equation:

$$(\mathbf{A}_0 + \Delta\mathbf{A}_0(t))^T\mathbf{P} + \mathbf{P}(\mathbf{A}_0 + \Delta\mathbf{A}_0(t)) + \mathbf{Q}_0 = \mathbf{0} \tag{2.4.12}$$

or $$(\mathbf{A}_0 + \mathbf{N}\Phi(t)\mathbf{M})^T\mathbf{P} + \mathbf{P}(\mathbf{A}_0 + \mathbf{N}\Phi(t)\mathbf{M}) + \mathbf{Q}_0 = \mathbf{0}$$

Since the closed loop system is a time-varying system, and it is impossible to find a time-invariant solution $\mathbf{P} > \mathbf{0}$ for the above equation, so the result of Lemma 2.3.2 can not be directly used here to solve the performance matrix of this uncertain system. However, since the time-varying uncertain matrix $\Phi(t)$ is constrained by $\bar{\sigma}(\Phi(t)) \le \varepsilon$, a bound matrix $\mathbf{P}_b$ can be found. This is the upper limit of all possible solutions for (2.4.12) over all admissible values of $\Phi(t)$ and all $t \in [0, \infty)$. The following Lemma will provide a method to find the cost bound for the uncertain closed loop system (2.4.1) over all admissible values of $\Phi(t)$.

Lemma 2.4.4 If for any positive definite matrix $\mathbf{P}$, there exists a bound function $\Theta(\mathbf{N},\mathbf{M},\mathbf{P})$, such that:

$$(\Delta\mathbf{A}_0(t))^T\mathbf{P}+\mathbf{P}(\Delta\mathbf{A}_0(t)) \le \Theta(\mathbf{N},\mathbf{M},\mathbf{P}) \tag{2.4.13}$$

Then there exists a cost performance bound $J_b = \mathbf{x}_0^T\mathbf{P}_b\mathbf{x}_0$ over all admissible values of $\Phi(t)$ and all $t \in [0, \infty)$.

i.e. $J \le J_b = \mathbf{x}_0^T\mathbf{P}_b\mathbf{x}_0$, where $\mathbf{P}_b$ is the positive definite solution of the following Riccati equation:

$$\mathbf{A}_0^T\mathbf{P}+\mathbf{P}\mathbf{A}_0+\Theta(\mathbf{N},\mathbf{M},\mathbf{P})+\mathbf{Q}_0=\mathbf{0} \tag{2.4.14}$$

Proof: Choosing $\mathbf{P}=\mathbf{P}_b$ and adding (2.4.13) to (2.4.14) it will follow that the following expression will be hold for all admissible values of $\Phi(t)$ and all time $t \in [0, \infty)$:

$$(\mathbf{A}_0+\Delta\mathbf{A}_0(t))^T\mathbf{P}_b+\mathbf{P}_b(\mathbf{A}_0+\Delta\mathbf{A}_0(t))+\mathbf{Q}_0 \le \mathbf{0} \tag{2.4.15}$$

this means that if (2.4.14) has a positive definite solution $\mathbf{P}_b$, this solution will also satisfy the expression (2.4.15) for all time $t \in [0, \infty)$.

Following we will prove that the value $\mathbf{x}_0^T\mathbf{P}_b\mathbf{x}_0$ is a bound for the cost $J = \mathbf{x}_0^T\mathbf{P}\mathbf{x}_0$ for all admissible values of $\Phi(t)$ and all time $t \in [0, \infty)$, i.e., $J \le J_b = \mathbf{x}_0^T\mathbf{P}_b\mathbf{x}_0$. From (2.4.15) we find that:

$$\mathbf{Q}_0 \le -(\mathbf{A}_0+\Delta\mathbf{A}_0(t))^T\mathbf{P}_b-\mathbf{P}_b(\mathbf{A}_0+\Delta\mathbf{A}_0(t)) \tag{2.4.16}$$

or

$$x^T\mathbf{Q}_0 x \le -x^T[(\mathbf{A}_0+\Delta\mathbf{A}_0(t))^T\mathbf{P}_b+\mathbf{P}_b(\mathbf{A}_0+\Delta\mathbf{A}_0(t))]x \tag{2.4.17}$$

so

$$J=\int_0^\infty x^T\mathbf{Q}_0 x dt \le -\int_0^\infty x^T[(\mathbf{A}_0+\Delta\mathbf{A}_0(t))^T\mathbf{P}_b+\mathbf{P}_b(\mathbf{A}_0+\Delta\mathbf{A}_0(t))]xdt$$

$$\begin{aligned} &\le -\int_0^\infty \frac{d}{dt}(x^T\mathbf{P}_b x)dt \\ &\le x^T(0)\mathbf{P}_b x(0)-x^T(\infty)\mathbf{P}_b x(\infty) \\ &\le x^T(0)\mathbf{P}_b x(0) \end{aligned} \tag{2.4.18}$$

Since (2.4.15) has a positive definite solution, according to the Lemma 2.4.2, the closed loop system should be stable, hence $\mathbf{x}(\infty)=\mathbf{0}$, the bound of cost performance can be found as $J=\mathbf{x}_0^T\mathbf{P}\mathbf{x}_0 \le \mathbf{x}_0^T\mathbf{P}_b\mathbf{x}_0$. ◆

From (i) of Lemma 2.4.1 it follows that for any given $\mathbf{P} > \mathbf{0}$, we always have that:

$$(\Delta\mathbf{A}_0(t))^T\mathbf{P}+\mathbf{P}(\Delta\mathbf{A}_0(t)) = (\mathbf{N}\Phi(t)\mathbf{M})^T\mathbf{P}+\mathbf{P}(\mathbf{N}\Phi(t)\mathbf{M})$$

so
$$(\Delta\mathbf{A}_0(t))^T\mathbf{P}+\mathbf{P}(\Delta\mathbf{A}_0(t)) \le \alpha\varepsilon^2\mathbf{P}\mathbf{N}\mathbf{N}^T\mathbf{P}+\frac{1}{\alpha}\mathbf{M}^T\mathbf{M} \tag{2.4.19}$$

Hence for uncertain system (2.4.1), a bound function can be found as:

$$\Theta(\mathbf{N},\mathbf{M},\mathbf{P}) = \alpha\varepsilon^2\mathbf{P}\mathbf{N}\mathbf{N}^T\mathbf{P}+\frac{1}{\alpha}\mathbf{M}^T\mathbf{M}$$

The cost bound J_b is the maximum possible cost for all admissible values of $\Phi(t)$, if J_b is acceptable, this means the degradation of the cost value is small enough over all admissible values of uncertain matrix $\Phi(t)$, thus the system has good performance robustness.

The performance robustness may be measured by a *performance degradation parameter* , or *performance robustness index* ρ, defined as:

$$\rho = \frac{\mathbf{x}_0^T(\mathbf{P}_b-\mathbf{P}_0)\mathbf{x}_0}{\mathbf{x}_0^T\mathbf{P}_0\mathbf{x}_0}\times 100\%$$

where $\mathbf{P}_0 > 0$ is the performance matrix of the certain part of the closed loop system as Lemma 2.3.2. This gives a measure of the possible relative cost variation across the admissible domain of uncertainties. It can be noted that if no uncertainty is presented, i.e., $\mathbf{N}\Phi(t)\mathbf{M}$=0, then $\rho = 0$, there is no performance degradation. For a given parametric uncertainty $\mathbf{N}\Phi(t)\mathbf{M}$, a relatively small performance bound J_b produces a small performance degradation and the system is said to have good performance robustness.

To design a controller which provides the system with good performance robustness means: to choose a fixed controller such that (1) the closed loop system has robust stability, (2) performance degradation rate ρ is as small as possible.

So in this section, we use the closed loop system with parametric uncertainty as an example to show the performance degradation of different controllers. The concept of good performance robustness was also presented. To conclude, a robust control system is required to have both good stability robustness and good performance robustness.

CHAPTER 3

ROBUSTNESS ANALYSIS OF SOME EXISTING CONTROLLER DESIGN TECHNIQUES

Unavoidable differences between mathematical models and real world systems can result in degradation of control-system performance including instability. Hence, a good technique of control system design should provide a certain level inherent robustness. In this chapter, robust stability conditions developed in §2.4.1 will be used to assess the stability robustness of some common control solutions applied to the nominal model. The chapter is divided between full state feedback control and dynamic output feedback control systems. For full state feedback control systems we have argued that robustness to parametric uncertainty should be studied and for dynamic output feedback control systems, robustness analysis to nonparametric uncertainty is necessary.

The following is a summary of some common control solutions that are declared as "Optimal controller design" for some particular control performance measures:

(1). For full state feedback control design, LQR design seeks to minimise the total transfer of energy from system input to output. H^∞ optimisation design seeks to minimise the peak in the frequency spectrum of the energy transfer. However, H_2/H^∞ optimisation design tries to find the optimal controller for a combination of these objectives.

(2). For dynamic output feedback control design, LQG design has the same objective as LQR. LQG/LTR tries to recover the stability robustness of the LQR design. H^∞ and H_2/H^∞ design tries to find the optimal controller for a combination of these objectives.

We will state, without giving derivation, how to implement the above modern control design techniques based on a nominal model. Then the inherent stability robustness analysis of the resulting systems will be examined according to the robustness principles developed in Chapter 2. Analysis results will tell us, based on the nominal model, which controller design techniques are robust, (i.e., some inherent robustness is offered by this design technique), which are not.

Analysis will also confirm some well-known results such as: the LQR and H^∞ design methods can provide some inherent robustness, the LQG optimal design can not. The LQG/LTR design can be used to recover the robustness of the LQR design for output feedback systems.

3.1 SYSTEMS WITH FULL STATE FEEDBACK CONTROLLERS

As noted in §2.2.4, for state feedback control design methods, it is only necessary to consider the parametric part of unknown uncertainty. To analyse the robustness of the closed loop system with full state feedback controller, the result of §2.4.1 is recalled here:

> **Remark 3.1.1** System $\dot{\mathbf{x}} = (\mathbf{A}_0 + \Delta\mathbf{A}_0)\mathbf{x}$ is robustly stable with $\Delta\mathbf{A}_0 = \mathbf{N}\Phi(t)\mathbf{M}$ (the parametric part of unknown uncertainty) and $\overline{\sigma}(\Phi(t)) \le \varepsilon$ if there exist two matrices $\mathbf{N}_0$, $\mathbf{M}_0$ and any $\alpha > 0$ such that:
>
> $$\left\|\mathbf{M}_0(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{N}_0\right\|_\infty < 1, \quad \text{and} \quad \alpha\varepsilon\mathbf{M}^T\mathbf{M} \le \mathbf{M}_0^T\mathbf{M}_0; \quad \frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T \le \mathbf{N}_0\mathbf{N}_0^T$$

So generally, to analyse the stability robustness of a closed loop system with full state feedback controller means to find the suitable matrices $\mathbf{N}_0$, $\mathbf{M}_0$ such that the following condition is satisfied:

$$\left\|\mathbf{M}_0(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{N}_0\right\|_\infty < 1$$

This result will be applied to some existing state feedback controller design techniques such as, LQR, H∞ and H2/H∞.

3.1.1 Linear Quadratic regulator design

Since the beginning of the sixties, Linear Quadratic Regulator design (LQR) has been viewed as an important design technique for linear system control. The associated Riccati equation solution provides the optimal state feedback controller that can minimise the cost function of the closed-loop system. The Riccati equation itself can be solved by some very efficient numerical procedures. For these reasons, LQR design has become very popular (Anderson and Moore, 1990).

Let us consider a system that has been introduced in §2.1.1, whose nominal model and initial condition are given as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}; \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{3.1.1}$$

Where the vector $\mathbf{x}$ is the state of the system (assumed to be available for control), the vector $\mathbf{u}$ is the control signal vector. Matrices $\mathbf{A}$ and $\mathbf{B}$ have compatible dimensions and the pair $(\mathbf{A}, \mathbf{B})$ is supposed to be stabilisable. The design objective here is to find a full state feedback controller that can stabilise the plant described by the nominal model (3.1.1) and minimise the quadratic cost function that has been given in chapter 2 as:

$$J = \int_0^{\infty}(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt \tag{3.1.2}$$

Where $\mathbf{Q} \geq \mathbf{0}$ and $\mathbf{R} > \mathbf{0}$ are weighting matrices. The solution of controller which can minimise cost function J is given by

$$\mathbf{u} = -\mathbf{K}\mathbf{x}; \quad \mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \tag{3.1.3}$$

Where **P** is the positive definite solution of the algebraic Riccati equation:

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{3.1.4}$$

For any initial condition $\mathbf{x}_0$, from Lemma 2.3.2, the minimal value of the cost function (3.1.2) is $J_0 = \mathbf{x}_0^T\mathbf{P}\mathbf{x}_0$.

Defining the closed loop system as

$$\mathbf{A}_0 = \mathbf{A} + \mathbf{B}\mathbf{K} = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \tag{3.1.5}$$

The above optimal closed loop system is stable with finite performance index if and only if: (pp. 48, Anderson & Moore, 1990)

$[\mathbf{A}, \mathbf{B}]$ is stabilisable and $[\mathbf{A}, \sqrt{\mathbf{Q}}]$ is detectable.

Now, let us consider the robustness criterion for standard LQR design technique. From (3.1.4) and (3.1.5) we find that:

$$\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\mathbf{A}_0 + \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}$$

According to Lemma 2.3.5 it follows that:

$$\left\|\sqrt{\mathbf{Q}}(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}\sqrt{\mathbf{R}^{-1}}\right\|_\infty < 1 \tag{3.1.6}$$

Subject to the Remark 3.1.1, the suitable matrices $\mathbf{N}_0$, $\mathbf{M}_0$ can be found that $\mathbf{M}_0 = \sqrt{\mathbf{Q}}$, and $\mathbf{N}_0 = \mathbf{B}\sqrt{\mathbf{R}^{-1}}$. Hence, if there exists a positive parameter α such that the following conditions are satisfied:

$$\alpha\varepsilon\mathbf{M}^T\mathbf{M} < \mathbf{Q}; \quad \frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T \leq \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \tag{3.1.7}$$

the LQR controller can provide the stability robustness guarantee to the uncertainty $\Delta\mathbf{A}_0 = \mathbf{N}\Phi(t)\mathbf{M}$ with $\bar{\sigma}(F(t)) \leq \varepsilon$.

To summarise the robustness analysis of LQR Design, it follows that:

(1). LQR Design can provide the minimal integral-quadratic performance for nominal system models.

(2). There are some inherent robustness properties in LQR Design and the robust stability condition is given as (3.1.7).

(3). Since the inherent robustness of LQR Design depends on weighting matrices **Q** and **R**, and these two matrices are normally used for tuning the cost performance, so the inherent robustness is coupled to the choice of the cost function and sometimes, "blindly" designing a LQR controller based on the nominal system does not guarantee to provide enough inherent robustness for the actual system.

3.1.2 H∞ controller design

When designing a control system, one often assumes the plant is subject to some inputs, such as disturbances and sensor noise. It is always desired to reduce the effect of these inputs on the outputs of the closed loop system, this disturbance or noise rejection is also a very important performance requirement of a control system. As mentioned in §2.3.4, H∞ norm offers a good performance measure for disturbances/noise rejection and hence, the standard H∞ optimisation design problem is to find a controller to minimise the H∞ norm of the transfer matrix from disturbances/noise to the outputs (i.e., best disturbance or noise rejection).

Consider a linear system described as (2.1.3) which has been introduced in §2.1.1 (Doyle et al. 1989):

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\omega + \mathbf{B}_2\mathbf{u} \\ \mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_1\mathbf{u} \end{cases} \tag{3.1.8}$$

Where **x** is the state vector (assumed to be available for feedback), **u** is the control signal, ϖ is the disturbance vector and **z** is the performance vector. The matrices **A** and $\mathbf{B}_2$ have compatible dimensions and the pair (**A**, $\mathbf{B}_2$) is supposed to be stabilisable, matrix $\mathbf{D}_1$ is required to be full column rank, and for simplicity, it is assumed that $\mathbf{D}_1^T\mathbf{C}_1 = \mathbf{0}$.

The design objective here is to find a controller **u=Kx** such that the H∞ norm bound of the transfer function from disturbances/noise ϖ to the performance vector **z** is minimal, i.e., to minimise γ_m such that:

$$\left\|\mathbf{T}_{\omega z}\right\|_\infty < \gamma_m$$

or

$$\left\|(\mathbf{C}_1 + \mathbf{D}_1\mathbf{K})(s\mathbf{I} - \mathbf{A} - \mathbf{B}_2\mathbf{K})^{-1}\mathbf{B}_1\right\|_\infty < \gamma_m \tag{3.1.9}$$

From Lemma 2.3.5 we know that expression (3.1.9) requires the existence of a positive definite matrix $\mathbf{P}$ which satisfies

$$(\mathbf{A}+\mathbf{B}_2\mathbf{K})^T\mathbf{P}+\mathbf{P}(\mathbf{A}+\mathbf{B}_2\mathbf{K})+\gamma_m^{-2}\mathbf{P}\mathbf{B}_1\mathbf{B}_1^T\mathbf{P}+\mathbf{K}^T\mathbf{D}_1^T\mathbf{D}_1\mathbf{K}+\mathbf{C}_1^T\mathbf{C}_1<\mathbf{0} \tag{3.1.10}$$

A controller that satisfies that is:

$$\mathbf{K}_m=-(\mathbf{D}_1^T\mathbf{D}_1)^{-1}\mathbf{B}_2^T\mathbf{P}_0 \tag{3.1.11}$$

Where $\mathbf{P}_0$ is the positive definite solution of the following expression:

$$\mathbf{A}^T\mathbf{P}_0+\mathbf{P}_0\mathbf{A}-\mathbf{P}_0\mathbf{B}_2(\mathbf{D}_1^T\mathbf{D}_1)^{-1}\mathbf{B}_2^T\mathbf{P}_0+\gamma_m^{-2}\mathbf{P}_0\mathbf{B}_1\mathbf{B}_1^T\mathbf{P}_0+\mathbf{C}_1^T\mathbf{C}_1<\mathbf{0} \tag{3.1.12}$$

γ_m may be reduced to find the optimal controller $\mathbf{K}_m$.

Now, let us consider the robustness criterion for this H∞ controller design technique, defining the closed loop system as:

$$\mathbf{A}_0=\mathbf{A}+\mathbf{B}_2\mathbf{K}_m=\mathbf{A}-\mathbf{B}_2(\mathbf{D}_1^T\mathbf{D}_1)^{-1}\mathbf{B}_2^T\mathbf{P}_0$$

then from (3.1.12) we find that $\mathbf{A}_0$ satisfies:

$$\mathbf{A}_0^T\mathbf{P}_0+\mathbf{P}_0\mathbf{A}_0+\mathbf{K}_m^T\mathbf{D}_1^T\mathbf{D}_1\mathbf{K}_m+\gamma_m^{-2}\mathbf{P}_0\mathbf{B}_1\mathbf{B}_1^T\mathbf{P}_0+\mathbf{C}_1^T\mathbf{C}_1<\mathbf{0} \tag{3.1.13}$$

From the result of Lemma 2.3.5 it follows that:

$$\left\|(\mathbf{C}_1+\mathbf{D}_1\mathbf{K}_m)(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B}_1\right\|_\infty<\gamma_m \tag{3.1.14}$$

From Corollary 2.3.5, the stability of the closed loop system can be ascertained by expression (3.1.14), and the minimal H∞ norm value is the minimal value of γ_m such that a positive definite solution can be found for the Riccati equation (3.1.12).

So according to the Remark 3.1.1 it can be found that $\mathbf{M}_0$ and $\mathbf{N}_0$ can be chosen as:

$$\mathbf{M}_0=\gamma_m^{-1/2}\begin{bmatrix}\mathbf{C}_1\\ \mathbf{D}_1\mathbf{K}_m\end{bmatrix}, \quad \text{and} \quad \mathbf{N}_0=\gamma_m^{-1/2}\mathbf{B}_1$$

Hence, if there exists a positive parameter α such that the following conditions are satisfied:

$$\alpha\varepsilon\mathbf{M}^T\mathbf{M}\le\gamma_m^{-1}(\mathbf{C}_1^T\mathbf{C}_1+\mathbf{K}_m^T\mathbf{D}_1^T\mathbf{D}_1\mathbf{K}_m); \quad \frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T\le\gamma_m^{-1}\mathbf{B}_1\mathbf{B}_1^T \tag{3.1.15}$$

The H∞ optimal controller can provide a stability robustness guarantee to the uncertainty $\Delta\mathbf{A}_0=\mathbf{N}\Phi(t)\mathbf{M}$ with $\bar{\sigma}(F(t))\le\varepsilon$.

To summarise the robustness analysis results for the H∞ Controller Design, it follows that:

(1). H∞ Optimal Design minimises the maximal singular value of the transfer function matrix from disturbances/noise to the performance vector. Hence the best disturbance/noise rejection in this sense has been given to the closed loop system.

(2). At the same time, H∞ Optimal Design provides a certain level inherent robustness for the closed loop system.

(3). Since the robustness of H∞ Optimal Design depends on performance matrices $\mathbf{C}_1$ and $\mathbf{D}_1$, and these two matrices are normally considered for the disturbance rejection, so the robustness of the controlled system is coupled to the disturbance rejection and sometimes, "blindly" applying H∞-norm Optimisation Design based on the nominal model does not guarantee to provide enough inherent robustness for the actual system.

3.1.3 Mixed H2/H∞ controllers design

Design of control systems almost always involves trade-offs among competing objectives. The desired controller is often required to meet several different performance and robustness goals, and normally all of these can not be met simultaneously. One method of studying the trade-off among competing objectives is a certain constrained optimal controller synthesis problem, so called mixed H2/H∞ synthesis problem.

Consider the linear time-invariant system shown by Fig.3.1, which is similar to the general description (2.1.3) but with two performance vectors:

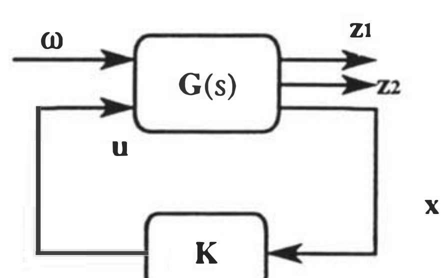

Fig. 3.1 The mixed H2/H∞ state feedback controlled system

Its nominal model can be described by the state equations:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_2\mathbf{u} + \mathbf{B}_1\omega \\ \mathbf{z}_1 = \mathbf{C}_1\mathbf{x} + \mathbf{D}_1\mathbf{u} \\ \mathbf{z}_2 = \mathbf{C}_2\mathbf{x} + \mathbf{D}_2\mathbf{u} \end{cases} \tag{3.1.16}$$

Where $\mathbf{z}_1$ and $\mathbf{z}_2$ are performance vectors. The matrices $\mathbf{A}$, $\mathbf{B}_2$ have compatible dimensions and the pair $(\mathbf{A}, \mathbf{B}_2)$ is supposed to be stabilisable, matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ are required to have full column rank, and for simplicity, it is assumed that:

$$\mathbf{D}_1^T\mathbf{C}_1 = \mathbf{0}; \quad \mathbf{D}_2^T\mathbf{C}_2 = \mathbf{0}.$$

The design objective of the H2/H∞ control system is to find a controller **u=Kx** such that:

(1). $\left\|\mathbf{T}_{\varpi z1}\right\|_\infty < \gamma_0$.

(2). $\left\|\mathbf{T}_{\varpi z2}\right\|_2 = \left\|(\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}_m)(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}_1\right\|_2$ is minimised subjected to (1).

From Lemma 2.3.5 we know that condition (1) means to find a control matrix $\mathbf{K}$ such that there exists a positive definite solution for the following expression:

$$(\mathbf{A} + \mathbf{B}_2\mathbf{K})^T\mathbf{P}_1 + \mathbf{P}_1(\mathbf{A} + \mathbf{B}_2\mathbf{K}) + \gamma_0^{-1}\mathbf{P}_1\mathbf{B}_1\mathbf{B}_1^T\mathbf{P}_1 + \gamma_0^{-1}\mathbf{K}^T\mathbf{D}_1^T\mathbf{D}_1\mathbf{K} + \gamma_0^{-1}\mathbf{C}_1^T\mathbf{C}_1 < \mathbf{0}$$

From Lemma 2.3.4 the condition (2) means that to find a control matrix $\mathbf{K}$ such that there exists a minimal positive definite solution for the following expression:

$$(\mathbf{A} + \mathbf{B}_2\mathbf{K})^T\mathbf{P}_2 + \mathbf{P}_2(\mathbf{A} + \mathbf{B}_2\mathbf{K}) + \mathbf{K}^T\mathbf{D}_2^T\mathbf{D}_2\mathbf{K} + \mathbf{C}_2^T\mathbf{C}_2 < \mathbf{0} \tag{3.1.17}$$

Since no analytic solution for the *optimal* H2/H∞ problem is available, consider the following sub-optimal design procedure that minimises an upper bound of H2 performance subject to the H∞ norm requirement.

A combined requirement that firstly guarantee the H∞ norm condition and also considers the H2 performance can be described as:

$$\begin{aligned}&(\mathbf{A} + \mathbf{B}_2\mathbf{K})^T\mathbf{P} + \mathbf{P}(\mathbf{A} + \mathbf{B}_2\mathbf{K}) + \gamma_0^{-1}\mathbf{P}\mathbf{B}_1\mathbf{B}_1^T\mathbf{P}\\ &\quad + \mathbf{K}^T(\gamma_0^{-1}\mathbf{D}_1^T\mathbf{D}_1 + \mathbf{D}_2^T\mathbf{D}_2)\mathbf{K} + (\gamma_0^{-1}\mathbf{C}_1^T\mathbf{C}_1 + \mathbf{C}_2^T\mathbf{C}_2) < \mathbf{0}\end{aligned} \tag{3.1.18}$$

Evidently for any $\mathbf{C}_2$ and $\mathbf{D}_2$, if (3.1.18) has a positive definite solution, this implies condition (1) can be satisfied. If we relax the H∞ norm constraint, i.e., $\gamma_0 = \infty$, then (3.1.18) becomes (3.1.17), i.e., pure H2 optimal design. For a finite γ_0, (3.1.18) gives an upper bound for the performance matrix of (3.1.17), i.e., $\mathbf{P}_2 \le \mathbf{P}$.

For a given H∞ norm bound γ_0, a sub-optimal solution for the mixed H2/H∞ controller that gives a minimal solution of $\mathbf{P}$ can be found for (3.1.18) as:

$$\mathbf{K}_m = -(\gamma_0^{-1}\mathbf{D}_1^T\mathbf{D}_1 + \mathbf{D}_2^T\mathbf{D}_2)^{-1}\mathbf{B}_2^T\mathbf{P}_0 \tag{3.1.19}$$

Where $\mathbf{P}_0 \le \mathbf{P}$ is the positive definite solution of the following equation:

$$\mathbf{A}^T\mathbf{P}_0 + \mathbf{P}_0\mathbf{A} - \mathbf{P}_0\mathbf{B}_2(\gamma_0^{-1}\mathbf{D}_1^T\mathbf{D}_1 + \mathbf{D}_2^T\mathbf{D}_2)^{-1}\mathbf{B}_2^T\mathbf{P}_0\gamma_0^{-1}\mathbf{P}_0\mathbf{B}_1\mathbf{B}_1^T\mathbf{P}_0 + (\gamma_0^{-1}\mathbf{C}_1^T\mathbf{C}_1 + \mathbf{C}_2^T\mathbf{C}_2) < \mathbf{0} \tag{3.1.20}$$

To consider the performance of the closed-loop system, let us define the closed loop system as: $\mathbf{A}_0 = \mathbf{A} + \mathbf{B}_2\mathbf{K}_m$, then from (3.1.20) it follows that $\mathbf{A}_0$ satisfies the following expression:

$$\begin{aligned}&\mathbf{A}_0^T\mathbf{P}_0 + \mathbf{P}_0\mathbf{A}_0 + \gamma_0^{-1}\mathbf{P}_0\mathbf{B}_1\mathbf{B}_1^T\mathbf{P}_0 + \gamma_0^{-1}(\mathbf{C}_1 + \mathbf{D}_1\mathbf{K}_m)^T(\mathbf{C}_1 + \mathbf{D}_1\mathbf{K}_m)\\ &\quad + (\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}_m)^T(\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}_m) < \mathbf{0}\end{aligned} \tag{3.1.21}$$

From Lemma 2.3.5 it follows that:

$$\left\| \begin{bmatrix} (\mathbf{C}_1 + \mathbf{D}_1\mathbf{K}_m) \\ \gamma_0^{1/2}(\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}_m) \end{bmatrix} (s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}_1 \right\|_\infty < \gamma_0 \tag{3.1.22}$$

So from Lemma 2.3.6 it is clear that $\left\|(\mathbf{C}_1 + \mathbf{D}_1\mathbf{K}_m)(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}_1\right\|_\infty < \gamma_0$.

Relating to §2.3.3, The H2 norm can be found for the sub-optimal control system as:

$$\left\|T_{\varpi z2}\right\|_2 = \left\|(\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}_m)(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{B}_1\right\|_2 \le \{\mathrm{Tr}(\mathbf{P}_0\mathbf{B}_1\mathbf{B}_1^T)\}^{1/2} \tag{3.1.23}$$

Since $\mathbf{P}_0 \le \mathbf{P}$, so (3.1.23) gives a minimal H2 performance bound subject to the condition $\left\|\mathbf{T}_{\varpi z1}\right\|_\infty < \gamma_0$. From expression (3.1.21), it is clear that there is a trade-off between $\mathbf{P}_0$ and the value γ_0. If a smaller γ_0 is chosen, then closed loop system will have good robustness but the cost performance will have more degradation, and if a larger γ_0 is chosen, the closed loop system will have better cost performance and the robustness will be degraded. When $\gamma_0 = \infty$, the equation (3.1.18) will be the same as (3.1.17), the H2/H∞ design becomes the standard LQR design.

Now, let us consider the robustness criterion for this control design method, according to the Remark 3.1.1 we know that

$$\mathbf{M}_0 = \begin{bmatrix} \gamma_0^{-1/2}(\mathbf{C}_1 + \mathbf{D}_1\mathbf{K}_m) \\ (\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}_m) \end{bmatrix}, \quad \text{and} \quad \mathbf{N}_0 = \gamma_0^{-1/2}\mathbf{B}_1$$

Hence, if there exists a positive parameter α such that the following conditions are satisfies:

and $$\alpha\varepsilon\mathbf{M}^T\mathbf{M} \le \gamma_0^{-1}(\mathbf{C}_1 + \mathbf{D}_1\mathbf{K}_m)^T(\mathbf{C}_1 + \mathbf{D}_1\mathbf{K}_m) + (\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}_m)^T(\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}_m) \tag{3.1.24}$$

$$\frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T \le \gamma_0^{-1}\mathbf{B}_1\mathbf{B}_1^T \tag{3.1.25}$$

The mixed H2/H∞ sub-optimal control system can provide the stability robustness guarantee to the uncertainty $\Delta\mathbf{A}_0 = \mathbf{N}\Phi(t)\mathbf{M}$ with $\overline{\sigma}(F(t)) \le \varepsilon$.

To summarise the robustness analysis result for the mixed H2/H∞ sub-optimal controller design, it follows that:

(1). H2/H∞ design provides an H∞ norm bound for the transfer function matrix from disturbances to the output, so a certain level of disturbance rejection will be provided.

(2). H2/H∞ design also provides a minimal H2 norm bound subject to an H∞ norm condition of the transfer function from inputs to the outputs, this means the closed loop system has sub-optimal cost performance.

(3). There is a trade-off between the H2 norm performance and H∞ norm performance, (or disturbance rejection and cost value). When $\gamma_0 = \infty$, the H2/H∞ design turns to standard LQR design.

(4). At the same time, similar to H∞ optimal control design theory, H2/H∞ provides some inherent robustness properties for the closed loop system, but this inherent robustness is also coupled to the weighting matrices of the performance vectors.

3.1.4 Discussions of full state feedback control systems

In this section, several full state feedback controller design methods are stated and their robustness criteria are assessed by the use of the principles of robustness derived in Chapter 2. Generally, since only the parametric part of the unknown uncertainty affects the closed-loop system with full state feedback controller, to analyse the robust stability of a closed loop system means to look for suitable matrices $\mathbf{N}_0$, $\mathbf{M}_0$ such that the following is satisfied:

$$\left\|\mathbf{M}_0(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{N}_0\right\|_\infty < 1$$

From the analysis of LQR design, H∞ optimal design and mixed H2/H∞ optimal design, we can always find some suitable matrices $\mathbf{N}_0$, $\mathbf{M}_0$ such that the above H∞ norm requirement can be satisfied. Hence the closed loop system with full state feedback controller can really provide some inherent stability robustness, that is, all full state feedback controller design techniques possess some robustness and at the same time they are also "optimal" for input output energy transfer or disturbance/noise rejection.

However, since these controller design techniques are based on the nominal system model and some performance requirements, the inherent robustness of the closed loop system is coupled to these performance requirements and sometimes, i.e., matrices $\mathbf{N}_0$, $\mathbf{M}_0$ depend on the parameters and performance weighting matrices. This means that "blindly" designing a controller does not guarantee to provide enough inherent robustness for the actual system.

Some trade-off relations are also very interesting for the full state feedback controller design, normally there are between the dynamic performance measure and robustness. A good controller design should suitably consider these trade-off. In chapter 4 a robust design technique for state feedback control systems will be presented which allows the performance and robustness requirements to be decoupled and a suitable compromise reached.

3.2 Systems with Dynamic Output Feedback Controllers

It has been shown that the closed loop system with full state feedback controller, such as LQR, H_∞ or H_2/H_∞, will normally provide good control performance (such as H_2, H_∞ performance) as well as some inherent robustness. However, as discussed in §2.1.2, it is not usually possible to implement the full state feedback solution. The reason is that it is not usually possible to measure all the state variables of the plant. This is why often the output controller must be used.

As noted in §2.2.4, for dynamic output feedback control designs, the nonparametric part of unknown uncertainty $\Delta\mathbf{G}(s)$ may be considered as a general case. A good controller design technique should provide some inherent robustness properties subject to $\Delta\mathbf{G}(s)$.

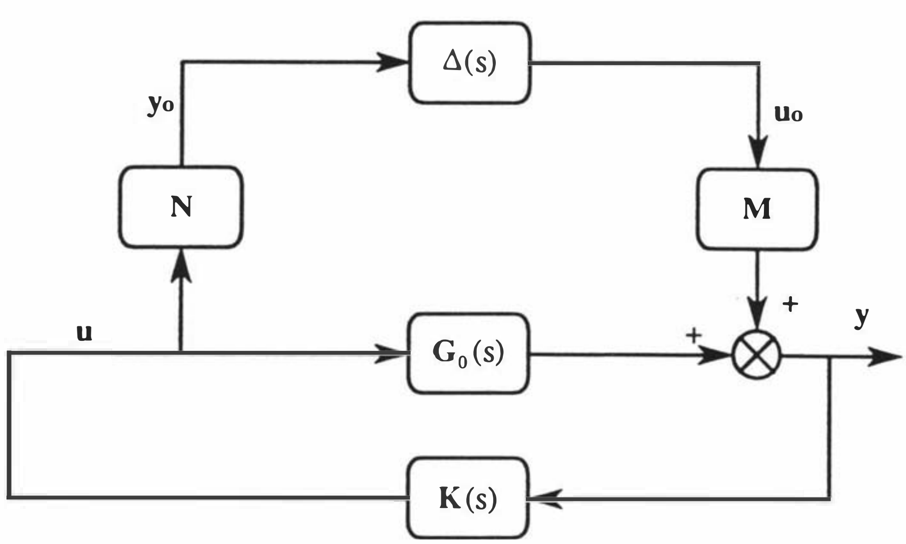

Fig. 3.2 The uncertain closed loop system

To analyse the robustness of the output feedback controlled system, the conclusion of §2.4.1 is recalled here and will be applied to some common control system design techniques.

Remark 3.2.1 For the uncertain closed loop system shown by Fig. 3.2, if the H∞ norm of $\mathbf{T}_{u_0y_0}(s) = \mathbf{C}_0(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{B}_0$ satisfies the condition of $\left\|\mathbf{T}_{u_0y_0}(j\omega)\right\|_\infty < \frac{1}{\eta}$, a stability robustness guarantee will be provided to the nonparametric uncertainty $\Delta\mathbf{G}(s) = \mathbf{N}\Delta(s)\mathbf{M}$ with $\left\|\Delta(j\omega)\right\|_\infty \le \eta$.

Where

$$G_0(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{C}_c \\ \mathbf{B}_c\mathbf{C} & \mathbf{A}_c \end{bmatrix},\quad \mathbf{C}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{N}\mathbf{C}_c \end{bmatrix},\quad \mathbf{B}_0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}\mathbf{B}_c \end{bmatrix}$$

and the dynamic output feedback controller $\mathbf{u}=\mathbf{K}(s)\mathbf{y}$ can be described as:

$$\begin{cases} \dot{\varsigma} = \mathbf{A}_c\varsigma + \mathbf{B}_c\mathbf{y} \\ \mathbf{u} = \mathbf{C}_c\varsigma \end{cases}$$

3.2.1 Linear Quadratic Gaussian (LQG) controllers design

The plant described by state-space form as (2.1.2) is recalled here:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{d} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{F}\mathbf{v} \end{cases} \tag{3.2.1}$$

d is the vector of Gaussian random disturbance processes, $\mathbf{v}$ is the vector of Gaussian random measurement noise processes. It is assumed **d** and $\mathbf{v}$ are uncorrelated, $\mathcal{E}[\mathbf{d}\mathbf{v}^T] = \mathbf{0}$, their mean values are zero, $\mathcal{E}[\mathbf{d}] = \mathcal{E}[\mathbf{v}] = \mathbf{0}$, and their covariance matrices are **W** and **V** respectively. It is also assumed that [**A**, **B**] is *controllable* and [**A**, **C**] is *observable*. The objective here is to find a feedback control law for the input **u** that will minimise the quadratic 'cost':

$$J = \lim_{t_0\to\infty} \frac{1}{t_0}\mathcal{E}\left\{\int_0^{t_0} (\mathrm{x}^T\mathrm{Q}\mathrm{x} + \mathrm{u}^T\mathrm{R}\mathrm{u})\, dt\right\} \tag{3.2.2}$$

Where $\mathcal{E}\{...\}$ denotes the 'expected' or mean value, $\mathbf{Q} \ge \mathbf{0}$, $\mathbf{R} > \mathbf{0}$ are weighting matrices. *This is known as the Linear Quadratic Gaussian (LQG) feedback design problem.*

The solution of this problem depends entirely on the four matrices **W**, **V**, **Q**, **R**. As we shall see, the LQG problem has a very complete theory, and its solution has some very attractive properties (Kalman, 1964), so it has been very attractive to control theorists since the 1960s. A separation principle holds for the solution of the LQG problem: it can be obtained as the solution of two separate sub-problems. The first of these is the optimal state estimation problem: given the model as above find the optimal estimate $\hat{\mathbf{x}}$ of the state $\mathbf{x}$ from observations of the inputs $\mathbf{u}$ and the outputs $\mathbf{y}$. The solution to this sub-problem is given by the Kalman filter, which is a special case of a state observer. The second sub-problem is the deterministic state feedback problem that is the same as LQR design. The separation principle says that the solution to the LQG problem is given by using a Kalman filter to estimate the state, then passing that estimate through the optimal state-feedback matrix, as if it were the true state, to form the controller input.

To solve the state feedback sub-problem it is first necessary to solve the following Riccati equation:

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}_c + \mathbf{P}_c\mathbf{A} - \mathbf{P}_c\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_c + \mathbf{Q} = \mathbf{0} \tag{3.2.3}$$

Which is to be solved for the matrix $\mathbf{P}_c$. As mention in §3.1.1, the optimal state feedback matrix is given simply by:

$$\mathbf{K}_c = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_c \tag{3.2.4}$$

Making the substitution $\mathbf{u} = -\mathbf{K}_c\mathbf{x}$ we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} = (\mathbf{A} - \mathbf{B}\mathbf{K}_c)\mathbf{x} \tag{3.2.5}$$

and it can be shown that the matrix $(\mathbf{A} - \mathbf{B}\mathbf{K}_c)$ has all its eigenvalues in the left half-plane, so that the state feedback scheme is stable.

The state estimation sub-problem is solved by the Kalman filter, which is a state observer with a particular feedback gain matrix, to obtain it, the following Riccati equation should be solved

$$\mathbf{A}^T\mathbf{P}_f + \mathbf{P}_f\mathbf{A} - \mathbf{P}_f\mathbf{C}^T(\mathbf{F}\mathbf{V}\mathbf{F}^T)^{-1}\mathbf{C}\mathbf{P}_f + \mathbf{E}\mathbf{W}\mathbf{E}^T = \mathbf{0} \tag{3.2.6}$$

and the feedback gain matrix can be found as

$$\mathbf{K}_f = \mathbf{P}_f\mathbf{C}^{\mathrm{T}}(\mathbf{F}\mathbf{V}\mathbf{F}^{\mathrm{T}})^{-1} \tag{3.2.7}$$

The Kalman filter itself has the state equation

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_f\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}_f\mathbf{y} \tag{3.2.8}$$

and it can be shown that all the eigenvalues of the matrix $(\mathbf{A} - \mathbf{K}_f\mathbf{C})$ line in the left half-plane, so that the Kalman filter is a stable system.

When the full LQG solution is implemented, namely the combination of a Kalman filter with optimal state feedback, then the state equation of the closed loop system becomes

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK}_c & \mathbf{BK}_c \\ \mathbf{0} & \mathbf{A}-\mathbf{K}_f\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{Ed} \\ \mathbf{Ed}-\mathbf{K}_f\mathbf{Fv} \end{bmatrix} \tag{3.2.9}$$

Where $\mathbf{e}=\mathbf{x}-\hat{\mathbf{x}}$ is the state estimation error.

Here we focus on the stability analysis of LQG design method, from the triangular nature of the state evolution matrix we can deduce that the closed loop eigenvalues of the whole scheme are union of the eigenvalues of $(\mathbf{A}-\mathbf{BK}_c)$ with those of $(\mathbf{A}-\mathbf{K}_f\mathbf{C})$. Hence they are all in the left half plane, and the whole system is stable. Notice that closed loop stability comes automatically with this design method.

The fact that the nominal closed loop system is guaranteed to be stable does not imply that we have a useful design, the measurement of robustness is also necessary. Since the full LQG solution consists of the combination of optimal state feedback with a Kalman filter, and from §3.1.1 we know that optimal state feedback control has good robustness, and evidently Kalman filter is a special form of optimal state feedback control, so both of them have well robustness properties, it might be expected that the full solution would inherit these good properties. Unfortunately this is not (Doyel and Stein, 1979).

Now, let us consider the robustness criterion for standard LQG design technique. For simplicity, the closed loop system equation (3.2.9) can be rewritten as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \mathbf{E}_0 \begin{bmatrix} \mathbf{d} \\ \mathbf{v} \end{bmatrix} \tag{3.2.10}$$

Since above system is stable, so from Lemma 2.3.1 we know that for any positive (or semi-positive) definite matrix $\mathbf{C}_0^T\mathbf{C}_0$, there exists a positive definite matrix $\mathbf{P}$ such that

$$\mathbf{A}_0^T\mathbf{P}+\mathbf{PA}_0+\mathbf{C}_0^T\mathbf{C}_0=\mathbf{0} \tag{3.2.11}$$

But during the LQG design, we have not been given the guarantee that there exists a positive definite solution for the following equation subject to any γ_0:

$$\mathbf{A}_0^T\mathbf{P}+\mathbf{PA}_0+\gamma_0^{-2}\mathbf{PB}_0\mathbf{B}_0^T\mathbf{P}+\mathbf{C}_0^T\mathbf{C}_0=\mathbf{0} \tag{3.2.12}$$

what we can guarantee is only equation (3.2.11) will have a positive solution, this means only when $\gamma_0=\infty$, we can guarantee (3.2.12) has a positive definite solution. (where $\mathbf{B}_0$ and $\mathbf{C}_0$ are defined in Remark 3.2.1)

The H^∞ norm bound of Remark 3.2.1 can only be found as

$$\left\|\mathbf{T}_{u_o y_o}(j\omega)\right\|_\infty < \frac{1}{\varepsilon} = \infty \tag{3.2.13}$$

Hence, from the result of Remark 3.2.1, there is no *robustness guarantee* for LQG design and its stability margins could be arbitrarily small, i.e., the size of uncertain matrix $\varepsilon \to 0$.

It should be also noted that there is no robustness guarantee does not mean no robustness exists for particular controlled systems, some may have but some have not.

To summarise the LQG Design, it follows that:

(1). LQG Design can provide the minimal integral-quadratic performance for the nominal plant, and the closed loop stability comes automatically.

(2). There is no robustness guarantee for LQG Design and its stability margins could be arbitrarily small.

3.2.2 LQG/LTR controllers design

As mentioned in the previous section, the standard LQG problem can be obtained as the solution of the optimal state estimation problem and the optimal state feedback problem, and both of these two sub-problems have good inherent robustness, but the LQG has not. To overcome this, the Loop Transfer Recovery (LTR) design procedure allows one to design a full modified LQG control system, and to approach the good robustness properties exhibited by either optimal state feedback, (or a Kalman filter). Hence there are two versions of LTR (Anderson and Moore, 1990): one approaches good robustness of the optimal feedback control, the other approaches the good robustness of the Kalman filter. The first version consists of the following two steps.

(1). **Design an optimal state feedback system.**
Using the given weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ to design a controller by the standard LQR design method, which is the same as (3.2.3) and (3.2.4). According the results of §3.1.1, good robustness properties come automatically at this stage.

(2). **Synthesise a Kalman filter in the following way.**
Set $\mathbf{W} = q\mathbf{I}$, and $\mathbf{V} = \mathbf{I}$, where q is a positive real number. Then q is increased, when q is large enough, the robustness of step (1) can be recovered.

The second version is the dual of the first one

(1). **Design a Kalman filter.**
Using the given weighting matrices **W** and **V** to design a Kalman filter which is the same as (3.2.6) and (3.2.7). According the results of §3.2.1, good robustness properties come automatically at this stage.

(2). **Synthesise an optimal state feedback system in the following way.**
Set **Q**=q**I**, and **R**=**I**, where q is a positive real number. Then q is increased, when q is large enough, the robustness of step (1) can be recovered.

We will use the first version as an example to show how the LTR design recovers the robustness of the state feedback controlled system. The modified state estimation sub-problem is solved by the Kalman filter:

$$\mathbf{P}_f\mathbf{A}^T + \mathbf{P}_f\mathbf{A} - \mathbf{P}_f\mathbf{C}^T\mathbf{C}\mathbf{P}_f + q\mathbf{I} = \mathbf{0} \tag{3.2.14}$$

and the feedback gain matrix can be found as

$$\mathbf{K}_f = \mathbf{P}_f\mathbf{C}^T$$

Then (3.2.14) can also be rewritten as:

$$\mathbf{P}_f\mathbf{A}_{02}^T + \mathbf{P}_f\mathbf{A}_{02} + \mathbf{P}_f\mathbf{C}^T\mathbf{C}\mathbf{P}_f + q\mathbf{I} = \mathbf{0} \tag{3.2.15}$$

where $\mathbf{A}_{02} = (\mathbf{A} - \mathbf{K}_f\mathbf{C})$.

Since the LQG/LTR design is actually the same as the optimal state feedback control with a particular high gain observer, as the gain of the Kalman filter is increased to infinity, the effect of nonparametric uncertainty $\Delta\mathbf{G}(s)$ and parametric uncertainty $\Delta\mathbf{C}$ will vanish, hence only parametric uncertainties $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ should be considered here.

In the following we will use the LQG/LTR controller to recover the robustness of the full state feedback LQR controller. Let us consider the system (3.2.1) with parametric uncertainties as:

$$\begin{cases} \dot{\mathbf{x}} = (\mathbf{A} + \Delta\mathbf{A})\mathbf{x} + (\mathbf{B} + \Delta\mathbf{B})\mathbf{u} + \mathbf{E}\mathbf{d} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{F}\mathbf{v} \end{cases} \tag{3.2.16}$$

Where uncertainties of this system can be represented as:

$$\Delta\mathbf{A}_0 = \Delta\mathbf{A} + \Delta\mathbf{B}\mathbf{K} = \mathbf{N}\Phi(t)\mathbf{M}$$

Recall the results in §3.1.1 it follows that if there exists a positive parameter α such that the following conditions are satisfied:

$$\alpha\varepsilon \mathbf{M}^T\mathbf{M} < \mathbf{Q}; \quad \frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T \le \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \tag{3.2.17}$$

then the full state feedback LQR control system can provide a stability robustness guarantee to the uncertainty $\Delta\mathbf{A}_0$ with $\overline{\sigma}(\Phi(t)) \le \varepsilon$,

Now, we will prove that if a LQR full state feedback controller can provide the robustness for the uncertainty $\Delta\mathbf{A}_0$, then a LQG/LTR controller can also do this.
Consider a LQR controller as:

$$\mathbf{K}_c = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_c$$

where $\mathbf{P}_c \ge \mathbf{0}$ is the solution of the following equation:

$$\mathbf{A}_{01}^T\mathbf{P}_c + \mathbf{P}_c\mathbf{A}_{01} + \mathbf{P}_c\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_c + \mathbf{Q} = \mathbf{0}$$

and $\mathbf{A}_{01} = (\mathbf{A} - \mathbf{B}\mathbf{K}_c)$. Suppose that condition (3.2.17) is satisfied, this means that LQR can provide robustness for the uncertainty $\Delta\mathbf{A}_0$, then compare the above Lyapunov equation and condition (3.2.17) it follows that:

$$\mathbf{A}_{01}^T\mathbf{P}_c + \mathbf{P}_c\mathbf{A}_{01} + \mathbf{P}_c\frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T\mathbf{P}_c + \alpha\varepsilon\mathbf{M}^T\mathbf{M} < \mathbf{0} \tag{3.2.18}$$

The state equations of the closed loop system can be found as:

$$\frac{d}{dt}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix} = \begin{bmatrix}\mathbf{A}+\Delta\mathbf{A}-\mathbf{B}\mathbf{K}_c & \mathbf{B}\mathbf{K}_c\\ \Delta\mathbf{A} & \mathbf{A}-\mathbf{K}_f\mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix} + \begin{bmatrix}\mathbf{E}\mathbf{d}\\ \mathbf{E}\mathbf{d}-\mathbf{K}_f\mathbf{F}\mathbf{v}\end{bmatrix} \tag{3.2.19}$$

Introducing a Lyapunov function for the closed loop system

$$\Pi(\mathbf{x},\mathbf{e}) = \begin{bmatrix}\mathbf{x}^T & \mathbf{e}^T\end{bmatrix}\begin{bmatrix}\mathbf{P}_c & \mathbf{0}\\ \mathbf{0} & \overline{\mathbf{P}}_f\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}$$

Where $\overline{\mathbf{P}}_f = \mathbf{P}_f^{-1}$ from (3.2.15), then we obtain, after standard manipulations, the following derivative of the Lyapunov function:

$$\dot{\Pi}(\mathbf{x},\mathbf{e}) = \begin{bmatrix}\mathbf{x}^T & \mathbf{e}^T\end{bmatrix}\begin{bmatrix}\mathbf{P}_c\mathbf{A}_{01} + \mathbf{A}_{01}^T\mathbf{P}_c + \mathbf{P}_c\Delta\mathbf{A} + \Delta\mathbf{A}^T\mathbf{P}_c & \mathbf{P}_c\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_c + \Delta\mathbf{A}^T\overline{\mathbf{P}}_f\\ \mathbf{P}_c\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_c + \overline{\mathbf{P}}_f\Delta\mathbf{A} & \overline{\mathbf{P}}_f\mathbf{A}_{02} + \mathbf{A}_{02}^T\overline{\mathbf{P}}_f\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}$$

By the use of Lemma 2.4.1 it is found that:

$$\dot{\Pi}(\mathbf{x},\mathbf{e}) \le \mathbf{x}^T\mathbf{M}_1\mathbf{x} + \mathbf{e}^T\mathbf{M}_2\mathbf{e} \tag{3.2.20}$$

Where

$$\mathbf{M}_1 = \mathbf{A}_{01}^T\mathbf{P}_c + \mathbf{P}_c\mathbf{A}_{01} + \mathbf{P}_c\frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T\mathbf{P}_c + (\alpha\varepsilon+\beta)\mathbf{M}^T\mathbf{M} \tag{3.2.21}$$

$$\mathbf{M}_2 = \overline{\mathbf{P}}_f(\mathbf{P}_f\mathbf{A}_{02}^T + \mathbf{P}_f\mathbf{A}_{02} + \mathbf{P}_f\mathbf{P}_c\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_c\mathbf{P}_f + \frac{\varepsilon}{\beta}\mathbf{N}\mathbf{N}^T)\overline{\mathbf{P}}_f \tag{3.2.22}$$

and α, β are scalar parameters introduced by Lemma 2.4.1. Substituting equation (3.2.15) in (3.2.22) gives

$$\mathbf{M}_2 = -\overline{\mathbf{P}}_f(\mathbf{P}_f\mathbf{C}^T\mathbf{C}\mathbf{P}_f + q\mathbf{I} - \mathbf{P}_f\mathbf{P}_c\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_c\mathbf{P}_f - \frac{\varepsilon}{\beta}\mathbf{N}\mathbf{N}^T)\overline{\mathbf{P}}_f \quad (3.2.23)$$

From (3.2.21) and (3.2.18), we can always find a small enough β such that:

$$\mathbf{M}_1 = \mathbf{A}_{01}^T\mathbf{P}_c + \mathbf{P}_c\mathbf{A}_{01} + \mathbf{P}_c\frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T\mathbf{P}_c + (\alpha\varepsilon + \beta)\mathbf{M}^T\mathbf{M} < 0 \quad (3.2.24)$$

But for any small β when q is chosen large enough or $q \to \infty$, it always follows that:

$$\mathbf{M}_2 = -\overline{\mathbf{P}}_f(\mathbf{P}_f\mathbf{C}^T\mathbf{C}\mathbf{P}_f + q\mathbf{I} - \mathbf{P}_f\mathbf{P}_c\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_c\mathbf{P}_f - \frac{\varepsilon}{\beta}\mathbf{N}\mathbf{N}^T)\overline{\mathbf{P}}_f < 0 \quad (3.2.25)$$

So if the full state feedback LQR controller can guarantee the robust stability for the uncertain system (3.2.16), then the modified LQG controller can also guarantee the robust stability when $q \to \infty$, that is, the LQG/LTR design can recover the robustness of LQR design.

The LQG/LTR design is actually the same as the optimal state feedback controller with a particular *high gain observer*, or the optimal state estimation with *high gain state feedback*. The robustness recovery procedure actually transfers the effect of model uncertainties to system disturbances or performance weightings.

To summarise the LQG/LTR Design, it follows that:

(1). LQG/LTR design can recover the robustness of LQR design. The closed loop stability comes automatically.

(2). LQG/LTR design does not consider the integral-quadratic performance, since it uses the high observer or high gain state feedback, so the cost value could be much worse.

(3). In addition to the requirement for LQG/LTR design, the system must be minimal phase system.

3.2.3 H∞ controllers design

H∞ optimisation output feedback controller design is partly motivated by the shortcomings of LQG control design (Doyle *et al.*, 1989), the basic idea of H∞ optimisation design has been mentioned in §3.1.2. A standard compensated configuration that is widely used in the H∞ literature is shown in Fig. 3.3.

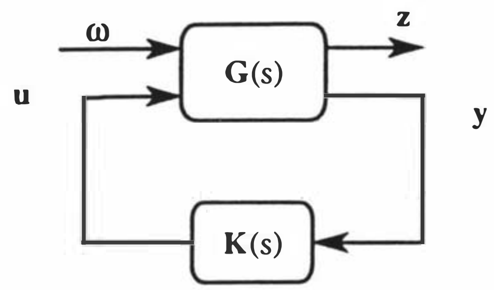

Fig 3.3 H∞ output feedback controlled system

The design objective here is to find a controller **K**(s) for the plant **G**(s) such that the transfer function characteristics from the external input vector **ω** to the performance vector **z** are desirable. The input vector **ω** may include, for example, reference input, disturbances and noise, the performance vector **z** may include errors, process outputs and control inputs. The internal compensated signals are represented by vectors **y** and **u**, and correspond to the sensor signals and actuator demands, respectively.

The plant **G**(s) of (2.1.3) is recalled here with the following state-space form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_2\mathbf{u} + \mathbf{B}_1\boldsymbol{\omega} \\ \mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_1\mathbf{u} \\ \mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{D}_2\boldsymbol{\omega} \end{cases} \tag{3.2.26}$$

It is assumed that $[\mathbf{A}, \mathbf{B}_2]$ is controllable, and $[\mathbf{A}, \mathbf{C}_2]$ is observable. For simplicity, we also assume:

(A1). $\mathbf{C}_1^T\mathbf{D}_1 = \mathbf{0}$, $\mathbf{D}_1^T\mathbf{D}_1 > \mathbf{0}$

(A2). $\mathbf{B}_1\mathbf{D}_2^T = \mathbf{0}$, $\mathbf{D}_2\mathbf{D}_2^T > \mathbf{0}$

The design objective here is to find a sub-optimal dynamic output feedback controller **K**(s) such that the following bound γ_m is minimal.

$$\left\|\mathbf{T}_{\omega z}\right\|_\infty < \gamma_m \tag{3.2.27}$$

The following results (Doyle *et al.* 1989) are normally used to solve the H∞ optimisation problem:

Lemma 3.3.2 There exists a controller such that the H∞ norm bound (3.2.27) is satisfied if and only if $\lambda_{max}(X_\infty Y_\infty) < \gamma_m^2$, and the two following Riccati equations have positive definite solutions.

$$A^T X_\infty + X_\infty A + X_\infty(\gamma_m^{-2} B_1 B_1^T - B_2(D_1^T D_1)^{-1} B_2^T) X_\infty + C_1^T C_1 = 0 \tag{3.2.28}$$

$$A Y_\infty + Y_\infty A^T + Y_\infty(\gamma_m^{-2} C_1^T C_1 - C_2^T(D_2 D_2^T)^{-1} C_2) Y_\infty + B_1 B_1^T = 0 \tag{3.2.29}$$

Moreover, when the above condition holds, one such controller is:

$$K_\infty(s, \gamma_m) = F_\infty(sI - A_\infty)^{-1} Z_\infty L_\infty \tag{3.2.30}$$

where

$$A_\infty = A + \gamma_m^{-2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$

$$F_\infty = -(D_1^T D_1)^{-1} B_2^T X_\infty,\quad L_\infty = -Y_\infty C_2^T (D_2 D_2^T)^{-1},\quad Z_\infty = (I - \gamma_m^{-2} Y_\infty X_\infty)^{-1}$$

Now for this standard H∞ optimisation design technique, let us consider the stability robustness of the controlled system subject to the nonparametric uncertainty by the use of Corollary 2.4.3.

Suppose for the system (3.2.26) we can find a controller as:

$$K(s) = C_c(sI - A_c)^{-1} B_c$$

or

$$\begin{cases} \dot{\zeta} = A_c \zeta + B_c y \\ u = C_c \zeta \end{cases} \tag{3.2.31}$$

such that the closed loop system satisfies the H∞ norm bound condition $\|T_{\omega z}\|_\infty < \gamma_m$. The closed loop system can then be described as:

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} A & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} + \begin{bmatrix} B_1 \\ B_c D_2 \end{bmatrix} \omega \\ z = \begin{bmatrix} C_1 & D_1 C_c \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} \end{cases} \tag{3.2.32}$$

So

$$T_{\omega z}(s) = \begin{bmatrix} C_1 & D_1 C_c \end{bmatrix} (sI - A_0)^{-1} \begin{bmatrix} B_1 \\ B_c D_2 \end{bmatrix} \tag{3.2.33}$$

where

$$A_0 = \begin{bmatrix} A & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix}$$

Then from the assumption (A1) and (A2) it follows that:

$$\mathbf{D}_1^T\mathbf{T}_{\omega z}(s)\mathbf{D}_2^T = \begin{bmatrix}\mathbf{0} & \mathbf{D}_1^T\mathbf{D}_1\mathbf{C}_c\end{bmatrix}(s\mathbf{I}-\mathbf{A}_0)^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{B}_c\mathbf{D}_2\mathbf{D}_2^T\end{bmatrix}$$

i.e.,

$$\mathbf{D}_1^T\mathbf{T}_{\omega z}(s)\mathbf{D}_2^T = \mathbf{D}_1^T\mathbf{D}_1\begin{bmatrix}\mathbf{0} & \mathbf{C}_c\end{bmatrix}(s\mathbf{I}-\mathbf{A}_0)^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{B}_c\end{bmatrix}\mathbf{D}_2\mathbf{D}_2^T$$

From assumption (A1) and (A2) it also follow that:

$$\begin{bmatrix}\mathbf{0} & \mathbf{C}_c\end{bmatrix}(s\mathbf{I}-\mathbf{A}_0)^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{B}_c\end{bmatrix} = (\mathbf{D}_1^T\mathbf{D}_1)^{-1}\mathbf{D}_1^T\mathbf{T}_{\omega z}(s)\mathbf{D}_2^T(\mathbf{D}_2\mathbf{D}_2^T)^{-1}$$

According to the definition in Remark 3.2.1, it can be shown that:

$$\mathbf{T}_{\mathbf{u}_o\mathbf{y}_o}(s) = \mathbf{N}\begin{bmatrix}\mathbf{0} & \mathbf{C}_c\end{bmatrix}(s\mathbf{I}-\mathbf{A}_0)^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{B}_c\end{bmatrix}\mathbf{M}$$

Hence

$$\mathbf{T}_{\mathbf{u}_o\mathbf{y}_o}(s) = \mathbf{N}(\mathbf{D}_1^T\mathbf{D}_1)^{-1}\mathbf{D}_1^T\mathbf{T}_{\omega z}(s)\mathbf{D}_2^T(\mathbf{D}_2\mathbf{D}_2^T)^{-1}\mathbf{M} \tag{3.2.34}$$

From the properties of norm in §2.3.4 we find that

$$\left\|\mathbf{T}_{\mathbf{u}_o\mathbf{y}_o}(s)\right\|_\infty \le \overline{\sigma}(\mathbf{N}(\mathbf{D}_1^T\mathbf{D}_1)^{-1}\mathbf{D}_1^T)\left\|\mathbf{T}_{\omega z}\right\|_\infty\overline{\sigma}(\mathbf{D}_2^T(\mathbf{D}_2\mathbf{D}_2^T)^{-1}\mathbf{M})$$

i.e.,

$$\left\|\mathbf{T}_{\mathbf{u}_o\mathbf{y}_o}(s)\right\|_\infty < \overline{\sigma}(\mathbf{N}(\mathbf{D}_1^T\mathbf{D}_1)^{-1}\mathbf{D}_1^T)\overline{\sigma}(\mathbf{D}_2^T(\mathbf{D}_2\mathbf{D}_2^T)^{-1}\mathbf{M})\gamma_m \tag{3.2.35}$$

So to conclude, for arbitrary constant matrices $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{N}$ and $\mathbf{M}$, it follows that:

$$\left\|\mathbf{T}_{\mathbf{u}_o\mathbf{y}_o}(s)\right\|_\infty \le \delta\left\|\mathbf{T}_{\omega z}\right\|_\infty < \delta\gamma_m$$

where δ is a positive scalar parameter. So any controller which can minimise the bound γ_m of $\|\mathbf{T}_{wz}\|_\infty$ will also minimise the bound of $\|\mathbf{T}_{u_0y_0}\|_\infty$, and from Remark 3.2.1, this

controller will provide some robustness for the closed loop system. Furthermore,

If $\mathbf{N}=\mathbf{D}_1$ and $\mathbf{M}=\mathbf{D}_2$, then $\left\|\mathbf{T}_{\mathbf{u}_o\mathbf{y}_o}(s)\right\|_\infty \le \left\|\mathbf{T}_{\omega z}\right\|_\infty < \gamma_m$

If $\mathbf{D}_1 = \begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix}$ and $\mathbf{D}_2 = \begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}$, then $\left\|\mathbf{T}_{\mathbf{u}_o\mathbf{y}_o}(s)\right\|_\infty = \left\|\mathbf{T}_{\omega z}\right\|_\infty < \gamma_m$

To summarise the H∞ Optimal Controller Design, it follows that:

(1). H∞ Control Design minimises the maximum singular values of the transfer function matrix from disturbances to the output, so disturbance/noise rejection is optimal in this sense.

(2). At the same time, H∞ Control Design provides good inherent robustness for the closed loop system subject to unknown uncertainty. However, from (3.2.35), this inherent robustness is coupled with the weighting matrices of performance vector.

(3). There is no consideration of H2 performance.

3.2.4 Mixed H2/H∞ controllers design

Similar to state feedback H2/H∞ controller design of §3.1.3, we consider the following system that is described as:

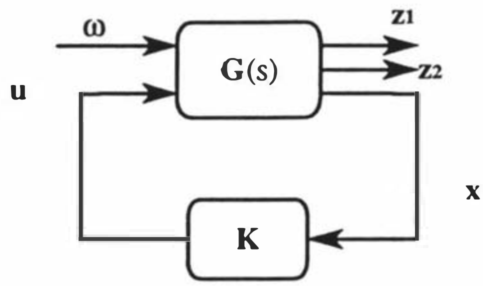

Fig. 3.4 The mixed H2/H∞ output feedback controlled system

The mixed H2/H∞ problem is to find a dynamic output feedback controller **K**(s) such that

(1). $\left\|T_{\omega z_1}\right\|_\infty < \gamma_0$.

(2). $\left\|T_{\omega z_2}\right\|_2$ can be minimised subject to (1).

Currently, no analytic solution to this problem is known. Only some attempts have been made to solve "modified" versions of optimisation problem. (Bernstein 1989 and Mustafa 1990)

3.2.5 Discussions of output feedback control systems

In this section, several output-feedback controller design methods are stated, without derivation, and their robustness criteria are assessed by the use of to the principle of robustness derived in Chapter 2.

Generally, to analyse the robust stability of a closed loop system with output-feedback controller means to find an H∞ norm bound for $\left\|\mathbf{T}_{u_o y_o}(s)\right\|_\infty$. It is found that although the LQG design can provide the minimal integral-quadratic performance for the nominal plant, since we cannot find a finite H∞ norm bound for $\left\|\mathbf{T}_{u_o y_o}(s)\right\|_\infty$, so there is no robustness guarantee for LQG design and its stability margins could be arbitrarily small.

To overcome this disadvantage of standard LQG design, a LQG/LTR can be used to recover the robustness of LQR design. But since it uses the high gain observer (or high gain state feedback), there is no consideration the integral-quadratic performance, so the cost value could be much worse. The system is only applicable to minimal phase systems. The H^∞ optimisation design can minimise the maximal singular value of the transfer function matrix from disturbance to the output. At the same time, it can also provide some good inherent robustness for the closed loop system subject to the unknown uncertainty, however, this inherent robustness is coupled with the weighting matrices of performance vector. The mixed H_2/H^∞ design can be used to find a dynamic output feedback controller $\mathbf{K}(s)$ such that $\|T_{\omega z_2}\|_2$ can be reduced subject to $\|T_{\omega z_1}\|_\infty < \gamma_0$. Unfortunately, no analytic solution to this problem is currently known.

There is a trade-off between dynamic performance and robustness, a good controller design technique should consider this trade-off. From the analysis and synthesis of a series of control design techniques based on nominal models of the system, it can be found that some of these methods possess very good inherent robustness properties and hence, they are powerful tools for practising control engineers. However, they have some limitations that should be appreciated. The main problem is that inherent robustness of closed loop system is normally coupled with the performance weighting matrices, and also the potential for conservatism arises because all uncertainties are described as unknown uncertainty. In practice, such uncertainty description is very general and imprecise, if this can be avoided the design will be less conservative.

CHAPTER 4

THE DESIGN OF STATE-FEEDBACK OPTIMAL ROBUST LQR

> For an uncertain system described by a *nominal model* and *modelled parametric uncertainties*, *Robust Control Design* means to design a **fixed** state feedback controller that can stabilise the closed loop system subject to these parametric uncertainties and also provide some inherent robustness to *residual unknown uncertainty*. At the same time, satisfactory closed loop performance for all "admissible" plant is sought.

In §2.2.3 modelling of uncertain systems was addressed and two formats were expounded. The first format was to represent the system by a nominal model and some unknown uncertainty. For this case the controller is designed with respect to the nominal model alone. A robustness condition has been found in Lemma 2.4.2 and robustness analysis of state feedback controller designs was addressed in §3.1.

The second case assumes that some knowledge of the structure and magnitude of the uncertainties are known, thus the system may be represented by a nominal model, some modelled uncertainties and some residual unknown uncertainties. To avoid conservative design and analysis, the uncertainty should be described parametrically when possible, minimising the requirement for robustness to residual unknown uncertainty. For state feedback systems, it was argued in §2.2.4 that the modelled uncertainty and unknown residual uncertainty should be described by parametric uncertainty models. For such an uncertain system with a quadratic cost function, an optimal full state feedback robust controller design methodology is presented in this chapter which offers both good stability robustness and good performance robustness. Robust stability is guaranteed for all admissible uncertainties and the cost performance is guaranteed to lie within a specific bound and furthermore, the worst case performance degradation is also proved to be minimal. There is an inherent trade off between stability robustness and performance robustness, and this may be illustrated by considering the designs resulting from varying the magnitude of the admissible domain of uncertainty. Similar to normal LQR design, a certain level inherent robustness properties are also provided with respect to the residual unknown uncertainty.

The approach is presented for both norm bounded and matched norm bounded formats of uncertainty in both system and input matrices but is readily extendible to other formats. The methodology is an extension of the original work on guaranteed cost control (Chang and Peng, 1972) which was further pursued by Petersen (1992). These papers considered only uncertainty in the system matrix, Petersen (1994) extended this to cover uncertainty in the input matrix but only for the matched norm bounded case. The methodology is developed in this chapter using a new result for the comparison of the solutions of two related Riccati equations which, in the author's opinion, gives a significantly simpler approach than Petersen (1994).

An equivalence is established between the robust LQR approach and the H^∞ control approach for a suitably scaled version of the system. Hence, a complete solution to the robust LQR design problem can be obtained through existing H^∞ control techniques. The existence of a solution is equivalent to the existence of an H^∞ optimal controller and solutions have been obtained for a reasonably broad range of examples studied. If the optimal robust LQR solution does not exist, then by releasing the cost requirement and employing Lyapunov stability theory a robust stabilising controller could be looked for.

After the problem statement, a method will be developed for the system with uncertainties which can be described by a norm bounded structure (4.1.2) in §4.2, then as a special case of this the matched norm bounded case (4.1.3) will be studied. The results allow a quantitative argument describing the trade off between stability robustness and performance robustness to be presented. In §4.3, it will be demonstrated how the RLQR design method can be posed as an H^∞ control design problem for a scaled certain system hence allowing the existing H^∞ numerical techniques to be used. Example systems will be used in §4.4 to illustrate the implementation of the methodology.

4.1 Problem Statement

As discussed in §2.2.4, only parametric uncertainties $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ will affect the robustness of state feedback control system, hence the uncertain systems to be studied in this chapter will be described by the following state-space representation:

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \Delta\mathbf{A}(t))\mathbf{x}(t) + (\mathbf{B} + \Delta\mathbf{B}(t))\mathbf{u}(t); \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{4.1.1}$$

Where $\Delta\mathbf{A}(t), \Delta\mathbf{B}(t)$ are time-varying matrices which describe the parametric uncertainties in system matrix $\mathbf{A}$ and input matrix $\mathbf{B}$, they are constrained to lie within an admissible domain. According to the discussions in §2.2.1, this domain may be bounded by some singular values as:

$$\Pi = \left\{ \begin{array}{l} \Delta\mathbf{A} = \mathbf{N}_a \Phi_a(t) \mathbf{M}_a \;\; \overline{\sigma}(\Phi_a(t)) \le 1 \\ \Delta\mathbf{B} = \mathbf{N}_b \Phi_b(t) \mathbf{M}_b \;\; \overline{\sigma}(\Phi_b(t)) \le 1 \end{array} \right\} \tag{4.1.2}$$

where $\mathbf{N}_a$, $\mathbf{N}_b$, $\mathbf{M}_a$, $\mathbf{M}_b$ are given constant matrices.

The full state feedback control law $\mathbf{u} = \mathbf{Kx}$ will be considered in this chapter and the cost performance is assessed by the quadratic cost criterion (2.3.5) as:

$$J = \int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt \tag{4.1.3}$$

where $\mathbf{R}$ and $\mathbf{Q}$ are weighting matrices which are assumed to be positive definite matrices.

The methodology described in this chapter aims to design a controller $\mathbf{u} = \mathbf{Kx}$ which can stabilise the uncertain system (4.1.1) and provide a minimal bound for the performance index (4.1.3) for all admissible values of $\Delta\mathbf{A}, \Delta\mathbf{B}$. Like the optimal LQR control design discussed in §3.1.1, the optimal robust LQR controller should also have some inherent robustness properties for residual unknown uncertainty.

A important part of the problem is to choose suitable structural matrices $\mathbf{N}_a, \mathbf{N}_b, \mathbf{M}_a, \mathbf{M}_b$ and bounding matrices $\mathbf{\Phi}_a(t), \mathbf{\Phi}_b(t)$ to represent the given uncertainty model $\Delta\mathbf{A}, \Delta\mathbf{B}$ in a precise way. A precise description will lead to a less conservative robust controller. In general this is a complex problem and no generally applicable algorithms are known. A commonly used formulation is to 'match' the uncertainty descriptions for $\Delta\mathbf{A}, \Delta\mathbf{B}$ by choosing:

$$\mathbf{N}_a = \mathbf{N}_b = \mathbf{N}, \quad \mathbf{\Phi}_a(t) = \mathbf{\Phi}_b(t) = \mathbf{\Phi}(t)$$

This gives the special case of *matched norm bounded uncertainty*, which may be described by:

$$\Pi_m = \{[\Delta\mathbf{A}, \Delta\mathbf{B}] = \mathbf{N}\mathbf{\Phi}(t)[\mathbf{M}_a, \mathbf{M}_b] : \overline{\sigma}(\mathbf{\Phi}(t)) \le 1\} \tag{4.1.4}$$

This issue is further discussed in §4.2.3.

4.2 Robust Optimal Controller Design

For a given controller $\mathbf{u}=\mathbf{Kx}$ and uncertain system (4.1.1), the closed loop system can be described as:

$$\dot{\mathbf{x}}(t) = (\mathbf{A}_0 + \Delta\mathbf{A}_0)\mathbf{x}(t) \tag{4.2.1}$$

where

$$\mathbf{A}_0 = \mathbf{A} + \mathbf{BK}, \qquad \Delta\mathbf{A}_0 = \Delta\mathbf{A} + \Delta\mathbf{B}\mathbf{K} \tag{4.2.2}$$

Since the uncertainties ($\Delta\mathbf{A}, \Delta\mathbf{B}$) are constrained, a performance bound J_b can be found which is a bound for the cost over all admissible values of ($\Delta\mathbf{A}, \Delta\mathbf{B}$). The following will provide a method, for systems with unmatched norm bound uncertainty (4.1.2) or matched norm bound uncertainty (4.1.3), to find the performance bound by the use of the result of Lemma 2.4.4.

4.2.1 Systems with norm bounded uncertainty

For the uncertain system (4.1.1) with a controller **u=Kx** and unmatched norm bound uncertainty (4.1.2), it follows that:

$$\Delta\mathbf{A}_0 = \Delta\mathbf{A} + \Delta\mathbf{BK} = \mathbf{N}_a\Phi_a(t)\mathbf{M}_a + \mathbf{N}_b\Phi_b(t)\mathbf{M}_b\mathbf{K} \tag{4.2.3}$$

and after choosing:

$$\mathbf{Q}_0 = \mathbf{Q} + \mathbf{K}^T\mathbf{RK} \tag{4.2.4}$$

The performance bound of the closed loop system can be found by the use of the result of Lemma 2.4.4.

To determine a bound for the uncertain $\Delta\mathbf{A}_0$, the result of Lemma 2.4.1 can be used here, it follows that for any positive definite matrix **P**, constant matrix **K** and scaling parameters $\alpha_1, \alpha_2 > 0$

since $\quad \Delta\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A}_0 = \Delta\mathbf{A}^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A} + (\Delta\mathbf{BK})^T\mathbf{P} + \mathbf{P}\Delta\mathbf{BK}$

and $\quad \Delta\mathbf{A}^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A} = (\mathbf{N}_a\Phi_a(t)\mathbf{M}_a)^T\mathbf{P} + \mathbf{P}(\mathbf{N}_a\Phi_a(t)\mathbf{M}_a) \le \alpha_1\mathbf{PN}_a\mathbf{N}_a^T\mathbf{P} + \frac{1}{\alpha_1}\mathbf{M}_a^T\mathbf{M}_a$

$$(\Delta\mathbf{BK})^T\mathbf{P} + \mathbf{P}\Delta\mathbf{BK} = (\mathbf{N}_b\Phi_b(t)\mathbf{M}_b\mathbf{K})^T\mathbf{P} + \mathbf{P}(\mathbf{N}_b\Phi_b(t)\mathbf{M}_b\mathbf{K})$$

So $\quad (\Delta\mathbf{BK})^T\mathbf{P} + \mathbf{P}\Delta\mathbf{BK} \le \alpha_2\mathbf{PN}_b\mathbf{N}_b^T\mathbf{P} + \frac{1}{\alpha_2}\mathbf{K}^T\mathbf{M}_b^T\mathbf{M}_b\mathbf{K}$

Then a bound function Θ can be found as:

$$\Delta\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A}_0 \le \mathbf{PW}_1\mathbf{P} + \mathbf{W}_2 + \mathbf{K}^T\mathbf{W}_3\mathbf{K} \stackrel{\Delta}{=} \Theta(\mathbf{M}_a, \mathbf{M}_b, \mathbf{N}_a, \mathbf{N}_b, \mathbf{P}, \mathbf{K}, \alpha_1, \alpha_2) \tag{4.2.5}$$

where $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ are defined by:

$$\mathbf{W}_1 = \alpha_1\mathbf{N}_a\mathbf{N}_a^T + \alpha_2\mathbf{N}_b\mathbf{N}_b^T, \quad \mathbf{W}_2 = \frac{\mathbf{M}_a^T\mathbf{M}_a}{\alpha_1}, \quad \mathbf{W}_3 = \frac{\mathbf{M}_b^T\mathbf{M}_b}{\alpha_2} \tag{4.2.6}$$

It should be noted that the subsequent results may be sensitive to the values chosen for α_1 and α_2. These effectively describe the particular factorisation of the uncertainty and may be searched to improve the results. Thus by the use of the uncertain bound (4.2.5) in Lemma 2.4.4, a performance bound for the uncertain system can be found.

Lemma 4.2.1 Consider the uncertain system (4.1.1) with uncertainty bound (4.1.2) and cost performance index (4.1.3). For any stabilising controller **u=Kx**, if the following Riccati equation has a positive solution, $\mathbf{P}_b > 0$,

$$(\mathbf{A}+\mathbf{BK})^T\mathbf{P}_b + \mathbf{P}_b(\mathbf{A}+\mathbf{BK}) + \mathbf{P}_b\mathbf{W}_1\mathbf{P}_b + \mathbf{W}_2 + \mathbf{K}^T(\mathbf{R}+\mathbf{W}_3)\mathbf{K} + \mathbf{Q} = \mathbf{0} \qquad (4.2.7)$$

then $J_b = \mathbf{x}_0^T\mathbf{P}_b\mathbf{x}_0$ is a bound for the cost values for all admissible $(\Delta\mathbf{A}, \Delta\mathbf{B})$ such that: $J \le J_b$ (4.2.8)

Proof: This results can be found by the choosing $\mathbf{A}_0 = \mathbf{A}+\mathbf{BK}$, $\mathbf{Q}_0 = \mathbf{Q}+\mathbf{K}^T\mathbf{RK}$ and using bound (4.2.5) in Lemma 2.4.4. ◆

The optimal robust controller which minimises this performance bound is sought; to describe the effect of the controller on the performance bound, (4.2.7) is rewritten as:

$$\mathbf{A}^T\mathbf{P}_b + \mathbf{P}_b\mathbf{A} - \mathbf{P}_b\mathbf{B}\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_b + \mathbf{P}_b\mathbf{W}_1\mathbf{P}_b + \mathbf{Z}(\mathbf{K},\mathbf{P}_b) + \overline{\mathbf{Q}} = \mathbf{0} \qquad (4.2.9)$$

where $\overline{\mathbf{R}} = \mathbf{R}+\mathbf{W}_3$, $\overline{\mathbf{Q}} = \mathbf{W}_2 + \mathbf{Q}$ (4.2.10)

and $\mathbf{Z}(\mathbf{K},\mathbf{P}_b) = [\mathbf{K}+\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_b]^T\overline{\mathbf{R}}[\mathbf{K}+\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_b] \ge \mathbf{0}$ (4.2.11)

Now the effect of the controller is expressed explicitly through $\mathbf{Z}(\mathbf{K},\mathbf{P}_b)$. A new property of solutions of modified Riccati equations developed here enables the effect on the performance bound of the term $\mathbf{Z}(\mathbf{K},\mathbf{P}_b)$ to be quantified.

Lemma 4.2.2. For the following two Riccati equations:

$$\mathbf{A}^T\mathbf{P}_1 + \mathbf{P}_1\mathbf{A} - \mathbf{P}_1\mathbf{B}\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_1 + \mathbf{P}_1\mathbf{N}\mathbf{N}^T\mathbf{P}_1 + \mathbf{Z}(\mathbf{K}_1,\mathbf{P}_1) + \mathbf{Q} = 0 \qquad (4.2.12)$$

$$\mathbf{A}^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{A} - \mathbf{P}_2\mathbf{B}\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{N}\mathbf{N}^T\mathbf{P}_2 + \mathbf{Z}(\mathbf{K}_2,\mathbf{P}_2) + \mathbf{Q} = 0 \qquad (4.2.13)$$

with $\mathbf{Z}(\mathbf{K}_1,\mathbf{P}) \ge \mathbf{Z}(\mathbf{K}_2,\mathbf{P})$ for any positive definite matrix **P**, if equation (4.2.12) has a positive definite solution, $\mathbf{P}_1 > \mathbf{0}$, then

(i) equation (4.2.13) will have a positive definite solution, $\mathbf{P}_2 > \mathbf{0}$.

(ii) $\mathbf{P}_2 \le \mathbf{P}_1$

Furthermore, it follows that for all **K** for which (4.2.12) has a positive definite solution, $\mathbf{K}_0$ will give a minimal solution if $\mathbf{Z}(\mathbf{K},\mathbf{P}) \ge \mathbf{Z}(\mathbf{K}_0,\mathbf{P})$ for all **P**.

Proof: Defining

$$\mathbf{H}(\mathbf{P}) = \mathbf{Z}(\mathbf{K}_1,\mathbf{P}) - \mathbf{Z}(\mathbf{K}_2,\mathbf{P}) \ge \mathbf{0}$$

then (4.2.12) can be written as:

$$\mathbf{A}^T\mathbf{P}_1 + \mathbf{P}_1\mathbf{A} - \mathbf{P}_1\mathbf{B}\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_1 + \mathbf{P}_1\mathbf{N}\mathbf{N}^T\mathbf{P}_1 + \mathbf{Z}(\mathbf{K}_2,\mathbf{P}_1) + \mathbf{H}(\mathbf{P}_1) + \mathbf{Q} = 0$$

Substituting for $\mathbf{Z}(\mathbf{K}_2,\mathbf{P}_1)$ from (4.2.11) gives:

$$(\mathbf{A}+\mathbf{B}\mathbf{K}_2)^T\mathbf{P}_1+\mathbf{P}_1(\mathbf{A}+\mathbf{B}\mathbf{K}_2)+\mathbf{P}_1\mathbf{N}\mathbf{N}^T\mathbf{P}_1+\mathbf{K}_2^T\overline{\mathbf{R}}\mathbf{K}_2+\mathbf{H}(\mathbf{P}_1)+\mathbf{Q}=0 \qquad (4.2.14)$$

Similarly, (4.2.13) can be written as

$$(\mathbf{A}+\mathbf{B}\mathbf{K}_2)^T\mathbf{P}_2+\mathbf{P}_2(\mathbf{A}+\mathbf{B}\mathbf{K}_2)+\mathbf{P}_2\mathbf{N}\mathbf{N}^T\mathbf{P}_2+\mathbf{K}_2^T\overline{\mathbf{R}}\mathbf{K}_2+\mathbf{Q}=0 \qquad (4.2.15)$$

Defining that $\mathbf{A}_0=\mathbf{A}+\mathbf{B}\mathbf{K}_2$ and $\mathbf{Q}_0=\mathbf{K}_2^T\mathbf{R}\mathbf{K}_2+\mathbf{Q}$, (4.2.14) and (4.2.15) may be rewritten as:

$$\mathbf{A}_0^T\mathbf{P}_1+\mathbf{P}_1\mathbf{A}_0+\mathbf{P}_1\mathbf{N}\mathbf{N}^T\mathbf{P}_1+\mathbf{H}(\mathbf{P}_1)+\mathbf{Q}_0=0 \qquad (4.2.16)$$

and $$\mathbf{A}_0^T\mathbf{P}_2+\mathbf{P}_2\mathbf{A}_0+\mathbf{P}_2\mathbf{N}\mathbf{N}^T\mathbf{P}_2+\mathbf{Q}_0=0 \qquad (4.2.17)$$

i) Since $\mathbf{H}(\mathbf{P})\geq\mathbf{0}$ for any positive definite matrix $\mathbf{P}$, so we get that $\mathbf{H}(\mathbf{P}_1)\geq 0$, and also $\mathbf{Q}_0>0$ since $\mathbf{Q}>\mathbf{0}$, so $\mathbf{A}_0$ and $\begin{bmatrix}\mathbf{Q}_0^{1/2}\\ \mathbf{H}^{1/2}(\mathbf{P}_1)\end{bmatrix}$ must be observable. From the results of Lemma 2.3.5 it follows that if equation (4.2.16) has a positive definite solution, then

$$\left\|\begin{bmatrix}\mathbf{Q}_0^{1/2}\\ \mathbf{H}^{1/2}(\mathbf{P}_1)\end{bmatrix}(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{N}\right\|_\infty<1$$

So from Lemma 2.3.6 it follows that

$$\left\|\mathbf{Q}_0^{1/2}(s\mathbf{I}-\mathbf{A}_0)^{-1}\mathbf{N}\right\|_\infty<1$$

Then from the results of Lemma 2.3.5 it is found that if equation (4.2.16) has a positive definite solution, $\mathbf{P}_1>\mathbf{0}$, the equation (4.2.17) will have a positive definite solution, $\mathbf{P}_2>\mathbf{0}$.

(ii) Equations (4.2.16) and (4.2.17) may be written as:

$$\begin{aligned}&(\mathbf{A}_0+\mathbf{N}\mathbf{N}^T\mathbf{P}_2)^T\mathbf{P}_1+\mathbf{P}_1(\mathbf{A}_0+\mathbf{N}\mathbf{N}^T\mathbf{P}_2)+\mathbf{P}_1\mathbf{N}\mathbf{N}^T\mathbf{P}_1\\ &\quad-\mathbf{P}_1\mathbf{N}\mathbf{N}^T\mathbf{P}_2-\mathbf{P}_2\mathbf{N}\mathbf{N}^T\mathbf{P}_1+\mathbf{H}(\mathbf{P}_1)+\mathbf{Q}_0=0\end{aligned} \qquad (4.2.18)$$

$$(\mathbf{A}_0+\mathbf{N}\mathbf{N}^T\mathbf{P}_2)^T\mathbf{P}_2+\mathbf{P}_2(\mathbf{A}_0+\mathbf{N}\mathbf{N}^T\mathbf{P}_2)-\mathbf{P}_2\mathbf{N}\mathbf{N}^T\mathbf{P}_2+\mathbf{Q}_0=0 \qquad (4.2.19)$$

subtracting (4.2.19) from (4.2.18) yields:

$$(\mathbf{A}_0+\mathbf{N}\mathbf{N}^T\mathbf{P}_2)^T(\mathbf{P}_1-\mathbf{P}_2)+(\mathbf{P}_1-\mathbf{P}_2)(\mathbf{A}_0+\mathbf{N}\mathbf{N}^T\mathbf{P}_2)+(\mathbf{P}_1-\mathbf{P}_2)\mathbf{N}\mathbf{N}^T(\mathbf{P}_1-\mathbf{P}_2)+\mathbf{H}(\mathbf{P}_1)=\mathbf{0}$$

So $$(\mathbf{A}_0+\mathbf{N}\mathbf{N}^T\mathbf{P}_2)^T(\mathbf{P}_1-\mathbf{P}_2)+(\mathbf{P}_1-\mathbf{P}_2)(\mathbf{A}_0+\mathbf{N}\mathbf{N}^T\mathbf{P}_2)\leq\mathbf{0}$$

Since equation (4.2.17) has a positive definite solution, $\mathbf{P}_2>\mathbf{0}$, it is well known from the Lemma 1 of Doyle et al. (1989) that $\mathbf{A}_0+\mathbf{N}\mathbf{N}^T\mathbf{P}_2$ is stable, then from Lemma 2.3.1 it can be deduced that since: $\mathbf{H}(\mathbf{P}_1)\geq 0$

then $(\mathbf{P}_1 - \mathbf{P}_2) \geq 0$ i.e. $\mathbf{P}_1 \geq \mathbf{P}_2$. ◆

Hence, from Lemma 4.2.2, a controller $\mathbf{K}_0$ which minimises $\mathbf{Z}(\mathbf{K},\mathbf{P}_b)$ for all $\mathbf{P}_b$ will give a minimal solution to (4.2.9) and hence a minimal performance bound. It is clear from (4.2.11) that $\mathbf{Z}(\mathbf{K},\mathbf{P}_b)$ is minimised with a value of zero by choosing $\mathbf{K} = \mathbf{K}_0 = -\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_b$. Making this substitution in (4.2.9) yields a modified Riccati expression for the minimal performance bound and leads to the following theorem.

Theorem 4.2.3. For the uncertain system (4.1.1) with uncertainty bound (4.1.2), if the following Riccati equation

$$\mathbf{A}^T\mathbf{P}_m + \mathbf{P}_m\mathbf{A} - \mathbf{P}_m\mathbf{B}\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_m + \mathbf{P}_m\mathbf{W}_1\mathbf{P}_m + \overline{\mathbf{Q}} = \mathbf{0} \qquad (4.2.20)$$

has a positive definite solution, $\mathbf{P}_m > 0$, then $\mathbf{P}_m \leq \mathbf{P}_b$ for any controller $\mathbf{K}$ for which the Riccati equation (4.2.7) has a positive definite solution $\mathbf{P}_b$. Furthermore, choosing:

$$\mathbf{K}_r = -\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_m \qquad (4.2.21)$$

will stabilise the uncertain system (4.1.1) for all admissible $(\Delta\mathbf{A}, \Delta\mathbf{B})$ and provide a minimal performance bound.

Proof: The expressions for the minimal performance bound and optimal controller are self evident from previous arguments. The same method as the proof of Lemma 2.4.2 will be used here to prove asymptotic stability of the uncertain system (4.1.1) with the optimal controller (4.2.21). From the definition of (4.2.21), equation (4.2.20) can be rewritten as:

$$(\mathbf{A} + \mathbf{B}\mathbf{K}_r)^T\mathbf{P}_m + \mathbf{P}_m(\mathbf{A} + \mathbf{B}\mathbf{K}_r) + \mathbf{P}_m\mathbf{W}_1\mathbf{P}_m + \mathbf{K}_r^T\overline{\mathbf{R}}\mathbf{K}_r + \overline{\mathbf{Q}} = \mathbf{0}$$

So equation (4.2.20) has a positive definite solution implies that the above equation will also have a positive definite solution $\mathbf{P}_m > 0$. From the uncertainty constraint (4.2.5) it follows that:

$$\Delta\mathbf{A}_0^T\mathbf{P}_m + \mathbf{P}_m\Delta\mathbf{A}_0 \leq \mathbf{P}_m\mathbf{W}_1\mathbf{P}_m + \mathbf{W}_2 + \mathbf{K}_r^T\mathbf{W}_3\mathbf{K}_r$$

Adding this expression to the above equation it follows that:

$$\mathbf{A}_c^T\mathbf{P}_m + \mathbf{P}_m\mathbf{A}_c + \mathbf{K}_r^T\mathbf{R}\mathbf{K}_r + \mathbf{Q} \leq \mathbf{0}$$

Where $\mathbf{A}_c = \mathbf{A} + \mathbf{B}\mathbf{K}_r + \Delta\mathbf{A} + \Delta\mathbf{B}\mathbf{K}_r$ is the closed loop system matrix. So this positive definite solution $\mathbf{P}_m > 0$ satisfies:

$$\mathbf{A}_c^T\mathbf{P}_m + \mathbf{P}_m\mathbf{A}_c \leq -\mathbf{K}_r^T\mathbf{R}\mathbf{K}_r - \mathbf{Q} < 0$$

So from the definition of the Lemma 2.3.1 it follows that the uncertain closed loop system is quadratically stable. ◆

Hence, for a system with unmatched norm bounded uncertainties, a robust optimal LQ control law is given by (4.2.21) which requires the solution of a modified Riccati equation (4.2.20), this controller stabilises the uncertain system and provides a minimal performance bound. At the same time, since the performance bound is dependent upon the particular factorisation of the uncertainty, α_1 and α_2 may be searched to improve the results.

Furthermore, from the proof of the above Theorem we know that there exists a $\mathbf{P}_m > 0$ such that:

$$\mathbf{A}_c^T\mathbf{P}_m + \mathbf{P}_m\mathbf{A}_c + \mathbf{K}_r^T\mathbf{R}\mathbf{K}_r + \mathbf{Q} \le \mathbf{0}$$

i.e. $$\mathbf{A}_c^T\mathbf{P}_m + \mathbf{P}_m\mathbf{A}_c + \mathbf{P}_m\mathbf{B}\overline{\mathbf{R}}^{-1}\mathbf{R}\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_m + \mathbf{Q} \le \mathbf{0}$$

Then from Lemma 2.3.5, the active closed loop system satisfies:

$$\left\|\mathbf{Q}^{1/2}(s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}\overline{\mathbf{R}}^{-1}\mathbf{R}^{1/2}\right\|_\infty < 1$$

Relating this to the result of §3.1.1 of the standard LQR design, it is found that, in the optimal RLQR design, there are also some inherent robustness properties for residual unknown uncertainty of the system.

4.2.2 Systems with matched norm bounded uncertainty

In this section we consider the case when the parametric uncertainties of the system matrix and the input matrix are matched as (4.1.2). The closed loop system of uncertain system (4.1.1) is also described by (4.2.1) and again the uncertain terms may be bounded to yield an equation for the performance bound.

For the uncertain system (4.1.1) with a stabilising controller **u=Kx** and norm bound uncertainty (4.1.4), it follows that:

$$\Delta\mathbf{A}_0 = \Delta\mathbf{A} + \Delta\mathbf{B}\mathbf{K} = \mathbf{N}\boldsymbol{\Phi}(t)(\mathbf{M}_a + \mathbf{M}_b\mathbf{K})$$

and also choosing: $\mathbf{Q}_0 = \mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}$, then the performance bound of the closed loop system can be found by the use of the result of Lemma 2.4.4. To determine a bound for the uncertain $\Delta\mathbf{A}_0$, the result of Lemma 2.4.1 can be used here, it follows that for any positive definite matrix **P**, constant matrix **K** and scaling parameters $\alpha > 0$, it follows that:

$$\begin{aligned}\Delta\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A}_0 &\le \Delta\mathbf{A}^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A} + \mathbf{P}\Delta\mathbf{B}\mathbf{K} + \mathbf{K}^T\Delta\mathbf{B}^T\mathbf{P} \\ &= (\mathbf{M}_a + \mathbf{M}_b\mathbf{K})^T\boldsymbol{\Phi}^T(t)\mathbf{N}^T\mathbf{P} + \mathbf{P}\mathbf{N}\boldsymbol{\Phi}(t)(\mathbf{M}_a + \mathbf{M}_b\mathbf{K})\end{aligned}$$

So by the use of Lemma 2.4.1 a bound function Θ_M can be found as:

$$\begin{aligned}\Delta A_0^T P + P\Delta A_0 &\le \alpha PNN^T P + \frac{1}{\alpha}(M_a + M_b K)^T (M_a + M_b K) \\ &\stackrel{\Delta}{=} \Theta_M(M_a, M_b, N, P, K, \alpha)\end{aligned} \tag{4.2.22}$$

Thus using similar steps as §4.2.1, i.e., replacing the uncertain bound in (2.4.22) by the bound (4.2.22), then directly using Lemma 2.4.4, the performance bound matrix $\mathbf{P}_b$ is the solution of the following Riccati equation:

$$\begin{aligned}&(A + BK)^T P_b + P_b(A + BK) + \alpha P_b NN^T P_b \\ &+ \frac{1}{\alpha}(M_a + M_b K)^T (M_a + M_b K) + Q + K^T RK = 0\end{aligned} \tag{4.2.23}$$

Defining:

$$Z(K, P_b) = [(P_b B + \frac{1}{\alpha} M_a^T M_b)\Sigma + K^T]\Sigma^{-1}[(P_b B + \frac{1}{\alpha} M_a^T M_b)\Sigma + K^T]^T \ge 0$$

(4.2.23) can be rewritten as

$$A_1^T P_b + P_b A_1 - P_b B\Sigma B^T P_b + \alpha P_b NN^T P_b + Z(K, P_b) + Q_1 = 0 \tag{4.2.24}$$

where

$$A_1 = A - \frac{1}{\alpha} B\Sigma M_b^T M_a,\ Q_1 = \frac{1}{\alpha} M_a^T (I - M_b \Sigma M_b^T) M_a + Q \text{ and } \Sigma = (R + \frac{1}{\alpha} M_b^T M_b)^{-1}$$

Again Lemma 4.2.2 may be employed and leads to the following result:

Theorem 4.2.4. For the uncertain system (4.1.1) with uncertainty bound (4.1.4), if the following Riccati equation

$$A_1^T P_m + P_m A_1 - P_m B\Sigma B^T P_m + \alpha P_m NN^T P_m + Q_1 = 0 \tag{4.2.25}$$

has a positive definite solution, $\mathbf{P}_m > 0$, then $\mathbf{P}_m \le \mathbf{P}_b$ for any controller $\mathbf{K}$ for which the Riccati equation (4.2.23) has a positive definite solution $\mathbf{P}_b$. Furthermore, choosing:

$$K = K_r = -\frac{\Sigma}{\alpha}(\alpha B^T P_m + M_b^T M_a) \tag{4.2.26}$$

will stabilise the uncertain system (4.1.1) for all admissible $(\Delta A, \Delta B)$ and provide a minimal performance bound.

Proof: The proof of this Theorem is similar to the proof of Theorem 4.2.3. ◆

Hence, for a system with matched norm bounded uncertainties, a robust optimal LQ control law is given by (4.2.26) which requires the solution of a modified Riccati equation

(4.2.25), this controller stabilises the uncertain system and provides a minimal performance bound. As before, α may be searched to improve the results.

4.2.3 The choice of an appropriate uncertainty description

For a given uncertainty model $(\Delta\mathbf{A}, \Delta\mathbf{B})$, the choice of a suitable description Π (4.1.2) or Π_M (4.1.4) is critical to the design of less conservative RLQR controllers. In general this is a complex task and no general algorithms are available, so some guidelines are given in this section. In §4.1, the matched norm bounded uncertainty format was introduced, however it will be shown that it is not always possible to represent the uncertainty as this format without unnecessarily increasing the dimensions and singular values of $\mathbf{N}_a, \Phi_a(t), \mathbf{M}_a$ and $\mathbf{N}_b, \Phi_b(t), \mathbf{M}_b$, such a increase would lead to an imprecise description. However, it is also shown that if a matched norm bounded format is available, and giving no unnecessary increase of the dimensions and singular values of uncertainties, then using Theorem 4.2.4 will give a less conservative design than Theorem 4.2.3.

Conjecture 1: In general, a description which can make the dimensions and singular values of the matrices $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ in (4.2.6) as small as possible will give a less conservative RLQR controller and also with a lower performance bound. This result is evident from the proof of Lemma 4.2.2: consider equations (4.2.16) and (4.2.17), the comparison result shows that if equation (4.2.16) has a positive definite solution, $\mathbf{P}_1 > \mathbf{0}$, the equation (4.2.17) will have a positive definite solution, $\mathbf{P}_2 > \mathbf{0}$, and $\mathbf{P}_2 \le \mathbf{P}_1$. From this comparison it can be deduced that less dimensions and singular values of the matrices $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ will give less conservative design.

Conjecture 2: Not every uncertainty $(\Delta\mathbf{A}, \Delta\mathbf{B})$ can be described in the matched norm bounded format (without unnecessarily increasing the dimensionalty). This is illustrated by the following simple example, consider

$$\Delta\mathbf{A} = \begin{bmatrix} 0 & 0 \\ \varphi_1 & -\varphi_2 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

where $|\varphi_1| \le 1$ and $|\varphi_2| \le 1$. Let us attempt to represent this with the matched norm bounded format (4.1.4), for a minimal dimension representation we should choose

$$\Phi = \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix}, \quad \text{and} \quad \bar{\sigma}(\Phi) < 1$$

Then $\mathbf{N}$, $\Phi(t)$ and $\mathbf{M}_b$ can be chosen as:

$$\Delta\mathbf{B} = \begin{bmatrix} 1/\rho_1 & 0 \\ 0 & 1/\rho_2 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix} \cdot \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$$

where ρ_1, ρ_2 are arbitrary positive scalar parameters. Then to represent $\Delta\mathbf{A}$ in the matched format it is required that

$$\Delta\mathbf{A} = \begin{bmatrix} 1/\rho_1 & 0 \\ 0 & 1/\rho_2 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix} \cdot \mathbf{M}_a$$

Since the elements of suitable $\mathbf{M}_a$ must be constant and not functions of φ_1 and φ_2 and it can be shown that no suitable solutions exist, this uncertainty can not be represented in a minimal dimension matched norm bounded format. It may be possible to represent $(\Delta\mathbf{A}, \Delta\mathbf{B})$ in matched format by increasing the dimensions of $\mathbf{N}, \mathbf{\Phi}(t), \mathbf{M}_a$ and $\mathbf{N}, \mathbf{\Phi}(t), \mathbf{M}_b$, but this will give less precise description.

Conjecture 3: When a particular matched norm bounded description Π_m (4.1.4) of $(\Delta\mathbf{A}, \Delta\mathbf{B})$ is available it is found that using Theorem 4.2.4 (specifically developed for this format) will give a less conservative design than using the general Theorem 4.2.3.

Consider a uncertainty domain as Π (4.1.2), assume that the uncertainty is matched so we may write:

$$\mathbf{N}_a = \mathbf{N}_b = \mathbf{N}, \quad \text{and} \quad \mathbf{\Phi}_a(t) = \mathbf{\Phi}_b(t) = \mathbf{\Phi}(t)$$

Following the method for the general unmatched description we get an uncertainty bound function Θ (4.2.5) as

$$\Theta = (\alpha_1 + \alpha_2)\mathbf{PNN}^T\mathbf{P} + \frac{\mathbf{M}_a^T\mathbf{M}_a}{\alpha_1} + \frac{\mathbf{K}^T\mathbf{M}_b^T\mathbf{M}_b\mathbf{K}}{\alpha_2}$$

or by the specific method for matched uncertainty we get a bound function Θ_m (4.2.22) as

$$\Theta_m = \alpha\mathbf{PNN}^T\mathbf{P} + \frac{1}{\alpha}(\mathbf{M}_a + \mathbf{M}_b\mathbf{K})^T(\mathbf{M}_a + \mathbf{M}_b\mathbf{K})$$

for any given α_1 and α_2, we can choose $\alpha = \alpha_1 + \alpha_2$ then:

$$\begin{aligned} \Theta - \Theta_m &= \frac{\mathbf{M}_a^T\mathbf{M}_a}{\alpha_1} + \frac{\mathbf{K}^T\mathbf{M}_b^T\mathbf{M}_b\mathbf{K}}{\alpha_2} - \frac{1}{\alpha}(\mathbf{M}_a + \mathbf{M}_b\mathbf{K})^T(\mathbf{M}_a + \mathbf{M}_b\mathbf{K}) \\ &= \frac{1}{\alpha_1 + \alpha_2}\left[\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\mathbf{M}_a - \sqrt{\frac{\alpha_1}{\alpha_2}}\mathbf{M}_b\mathbf{K}\right)^T\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\mathbf{M}_a - \sqrt{\frac{\alpha_1}{\alpha_2}}\mathbf{M}_b\mathbf{K}\right)\right] \geq \mathbf{0} \end{aligned}$$

Hence, if a matched norm bounded format description is available Theorem 4.2.4 will give a less conservative controller i.e. it will have a lower performance bound.

Overall the choice of the 'best' description of the uncertainty is quite a complex problem to which no general solution is known. However, some guidelines are firstly to choose a description with minimal dimensioned structural matrices and secondly, if it is possible within this constraint, choose a matched norm bounded format and employ the special method (Theorem 4.2.4) for this case.

4.2.4 Relationship between stability and performance robustness

Let us now consider the trade off between stability robustness and performance robustness. The norm bounded uncertainty description is used here but similar arguments apply for the matched norm bounded case. The cost bound $\mathbf{x}_0^T\mathbf{P}_b\mathbf{x}_0$ is a bound for the maximum possible cost for all admissible values of $(\Delta\mathbf{A}, \Delta\mathbf{B})$. As defined in §2.4.2, the performance robustness may be measured by performance degradation parameter ρ as:

$$\rho = \frac{\mathbf{x}_0^T(\mathbf{P}_b - \mathbf{P}_0)\mathbf{x}_0}{\mathbf{x}_0^T\mathbf{P}_0\mathbf{x}_0} \times 100\% \tag{4.2.27}$$

where $\mathbf{P}_0 > 0$ is the performance matrix of the certain part of the system (4.1.1) when the optimal LQR controller for the certain part is employed. For some given parametric uncertainties $\Delta\mathbf{A}, \Delta\mathbf{B}$, a relatively small performance bound $\mathbf{P}_b$ produces a small performance degradation rate and the system is said to have good performance robustness and furthermore, the minimal performance bound $\mathbf{P}_m$ gives a minimal performance degradation rate, ρ_m.

It is desirable that a control system possesses both good stability robustness and performance robustness. Stability robustness may be measured by the size of the admissible domain specified for the uncertainties, $(\Delta\mathbf{A}, \Delta\mathbf{B})$ and to increase the size of this domain $\mathbf{W}_1, \mathbf{W}_2$ or $\mathbf{W}_3$ should be increased. However, from (4.2.20) it can be deduced that $\mathbf{P}_m$ and thus ρ_m will also be increased and hence performance robustness will be reduced. Thus, it is clear that there is an inherent trade off between stability robustness and performance robustness; if better stability robustness is required, performance robustness must be reduced and conversely, if better performance robustness is required, stability robustness must be reduced.

4.3 RLQR Design Using H∞ Design Techniques

The fast development of H∞ optimisation design has produced many good design techniques and tools. For easy computation of the robust LQR controller design, H∞ controller design techniques may be applied to a suitably modified system. Firstly the H∞

techniques will be introduced then they will be applied to uncertain systems for both norm bounded and matched norm bounded uncertainties.

4.3.1 H∞ controller design techniques

The design of full state feedback H∞ optimisation controller and the concept of quadratic stabilisation with an H∞ norm bound have been discussed in §3.1.2. Its result can be recalled here: consider the system described as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_1\omega(t) + \mathbf{B}_2\mathbf{u}(t) \\ \mathbf{z}(t) = \mathbf{C}_1\mathbf{x}(t) + \mathbf{D}_1\mathbf{u}(t) \end{cases} \tag{4.3.1}$$

with $\Omega = \mathbf{D}_1^T\mathbf{D}_1 > 0$

Remark 4.3.1. A full state feedback controller can stabilise the system and satisfy the H∞ norm bound condition

$$\|\mathbf{T}_{\omega z}\|_\infty < \gamma_0 \tag{4.3.2}$$

if and only if the following modified Riccati equation has a positive definite solution **P>0**,

$$(\mathbf{A} - \mathbf{B}_2\Omega^{-1}\mathbf{C}_1^T\mathbf{D}_1)^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}_2\Omega^{-1}\mathbf{C}_1^T\mathbf{D}_1) - \mathbf{P}\mathbf{B}_2\Omega^{-1}\mathbf{B}_2^T\mathbf{P}$$
$$+\frac{1}{\gamma_0^2}\mathbf{P}\mathbf{B}_1\mathbf{B}_1^T\mathbf{P} + \mathbf{C}_1^T(\mathbf{I} - \mathbf{D}_1\Omega^{-1}\mathbf{D}_1^T)\mathbf{C}_1 = \mathbf{0} \tag{4.3.3}$$

the required control law can be constructed as: $\mathbf{u} = \mathbf{K}_f\mathbf{x}$ with

$$\mathbf{K}_f = -\Omega^{-1}(\mathbf{B}_2^T\mathbf{P} + \mathbf{D}_1^T\mathbf{C}_1) \tag{4.3.4}$$

Hence, to design a controller which satisfies the H∞ norm bound (4.3.2) and stabilises the system, use the positive definite solution of (4.3.3) in the expression (4.3.4).

4.3.2 RLQR design: norm bounded uncertainty case

Using H∞ controller design techniques, a RLQR controller for an uncertain system (4.1.1) with norm bounded uncertainties (4.1.2) and quadratic cost function (4.1.3) is found. Consider the following modified system with particular disturbance vector and performance vector:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_1\omega(t) + \mathbf{B}_2\mathbf{u}(t) \\ \mathbf{z}(t) = \mathbf{C}_1\mathbf{x}(t) + \mathbf{D}_1\mathbf{u}(t) \end{cases} \tag{4.3.5}$$

where $\mathbf{A}$ and $\mathbf{B}_2 = \mathbf{B}$ are the system and input matrices of the certain part of (4.1.1) and the uncertainty bound and cost function are encoded into the modified system thus:

$$\mathbf{B}_1 = \left[\sqrt{\alpha_1}\mathbf{N}_a \quad \sqrt{\alpha_2}\mathbf{N}_b\right], \quad \mathbf{C}_1 = \begin{bmatrix} \mathbf{0} \\ \overline{\mathbf{M}} \\ \mathbf{M}_a/\sqrt{\alpha_1} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} \overline{\mathbf{N}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{M}_b/\sqrt{\alpha_2} \end{bmatrix}$$

where $\overline{\mathbf{M}}$ and $\overline{\mathbf{N}}$ are related to the weighting matrices $\mathbf{Q} > \mathbf{0}$ and $\mathbf{R}>\mathbf{0}$ in the cost function by $\mathbf{Q} = \overline{\mathbf{M}}^T\overline{\mathbf{M}}$ and $\mathbf{R} = \overline{\mathbf{N}}^T\overline{\mathbf{N}}$.

The following theorem states that employing the H∞ design technique to the modified system will produce a RLQR controller for the uncertain system.

Theorem 4.3.2. Employing the H∞ controller design technique, as described in Remark 4.3.1, with $\gamma_0 = 1$ to the modified system (4.3.5) produces the same controller as the RLQR design technique for any given uncertain system (4.1.1) with norm bounded uncertainty (4.1.2) and quadratic cost function (4.1.3).

Proof: From Remark 4.3.1, the H∞ controller is given by

$$\mathbf{K}_f = -\Omega^{-1}(\mathbf{B}_2^T\mathbf{P} + \mathbf{D}_1^T\mathbf{C}_1) \tag{4.3.6}$$

where $\mathbf{P}$ is the positive definite solution of

$$\begin{aligned}(\mathbf{A} - \mathbf{B}_2\Omega^{-1}\mathbf{C}_1^T\mathbf{D}_1)^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}_2\Omega^{-1}\mathbf{C}_1^T\mathbf{D}_1) - \mathbf{P}\mathbf{B}_2\Omega^{-1}\mathbf{B}_2^T\mathbf{P} \\ + \mathbf{P}\mathbf{B}_1\mathbf{B}_1^T\mathbf{P} + \mathbf{C}_1^T(\mathbf{I} - \mathbf{D}_1\Omega^{-1}\mathbf{D}_1^T)\mathbf{C}_1 = \mathbf{0}\end{aligned} \tag{4.3.7}$$

Since from the definitions for the modified system (4.3.5)

$$\Omega = \mathbf{D}_1^T\mathbf{D}_1 = \mathbf{R} + \mathbf{M}_b^T\mathbf{M}_b/\alpha_2 \quad \text{and} \quad \mathbf{C}_1^T\mathbf{C}_1 = \mathbf{Q} + \mathbf{M}_a^T\mathbf{M}_a/\alpha_1, \quad \mathbf{C}_1^T\mathbf{D}_1 = \mathbf{0}$$

(4.3.7) can be reduced to

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}_2\Omega^{-1}\mathbf{B}_2^T\mathbf{P} + \mathbf{P}\mathbf{B}_1\mathbf{B}_1^T\mathbf{P} + \mathbf{C}_1^T\mathbf{C}_1 = \mathbf{0} \tag{4.3.8}$$

which can be shown to be identical to (4.2.14) and (4.3.6) can be written as

$$\mathbf{K}_f = -\Omega^{-1}\mathbf{B}_2^T\mathbf{P} \tag{4.3.9}$$

which is identical to (4.2.15). Hence the controller is identical to that produced by the RLQR method in Theorem 4.2.3. ◆

So for $\mathbf{Q} > \mathbf{0}$, there exists an optimal robust LQR solution for uncertain system (4.1.1) if and only if there exists an H∞ optimal controller with H∞ norm bound $\gamma_0 = 1$.

If an optimal robust LQR solution does not exist, i.e., there does not exist a positive solution for equation (4.2.20) or (4.3.3), then we could turn to look for a robust stabilising controller. Consider another system described as (4.3.1) but choose

$$\mathbf{C}_1 = \begin{bmatrix} \mathbf{M}_a/\sqrt{\alpha_1} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_b/\sqrt{\alpha_2} \end{bmatrix} \tag{4.3.10}$$

The following Lemma states that employing the H∞ design technique to the modified system will produce a robust stabilising controller for the uncertain system.

Lemma 4.3.3. Employing the H∞ controller design technique with $\gamma_0 = 1$ to the modified system (4.3.5) subject to (4.3.10), a robust stabilising controller for any given uncertain system (4.1.1) with norm bounded uncertainty (4.1.2) and quadratic cost function (4.1.3) will be produced.

Proof: The proof is similar to that of Theorem 4.3.2. The H∞ controller which can stabilise system (4.3.5) with $\|\mathbf{T}_{\omega z}\| < 1$ is given by

$$\mathbf{K}_f = -\Omega^{-1}(\mathbf{B}_2^T\mathbf{P} + \mathbf{D}_1^T\mathbf{C}_1) \tag{4.3.11}$$

where $\mathbf{P}$ is the positive definite solution of

$$\begin{aligned}&(\mathbf{A} - \mathbf{B}_2\Omega^{-1}\mathbf{C}_1^T\mathbf{D}_1)^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}_2\Omega^{-1}\mathbf{C}_1^T\mathbf{D}_1) - \mathbf{P}\mathbf{B}_2\Omega^{-1}\mathbf{B}_2^T\mathbf{P} \\ &\quad + \mathbf{P}\mathbf{B}_1\mathbf{B}_1^T\mathbf{P} + \mathbf{C}_1^T(\mathbf{I} - \mathbf{D}_1\Omega^{-1}\mathbf{D}_1^T)\mathbf{C}_1 < \mathbf{0}\end{aligned} \tag{4.3.12}$$

Since from the definitions for the modified system (4.3.5) subject to (4.3.10)

$$\Omega = \mathbf{D}_1^T\mathbf{D}_1 = \mathbf{M}_b^T\mathbf{M}_b/\alpha_2 \quad \text{and} \quad \mathbf{C}_1^T\mathbf{C}_1 = \mathbf{M}_a^T\mathbf{M}_a/\alpha_1, \quad \text{so} \quad \mathbf{C}_1^T\mathbf{D}_1 = \mathbf{0}$$

(4.3.12) can be reduced to

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}_2\Omega^{-1}\mathbf{B}_2^T\mathbf{P} + \mathbf{P}\mathbf{B}_1\mathbf{B}_1^T\mathbf{P} + \mathbf{C}_1^T\mathbf{C}_1 < \mathbf{0}$$

from (4.3.11) we find that $\mathbf{K}_f = -\Omega^{-1}\mathbf{B}_2^T\mathbf{P}$ and the above expression can be rewritten as:

$$(\mathbf{A} + \mathbf{B}_2\mathbf{K}_f)^T\mathbf{P} + \mathbf{P}(\mathbf{A} + \mathbf{B}_2\mathbf{K}_f) + \mathbf{K}_f^T\Omega\mathbf{K}_f + \mathbf{P}\mathbf{B}_1\mathbf{B}_1^T\mathbf{P} + \mathbf{C}_1^T\mathbf{C}_1 < \mathbf{0} \tag{4.3.13}$$

Hence

$$\mathbf{A}_c^T\mathbf{P} + \mathbf{P}\mathbf{A}_c < \mathbf{0}$$

where $\mathbf{A}_c = \mathbf{A} + \mathbf{B}_2\mathbf{K}_f + \Delta\mathbf{A} + \Delta\mathbf{B}\mathbf{K}_f$

From the definition of the Lemma 2.3.1 it follows that the closed-loop system matrix $\mathbf{A}_c$ is quadratically stable. ◆

4.3.3 RLQR design: matched norm bounded uncertainty case

Now consider the matched norm bounded case, using the same modified system as (4.3.5) with following definitions:

$$\mathbf{B}_1 = \sqrt{\gamma}\mathbf{N}, \quad \mathbf{C}_1 = \begin{bmatrix} \mathbf{0} \\ \overline{\mathbf{M}} \\ \mathbf{M}_a/\sqrt{\alpha_1} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} \overline{\mathbf{N}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{M}_b/\sqrt{\alpha_2} \end{bmatrix} \geq \mathbf{0} \tag{4.3.14}$$

The following theorem states that employing the H∞ design technique to the modified system will produce a RLQR controller for the uncertain system.

Corollary 4.3.4. Employing the H∞ controller design technique with $\gamma_0 = 1$ to the modified system (4.3.5) subject to (4.3.14) produces the same controller as the RLQR design technique for any given uncertain system (4.1.1) with matched norm bounded uncertainty (4.1.4) and quadratic cost function (4.1.3).

If an optimal robust LQR solution does not exit, i.e., there does not exist a positive solution for equation (4.2.20) or (4.3.3), then we could turn to look for a robust stabilising controller. Consider another system described as (4.3.5) but choose

$$\mathbf{B}_1 = \sqrt{\gamma}\mathbf{N}, \quad \mathbf{C}_1 = \begin{bmatrix} \mathbf{M}_a/\sqrt{\alpha_1} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_b/\sqrt{\alpha_2} \end{bmatrix} \tag{4.3.15}$$

The following Corollary states that employing the H∞ design technique to the modified system will produce a robust stabilising controller for the uncertain system.

Corollary 4.3.5. Employing the H∞ controller design technique with $\gamma_0 = 1$ to the modified system (4.3.5) subject to (4.3.15) produces a robust stabilising controller for any given uncertain system (4.1.1) with norm bounded uncertainty (4.1.2) and quadratic cost function (4.1.3).

4.4 Example Applications of the Method

Examples are given here to demonstrate the application of the method and illustrate its effectiveness in producing good performance robustness and furthermore to show that care should be taken when formatting an uncertainty description. Example 4.1 shows that the robust LQR control design has significantly better performance robustness for an uncertain system than the standard LQR method. Example 4.2 shows that different descriptions of the same uncertainty will yield different robust LQR control laws and that a more precise uncertainty description will give improved performance robustness.

Example 4.1 Consider the following uncertain system, performance index and particular initial condition:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1+0.8\varphi_1(t) & -3-0.6\varphi_2(t) \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbf{u}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$J = \int_0^\infty (\mathbf{x}^T\mathbf{x} + \mathbf{u}^T\mathbf{u})dt, \qquad \varphi_1^2(t) \le 1, \qquad \varphi_2^2(t) \le 1$$

A full state-feedback control law is to be designed, the uncertainty constraint can be described using the norm bounded uncertainty format (4.1.2) with $\Delta\mathbf{B} = \mathbf{0}$, and $\Delta\mathbf{A} = \mathbf{N}_a\mathbf{\Phi}_a(t)\mathbf{M}_a$, where

$$\mathbf{N}_a = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{M}_a = \begin{bmatrix} 0.8 & 0 \\ 0 & -0.6 \end{bmatrix}, \qquad \mathbf{\Phi}_a(t) = \begin{bmatrix} \varphi_1(t) & 0 \\ 0 & \varphi_2(t) \end{bmatrix}$$

with $\overline{\sigma}(\mathbf{\Phi}_a(t)) \le 1$.

The application of Theorem 4.2.3 provides the optimal RLQR controller, $\mathbf{u} = \mathbf{K}_r\mathbf{x}$ with

$$\mathbf{K}_r = -\overline{\mathbf{R}}^{-1}\mathbf{B}^T\mathbf{P}_m = [-1.15, -0.67]$$

The best value for the scalar parameter was found to be α_1=0.56 for this example (α_2 is redundant here since $\Delta\mathbf{B} = \mathbf{0}$). For comparison, by standard LQR design (3.1.2), a controller $\mathbf{u} = -\mathbf{K}_0\mathbf{x}$ can be found for the *certain part of the system* with:

$$\mathbf{K}_0 = [0.414, 0.290]$$

From the summary of the performance of the two controllers in Table 4.1 it can be seen that, as expected, the LQR controller offers superior performance for the certain part of the system (it is in fact optimal for this case). However, when the uncertainty is considered, the performance bound for the RLQR controller is significantly lower and hence an improved performance robustness index is achieved.

	Performance (Bound) for LQR ($\mathbf{K}_0$)	Performance (Bound) for RLQR ($\mathbf{K}_r$)
Certain System $\mathbf{x}_0^T\mathbf{P}\mathbf{x}_0$	2.77	3.38
Uncertain System $\mathbf{x}_0^T\mathbf{P}_b\mathbf{x}_0$	12.22	6.99
Performance Degradation Parameter, ρ	341.16%	152.35%

Table 4.1. Comparison of performance of LQR and RLQR controllers.

Example 4.2 Consider the following uncertain system, performance index and particular initial condition:

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \Delta\mathbf{A})\mathbf{x}(t) + (\mathbf{B} + \Delta\mathbf{B})\mathbf{u}(t)$$

$$J = \int_0^\infty (\mathbf{x}^T\mathbf{x} + \mathbf{u}^T\mathbf{u})dt, \qquad \varphi^2(t) \le 1, \quad \mathbf{x}_0 = \begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}^T$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.1 & --0.2 & 0.3 \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5\varphi(t) \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1.5 \end{bmatrix}^T, \quad \Delta\mathbf{B} = \begin{bmatrix} 0 & 0 & 0.5\varphi(t) \end{bmatrix}^T$$

Controllers designed for various descriptions of this uncertainty will be compared for all cases $\mathbf{\Phi}_a = \mathbf{\Phi}_b = \mathbf{\Phi} = \varphi\mathbf{I}$, then $\overline{\sigma}(\mathbf{\Phi}) \le 1$

(i) Norm bounded format 1: the uncertainty constraint can be described by:

$$\Delta\mathbf{A} = \mathbf{N}_a\mathbf{\Phi}(t)\mathbf{M}_a, \quad \Delta\mathbf{B} = \mathbf{N}_b\mathbf{\Phi}(t)\mathbf{M}_b, \quad \mathbf{\Phi} = \varphi\mathbf{I}$$

where $\mathbf{N}_a = 0.5\mathbf{I}, \quad \mathbf{N}_b = 0.5\mathbf{I}, \quad \mathbf{M}_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_b = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$

From (4.2.20) and (4.2.21) the optimal RLQR controller, $\mathbf{K}_r$, is found where

$$\mathbf{K}_r = [-0.9 \ -4.59 \ -4.27]$$

and a minimal performance bound of 537.8 for the particular initial condition. The 'best' values for the scalar parameters were found to be $\alpha_1 = 0.025$ and $\alpha_2 = 0.023$.

(ii) Norm bounded format 2: In this case a description which can make the dimensions and singular values of matrices $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ defined in (4.2.6) less then format 1 will be used to illustrate the *Conjecture* 1. The result will show that this format will give less conservative design. Consider a new description of the uncertainties as:

$$\mathbf{N}_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{N}_b = \mathbf{N}_a, \quad \mathbf{M}_a = 0.5\mathbf{N}_a, \quad \mathbf{M}_b = \begin{bmatrix} 0 & 0 & 0.5 \end{bmatrix}^T$$

Then from (4.2.20) and (4.2.21) the optimal RLQR controller, $\mathbf{K}_r$, is found where

$$\mathbf{K}_r = [-0.93\ -2.46\ -3.27]$$

and a minimal performance bound of 19.31 for the particular initial condition. The 'best' values for the scalar parameters were found to be $\alpha_1 = 0.26$ and $\alpha_2 = 0.35$. This performance bound is significantly lower than that achieved using the norm bounded format (i), so the Conjecture 1 is illustrated.

(iii) Matched norm bounded format: It is clear that these formats are matched norm bounded format, so let us use the special description (4.2.4) with

$$\mathbf{N} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_a = 0.5\mathbf{N}, \quad \mathbf{M}_b = \begin{bmatrix} 0 & 0 & 0.5 \end{bmatrix}^T$$

From (4.2.25) and (4.2.26) an optimal RLQR controller, $\mathbf{K}_r$, is found where

$$\mathbf{K}_r = [-0.79 \quad -1.96 \quad -2.16]$$

and a minimal performance bound of 16.06 for the particular initial condition. The 'best' value for the scalar parameter was found to be α=0.52 in this case. This performance bound is significantly lower than that achieved using a norm bounded format to describe the same uncertainty constraint and in general, if it is possible to describe the uncertainty with a matched norm bounded format then this will result in better performance than if a norm bounded format is used.

4.5 DISCUSSION

A robust LQR design methodology is presented which guarantees both closed loop stability for all admissible uncertainties and provides a minimal performance bound. The inherent trade off between stability robustness and performance robustness can be illustrated by considering the effect of increasing the magnitude of the uncertainty domain. It is shown how the RLQR design problem may be presented as an H^{∞} design problem for a scaled version of the nominal system and that for suitable choices of the disturbance and performance vector identical controllers are produced. This enables the numerical techniques developed for the solution of H^{∞} problems to be employed to implement the RLQR method described here. The performance robustness is sensitive to the precise format in which the uncertainty bound is described and it is shown that a format giving a precise description will give good robust performance.

CHAPTER 5

STABILITY AND PERFORMANCE ROBUSTNESS ANALYSIS

The stability robustness analysis of full state feedback controlled systems has been discussed in chapter 3. It was shown that existing design techniques offer some inherent stability robustness to unknown residual uncertainty. This chapter will focus on the analysis of the robustness of full state feedback controlled systems with modelled parametric uncertainties; both the ability to remain stable and maintain a prescribed level of performance are assessed. Some robustness bounds are developed for a given closed loop system, these bounds describe the largest magnitude of uncertainty for which the system can both be guaranteed to remain stable and to satisfy a given performance criterion. Furthermore, a maximal robustness bound will be developed in this chapter. This is a sufficient condition for the existence of an Robust LQR controller produced by the technique of chapter 4. The adoption of the resulting controller will guarantee closed loop stability and the adherence of the performance criterion for all admissible uncertainties. This robustness analysis technique enables the trade off between performance and robustness to be quantitatively assessed.

A standard approach in robust controller design is to use the magnitude of the uncertainty bound as a design variable, thus for all admissible uncertainties a (minimal) performance bound is offered. The robustness analysis technique developed here permits an alternative approach to controller design: a performance criterion may be specified and a robustness bound found which may then be used to specify the uncertainty magnitude for the design procedure. Thus a robust controller is designed using the performance criterion as a design variable and a (maximal) robustness bound is offered. It should be noted that though this approach may be taken iteratively using the standard approach, this method permits a direct one-step solution. It is also known that there is an inherent trade off between the level of performance that may be guaranteed and the magnitude of uncertainty for which such a guarantee is valid, ie. a trade off between robustness (magnitude of uncertainty bound) and performance (worst case performance bound). This robustness analysis technique provides a succinct method to quantitatively assess this trade off.

A recent technique paper published by Neto (1992) derives robustness bounds with respect to a parametric uncertainty, for uncertainties within these bounds stability is guaranteed. This concept is extended in this chapter to permit the guarantee of a performance criterion and generalised for a larger class of uncertainty structures. It will also be shown that Neto's result is a special cases when the requirement of performance robustness is released. The methodology described here is also described in the paper of Wei & Marsh (1995b).

5.1 Problem Statement

From the discussion in §2.2.4, only parametric uncertainties $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ affect the robustness of state feedback control system, the nonparametric uncertainties do not affect the robustness. Hence the uncertain systems to be studied here can be described by the following state-space representation:

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \Delta\mathbf{A})\mathbf{x}(t) + (\mathbf{B} + \Delta\mathbf{B})\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{5.1.1}$$

As in definition (4.1.2), uncertainties $\Delta\mathbf{A}, \Delta\mathbf{B}$ are norm bounded time-varying matrices which are assumed to lie in a measurable domain defined as:

$$\Pi = \left\{ \begin{array}{l} \Delta\mathbf{A} = \mathbf{N}_a \Phi_a(t) \mathbf{M}_a \;\; \bar{\sigma}(\Phi_a(t)) \le \varepsilon \\ \Delta\mathbf{B} = \mathbf{N}_b \Phi_b(t) \mathbf{M}_b \; : \bar{\sigma}(\Phi_b(t)) \le \varepsilon \end{array} \right\} \tag{5.1.2}$$

where $\mathbf{N}_a$, $\mathbf{N}_b$, $\mathbf{M}_a$, $\mathbf{M}_b$ are constant matrices describing the structure of the uncertainty, they could be identity matrices if no information of the uncertainty structure is known. The size of the uncertainty domain is described by a single parameter, ε, this is called the *robustness parameter*. In chapter 4, to design a robust controller, the magnitude of the uncertainty bound is given, so by suitable choice of $\mathbf{N}_a$, $\mathbf{N}_b$, $\mathbf{M}_a$, $\mathbf{M}_b$, the value of ε can be chosen as 1, but in this chapter, ε will be a variable.

The full state feedback control law $\mathbf{u} = \mathbf{K}\mathbf{x}$ will be considered in this chapter, and the cost performance is assessed by the quadratic cost criterion (2.3.5) as:

$$J = \int_0^\infty (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u}) dt \tag{5.1.3}$$

Where Q and R are assumed to be positive definite matrices. The performance of the uncertain system cannot generally be evaluated since the precise system description is unknown and it is common to consider a performance bound J_b. In this study it is proposed to characterise the performance of the uncertain system by a *performance parameter*, β, such that the following cost criterion is guaranteed to be satisfied for all admissible uncertainties:

$$J_b \le \beta J_0 \tag{5.1.4}$$

where J_0 is the optimal cost performance of the certain system, (when $\Delta\mathbf{A} = \Delta\mathbf{B} = 0$). So in this case, the performance degradation parameter is required to satisfy:

$$\rho \le (\beta - 1) \times 100\% .$$

The first objective relates to the analysis of the robustness of a given system; for a given uncertain system (5.1.1), controller $\mathbf{u}=\mathbf{K}\mathbf{x}$ and performance parameter β, determine a bound ε_K for ε for which the performance adheres to (5.1.4). This bound is called the *robustness bound* and a simple expression for it is developed in §5.2.1. It should be noted that this guarantee of adherence to (5.1.4) implies that the cost will be finite which is sufficient to guarantee closed loop stability, i.e., robustness requirements of stability and performance could be satisfied. Furthermore, if the guarantee of closed loop stability is the sole goal then a *stability robustness bound* should be found for this given controller by considering the limit as β tends to infinity.

The RLQR controller design methodology of chapter 4 will produce, if it exists, a controller which will provide a minimal performance bound. The second objective is, for a given performance parameter β, find a maximal robustness bound ε_M for ε for which the existence of an RLQR controller is guaranteed and furthermore, show that such a controller will guarantee the adherence of (5.1.4) for any $\varepsilon \leq \varepsilon_M$. It will further be shown that this bound is maximal, ie. $\varepsilon_M \geq \varepsilon_K$. This bound is developed in §5.4 and can be described by a simple expression dependent only on the system parameters and the optimal LQR controller for the certain system.

5.2 Robustness Bound for a Given Controller

For a given uncertain system (5.1.1), controller **u=Kx** and performance parameter β, a *robustness bound* ε_K for ε will be determined for which the performance adheres to (5.1.4). It will also be shown that this guarantee of adherence to (5.1.4) implies that the cost will be finite which is sufficient to guarantee closed loop stability.

Let us firstly recall the optimal LQR design for the certain part of system (5.1.1) from §3.1.1, then a minimal value $\mathbf{P}_0$ for $\mathbf{P}$ can be found by selecting:

$$\mathbf{K} = \mathbf{K}_0 = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_0 \tag{5.2.1}$$

where $\mathbf{P}_0$ is the unique positive definite solution of algebraic Riccati equation:

$$\mathbf{A}^T\mathbf{P}_0 + \mathbf{P}_0\mathbf{A} - \mathbf{P}_0\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_0 + \mathbf{Q} = 0 \tag{5.2.2}$$

and the optimal cost is given by

$$\mathbf{J}_0 = \mathbf{x}_0^T\mathbf{P}_0\mathbf{x}_0 \tag{5.2.3}$$

As in §4.2, for the uncertain system (5.1.1) with control matrix $\mathbf{K}$, the result of Lemma 2.4.1 can be used here to find the performance bound for the uncertain system. Since

$$\Delta\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A}_0 = \Delta\mathbf{A}^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A} + (\Delta\mathbf{B}\mathbf{K})^T\mathbf{P} + \mathbf{P}\Delta\mathbf{B}\mathbf{K}$$

According to the result of Lemma 2.4.1, for any positive definite matrix $\mathbf{P}$, constant matrix $\mathbf{K}$ and set of positive scalars $\{\alpha_1, \alpha_2, \beta\}$, it follows that:

$$\Delta A^T P + P\Delta A = (N_a \Phi_a(t) M_a)^T P + P(N_a \Phi_a(t) M_a)$$
$$\leq \varepsilon^2 \frac{\alpha_1}{\beta} P N_a N_a^T P + \frac{\beta}{\alpha_1} M_a^T M_a$$

and

$$(\Delta BK)^T P + P\Delta BK = (N_b \Phi_b(t) M_b K)^T P + P(N_b \Phi_b(t) M_b K)$$
$$\leq \varepsilon^2 \frac{\alpha_2}{\beta} P N_b N_b^T P + \frac{\beta}{\alpha_2} K^T M_b^T M_b K$$

so

$$(\Delta A + \Delta BK)^T P + P(\Delta A + \Delta BK)$$
$$\leq \varepsilon^2 P(\frac{\alpha_1}{\beta} N_a N_a^T + \frac{\alpha_2}{\beta} N_b N_b^T)P + \frac{\beta}{\alpha_1} M_a^T M_a + \frac{\beta}{\alpha_2} K^T M_b^T M_b K$$

where W_1, W_2 and W_3 are defined by:

$$W_1 = \frac{\varepsilon^2}{\beta}(\alpha_1 N_a N_a^T + \alpha_2 N_b N_b^T), \quad W_2 = \frac{\beta}{\alpha_1} M_a^T M_a, \quad W_3 = \frac{\beta}{\alpha_2} M_b^T M_b \tag{5.2.4}$$

So a bound function Θ can be found as:

$$\Delta A_0^T P + P\Delta A_0 \leq PW_1P + W_2 + K^T W_3 K \stackrel{\Delta}{=} \Theta(M_a, M_b, N_a, N_b, P) \tag{5.2.5}$$

Thus providing an alternative description of the admissible domain of $\Delta A, \Delta B$. It should be noted however, that the size of the domain is still described by the robustness parameter ε.

According to Lemma 4.2.1, a performance bound for the uncertain system may be found using the Lemma 2.4.4 to the uncertain system (5.1.1) with cost performance index (5.1.3) and a given stabilising control law **u=Kx**, if the following Riccati equation

$$(A+BK)^T P_b + P_b(A+BK) + P_b W_1 P_b + W_2 + Q + K^T(R+W_3)K = 0 \tag{5.2.6}$$

has a positive solution $P_b>0$ for all admissible ($\Delta A, \Delta B$), then the cost is bounded by

$$J \leq J_b = x_0^T P_b x_0 \tag{5.2.7}$$

Before the main result, it is necessary to introduce a new lemma which is used throughout this chapter to compare the solutions of two modified Riccati equations.

Lemma 5.2.1 For the following two modified Riccati expressions with any $\mathbf{T_0} > \mathbf{0}$:

$$\mathbf{A^T P_1 + P_1 A - P_1 M M^T P_1 + P_1 N N^T P_1 + T_0 \le 0} \tag{5.2.8}$$

$$\mathbf{A^T P_2 + P_2 A - P_2 M M^T P_2 + P_2 N N^T P_2 + T_0 = 0} \tag{5.2.9}$$

if there exists $\mathbf{P_1} > \mathbf{0}$ which satisfies inequality (5.2.8), then

(i) Equation (5.2.9) will have a positive definite solution $\mathbf{P_2} > \mathbf{0}$.

(ii) $\mathbf{P_2} \le \mathbf{P_1}$

Proof: Lemma 4.2.2 can be used to prove this result.
Defining $\mathbf{L^T L} \ge \mathbf{0}$, $\mathbf{T_0} = \mathbf{Q}$, $\mathbf{M M^T} = \mathbf{B \overline{R}^{-1} B^T}$, so (5.2.8) and (5.2.9) can be rewritten as

$$\mathbf{A^T P_1 + P_1 A - P_1 B \overline{R}^{-1} B^T P_1 + P_1 N_1 N_1^T P_1 + Q + L^T L = 0} \tag{5.2.10}$$

$$\mathbf{A^T P_2 + P_2 A - P_2 B \overline{R}^{-1} B^T P_2 + P_2 N N^T P_2 + Q = 0} \tag{5.2.11}$$

Recall definition (4.2.11) as:

$$\mathbf{Z(K,P) = [K + \overline{R}^{-1} B^T P]^T \overline{R} [K + \overline{R}^{-1} B^T P] \ge 0} \tag{5.2.12}$$

After choosing:

$$\mathbf{K_1 = \overline{R}^{-\frac{1}{2}} L - \overline{R}^{-1} B^T P} \tag{5.2.13}$$

$$\mathbf{K_2 = -\overline{R}^{-1} B^T P} \tag{5.2.14}$$

then

$$\mathbf{Z_1(K_1, P) = L^T L} \text{ and } \mathbf{Z_2(K_2, P) = 0}$$

So for any $\mathbf{P} > \mathbf{0}$, it follows that

$$\mathbf{Z_1(K_1, P) \ge Z_2(K_2, P)} \tag{5.2.15}$$

then (5.2.10) and (5.2.11) can also be rewritten as

$$\mathbf{A^T P_1 + P_1 A - P_1 B \overline{R}^{-1} B^T P_1 + P_1 N_1 N_1^T P_1 + Q + Z_1(K_1, P_1) = 0} \tag{5.2.16}$$

$$\mathbf{A^T P_2 + P_2 A - P_2 B \overline{R}^{-1} B^T P_2 + P_2 N N^T P_2 + Q + Z_2(K_2, P_2) = 0} \tag{5.2.17}$$

Hence Lemma 5.2.1 can now be proved using (5.2.16), (5.2.17) and (5.2.15) to Lemma 4.2.2. ◆

Following two kinds of bound will be studied, one is robustness bound which consider both robustness of stability and performance, the other is the robust stability bound which only consider the robust stability.

5.2.1 Robustness bound

In the following Theorem the results of Lemma 5.2.1 are used to derive an expression for the robustness bound ε_K:

Theorem 5.2.2 For the uncertain system (5.1.1) with a given control law **u=Kx** and a given *finite positive parameter* β such that $\mathbf{A}_0 = \mathbf{A} + \mathbf{BK}$ is stable, if there exist positive parameters α_1, α_2 such that $\mathbf{\Phi}_K > \mathbf{0}$ and the robustness parameter ε satisfies:

$$\varepsilon \le \varepsilon_K \tag{5.2.18}$$

then: (1) the uncertain system will be robustly stabilised by this given controller.

(2) $\mathbf{P}_b \le \beta \mathbf{P}_0$, i.e., $J_b \le \beta J_0$

where ($\overline{\lambda}$ denotes maximum eigenvalue)

$$\varepsilon_K = 1/\overline{\lambda}^{1/2}(\Omega \Phi_K^{-1}) \tag{5.2.19}$$

$$\Omega = \mathbf{P}_0(\alpha_1 \mathbf{N}_a \mathbf{N}_a^T + \alpha_2 \mathbf{N}_b \mathbf{N}_b^T)\mathbf{P}_0 \tag{5.2.20}$$

$$\begin{aligned}\Phi_K = {} & (1-\frac{1}{\beta})\mathbf{K}^T\mathbf{RK} + (1-\frac{1}{\beta})\mathbf{Q} - \frac{1}{\alpha_1}\mathbf{M}_a^T\mathbf{M}_a \\ & -(\mathbf{K}-\mathbf{K}_0)^T\mathbf{R}(\mathbf{K}-\mathbf{K}_0) - \frac{1}{\alpha_2}\mathbf{K}^T\mathbf{M}_b^T\mathbf{M}_b\mathbf{K}\end{aligned} \tag{5.2.21}$$

Proof: Firstly it is assumed that the robustness parameter ε satisfies (5.2.18) and then it is shown that this guarantees $\mathbf{P}_b \le \beta \mathbf{P}_0$ and hence (5.1.4) holds.

From Theorem 7.7.3 of Horn and Johnson (1991) if **A>0** and $\mathbf{B} \ge 0$, then $\mathbf{B} \le \mathbf{A}$ if and only if $\overline{\lambda}(\mathbf{BA}^{-1}) \le 1$. So the definition (5.2.19) enables the condition (5.2.18) for ε to be expressed as

$$\varepsilon^2 \Omega \le \Phi_K \tag{5.2.22}$$

From (5.2.6), (5.2.20) and (5.2.21) it follows directly that

$$\varepsilon^2 \Omega = \beta \mathbf{P}_0 \mathbf{W}_1 \mathbf{P}_0$$

and

$$\Phi_K = (1-\frac{1}{\beta})\mathbf{K}^T\mathbf{RK} + (1-\frac{1}{\beta})\mathbf{Q} - \frac{\mathbf{W}_2}{\beta} - (\mathbf{K}-\mathbf{K}_0)^T\mathbf{R}(\mathbf{K}-\mathbf{K}_0) - \frac{1}{\beta}\mathbf{K}^T\mathbf{W}_3\mathbf{K}$$

substituting this into (5.2.22) gives

$$\beta^2\mathbf{P}_0\mathbf{W}_1\mathbf{P}_0 + \mathbf{W}_2 + \mathbf{K}^T\mathbf{W}_3\mathbf{K} \le (\beta-1)\mathbf{K}^T\mathbf{RK} - \beta(\mathbf{K}-\mathbf{K}_0)^T\mathbf{R}(\mathbf{K}-\mathbf{K}_0) + (\beta-1)\mathbf{Q} \tag{5.2.23}$$

and collecting terms

$$\begin{aligned}& \beta^2\mathbf{P}_0\mathbf{W}_1\mathbf{P}_0 + \mathbf{W}_2 + \mathbf{K}^T(\mathbf{W}_3+\mathbf{R})\mathbf{K} + \mathbf{Q} + \beta\mathbf{K}^T\mathbf{B}^T\mathbf{P}_0 + \beta\mathbf{P}_0\mathbf{BK} \le \\ & \quad -\beta\mathbf{P}_0\mathbf{BR}^{-1}\mathbf{B}^T\mathbf{P}_0 + \beta\mathbf{Q}\end{aligned} \tag{5.2.24}$$

From (5.2.2) it follows that for any finite parameter β:

$$\beta(\mathbf{A}^T\mathbf{P}_0 + \mathbf{P}_0\mathbf{A} - \mathbf{P}_0\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_0 + \mathbf{Q}) = 0 \tag{5.2.25}$$

allowing (5.2.24) to be simplified to

$$(\mathbf{A}+\mathbf{BK})^T\beta\mathbf{P}_0 + \beta\mathbf{P}_0(\mathbf{A}+\mathbf{BK}) + \beta^2\mathbf{P}_0\mathbf{W}_1\mathbf{P}_0 + \mathbf{W}_2 + \mathbf{Q} + \mathbf{K}^T(\mathbf{W}_3+\mathbf{R})\mathbf{K} \le 0 \tag{5.2.26}$$

The preceding steps are simply a transformation of the condition for ε in (5.2.18); ε is expressed through $\mathbf{W}_1$ and for any ε satisfying (5.2.18) the inequality (5.2.26) will hold. This inequality may be considered as a Riccati expression in $\beta\mathbf{P}_0$ similar to (5.2.10) which is positive by definition (5.2.2).

Let us now consider the performance bound for the system for any given value of ε. The uncertainty bound can be described through (5.2.6) and (5.2.7) and from Corollary 4.2.1, if the following Riccati equation has a solution $\mathbf{P}_b > 0$,

$$(\mathbf{A}+\mathbf{BK})^T\mathbf{P}_b + \mathbf{P}_b(\mathbf{A}+\mathbf{BK}) + \mathbf{P}_b\mathbf{W}_1\mathbf{P}_b + \mathbf{W}_2 + \mathbf{Q} + \mathbf{K}^T(\mathbf{W}_3+\mathbf{R})\mathbf{K} = 0 \tag{5.2.27}$$

then $\mathbf{P}_b$ is a performance bound matrix.

By setting **M=0**, Lemma 5.2.1 can now be used to compare the Riccati expressions (5.2.26) and (5.2.27); since for an ε satisfying (5.2.18) there exists $\beta\mathbf{P}_0>0$ which satisfies (5.2.26), then equation (5.2.27) will have a positive solution $\mathbf{P}_b > 0$, and furthermore,

$$\mathbf{P}_b \le \beta\mathbf{P}_0 \tag{5.2.28}$$

So to summarise, if ε satisfies (5.2.18) then (5.2.26) will hold and (5.2.28) is implied which is sufficient to guarantee (5.1.4) for any $\mathbf{x}_0$. Furthermore, if (5.2.27) has a positive definite solution, then this solution also satisfies:

$$(\mathbf{A}+\mathbf{BK}+\Delta\mathbf{A}+\Delta\mathbf{BK})^T\mathbf{P}_b + \mathbf{P}_b(\mathbf{A}+\mathbf{BK}+\Delta\mathbf{A}+\Delta\mathbf{BK}) < \mathbf{0}$$

Above expression can be obtained by adding (5.2.27) to (5.2.7), from Corollary 2.3.1 it follows that if (5.2.27) has a positive definite solution, then the uncertain system (5.1.1) can be robustly stabilised by this given controller **u=Kx**. ◆

Thus, for a given full state feedback control law, the performance robustness bound ε_K is given in (5.2.19). The scalar parameters α_1 and α_2 may be searched to maximise ε_K.

5.2.2 Robust stability bound

Robust stability is often a core goal of control design, so it is necessary to consider the bound of uncertainty for which a given controller can guarantee robust stability. Firstly, let us consider the relationship between the robustness bound ε_K and the performance parameter β, the following result shows that if β is increased ε_K also increases.

Lemma 5.2.3 For a given β if there exist γ_1, γ_2 such that $\Phi_K(\beta) > \mathbf{0}$, then for any $\beta_1 > \beta$ these γ_1, γ_2 give $\Phi_K(\beta_1) > \mathbf{0}$ and $\varepsilon_K(\beta_1) > \varepsilon_K(\beta)$.

Proof: Since $\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K} > \mathbf{0}$ and from (5.2.21) it follows that

$$\Phi_K(\beta_1) - \Phi_K(\beta) = (\frac{1}{\beta} - \frac{1}{\beta_1})(\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}) \geq 0 \qquad (5.2.29)$$

so If $\beta \leq \beta_1$, then $\Phi_K(\beta_1) \geq \Phi_K(\beta)$.

From the definition in (5.2.20) and since and $\Omega \geq 0$, $\Phi_K(\beta_1)$ and $\Phi_K(\beta)$ are positive definite matrices, it follows that

$$\begin{aligned} \Phi_K(\beta_1) \geq \Phi_K(\beta) \quad &\Rightarrow \quad \Phi_K^{-1}(\beta_1) \leq \Phi_K^{-1}(\beta) \\ &\Rightarrow \quad \Omega^{1/2}\Phi_K^{-1}(\beta_1)\Omega^{1/2} \leq \Omega^{1/2}\Phi_K^{-1}(\beta)\Omega^{1/2} \end{aligned}$$

hence

$$\bar{\lambda}(\Omega^{1/2}\Phi_K^{-1}(\beta_1)\Omega^{1/2}) \leq \bar{\lambda}(\Omega^{1/2}\Phi_K^{-1}(\beta)\Omega^{1/2})$$

and

$$\bar{\lambda}(\Omega\Phi_K^{-1}(\beta_1)) \leq \bar{\lambda}(\Omega\Phi_K^{-1}(\beta))$$

Hence from the definition (5.2.19) it can be deduced that $\varepsilon_K(\beta) \leq \varepsilon_K(\beta_1)$. ◆

From this result it is clear that the robustness bound for a given controller is maximised in the limit as β tends to infinity. No performance guarantee may be given as Theorem 5.2.2 applies for finite β only, but the following Lemma will show that for any finite β and $\varepsilon \leq \varepsilon_K$ the cost value is guaranteed to be finite, *this is sufficient to guarantee closed loop stability*, hence a *stability robustness bound* is presettled.

Lemma 5.2.4 For the uncertain system (5.1.1) with a given control law **u=Kx** such that $\mathbf{A}_0 = \mathbf{A} + \mathbf{BK}$ is stable, if there exist positive parameters α_1, α_2 such that $\Phi_K > 0$ and the robustness parameter ε satisfies

$$\varepsilon < \varepsilon_{KS} \tag{5.2.30}$$

then the given control law can stabilise the uncertain system.

Where Ω is defined as in (5.2.20) and

$$\varepsilon_{KS} = \lim_{\beta\to\infty}(\varepsilon_K) = 1\Big/\bar{\lambda}^{\frac{1}{2}}(\Omega\Phi_{KS}^{-1}) \tag{5.2.31}$$

$$\begin{aligned}\Phi_{KS} = \lim_{\beta\to\infty}(\Phi_K) = \mathbf{K}^T\mathbf{RK} + \mathbf{Q} - \frac{1}{\alpha_1}\mathbf{M}_a^T\mathbf{M}_a - \frac{1}{\alpha_2}\mathbf{K}^T\mathbf{M}_b^T\mathbf{M}_b\mathbf{K} \\ -(\mathbf{K}-\mathbf{K}_0)^T\mathbf{R}(\mathbf{K}-\mathbf{K}_0)\end{aligned} \tag{5.2.32}$$

Proof: The initial stages of the proof of Theorem 5.2.2 may be repeated to show that $\varepsilon < \varepsilon_{KS} \Rightarrow \mu^2\Omega < \Phi_{KS}$. Substituting for Ω and Φ_{KS} from (5.2.20) and (5.2.32) respectively gives

$$\begin{aligned}\mu^2\mathbf{P}_0(\alpha_1\mathbf{N}_a\mathbf{N}_a^T + \alpha_2\mathbf{N}_b\mathbf{N}_b^T)\mathbf{P}_0 + \frac{1}{\alpha_1}\mathbf{M}_a^T\mathbf{M}_a + \frac{1}{\alpha_2}\mathbf{K}^T\mathbf{M}_b^T\mathbf{M}_b\mathbf{K} \\ < \mathbf{K}^T\mathbf{RK} + \mathbf{Q} - (\mathbf{K}-\mathbf{K}_0)^T\mathbf{R}(\mathbf{K}-\mathbf{K}_0)\end{aligned} \tag{5.2.33}$$

From Lemma 2.4.1 we can derive

$$\begin{aligned}&(\Delta\mathbf{A} + \Delta\mathbf{BK})^T\mathbf{P}_0 + \mathbf{P}_0(\Delta\mathbf{A} + \Delta\mathbf{BK}) \\ &\quad \le \mu^2\mathbf{P}_0(\alpha_1\mathbf{N}_a\mathbf{N}_a^T + \alpha_2\mathbf{N}_b\mathbf{N}_b^T)\mathbf{P}_0 + \frac{1}{\alpha_1}\mathbf{M}_a^T\mathbf{M}_a + \frac{1}{\alpha_2}\mathbf{K}^T\mathbf{M}_b^T\mathbf{M}_b\mathbf{K}\end{aligned}$$

hence from (5.2.33)

$$(\Delta\mathbf{A} + \Delta\mathbf{BK})^T\mathbf{P}_0 + \mathbf{P}_0(\Delta\mathbf{A} + \Delta\mathbf{BK}) < \mathbf{K}^T\mathbf{RK} + \mathbf{Q} - (\mathbf{K}-\mathbf{K}_0)^T\mathbf{R}(\mathbf{K}-\mathbf{K}_0)$$

expanding and substituting for $\mathbf{K}_0$ from (5.2.1) gives

$$(\Delta\mathbf{A} + \Delta\mathbf{BK} + \mathbf{BK})^T\mathbf{P}_0 + \mathbf{P}_0(\Delta\mathbf{A} + \Delta\mathbf{BK} + \mathbf{BK}) < \mathbf{Q} - \mathbf{P}_0\mathbf{BR}^{-1}\mathbf{B}^T\mathbf{P}_0$$

Adding $\mathbf{A}^T\mathbf{P}_0 + \mathbf{P}_0\mathbf{A}$ to both sides and substituting from (5.2.2) gives

$$\mathbf{A}_c{}^T\mathbf{P}_0 + \mathbf{P}_0\mathbf{A}_c < \mathbf{A}^T\mathbf{P}_0 + \mathbf{P}_0\mathbf{A} + \mathbf{Q} - \mathbf{P}_0\mathbf{BR}^{-1}\mathbf{B}^T\mathbf{P}_0 = \mathbf{0} \tag{5.2.34}$$

Proposing $V = \mathbf{x}^T\mathbf{P}_0\mathbf{x} > 0$ as a Lyapunov function; since $\dot{\mathbf{x}} = \mathbf{A}_c\mathbf{x}$ and from (5.2.34) it can be deduced that $\dot{V} = \mathbf{x}^T(\mathbf{A}_c{}^T\mathbf{P}_0 + \mathbf{P}_0\mathbf{A}_c)\mathbf{x} < 0$, hence V is a valid Lyapunov function and the system is guaranteed to be stable. (also from Corollary 2.3.1) ◆

NB. An alternative proof is possible by extending Theorem 5.2.2 to show that $\varepsilon < \varepsilon_K(\beta) \Rightarrow \mathbf{P}_b < \beta\mathbf{P}_0$, hence for $\varepsilon < \lim_{\beta\to\infty}(\varepsilon_K(\beta))$, $\mathbf{P}_b$ is finite and the system is stable.

So, a stability robustness bound will be found from the above Lemma, and from the result of Lemma 5.3.2, it always follows that: $\varepsilon_K \le \varepsilon_{KS}$.

5.3 ANALYSIS AND SYNTHESIS

Two particular controllers: optimal LQR and optimal robust LQR controller will be used in this section to demonstrate the application of results derived in above section. Their robustness bounds and robust stability bounds will be found for the uncertain system (5.1.1) with norm bounded uncertainties described as (5.1.2). These bounds will also be related to the robustness condition of §3.1.1.

5.3.1 Robustness bound for optimal LQR control system

The optimal LQR controller for the certain system is commonly proposed, hence it is very interesting to study it's stability robustness and performance degradation and relate it to the stability robustness analysis described in this chapter. Here the optimal LQR controller is used to evaluate the robustness bound ε_L and the stability robustness bound ε_{LS} for this special case. The following results can be found directly from Theorem 5.2.2 by choosing the given controller as the optimal LQR controller (5.2.1).

Corollary 5.3.1 For the uncertain system (5.1.1) with optimal LQR controller (5.2.1), if there exist positive parameters α_1, α_2 such that $\Phi_L > 0$ and the robustness parameter ε satisfies

$$\varepsilon \le \varepsilon_L \tag{5.3.1}$$

then: (1) uncertain system will be robustly stabilised by LQR controller.

(2) $\mathbf{P}_b \le \beta \mathbf{P}_0$, i.e., $J_b \le \beta J_0$

Where Ω was defined in (5.2.20) and

$$\varepsilon_L = 1/\bar{\lambda}^{1/2}(\Omega \Phi_L^{-1}) \tag{5.3.2}$$

$$\Phi_L = (1-\frac{1}{\beta})\mathbf{K}_0^T\mathbf{R}\mathbf{K}_0 + (1-\frac{1}{\beta})\mathbf{Q} - \frac{1}{\alpha_1}\mathbf{M}_a^T\mathbf{M}_a - \frac{1}{\alpha_2}\mathbf{K}_0^T\mathbf{M}_b^T\mathbf{M}_b\mathbf{K}_0 \tag{5.3.3}$$

Furthermore, a *stability robustness bound* for LQR controller can be found as:

$$\varepsilon \le \varepsilon_{LS} \tag{5.3.4}$$

where

$$\varepsilon_{LS} = \lim_{\beta \to \infty}(\varepsilon_L) = 1/\bar{\lambda}^{1/2}(\Omega \Phi_{LS}^{-1}) \tag{5.3.5}$$

$$\Phi_{LS} = \lim_{\beta\to\infty}(\Phi_L) = \mathbf{K}_0^T\mathbf{R}\mathbf{K}_0 + \mathbf{Q} - \frac{1}{\alpha_1}\mathbf{M}_a^T\mathbf{M}_a - \frac{1}{\alpha_2}\mathbf{K}_0^T\mathbf{M}_b^T\mathbf{M}_b\mathbf{K}_0$$

This stability robustness bound may be related to the stability robustness analysis method developed in §3.1.1. If the robustness parameter ε satisfies the stability robustness bound (5.3.4) then:

$$\varepsilon^2\Omega \le \Phi_{LS} \quad \text{i.e.}$$

$$\varepsilon^2\mathbf{P}_0(\alpha_1\mathbf{N}_a\mathbf{N}_a^T + \alpha_2\mathbf{N}_b\mathbf{N}_b^T)\mathbf{P}_0 \le \mathbf{K}_0^T\mathbf{R}\mathbf{K}_0 + \mathbf{Q} - \frac{1}{\alpha_1}\mathbf{M}_a^T\mathbf{M}_a - \frac{1}{\alpha_2}\mathbf{K}_0^T\mathbf{M}_b^T\mathbf{M}_b\mathbf{K}_0 \tag{5.3.6}$$

Defining the uncertainty description of the closed loop system as:

$$\Delta\mathbf{A}_0 = \mathbf{N}_a\Phi_a(t)\mathbf{M}_a + \mathbf{N}_b\Phi_b(t)\mathbf{M}_b\mathbf{K} \stackrel{\Delta}{=} \mathbf{N}\Phi(t)\mathbf{M}$$

where $\mathbf{N} \stackrel{\Delta}{=} \sqrt{\alpha_1}\begin{bmatrix}\mathbf{N}_a & \sqrt{\frac{\alpha_2}{\alpha_1}}\mathbf{N}_b\end{bmatrix}$, $\Phi(t) \stackrel{\Delta}{=} \begin{bmatrix}\Phi_a(t) & 0 \\ 0 & \Phi_b(t)\end{bmatrix}$, $\mathbf{M} \stackrel{\Delta}{=} \sqrt{\frac{1}{\alpha_1}}\begin{bmatrix}\mathbf{M}_a \\ \sqrt{\frac{\alpha_1}{\alpha_2}}\mathbf{M}_b\mathbf{K}_0\end{bmatrix}$

Using the above definitions, (5.3.6) can be rewritten as:

$$\alpha_1\varepsilon^2\mathbf{P}_0\mathbf{N}\mathbf{N}^T\mathbf{P}_0 + \frac{1}{\alpha_1}\mathbf{M}^T\mathbf{M} \le \mathbf{K}_0^T\mathbf{R}\mathbf{K}_0 + \mathbf{Q}$$

From the definition for $\mathbf{K}_0$ (5.2.1) it follows that:

$$\alpha_1\varepsilon^2\mathbf{P}_0\mathbf{N}\mathbf{N}^T\mathbf{P}_0 + \frac{1}{\alpha_1}\mathbf{M}^T\mathbf{M} \le \mathbf{P}_0\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_0 + \mathbf{Q} \tag{5.3.7}$$

This is directly comparable with the robust stability condition (3.1.7).

$$\alpha\varepsilon\mathbf{M}^T\mathbf{M} < \mathbf{Q}; \quad \frac{\varepsilon}{\alpha}\mathbf{N}\mathbf{N}^T \le \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T$$

It is clear that above condition could be a special case of (5.3.7) with $\alpha_1 = \frac{1}{\alpha\varepsilon}$. If there exists a positive α such that above conditions are satisfied, then condition (5.3.7) will also be satisfied. It can also be shown that the stability robustness bound ε_{LS} which is given in (5.3.4) for the uncertain system with the optimal LQR controller is similar to that developed in Neto et al. (1992). Thus the result described here is less conservative than traditional method the described in §3.1.1.

5.3.2 Robustness bound for optimal robust LQR control system

A optimal robust LQR controller design method has been developed in chapter 4 and it is proved that RLQR can provide both good stability and performance robustness. In this section, robustness analysis will be made for the uncertain system (5.1.1) with the optimal RLQR controller (4.2.26). Since the design of RLQR controller requires the specification of a bound of uncertain parameter, so it will be very useful and interesting to find a bound for the uncertain parameter such that there exists a optimal RLQR controller which can stabilise the uncertain system (5.1.1) and satisfy the performance degradation requirement.

Consider the performance bound for the system for any given value of ε; for a given controller **u=Kx**, the performance bound will be given by the solution $\mathbf{P}_b$ of (5.2.8) if $\mathbf{P}_b > \mathbf{0}$. The RLQR controller $\mathbf{u} = \mathbf{K}_r\mathbf{x}$ for the uncertain system (5.1.1) is given (Theorem 4.3.4) by

$$\mathbf{K}_r = -(\mathbf{R} + \mathbf{W}_3)^{-1}\mathbf{B}^T\mathbf{P}_m \tag{5.3.8}$$

if the following equation has a positive definite solution $\mathbf{P}_m > \mathbf{0}$:

$$\mathbf{A}^T\mathbf{P}_m + \mathbf{P}_m\mathbf{A} - \mathbf{P}_m\mathbf{B}(\mathbf{R} + \mathbf{W}_3)^{-1}\mathbf{B}^T\mathbf{P}_m + \mathbf{P}_m\mathbf{W}_1\mathbf{P}_m + \mathbf{W}_2 + \mathbf{Q} = \mathbf{0} \tag{5.3.9}$$

NB. It can also be shown that $\mathbf{P}_m$ is a minimal performance bound matrix, ie. $\mathbf{P}_m \le \mathbf{P}_b$.

Then the robustness bound ε_{RL} is developed in the following theorem:

Theorem 5.3.2 For the uncertain system (5.1.1) and performance parameter β, if there exist positive parameters α_1, α_2 such that $\Phi_{RL} > 0$ and the robustness parameter ε satisfies

$$\varepsilon \le \varepsilon_{RL} \tag{5.3.10}$$

then an RLQR controller (5.3.8) exists such that:

(1) the uncertain system will be robustly stabilised by LQR controller.

(2) $\mathbf{P}_b \le \beta\mathbf{P}_0$, i.e., $J_b \le \beta J_0$

Where Ω is defined as in (5.2.20) and

$$\varepsilon_{RL} = 1/\bar{\lambda}^{1/2}(\Omega\Phi_{RL}^{-1}) \tag{5.3.11}$$

$$\Phi_{RL} = (1 - \frac{1}{\beta})\mathbf{Q} + \mathbf{K}_0^T\mathbf{R}(\frac{\mathbf{R}}{\beta} + \frac{\mathbf{M}_b^T\mathbf{M}_b}{\alpha_2})^{-1}\mathbf{R}\mathbf{K}_0 - \mathbf{K}_0^T\mathbf{R}\mathbf{K}_0 - \frac{\mathbf{M}_a^T\mathbf{M}_a}{\alpha_1} \tag{5.3.12}$$

Proof: The proof is similar to that for Theorem 5.2.2, firstly it is assumed that the robustness parameter ε satisfies (5.3.10) and then it is shown that this guarantees the existence of an RLQR controller which if employed guarantees that (5.1.4) will be satisfied.

Following the initial steps of the proof of Theorem 5.2.2, condition (5.3.10) may be rewritten as

$$\varepsilon^2\Omega \le \Phi_{RL} \tag{5.3.13}$$

From (5.2.20) and (5.2.6) it follows that

$$\varepsilon^2\Omega = \beta \mathbf{P}_0 \mathbf{W}_1 \mathbf{P}_0$$

and (5.3.12) gives

$$\Phi_{RL} = (1-\frac{1}{\beta})\mathbf{Q} + \beta \mathbf{K}_0^T \mathbf{R}(\mathbf{R}+\mathbf{W}_3)^{-1}\mathbf{R}\mathbf{K}_0 - \mathbf{K}_0^T\mathbf{R}\mathbf{K}_0 - \frac{\mathbf{W}_2}{\beta}$$

Substituting these into (5.3.13) gives

$$\beta^2\mathbf{P}_0\mathbf{W}_1\mathbf{P}_0 - \beta^2\mathbf{P}_0\mathbf{B}(\mathbf{R}+\mathbf{W}_3)^{-1}\mathbf{B}^T\mathbf{P}_0 + \mathbf{W}_2 + \mathbf{Q} \le -\beta\mathbf{K}_0^T\mathbf{R}\mathbf{K}_0 + \beta\mathbf{Q}$$

again (5.2.25) allows this to be simplified to

$$\mathbf{A}^T\beta\mathbf{P}_0 + \beta\mathbf{P}_0\mathbf{A} - \beta^2\mathbf{P}_0\mathbf{B}(\mathbf{R}+\mathbf{W}_3)^{-1}\mathbf{B}^T\mathbf{P}_0 + \beta^2\mathbf{P}_0\mathbf{W}_1\mathbf{P}_0 + \mathbf{W}_2 + \mathbf{Q} \le 0 \tag{5.3.14}$$

which is again simply a transformation of the condition for ε (5.3.10) and may be considered as a Riccati expression in $\beta\mathbf{P}_0$ similar to (5.2.10).

Lemma 5.2.1 can again be used to compare the modified Riccati expressions (5.3.14) and (5.3.9); since for ε satisfying (5.3.10) there exists $\beta\mathbf{P}_0>0$ which satisfies (5.3.14) then part (i) states that equation (5.3.9) will have a positive solution $\mathbf{P}_m>\mathbf{0}$ which guarantees the existence of the RLQR controller and furthermore part (ii) gives

$$\mathbf{P}_m \le \beta\mathbf{P}_0 \tag{5.3.15}$$

So if ε satisfies (5.3.12) then (5.3.14) will hold and (5.3.15) is implied which is sufficient to guarantee (5.1.4) for any $\mathbf{x}_0$. ◆

If stability robustness is only considered the stability robustness bound ε_{RSL} can again be found such that if $\varepsilon \le \varepsilon_{RSL}$ then an RLQR controller exists such that the uncertain system will be robustly stabilised.

Corollary 5.3.2 If there exist positive parameters α_1, α_2 such that $\Phi_{RSL} > 0$ then there exists a maximal stability robustness bound ε_{RSL} given by

$$\varepsilon_{RSL} = \lim_{\beta\to\infty}(\varepsilon_{RL}) = \frac{1}{\overline{\lambda}^{1/2}(\Omega\Phi_{RSL}^{-1})} \tag{5.3.16}$$

such that if $\varepsilon < \varepsilon_{RSL}$ then the uncertain system is guaranteed to be stable, where

$$\Phi_{RSL} = \lim_{\beta\to\infty}(\Phi_{RL}) = \mathbf{Q} + \mathbf{K}_0^T\mathbf{R}\left(\frac{\mathbf{M}_b^T\mathbf{M}_b}{\alpha_2}\right)^{-1}\mathbf{R}\mathbf{K}_0 - \mathbf{K}_0^T\mathbf{R}\mathbf{K}_0 - \frac{\mathbf{M}_a^T\mathbf{M}_a}{\alpha_1} \tag{5.3.17}$$

β and Ω was defined in (5.2.20)

Proof: Firstly we can prove that for given β if there exist α_1, α_2 such that $\Phi_{RSL} > 0$ then for any $\varepsilon_{RLS}(\beta_1) < \varepsilon_{RLS}(\beta)$ implies $\beta_1 < \beta$.

From the proof of Lemma 5.2.3, $\varepsilon_{RLS}(\beta_1) < \varepsilon_{RLS}(\beta)$ implies that $\Phi_{RLS}(\beta_1) < \Phi_{RLS}(\beta)$, i.e., $\Phi_{RLS}(\beta_1) - \Phi_{RLS}(\beta) < 0$, from the definition (5.3.20) it can be shown that:

$$(\frac{1}{\beta} - \frac{1}{\beta_1})\mathbf{Q} + \mathbf{K}_0^T\mathbf{R}[(\frac{\mathbf{R}}{\beta_1} + \frac{\mathbf{M}_b^T\mathbf{M}_b}{\alpha_2})^{-1} - (\frac{\mathbf{R}}{\beta} + \frac{\mathbf{M}_a^T\mathbf{M}_a}{\alpha_1})^{-1}]\mathbf{R}\mathbf{K}_0 < 0$$

and above relationship can only implies $\beta_1 < \beta$. Hence, for a ε with $\varepsilon < \varepsilon_{RLS}$, there will exist a finite β such that a RLQR controller can be designed for the uncertain system, which can guarantee the closed loop system to be stable and the cost criterion (5.1.4) will also be satisfied. ◆

5.4 The Maximal Robustness Bound

In this section a maximal robustness bound ε_M is developed for a given system and performance parameter β. It will be proved that the maximal robustness bound ε_M is provided by a RLQR controller. For any $\varepsilon \le \varepsilon_M$ a suitable controller is shown to exist and if employed will guarantee that the cost criterion (5.1.4) is satisfied. This robustness bound is independent of the actual controller employed and is shown to be a maximal bound, ie. $\varepsilon_M \ge \varepsilon_K$

It will now be shown that this robustness bound is maximal for all controllers.

Theorem 5.4.1 For the uncertain system (5.1.1) and performance parameter β, a *maximal robustness bound* ε_M can be found as

$$\varepsilon_M = \varepsilon_{RL} \ge \varepsilon_K \tag{5.4.1}$$

Proof: From (5.2.21) and (5.3.3), also by the use of (5.2.6), a similar relationship to (5.2.30) can be deduced

$$\begin{aligned}\Phi_{RL} - \Phi_K &= \mathbf{K}_0\mathbf{R}^{-1}(\mathbf{R}+\mathbf{W}_3)^{-1}\mathbf{R}^{-1}\mathbf{K}_0 - \mathbf{K}_0^T\mathbf{R}\mathbf{K} - \mathbf{K}^T\mathbf{R}\mathbf{K}_0 + \mathbf{K}^T(\mathbf{R}+\mathbf{W}_3)\mathbf{K}\\ &= \{\mathbf{K} - (\mathbf{R}+\mathbf{W}_3)^{-1}\mathbf{K}_0\}^T(\mathbf{R}+\mathbf{W}_3)\{\mathbf{K} - (\mathbf{R}+\mathbf{W}_3)^{-1}\mathbf{K}_0\}\end{aligned}$$

so $\quad \Phi_{RL} - \Phi_K \ge 0$

and a similar method to that used to prove Lemma 5.2.1 can be used to show:

$$\varepsilon_M = \varepsilon_{RL} \ge \varepsilon_K.$$ ◆

As for ε_K, it may be deduced that if β is increased ε_M also increases.

Lemma 5.4.2 For given β if there exist α_1, α_2 such that $\Phi_M(\beta) > 0$ then for any $\beta_1 > \beta$ these α_1, α_2 give $\Phi_M(\beta_1) > 0$ and $\varepsilon_M(\beta_1) > \varepsilon_M(\beta_2)$.

where $\Phi_M(\beta) = \Phi_{RL}(\beta)$

Proof: The proof follows that of Lemma 5.2.3, from the definition (4.6) it can be shown that $\Phi_M(\beta_1) > \Phi_M(\beta)$ which implies that $\varepsilon_M(\beta_1) > \varepsilon_M(\beta)$. ◆

If stability robustness is only considered the *maximal stability robustness bound* ε_M can again be found as

$$\varepsilon_{MS} = \varepsilon_{RSL} \geq \varepsilon_{KS} \tag{5.4.2}$$

The maximal bound of uncertain parameter such that there exist a robust LQR controller is ε_{MS}, and it always follows that:

$$\varepsilon_{MS} \geq \varepsilon_L \tag{5.4.3}$$

The trade off between performance and robustness of an uncertain system is illustrated in Figure 5.1. The maximal bound of the robustness parameter ε_M is plotted against the performance parameter β as the independent variable. This describes the robustness of the system by specifying the maximum size of the uncertainty for which the given performance criteria can be guaranteed.

Let us consider the robustness bound as β varies. For β just greater than 1 a strong cost criterion is specified. Here the performance bound of the uncertain system is only permitted to be slightly greater than the optimal performance of the certain system. For such performance parameters a small ε_M results. Hence, only for small uncertainties may the cost criterion be guaranteed. Conversely, for larger β a weaker cost criterion is specified and so the performance bound of the uncertain system is permitted to be higher, consequently larger ε_M result. In the limit as β tends to infinity, no performance criterion is given and the sole requirement is closed loop stability. ε_M will asymptotically approach its maximum value ε_M.

This analysis enables us to analyse the trade off between performance and robustness and choose a suitable value for β as a design criteria. There are two conceptually differing approaches to robust controller design. The one proposed in chapter 4 assumes that the design criteria will be posed as an uncertainty bound description ie. given ε, for which a controller is developed which offers a minimal performance bound for all admissible uncertainties. Thus ε is a design variable and a controller offering a maximal performance and hence maximal β results.

An alternative approach is to specify a performance criteria via β as a design criteria for the controller. To facilitate this, the method described here may be used to first calculate ε_M which may then be used as a design variable in the RLQR design method in chapter 4. The resulting controller is guaranteed to satisfy the specified performance criteria. It should be noted that the resulting performance bound may be significantly less than that given in the design criteria, hence the controller design may be conservative.

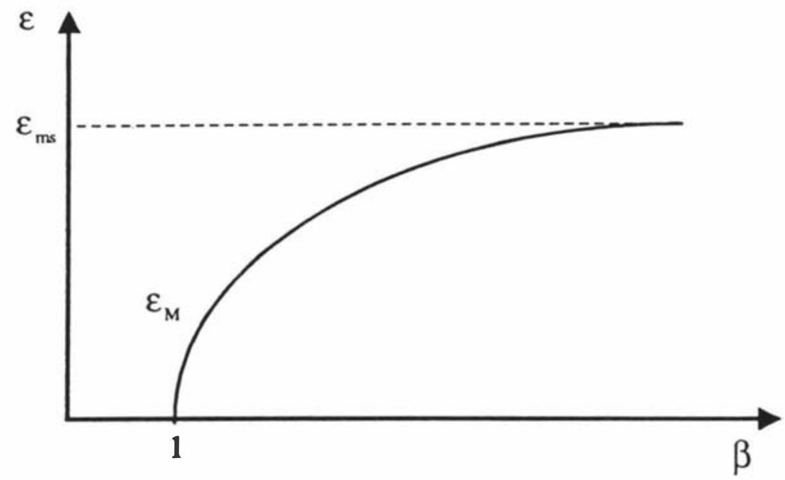

Fig. 5.1 Maximal Robustness Bound .V. Performance Parameter.

5.5 APPLICATION AND EXAMPLE

An example uncertain system is used to illustrate the calculation of robustness bounds for given values of performance parameter. Maximal robustness bounds and those offered by the optimal LQR controller for the underlying certain system will be compared. The trends over values of β demonstrate the trade off between performance and robustness.

The example uncertain system is a member of the class (5.2.1) with

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & 3\end{bmatrix},\quad \Delta\mathbf{A}(t)=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & -0.1\Lambda(t) & 0.2\Lambda(t)\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}0\\ 0\\ 1.5\end{bmatrix},\quad \Delta\mathbf{B}(t)=\begin{bmatrix}0\\ 0\\ 0.2\Lambda(t)\end{bmatrix}$$

The uncertainties are parameterised by a single time-varying process $\Lambda(t)$, which is bounded as $|\Lambda(t)|\le\varepsilon$, thus the admissible domain of uncertainty may be described by Π with $\mathbf{M}_a=\mathbf{M}_b=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$, $\mathbf{N}_a=\begin{bmatrix}0 & -0.1 & 0.2\end{bmatrix}$, $\mathbf{N}_b=0.2$

In the cost function, **Q** and **R** are appropriately sized identity matrices. Consider the particular case β=10, for

$$\mathbf{K}=\mathbf{K}_0=[-0.54, -2.16, -4.81]$$

a robustness bound is given by $\varepsilon_K = 1/\overline{\lambda}^{1/2}(\Omega\Phi_K^{-1})$ for $\Phi_K > 0$.

A gradient optimisation is performed over γ_1, γ_2 to maximise ε_K, this results in ε_K= 2.72 for γ_1=0.048 and γ_2=0.11. To validate the performance criteria, a performance bound $\mathbf{P}_b > 0$ may be evaluated from (5.2.8) using these values of γ_1, γ_2 and β, this will satisfy $\beta\mathbf{P}_0 - \mathbf{P}_b \geq 0$. The maximal robustness bound is given by $\varepsilon_M = 1/\overline{\lambda}^{1/2}(\Omega\Phi_M^{-1})$ for $\Phi_M > 0$, optimisation over γ_1, γ_2 results in $\varepsilon_M = 4.64$ for γ_1=0.042 and γ_2=0.25. To validate the existence of an RLQR controller Theorem 4.1 is employed for $\varepsilon=\varepsilon_M$= 4.64, choosing similar values for γ_1, γ_2 and β, gives a positive definite solution to (5.3.9) which is a minimal performance bound and $\mathbf{K}_r$=[-0.27, -5.93, -17.1], furthermore the performance criterion is satisfied as $\beta\mathbf{P}_0 - \mathbf{P}_m \geq 0$.

Values of ε_K and ε_M for various β are given in the following Table, this illustrates the trade off between performance and robustness. For small β, e.g. β=1.1, only small uncertainties are permitted and the robustness of the LQR controller is near maximal, however for larger β, larger uncertainties are permitted and the RLQR controller clearly offers significantly greater robustness. Furthermore, as $\beta \to \infty$, ε_K and ε_M approach the respective stability robustness bounds of ε_{KS}=3.02 and ε_{MS}=7.50.

β	1.01	1.1	1.5	2	5	10	10^5	$\lim_{\beta\to\infty}$
ε_K	0.03	0.27	1.00	1.51	2.42	2.72	3.02	3.02
ε_M	0.03	0.28	1.14	1.84	3.64	4.64	7.47	7.50

These maximal robustness bounds, as well as the trade-off between performance and robustness are also shown in Fig. 5.2.

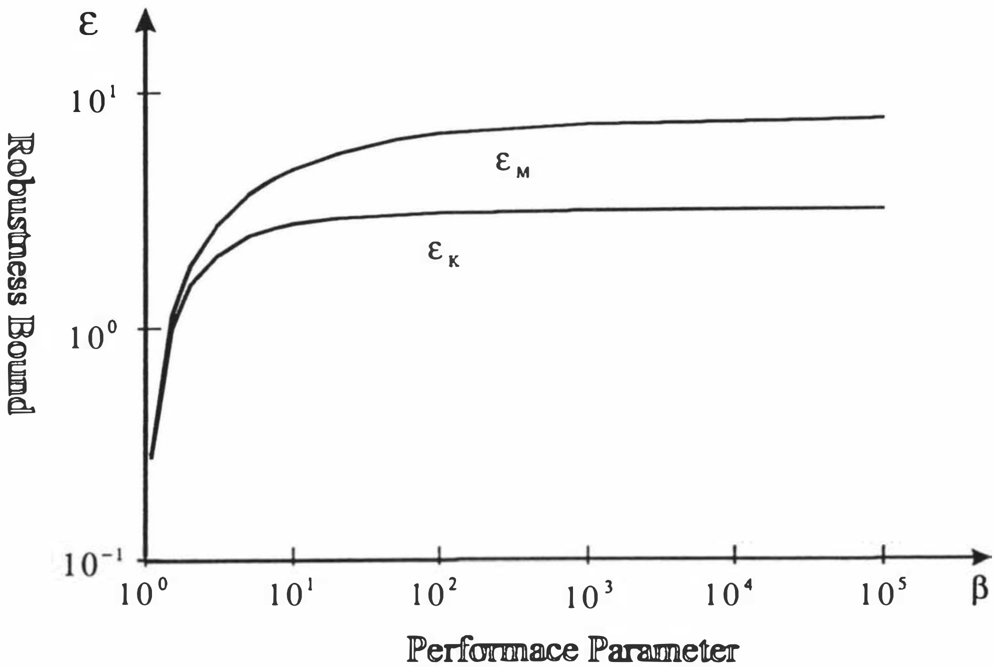

Fig. 5.2 Robustness Bound .V. Performance Parameter for Example System.

5.6 DISCUSSION

A robustness analysis procedure for a given closed loop system is presented. This produces a robustness bound with respect to a performance criterion such that for any uncertainty within this bound it is guaranteed that the performance criterion will be met. An expression for a maximal robustness bound for a given system subject to a performance criterion is developed. This bound is sufficient to guarantee the existence of an RLQR controller which enables the adherence of the performance criterion to be guaranteed.

The inherent trade off between robustness and performance may be quantitatively assessed using this robustness analysis method. It is shown, both in general and for a specific example system, that if greater performance bounds are permitted then larger robustness bounds result. Conversely, if lower performance bounds are demanded then smaller robustness bounds result. An alternative approach to robust controller design is proposed which effectively uses a performance criterion as the design parameter and is valid for uncertainties within a resulting (maximal) robustness bound.

CHAPTER 6

OPTIMAL ROBUST OUTPUT FEEDBACK LQG CONTROLLER DESIGN

From chapter 3, the standard LQG control design theory can not guarantee good stability robustness and its stability margin can be arbitrarily small, the existence of system uncertainty may degrade the system performance or even destabilise the closed loop system. So in recent years, increasing attention has been paid to the design of dynamic output feedback controllers that can stabilise the uncertain system and guarantee the cost performance, the H^∞ performance or both to lie within a specified bound. For the system with modelled uncertainties and residual unknown uncertainty, a design technique of the robust state feedback controller (RLQR) has been presented in chapter 4. It is proved that this robust LQR can provide optimal robustness of stability and performance. However, since it is often not possible to measure all state variables, the output feedback controller is more practicable.

In this chapter, a new design methodology has been developed for robust dynamic output feedback controllers that is applicable for systems with parametric uncertainty. The method guarantees robust stability of the system for all uncertainties within a given admissible domain, the magnitude of this domain is treated as a design parameter. As discussed in §2.2.4, for simplicity, the residual unknown uncertainty is considered to be nonparametric, robustness subject to it may be measured by the H^∞ norm bound of the system. Thus the desired H^∞ norm bound is also treated as a design parameter for the method. Cost performance with respect to a quadratic cost function is explicitly considered and the problem is initially posed in the LQG format. A relationship established between LQG solutions and H^∞ solutions allows the robust LQG design problem to be translated to an H^∞ problem with explicit reference to the cost function. Thus the controller is designed by employing the established solution technique to the H^∞ problem for the specified level of nonparametric robustness. System performance is also inherently considered in the design with respect to a quadratic cost function. Though no optimality is proved, informal analysis and example applications have produced good performance subject to the robustness constraints specified. In essence the method enables the designer to trade off robustness to parametric uncertainty, robustness to nonparametric uncertainty and cost performance.

After the problem statement, a robust LQG design method will be developed for the system with only nominal model in §6.2, then the system with parametric uncertainties will be considered in §6.3, the robust LQG controller will be found for the system with norm bounded or matched norm bounded uncertainties, it will also be demonstrated how the RLQG controller design method can be posed as an H^∞ control design problem for a scaled certain system hence allowing the existing H^∞ numerical techniques to be used. In §6.4, example systems will be used to illustrate the implementation of the methodology.

6.1 PROBLEM STATEMENT

As mentioned in §2.2.3, both parametric and nonparametric uncertainties effect the stability and performance robustness of dynamic output feedback controlled systems, to avoid conservative design, for a system with modelled (parametric) uncertainty, robust controller design should refer to the uncertainty model. The system with modelled uncertainty to be studied may be described by the following state-space representation:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{A}+\Delta\mathbf{A}(t))\mathbf{x}(t)+(\mathbf{B}_2+\Delta\mathbf{B}(t))\mathbf{u}(t)+\mathbf{E}\mathbf{d} \\ \mathbf{y}(t) &= (\mathbf{C}_2+\Delta\mathbf{C}(t))\mathbf{x}(t)+\mathbf{F}\mathbf{v} \end{aligned} \tag{6.1.1}$$

Where $\mathbf{x}\in\Re^n$ is the state vector, $\mathbf{u}\in\Re^m$ is the control vector, $\mathbf{y}\in\Re^r$ is the observation vector, $\mathbf{d}$ is the vector of disturbance inputs and $\mathbf{v}$ is the vector of measurement noises. The system is linear and all disturbance and noise processes are assumed to be uncorrelated Gaussian white noise processes, and have covariances $\mathbf{W}$ and $\mathbf{V}$ respectively.

Related to §2.2.1, the plant uncertainties $\Delta\mathbf{A}(t)$, $\Delta\mathbf{B}(t)$, $\Delta\mathbf{C}(t)$ are assumed to lie within an admissible domain and may be described by:

$$\Pi = \begin{cases} \Delta\mathbf{A}(t) = \mathbf{N}_a\Phi_1(t)\mathbf{M}_a \\ \Delta\mathbf{B}(t) = \mathbf{N}_b\Phi_2(t)\mathbf{M}_b \\ \Delta\mathbf{C}(t) = \mathbf{N}_c\Phi_3(t)\mathbf{M}_c \end{cases} \tag{6.1.2}$$

Where $\mathbf{N}_a,\mathbf{M}_a,\mathbf{N}_b,\mathbf{M}_b,\mathbf{N}_c,\mathbf{M}_c$ are constant matrices describing the structure of the admissible domain and the magnitude is constrained by $\overline{\sigma}(\Phi_i(t))\le\varepsilon$ for i=1, 2 and 3, where, $\overline{\sigma}$ denotes maximum singular value. As a special case of this, matched norm bounded uncertainties may be described by

$$\Pi = \begin{cases} [\Delta\mathbf{A}(t), \Delta\mathbf{B}(t)] = \mathbf{N}_{ab}\Phi(t)[\mathbf{M}_a, \mathbf{M}_b] \\ \Delta\mathbf{C}(t) = \mathbf{N}_c\Phi(t)\mathbf{M}_c \end{cases} \tag{6.1.3}$$

with $\overline{\sigma}(\Phi(t))\le\varepsilon$.

The controller is to be designed with reference to a quadratic cost criterion of the form:

$$J = \lim_{T_0\to\infty}\frac{1}{T_0}\mathcal{E}\int_0^{T_0}(\mathbf{x}^T\mathbf{Q}\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u})dt \tag{6.1.4}$$

Where $\mathbf{R}>\mathbf{0}$ and $\mathbf{Q}\ge\mathbf{0}$.

The design criteria are to produce a dynamic output feedback controller that will guarantee stability for all admissible values of parametric uncertainties $\Delta\mathbf{A}(t)$, $\Delta\mathbf{B}(t)$ and $\Delta\mathbf{C}(t)$, to guarantee an $H\infty$ norm bound such that a prescribed level of nonparametric plant uncertainty $\Delta G(s)$ may be tolerated. Furthermore, the reference cost function is implicitly considered in the controller synthesis and good cost performance is sought subject to the above robustness constraints.

6.2 Design of Robust LQG Controllers Based on Nominal Models

In §3.2.1 and §3.2.3, the LQG and H∞ controller design techniques are presented, and from the robustness analysis in §2.4.1, it follows that the robustness of the closed loop system against nonparametric uncertainty can be measured by the H∞ norm bound of the transfer function $\mathbf{T}_{u_o y_o}(s)$, the smaller $\left\|\mathbf{T}_{u_o y_o}(s)\right\|_\infty$ the greater the robustness. In this section, a design approach is proposed in this section which offers a compromise between cost performance and robustness against nonparametric uncertainty, and it is also found that the LQG and H∞ controller design techniques are equivalent under certain conditions. This leads to the development of a robust LQG controller design approach that enables a compromising controller to be found.

6.2.1 Comparison of LQG and H∞ designs

The certain part of the system (6.1.1) will be considered here

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}+\mathbf{B}_2\mathbf{u}+\mathbf{E}\mathbf{d}\\ \mathbf{y} &= \mathbf{C}_2\mathbf{x}+\mathbf{F}\mathbf{v}\end{aligned} \tag{6.2.1}$$

A stabilising controller that minimises the cost function (6.1.4) can be found by the LQG design method in §3.2.1. By the use of the separation principle, a dynamic LQG output feedback controller is combined by a Kalman filter and a full state feedback controller as:

$$\mathbf{K}_{LQG}(s)=\mathbf{K}_c(s\mathbf{I}-\mathbf{A}_c)^{-1}\mathbf{K}_f \tag{6.2.2}$$

where $\mathbf{A}_c=\mathbf{A}+\mathbf{B}_2\mathbf{K}_c-\mathbf{K}_f\mathbf{C}_2,\quad \mathbf{K}_c=-\mathbf{R}^{-1}\mathbf{B}_2^T\mathbf{P}_c,\quad \mathbf{K}_f=\mathbf{P}_f\mathbf{C}_2^T(\mathbf{F}\mathbf{V}\mathbf{F}^T)^{-1}$

$$\mathbf{P}_c:\quad \mathbf{A}^T\mathbf{P}_c+\mathbf{P}_c\mathbf{A}-\mathbf{P}_c\mathbf{B}_2\mathbf{R}^{-1}\mathbf{B}_2^T\mathbf{P}_c+\mathbf{Q}=\mathbf{0}$$

$$\mathbf{P}_f:\quad \mathbf{A}\mathbf{P}_f+\mathbf{P}_f\mathbf{A}^T-\mathbf{P}_f\mathbf{C}_2^T(\mathbf{F}\mathbf{V}\mathbf{F}^T)^{-1}\mathbf{C}_2\mathbf{P}_f+\mathbf{E}\mathbf{W}\mathbf{E}^T=\mathbf{0}$$

For comparison the H∞ in (6.2.1), this system may be described in the format used for the standard H∞ design method (3.2.26) by choosing $\omega=\begin{bmatrix}\mathbf{d}^T & \upsilon^T\end{bmatrix}^T$ and the noise/disturbance input matrices with respect to the LQG format as:

$$\mathbf{B}_1=\begin{bmatrix}\sqrt{\mathbf{E}\mathbf{W}\mathbf{E}^T} & \mathbf{0}\end{bmatrix},\quad \mathbf{D}_2=\begin{bmatrix}\mathbf{0} & \sqrt{\mathbf{F}\mathbf{V}\mathbf{F}^T}\end{bmatrix} \tag{6.2.3}$$

So

$$\mathbf{B}_1^T\mathbf{D}_2=\mathbf{0},\quad \mathbf{B}_1\mathbf{B}_1^T=\mathbf{E}\mathbf{W}\mathbf{E}^T,\quad \mathbf{D}_2\mathbf{D}_2^T=\mathbf{F}\mathbf{V}\mathbf{F}^T$$

Consider also the selection of a particular performance vector $\mathbf{z}$ with reference to the cost function such that:

$$\mathbf{C}_1 = \begin{bmatrix} \sqrt{\mathbf{Q}} \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{D}_1 = \begin{bmatrix} \mathbf{0} \\ \sqrt{\mathbf{R}} \end{bmatrix} \tag{6.2.4}$$

So $$\mathbf{C}_1^T\mathbf{D}_1 = \mathbf{0}, \quad \mathbf{C}_1^T\mathbf{C}_1 = \mathbf{Q}, \quad \mathbf{D}_1^T\mathbf{D}_1 = \mathbf{R} \tag{6.2.5}$$

It may be noted that the constraints (A1) and (A2) in §3.2.3 are satisfied. The H∞ design method provides a controller that will stabilise the system (3.2.26) and satisfy the following H∞ norm bound:

$$\left\|\mathbf{T}_{\omega z}\right\|_\infty < \gamma_0 \tag{6.2.6}$$

where γ_0 may be treated as a design parameter. Then according to Lemma 3.3.2, $\gamma_0 = \infty$ implies $\mathbf{P}_c = \mathbf{X}_\infty$, $\mathbf{P}_f = \mathbf{Y}_\infty$, so $\mathbf{A}_c = \mathbf{A}_\infty$, $\mathbf{K}_c = \mathbf{F}_\infty$, $\mathbf{K}_f = \mathbf{L}_\infty$ Thus the LQG controller solution (6.2.2) is a special case of the H∞ controller solution for this particular performance vector and disturbance/noise vector, giving an equivalent solution when $\gamma_0 = \infty$.

Reference to Corollary 2.4.3 illustrates that the LQG solution offers no stability robustness guarantee against nonparametric uncertainty, this is consistent with the analysis in chapter 3.

6.2.2 Proposed robust LQG design approach

It is proposed that a compromise between cost performance and robustness to nonparametric uncertainty may be found by employing the H∞ design technique to the system with particular performance vector (6.2.4) for a finite H∞ norm bound γ_0. The robustness analysis results in Corollary 2.4.3 and (6.2.5) have shown that γ_0 gives a measure of nonparametric robustness, with a lower bound relating to greater robustness. It has also been shown that if the bound is chosen to be infinite then the solution is equivalent to that of the LQG approach that is known to be cost optimal. It has been found from example applications that the cost monotonically increases as γ_0 is reduced. Thus, it is proposed that suitable choice of γ_0 can yield a suitable compromise controller solution. This is consistent with (Mustafa and Glover, 1990) where this result is the subject of an unproved conjecture. To complete the analysis, a result is given to calculate the cost performance for any given dynamic output controller:

Lemma 6.2.1 For the system (6.2.1) and a given stabilising controller $\mathbf{K}(s) = \mathbf{C}_c(s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}_c$, the cost performance (6.1.4) $\mathbf{J}(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c)$ is given by $\mathbf{tr}(\overline{\mathbf{P}}\overline{\mathbf{R}})$

where $\overline{\mathbf{P}}$: $\mathbf{A}_0\overline{\mathbf{P}} + \overline{\mathbf{P}}\mathbf{A}_0^T + \overline{\mathbf{V}} = \mathbf{0}$

$$\overline{\mathbf{R}} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_c^T\mathbf{R}\mathbf{C}_c \end{bmatrix}, \quad \overline{\mathbf{V}} = \begin{bmatrix} \mathbf{E}\mathbf{W}\mathbf{E}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_c\mathbf{F}\mathbf{V}\mathbf{F}^T\mathbf{B}_c^T \end{bmatrix}$$

and $\mathbf{A}_0$ is the system matrix of the closed loop system, $\mathbf{A}_0 = \begin{bmatrix} \mathbf{A} & \mathbf{B}_2\mathbf{C}_c \\ \mathbf{B}_c\mathbf{C}_2 & \mathbf{A}_c \end{bmatrix}$

Proof: This result can be directly found from Lemma 2.3.3 by using the above $\mathbf{A}_0$ and substituting $\overline{\mathbf{R}}$ for $\overline{\mathbf{Q}}_0$, and $\overline{\mathbf{V}}$ for $\mathbf{E}\mathbf{W}\mathbf{E}^T$.

6.3. Design of Robust LQG Controllers

In this section, the robust LQG approach will be extended to systems with parametric uncertainty. In addition to a compromise between cost performance and nonparametric robustness, it is required to guarantee robust stability such that the system is guaranteed to remain stable for all admissible parametric uncertainty. The basic approach is similar to that proposed in the previous section where the problem is translated into the H^∞ format with a performance vector chosen with respect to the quadratic cost function. To accommodate the parametric uncertainty, the disturbance/noise and performance vectors of the H^∞ description are suitably appended. It is then shown how robust stability and nonparametric robustness may be guaranteed by an H^∞ controller designed for this appended system. The H^∞ norm bound specified relates to the magnitude of the admissible domain and for ease of explanation the result is initially expounded for a simple relationship.

As mentioned in §6.1, there exist two formats of parametric uncertainty descriptions; norm bounded uncertainty (6.1.2) and matched norm bounded uncertainty (6.1.3). They will be considered separately in this chapter.

6.3.1 Systems with norm bounded uncertainty

By using the definitions in (6.2.3) and choosing the performance vector as (6.2.4) the system with parametric uncertainty (6.1.1) may be written in the H^∞ format as:

$$\begin{cases} \dot{\mathbf{x}} = (\mathbf{A}+\Delta\mathbf{A})\mathbf{x}+(\mathbf{B}_2+\Delta\mathbf{B})\mathbf{u}+\mathbf{B}_1\omega \\ \mathbf{z} = \mathbf{C}_1\mathbf{x}+\mathbf{D}_1\mathbf{u} \\ \mathbf{y} = (\mathbf{C}_2+\Delta\mathbf{C})\mathbf{x}+\mathbf{D}_2\omega \end{cases} \tag{6.3.1}$$

Consider also a certain system with appended disturbance/noise input vector, $\overline{\omega}$ and performance vector, $\overline{\mathbf{z}}$.

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}+\mathbf{B}_2\mathbf{u}+\overline{\mathbf{B}}_1\overline{\omega} \\ \overline{\mathbf{z}} &= \overline{\mathbf{C}}_1\mathbf{x}+\overline{\mathbf{D}}_1\mathbf{u} \\ \mathbf{y} &= \mathbf{C}_2\mathbf{x}+\overline{\mathbf{D}}_2\overline{\omega} \end{aligned} \tag{6.3.2}$$

where

$$\overline{\mathbf{B}}_1 = \begin{bmatrix} \mathbf{B}_{1n} & \mathbf{0}^{nb1\times md2} \end{bmatrix}, \qquad \overline{\mathbf{D}}_2 = \begin{bmatrix} \mathbf{0}^{nd2\times mb1} & \mathbf{D}_{2n} \end{bmatrix} \text{ and}$$

$$\overline{\mathbf{C}}_1 = \begin{bmatrix} \mathbf{C}_{1n} \\ \mathbf{0}^{nd1\times mc1} \end{bmatrix}, \quad \overline{\mathbf{D}}_1 = \begin{bmatrix} \mathbf{0}^{nc1\times md1} \\ \mathbf{D}_{1n} \end{bmatrix}$$

and (with reference to definition (6.1.2))

$$\mathbf{B}_{1n} = \begin{bmatrix} \dfrac{\mathbf{N}_a}{\alpha_a} & \dfrac{\mathbf{N}_b}{\alpha_b} & \mathbf{B}_1 \end{bmatrix} \in \Re^{nb1\times mb1}, \qquad \mathbf{D}_{2n} = \begin{bmatrix} \dfrac{\mathbf{N}_c}{\alpha_c} & \mathbf{D}_2 \end{bmatrix} \in \Re^{nd2\times md2}$$

$$\mathbf{C}_{1n} = \begin{bmatrix} \alpha_a\mathbf{M}_a \\ \alpha_c\mathbf{M}_c \\ \mathbf{C}_1 \end{bmatrix} \in \Re^{nc1\times mc1}, \qquad \mathbf{D}_{1n} = \begin{bmatrix} \alpha_b\mathbf{M}_b \\ \mathbf{D}_1 \end{bmatrix} \in \Re^{nd1\times md1}$$

$nb1$, $mb1$ denote the dimensions of $\mathbf{B}_{1n}$ etc. and $\alpha_a, \alpha_b, \alpha_c$ are positive scalar parameters. Then the robust stability and nonparametric robustness of the system with parametric uncertainty (6.3.1) may be related to the H^∞ norm bound of the appended certain system (6.3.2):

> **Theorem 6.3.1** If a linear dynamic output feedback controller $\mathbf{K}(s)$ can stabilise the appended certain system (6.3.2) with $\|\mathbf{T}_{\overline{\omega}\overline{z}}\|_\infty < 1/\varepsilon$, then $\mathbf{K}(s)$ can stabilise and guarantee $\|\mathbf{T}_{\omega z}\|_\infty < 1/\varepsilon$ for the uncertain system (6.3.1), for all admissible uncertainties (6.1.2).

Proof: It is supposed that the controller $\mathrm{K}(s)$ can be realised as:

$$\mathbf{K}(s) = \mathbf{C}_c(s\mathbf{I}-\mathbf{A}_c)^{-1}\mathbf{B}_c \tag{6.3.3}$$

Applying this controller to system (6.3.2), the closed loop system can be described as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathrm{A}_0\mathbf{x}+\overline{\mathbf{B}}_0\overline{\omega} \\ \overline{\mathbf{z}} &= \overline{\mathbf{C}}_0\mathbf{x} \end{aligned}$$

with

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{A} & \mathbf{B}_2\mathbf{C}_c \\ \mathbf{B}_c\mathbf{C}_2 & \mathbf{A}_c \end{bmatrix},\quad \overline{\mathbf{B}}_0 = \begin{bmatrix} \overline{\mathbf{B}}_1 \\ \mathbf{B}_c\overline{\mathbf{D}}_2 \end{bmatrix},\quad \overline{\mathbf{C}}_0 = \begin{bmatrix} \overline{\mathbf{C}}_1 & \overline{\mathbf{D}}_1\mathbf{C}_c \end{bmatrix} \tag{6.3.4}$$

or by the transfer function:

$$\overline{\mathbf{G}}_0(s) = \overline{\mathbf{C}}_0(s\mathbf{I} - \mathbf{A}_0)^{-1}\overline{\mathbf{B}}_0 = \frac{\overline{\mathbf{Z}}(s)}{\overline{\omega}(s)} = \mathbf{T}_{\overline{\omega z}}$$

From Lemma 2.3.5, if $\|\mathbf{T}_{\overline{\omega z}}\|_\infty < 1/\varepsilon$, then there exits a positive definite solution for the following Riccati expression:

$$\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\mathbf{A}_0 + \varepsilon^2\mathbf{P}\overline{\mathbf{B}}_0\overline{\mathbf{B}}_0^T\mathbf{P} + \overline{\mathbf{C}}_0^T\overline{\mathbf{C}}_0 < \mathbf{0} \tag{6.3.5}$$

Now consider the application of controller (6.3.3) to system (6.3.1), the resulting closed loop system can be described as:

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A}_0 + \Delta\mathbf{A}_0)\mathbf{x} + \mathrm{B}_0\overline{\omega} \\ \mathbf{z} &= \mathrm{C}_0\mathbf{x} \end{aligned} \tag{6.3.6}$$

with

$$\Delta\mathbf{A}_0 = \begin{bmatrix} \Delta\mathbf{A} & \Delta\mathbf{B}\mathbf{C}_c \\ \mathbf{B}_c\Delta\mathbf{C} & \mathbf{0} \end{bmatrix},\quad \mathbf{B}_0 = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_c\mathbf{D}_2 \end{bmatrix},\qquad \mathbf{C}_0 = \begin{bmatrix} \mathbf{C}_1 & \mathbf{D}_1\mathbf{C}_c \end{bmatrix} \tag{6.3.7}$$

or by the transfer function:

$$\mathbf{G}_0(s) = \mathbf{C}_0(s\mathbf{I} - \mathbf{A}_0 - \Delta\mathbf{A}_0)^{-1}\mathbf{B}_0 = \frac{\mathbf{Z}(s)}{\omega(s)} = \mathbf{T}_{\omega z} \tag{6.3.8}$$

From Lemma 2.3.5 we know that a sufficient condition to guarantee $\|\mathbf{T}_{\omega z}\|_\infty < 1/\varepsilon$ and thus, from Lyapunov stability theory in §2.3.2, robust stability of the system described by (6.3.8) is that there exists a positive definite solution to the following Riccati expression:

$$(\mathbf{A}_0 + \Delta\mathbf{A}_0)^T\mathbf{P} + \mathbf{P}(\mathbf{A}_0 + \Delta\mathbf{A}_0) + \varepsilon^2\mathbf{P}\mathbf{B}_0\mathbf{B}_0^T\mathbf{P} + \mathbf{C}_0^T\mathbf{C}_0 < \mathbf{0} \tag{6.3.9}$$

From definition (6.3.7), the uncertain terms in (6.3.9) may be expanded as:

$$\begin{aligned} \Delta\mathbf{A}_0^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A}_0 &= \begin{bmatrix} \Delta\mathbf{A} & \Delta\mathbf{B}\mathbf{C}_c \\ \mathbf{B}_c\Delta\mathbf{C} & \mathbf{0} \end{bmatrix}^T \mathbf{P} + \mathbf{P}\begin{bmatrix} \Delta\mathbf{A} & \Delta\mathbf{B}\mathbf{C}_c \\ \mathbf{B}_c\Delta\mathbf{C} & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \Delta\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T \mathbf{P} + \mathbf{P}\begin{bmatrix} \Delta\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \Delta\mathbf{B}\mathbf{C}_c \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T \mathbf{P} \\ &\quad + \mathbf{P}\begin{bmatrix} \mathbf{0} & \Delta\mathbf{B}\mathbf{C}_c \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_c\Delta\mathbf{C} & \mathbf{0} \end{bmatrix}^T \mathbf{P} + \mathbf{P}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_c\Delta\mathbf{C} & \mathbf{0} \end{bmatrix} \end{aligned} \tag{6.3.10}$$

Substituting (6.1.2) in to (6.3.10) it follows that

$$\Delta A_0^T P + P\Delta A_0 = \begin{bmatrix} M_a & 0 \end{bmatrix}^T \Phi_1^T(t) \begin{bmatrix} N_a \\ 0 \end{bmatrix}^T P + P \begin{bmatrix} N_a \\ 0 \end{bmatrix} \Phi_1(t) \begin{bmatrix} M_a & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & M_b C_c \end{bmatrix}^T \Phi_2^T(t) \begin{bmatrix} N_b \\ 0 \end{bmatrix}^T P + P \begin{bmatrix} N_b \\ 0 \end{bmatrix} \Phi_2(t) \begin{bmatrix} 0 & M_b C_c \end{bmatrix}$$

$$+ \begin{bmatrix} M_c & 0 \end{bmatrix}^T \Phi_3^T(t) \begin{bmatrix} 0 \\ B_c N_c \end{bmatrix}^T P + P \begin{bmatrix} 0 \\ B_c N_c \end{bmatrix} \Phi_3(t) \begin{bmatrix} M_c & 0 \end{bmatrix}$$

By the application of Lemma 2.4.1 to each pair of terms on the right hand side of above and subsequent collecting of terms yields the following inequality:

$$\Delta A_0^T P + P\Delta A_0 \le \varepsilon^2 P \Psi_1 P + \Psi_2 \tag{6.3.11}$$

where

$$\Psi_1 = \begin{bmatrix} \dfrac{N_a N_a^T}{\alpha_a^2} + \dfrac{N_b N_b^T}{\alpha_b^2} & 0 \\ 0 & \dfrac{B_c N_c N_c^T B_c^T}{\alpha_c^2} \end{bmatrix} \ge 0$$

$$\Psi_2 = \begin{bmatrix} \alpha_a^2 M_a^T M_a + \alpha_c^2 M_c^T M_c & 0 \\ 0 & \alpha_b^2 C_c^T M_b^T M_b C_c \end{bmatrix} \ge 0$$

From the definitions in (6.3.4) and (6.3.7) it can be shown that

$$\Psi_1 = \overline{B}_0 \overline{B}_0^T - B_0 B_0^T, \quad \Psi_2 = \overline{C}_0 \overline{C}_0^T - C_0^T C_0 \tag{6.3.12}$$

Substituting in (6.3.11) gives:

$$\Delta A_0^T P + P\Delta A_0 \le \varepsilon^2 P \left(\overline{B}_0 \overline{B}_0^T - B_0 B_0^T \right) P + \overline{C}_0 \overline{C}_0^T - C_0^T C_0$$

This inequality is valid for any positive definite $\mathbf{P}$ and addition to inequality (6.3.5) with subsequent cancellation of terms gives:

$$(A_0 + \Delta A_0)^T P + P(A_0 + \Delta A_0) + \varepsilon^2 P B_0 B_0^T P + C_0^T C_0 < 0 \tag{6.3.13}$$

This expression is identical to (6.3.9). Thus if there exists a positive definite solution to (6.3.5) then a positive definite solution exists for (6.3.13) and hence (6.3.9). Thus $\|T_{\overline{\omega}\overline{z}}\|_\infty < 1/\varepsilon$ for (6.3.2) is sufficient to guarantee $\|T_{\omega z}\|_\infty < 1/\varepsilon$ and stability for (6.3.1). ◆

Hence, a controller may be designed by standard H∞ techniques for such an appended system that will guarantee the robust stability and provide an H∞ norm bound for a system with parametric uncertainty. However, thus far there is a fixed relationship between the given magnitude of parametric uncertainty and the H∞ norm bound produced. Since the H∞ norm bound is known to dictate the compromise between nonparametric robustness and cost performance such a fixed relationship may not be desirable. A more flexible approach may be offered by decoupling the size of the H∞ norm bound from the magnitude of the admissible domain of parametric uncertainty effectively offering the designer an extra degree of freedom. It is shown how this decoupling may be achieved by suitable description of the given admissible domain. Thus for any given admissible domain of parametric uncertainty an arbitrary H∞ norm bound may be specified enabling it to be treated as an independent design parameter.

Consider a given admissible domain of parametric uncertainty (6.1.2), for simplicity, consider only $\Delta A(t)$: (Similar steps may be applied to $\Delta B(t)$, $\Delta C(t)$)

$$\Delta\mathbf{A}(t) = \mathbf{N}_a\Phi_1(t)\mathbf{M}_a, \quad \overline{\sigma}(\Phi_1(t)) \le \varepsilon \tag{6.3.14}$$

This may be equivalently described by

$$\Delta\mathbf{A}(t) = \delta\mathbf{N}_a\frac{\Phi_1(t)}{\delta}\mathbf{M}_a = \overline{\mathbf{N}}_a\overline{\Phi}_1(t)\mathbf{M}_a, \qquad \overline{\sigma}(\overline{\Phi}_1(t)) \le \frac{1}{\gamma_o} \tag{6.3.15}$$

where

$$\overline{\mathbf{N}}_a = \delta\mathbf{N}_a, \quad \overline{\Phi}(t) = \frac{\Phi(t)}{\delta}, \qquad \gamma_o = \frac{\delta}{\varepsilon}$$

For an appended system created in a similar manner to system (6.3.2) with respect to this description, employing Theorem 6.3.1 states that a controller that provides an H∞ norm bound of γ_0 for the appended system is able to guarantee robust stability for this admissible domain and an H∞ norm bound of γ_0. Thus by suitable choice of δ, an arbitrary H∞ norm bound may be chosen for any given admissible domain. The choice of δ will dictate $\overline{\mathbf{N}}_a$ (similarly $\overline{\mathbf{N}}_b$ and $\overline{\mathbf{N}}_c$) which are used in the appended system and hence dictate the controller produced.

If no robust LQG controller exists, i.e., there does not exist a linear dynamic output feedback controller that can stabilise the appended certain system (6.3.2) with $\|\mathbf{T}_{\overline{\omega}\overline{z}}\|_\infty < \frac{1}{\varepsilon}$, it is possible to omit the consideration of the performance robustness and focus on the stability robustness. A new appended system should be used in this case:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_2\mathbf{u} + \overline{\mathbf{B}}_1\omega \\ \overline{\mathbf{z}} &= \overline{\mathbf{C}}_1\mathbf{x} + \overline{\mathbf{D}}_1\mathbf{u} \\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \overline{\mathbf{D}}_2\omega \end{aligned} \tag{6.3.16}$$

where

$$\overline{\mathbf{B}}_1 = \begin{bmatrix}\mathbf{B}_{1n} & \mathbf{0}^{nb1\times md2}\end{bmatrix}, \qquad \overline{\mathbf{D}}_2 = \begin{bmatrix}\mathbf{0}^{nd2\times mb1} & \mathbf{D}_{2n}\end{bmatrix}$$

$$\overline{\mathbf{C}}_1 = \begin{bmatrix}\mathbf{C}_{1n} \\ \mathbf{0}^{nd1\times mc1}\end{bmatrix}, \quad \overline{\mathbf{D}}_1 = \begin{bmatrix}\mathbf{0}^{nc1\times md1} \\ \mathbf{D}_{1n}\end{bmatrix}$$

$$\mathbf{B}_{1n} = \begin{bmatrix} \mathbf{N}_a & \mathbf{N}_b \\ \alpha_a & \alpha_b \end{bmatrix} \in \Re^{nb1 \times mb1}, \qquad \mathbf{D}_{2n} = \begin{bmatrix} \mathbf{N}_c & \eta\mathbf{D}_2 \\ \alpha_c & \end{bmatrix} \in \Re^{nd2 \times md2},$$

$$\mathbf{C}_{1n} = \begin{bmatrix} \alpha_a \mathbf{M}_a \\ \alpha_c \mathbf{M}_c \end{bmatrix} \in \Re^{nc1 \times mc1}, \qquad \mathbf{D}_{1n} = \begin{bmatrix} \alpha_b \mathbf{M}_b \\ \eta\mathbf{D}_1 \end{bmatrix} \in \Re^{nd1 \times md1}$$

where $\eta > 0$ is a very small parameter which is used to guarantee that (6.3.12) is a non-singular system, i.e.,

$$\eta^2 \mathbf{D}_1^T \mathbf{D}_1 > \mathbf{0}, \quad \eta^2 \mathbf{D}_2 \mathbf{D}_2^T > \mathbf{0}$$

So essentially, the explicit reference to the LQG cost function is removed. Then a stabilising dynamic output controller for the uncertain system (6.3.1) can be found from the following result:

Lemma 6.3.2 If a linear dynamic output feedback controller $\mathbf{K}(s)$ can stabilise the appended certain system (6.3.16) with $\|\mathbf{T}_{\overline{\omega z}}\|_\infty < 1/\varepsilon$, then $\mathbf{K}(s)$ can stabilise and guarantee $\|\mathbf{T}_{\omega z}\|_\infty < 1/\varepsilon$ for the uncertain system (6.3.1), for all admissible uncertainties (6.1.2).

Proof: The proof of this Lemma is similar to the proof of Theorem 6.3.1.

It should be noted that the scalar parameters $\alpha_a, \alpha_b, \alpha_c$ may be searched to find the controller that offers the best cost performance subject to the given robustness constraints. A bound for the cost performance of the uncertain system may be calculated using the following Lemma.

Lemma 6.3.3 For an uncertain system (6.3.1) with admissible domain of parametric uncertainty (6.1.2) employing any given stabilising controller as (6.3.3), a bound $J_b(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c)$ for the cost performance (6.1.4) is given by:

$$J_b(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c) = \mathrm{tr}(\mathbf{P}_b \mathbf{B}_0 \mathbf{B}_0^T) \tag{6.3.17}$$

Where $\mathbf{P}_b$ it the positive definite solution of the following equation:

$$\mathbf{A}_0^T \mathbf{P}_b + \mathbf{P}_b \mathbf{A}_0 + \varepsilon^2 \mathbf{P}_b \Psi_1 \mathbf{P}_b + \overline{\mathbf{C}}_0^T \overline{\mathbf{C}}_0 = \mathbf{0} \tag{6.3.18}$$

and $\mathbf{B}_0$, Ψ_1 and $\overline{\mathbf{C}}_0$ are defined in (6.3.7), (6.3.11) and (6.3.4) respectively.

Proof: The deployment a controller as (6.3.3) to system (6.3.1) results in an uncertain closed loop system as given in (6.3.6). A fundamental result in robust optimal control (Doyle et al., 1989), states that the cost performance of such a closed loop system may be given by:

$$J(\mathbf{A}_c,\mathbf{B}_c,\mathbf{C}_c)=\mathrm{tr}(\mathbf{P}\mathbf{B}_0\mathbf{B}_0^T) \tag{6.3.19}$$

where $\mathbf{P}$ is the positive definite solution of the following Riccati equation:

$$(\mathbf{A}_0+\Delta\mathbf{A}_0)^T\mathbf{P}+\mathbf{P}(\mathbf{A}_0+\Delta\mathbf{A}_0)+\mathbf{C}_0^T\mathbf{C}_0=\mathbf{0}$$

Expanding and substituting from (6.3.12) this can be expressed as

$$\mathbf{A}_0^T\mathbf{P}+\mathbf{P}\mathbf{A}_0+\Delta\mathbf{A}_0^T\mathbf{P}+\mathbf{P}\Delta\mathbf{A}_0-\Psi_2+\overline{\mathbf{C}}_0^T\overline{\mathbf{C}}_0=\mathbf{0} \tag{6.3.20}$$

Since the uncertainty $\Delta\mathbf{A}_0$ is a time-varying matrix, and it is impossible to find a time-invariant solution for $\mathbf{P}>\mathbf{0}$ for (6.3.20). However, since the time-varying uncertain matrix $\Phi_i(t)$ is constrained by $\overline{\sigma}(\Phi_i(t))\leq\varepsilon$, a bound matrix $\mathbf{P}_b$ can be found. This is the upper limit of all possible solutions for (6.3.20) over all admissible values of $\Phi_i(t)$ and all $t\in[0,\infty)$.

From (6.3.11) we know that for any positive definite matrix $\mathbf{P}_0$, it follows that

$$\Delta\mathbf{A}_0^T\mathbf{P}_0+\mathbf{P}_0\Delta\mathbf{A}_0\leq\varepsilon^2\mathbf{P}_0\Psi_1\mathbf{P}_0+\Psi_2\stackrel{\Delta}{=}\Theta(\mathbf{P}_0,\Psi_1,\Psi_2)$$

Then from the result of Lemma 2.4.4 it can be shown that (6.3.17) is a bound for the cost of the uncertain system (6.3.19). ◆

It should be noted that the cost bound does not depend upon the particular description of the given admissible domain as any scaling factor δ introduced as in (6.3.15) would have a self-cancelling effect on ε and Ψ_1 in (6.3.18). However, the scalar parameters $\alpha_a,\alpha_b,\alpha_c$ used to define Ψ_1 and $\overline{\mathbf{C}}_0$ in (6.3.18) do effect the resulting cost bound and may be searched to find a minimal cost bound. The search over these parameters is purely to find a minimal cost bound for a given controller and is independent of the search performed during the controller design.

In summary, to design a robust hybrid LQG/H∞ controller for system (6.3.1) with admissible domain of parametric uncertainty (6.1.2) the following steps should be followed:

i) Choose values for the design parameters: the magnitude of the admissible domain, ε and the desired H∞ norm bound γ_0.

ii) Choose a suitable description of the admissible domain by suitable choice of δ (6.3.15).

iii) Create an appended system similar to (6.3.2)

iv) Apply the standard H∞ design techniques to the appended system for the chosen value of γ_0.

During step iv) the scaling parameters $\alpha_a, \alpha_b, \alpha_c$ may be searched to provide a controller that offers the best cost performance subject to the robustness constraints given in step i). A bound for the cost performance of any given controller may be found by employing Lemma 6.3.2 and again searching the scaling parameters will yield the minimal bound. Thus to find the controller that offers the best cost performance a nested search of scaling parameters is required.

Thus a controller is produced which offers robust stability to parametric uncertainties within the admissible domain of magnitude ε and nonparametric robustness as measured by the H^∞ norm bound γ_0. Furthermore, cost performance is implicitly considered by choosing the performance vector with respect to the quadratic cost function. Example applications have shown that for a given admissible domain magnitude the cost performance is monotonically decreasing for increased H^∞ norm bound, however again complexity of the relationship in Lemma 6.3.2 has prevented the establishment of a general result.

6.3.2 Systems with matched norm bounded uncertainty

System uncertainty can often be described in matched norm bounded format, to avoid conservative design, a different controller should be designed for such special cases. To design a robust LQG controller for the uncertain system (6.3.1) with matched uncertainties (6.1.3), we again consider a certain system with appended disturbance/noise input vector, $\overline{\boldsymbol{\omega}}$ and performance vector, $\overline{\mathbf{z}}$.

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_2\mathbf{u} + \overline{\mathbf{B}}_1\overline{\boldsymbol{\omega}} \\
\overline{\mathbf{z}} &= \overline{\mathbf{C}}_1\mathbf{x} + \overline{\mathbf{D}}_1\mathbf{u} \\
\mathbf{y} &= \mathbf{C}_2\mathbf{x} + \overline{\mathbf{D}}_2\overline{\boldsymbol{\omega}}
\end{aligned} \tag{6.3.21}$$

where

$$\overline{\mathbf{B}}_1 = \begin{bmatrix}\mathbf{B}_{1n} & \mathbf{0}^{nb1\times md2}\end{bmatrix},\quad \overline{\mathbf{D}}_2 = \begin{bmatrix}\mathbf{0}^{nd2\times mb1} & \mathbf{D}_{2n}\end{bmatrix},\quad \overline{\mathbf{C}}_1 = \begin{bmatrix}\mathbf{C}_{1n} \\ \mathbf{0}^{nd1\times mc1}\end{bmatrix},\quad \overline{\mathbf{D}}_1 = \begin{bmatrix}\mathbf{0}^{nc1\times md1} \\ \mathbf{D}_{1n}\end{bmatrix}$$

$$\mathbf{B}_{1n} = \begin{bmatrix}\dfrac{\mathbf{N}_{ab}}{\alpha_{ab}} & \mathbf{B}_1\end{bmatrix} \in \Re^{nb1\times mb1},\qquad \mathbf{D}_{2n} = \begin{bmatrix}\dfrac{\mathbf{N}_c}{\alpha_c} & \mathbf{D}_2\end{bmatrix} \in \Re^{nd2\times md2},$$

$$\mathbf{C}_{1n} = \begin{bmatrix}\alpha_a\mathbf{M}_a \\ \alpha_c\mathbf{M}_c \\ \mathbf{C}_1\end{bmatrix} \in \Re^{nc1\times mc1},\qquad \mathbf{D}_{1n} = \begin{bmatrix}\alpha_b\mathbf{M}_b \\ \mathbf{D}_1\end{bmatrix} \in \Re^{nd1\times md1}$$

Then the robust stability and nonparametric robustness of the system with parametric uncertainty (6.3.1) may be related to the H^∞ norm bound of the appended certain system (6.3.21):

Theorem 6.3.4 If a linear dynamic output feedback controller $\mathbf{K}(s)$ can stabilise the appended certain system (6.3.21) with $\|\mathbf{T}_{\overline{\omega z}}\|_\infty < 1/\varepsilon$, then $\mathbf{K}(s)$ can stabilise and guarantee $\|\mathbf{T}_{\omega z}\|_\infty < 1/\varepsilon$ for the uncertain system (6.3.1), for all admissible uncertainties (6.1.3).

Proof: From Lemma 2.3.5 we know that a sufficient condition to guarantee $\|\mathbf{T}_{\omega z}\|_\infty < 1/\varepsilon$ and furthermore robust stability of the system described by (6.3.21) is that there exists a positive definite solution to the following Riccati expression:

$$(\mathbf{A}_0+\Delta\mathbf{A}_0)^T\mathbf{P}+\mathbf{P}(\mathbf{A}_0+\Delta\mathbf{A}_0)+\varepsilon^2\mathbf{P}\mathbf{B}_0\mathbf{B}_0^T\mathbf{P}+\mathbf{C}_0^T\mathbf{C}_0<\mathbf{0} \tag{6.3.22}$$

From definition (6.3.7), the uncertain terms in (6.3.22) may be expanded as:

$$\begin{aligned}\Delta\mathbf{A}_0^T\mathbf{P}+\mathbf{P}\Delta\mathbf{A}_0 &= \begin{bmatrix}\Delta\mathbf{A} & \Delta\mathbf{B}\mathbf{C}_c\\ \mathbf{B}_c\Delta\mathbf{C} & \mathbf{0}\end{bmatrix}^T\mathbf{P}+\mathbf{P}\begin{bmatrix}\Delta\mathbf{A} & \Delta\mathbf{B}\mathbf{C}_c\\ \mathbf{B}_c\Delta\mathbf{C} & \mathbf{0}\end{bmatrix}\\ &= \begin{bmatrix}\Delta\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}^T\mathbf{P}+\mathbf{P}\begin{bmatrix}\Delta\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{0} & \Delta\mathbf{B}\mathbf{C}_c\\ \mathbf{0} & \mathbf{0}\end{bmatrix}^T\mathbf{P}\\ &\quad+\mathbf{P}\begin{bmatrix}\mathbf{0} & \Delta\mathbf{B}\mathbf{C}_c\\ \mathbf{0} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{B}_c\Delta\mathbf{C} & \mathbf{0}\end{bmatrix}^T\mathbf{P}+\mathbf{P}\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{B}_c\Delta\mathbf{C} & \mathbf{0}\end{bmatrix}\end{aligned} \tag{6.3.23}$$

Substituting (6.1.3) in to (6.3.23) it follows that

$$\begin{aligned}\Delta\mathbf{A}_0^T\mathbf{P}+\mathbf{P}\Delta\mathbf{A}_0 &= [\mathbf{M}_{ab}\quad \mathbf{M}_b\mathbf{C}_c]^T\Phi^T(t)\begin{bmatrix}\mathbf{N}_a\\ \mathbf{0}\end{bmatrix}^T\mathbf{P}+\mathbf{P}\begin{bmatrix}\mathbf{N}_{ab}\\ \mathbf{0}\end{bmatrix}\Phi(t)[\mathbf{M}_a\quad \mathbf{M}_b\mathbf{C}_c]\\ &\quad+[\mathbf{M}_c\quad \mathbf{0}]^T\Phi^T(t)\begin{bmatrix}\mathbf{0}\\ \mathbf{B}_c\mathbf{N}_c\end{bmatrix}^T\mathbf{P}+\mathbf{P}\begin{bmatrix}\mathbf{0}\\ \mathbf{B}_c\mathbf{N}_c\end{bmatrix}\Phi(t)[\mathbf{M}_c\quad \mathbf{0}]\end{aligned}$$

By the application of Lemma 2.4.1 to each pair of terms on the right hand side of above and subsequent collecting of terms yields the following inequality:

$$\Delta\mathbf{A}_0^T\mathbf{P}+\mathbf{P}\Delta\mathbf{A}_0 \le \varepsilon^2\mathbf{P}\Psi_1\mathbf{P}+\Psi_2 \tag{6.3.24}$$

where

$$\Psi_1=\begin{bmatrix}\dfrac{\mathbf{N}_{ab}\mathbf{N}_{ab}^T}{\alpha_{ab}^2} & \mathbf{0}\\ \mathbf{0} & \dfrac{\mathbf{B}_c\mathbf{N}_c\mathbf{N}_c^T\mathbf{B}_c^T}{\alpha_c^2}\end{bmatrix}\ge\mathbf{0}$$

$$\Psi_2 = \begin{bmatrix} \alpha_{ab}^2 \mathbf{M}_a^T \mathbf{M}_a + \alpha_c^2 \mathbf{M}_c^T \mathbf{M}_c & \alpha_{ab}^2 \mathbf{M}_a^T \mathbf{M}_b \mathbf{C}_c \\ \alpha_{ab}^2 \mathbf{C}_c^T \mathbf{M}_b^T \mathbf{M}_c & \alpha_{ab}^2 \mathbf{C}_c^T \mathbf{M}_b^T \mathbf{M}_b \mathbf{C}_c \end{bmatrix} \geq \mathbf{0}$$

From the definitions in (6.3.4) and (6.3.7) it can be shown that

$$\Psi_1 = \overline{\mathbf{B}}_0 \overline{\mathbf{B}}_0^T - \mathbf{B}_0 \mathbf{B}_0^T, \quad \Psi_2 = \overline{\mathbf{C}}_0 \overline{\mathbf{C}}_0^T - \mathbf{C}_0^T \mathbf{C}_0 \tag{6.3.25}$$

Then same method as the proof of Theorem 6.3.1 can be used here to complete the proof.

◆

A bound for the cost performance of the uncertain system may be calculated using the following Lemma.

Lemma 6.3.5 For an uncertain system (6.3.21) with admissible domain of parametric uncertainty (6.1.3) employing any given stabilising controller as (6.3.3), a bound $J_b(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c)$ for the cost performance (6.1.4) is given by:

$$J_b(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c) = \mathrm{tr}(\mathbf{P}_b \mathbf{B}_0 \mathbf{B}_0^T)$$

Where $\mathbf{P}_b$ it the positive definite solution of the following equation:

$$\mathbf{A}_0^T \mathbf{P}_b + \mathbf{P}_b \mathbf{A}_0 + \varepsilon^2 \mathbf{P}_b \Psi_1 \mathbf{P}_b + \overline{\mathbf{C}}_0^T \overline{\mathbf{C}}_0 = \mathbf{0}$$

6.4 Examples

Two example applications are considered here to illustrate the methodology. In Example 6.1, the Robust LQG method is applied to a certain system and it is shown how the method enables a compromise to be found between cost performance and nonparametric robustness. In Example 6.2 robust LQG method is applied to a system with parametric uncertainties and it is shown how a compromise may be found between parametric robustness, nonparametric robustness a cost performance.

Example 6.1

Consider a certain system that may be described by (6.2.1) with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 \\ 1.2 \\ 2.5 \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{F} = 1$$

The variances of the noise and disturbance processes are both assumed to be unity: **W**=1, **V**=1. The weighting matrices in the cost function (6.1.4) are given as:

$$\mathbf{R} = 1, \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

To implement the method the system is first expressed in the H∞ format (3.2.26) by choosing $\mathbf{B}_1$ and $\mathbf{D}_2$ as (6.2.3) and defining the performance vector by the choice of $\mathbf{C}_1$ and $\mathbf{D}_1$ as (6.2.4). Robust LQG controllers may be designed by employing the standard H∞ design technique Lemma 3.3.2 (Doyle et al., 1989) to this system for suitable values of γ_0. The two extremes are: the LQG controller, $\gamma_0 = \infty$, giving optimal cost performance and; the H∞ sub-optimal controller for which the smallest permissible value of γ_0 (found to be $\gamma_0 = 6.02$) is used giving maximal nonparametric robustness. Several values of γ_0 are selected, choosing γ_0 between these values produces a compromising controller. The actual nonparametric robustness and cost performance may be analysed by minimising γ_0 in Lemma 2.3.5 and Lemma 6.2.1 respectively.

The performance and robustness of the LQG controller and H∞ output controller are shown in Table 6.1:

Controllers	Cost J	Guaranteed H∞ norm bound	Actual H∞ norm
LQG	49.98	∞	8.23
H∞	5884	6.02	6.01
Robust LQG 1	118.1	6.5	6.23
Robust LQG 2	75.89	7	6.77
Robust LQG 3	53.74	10	8.47

Table 6.1 Robust LQG Design

The relationship between the H∞ norm constraint and the cost performance is shown in Fig. 6.2.

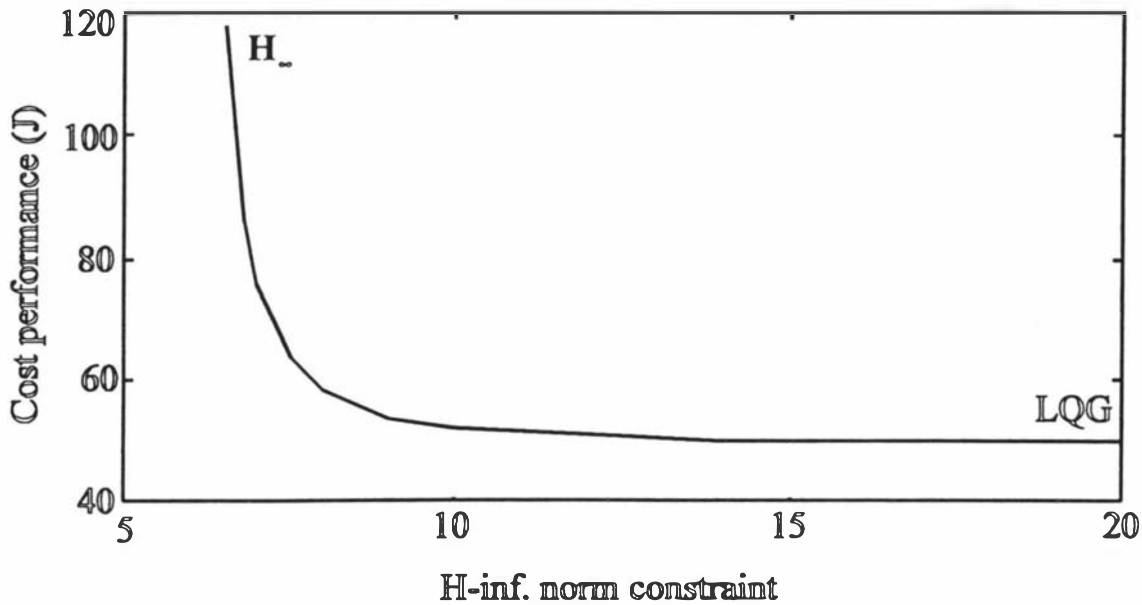

Fig. 6.1 The Cost performance Via H∞ norm Constraint.

Example 6.2

Consider an uncertain system with parametric uncertainty as

$$\begin{cases} \dot{\mathbf{x}}(t) = (\mathbf{A} + \Delta\mathbf{A}(t))\mathbf{x}(t) + (\mathbf{B}_2 + \Delta\mathbf{B}(t))\mathbf{u}(t) + \mathbf{Ed} \\ \mathbf{y}(t) = (\mathbf{C}_2 + \Delta\mathbf{C}(t))\mathbf{x}(t) + \mathbf{Fv} \end{cases}$$

with the parameter matrices as:

$$\mathbf{A} = \begin{bmatrix} -1 & -3 \\ 1.5 & -2 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

assume that the uncertainties of system can be modelled as:

$$\Delta\mathbf{A} = \begin{bmatrix} 1.1\varphi_1(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0.05\varphi_2(t) \\ 0 \end{bmatrix}, \quad \Delta\mathbf{C} = \begin{bmatrix} 0.2\varphi_3(t) & 0 \\ 0 & 0.1\varphi_3(t) \end{bmatrix}$$

where $\varphi_i(t), \{i = 1,2,3\}$ are uncertain values varying in $[-\varepsilon, \varepsilon]$.

The cost performance index used here is

$$\mathbf{J} = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u}) dt$$

with the weighting matrices and the noise covariances as:

$$\mathbf{Q} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \mathbf{R} = 10, \quad \mathbf{W} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

The cost value of certain part can be found to be: $\mathbf{J}_0 = 0.23$

(1). **Datum Case.**
As a datum, it is assumed that the magnitude of the admissible domain of parametric uncertainty is $\varepsilon=1$, as described in section 6.2.1, the system may be translated to H^∞ format (6.3.2). To provide robustness to nonparametric uncertainty, a H^∞ norm bound of 1 is required, i.e.,

$$\|\mathbf{T}_{\omega z}\|_\infty < 1$$

Then, a nominal description of the structure of the admissible domain of uncertainty may be given as the form of (6.1.3) with:

$$\mathbf{N}_a = \begin{bmatrix} 1.1 \\ 0 \end{bmatrix}, \quad \mathbf{M}_a = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{N}_b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{M}_b = 0.05$$

$$\mathbf{N}_c = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \mathbf{M}_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad |\varphi_i(t)| \le \varepsilon = 1, \quad i = 1,2,3$$

Hence an appended system is created in H^∞ format as (6.3.2) and a controller designed with the standard H^∞ technique Lemma 3.3.2 (Doyle et al., 1989) for $\gamma_0 = 1$. A search over the scalar parameters yielded the best controller for:

$$\alpha_a = 2.5, \quad \alpha_b = 0.3, \quad \alpha_c = 11$$

then a robust stabilising controller

$$\mathbf{A}_c = \begin{bmatrix} -6.47 & -3.89 \\ 1.02 & -3.01 \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} 1.11 & 0.01 \\ 0.03 & 0.23 \end{bmatrix}, \quad \mathbf{C}_c = \begin{bmatrix} -0.35 & -0.88 \end{bmatrix}$$

can be found such that closed loop system satisfies $\|\mathbf{T}_{\overline{\omega z}}\|_\infty < 1$, the cost performance bound is $J_b = 0.456$.

(2). **Increasing parametric uncertainties:**
To illustrate how the parametric robustness may be increased, the magnitude of the admissible domain of parametric uncertainty is increased to $\varepsilon=1.2$. So still considering the same structure uncertainties

$$\Delta\mathbf{A} = \begin{bmatrix} 1.1\varphi_1(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} 0.05\varphi_2(t) \\ 0 \end{bmatrix}, \quad \Delta\mathbf{C} = \begin{bmatrix} 0.2\varphi_3(t) & 0 \\ 0 & 0.1\varphi_3(t) \end{bmatrix}$$

but $\varphi_i(t)$ is varying in [-1.2, 1.2], i=1, 2 and 3, and the robustness requirement for non-parametric uncertainty is also

$$\|\mathbf{T}_{\omega z}\|_\infty < 1$$

To achieve the datum H^∞ norm bound the admissible domain may be redescribed as (6.3.15) with $\delta = 1.2$, thus uncertainties can be redescribed as:

$$\overline{\mathbf{N}}_a = 1.2\mathbf{N}_a = \begin{bmatrix} 1.32 \\ 0 \end{bmatrix}, \quad \overline{\mathbf{N}}_b = 1.2\mathbf{N}_b = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix}, \quad \overline{\mathbf{N}}_c = 1.2\mathbf{N}_c = \begin{bmatrix} 0.24 & 0 \\ 0 & 0.12 \end{bmatrix}$$

$$|\overline{\varphi}_i(t)| = \left|\frac{\varphi_i(t)}{1.2}\right| \le \varepsilon = 1, \quad i = 1,2,3$$

Again a standard H∞ controller is designed for the suitably appended system $\gamma_0 = 1$. A search over the scalar parameters yielded the best controller for:

$$\alpha_a = 2.5, \quad \alpha_b = 0.3, \quad \alpha_c = 10$$

then a robust stabilising controller

$$\mathbf{A}_c = \begin{bmatrix} -6.75 & -4.34 \\ 1.101 & -3.34 \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} 1.22 & 0.03 \\ 0.06 & 0.22 \end{bmatrix}, \quad \mathbf{C}_c = \begin{bmatrix} -0.20 & -1.32 \end{bmatrix}$$

can be found such that closed loop system has $\|\mathbf{T}_{\overline{\omega z}}\|_\infty < 1$, the cost performance bound is $J_b = 0.5681$.

(3). **Increasing nonparametric uncertainty:**

To illustrate how the nonparametric robustness may be increased, the required H∞ norm bound is reduced to $\gamma_0 = 0.6$, the datum admissible domain is retained, i.e., the robustness requirement for nonparametric uncertainty as

$$\|\mathbf{T}_{\omega z}\|_\infty < 0.6$$

To achieve this the admissible domain may be described as (6.3.15) with $\delta = 0.6$, thus uncertainties can be redescribed as:

$$\overline{\mathbf{N}}_a = 0.6\mathbf{N}_a = \begin{bmatrix} 0.66 \\ 0 \end{bmatrix}, \quad \overline{\mathbf{N}}_b = 0.6\mathbf{N}_b = \begin{bmatrix} 0.6 \\ 0 \end{bmatrix}, \quad \overline{\mathbf{N}}_c = 0.6\mathbf{N}_c = \begin{bmatrix} 0.12 & 0 \\ 0 & 0.06 \end{bmatrix}$$

$$|\overline{\varphi}_i(t)| = \left|\frac{\varphi_i(t)}{0.6}\right| \le \varepsilon = \frac{1}{0.6}, \quad i = 1,2,3$$

Now a standard H∞ controller is designed for the suitably appended system with $\gamma_0 = 0.6$. A search over the scalar parameters yielded the best controller for:

$$\alpha_a = 2.5, \quad \alpha_b = 0.3, \quad \alpha_c = 11$$

then a robust stabilising controller

$$\mathbf{A}_c = \begin{bmatrix} -5.68 & -7.12 \\ 2.95 & -6.18 \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} 1.22 & -0.04 \\ -0.13 & 0.31 \end{bmatrix}, \quad \mathbf{C}_c = \begin{bmatrix} 0.85 & -4.13 \end{bmatrix}$$

can be found such that closed loop system has $\|\mathbf{T}_{\overline{\omega z}}\|_\infty < 0.6$, the cost performance bound is $J_b = 0.5139$.

Performance Summary:

The results are summarised in following table, it is illustrated how either the parametric robustness (Case 2) or the nonparametric robustness (Case 3) may be increased compared to the datum (Case 1) but this will result in inferior cost performance.

	ε for Design	γ_0 for design	J_b
Case 1	1	1	0.454
Case 2	1.2	1	0.568
Case 3	1	0.6	0.514

6.5 DISCUSSION

A controller design approach is illustrated in this section that offers a compromise between parametric robustness, nonparametric robustness and cost performance. Magnitude of the admissible domain of parametric uncertainty and an H^∞ norm bound relating to the permissible size of nonparametric uncertainty are treated as design parameters and the system is guaranteed to be stable subject to these constraints. The controller is realised by the solution of a standard H^∞ problem constructed with explicit reference to a performance cost function.

It is also clear that if we use nominal model only, the designed control system must allow for large unknown residual uncertainty, so a low H^∞ norm bound is required. However, if we design controller based on both nominal model and parametric uncertainties, the unknown residual uncertainty should be smaller, so a high H^∞ norm bound is allowed.

CHAPTER 7

CONCLUSIONS AND FUTURE WORK

This thesis is a partial result of my three years research work on robust control system design and analysis. The core of the thesis can be divided into two fields. One is the development of some robust controller design techniques for systems subject to uncertainties, such that the stability of the controlled system can be guaranteed and the performance degradation is minimal. The another is to analyse the robust properties of a given controlled system, such that a robust stability bound of uncertainties can be determinated subject to the requirements of robust stability and a specific performance degradation rate.

To conclude this thesis, a summary of results and discussions will be given in the next section. The robust controller design and robustness analysis methodologies presented aim to be less conservative than traditional methods. Two of the major weakness remaining in the field at present are: for robust controller design, only the worst case uncertainty is considered, this may make the controlled system unnecessarily conservative. Secondly, for robustness analysis, the criteria for adherence to performance specifications are generally sufficient not necessary, again tending to unnecessary conservativeness. To advance these areas, some interesting and significant future work will be discussed.

7.1 Summary of Results and Discussions

Robust controller design and robustness analysis addressees a broad range of problems which are considered for state feedback and dynamic output feedback systems separately. Two robustness measures, stability robustness and performance robustness, are considered in the robust design and analysis procedures.

7.1.1 Fundamentals: Description of System Models and Associated Uncertainties

The system used for robust controller design or robustness analysis may be described either by a nominal model alone or a nominal model and an uncertainty model. Since it is impossible to a account for all uncertainty, there always exists some residual unknown uncertainty. A good robust controller design technique should provide a certain level of inherent robustness to such unknown residual uncertainty. By representing the majority of the uncertainty precisely with an uncertainty model, the level of robustness to residual uncertainty may be reduced, this will provide a more focused, less conservative design.

The description of the uncertainty is critical and will have a direct effect on the conservativeness of the controller or robustness bound produced. Uncertainties may be described parametrically or nonparametrically, with or without some fixed structure. There is great potential for conservativeness as both types of uncertainty description are worst case: Nonparametric uncertainties are characterised by a H^∞ norm bound and parametric uncertainty by a maximum singular value. Thus any uncertainty included in the admissible domain unnecessarily may have a significant effect.

Some attention has been paid to the choice of uncertainty description for structured parametric uncertainties. Overall the choice of the 'best' description of the uncertainty is quite a complex problem to which no general solution is known. However, some guidelines are firstly, to choose a description with minimal dimensioned structural matrices and secondly, if it is possible within this constraint, choose a matched norm bounded format and employ the special methods for this case.

It is noted that even for a fixed structured parametric uncertainty: $\Delta\mathbf{A} = \mathbf{N}\Phi(t)\mathbf{M}$, there are some different ways to describe it, this is because we can always choose different parameters to give a description as: $\Delta\mathbf{A} = (\alpha\mathbf{N})\Phi(t)(\mathbf{M}/\alpha)$. This method has been used throughout this thesis. It is illustrated by examples that the best value of α can be found by the searching to optimise the particular goal. For systems with two or more uncertainty items, there could exist two or more scaling parameters.

For state feedback control systems, only parametric uncertainties effect the closed loop system performance, if the state feedback controller can robustly stabilise the system with parametric uncertainties, then for any bounded (i.e., stable) nonparametric uncertainty, the stability robustness of the closed loop system will also be guaranteed. However, for systems with output feedback controllers, both parametric and nonparametric uncertainties in the system will effect the closed loop system performance. Thus, to guarantee robustness both parametric and nonparametric uncertainty need be considered. So state feedback and output feedback systems are treated separately in this thesis.

7.1.2 Robust Controller Design

1). Robust LQR Design for State Feedback Controlled Systems

A RLQR design methodology is presented in this thesis which both guarantees good stability robustness and performance robustness. Stability robustness of the closed loop

system is guaranteed for all admissible parametric uncertainties and the cost performance is guaranteed to lie within a specified bound and degradation of performance is proved to be minimal.
The robust LQR methodology can also be implemented by employing existing H^∞ techniques on a scaled system, numerical tools for the solution of such problems are now commonly available. There is an inherent trade off between the stability robustness and performance robustness. As the size of the admissible domain of uncertainty is increased, the stability robustness (range of uncertainty for which system is guaranteed to remain stable) should be naturally increased. However, the performance degradation will also increase, hence the performance robustness decreases. Finally, as mentioned in section 7.1, the performance robustness is sensitive to the description of uncertainties and the selection of scaling parameters.

2). Robust LQG Design for Dynamic Output Feedback Controlled Systems
To overcome the inadequate robustness of traditional LQG design, a RLQG design methodology is presented which both guarantees good stability robustness and performance robustness. The RLQG controller is designed with respect to both a parametric uncertainty and unknown residual uncertainty. The residual unknown uncertainty is modelled nonparametrically, robustness subject to it may be measured by the H^∞ norm bound of the system. The desired H^∞ norm bound and the magnitude of the admissible domain of parametric uncertainty are treated as design parameters. An interconnection between standard H^∞ design for a certain system and RLQG design for an uncertain system has been established. On the basis of on this interconnection and the use of H^∞ design theory for a scaled certain system, the RLQG controller can guarantee closed loop stability for all admissible uncertainties and provide a cost performance bound. It is shown that an inherent trade off exists between stability robustness and performance robustness.

Both RLQR and RLQG robust controller design techniques are based on the worst case of admissible uncertainty. Hence the performance bound, is also considered as the worst case and often, the real performance value could be much less than this bound.

7.1.3 Robustness Analysis

1). Stability robustness analysis:

On the basis of robustness principles, a systematic framework is constructed to analyse the stability robustness of some current modern control design methods such as LQR design, H^∞ design and H_2/H^∞ design for state feedback control systems and LQG design, LQG/LTR design, H^∞ design and H_2/H^∞ design for dynamic output feedback control systems. These methods are normally applied to systems which are only described by nominal models. According to previous discussions, for state feedback controlled systems, it is reasonable to consider unknown parametric uncertainty only. However, for output feedback controlled systems, both unknown parametric and nonparametric uncertainties should be considered, but for simplicity, all uncertainty may be represented by a nonparametric model.

When the robustness of LQR design, H^∞ design, H_2/H^∞ design and LQG/LTR design is analysed, if the H^∞ norm bound of the appropriate closed loop system is finite, then the controlled system has some inherent robustness to unknown residual uncertainty. However the H^∞ norm bound of the LQG design is infinite, this implies that there is no inherent robustness. It is also found that although LQR design, H^∞ design, H_2/H^∞ design and LQG/LTR design can have good inherent robustness to unknown uncertainty, this robustness is coupled with other design parameters. For example, in LQR design the robustness is coupled with the weighting matrices of the cost function, and in H^∞ design it coupled with the weighting matrices of the disturbance and performance vectors. So it is quite possible that there is insufficient inherent robustness when we design a robust control system based on the nominal system model alone, and furthermore, the design result could also be conservative since no information of the uncertainty is used.

2). Performance robustness analysis:

Stability robustness is not enough for a good robust control system design, in most cases, long before the onset of instability, the closed loop performance will degrade to the point of unacceptability, hence the variation of the control performance should be also concerned for all admissible uncertainties. It is clear that for an uncertain system it is impossible to find the exact performance value, however, since the uncertainties considered here are constrained by some kind of bound, a performance bound of the closed loop system can be found over all admissible uncertainties. The performance robustness considered in the thesis is the measure of maximal performance degradation, hence to design an "optimal" robust controller means to minimise the performance degradation, i.e., to minimise the performance bound over all admissible uncertainties.

3). Robust uncertain bounds:

The robustness analysis is also considered for systems with parametric uncertainties. The relationship between the size of uncertainties and both stability and performance robustness is studied. For the robust stability and a given performance degradation requirement, some uncertainty bounds were found for a general given controller. The results are also applied to LQR controllers and RLQR controllers. It is evident that for any given robustness requirements, the maximal robustness bound can be provided by RLQR design. An expression for a maximal robustness bound for a given system subject to a performance criterion is developed which is controller independent. This bound is sufficient to guarantee the existence of an RLQR controller which enables the adherence of the performance criterion to be guaranteed. The stability robustness bound can be obtained by relaxing the performance degradation requirement. It is shown that for the particular case of stability robustness bounds of LQR controllers this method agrees with a previous result by Neto et. al. (1992), and furthermore, the maximal stability robustness bound is provided by RLQR design.

An alternative approach to robust controller design is proposed which effectively uses a performance criterion as the design parameter and the controller produced will satisfy this performance criterion subject to uncertainties within a resulting (maximal) robustness bound. The inherent trade off between robustness and performance may be quantitatively assessed using this robustness analysis method. It is shown, both for general and a specific example system, that if greater performance bounds are permitted then larger robustness bounds result. Conversely, if lower performance bounds are demanded then smaller robustness bounds result.

7.2 Conclusions

This thesis reviews and develops the fundamentals of robust controller design and analysis, the contributions can be divided into four areas which relate to full state feedback and dynamic output feedback; based on a given nominal system model alone and both nominal system model and uncertainty model. Methods of robust controller design and robustness analysis have been addressed for each area.

Many techniques, such as LQR, H_∞ or H_2/H_∞ methods, exist to design full state feedback controllers based on *a nominal system model*. These techniques have also been shown to provide inherent robustness for controlled systems. However, the level of inherent robustness is coupled with design parameters relating to nominal system performance. For given controlled systems, some robustness conditions were developed to determine their inherent robustness such that if they were satisfied, the robust stability to a given uncertainty would be guaranteed. Thus, it is found that all existing full state feedback control design techniques will provide a certain level of inherent robustness.

For systems with both parametric uncertainties and some unknown uncertainty, a technique to design optimal full state feedback robust controller, the robust LQR design methodology, is presented which both guarantees closed loop stability for all admissible parametric uncertainties and provides a minimal performance bound. At the same time, the RLQR can also provide some inherent robustness for the unknown uncertainty. The inherent trade off between stability robustness and performance robustness can be illustrated by considering the effect of increasing the magnitude of the uncertainty domain. It has also been shown that this methodology can be implemented by employing existing H_∞ techniques on a scaled system. The performance robustness is sensitive to the precise format in which the uncertainty bound is described and it is shown that a format giving a precise description will give good robust performance.

In the face of both parametric uncertainty and some unknown uncertainties, a robustness analysis procedure for a given full state feedback controlled system is presented. This produces a robustness bound with respect to the given controller for a given performance degradation requirement such that for any uncertainty within this bound it is guaranteed that the performance degradation will meet the requirement. An expression for a maximal robustness bound for a given system subject to a performance criterion is developed which is controller independent. This bound is sufficient to guarantee the existence of an RLQR controller which enables the adherence of the performance criterion to be guaranteed. The inherent trade off between robustness and performance may be quantitatively assessed using this robustness analysis method. It is shown, both in general and for a specific example system, that if greater performance bounds are permitted then larger robustness bounds result. Conversely, if lower performance bounds are demanded then smaller robustness bounds result. An alternative approach to robust controller design is proposed which effectively uses a performance criterion as the design parameter and is valid for uncertainties within a resulting (maximal) robustness bound.

Applying the robustness measures developed in thesis to determine the inherent robustness of existing controller design methods provides good agreement with some well-known result: full state feedback controllers have good inherent robustness; the LQG has poor robustness and the LQG/LTR method enables the robustness properties of the LQR method to be recovered by the deployment of a high gain observer or controller. $H\infty$ methods can offer good robustness properties. However, for all these methods, the cost performance is compromised to achieve desired robustness in non-systematic way.

For a nominal system and unknown uncertainty, the RLQG design method, which uses the cost function weighting matrices to determine suitable definitions for the disturbance and performance vectors of a related $H\infty$ problem, is employed to overcome the robustness shortage of LQG design. The level of inherent robustness to unknown uncertainties may be evaluated by calculating a suitable $H\infty$ norm bound for a scaled $H\infty$ control system. By increasing the level of inherent robustness, cost performance degradation will also be increased. The technique is similar to minimum Entropy $H\infty$ controller design but is motivated from the perspective of making the LQG method more robust. Given some parametric uncertainty the RLQG controller design provides both good stability and cost performance robustness for all admissible values of uncertainty.

Thus the RLQG offers a compromise between parametric robustness, nonparametric robustness and cost performance. Magnitude of the admissible domain of parametric uncertainty and an $H\infty$ norm bound relating to the permissible size of nonparametric uncertainty are treated as design parameters and the system is guaranteed to be stable subject to these constraints. The controller is realised by the solution of a standard $H\infty$ problem constructed with implicit reference to a performance cost function. These results allowing extension to systems with parametric uncertainty permit the explicit use of a parametric uncertainty model in the design. This offers the designer greater flexibility to reach a compromise between cost performance and robustness.

7.3 Future Work

In author's option, a main area requiring attention is the reduction of the conservatism of the current robust design and analysis methods. This will make robust control systems more practical and reasonable. Since the current control design methods consider only the worst cases of uncertainties, the conservative design of both stability and performance robustness result, hence some new robustness principles are desired to reduce this conservatism. The main aim in robustness analysis should be to find less conservative, sufficient and necessary robustness criteria for given controlled systems.

Some specific work related to optimal robust LQR and LQG controller design still needs to be done. For robust LQR design, it is important to find existence conditions for the controller solution. Also, since the uncertainty description has a significant effect on the robust controller design, a method to describe uncertainty optimally would be very useful. It has been shown that the performance of the closed loop system is sensitive to the selection of the scaling parameters, so it is necessary to develop a method to choose these parameters optimally. For the RLQG design, a simple existence condition for the controller solution will be very useful for the designer. At the same time, the optimality of the cost value subject to H^∞ norm constraint needs to be studied. To complete the methodology, it is also necessary to develop a method to choose the scaling parameters optimally.

To advance the field, the methods need to be tested by application to realistic problems. For a particular uncertain system with output feedback controller, several design methods such as LQG, H^∞ optimal, LQG/LTR and Robust LQG can be used and some comparison made. Stability and performance robustness could be two measures of this comparison. To test this, some benchmark problems (Nie and Bernstein, 1991) could be studied.

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