A NON-LINEAR BOUNDARY VALUE PROBLEM ARISING IN THE THEORY OF THERMAL EXPLOSIONS

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ABSTRACT

When a heat-producing chemical reaction takes place within a confined region, then under certain circumstances a thermal explosion will occur. In investigating from a theoretical viewpoint the conditions under which this happens, it is necessary to study the behaviour of the solution of a certain non-linear parabolic initial-boundary value problem.

A frequently used approach is to study the problem indirectly, by investigating whether positive steady-state solutions exist; the underlying assumption is that positive steady-state solutions exist if and only if a thermal explosion does not occur. The main theme of this thesis is the development and application of an alternative direct approach to the problem, involving the construction of upper and lower solutions for the parabolic problem and the application of appropriate comparison theorems. The assumption here is that a thermal explosion will not occur if and only if the solution of the parabolic problem remains bounded for all positive time.

Following three chapters of introductory material, Chapter 4 contains a survey of some of the important known results concerning the existence of positive steady-state solutions, especially those dealing with the effect on the theory of different assumptions as to the rate at which heat is produced in the reaction.

The comparison theorems that are used in the alternative approach, which are modified versions of known results, are proved in Chapter 5.

In Chapter 6, the equivalence of the two criteria mentioned above for the occurrence or non-occurrence of a thermal explosion is established under fairly general conditions. Also in this chapter, a critical value λ^* is defined for a parameter λ appearing in the problem, such that a thermal explosion will not occur if the value of λ is smaller than λ^*, but will occur if the value of λ is greater than λ^*.

In Chapter 7, upper and lower solutions are constructed for the time-dependent problem under a variety of assumptions as to the rate at which heat is produced in the reaction, and these are used to obtain a number of theorems concerning the behaviour of the solution of the problem, especially as the time variable tends to infinity. The information obtained from these theorems is related to and compared with that known from investigations of the existence of positive steady-state solutions. In conclusion, a theorem is proved concerning the effect of reactant consumption on the theory. This is examined in the light of

some recent research, and an apparent defect which is thereby revealed in the usual criteria for the occurrence of a thermal explosion is discussed.

The theorems of Chapter 7 are employed in Chapter 8 to obtain rigorously derived bounds for the critical parameter λ^*, for a number of different shapes of the region in which the reaction takes place; these bounds are compared with known estimates for λ^* obtained using an empirically derived formula.

The thesis concludes, in Chapter 9, by using the methods of Chapters 7 and 8 to obtain some results concerning the case where the boundary condition is non-linear.

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CONTENTS

INDEX TO THEOREMS

1 INTRODUCTION

The mathematical problem discussed in this thesis arises from a topic in chemical kinetics, the study of the evolution in time of chemically reacting systems. Suppose we are dealing with a heat-producing reaction taking place in a confined region. For simplicity, we suppose for the time being that there is no consumption of reactant. If the heat produced by the reaction cannot all be removed at the boundary of the region, the temperature of the reactant will rise, leading to an increase in the reaction rate, in turn producing more heat. In practice, one of two things then happens. Either the rate of increase of temperature gradually diminishes and the system approaches a steady state, or the temperature increases rapidly and without limit, and what is usually called a *thermal explosion* takes place. An elementary discussion of this phenomenon is given by Boudart[5, pp.160-163], and additional information may be found in the book by Bradley[6, especially pp.2,8-15]. The problem with which we shall be concerned is that of determining whether or not a thermal explosion will take place in a given situation.

Suppose we have a heat-producing reaction taking place within a region V bounded by a surface S. We shall continue to ignore for the time being the effect of reactant consumption, which will be commented on in Ch.7, and we also assume that the thermal parameters of the system are constant in space and time. Then (see, for example, Ozisik's book[26, p.6]) the system is described by a differential equation of the form

$$K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \rho c \frac{\partial u}{\partial t} + f(u) = 0$$

for (x,y,z) in the interior of V, and time $t > 0$, together with the initial-boundary conditions

$$K \frac{\partial u}{\partial \nu} + Hg(u) = 0 \quad \text{for } (x,y,z) \text{ on } S \text{ and } t > 0$$

$$u = 0 \quad \text{for } (x,y,z) \text{ on } V \cup S \text{ when } t = 0.$$

Here $u(x,y,z,t)$ is the difference between the temperature T at any point and the ambient temperature T_a, K, ρ and c are the thermal conductivity, density and specific heat respectively of the reactant, H is the surface heat transfer coefficient and $\frac{\partial u}{\partial \nu}$ is the outward normal derivative to S. We shall say that thermal explosion takes place if $u(x,y,z,t) \to \infty$ as $t \to \infty$ or if $u(x,y,z,t) \to \infty$ as $t \to \tau-$ (τ finite).

The form of the heat-generation function f is still a matter for debate. It is proportional to the rate at which the reaction takes place, and the classical form for it is the empirical one due to Arrhenius:

$$f(u) = C_1\exp(-\frac{C_2}{u + T_a})$$

$$= C_1\exp(-\frac{C_2}{T}) \quad (C_1,C_2 > 0 \text{ and independent of } u).$$

Later theories lead to the replacement of the constant C_1 by an expression of the form $C_3T^{n/2}$ where n is a positive integer which depends on the nature of the reaction and can at present be found only empirically (see, for example, the books by Glasstone and Lewis[15, pp.626-638] and Kaufman[17, pp.198-214 and 233-240]).

In work on the theory of thermal explosions, however, it is usual to use the so-called Frank-Kamenetskii approximation for f, introduced by D.A.Frank-Kamenetskii[13]:

$$f(u) = C_4e^{C_5u} \quad (C_4,C_5 > 0).$$

While this is indeed an approximation to the Arrhenius expression for f(u) when u is small, the theoretical justification for using it in the study of thermal explosions (where large values of u occur) is unclear. It may well be that for large values of u the Arrhenius expression is no longer valid, and the Frank-Kamenetskii expression is in fact more accurate. Alternatively, it may be that situations in which the use of the Frank-Kamenetskii approximation would lead to significantly inaccurate answers have not yet arisen in practice. We shall be particularly concerned in this thesis with the effect on the theory of different assumptions as to the form of f.

Various assumptions may also be made about the form of the function g which appears in the boundary condition. The usual approach is to assume that heat loss at the boundary follows Newton's law of cooling, so that $g(u) = u$ and the boundary condition is linear. Most of the discussion in this thesis is concerned with the linear boundary condition, but in Ch.9 we shall discuss the effect of assuming a non-linear boundary condition. There are two non-linear boundary conditions which arise naturally, corresponding to different cooling processes at the boundary. If cooling at the boundary is by natural convection, then $g(u) = u^{5/4}$, while if cooling is by thermal radiation, then $g(u) = (u+T_a)^4 - T_a^4$ and $H = \sigma\varepsilon$ where σ is the Stefan-Boltzmann constant and ε is the emissivity of the surface (see Ozisik's book[26, pp.7-9 and 348-349]). The discussion in Ch.9 covers more general non-linear boundary conditions as well as these two conditions in particular.

The customary method of tackling the problem of whether or not a thermal explosion will take place is to equate the absence of thermal explosion with the existence of positive stable steady-state solutions, i.e. solutions of the time-independent equation

$$K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + f(u) = 0 \quad \text{for } (x,y,z) \text{ in the interior of } V$$

together with the boundary condition

$$K\frac{\partial u}{\partial \nu} + Hg(u) = 0 \quad \text{for } (x,y,z) \text{ on } S.$$

The underlying assumption here is that if positive stable steady-state solutions $u(x,y,z)$ exist, then the solution of the original time-dependent problem will approach one of these steady states as $t \to \infty$, and so explosion will not take place. We shall show in Ch.6 how this assumption may be justified mathematically.

A discussion of this question of the existence of positive steady-state solutions, treated from the chemist's point of view, is given by Boddington, Gray and Harvey[4]. These authors, using the linear boundary condition and the Frank-Kamenetskii approximation for f, with a change to a suitably chosen new variable θ proportional to u, obtain the equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} + \gamma e^{\theta} = 0 \quad \text{for } (x,y,z) \text{ in the interior of } V$$

together with the boundary condition

$$K\frac{\partial \theta}{\partial \nu} + H\theta = 0 \quad \text{for } (x,y,z) \text{ on } S.$$

Here γ is a parameter whose value depends on the physical and chemical properties of the reactant and on the ambient temperature. A positive stable solution of this steady-state problem is known to exist if and only if γ is less than or equal to a critical value denoted by γ_{crit} (we shall discuss this point in more detail in Ch.4). Thus, if the mathematical formulation of the problem is a reasonably accurate model of the physical situation, a thermal explosion will occur if $\gamma > \gamma_{crit}$ but not if $\gamma \leq \gamma_{crit}$. The value of γ_{crit} depends upon the shape of V.

The authors are chiefly concerned with methods of determining, or determining approximately, the values of γ_{crit} for various shapes V, using a combination of analytical and empirical methods. In Ch.8, we shall apply the methods developed in this thesis to the problem of obtaining lower and upper bounds for γ_{crit}, for various shapes V, and compare the bounds so obtained with the estimates given by Boddington, Gray and Harvey.

In treating the thermal explosion problem from the mathematician's point of view, we shall work for the most part with equations more general than those discussed so far. Letting x denote the n-dimensional vector $(x_1,x_2,\dots x_n)$, we consider the equation

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} + \lambda f(x,t,u) = 0$$

for x in the interior of an n-dimensional region V and $t > 0$, together with the initial-boundary conditions

$$d_0(x,t)g(u) + d_1(x,t)\frac{\partial u}{\partial n} = 0 \quad \text{for x on the boundary S of V and } t > 0$$

$$u = u_0(x) \quad \text{for x on } V \cup S \text{ when } t = 0$$

where the differential operator in the first equation is uniformly parabolic, $\frac{\partial u}{\partial n}$ denotes an arbitrary (not necessarily normal) outward directional derivative, and appropriate conditions are imposed upon the coefficients a_{ij}, b_i, c, d_0, d_1 and the functions f and g.

The corresponding time-independent (i.e. steady-state) problem is described by an equation of the form

$$\sum_{i,j=1}^{n} \hat{a}_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \hat{b}_i(x)\frac{\partial u}{\partial x_i} + \hat{c}(x)u + \lambda\hat{f}(x,u) = 0$$

for x in the interior of V, together with the boundary condition

$$\hat{d}_0(x)g(u) + \hat{d}_1(x)\frac{\partial u}{\partial n} = 0 \quad \text{for x on S}$$

where the coefficients are the limits, as $t \to \infty$, of the corresponding time-dependent coefficients, and $\hat{f}(x,u)$ is the limit of $f(x,t,u)$.

We shall discuss in Ch.4 some of the more important results that have been obtained on the existence of positive stable steady-state solutions. In Chs.7, 8 and 9 we shall employ an alternative method of investigating the behaviour of $u(x,t)$ as $t \to \infty$, by using the comparison theorems proved in Ch.5 to directly attack the original time-dependent equation. We should mention here that there are indications that in certain cases neither of these approaches to the thermal explosion problem is adequate; some remarks on this point appear at the end of Ch.7.

2 AN EXAMPLE

Before proceeding, we give a simple example of the sort of equation we shall be studying. We shall be using this example from time to time for illustrative purposes and as a counter-example.

Consider first the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + ku + \lambda = 0 \ (-1 < x < 1,\ t > 0)$$

where we assume $k > 0$, $\lambda > 0$; further, $u(x,t)$ satisfies the initial-boundary conditions

$$u(x,0) = 0 \text{ for } -1 \le x \le 1$$

$$u(-1,t) = 0 \text{ for } t \ge 0$$

$$u(1,t) = 0 \text{ for } t \ge 0.$$

Using Laplace transform techniques (see Appendix for details) it may be shown that if $k \neq \frac{(2n+1)^2\pi^2}{4}$ for $n = 0,1,2,\ldots.$, the above problem has the solution

$$u(x,t) = \frac{4\lambda}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{\{k - \frac{(2n+1)^2\pi^2}{4}\}(2n+1)} \cos \frac{(2n+1)\pi x}{2}\, e^{(k - \frac{(2n+1)^2\pi^2}{4})t}$$

$$+ \frac{\lambda \cos \sqrt{k}x}{k \cos \sqrt{k}} - \frac{\lambda}{k}$$

while if $k = \frac{(2N+1)^2\pi^2}{4}$ for some $N = 0,1,2,\ldots.$, the above problem has the solution

$$u(x,t) = \frac{4\lambda}{\pi} \sum_{n \neq N} \frac{(-1)^n}{\{k - \frac{(2n+1)^2\pi^2}{4}\}(2n+1)} \cos \frac{(2n+1)\pi x}{2}\, e^{(k - \frac{(2n+1)^2\pi^2}{4})t}$$

$$+ \frac{3\lambda(-1)^N}{\pi k(2N+1)} \cos \frac{(2N+1)\pi x}{2} + \frac{\lambda(-1)^N x}{k} \sin \frac{(2N+1)\pi x}{2}$$

$$+ \frac{\lambda\pi}{k}(-1)^N(2N+1)t \cos \frac{(2N+1)\pi x}{2} - \frac{\lambda}{k}.$$

From this we see that, regardless of the value of $\lambda > 0$, we have:

If $0 < k < \frac{\pi^2}{4}$, then $u(x,t) \to \frac{\lambda}{k}\{\frac{\cos \sqrt{k}x}{\cos \sqrt{k}} - 1\}$ as $t \to \infty$.

If $k \ge \frac{\pi^2}{4}$, then $u(x,t)$ is unbounded as $t \to \infty$.

The corresponding steady-state problem is

$$\frac{d^2 u}{dx^2} + ku + \lambda = 0 \ (-1 < x < 1)$$

where $k > 0$, $\lambda > 0$, and the boundary conditions are $u(-1) = u(1) = 0$. In this case (again see Appendix for details) the situation is as follows:

If $k \neq \frac{m^2\pi^2}{4}$ for any $m = 1,2,3,\ldots$, the solution is

$$u(x) = \frac{\lambda}{k}\left[\frac{\cos\sqrt{k}x}{\cos\sqrt{k}} - 1\right].$$

If $k = n^2\pi^2$ for some $n = 1,2,3,\ldots$, the solution is

$$u(x) = \frac{\lambda}{k}\left[\frac{\cos\sqrt{k}x}{\cos\sqrt{k}} - 1\right] + B\sin\sqrt{k}x$$

where B is arbitrary.

If $k = \frac{(2n+1)^2\pi^2}{4}$ for some $n = 0,1,2,\ldots$, no solution exists.

In particular, if $0 < k < \frac{\pi^2}{4}$ then the steady-state problem has the positive solution $u(x) = \frac{\lambda}{k}\left[\frac{\cos\sqrt{k}x}{\cos\sqrt{k}} - 1\right]$, which is also the limit of the solution $u(x,t)$ of the time-dependent problem as $t \to \infty$. For larger values of k, positive solutions of the steady-state problem do not exist; those solutions which do exist can easily be seen to be negative for certain values of x in $(-1,1)$.

If we now take $k = \lambda$, the differential equation becomes

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + \lambda(u+1) = 0$$

with boundary conditions as before. This is a simple example of the sort of equation we wish to study, with $f(x,t,u) = u+1$ in this case, using the notation introduced in Ch.1. In this example, we have that if $0 < \lambda < \frac{\pi^2}{4}$, the solution $u(x,t)$ tends to the (positive) solution of the corresponding steady-state problem as $t \to \infty$, while if $\lambda \geq \frac{\pi^2}{4}$ the solution $u(x,t)$ is unbounded as $t \to \infty$, and the steady-state problem has no positive solutions. So we have a critical value $\frac{\pi^2}{4}$ for λ; if λ is greater than or equal to this critical value, the solution "explodes". In the sequel, we shall investigate for which choices of the function f behaviour similar to this occurs, and also investigate the sort of behaviour which occurs for other choices of the function f.

We mention in passing that the original example with $k \neq \lambda$ will be needed as a counter-example later.

3 DEFINITIONS AND NOTATION

It is convenient to collect here some notational conventions and basic definitions. Firstly, the following notation will be used throughout the rest of this thesis:

V denotes a bounded, open, connected set of points in n-dimensional real Euclidean space E_n.

$x = (x_1, x_2, \ldots x_n)$ denotes a point in E_n.

$D_T = \{(x,t): x \in V,\ 0 < t \leq T\}$, regarded as a subset of E_{n+1}.

$D = \{(x,t): x \in V,\ t > 0\}$, also regarded as a subset of E_{n+1}.

$\overline{}$ denotes closure.

We shall next define several important function spaces. We follow, with some modifications, the definitions used by Ladyzenskaja, Solonnikov and Ural'ceva[21, pp.2-10]. In framing these definitions, we shall write

$$D_x^{\ell} u = \frac{\partial^{|\ell|} u}{\partial x_1^{\ell_1} \partial x_2^{\ell_2} \ldots \partial x_n^{\ell_n}}$$

where $\ell = (\ell_1, \ell_2, \ldots \ell_n)$, the $\ell_i (i = 1,2,\ldots n)$ being non-negative integers, and $|\ell| = \ell_1 + \ell_2 + \ldots + \ell_n$. We shall use $\sum\limits_{|\ell|=k}$ to denote summation over all derivatives of a given order k. We shall also write, for any non-negative integer r, $D_t^r v = \dfrac{\partial^r v}{\partial t^r}$.

Now let k be a non-negative integer and α a real number with $0 < \alpha < 1$. We say a function $u:\overline{V} \to R$ satisfies a Hölder condition with exponent α on $\overline{V}$ if $\langle u \rangle_V^{(\alpha)}$ is finite, where

$$\langle u \rangle_V^{(\alpha)} = \sup_{\substack{x,y \in \overline{V} \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}.$$

For any function $u:\overline{V} \to R$ which has continuous derivatives up to order k, and whose derivatives of order k satisfy a Hölder condition with exponent α on $\overline{V}$, we define:

$$|u|_V^{(0)} = \sup_{x \in \overline{V}} |u(x)|.$$

$$|u|_V^{(j)} = \sum_{|\ell|=j} |D_x^{\ell} u|_V^{(0)} \quad \text{for } j = 1,2,\ldots .$$

$$|u|_V^{(k+\alpha)} = \sum_{|\ell|=k} \langle D_x^{\ell} u\rangle_V^{(\alpha)} + \sum_{j=0}^{k} |u|_V^{(j)}.$$

The Hölder space $C^{k+\alpha}(\overline{V})$ is the space of all functions $u:\overline{V} \to R$ for which $|u|_V^{(k+\alpha)}$ is finite; with $|u|_V^{(k+\alpha)}$ as norm, the space $C^{k+\alpha}(\overline{V})$ is a Banach space.

Again let k be a non-negative integer and α a real number with $0 < \alpha < 1$. For any function $u:\overline{D}_T \to R$, we define Hölder constants $\langle u\rangle_{x,D_T}^{(\alpha)}$ and $\langle u\rangle_{t,D_T}^{(\alpha)}$ thus:

$$\langle u\rangle_{x,D_T}^{(\alpha)} = \sup_{\substack{(x,t),(y,t)\in\overline{D}_T \\ x\neq y}} \frac{|u(x,t) - u(y,t)|}{|x - y|^{\alpha}}.$$

$$\langle u\rangle_{t,D_T}^{(\alpha)} = \sup_{\substack{(x,t),(x,\tau)\in\overline{D}_T \\ t\neq \tau}} \frac{|u(x,t) - u(x,\tau)|}{|t - \tau|^{\alpha}}.$$

For any function $u:\overline{D}_T \to R$ which has continuous derivatives of the form $D_t^r D_x^s u$, where $2r + |s| \leq k$, we define:

$$\|u\|_{D_T}^{(0)} = \sup_{(x,t)\in\overline{D}_T} |u(x,t)|.$$

$$\|u\|_{D_T}^{(j)} = \sum_{2r+|s|=j} \|D_t^r D_x^s u\|_{D_T}^{(0)} \quad \text{for } j = 1,2,\ldots .$$

$$\|u\|_{D_T}^{(k+\alpha)} = \sum_{2r+|s|=k} \langle D_t^r D_x^s u\rangle_{x,D_T}^{(\alpha)} + \sum_{0<k+\alpha-2r-|s|<2} \langle D_t^r D_x^s u\rangle_{t,D_T}^{\left(\frac{k+\alpha-2r-|s|}{2}\right)} + \sum_{j=0}^{k} \|u\|_{D_T}^{(j)}.$$

The Hölder space $H^{k+\alpha}(\overline{D}_T)$ is the space of all functions $u:\overline{D}_T \to R$ for which $\|u\|_{D_T}^{(k+\alpha)}$ is finite; with $\|u\|_{D_T}^{(k+\alpha)}$ as norm, the space $H^{k+\alpha}(\overline{D}_T)$ is a Banach space.

For any real $q \geq 1$, we define the Banach space $L_q(V)$ in the usual way to be the space consisting of all real measurable functions on V with finite norm

$$\|u\|_{q,V} = \left[\int_V |u(x)|^q dx\right]^{\frac{1}{q}}.$$

We say that $u \in L_q(V)$ has an L_q-derivative on V with respect to x_i if there exists $v \in L_q(V)$ such that, for all functions φ which are infinitely differentiable on $\overline{V}$ and vanish on the boundary of V,

$$\int_V v(x)\varphi(x)dx = -\int_V u(x)\,\frac{\partial\varphi}{\partial x_i}\,dx$$

(the fact that the integral on the right is finite follows from Hölder's inequality). We write $v = \frac{\partial u}{\partial x_i}$. Higher order L_q-derivatives of u are defined iteratively. Integration by parts shows that when u has a classical derivative $\frac{\partial u}{\partial x_i} \in L_q(V)$, then the L_q-derivative of u coincides with the classical derivative.

For q as above and k a non-negative integer, we define the Sobolev space $S_{k,q}(V)$ to be the space of all functions $u \in L_q(V)$ having L_q-derivatives on V up to order k, so that $|u|_V^{(k,q)}$ is finite, where:

$$|u|_V^{(k,q)} = \sum_{j=0}^{k}\left\{\sum_{|\ell|=j} \|D_x^{\ell}u\|_{q,V}\right\}.$$

With $|u|_V^{(k,q)}$ as norm, the space $S_{k,q}(V)$ is a Banach space.

For any real $q \geq 1$, we define the Banach space $L_q(D_T)$ to be the space consisting of all real measurable functions on D_T with finite norm

$$\|u\|_{q,D_T} = \left[\int_0^T \int_V |u(x,t)|^q dx\, dt\right]^{\frac{1}{q}}.$$

For q as above and k a non-negative integer, we define the Sobolev space $W_{k,q}(D_T)$ to be the space of all functions $u \in L_q(D_T)$ having L_q-derivatives on D_T of the form $D_t^r\, D_x^s u$ for any r and s satisfying $2r + |s| \leq k$, so that $\|u\|_{D_T}^{(k,q)}$ is finite, where:

$$\|u\|_{D_T}^{(k,q)} = \sum_{j=0}^{k}\left\{\sum_{2r+|s|=j} \|D_t^r\, D_x^s u\|_{q,D_T}\right\}.$$

With $\|u\|_{D_T}^{(k,q)}$ as norm, the space $W_{k,q}(D_T)$ is a Banach space.

For k a non-negative integer and α real with $0 < \alpha < 1$, we say that a surface $S \subset E_n$ is of class $C^{k+\alpha}$ if S can be covered by a finite number m of neighbourhoods S_j, j=1,2,... m (i.e. the S_j are the intersections of open n-balls with S), and each S_j can be globally represented by an equation of the form

$$x_{i_j} = X_j(x_1,\ldots x_{i_j-1}, x_{i_j+1},\ldots x_n)$$

for $(x_1,\ldots x_{i_j-1},x_{i_j+1},\ldots x_n)$ in some bounded, open, connected domain $\Omega_j \subset E_{n-1}$, where $X_j \in C^{k+\alpha}(\overline{\Omega}_j)$ for each $j = 1,2,\ldots m$.

If S is a surface of class $C^{k+\alpha}$, we say that a function $u:S \to R$ is of class $C^{\ell+\alpha}(S)$ with $\ell \le k$ if, for each $j = 1,2,\ldots m$, the function $u_j:\overline{\Omega}_j \to R$ defined by

$$u_j(x_1,\ldots x_{i_j-1},x_{i_j+1},\ldots x_n)$$
$$= u(x_1,\ldots x_{i_j-1},X_j(x_1,\ldots x_{i_j-1},x_{i_j+1},\ldots x_n),x_{i_j+1},\ldots x_n)$$

is an element of the space $C^{\ell+\alpha}(\overline{\Omega}_j)$. We define $|u|_S^{(\ell+\alpha)}$ to be $\max\{|u_j|_{\Omega_j}^{(\ell+\alpha)}: j = 1,2,\ldots m\}$.

Suppose $S \subset E_n$ is a surface of class $C^{k+\alpha}$ as described in the previous paragraph. We shall say that a function $u:S\times[0,T] \to R$ is of class $H^{\ell+\alpha}(S\times[0,T])$ with $\ell \le k$ if, for each $j = 1,2,\ldots m$, the function $u_j:\overline{\Omega}_j\times[0,T] \to R$ defined by

$$u_j(x_1,\ldots x_{i_j-1},x_{i_j+1},\ldots x_n,t)$$
$$= u(x_1,\ldots x_{i_j-1},X_j(x_1,\ldots x_{i_j-1},x_{i_j+1},\ldots x_n),x_{i_j+1},\ldots x_n,t)$$

is an element of $H^{\ell+\alpha}(\overline{\Omega}_j\times[0,T])$. We define $\|u\|_{S\times(0,T]}^{(\ell+\alpha)}$ to be $\max\{\|u_j\|_{\Omega_j\times(0,T]}^{(\ell+\alpha)}: j = 1,2,\ldots m\}$.

With V as defined at the beginning of this chapter, we adopt the following further notation as standard throughout the rest of this thesis:

∂V denotes the boundary of V, and is always assumed to be a surface of class $C^{2+\alpha}$ for some α with $0 < \alpha < 1$.

$S_T = \{(x,t): x \in \partial V,\ 0 < t \le T\}$.

$S = \{(x,t): x \in \partial V,\ t > 0\}$.

We shall denote by Lu the expression

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\,\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\,\frac{\partial u}{\partial x_i} + c(x,t)u$$

where the coefficients a_{ij}, b_i and c are assumed to be continuous real functions on $\overline{D}_T$ for all $T > 0$, and $a_{ij} = a_{ji}$ for $i,j = 1,2,\dots n$. Stronger assumptions regarding these coefficients will be made from time to time as needed. The differential operator L is assumed to be uniformly elliptic for each $T > 0$, i.e. there exists for each $T > 0$ an $A > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq A \sum_{i=1}^{n} \xi_i^2$$

for all real vectors $\xi = (\xi_1,\xi_2,\dots \xi_n)$ and all $(x,t) \in \overline{D}_T$.

We shall denote by $\hat{L}u$ the expression

$$\sum_{i,j=1}^{n} \hat{a}_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \hat{b}_i(x) \frac{\partial u}{\partial x_i} + \hat{c}(x)u$$

where the coefficients $\hat{a}_{ij}$, $\hat{b}_i$ and $\hat{c}$ are assumed to be continuous real functions on $\overline{V}$ unless stronger assumptions are needed, $\hat{a}_{ij} = \hat{a}_{ji}$ for $i,j = 1,2,\dots n$, and the differential operator $\hat{L}$ is assumed to be uniformly elliptic on $\overline{V}$ in a sense similar to that defined for the operator L, but now A does not depend on T.

We shall denote by $B_{lin}u$ the expression

$$d_0(x,t)u + d_1(x,t) \frac{\partial u}{\partial n}$$

and by $B_{gen}u$ the expression

$$d_0(x,t)g(u) + d_1(x,t) \frac{\partial u}{\partial n}$$

where d_0 and d_1 are assumed to be non-negative, continuous real functions on $\overline{S}_T$ for all $T > 0$ unless stronger assumptions are needed. The function $g:R \to R$ is assumed to be strictly increasing. Further, n denotes an outwardly directed, nowhere tangential unit vector field on ∂V of the form $n(x) = (n_1(x),n_2(x),\dots n_n(x))$ where $n_1,n_2,\dots n_n$ are of class $C^{1+\alpha}(\partial V)$. We shall denote the outward unit normal vector to ∂V by $\nu(x)$; this is of course a particular case of $n(x)$. $\frac{\partial u}{\partial n}$ denotes the directional derivative $\sum_{i=1}^{n} n_i(x) \frac{\partial u}{\partial x_i}$.

We shall denote by $\hat{B}_{lin}u$ the expression

$$\hat{d}_0(x)u + \hat{d}_1(x) \frac{\partial u}{\partial n}$$

and by $\hat{B}_{gen}u$ the expression

$$\hat{d}_0(x)g(u) + \hat{d}_1(x)\ \frac{\partial u}{\partial n}$$

where $\hat{d}_0$ and $\hat{d}_1$ are assumed to be non-negative, continuous real functions on ∂V unless stronger assumptions are needed, and the notation is in other respects the same as that defined in the previous paragraph.

4 THE STEADY-STATE PROBLEM - A SURVEY

As remarked in Ch.1, the question of the existence of positive stable steady-state solutions for the heat-generation problem has attracted much attention. In particular, the results of Keller and Cohen[19], Keener and Keller[18] and Amann[2] give quite a good picture of the relation between the form of the function $\hat{f}$ (introduced in Ch.1) and the existence or non-existence of positive stable steady-state solutions. We shall examine this picture, and then later, in Ch.7, compare it with the picture obtained by considering the related time-dependent problem.

All the above authors restrict themselves to the real self-adjoint problem described by the equation

$$\left.\begin{aligned} &\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\hat{a}_{ij}(x)\frac{\partial u}{\partial x_j}\right) - \hat{a}_0(x)u + \lambda\hat{f}(x,u) = 0 \quad \text{for } x \in V \\ &\text{together with the boundary condition} \\ &\hat{d}_0(x)u + \hat{d}_1(x)\sum_{i,j=1}^{n} \nu_i(x)\hat{a}_{ij}(x)\frac{\partial u}{\partial x_j} = 0 \quad \text{for } x \in \partial V. \end{aligned}\right\} \quad (1)$$

In the first equation it is required that $\hat{a}_{ij} = \hat{a}_{ji} \in C^{1+\alpha}(\overline{V})$ for $i,j = 1,2,\ldots n$ and some α with $0 < \alpha < 1$, $\hat{a}_0 \in C^{\alpha}(\overline{V})$ and $\hat{a}_0(x) \geq 0$ for all $x \in \overline{V}$, and the differential operator is uniformly elliptic. Thus the equation is a special case of the equation $\hat{L}u + \lambda\hat{f}(x,u) = 0$. In the boundary condition, it is required that $\hat{d}_0$ and $\hat{d}_1$ be non-negative functions of class $C^{1+\alpha}(\partial V)$, ∂V being a surface of class $C^{2+\alpha}$. With $\nu(x)$ denoting as usual the outward unit normal vector to ∂V, the quantity $\sum_{i,j=1}^{n} \nu_i(x)\hat{a}_{ij}(x)\frac{\partial u}{\partial x_j}$ is the *conormal derivative*; this is a directional derivative of the form $\frac{\partial u}{\partial n}$ where

$$n_j(x) = \sum_{i=1}^{n} \nu_i(x)\hat{a}_{ij}(x) \quad (j = 1,2,\ldots n)$$

so that, for $j = 1,2,\ldots n$, n_j is of class $C^{1+\alpha}(\partial V)$. Thus, for all $x \in \partial V$:

$$\begin{aligned} n(x).\nu(x) &= \sum_{i,j=1}^{n} \hat{a}_{ij}(x)\nu_i(x)\nu_j(x) \\ &\geq A \sum_{i=1}^{n} [\nu_i(x)]^2 = A \quad \text{for some } A > 0 \text{ independent} \end{aligned}$$

of x, since the differential operator in (1) is uniformly elliptic. It follows that $n(x)$ is an outwardly directed, non-tangential vector for each $x \in \partial V$. Thus the boundary condition is a special case of the condition $\hat{B}_{lin}u = 0$. Finally, it is assumed that $\partial V = S_1 \cup S_2$ where S_1 has positive measure and:

$$\hat{d}_0(x) > 0,\ \hat{d}_1(x) = 0 \quad \text{for all } x \in S_1.$$

$$\hat{d}_0(x) \geq 0,\ \hat{d}_1(x) > 0 \quad \text{for all } x \in S_2.$$

The condition that S_1 should have positive measure is needed in order to apply a certain uniqueness theorem based on the generalised maximum principle; one form of this theorem is given by Protter and Weinberger [28, Ch.2, Theorem 12].

It should be noted that, apart perhaps from this last restriction on the boundary condition, the original steady-state heat-generation problem in the form studied by Boddington, Gray and Harvey[4] is a special (three-dimensional) case of this general self-adjoint problem in which $\hat{a}_{ii}(x) = 1 (i = 1,2,3)$ and $\hat{a}_{ij}(x) = 0 (i \neq j)$ for all $x \in \overline{V}$.

Following Keller and Cohen[19], we refer to the set of values of λ for which positive solutions $u(\lambda;x)$ of (1) exist as the *spectrum* of (1), and denote the least upper bound of this spectrum by λ^*. Keller and Cohen begin by assuming that $\hat{f}$ satisfies the following hypotheses:

H_0: $\hat{f}$ is continuous for $x \in V$, $u \geq 0$.

H_1: $\hat{f}(x,0) > 0$ for $x \in V$.

H_2: $\hat{f}(x,v) > \hat{f}(x,u)$ on V if $v > u \geq 0$.

With these hypotheses, Keller and Cohen are able to prove the following:

(i) Only positive λ can be in the spectrum of (1).

(ii) For every $\lambda > 0$ in the spectrum of (1), there exists a positive solution $u_{min}(\lambda;x)$ of (1) which is minimal, i.e. which is such that $u_{min}(\lambda;x) \leq u(\lambda;x)$ on V for any positive solution $u(\lambda;x)$ of (1).

(iii) If $\lambda' > 0$ is in the spectrum of (1), then all λ satisfying $0 < \lambda \leq \lambda'$ are in the spectrum, and $u_{min}(\lambda;x)$ is an increasing function of λ for each $x \in V$ and $0 < \lambda \leq \lambda'$.

(iv) If there exists a positive function F on V such that $\hat{f}(x,u) < F(x)$ for all $u > 0$ and all $x \in V$, then all $\lambda > 0$ are in the spectrum of (1), i.e. a finite λ^* does not exist.

(v) If there exist positive functions F, ρ such that for all $u > 0$ and all $x \in V$, $\hat{f}(x,u) < F(x) + \rho(x)u$, then the spectrum of (1) contains all λ such that $0 < \lambda < \mu_1\{\rho\}$, where $\mu_1\{\rho\}$ denotes the principal eigenvalue of

$$\left.\begin{aligned} &\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\hat{a}_{ij}(x)\frac{\partial v}{\partial x_j}\right) - \hat{a}_0(x)v + \mu\rho(x)v = 0 \quad \text{on } V \\ &\hat{d}_0(x)v + \hat{d}_1(x)\sum_{i,j=1}^{n} \nu_i(x)\hat{a}_{ij}(x)\frac{\partial v}{\partial x_j} = 0 \quad \text{on } \partial V. \end{aligned}\right\}.$$

Thus $\lambda^* \geq \mu_1\{\rho\}$.

If, on the other hand, $\hat{f}(x,u)$ satisfies $\hat{f}(x,u) > F(x) + \rho(x)u$ on V, for all $u > 0$, then $\lambda^* \leq \mu_1\{\rho\}$.

Assuming that $\hat{f}$ satisfies H_0 (and possibly H_2) but not H_1, Keller and Cohen prove also:

(vi) If there exists a positive ρ such that $\hat{f}(x,u) < \rho(x)u$ on V, for all $u > 0$, then no λ such that $0 < \lambda < \mu_1\{\rho\}$ is in the spectrum of (1).

An important point that emerges from results (i), (iii) and (iv) is that, assuming hypotheses H_0, H_1, H_2, positive solutions of (1), if they exist at all, exist for λ on an interval of one of the forms $0 < \lambda < \lambda^*$, $0 < \lambda \leq \lambda^*$ or $\lambda > 0$.

Keller and Cohen next introduce the strong monotonicity condition

$H_{2'}$: $\hat{f}_u(x,u) > 0$ and continuous on V for $u > 0$.

On the assumption that $\hat{f}$ satisfies H_0, H_1 and $H_{2'}$, and that (1) has positive solutions for all λ such that $0 < \lambda < \lambda^*$, they then prove that each λ in this interval satisfies $\lambda < \mu_1(\lambda)$ where $\mu_1(\lambda)$ is $\mu_1\{\rho\}$ as defined above, with $\rho(x) \equiv \hat{f}_u(x,u_{min}(\lambda;x))$. Thus $\mu_1(\lambda)$ is the principal eigenvalue of the linearization of (1).

Following Keller and Cohen, we say that $\hat{f}$ is *concave* if it satisfies $H_{2'}$ and in addition

H_{3a}: $\hat{f}_u(x,u) < \hat{f}_u(x,v)$ on V if $u > v \geq 0$

and we say that $\hat{f}$ is *convex* if it satisfies $H_{2'}$ and in addition

H_{3b}: $\hat{f}_u(x,u) > \hat{f}_u(x,v)$ on V if $u > v \geq 0$.

Keller and Cohen then obtain the following results:

(vii) If $\hat{f}$ satisfies H_0, H_1 and is $\left\{\begin{matrix}\text{concave}\\ \text{convex}\end{matrix}\right\}$, and if (1) has the spectrum $0 < \lambda < \lambda^*$ or $0 < \lambda \leq \lambda^*$, then $\mu_1(\lambda)$ is an $\left\{\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}\right\}$ function of λ on this interval. Furthermore, if $\hat{f}$ is concave then $\mu_1(\lambda) < \lambda^*$ for $0 < \lambda < \lambda^*$, and if $\hat{f}$ is convex then $\mu_1(\lambda) > \lambda^*$ for $0 < \lambda < \lambda^*$.

(viii) If $\hat{f}$ satisfies H_0, H_1 and is concave, then $\lim\limits_{\lambda \uparrow \lambda^*} \mu_1(\lambda) = \lambda^*$ and λ^* is not a point of the spectrum. Thus the spectrum must take one of the forms $0 < \lambda < \lambda^*$ or $\lambda > 0$. Furthermore, there is exactly one positive solution of (1) for each λ in the spectrum.

(ix) If $\hat{f}$ satisfies H_0, H_1 and is concave, and if in addition $\lim\limits_{u \to \infty} \hat{f}_u(x,u) = \rho(x)$ on V, then $\lambda^* = \mu_1\{\rho\}$ where we adopt the convention that $\mu_1\{\rho\} = \infty$ if $\rho(x) \equiv 0$.

Thus Keller and Cohen obtain a reasonably complete picture of the situation in the case of concave $\hat{f}$, but rather less information in the case of convex $\hat{f}$. In the case of convex $\hat{f}$, note that it is known in certain special cases that the positive solutions for all λ in the interior of the spectrum are non-unique (see the paper by Laetsch[23]).

Keller and Cohen conclude by discussing the question of stability. For any λ in the spectrum of (1), they define a steady-state solution $u(\lambda;x)$ to be *stable* if, roughly, any solution of the time-dependent problem which satisfies an initial condition of the form

$$u_0(x) = u(\lambda;x) + \epsilon v(x)$$

decays exponentially in t to $u(\lambda;x)$, to first order in ϵ. If one of two stable steady-state solutions is such that this exponential decay described above is more rapid than in the case of the other steady-state solution, then the first solution is said to be *relatively more stable* than the second. Keller and Cohen then prove the following:

(x) Suppose $\hat{f}$ satisfies H_0, H_1 and H_2', and is such that (1) has a non-empty spectrum. Then, for $0 < \lambda < \lambda^*$, the minimal positive solution of (1) is stable. If, in addition, $\hat{f}$ is convex, the minimal positive solution for a given λ is relatively more stable than any other positive solution for the same λ (if $\hat{f}$ is concave, we already know by (viii) that the minimal positive solution is in fact the only positive solution for a given λ). Finally, if $\hat{f}$ is $\left\{\begin{matrix}\text{concave}\\ \text{convex}\end{matrix}\right\}$, the relative stability of the minimal positive solutions $\left\{\begin{matrix}\text{increases}\\ \text{decreases}\end{matrix}\right\}$ as λ increases, on $0 < \lambda < \lambda^*$.

The case where $\hat{f}$ is convex is studied in more detail by Keener and Keller[18]. They use the following strong convexity condition:

$H_{(3b)'}$: $\hat{f}_{uu}(x,u) > 0$ and continuous on V for $u > 0$.

A solution $u(\lambda;x)$ of (1) is said by Keener and Keller to be *non-isolated* if the linearization of (1) about that solution, i.e. the problem

$$\left.\begin{aligned} &\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\hat{a}_{ij}(x)\frac{\partial v}{\partial x_j}\right) - \hat{a}_0(x)v + \lambda\hat{f}_u(x,u(\lambda;x))v = 0 \quad \text{on } V \\ &\hat{d}_0(x)v + \hat{d}_1(x)\sum_{i,j=1}^{n} \nu_i(x)\hat{a}_{ij}(x)\frac{\partial v}{\partial x_j} = 0 \quad \text{on } \partial V \end{aligned}\right\} \qquad (2)$$

has a non-trivial solution. A solution $u(\lambda;x)$ of (1) is said to be a *principal non-isolated solution* if it is a non-isolated solution for which (2) has a positive solution. Keener and Keller then prove the following fundamental result:

Let H_0, H_1, $H_{2'}$ and $H_{(3b)'}$ hold, and for $\lambda = \lambda_0 > 0$, let (1) have a positive principal non-isolated solution, $u(\lambda_0;x) > 0$ on V. Then:

(a) $\lambda_0 = \lambda^*$, and $u(\lambda_0;x)$ is the unique positive solution of (1) for $\lambda = \lambda^*$;

(b) A minimal positive solution of (1) exists for all $\lambda \in (0,\lambda^*)$, and no positive solutions exist for $\lambda > \lambda^*$;

(c) For some sufficiently small $\Delta > 0$, a pair of positive solutions of (1) exists for each $\lambda \in [\lambda^*-\Delta,\lambda^*)$.

After this, the major question remaining for Keener and Keller to deal with is that of the existence of a positive principal non-isolated solution. They require for this the following hypothesis of asymptotic linearity:

H_4: $\lim\limits_{u\to\infty}\left\{\frac{\hat{f}(x,u) - [F(x) + uG(x)]}{u}\right\} = 0$ on V, where $G(x) > 0$ on V.

To complete their proof, they also require an hypothesis H_5 of a rather technical nature, which need not be given here, since Amann[2] has shown that it can be dispensed with. On the assumption that $\hat{f}$ satisfies H_0, H_1, $H_{2'}$, $H_{(3b)'}$, H_4 and H_5, Keener and Keller prove that a positive principal non-isolated solution of (1) does in fact exist for some positive λ_0. This shows that for such $\hat{f}$ the spectrum of (1) is an interval of the form $0 < \lambda \le \lambda^*$.

Subsequently, Amann[2] has shown that the crucial hypothesis in the work of Keener and Keller is that of asymptotic linearity. Amann assumes only that $\hat{f}$ satisfies the following conditions:

1) $\hat{f}(x,u) > 0$ for $x \in \overline{V}$, $u > 0$.
2) $\lim_{u\to\infty} \hat{f}_u(x,u) = \hat{f}_\infty(x)$ exists uniformly for $x \in \overline{V}$, and $\hat{f}_\infty(x) > 0$ for $x \in \overline{V}$ (this is the crucial assumption of asymptotic linearity).

Amann denotes by λ_∞ the principal eigenvalue $\mu_1\{\hat{f}_\infty\}$ (in the notation of Keller and Cohen introduced earlier). He is then able to prove the following comprehensive theorem:

There exists $\lambda^* > 0$ such that for every $\lambda \in (0,\lambda^*)$, (1) has a minimal positive solution $u_{min}(\lambda;x)$, and (1) has no solution for $\lambda > \lambda^*$. (1) has a solution for $\lambda = \lambda^*$, in fact a minimal positive solution $u_{min}(\lambda^*;x)$, if and only if $\{\|u_{min}(\lambda;x)\|_{C(\overline{V})} : 0 < \lambda < \lambda^*\}$ is bounded. This is the case if and only if $u^*(x) = \lim_{\lambda\to\lambda^*-} u_{min}(\lambda;x)$ exists in $C^{2+\alpha}(\overline{V})$, in which case $u^*(x) = u_{min}(\lambda^*;x)$.

Further, we have $0 < \lambda_\infty \leq \lambda^*$. If $\lim_{\lambda\to\lambda^*-} \|u_{min}(\lambda;x)\|_{C(\overline{V})} = \infty$, then $\lambda_\infty = \lambda^*$ (compare this with result (ix) of Keller and Cohen for concave $\hat{f}$). On the other hand, if $\lambda_\infty < \lambda^*$, then (1) has a minimal positive solution for $\lambda = \lambda^*$, and for every $\lambda \in (\lambda_\infty,\lambda^*)$, (1) has at least two distinct positive solutions.

Finally, suppose there exists a positive function y, continuous on $\overline{V}$, and a constant $\rho > 0$, such that for all $x \in V$ and all $u \geq \rho$,

$$\hat{f}(x,u) - \hat{f}_u(x,u)u \leq -y(x).$$

Then $\lambda_\infty < \lambda^*$. Note that this condition is related to convexity, since it implies that for every $x \in V$ and for $u \geq \rho$, the tangent to the graph of $\hat{f}(x,u)$ against u intersects the negative y-axis.

Amann's theorem therefore gives a rather complete picture of the situation for the case where $\hat{f}$ is asymptotically linear, thus filling in the picture drawn by Keller and Cohen[19] and by Keener and Keller[18].

The example discussed in Ch.2 serves to illustrate one of the cases considered by Amann. In this example (the steady-state problem with $k = \lambda$) we have $\hat{f}(x,u) = u + 1$, which satisfies the conditions for Amann's theorem with $\hat{f}_\infty(x) = 1$. Ours is an example of the case dealt with by Amann in which $\lim_{\lambda\to\lambda^*-} u_{min}(\lambda;x)$ does not exist; in our example $u_{min}(\lambda;x)$ is $\dfrac{\cos\sqrt{\lambda}x}{\cos\sqrt{\lambda}} - 1$ and $\lambda^* = \dfrac{\pi^2}{4}$. In this case Amann's theorem tells us that $\lambda^* = \lambda_\infty = \mu_1\{\hat{f}_\infty\}$. Now $\mu_1\{\hat{f}_\infty\}$ in our example is the principal eigenvalue of the linear problem

$$\frac{d^2u}{dx^2} + \lambda u = 0, \quad u(1) = u(-1) = 0$$

which has eigenvalues $\lambda = \frac{m^2\pi^2}{4}$ ($m = 1,2,3,\ldots$), so that its principal eigenvalue is indeed $\frac{\pi^2}{4}$ as required. Note that, as required by Amann's theorem in this case, the spectrum is the open interval $(0,\frac{\pi^2}{4})$; no solution exists for $\lambda = \frac{\pi^2}{4}$.

5 COMPARISON THEOREMS FOR THE TIME-DEPENDENT PROBLEM

Comparison theorems, based on various versions of the maximum principle, are a standard tool in the study of differential equations. They are discussed in the books by Protter and Weinberger[28] and Friedman[14], and used by many authors, such as Chan[9], McNabb[25], Sattinger[33] and Wake[35]. The comparison theorems which will play a fundamental role in the rest of this thesis are based on those proved by McNabb, but they differ in certain important details, and so the proofs are given in full here. We require first a lemma due to Fejer[12], the proof of which we include for the sake of completeness.

LEMMA: *If* $g(x) = \sum_{i,k=1}^{n} g_{ik}x_ix_k$ *and* $h(x) = \sum_{i,k=1}^{n} h_{ik}x_ix_k$ *are two non-negative quadratic forms, with* $g_{ik} = g_{ki}$ *and* $h_{ik} = h_{ki}$ *for all* $i,k = 1,2,\dots n$, *then* $\sum_{i,k=1}^{n} g_{ik}h_{ik} \geq 0$.

Proof: The result is obvious if either g or h is identically zero; assume therefore that neither g nor h is identically zero. We shall first show that there are n linear forms $z_r(x) = \sum_{s=1}^{n} p_{rs}x_s$ (where $r = 1,2,\dots n$) with real coefficients p_{rs}, such that

$$g(x) = \sum_{r=1}^{n} z_r^2(x) = \sum_{r=1}^{n}\left(\sum_{s=1}^{n} p_{rs}x_s\right)^2 \qquad \dots\dots\dots(3)$$

We know that for all $i = 1,2,\dots n$, $g_{ii} \geq 0$, since g_{ii} is the value of g(x) when $x_j = 0 (j \neq i)$ and $x_i = 1$. Further, the coefficients g_{ii} cannot all be zero, for if they were, then since $g(x) \not\equiv 0$, there must be at least one $g_{ij} \neq 0 (i \neq j)$. In that case a negative value of g(x) could be obtained by choosing $x_i = 1$, $x_j = \pm 1$ (depending on the sign of g_{ij}) and $x_k = 0$ for $k \neq i,j$. Thus, for at least one $i = 1,2,\dots n$, we must have $g_{ii} > 0$. Without loss of generality, assume $g_{11} > 0$.

Now write:

$$p_{11} = (g_{11})^{\frac{1}{2}}.$$

$$p_{11}p_{1s} = g_{1s} \quad (s = 2,3,\dots n).$$

$$z_1(x) = \sum_{s=1}^{n} p_{1s}x_s.$$

$$g^{(1)}(x) = g(x) - z_1^2(x).$$

It is easily seen that the quadratic form $g^{(1)}(x)$ is independent of x_1. Furthermore, it is non-negative, for suppose we could obtain a negative value for $g^{(1)}(x)$ by taking $x_i = a_i (i = 2,3,\dots n)$. Put

$$a_1 = \frac{-1}{p_{11}}\left(\sum_{s=2}^{n} p_{1s}a_s\right).$$

Then, writing $a = (a_1,a_2,\dots a_n)$, we have $z_1(a) = 0$ and so we obtain the contradiction $g(a) = g^{(1)}(a) < 0$.

If $g^{(1)}(x)$ is identically zero, then (3) is proved already, since we may take $z_r(x) \equiv 0$ for $r = 2,3,\dots n$, and we have shown $g(x) = z_1^2(x)$. If not, we can carry out for $g^{(1)}(x)$ a construction similar to that carried out for $g(x)$, obtaining a linear form $z_2(x) = \sum_{s=2}^{n} p_{2s}x_s$ such that $g^{(2)}(x) = g^{(1)}(x) - z_2^2(x) = g(x) - [z_1^2(x) + z_2^2(x)]$ is a non-negative quadratic form independent of both x_1 and x_2. Continuing thus, after n steps we will obtain $g^{(n)}(x) \equiv 0$ and so $g(x) = \sum_{r=1}^{n} z_r^2(x)$, which proves (3).

It then follows from (3) that $g_{ik} = \sum_{r=1}^{n} p_{ri}p_{rk}$ $(i,k = 1,2,\dots n)$.

Similarly, we obtain $h_{ik} = \sum_{s=1}^{n} q_{si}q_{sk}$ $(i,k = 1,2,\dots n)$.

$$\text{Hence}\quad \sum_{i,k=1}^{n} g_{ik}h_{ik} = \sum_{i,k=1}^{n} \left\{\left(\sum_{r=1}^{n} p_{ri}p_{rk}\right)\left(\sum_{s=1}^{n} q_{si}q_{sk}\right)\right\}$$

$$= \sum_{i,k=1}^{n} \left\{\sum_{r,s=1}^{n} p_{ri}q_{si}p_{rk}q_{sk}\right\}$$

$$= \sum_{i,k=1}^{n} \left(\sum_{j=1}^{n} p_{rj}q_{sj}\right)^2 \geq 0.$$

We prove now the first comparison theorem we require; the method is similar to that used by McNabb[25, Theorem 1].

THEOREM 1: *Suppose that*

(a) *The functions* u_1 *and* u_2 *are defined and continuous in* $\overline{D}_T$, *their first-order* x_i*-derivatives exist in* $\overline{D}_T$, *their second-order* x_i*-derivatives exist and are continuous in* D_T, *and their first-order* t*-derivatives exist in* D_T.

(b) *For all* $(x,t) \in D_T$, $Lu_1 - \frac{\partial u_1}{\partial t} + f(x,t,u_1) > Lu_2 - \frac{\partial u_2}{\partial t} + f(x,t,u_2)$.

(c) $u_1(x,0) < u_2(x,0)$ *for all* $x \in \overline{V}$.

(d) *For all* $(x,t) \in S_T$, $B_{gen}u_1 < B_{gen}u_2$.

Then $u_1(x,t) < u_2(x,t)$ *for all* $(x,t) \in \overline{D}_T$.

Proof: Suppose, on the contrary, that there is a point P in $\overline{D}_T$ where $u_1 \geq u_2$. Then (by the continuity of u_1 and u_2) there is a point $(x', T') \in \overline{D}_T$ such that $u_1 \leq u_2$ in $\overline{D}_{T'}$, while $u_1 = u_2$ at (x', T'), $T' > 0$. So $v(x,t) = u_1(x,t) - u_2(x,t)$ has a maximum of zero in $\overline{D}_{T'}$ at the point (x', T').

Suppose first that we may choose $(x', T') \in D_{T'}$, i.e. such that x' is not on the boundary ∂V. Then the quadratic form

$$\sum_{i,j=1}^{n} D_{x_i} D_{x_j} v(x', T') x_i x_j$$

is non-positive. Since the quadratic form $\sum_{i,j=1}^{n} a_{ij}(x', T') x_i x_j$ is non-negative by definition of L, it follows by the lemma that:

$$-\sum_{i,j=1}^{n} a_{ij}(x', T') D_{x_i} D_{x_j} v(x', T') \geq 0$$

$$\text{i.e.} \quad \sum_{i,j=1}^{n} a_{ij}(x', T') D_{x_i} D_{x_j} v(x', T') \leq 0.$$

Further, for each $i = 1,2,\ldots n$, $D_{x_i} v(x', T') = 0$, and also $v(x', T') = 0$. Thus $Lv \leq 0$ at the point (x', T'). Since also $D_t v(x', T') \geq 0$, it follows that $Lv - \frac{\partial v}{\partial t} \leq 0$ at the point (x', T').

$$\therefore \quad Lu_1 - \frac{\partial u_1}{\partial t} \leq Lu_2 - \frac{\partial u_2}{\partial t} \quad \text{at the point } (x', T').$$

Finally, since $u_1(x', T') = u_2(x', T')$, we have that at the point (x', T'), $Lu_1 - \frac{\partial u_1}{\partial t} + f(x,t,u_1) \leq Lu_2 - \frac{\partial u_2}{\partial t} + f(x,t,u_2)$, contradicting hypothesis (b).

If x' cannot be chosen away from ∂V, then we must have $u_1 = u_2$ at (x', T') with $x' \in \partial V$, and $u_1 < u_2$ in $D_{T'}$.

$$\text{Thus } \frac{\partial}{\partial n}(u_1 - u_2) \leq 0 \text{ at } (x', T').$$

i.e. $B_{gen}u_2 \leq B_{gen}u_1$ at the point (x', T'), since $u_1 = u_2$ at (x', T'). This contradicts hypothesis (c), and so the proof of the theorem is complete.

In the case where the function $f(x,t,u)$ satisfies a uniform Lipschitz condition in u on any finite interval, a stronger comparison theorem can be proved. The method of proof is an extension of that used by McNabb[25, Theorem 2].

THEOREM 2: *Suppose that*

(a) *The functions* u_1 *and* u_2 *are defined and continuous in* $\overline{D}_T$, *their first-order* x_i*-derivatives exist in* $\overline{D}_T$, *their second-order* x_i*-derivatives exist and are continuous in* D_T, *and their first-order* t*-derivatives exist in* D_T.

(b) *For all* $(x,t) \in D_T$, $Lu_1 - \frac{\partial u_1}{\partial t} + f(x,t,u_1) \geq Lu_2 - \frac{\partial u_2}{\partial t} + f(x,t,u_2)$.

(c) $u_1(x,0) \leq u_2(x,0)$ *for all* $x \in \overline{V}$.

(d) *The coefficient* $d_0(x,t)$ *of* $g(u)$ *in* $B_{gen}u$ *is strictly positive for all* $(x,t) \in S_T$.

(e) *For all* $(x,t) \in S_T$, $B_{gen}u_1 \leq B_{gen}u_2$.

(f) *On any finite interval* $[a,b]$, *the function* $f(x,t,u)$ *satisfies a uniform Lipschitz condition in* u, *i.e. there exists a constant* $M_{[a,b]} > 0$ *(depending on the interval* $[a,b]$*) such that*

$$|f(x,t,u_1) - f(x,t,u_2)| \leq M_{[a,b]}|u_1 - u_2|$$

for all $u_1,u_2 \in [a,b]$ *and all* $(x,t) \in \overline{D}_T$.

Then $u_1(x,t) \leq u_2(x,t)$ *for all* $(x,t) \in \overline{D}_T$.

Proof: Since u_2 is continuous in $\overline{D}_T$, it is bounded there. Let $m_1 = \inf_{(x,t)\in\overline{D}_T} u_2(x,t)$ and $m_2 = \sup_{(x,t)\in\overline{D}_T} u_2(x,t)$. Choose $M' > 0$ such that

$M' > \sup_{(x,t)\in\overline{D}_T} \{M_{[m_1,m_2+1]} + c(x,t)\}$ (recall that $c(x,t)$ is the coefficient of u in Lu). For all $(x,t) \in \overline{D}_T$ and all $\lambda \in [0,1]$, define $U_\lambda(x,t) = u_2(x,t) + \lambda e^{M'(t-T)}$. Then for all $(x,t) \in \overline{D}_T$ and all $\lambda \in (0,1]$, $m_1 \leq u_2(x,t) < U_\lambda(x,t) \leq m_2 + 1$. Thus, for all $(x,t) \in D_T$ and all $\lambda \in (0,1]$, we have:

$$[LU_\lambda - \frac{\partial U_\lambda}{\partial t} + f(x,t,U_\lambda)] - [Lu_2 - \frac{\partial u_2}{\partial t} + f(x,t,u_2)]$$

$$= f(x,t,U_\lambda) - f(x,t,u_2) + \lambda c(x,t)e^{M'(t-T)} - \lambda M' e^{M'(t-T)}$$

$$\leq M_{[m_1,m_2+1]}\lambda e^{M'(t-T)} + \lambda c(x,t)e^{M'(t-T)} - \lambda M' e^{M'(t-T)} \quad \text{by (f)}$$

$$= \lambda e^{M'(t-T)}\{M_{[m_1,m_2+1]} + c(x,t) - M'\} < 0.$$

Thus, using hypothesis (b), we have that for all $(x,t) \in D_T$ and all $\lambda \in (0,1]$,

$$Lu_1 - \frac{\partial u_1}{\partial t} + f(x,t,u_1) > LU_\lambda - \frac{\partial U_\lambda}{\partial t} + f(x,t,U_\lambda).$$

Also, for all $(x,t) \in S_T$ and all $\lambda > 0$,

$$\begin{aligned} B_{gen}U_\lambda - B_{gen}u_2 &= d_0(x,t)\{g(U_\lambda) - g(u_2)\} \\ &> 0 \end{aligned}$$

by hypothesis (d) and the fact that g is strictly increasing by definition of B_{gen}.

Thus, using hypothesis (e), we have:

$$B_{gen}u_1 < B_{gen}U_\lambda \quad \text{for all } (x,t) \in S_T \text{ and all } \lambda > 0.$$

Further, we have that $u_1(x,0) \leq u_2(x,0) < U_\lambda(x,0)$ for all $x \in \overline{V}$ and all $\lambda > 0$. It follows by Theorem 1 that, for all $\lambda \in (0,1]$, $u_1(x,t) < U_\lambda(x,t)$ for all $(x,t) \in \overline{D}_T$. Since $U_\lambda(x,t) \to u_2(x,t)$ as $\lambda \to 0+$ for each $(x,t) \in \overline{D}_T$, it follows that $u_1(x,t) \leq u_2(x,t)$ for all $(x,t) \in \overline{D}_T$, as required.

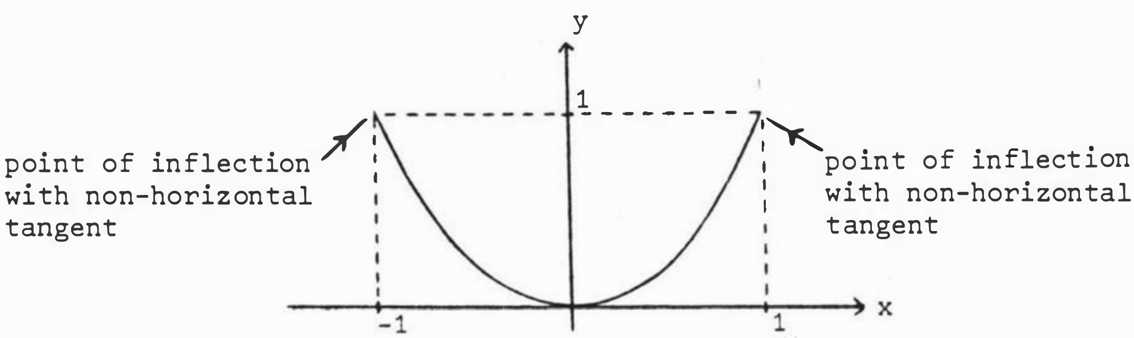
Notes: (i) If, in the statement of Theorem 2, we omit hypothesis (f), that $f(x,t,u)$ should satisfy a uniform Lipschitz condition in u on any finite interval, then the theorem fails to hold, as the following counter-example demonstrates.

Take Lu to be $\dfrac{\partial^2 u}{\partial x^2}$, V to be $\{x: -1 < x < 1\}$, $B_{gen}u$ to be $u + \dfrac{\partial u}{\partial \nu} = u + x\dfrac{\partial u}{\partial x}$ and $f(x,t,u)$ to be $576e^{|u|^{\frac{1}{2}}}$.

Consider first the function $y(x) = \frac{1}{5}(x^2-1)(5-x^2) + 1$.

$$\begin{aligned} \text{Here } y'(x) &= \tfrac{1}{5}\{(x^2-1)(-2x) + (5-x^2)2x\} \\ &= \tfrac{4x}{5}(3-x^2). \\ y''(x) &= \tfrac{4}{5}(3-x^2) + \tfrac{4x}{5}(-2x) \\ &= \tfrac{12}{5}(1-x^2). \end{aligned}$$

For $-1 \leq x \leq 1$, the graph of $y(x)$ is as follows:



Next we define $u_1(x,t) = t^2$, $u_2(x,t) = t^2y(x)$ for $t \ge 0$, $-1 \le x \le 1$. Then:

(a) $u_1(x,0) = 0 = u_2(x,0)$, so certainly $u_1(x,0) \le u_2(x,0)$ for $-1 \le x \le 1$.

(b) $Lu_1 - \frac{\partial u_1}{\partial t} + f(x,t,u_1) = -2t + 576e^{\sqrt{t}}$.

$$Lu_2 - \frac{\partial u_2}{\partial t} + f(x,t,u_2) = t^2y''(x) - 2ty(x) + 576e^{\sqrt{t}\{y(x)\}^{\frac{1}{4}}}.$$

$$\therefore\ [Lu_1 - \frac{\partial u_1}{\partial t} + f(x,t,u_1)] - [Lu_2 - \frac{\partial u_2}{\partial t} + f(x,t,u_2)]$$

$$= 2t\{y(x)-1\} - t^2y''(x) + 576\{e^{\sqrt{t}} - e^{\sqrt{t}\{y(x)\}^{\frac{1}{4}}}\} = A(x,t), \text{ say.}$$

We have that $A(x,0) = 0$ for $-1 \le x \le 1$. Assume now that $t > 0$ and $-1 < x < 1$. Then:

$$\frac{\partial}{\partial t} A(x,t) = 2\{y(x)-1\} - 2ty''(x) + 576\left\{\frac{e^{\sqrt{t}}}{2\sqrt{t}} - \frac{\{y(x)\}^{\frac{1}{4}}}{2\sqrt{t}}\, e^{\sqrt{t}\{y(x)\}^{\frac{1}{4}}}\right\} \qquad (4)$$

For $-1 < x < 1$, we have $0 \le y(x) < 1$, so $0 \le \{y(x)\}^{\frac{1}{4}} < 1$. Put $\{y(x)\}^{\frac{1}{4}} = 1 - \delta(x)$ where $0 < \delta(x) \le 1$.

$$\text{Then } y(x) = \{1-\delta(x)\}^4 = 1 - 4\delta(x) + 6\delta^2(x) - 4\delta^3(x) + \delta^4(x)$$

$$= 1 - \tfrac{1}{5}(1-x^2)(5-x^2).$$

$$\therefore\ \delta(x)\{4 - 6\delta(x) + 4\delta^2(x) - \delta^3(x)\} = \tfrac{1}{5}(1-x^2)(5-x^2).$$

$$\therefore\ \delta(x) = \frac{(1-x^2)(5-x^2)}{5\{4 - 6\delta(x) + 4\delta^2(x) - \delta^3(x)\}}$$

$$> \frac{(1-x^2)(5-x^2)}{40} \quad \text{for } -1 < x < 1, \text{ since then } 0 < \delta(x) \le 1.$$

Thus (4) becomes:

$$\frac{\partial}{\partial t} A(x,t) = \tfrac{2}{5}(x^2-1)(5-x^2) - \tfrac{24t}{5}(1-x^2) + \frac{288}{\sqrt{t}}\{e^{\sqrt{t}} - \{1-\delta(x)\}e^{\sqrt{t}\{1-\delta(x)\}}\}$$

$$= \tfrac{2}{5}(x^2-1)(5-x^2) - \tfrac{24t}{5}(1-x^2) + \frac{288e^{\sqrt{t}}}{\sqrt{t}}\{1 - [1-\delta(x)]e^{-\delta(x)\sqrt{t}}\}$$

$$\ge \tfrac{2}{5}(x^2-1)(5-x^2) - \tfrac{24t}{5}(1-x^2) + \frac{288e^{\sqrt{t}}}{\sqrt{t}}\{\delta(x)\}$$

$$> \tfrac{2}{5}(x^2-1)(5-x^2) - \tfrac{24t}{5}(1-x^2) + \frac{288e^{\sqrt{t}}}{\sqrt{t}}\left\{\frac{(1-x^2)(5-x^2)}{40}\right\} \quad \text{from above.}$$

Further, for $t > 0$, $e^{\sqrt{t}} = 1 + \sqrt{t} + \frac{t}{2!} + \frac{t\sqrt{t}}{3!} + \ldots$

$$> \sqrt{t} + \frac{t\sqrt{t}}{6}.$$

$$\therefore\ \frac{\partial}{\partial t} A(x,t) > \tfrac{2}{5}(x^2-1)(5-x^2) - \tfrac{24t}{5}(1-x^2) + \tfrac{72}{10}(1+\tfrac{t}{6})(1-x^2)(5-x^2)$$

$$= \tfrac{2}{5}(1-x^2)(5-x^2)\{-1+18\} + \tfrac{24t}{5}(1-x^2)\{-1 + \tfrac{1}{4}(5-x^2)\}.$$

The first term is obviously positive for $-1 < x < 1$; the second is also positive, since $5-x^2 > 4$ for $-1 < x < 1$. Thus, for $-1 < x < 1$, $A(x,0) = 0$ and $\frac{\partial}{\partial t} A(x,t) > 0$ for $t > 0$. It follows that $A(x,t) > 0$ for $t > 0$ and $-1 < x < 1$.

$$\therefore \quad Lu_1 - \frac{\partial u_1}{\partial t} + f(x,t,u_1) > Lu_2 - \frac{\partial u_2}{\partial t} + f(x,t,u_2) \quad \text{for } t > 0,\ -1 < x < 1.$$

(c) For $x = \pm 1$, $B_{gen}u_1 = u_1 + x\,\frac{\partial u_1}{\partial x} = t^2$.

$$\begin{aligned} B_{gen}u_2 = u_2 + x\,\frac{\partial u_2}{\partial x} &= t^2 y(x) + xt^2 y'(x) \\ &= t^2 y(x) + \frac{4}{5}\, t^2 x^2 (3-x^2) \\ &= t^2 + \frac{8t^2}{5} \quad \text{since } x^2 = 1,\ y(x) = 1. \end{aligned}$$

$$\therefore \quad B_{gen}u_1 < B_{gen}u_2 \quad \text{for } x = \pm 1,\ t > 0.$$

Thus all the hypotheses of Theorem 2 except hypothesis (f) are satisfied for any $T > 0$ (indeed, we have rather more than is required, since the inequalities in (b) and (c) of the counter-example are strict $>$ and $<$ rather than $\geq$ and $\leq$ as in the theorem). Hypothesis (f) is not satisfied, since the function $e^{u^{1/4}}$ does not satisfy a Lipschitz condition on any interval $[0,a]$ with $a > 0$, as its derivative $\frac{1}{4} u^{-3/4} e^{u^{1/4}}$ is unbounded on any such interval. And the conclusion that $u_1(x,t) \leq u_2(x,t)$ for all $t \geq 0$, $-1 \leq x \leq 1$ is false, since $0 \leq y(x) < 1$ if $-1 < x < 1$, and so $u_1(x,t) > u_2(x,t)$ for all $t > 0$ and $-1 < x < 1$.

(ii) Hypothesis (f) of Theorem 2 can, however, be replaced by a condition that $f(x,t,u)$ be monotone decreasing in u; this allows the proof to be slightly simplified. This version of the theorem is of little or no interest for the present discussion, since in this thesis the function $f(x,t,u)$ is generally assumed to be monotone increasing in u, this being the case in the heat-generation problem which motivates the whole discussion. Note that a comparison theorem involving monotone decreasing f is proved by Chan[9, Theorem 1] and used to derive some interesting existence and uniqueness theorems.

6 RELATIONS BETWEEN SOLUTIONS OF THE TIME-DEPENDENT AND STEADY-STATE PROBLEMS

It will be recalled that in Ch.1 we discussed two possible approaches to the problem of determining whether or not a thermal explosion will take place in a given situation. The usual approach is to argue that a thermal explosion will take place if the equation describing the system has no positive steady-state solutions. The approach used in this thesis is to argue that a thermal explosion will take place if the solution of the time-dependent equation is unbounded as $t \to \infty$ or as t tends to some finite value. We wish to show in this chapter that, under fairly wide conditions, these two approaches are mathematically equivalent. Accordingly, we wish to investigate the relation between the boundedness over all time of the solution of the time-dependent problem

$$Lu - \frac{\partial u}{\partial t} + f(x,t,u) = 0 \quad \text{for } (x,t) \in D$$

$$B_{lin}u = 0 \quad \text{for } (x,t) \in S$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \overline{V}$$

and the existence of positive solutions of the corresponding steady-state problem

$$\hat{L}u + \hat{f}(x,u) = 0 \quad \text{for } x \in V$$

$$\hat{B}_{lin}u = 0 \quad \text{for } x \in \partial V$$

where the coefficients are the limits, as $t \to \infty$, of the corresponding time-dependent coefficients, and $\hat{f}(x,u)$ is the limit as $t \to \infty$ of $f(x,t,u)$. We restrict the discussion to the linear boundary condition because the fundamental theory on which this chapter is based is not available for the non-linear boundary condition. The parameter λ which appeared in the equations mentioned in Ch.1 is, for the purposes of the present discussion, absorbed into the functions f and $\hat{f}$. It will reappear later.

In the book by Friedman[14, Ch.6] there are some important theorems concerning the case where f and $\hat{f}$ are independent of u. Friedman uses less general boundary conditions than ours, but his methods are easily adapted to our boundary conditions, as we shall show later.

With certain restrictions on the coefficients, Friedman proves that, if it is known that the steady-state problem has a *unique* solution, then the solution of the time-dependent problem will tend to this steady-state solution as $t \to \infty$. Reynolds[29] extends Friedman's method to prove a similar theorem for the case where f and $\hat{f}$ are dependent on u (Reynolds' theorem is rather general, since it allows for non-linearity of the differential operators as well as the functions f and $\hat{f}$). However, Reynolds' result is not quite what we want, since we are frequently concerned with situations where the steady-state problem is known to have multiple solutions, as we have seen in Ch.4.

It should also be mentioned that Liapunov methods have been used to study problems of this type, for example by Chafee and Infante[8] in the case of a special one-dimensional problem.

The technique which was found to be appropriate for our purposes was that of *monotone iteration*, introduced by Courant[11, pp.370,371] and developed further by Cohen[10] and others. Using this technique, Sattinger[31, 32] has proved two existence theorems, for parabolic and elliptic problems, which are of great value in our present study, and which we will now discuss in some detail.

<u>Monotone Iteration:</u>

We consider first the parabolic initial-boundary value problem

$$\left.\begin{aligned} Lu - \frac{\partial u}{\partial t} + f(x,t,u) = 0 \quad &\text{for } (x,t) \in D_T \\ B_{lin}u = 0 \quad &\text{for } (x,t) \in S_T \\ u(x,0) = u_0(x) \quad &\text{for } x \in \overline{V} \end{aligned}\right\} \qquad .....(5)$$

where L and B_{lin} are as defined in Ch.3, with the additional assumptions that $a_{ij} = a_{ji} \in H^{\alpha}(\overline{D}_T)$, $b_i \in H^{\alpha}(\overline{D}_T)$ and $c \in H^{\alpha}(\overline{D}_T)$ for all $T > 0$; also d_0 and d_1 are of class $H^{1+\alpha}(\overline{S}_T)$ for all $T > 0$. We assume also that $u_0 \in C^{2+\alpha}(\overline{V})$ and f is continuous for $(x,t) \in \overline{D}_T$ and at least some u-interval.

We call $\varphi(x,t)$ an *upper solution* for (5) if φ is continuous in $\overline{D}_T$, has continuous first-order x_i-derivatives in $\overline{D}_T$, continuous second-order x_i-derivatives in D_T and continuous first-order t-derivatives in D_T, and satisfies:

$$L\varphi - \frac{\partial \varphi}{\partial t} + f(x,t,\varphi) \le 0 \quad \text{for } (x,t) \in D_T$$

$$B_{lin}\varphi \ge 0 \quad \text{for } (x,t) \in S_T$$

$$\varphi(x,0) \ge u_0(x) \quad \text{for } x \in \overline{V}.$$

We call $\varphi(x,t)$ a *strict upper solution* for (5) if φ is continuous in $\overline{D}_T$, has continuous first-order x_i-derivatives in $\overline{D}_T$, continuous second-order x_i-derivatives in D_T and continuous first-order t-derivatives in D_T, and satisfies:

$$L\varphi - \frac{\partial\varphi}{\partial t} + f(x,t,\varphi) < 0 \quad \text{for } (x,t) \in D_T$$

$$B_{lin}\varphi > 0 \quad \text{for } (x,t) \in S_T$$

$$\varphi(x,0) > u_0(x) \quad \text{for } x \in \overline{V}.$$

The terms *lower solution* and *strict lower solution* are defined analogously by reversing the inequalities in the above definitions.

By a *solution* of (5) we shall understand a classical solution $u(x,t)$ of (5) which is continuous in $\overline{D}_T$, has continuous first-order x_i-derivatives in $\overline{D}_T$, continuous second-order x_i-derivatives in D_T and continuous first-order t-derivatives in D_T. It follows from Theorem 1 that if φ is a strict upper solution for (5) and u a solution of (5), then $\varphi(x,t) > u(x,t)$ for all $(x,t) \in \overline{D}_T$. If $f(x,t,u)$ satisfies a uniform Lipschitz condition in u on any finite interval, and the coefficient $d_0(x,t)$ of u in $B_{lin}u$ is strictly positive for all $(x,t) \in S_T$, and if φ is an upper solution for (5) and u a solution of (5), then Theorem 2 shows that $\varphi(x,t) \geq u(x,t)$ for all $(x,t) \in \overline{D}_T$. Analogous results with reversed inequalities hold for lower solutions.

Note that, in the case where f and d_0 satisfy the hypotheses of Theorem 2, we can use Theorem 2 to prove that the solution of (5), if it exists, is unique; if u_1 and u_2 are both solutions of (5), then by Theorem 2, $u_1(x,t) \leq u_2(x,t)$ and $u_2(x,t) \leq u_1(x,t)$ for all $(x,t) \in \overline{D}_T$, whence $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \overline{D}_T$.

We shall now give a detailed proof of Sattinger's existence theorem for parabolic problems[32, Theorem 2.3.2] to illustrate the method of monotone iteration, and also for the sake of completeness, since the proof is not given in detail by Sattinger, who proves in detail the corresponding theorem for elliptic problems. We require first a number of lemmas.

LEMMA 1: *If* u_0 *satisfies the boundary condition, i.e. if* $B_{lin}u_0 = 0$ *for* $t = 0$ *and all* $x \in \partial V$, *then for any* $g \in H^{\alpha}(\overline{D}_T)$, *the problem*

$$Lu - \frac{\partial u}{\partial t} = g(x,t) \quad \textit{for } (x,t) \in D_T$$

$$B_{lin}u = 0 \quad \textit{for } (x,t) \in S_T$$

$$u(x,0) = u_0(x) \quad \textit{for } x \in \overline{V}$$

has a unique solution $u \in H^{2+\alpha}(\overline{D}_T)$ *with*

$$\|u\|_{D_T}^{(2+\alpha)} \leq c_1(\|g\|_{D_T}^{(\alpha)} + |u_0|_V^{(2+\alpha)})$$

where c_1 *does not depend on* g *or* u_0.

Proof: This is a special case of Theorem 5.3 on p.320 of Ladyzenskaja, Solonnikov and Ural'ceva[21].

LEMMA 2: *Suppose* $q > 1$. *For any* $h \in L_q(D_T)$, *the problem*

$$Lu - \frac{\partial u}{\partial t} = h(x,t) \quad \textit{for } (x,t) \in D_T$$

$$B_{lin}u = 0 \quad \textit{for } (x,t) \in S_T$$

$$u(x,0) = 0 \quad \textit{for } x \in \overline{V}$$

has a unique (not necessarily classical) solution $u \in W_{2,q}(D_T)$ *with*

$$\|u\|_{D_T}^{(2,q)} \leq c_2 \|h\|_{q,D_T}$$

where c_2 *does not depend on* h.

Proof: This is a special case of a theorem analogous to Theorem 9.1 on p.341 of Ladyzenskaja, Solonnikov and Ural'ceva[21], but with a different boundary condition; see p.351 of the same reference.

Definition: We say that V satisfies the *cone condition* if there exists a fixed finite cone K such that, no matter at what point of $\overline{V}$ its vertex is placed, the cone can be swung so that all of it is contained in $\overline{V}$.

LEMMA 3: *If* V *satisfies the cone condition, if* $u \in H^{\alpha}(\overline{D}_T)$ *and also* $u \in W_{2,q}(D_T)$ *for some* q *such that* $q > \frac{n+2}{2}$ *and* $0 < \alpha < 2 - \frac{n+2}{q}$, *then*

$$\|u\|_{D_T}^{(\alpha)} \leq c_3 \|u\|_{D_T}^{(2,q)}$$

where c_3 *does not depend on* u.

Proof: If we write $\langle\langle u \rangle\rangle_{D_T}^{(j,q)} = \sum_{2r+|s|=j} \|D_t^r D_x^s u\|_{q,D_T}$ for j a non-negative integer, and also write

$$\langle u \rangle_{D_T}^{(\lambda)} = \sum_{2r+|s|=[\lambda]} \langle D_t^r D_x^s u \rangle_{x,D_T}^{(\lambda-[\lambda])} + \sum_{0<\lambda-2r-|s|<2} \langle D_t^r D_x^s u \rangle_{t,D_T}^{\left(\frac{\lambda-2r-|s|}{2}\right)}$$

for $\lambda > 0$

$$\langle u \rangle_{D_T}^{(0)} = \sup_{(x,t)\in\overline{D}_T} |u(x,t)|$$

then by the second part of Lemma 3.3 on p.80 of Ladyzenskaja, Solonnikov and Ural'ceva[21] we have that for any $u \in W_{2,q}(D_T)$:

$$\langle u\rangle_{D_T}^{(\lambda)} \leq c_4^{(\lambda)} \langle\!\langle u\rangle\!\rangle_{D_T}^{(2,q)} + c_5^{(\lambda)} \|u\|_{q,D_T} \qquad \ldots\ldots\ldots\ldots(6)$$

if $0 \leq \lambda < 2 - \frac{n+2}{q}$, where $c_4^{(\lambda)}$ and $c_5^{(\lambda)}$ depend on n, q, T, λ and the dimensions of the cone K, but not on u.

Now $\|u\|_{D_T}^{(\alpha)} = \langle u\rangle_{D_T}^{(\alpha)} + \sup_{(x,t)\in\overline{D}_T} |u(x,t)|$

$$\leq c_4^{(\alpha)} \langle\!\langle u\rangle\!\rangle_{D_T}^{(2,q)} + c_5^{(\alpha)} \|u\|_{q,D_T} + \langle u\rangle_{D_T}^{(0)} \quad \text{by (6) with } \lambda=\alpha$$

$$\leq c_4^{(\alpha)} \langle\!\langle u\rangle\!\rangle_{D_T}^{(2,q)} + c_5^{(\alpha)} \|u\|_{q,D_T} + c_4^{(0)} \langle\!\langle u\rangle\!\rangle_{D_T}^{(2,q)} + c_5^{(0)} \|u\|_{q,D_T} \quad \text{by (6) with } \lambda=0$$

$$\leq c_3\{ \langle\!\langle u\rangle\!\rangle_{D_T}^{(2,q)} + \|u\|_{q,D_T}\}$$

where $c_3 = \max\{c_4^{(\alpha)} + c_4^{(0)}, c_5^{(\alpha)} + c_5^{(0)}\}$ does not depend on u. The lemma follows since $\|u\|_{D_T}^{(2,q)} = \sum_{j=0}^{2} \langle\!\langle u\rangle\!\rangle_{D_T}^{(j,q)}$

$$= \|u\|_{q,D_T} + \langle\!\langle u\rangle\!\rangle_{D_T}^{(1,q)} + \langle\!\langle u\rangle\!\rangle_{D_T}^{(2,q)}.$$

LEMMA 4: *If* $f,g \in H^{\alpha}(\overline{D}_T)$, *then* $fg \in H^{\alpha}(\overline{D}_T)$, *and* $\|fg\|_{D_T}^{(\alpha)} \leq \|f\|_{D_T}^{(\alpha)} \|g\|_{D_T}^{(\alpha)}$.

Proof: $\|fg\|_{D_T}^{(\alpha)} = \sup_{(x,t)\in\overline{D}_T} |f(x,t)g(x,t)|$

$$+ \sup_{\substack{(x,t),(y,t)\in\overline{D}_T\\ x\neq y}} \frac{|f(x,t)g(x,t) - f(y,t)g(y,t)|}{|x-y|^{\alpha}}$$

$$+ \sup_{\substack{(x,t),(x,\tau)\in\overline{D}_T\\ t\neq\tau}} \frac{|f(x,t)g(x,t) - f(x,\tau)g(x,\tau)|}{|t-\tau|^{\alpha/2}}$$

$$\leq \sup_{(x,t)\in\overline{D}_T} |f(x,t)| \sup_{(x,t)\in\overline{D}_T} |g(x,t)|$$

$$+ \sup_{\substack{(x,t),(y,t)\in\overline{D}_T\\ x\neq y}} \frac{|f(x,t)g(x,t)-f(x,t)g(y,t)+f(x,t)g(y,t)-f(y,t)g(y,t)|}{|x-y|^{\alpha}}$$

$$+ \sup_{\substack{(x,t),(x,\tau)\in\overline{D}_T\\ t\neq\tau}} \frac{|f(x,t)g(x,t)-f(x,t)g(x,\tau)+f(x,t)g(x,\tau)-f(x,\tau)g(x,\tau)|}{|t-\tau|^{\alpha/2}}$$

$$\leq \sup_{(x,t)\in\overline{D}_T} |f(x,t)| \left\{ \sup_{(x,t)\in\overline{D}_T} |g(x,t)| + \sup_{\substack{(x,t),(y,t)\in\overline{D}_T \\ x\neq y}} \frac{|g(x,t) - g(y,t)|}{|x - y|^\alpha} + \sup_{\substack{(x,t),(x,\tau)\in\overline{D}_T \\ t\neq\tau}} \frac{|g(x,t) - g(x,\tau)|}{|t - \tau|^{\alpha/2}} \right\}$$

$$+ \sup_{(x,t)\in\overline{D}_T} |g(x,t)| \left\{ \sup_{\substack{(x,t),(y,t)\in\overline{D}_T \\ x\neq y}} \frac{|f(x,t) - f(y,t)|}{|x - y|^\alpha} + \sup_{\substack{(x,t),(x,\tau)\in\overline{D}_T \\ t\neq\tau}} \frac{|f(x,t) - f(x,\tau)|}{|t - \tau|^{\alpha/2}} \right\}$$

$\leq \|f\|_{D_T}^{(\alpha)} \|g\|_{D_T}^{(\alpha)}$ as required.

Definition: Given two functions $f:\overline{D}_T \times [a,b] \to R$ and $u:\overline{D}_T \to [a,b]$, we define the function $f[u]:\overline{D}_T \to R$ by

$$f[u](x,t) = f(x,t,u(x,t)) \quad \text{for all } (x,t) \in \overline{D}_T.$$

LEMMA 5: (a) *If the function* $f(x,t,u)$ *is uniformly Lipschitz in* (x,t) *and in* u *for* $a \leq u \leq b$ *and* $(x,t) \in \overline{D}_T$, *and if the function* $u(x,t)$ *is such that* $u \in H^\alpha(\overline{D}_T)$ *and* $a \leq u(x,t) \leq b$ *for all* $(x,t) \in \overline{D}_T$, *then* $f[u] \in H^\alpha(\overline{D}_T)$.

(b) *If, in addition to the hypotheses of* (a), *we have that* $u(x,t) = u_\tau(x,t) = A(x,t) + \tau B(x,t)$ *is a linear function of the parameter* τ, *where* $0 \leq \tau \leq 1$ *and* $A,B \in H^\alpha(\overline{D}_T)$, *then for all* $\tau \in [0,1]$ *we have* $\|f[u_\tau]\|_{D_T}^{(\alpha)} \leq M_1 + M_2(\|A\|_{D_T}^{(\alpha)} + \|B\|_{D_T}^{(\alpha)})$ *where* M_1 *and* M_2 *are independent of* τ *and* u_τ.

(c) *If the first partial derivative* f_u *of* f *is uniformly Lipschitz in* (x,t) *and in* u *for* $a \leq u \leq b$ *and* $(x,t) \in \overline{D}_T$, *and if* $u,v \in H^\alpha(\overline{D}_T)$ *and* $a \leq u(x,t),v(x,t) \leq b$ *for all* $(x,t) \in \overline{D}_T$, *then*

$$\|f[u] - f[v]\|_{D_T}^{(\alpha)} \leq (K_1 + K_2\|u\|_{D_T}^{(\alpha)} + K_3\|v\|_{D_T}^{(\alpha)})\|u-v\|_{D_T}^{(\alpha)}$$

where K_1, K_2, K_3 *are independent of* u *and* v.

Proof: (a) $\|f[u]\|_{D_T}^{(\alpha)} = \sup_{(x,t)\in\overline{D}_T} |f[u](x,t)|$

$$+ \sup_{\substack{(x,t),(y,t)\in\overline{D}_T \\ x\neq y}} \frac{|f[u](x,t)-f[u](y,t)|}{|x - y|^\alpha} + \sup_{\substack{(x,t),(x,\tau)\in\overline{D}_T \\ t\neq\tau}} \frac{|f[u](x,t)-f[u](x,\tau)|}{|t - \tau|^{\alpha/2}}$$

$$\le \sup_{(x,t)\in\overline{D}_T} |f(x,t,u(x,t))|$$

$$+ \sup_{\substack{(x,t),(y,t)\in\overline{D}_T\\ x\neq y}} \frac{|f(x,t,u(x,t))-f(y,t,u(x,t))|+|f(y,t,u(x,t))-f(y,t,u(y,t))|}{|x-y|^{\alpha}}$$

$$+ \sup_{\substack{(x,t),(x,\tau)\in\overline{D}_T\\ t\neq \tau}} \frac{|f(x,t,u(x,t))-f(x,\tau,u(x,t))|+|f(x,\tau,u(x,t))-f(x,\tau,u(x,\tau))|}{|t-\tau|^{\alpha/2}}.$$

Because of the uniform Lipschitz conditions satisfied by f, we have that there exist constants K_1 and K_2 such that:

$$\|f[u]\|^{(\alpha)}_{D_T} \le \sup_{(x,t)\in\overline{D}_T} |f(x,t,u(x,t))|$$

$$+ \sup_{\substack{(x,t),(y,t)\in\overline{D}_T\\ x\neq y}} \frac{K_1|(x,t)-(y,t)| + K_2|u(x,t)-u(y,t)|}{|x-y|^{\alpha}}$$

$$+ \sup_{\substack{(x,t),(x,\tau)\in\overline{D}_T\\ t\neq \tau}} \frac{K_1|(x,t)-(x,\tau)| + K_2|u(x,t)-u(x,\tau)|}{|t-\tau|^{\alpha/2}}$$

$$\le \sup_{(x,t)\in\overline{D}_T} |f(x,t,u(x,t))| + \sup_{\substack{x,y\in\overline{V}\\ x\neq y}} K_1|x-y|^{1-\alpha}$$

$$+ \sup_{\substack{0\le t,\tau\le T\\ t\neq\tau}} K_1|t-\tau|^{1-\frac{\alpha}{2}} + K_2\|u\|^{(\alpha)}_{D_T}.$$

Since we are assuming throughout that $0 < \alpha < 1$, it follows that $\|f[u]\|^{(\alpha)}_{D_T}$ is finite if $\|u\|^{(\alpha)}_{D_T}$ is finite, which proves (a).

(b) This follows from the above argument, since

(i) $\sup_{(x,t)\in\overline{D}_T} |f(x,t,u_\tau(x,t))| \le \sup_{\substack{(x,t)\in\overline{D}_T\\ a\le u\le b}} |f(x,t,u)|$, independent of τ and u_τ.

(ii) $\|u_\tau\|^{(\alpha)}_{D_T} \le \|A\|^{(\alpha)}_{D_T} + \tau\|B\|^{(\alpha)}_{D_T} \le \|A\|^{(\alpha)}_{D_T} + \|B\|^{(\alpha)}_{D_T}$, independent of τ, if $0 \le \tau \le 1$.

(c) We have that, for all $(x,t) \in \overline{D}_T$,

$$f[u](x,t) - f[v](x,t) = f(x,t,u(x,t)) - f(x,t,v(x,t))$$

$$= \int_0^1 f_u(x,t,u(x,t)+\tau\{u(x,t)-v(x,t)\})(u(x,t)-v(x,t))d\tau$$

$$= \int_0^1 \{f_u[u+\tau(u-v)](u-v)\}(x,t)d\tau$$

and also that by parts (a) and (b),

$$\|f_u[u + \tau(u-v)]\|_{D_T}^{(\alpha)} \le M_1 + M_2(\|u\|_{D_T}^{(\alpha)} + \|u-v\|_{D_T}^{(\alpha)})$$

$$\le K_1 + K_2\|u\|_{D_T}^{(\alpha)} + K_3\|v\|_{D_T}^{(\alpha)}$$

where K_1, K_2, K_3 are independent of u and v, and of τ.

$$\therefore\ \|f[u] - f[v]\|_{D_T}^{(\alpha)} = \left\|\int_0^1 f_u[u + \tau(u-v)](u-v)d\tau\right\|_{D_T}^{(\alpha)}$$

$$\le \int_0^1 \|f_u[u + \tau(u-v)]\|_{D_T}^{(\alpha)}\|u-v\|_{D_T}^{(\alpha)}d\tau \quad \text{using Lemma 4}$$

$$\le (K_1 + K_2\|u\|_{D_T}^{(\alpha)} + K_3\|v\|_{D_T}^{(\alpha)}) \int_0^1 \|u-v\|_{D_T}^{(\alpha)}d\tau$$

$$= (K_1 + K_2\|u\|_{D_T}^{(\alpha)} + K_3\|v\|_{D_T}^{(\alpha)})\ \|u-v\|_{D_T}^{(\alpha)} \quad \text{as required.}$$

THEOREM 3: *For the initial-boundary value problem* (5), *we suppose, in addition to the assumptions already made, that:*

(i) $B_{lin}u_0 = 0$ *for* $t = 0$ *and all* $x \in \partial V$;

(ii) V *satisfies the cone condition*;

(iii) *the coefficient* $d_0(x,t)$ *of* u *in* $B_{lin}u$ *is strictly positive for all* $(x,t) \in S_T$;

(iv) *there exist upper and lower solutions* φ *and* ψ *for* (5), *with* $\psi(x,t) \le \varphi(x,t)$ *for all* $(x,t) \in \overline{D}_T$, *and* $\psi,\varphi \in H^\alpha(\overline{D}_T)$;

(v) f *satisfies a uniform Lipschitz condition in* u *on any finite u-interval, for* $(x,t) \in \overline{D}_T$, *and in* (x,t) *on* $\overline{D}_T$, *for*

$$\inf_{(x,t)\in\overline{D}_T} \psi(x,t) \le u \le \sup_{(x,t)\in\overline{D}_T} \varphi(x,t);$$

(vi) *the partial derivative* f_u *is uniformly Lipschitz in* (x,t) *and in* u *for* $\inf_{(x,t)\in\overline{D}_T} \psi(x,t) \le u \le \sup_{(x,t)\in\overline{D}_T} \varphi(x,t)$ *and* $(x,t) \in \overline{D}_T$.

Then there exists a unique solution $u \in H^{2+\alpha}(\overline{D}_T)$ *of* (5) *such that for all* $(x,t) \in \overline{D}_T$, $\psi(x,t) \le u(x,t) \le \varphi(x,t)$.

Proof: By hypothesis (vi), f_u is bounded for $(x,t) \in \overline{D}_T$ and $\inf_{(x,t)\in\overline{D}_T} \psi(x,t) \le u \le \sup_{(x,t)\in\overline{D}_T} \varphi(x,t)$. Fix Ω such that $f_u+c(x,t)+\Omega > 0$ for all $(x,t) \in \overline{D}_T$ and $\inf_{(x,t)\in\overline{D}_T} \psi(x,t) \le u \le \sup_{(x,t)\in\overline{D}_T} \varphi(x,t)$.

For any $u \in H^{\alpha}(\overline{D}_T)$ such that $\inf_{(x,t)\in\overline{D}_T} \psi(x,t) \leq u(x,t) \leq \sup_{(x,t)\in\overline{D}_T} \varphi(x,t)$ for all $(x,t) \in \overline{D}_T$, we define Tu by saying that $v = Tu$ if and only if:

$$\left.\begin{aligned} &(L - c(x,t) - \Omega)v - \frac{\partial v}{\partial t} = -\{f(x,t,u) + c(x,t)u + \Omega u\} \quad \text{for } (x,t) \in D_T \\ &B_{lin}v = 0 \quad \text{for } (x,t) \in S_T \\ &v(x,0) = u_0(x) \quad \text{for } x \in \overline{V}. \end{aligned}\right\}$$

Using hypothesis (v), it follows by Lemmas 4, 5(a) and 1 that Tu is uniquely defined for each u as specified above, and $Tu \in H^{2+\alpha}(\overline{D}_T)$.

(a) We show first that T is monotone, in the sense that if $u(x,t) \leq v(x,t)$ for all $(x,t) \in \overline{D}_T$, and Tu and Tv exist, then $(Tu)(x,t) \leq (Tv)(x,t)$ for all $(x,t) \in \overline{D}_T$.

Suppose then that $\inf_{(x,t)\in\overline{D}_T} \psi(x,t) \leq u(x,t) \leq v(x,t) \leq \sup_{(x,t)\in\overline{D}_T} \varphi(x,t)$ for all $(x,t) \in \overline{D}_T$, and $u,v \in H^{\alpha}(\overline{D}_T)$. Then:

$$(L - c(x,t) - \Omega)Tu - \frac{\partial}{\partial t}(Tu) = -\{f(x,t,u) + c(x,t)u + \Omega u\} \quad \text{for } (x,t) \in D_T;$$

$$(L - c(x,t) - \Omega)Tv - \frac{\partial}{\partial t}(Tv) = -\{f(x,t,v) + c(x,t)v + \Omega v\} \quad \text{for } (x,t) \in D_T;$$

$$B_{lin}(Tu) = B_{lin}(Tv) = 0 \quad \text{for } (x,t) \in S_T;$$

$$(Tu)(x,0) = (Tv)(x,0) = u_0(x) \quad \text{for } x \in \overline{V}.$$

Put $w = Tv - Tu$. Then:

$$\begin{aligned}(L - c(x,t) - \Omega)w - \frac{\partial w}{\partial t} &= -[f(x,t,v) + c(x,t)v + \Omega v \\ &\qquad - \{f(x,t,u) + c(x,t)u + \Omega u\}] \\ &\leq 0 \quad \text{for } (x,t) \in D_T,\end{aligned}$$

since, by the choice of Ω, $f(x,t,u) + c(x,t)u + \Omega u$ is an increasing function of u for $(x,t) \in \overline{D}_T$ and $\inf_{(x,t)\in\overline{D}_T} \psi(x,t) \leq u \leq \sup_{(x,t)\in\overline{D}_T} \varphi(x,t)$.

$$\text{Also,} \quad B_{lin}w = 0 \quad \text{for } (x,t) \in S_T;$$

$$w(x,0) = 0 \quad \text{for } x \in \overline{V}.$$

It follows by Theorem 2, taking $u_1(x,t) \equiv 0$, $u_2(x,t) \equiv w(x,t)$ in the notation of that theorem, that $w(x,t) \geq 0$ for all $(x,t) \in \overline{D}_T$.

i.e. $(Tv)(x,t) \geq (Tu)(x,t)$ for all $(x,t) \in \overline{D}_T$, as required.

(b) Now put $u_1 = T\varphi$. We prove first that $u_1(x,t) \leq \varphi(x,t)$ for all $(x,t) \in \overline{D}_T$. We have:

$$(L - c(x,t) - \Omega)u_1 - \frac{\partial u_1}{\partial t} = -\{f(x,t,\varphi) + c(x,t)\varphi + \Omega\varphi\} \quad \text{for } (x,t) \in D_T;$$

$$B_{lin}u_1 = 0 \quad \text{for } (x,t) \in S_T;$$

$$u_1(x,0) = u_0(x) \quad \text{for } x \in \bar{V}.$$

$$\therefore \quad (L - c(x,t) - \Omega)(u_1 - \varphi) - \frac{\partial}{\partial t}(u_1 - \varphi)$$

$$= (L - c(x,t) - \Omega)u_1 - \frac{\partial u_1}{\partial t} - \{(L - c(x,t) - \Omega)\varphi - \frac{\partial \varphi}{\partial t}\}$$

$$= - \{f(x,t,\varphi) + c(x,t)\varphi + \Omega\varphi\} - \{L\varphi - \frac{\partial \varphi}{\partial t}\} + c(x,t)\varphi + \Omega\varphi$$

$$= - \{L\varphi + f(x,t,\varphi) - \frac{\partial \varphi}{\partial t}\}$$

≥ 0 for $(x,t) \in D_T$ since φ is an upper solution for (5).

Also: $$B_{lin}(u_1 - \varphi) = -B_{lin}\varphi \leq 0 \quad \text{for } (x,t) \in S_T;$$

$$u_1(x,0) - \varphi(x,0) = u_0(x) - \varphi(x,0) \leq 0 \quad \text{for } x \in \bar{V}.$$

Applying Theorem 2, we obtain $u_1(x,t) - \varphi(x,t) \leq 0$ for all $(x,t) \in \bar{D}_T$, as required.

Similarly, if we put $v_1 = T\psi$, then $v_1(x,t) \geq \psi(x,t)$ for all $(x,t) \in \bar{D}_T$. Furthermore, since $\psi(x,t) \leq \varphi(x,t)$ for all $(x,t) \in \bar{D}_T$, it follows by the monotone property of T that $T\psi \leq T\varphi$ on $\bar{D}_T$, i.e. $v_1(x,t) \leq u_1(x,t)$ for all $(x,t) \in \bar{D}_T$. So we have:

$$\psi \leq v_1 \leq u_1 \leq \varphi \quad \text{on } \bar{D}_T.$$

Since $u_1 \in H^{2+\alpha}(\bar{D}_T) \subset H^{\alpha}(\bar{D}_T)$, we may define $u_2 = Tu_1$. Since $u_1 \leq \varphi$ on $\bar{D}_T$, we have $Tu_1 \leq T\varphi$ on $\bar{D}_T$, i.e. $u_2 \leq u_1$ on $\bar{D}_T$. Similarly, if we define $v_2 = Tv_1$, then $v_2 \geq v_1$ on $\bar{D}_T$. Since $v_1 \leq u_1$ on $\bar{D}_T$, it follows also that $Tv_1 \leq Tu_1$ on $\bar{D}_T$, i.e. $v_2 \leq u_2$ on $\bar{D}_T$. So we have:

$$\psi \leq v_1 \leq v_2 \leq u_2 \leq u_1 \leq \varphi \quad \text{on } \bar{D}_T.$$

Continuing thus, we obtain two sequences $\{u_n\}$, $\{v_n\}$, with $u_n, v_n \in H^{2+\alpha}(\bar{D}_T)$ for each n, and such that

$$\psi \leq v_1 \leq v_2 \leq \ldots . \leq u_2 \leq u_1 \leq \varphi \quad \text{on } \bar{D}_T.$$

(c) Since the sequences $\{u_n\}$ and $\{v_n\}$ are monotone and bounded, both converge pointwise. In particular, $\{u_n\}$ does so. Let $\bar{u}(x,t) = \lim_{n\to\infty} u_n(x,t)$ for $(x,t) \in \bar{D}_T$. Since, by hypothesis (v), $f(x,t,u)$ is continuous in u over the relevant u-interval, the sequence $\{f(x,t,u_n) + c(x,t)u_n + \Omega u_n\}$ converges pointwise to $f(x,t,\bar{u})+c(x,t)\bar{u}+\Omega\bar{u}$, for each $(x,t) \in \bar{D}_T$. Since $\psi \leq u_n, \bar{u} \leq \varphi$ on $\bar{D}_T$, it follows by Lebesgue's dominated convergence theorem that, for any $q > 1$,

$$\left\{\int_0^T \int_V |f(x,t,u_n)+c(x,t)u_n+\Omega u_n-\{f(x,t,\bar{u})+c(x,t)\bar{u}+\Omega\bar{u}\}|^q \, dx\, dt\right\}^{1/q} \to 0$$
as $n \to \infty$.

Thus the sequence $\{f[u_n] + cu_n + \Omega u_n\}$ converges in the $L_q(D_T)$ norm, i.e. this sequence is a Cauchy sequence in $L_q(D_T)$. Hence, if we write

$$h_{m,n}(x,t) = f(x,t,u_m(x,t)) + c(x,t)u_m(x,t) + \Omega u_m(x,t)$$
$$- \{f(x,t,u_n(x,t)) + c(x,t)u_n(x,t) + \Omega u_n(x,t)\} \quad \text{for all } (x,t) \in \bar{D}_T$$

then given any $\varepsilon > 0$, there exists a positive integer N_ε such that

$$m,n \geq N_\varepsilon \Rightarrow \|h_{m,n}\|_{q,D_T} < \varepsilon.$$

Now let $w_{m,n} = Tu_m - Tu_n = u_{m+1} - u_{n+1}$, then $w_{m,n} \in H^{2+\alpha}(\bar{D}_T)$ for each m, n. Further, $w_{m,n}$ satisfies

$$\left.\begin{aligned} (L - c(x,t) - \Omega)w_{m,n} - \frac{\partial}{\partial t}(w_{m,n}) &= - h_{m,n}(x,t) \quad \text{for } (x,t) \in D_T \\ B_{lin}w_{m,n} &= 0 \quad \text{for } (x,t) \in S_T \\ w_{m,n}(x,0) &= 0 \quad \text{for } x \in \bar{V}. \end{aligned}\right\}$$

Thus, by Lemma 2, $\|w_{m,n}\|_{D_T}^{(2,q)} \leq c_2\|h_{m,n}\|_{q,D_T}$ where c_2 does not depend on $h_{m,n}$. Since we may choose q such that $q > \frac{n+2}{2}$ and $0 < \alpha < 2 - \frac{n+2}{q}$ (where n here is the dimension of V), it then follows by Lemma 3 that $\|w_{m,n}\|_{D_T}^{(\alpha)} \leq c_3c_2\|h_{m,n}\|_{q,D_T}$ where c_3 does not depend on $h_{m,n}$.

$$\therefore \quad m,n \geq N_\varepsilon \Rightarrow \|w_{m,n}\|_{D_T}^{(\alpha)} < c_3c_2\varepsilon$$
$$\Rightarrow \|u_{m+1} - u_{n+1}\|_{D_T}^{(\alpha)} < c_3c_2\varepsilon.$$

Hence the sequence $\{u_n\}$ converges in the $H^\alpha(\bar{D}_T)$ norm, so that $\bar{u} \in H^\alpha(\bar{D}_T)$ and $\|u_n - \bar{u}\|_{D_T}^{(\alpha)} \to 0$ as $n \to \infty$. Thus, by hypothesis (vi) and Lemma 5(c),

$$\|f[u_n] - f[\bar{u}]\|_{D_T}^{(\alpha)} \to 0 \text{ as } n \to \infty.$$

If we now let $\bar{w}_n = Tu_n - T\bar{u}$, then $\bar{w}_n$ satisfies

$$\left.\begin{aligned} &(L - c(x,t) - \Omega)\bar{w}_n - \frac{\partial}{\partial t}(\bar{w}_n) \\ &= - [f(x,t,u_n) + c(x,t)u_n + \Omega u_n - \{f(x,t,\bar{u})+c(x,t)\bar{u}+\Omega\bar{u}\}] \quad \text{for } (x,t) \in D_T \\ &B_{lin}\bar{w}_n = 0 \quad \text{for } (x,t) \in S_T \\ &\bar{w}_n(x,0) = 0 \quad \text{for } x \in \bar{V}. \end{aligned}\right\}$$

It follows by hypothesis (v) and Lemmas 4, 5(a) and 1 that
$\bar{w}_n \in H^{2+\alpha}(\bar{D}_T)$ and

$$\|\bar{w}_n\|_{D_T}^{(2+\alpha)} \le c_1\|f[u_n] - f[\bar{u}] + c(u_n-\bar{u}) + \Omega(u_n-\bar{u})\|_{D_T}^{(\alpha)}$$

$$\le c_1\{\|f[u_n] - f[\bar{u}]\|_{D_T}^{(\alpha)} + (\|c\|_{D_T}^{(\alpha)} + |\Omega|)\|u_n-\bar{u}\|_{D_T}^{(\alpha)}\}$$

$$\to 0 \quad \text{as } n \to \infty.$$

Thus $\|Tu_n - T\bar{u}\|_{D_T}^{(2+\alpha)} \to 0$ as $n \to \infty$. Hence $T\bar{u} \in H^{2+\alpha}(\bar{D}_T)$, since we know $Tu_n = u_{n+1} \in H^{2+\alpha}(\bar{D}_T)$; and Tu_n converges to $T\bar{u}$ in $H^{2+\alpha}(\bar{D}_T)$, so that $(Tu_n)(x,t)$ certainly converges uniformly to $(T\bar{u})(x,t)$ on $\bar{D}_T$.

$$\therefore \quad \bar{u}(x,t) = \lim_{n\to\infty} u_n(x,t) = \lim_{n\to\infty} (Tu_{n-1})(x,t) = (T\bar{u})(x,t) \quad \text{for all } (x,t) \in \bar{D}_T.$$

It follows that $\bar{u} \in H^{2+\alpha}(\bar{D}_T)$. Also, $T\bar{u}$ satisfies

$$\left.\begin{aligned} (L - c(x,t) - \Omega)T\bar{u} - \frac{\partial}{\partial t}(T\bar{u}) &= -\{f(x,t,\bar{u}) + c(x,t)\bar{u} + \Omega\bar{u}\} \\ &\qquad \text{for } (x,t) \in D_T \\ B_{lin}(T\bar{u}) &= 0 \quad \text{for } (x,t) \in S_T \\ (T\bar{u})(x,0) &= u_0(x) \quad \text{for } x \in \bar{V}. \end{aligned}\right\}$$

Since $T\bar{u} = \bar{u}$, we have at once that $\bar{u}$ is a solution of (5), which proves the theorem; similarly one can show that $\lim_{n\to\infty} v_n = \bar{v} \in H^{2+\alpha}(\bar{D}_T)$ is a solution of (5). It follows by the uniqueness of the solution that $\bar{u} = \bar{v}$ on $\bar{D}_T$.

We next consider the elliptic boundary value problem

$$\left.\begin{aligned} \hat{L}u + \hat{f}(x,u) &= 0 \quad \text{for } x \in V \\ \hat{B}_{lin}u &= 0 \quad \text{for } x \in \partial V \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where $\hat{L}$ and $\hat{B}_{lin}$ are as defined in Ch.3, with the additional assumptions that $\hat{a}_{ij} = \hat{a}_{ji} \in C^{\alpha}(\bar{V})$, $\hat{b}_i \in C^{\alpha}(\bar{V})$ and $\hat{c} \in C^{\alpha}(\bar{V})$; also $\hat{d}_0$ and $\hat{d}_1$ are of class $C^{1+\alpha}(\partial V)$ and $\hat{f}$ is continuous for $x \in \bar{V}$ and at least some u-interval.

We call $\hat{\phi}(x)$ an *upper solution* for (7) if $\hat{\phi}$ is continuous in $\bar{V}$, has continuous first-order x_i-derivatives in $\bar{V}$, continuous second-order x_i-derivatives in V, and satisfies:

$$\hat{L}\hat{\phi} + \hat{f}(x,\hat{\phi}) \le 0 \quad \text{for } x \in V$$
$$\hat{B}_{lin}\hat{\phi} \ge 0 \quad \text{for } x \in \partial V.$$

The terms *strict upper solution*, *lower solution* and *strict lower solution* are defined analogously, as for problem (5), and a *solution* of (7) is understood to be a solution in a similar sense as for problem (5).

It is of interest to observe that comparison theorems analogous to Theorems 1 and 2 do not hold in this case. For example, consider the problem for which $\hat{L}u$ is $\frac{d^2u}{dx^2}$, $\hat{B}_{lin}u$ is u, $\hat{f}(x,u)$ is 8u+1 and V is $\{x: -1 < x < 1\}$. Take $u_1(x) = 4x^4 + 3$, $u_2(x) = 8x^2$ for all $x \in \overline{V}$.

Then $\hat{L}u_1 + \hat{f}(x,u_1) - \{\hat{L}u_2 + \hat{f}(x,u_2)\} = 48x^2 + 32x^4 + 25 - \{64x^2 + 17\}$

$$= 32x^4 - 16x^2 + 8$$
$$= 2(4x^2-1)^2 + 6 > 0 \quad \text{for all } x.$$

Thus $\hat{L}u_1 + \hat{f}(x,u_1) > \hat{L}u_2 + \hat{f}(x,u_2)$ for all $x \in V$. Also, when $x \in \partial V$, i.e. $x = \pm 1$, then $\hat{B}_{lin}u_1 = u_1 = 7$, $\hat{B}_{lin}u_2 = u_2 = 8$. Thus $\hat{B}_{lin}u_1 < \hat{B}_{lin}u_2$ for all $x \in \partial V$. If theorems analogous to Theorems 1 and 2 held, we would expect at least that $u_1(x) \le u_2(x)$ for all $x \in \overline{V}$. However, when $x = 0$, $u_1(=3) > u_2(=0)$, so theorems analogous to Theorems 1 and 2 do not hold in this case.

Hence, if there exist upper and lower solutions $\hat{\phi}$ and $\hat{\psi}$ for (7), then in contrast to the case of the parabolic problem (5), we cannot assert that every solution u of (7) must lie between $\hat{\phi}$ and $\hat{\psi}$ on $\overline{V}$. However, Sattinger's method of monotone iteration is still applicable, and shows the existence of at least one solution u of (7) lying between $\hat{\phi}$ and $\hat{\psi}$ on $\overline{V}$. In this connection, though, it is interesting to note that it has been shown (e.g. by Parter[27]) that in certain special cases there exist solutions of (7) which cannot be obtained by monotone iteration procedures. Such solutions are unstable in the sense of Keller and Cohen[19] (see Ch.4). A full discussion of these points is given by Amann[3].

We give now Sattinger's existence theorem[32, Theorem 2.3.1] for elliptic problems. The proof is similar to that of Theorem 3 and so is not given in detail, but an outline of the procedure is given, as this will be referred to in the sequel.

THEOREM 4: *For the boundary value problem* (7), *we suppose, in addition to the assumptions already made, that*:

(i) *there exist upper and lower solutions* $\hat{\phi}$ *and* $\hat{\psi}$ *for* (7), *with* $\hat{\psi}(x) \le \hat{\phi}(x)$ *for all* $x \in \overline{V}$ *and* $\hat{\psi}, \hat{\phi} \in C^{2+\alpha}(\overline{V})$;

(ii) $\hat{f}$ *and its first partial derivative* $\hat{f}_u$ *are both uniformly Lipschitz in* x *and in* u *for* $\inf_{x\in\overline{V}} \hat{\psi}(x) \le u \le \sup_{x\in\overline{V}} \hat{\phi}(x)$ *and* $x \in \overline{V}$.

Then there exists a solution $u \in C^{2+\alpha}(\overline{V})$ *of* (7) *such that for all* $x \in \overline{V}$, $\hat{\psi}(x) \le u(x) \le \hat{\phi}(x)$.

Proof: The method is similar to that used in the proof of Theorem 3. Fix $\Omega > 0$ such that $\hat{f}_u + \hat{c}(x) + \Omega > 0$ for $\inf_{x\in\overline{V}} \hat{\psi}(x) \le u \le \sup_{x\in\overline{V}} \hat{\phi}(x)$ and $x \in \overline{V}$. Then define Tu by saying that v = Tu if and only if:

$$(\hat{L} - \hat{c}(x) - \Omega)v = -\{\hat{f}(x,u) + \hat{c}(x)u + \Omega u\} \quad \text{for } x \in V;$$

$$\hat{B}_{lin} v = 0 \quad \text{for } x \in \partial V.$$

As in the proof of Theorem 3, define sequences $\{\hat{u}_n\}$ and $\{\hat{v}_n\}$ so that $\hat{\psi} \le \hat{v}_1 \le \hat{v}_2 \le \dots \le \hat{u}_2 \le \hat{u}_1 \le \hat{\phi}$ on $\overline{V}$, and $\hat{u}_n, \hat{v}_n \in C^{2+\alpha}(\overline{V})$ for all $n \ge 1$. It may then be shown that $\lim_{n\to\infty} \hat{u}_n = \hat{\bar{u}}$ and $\lim_{n\to\infty} \hat{v}_n = \hat{\bar{v}}$ are both solutions of (7), thus proving the theorem; note that in contrast to Theorem 3 we do not necessarily have $\hat{\bar{u}} = \hat{\bar{v}}$ on $\overline{V}$ in this case.

Asymptotic Behaviour of Solutions:

In order to apply the two preceding existence theorems to the problem of determining the connection between the boundedness over all time of the solution of (5) and the existence of positive solutions of (7), we require next two theorems concerning the asymptotic behaviour of solutions of linear systems. These are analogous to theorems proved by Friedman[14, Ch.6], but with different boundary conditions.

THEOREM 5: *Suppose that the coefficients* a_{ij}, b_i, c *in* Lu *are uniformly continuous and bounded in* $\overline{D}$, *the coefficients* d_0 *and* d_1 *in* $B_{lin}u$ *are uniformly continuous and bounded in* S, *and for some* $\mu_1 > 0$, $d_0(x,t) \ge \mu_1$ *for all* $(x,t) \in S$. *Suppose further that* u(x,t) *satisfies the differential equation*

$$Lu - \frac{\partial u}{\partial t} = g(x,t) \quad \textit{for } (x,t) \in D$$

where g *is continuous on* $\overline{D}$, *together with the boundary condition*

$$B_{lin}u = h(x,t) \quad \textit{for } (x,t) \in S$$

where h *is continuous on* S. *If* $\lim_{t\to\infty} g(x,t) = 0$, $\lim_{t\to\infty} h(x,t) = 0$ *and* $\lim_{t\to\infty} c(x,t) \le 0$ *uniformly on* $\overline{V}$, ∂V *and* $\overline{V}$ *respectively, then* $\lim_{t\to\infty} u(x,t) = 0$ *uniformly on* $\overline{V}$.

Proof: Consider the function $\varphi(x) = e^{\lambda R} - e^{\lambda x_1}$ (x_1 being the first component of x), where R is any positive number satisfying $R \ge 2x_1$ for all $x \in \overline{V}$, and λ is a positive constant to be determined later. Then $\varphi(x)$ satisfies:

$$L\varphi = -a_{11}(x,t)\lambda^2 e^{\lambda x_1} - b_1(x,t)\lambda e^{\lambda x_1} + c(x,t)(e^{\lambda R} - e^{\lambda x_1}).$$

Independently of the value of R, we choose λ sufficiently large so that $L\varphi(x) < -2e^{\lambda x_1} + c(x,t)(e^{\lambda R} - e^{\lambda x_1})$, for $(x,t) \in \overline{D}$. Since $\lim_{t\to\infty} c(x,t) \le 0$, it follows that for some $\bar{\sigma}$ sufficiently large,

$$c(x,t)(e^{\lambda R} - e^{\lambda x_1}) < e^{\lambda x_1} \quad \text{for } t > \bar{\sigma}, \text{ and all } x \in \overline{V}.$$

Letting $\theta = \inf_{x\in\overline{V}} e^{\lambda x_1}$, we then have:

$$L\varphi(x) < -\theta \quad \text{for } t > \bar{\sigma} \text{ and all } x \in \overline{V} \qquad \ldots\ldots\ldots\ldots(8)$$

Also, $B_{lin}\varphi(x) = d_0(x,t)(e^{\lambda R} - e^{\lambda x_1}) - \lambda d_1(x,t)n_1(x)e^{\lambda x_1}$

$> \mu_2$ for all $(x,t) \in \overline{D}$, for some positive μ_2, if R is sufficiently large. Choose R so that this is the case.

Now let $\theta_0 = \inf_{x\in\overline{V}} \varphi(x)$, $\theta_1 = \sup_{x\in\overline{V}} \varphi(x)$. Consider the function

$$\psi(x,t) = \varepsilon\frac{\varphi(x)}{\theta} + \varepsilon\frac{\varphi(x)}{\mu_2} + A\frac{\varphi(x)}{\theta_0}e^{-\xi(t-\sigma)} \quad \text{for } t > \sigma \ge \bar{\sigma}$$

where ε, ξ are positive constants and $A = \sup_{x\in\overline{V}} |u(x,\sigma)|$.

By (8): $L\psi(x,t) < -\varepsilon - \frac{\theta\varepsilon}{\mu_2} - \frac{\theta A}{\theta_0}e^{-\xi(t-\sigma)}$, for all $x \in V$.

Also, $\frac{\partial\psi}{\partial t} = -\frac{\xi A\varphi(x)}{\theta_0}e^{-\xi(t-\sigma)} > -\frac{\xi A\theta_1}{\theta_0}e^{-\xi(t-\sigma)}$ for all $x \in V$. If we take $\xi = \frac{\theta}{\theta_1}$, then $\frac{\partial\psi}{\partial t} > -\frac{\theta A}{\theta_0}e^{-\xi(t-\sigma)}$, and so:

$$L\psi(x,t) - \frac{\partial\psi}{\partial t} < -\varepsilon \quad \text{for } t > \sigma \text{ and } x \in V \qquad \ldots\ldots(9)$$

Clearly:
$$\psi(x,\sigma) > A \quad \text{for } x \in V \qquad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Also: $$B_{lin}\psi(x,t) > \frac{\varepsilon\mu_2}{\theta} + \varepsilon + \frac{A\mu_2}{\theta_0} e^{-\xi(t-\sigma)}$$

$$> \varepsilon \quad \text{for all } x \in \partial V,\ t > \sigma \qquad \ldots\ldots\ldots\ldots(11)$$

By hypothesis, for any $\varepsilon > 0$, there exists $\sigma(\varepsilon)$ such that $|g(x,t)| < \varepsilon$ and $|h(x,t)| < \varepsilon$ for $t > \sigma$; we may assume $\sigma(\varepsilon) \geq \bar{\sigma}$. By two applications of Theorem 1, using (9), (10) and (11), we have

$u(x,t) < \psi(x,t)$ and $-u(x,t) < \psi(x,t)$ for $(x,t) \in \bar{V}$ and $t \geq \sigma$.

$\therefore \quad |u(x,t)| < \psi(x,t)$ for $(x,t) \in \bar{V}$ and $t \geq \sigma$.

$\therefore$ For $(x,t) \in \bar{V}$ and $t \geq \sigma$, we have:

$$|u(x,t)| \leq A_1\varepsilon + A_2 e^{-\xi(t-\sigma)},$$ A_1 and A_2 positive constants, A_1 depending only on φ

$$\leq 2A_1\varepsilon \quad \text{if } t \geq \sigma - \frac{1}{\xi}\ln\left(\frac{A_1\varepsilon}{A_2}\right).$$

This completes the proof of Theorem 5.

For the next theorem, we need the following standard result (see, for example, the book by Ladyzenskaja and Ural'ceva[22, pp.137,138]).

LEMMA: *Suppose that the operators $\hat{L}$ and $\hat{B}_{lin}$ are as defined in Ch.3, with the additional assumptions that:*

(i) $\hat{a}_{ij} = \hat{a}_{ji} \in C^{\alpha}(\bar{V})$, $\hat{b}_i \in C^{\alpha}(\bar{V})$, $\hat{c} \in C^{\alpha}(\bar{V})$ *and* $\hat{c}(x) \leq 0$ *for all* $x \in \bar{V}$;

(ii) $\hat{d}_0$ *and* $\hat{d}_1$ *are of class* $C^{1+\alpha}(\partial V)$, *and there exists* $\mu_1 > 0$ *such that* $\hat{d}_0(x) \geq \mu_1$ *for all* $x \in \partial V$.

Then for any $g \in C^{\alpha}(\bar{V})$, *the boundary value problem*

$$\hat{L}v = \hat{g}(x) \quad \textit{for } x \in V$$
$$\hat{B}_{lin}v = 0 \quad \textit{for } x \in \partial V$$

has a unique solution $v \in C^{2+\alpha}(\bar{V})$. *Thus, certainly,* v *and all its first and second partial derivatives are bounded in* $\bar{V}$.

THEOREM 6: *Suppose that the operators* L *and* B_{lin} *satisfy the hypotheses of Theorem 5, and the operators* $\hat{L}$ *and* $\hat{B}_{lin}$ *satisfy the hypotheses of the Lemma. Suppose also that* $a_{ij}(x,t) \to \hat{a}_{ij}(x)$, $b_i(x,t) \to \hat{b}_i(x)$, $c(x,t) \to \hat{c}(x)$, $g(x,t) \to \hat{g}(x)$, $d_0(x,t) \to \hat{d}_0(x)$ *and* $d_1(x,t) \to \hat{d}_1(x)$ *as* $t \to \infty$, *uniformly in* $\bar{V}$; *here* g *is continuous on* $\bar{D}$ *and* $\hat{g} \in C^{\alpha}(\bar{V})$.

If $u(x,t)$ *is a solution of the boundary value problem*

$$\left.\begin{aligned} Lu - \frac{\partial u}{\partial t} &= g(x,t) \quad \textit{for } (x,t) \in D \\ B_{lin}u &= 0 \quad \textit{for } (x,t) \in S \end{aligned}\right\}$$

and $v(x)$ *is the unique solution of the boundary value problem*

$$\left.\begin{aligned} \hat{L}v &= \hat{g}(x) \quad \textit{for } x \in V \\ \hat{B}_{lin}v &= 0 \quad \textit{for } x \in \partial V \end{aligned}\right\}$$

then $u(x,t) \to v(x)$ *as* $t \to \infty$, *uniformly in* $\overline{V}$.

Proof: Put $w(x,t) = u(x,t) - v(x)$, for $(x,t) \in \overline{D}$. Then:

$$\begin{aligned} Lw - \frac{\partial w}{\partial t} &= Lu - \frac{\partial u}{\partial t} - (L-\hat{L})v - \hat{L}v \\ &= g(x,t) - \hat{g}(x) - (L-\hat{L})v \quad \text{for } (x,t) \in D. \\ B_{lin}w = - B_{lin}v &= (\hat{B}_{lin} - B_{lin})v \quad \text{for } (x,t) \in S. \end{aligned}$$

By virtue of the hypotheses of Theorem 6 and the boundedness of v and its first and second partial derivatives on $\overline{V}$ (see the Lemma), we may apply Theorem 5 and conclude that $\lim_{t\to\infty} w(x,t) = 0$, uniformly on $\overline{V}$, which proves Theorem 6.

We are now in a position to make a first statement about the relationship between solutions of the parabolic problem (5) and the elliptic problem (7). We shall assume for the purposes of this discussion that the operators L, $\hat{L}$, B_{lin} and $\hat{B}_{lin}$ satisfy the hypotheses of Theorems 5 and 6 in addition to the conditions imposed when describing problems (5) and (7). We assume also that $f(x,t,u) \to \hat{f}(x,u)$ as $t \to \infty$, uniformly in x for $x \in \overline{V}$ and in u on any bounded u-interval.

Suppose that we have upper and lower solutions $\varphi(x,t)$ and $\psi(x,t)$ for (5), for all $T > 0$, and upper and lower solutions $\hat{\varphi}(x)$ and $\hat{\psi}(x)$ for (7), such that $\varphi(x,t) \to \hat{\varphi}(x)$ and $\psi(x,t) \to \hat{\psi}(x)$ as $t \to \infty$, uniformly for $x \in \overline{V}$. Suppose also that the conditions for the monotone iteration theorems, Theorems 3 and 4, are satisfied. It is clear from the constructions used in these theorems (applied for arbitrarily large T) that we can, using induction, apply Theorem 6 to the function pairs $u_n(x,t)$, $\hat{u}_n(x)$ and $v_n(x,t)$, $\hat{v}_n(x)$, and deduce that for all positive integers n:

$$\left.\begin{aligned} u_n(x,t) &\to \hat{u}_n(x) \\ v_n(x,t) &\to \hat{v}_n(x) \end{aligned}\right\} \text{ as } t \to \infty, \text{ uniformly for } x \in \overline{V}.$$

We now suppose further that $\hat{u}(x)$ is the *only* solution of (7) lying between $\hat{\psi}(x)$ and $\hat{\phi}(x)$. This would be the case, for example, if the lower solution $\hat{\psi}(x)$ were positive and the hypotheses for result (viii) of Keller and Cohen (see Ch.4) were satisfied (this means, in particular, that $\hat{f}(x,u)$ would be concave in u).

Since $\hat{u}(x)$ is the only solution of (7) between $\hat{\psi}(x)$ and $\hat{\phi}(x)$, and the sequences $\{\hat{u}_n(x)\}$ and $\{\hat{v}_n(x)\}$ both converge uniformly to solutions of (7) lying between $\hat{\psi}(x)$ and $\hat{\phi}(x)$ by Theorem 4, it follows that $\hat{u}_n(x) \to \hat{u}(x)$ and $\hat{v}_n(x) \to \hat{u}(x)$ as $n \to \infty$, uniformly for $x \in \bar{V}$, and we know also that $\hat{v}_n(x) \leq \hat{u}(x) \leq \hat{u}_n(x)$ for all $x \in \bar{V}$ and all positive integers n.

Thus, given any $\epsilon > 0$, there exists a positive integer $N(\epsilon)$, independent of $x \in \bar{V}$, such that $|\hat{u}_n(x) - \hat{u}(x)| < \frac{\epsilon}{2}$ and $|\hat{v}_n(x) - \hat{u}(x)| < \frac{\epsilon}{2}$ whenever $n \geq N(\epsilon)$. Further, there exists $\tau(\epsilon,N(\epsilon))$ independent of $x \in \bar{V}$ such that $|u_N(x,t) - \hat{u}_N(x)| < \frac{\epsilon}{2}$ and $|v_N(x,t) - \hat{v}_N(x)| < \frac{\epsilon}{2}$ whenever $t > \tau(\epsilon,N(\epsilon))$.

$$\therefore \quad |u_N(x,t) - \hat{u}(x)| < \epsilon \quad \text{and} \quad |v_N(x,t) - \hat{u}(x)| < \epsilon \quad \text{whenever } t > \tau(\epsilon,N(\epsilon)).$$

$$\therefore \quad \hat{u}(x) - \epsilon < v_N(x,t) \leq \bar{u}(x,t) \leq u_N(x,t) < \hat{u}(x) + \epsilon \quad \text{whenever } t > \tau(\epsilon,N(\epsilon))$$

where $\bar{u}$ is the solution of (5) obtained in the proof of Theorem 3; note that by applying Theorem 3 for arbitrarily large T, we can show that $\bar{u}(x,t)$ exists for all $t \geq 0$.

$$\therefore \quad \bar{u}(x,t) \to \hat{u}(x) \text{ as } t \to \infty, \text{ uniformly for } x \in \bar{V}.$$

Thus, under the given conditions, the existence of exactly one solution to the steady-state problem (7) lying between $\hat{\psi}(x)$ and $\hat{\phi}(x)$ implies that, for any initial value $u_0(x)$ lying between $\psi(x,0)$ and $\phi(x,0)$, the unique solution $u(x,t)$ of (5) (for arbitrarily large T) will tend to the steady-state solution $\hat{u}(x)$ of (7) as $t \to \infty$, uniformly for $x \in \bar{V}$.

Let us now consider the special case where the coefficients and the function f in the parabolic problem are independent of t, so that we are concerned with the problem

$$\left.\begin{aligned} \hat{L}u - \frac{\partial u}{\partial t} + \hat{f}(x,u) &= 0 \quad \text{for } (x,t) \in D \\ \hat{B}_{lin}u &= 0 \quad \text{for } (x,t) \in S \\ u(x,0) &= u_0(x) \quad \text{for } x \in \bar{V} \end{aligned}\right\} \qquad \text{........(12)}$$

As a particular case of the preceding discussion, we obtain the following theorem.

THEOREM 7: *Suppose that*
(i) *the operators* $\hat{L}$ *and* $\hat{B}_{lin}$ *satisfy the hypotheses of Theorem 6 in addition to the conditions imposed when describing problem* (7);
(ii) *the initial value* $u_0(x)$ *in* (12) *is non-negative for all* $x \in \overline{V}$, *and there exists a non-negative solution* μ *of* (7) *such that* $\mu(x) \geq u_0(x) \geq 0$ *for all* $x \in \overline{V}$;
(iii) *there is no solution* u *of* (7), *different from* μ, *such that* $0 \leq u(x) \leq \mu(x)$ *for all* $x \in \overline{V}$;
(iv) $\hat{f}(x,0) \geq 0$ *for all* $x \in V$;
(v) $\hat{L}$, $\hat{B}_{lin}$, $\hat{f}$, u_0 *and* V *satisfy the hypotheses of the monotone iteration theorems, Theorems 3 and 4.*
Then there exists a unique solution $\bar{u}$ *of* (12) *such that* $\bar{u}(x,t) \to \mu(x)$ *as* $t \to \infty$, *uniformly for* $x \in \overline{V}$.
Proof: Bearing in mind our assumptions that $\mu(x) \geq u_0(x) \geq 0$ for all $x \in \overline{V}$ and $\hat{f}(x,0) \geq 0$ for all $x \in V$, it is obvious that $\mu(x)$ is an upper solution and 0 a lower solution for both (7) and (12). The theorem then follows from the preceding discussion.
Note: A crucial hypothesis in Theorem 7 is the existence of a minimal non-negative solution μ of (7); as discussed in Ch.4, such a minimal non-negative solution does exist under a wide range of conditions.

Let us now try to relate the preceding material to the thermal explosion problem with which we began. Consider the time-dependent problem

$$\left.\begin{aligned} \hat{L}u - \frac{\partial u}{\partial t} + \lambda \hat{f}(x,u) &= 0 \quad \text{for } (x,t) \in D \\ \hat{B}_{lin}u &= 0 \quad \text{for } (x,t) \in S \\ u(x,0) &= 0 \quad \text{for } x \in \overline{V} \end{aligned}\right\} \qquad \ldots\ldots\ldots(13)$$

and its related steady-state problem

$$\left.\begin{aligned} \hat{L}u + \lambda \hat{f}(x,u) &= 0 \quad \text{for } x \in V \\ \hat{B}_{lin}u &= 0 \quad \text{for } x \in \partial V \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots(14)$$

where $\hat{L}$, $\hat{B}_{lin}$, $\hat{f}$ and V are assumed to satisfy the hypotheses of Theorem 7. These problems differ from problems (12) and (7) only in the re-introduction of the parameter λ and the fact that $u_0 = 0$, and represent a modest generalisation of the original heat-generation problem.

If we adopt the orthodox approach and equate the existence of positive steady-state solutions with the absence of a thermal explosion, then it is natural to define the *critical parameter* λ^* for the pair of problems (13), (14) to be the least upper bound of the set of positive values of λ for which positive solutions of (14) exist (if the set is not bounded above, we can take λ^* to be infinite). This conforms with the notation used by Keller and Cohen[19] and is also the definition used by Boddington, Gray and Harvey[4]. As discussed in Ch.4, if we assume that $\hat{L}$ and $\hat{B}_{lin}$ are of the rather special form described in that chapter, it is known that if $\hat{f}$ is continuous, positive, and strictly increasing in u, for $x \in V$ and $u \geq 0$ (these being hypotheses H_0, H_1 and H_2 of Keller and Cohen[19]), then positive steady-state solutions, if they exist at all, occur for all λ in $(0,\lambda^*)$ and for no λ greater than λ^*. Thus, by the orthodox criterion, "explosion" occurs if $\lambda > \lambda^*$ but not if $0 < \lambda < \lambda^*$.

Indeed, under the same conditions on $\hat{f}$, it is known, as has been remarked, that a *minimal* positive solution of (14) exists if $0 < \lambda < \lambda^*$. It then follows from Theorem 7 that if $0 < \lambda < \lambda^*$, (13) has a unique solution $u(x,t)$ which is bounded for $t \geq 0$ and tends to the minimal positive solution of (14) as $t \to \infty$.

What of the reverse implication? Suppose we know that (13) has a solution $u(x,t)$ which is bounded for $t \geq 0$; does it tend to a solution of (14) as $t \to \infty$? To deal with this question, we require two theorems analogous to Theorems 2.5.1 and 2.6.1 of Sattinger[32], but with different boundary conditions. The proofs are included for the sake of completeness; they are analogous to those used by Sattinger.

THEOREM 8: *Suppose that there exists a lower solution* $\hat{\psi}(x)$ *for the elliptic problem* (7), *and a solution* $u(x,t)$ *of the particular parabolic problem* (12) *with* $u_0 = \hat{\psi}$. *Suppose also that the coefficient* $\hat{d}_0(x)$ *in* $\hat{B}_{lin}$ *is strictly positive for all* $x \in \partial V$, *and the function* $\hat{f}(x,u)$ *satisfies a uniform Lipschitz condition in* u *on any finite* u*-interval, and has partial derivative* $\hat{f}_u$ *continuous for all* $x \in \overline{V}$ *and all real* u. *Then* $\frac{\partial u}{\partial t} \geq 0$ *for all* $(x,t) \in \overline{D}$.

Proof: For any $(x,t) \in \overline{D}$, put $w_h(x,t) = \frac{u(x,t+h) - u(x,t)}{h}$, where $h > 0$. Then if $(x,t) \in D$, we have:

$$\hat{L}w_h + \frac{\hat{f}(x,u(x,t+h)) - \hat{f}(x,u(x,t))}{h} - \frac{\partial w_h}{\partial t} = 0.$$

Now $$\hat{f}(x,b) - \hat{f}(x,a) = \int_a^b \hat{f}_u(x,u)du$$

$$= \int_0^1 \hat{f}_u(x, \tau b+(1-\tau)a)(b-a)d\tau.$$

Let $b = u(x,t+h)$ and $a = u(x,t)$, where $(x,t) \in D$. Then:

$$\frac{\hat{f}(x,u(x,t+h)) - \hat{f}(x,u(x,t))}{h} = \xi(x,t,h)w_h(x,t)$$

where $\xi(x,t,h) = \int_0^1 \hat{f}_u(x, \tau u(x,t+h)+(1-\tau)u(x,t))d\tau$. Hence $w_h(x,t)$ satisfies:

$$\hat{L}w_h + \xi w_h - \frac{\partial w_h}{\partial t} = 0 \quad \text{for } (x,t) \in D.$$

Also:

$$\hat{B}_{lin}w_h = \frac{1}{h}\{\hat{B}_{lin}u(x,t+h) - \hat{B}_{lin}u(x,t)\}$$
$$= 0 \quad \text{for } (x,t) \in S.$$
$$w_h(x,0) = \frac{u(x,h) - u(x,0)}{h}$$
$$= \frac{u(x,h) - \hat{\psi}(x)}{h} \geq 0 \quad \text{for } x \in \overline{V}, \text{ since } \hat{\psi} \text{ is}$$

also a lower solution for (12), and so $u(x,h) \geq \hat{\psi}(x)$ for all $x \in \overline{V}$ and $h > 0$ by Theorem 2. Applying Theorem 2 again, we see that $w_h(x,t) \geq 0$ for all $(x,t) \in \overline{D}$, and all $h > 0$. It follows that $\frac{\partial u}{\partial t} = \lim_{h\to 0+} w_h(x,t) \geq 0$ for all $(x,t) \in \overline{D}$.

Note: An analogous theorem holds if we have an upper solution $\hat{\phi}$ for (7) and a solution u of (12) with $u_0 = \hat{\phi}$; in that case $\frac{\partial u}{\partial t} \leq 0$ for all $(x,t) \in \overline{D}$.

THEOREM 9: *If we make the same assumptions as for Theorem 8, and suppose in addition that for some constant* K, $u(x,t) \leq K$ *for all* $(x,t) \in \overline{D}$, *then* $\lim_{t\to\infty} u(x,t) = \hat{u}(x)$ *exists for all* $x \in \overline{V}$, *and* $\hat{u}$ *is equal a.e. to a classical solution of the elliptic problem* (7).

Proof: We use the inner product notation to denote the usual $L_2(V)$ inner product for real functions, i.e. $(f,g) = \int_V fg\, dx$. Also, given two functions $\hat{f}:\overline{V} \times [a,b] \to R$ and $u:\overline{D} \to [a,b]$, we define the function $\hat{f}[u]:\overline{D} \to R$ by $\hat{f}[u](x,t) = \hat{f}(x,u(x,t))$ for all $(x,t) \in \overline{D}$.

Now consider the operator $L_1 = \hat{L} - \hat{c}(x)$, understood to have as domain the set of all $u \in S_{2,2}(V)$ satisfying $\hat{B}_{lin}u = 0$. Lions and Magenes[24, Vol.I, pp.114-121] describe the construction of the adjoint operator L_1^* and also the adjoint domain consisting of all $u \in S_{2,2}(V)$ satisfying an appropriate adjoint boundary condition $Cu = 0$. As shown by Lions and Magenes[24, Vol.I, Ch.2, Theorem 2.1, Corollary 2.1 and Remark 2.2 on pp.119,120], if $u \in$ domain L_1 and $v \in$ domain L_1^*, then $(L_1u,v) = (u,L_1^*v)$.

Now let ξ be in the domain of L_1^*. We know that $u(x,t)$ is in the domain of L_1 as a function of x, for each $t \geq 0$. Write

$$f_1(x,u) = \hat{f}(x,u) + \hat{c}(x)u.$$

Then since $\qquad \hat{L}u - \frac{\partial u}{\partial t} + \hat{f}(x,u) = 0 \quad \text{for } (x,t) \in D$

we have $(L_1 u,\xi) + (f_1[u],\xi) - (\xi,u_t) = 0$ for all $t > 0$.

i.e. $(u,L_1^*\xi) + (f_1[u],\xi) - (\xi,u_t) = 0$ for all $t > 0$.

So $\frac{1}{T}\int_0^T (u,L_1^*\xi)dt + \frac{1}{T}\int_0^T (f_1[u],\xi)dt - \frac{1}{T}\int_0^T (\xi,u_t)dt = 0$ for all $T > 0$.

Now let $T \to \infty$. Since $u(x,t)$ is non-decreasing in t by Theorem 8, and bounded above by K for all $(x,t) \in \overline{D}$ by hypothesis, it follows that $\lim_{t\to\infty} u(x,t) = \hat{u}(x)$ exists for all $x \in \overline{V}$, and hence that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T u(x,t)dt = \hat{u}(x) \quad \text{for all } x \in \overline{V}.$$

So $\lim_{T\to\infty} \frac{1}{T}\int_0^T (u,L_1^*\xi)dt = \lim_{T\to\infty} (\frac{1}{T}\int_0^T u(x,t)dt, L_1^*\xi)$ by interchanging the order of integration, since L_1^* is independent of t

$$= (\hat{u},L_1^*\xi).$$

Here the interchange of the order of integration follows by Fubini's Theorem, and the final step follows from the Lebesgue dominated convergence theorem and the fact that $\hat{\psi}(x) \leq u(x,t) \leq K$ for all $(x,t) \in \overline{D}$.

Similarly, using the fact that $f_1(x,u)$ is continuous in u, we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (f_1[u],\xi)dt = (f_1[\hat{u}],\xi).$$

Also, $\lim_{T\to\infty} \frac{1}{T}\int_0^T (\xi,u_t)dt = \lim_{T\to\infty} (\xi,\frac{1}{T}\int_0^T u_t(x,t)dt)$

$$= \lim_{T\to\infty} (\xi,\frac{u(x,T) - u(x,0)}{T})$$

$= (\xi,0) = 0$ since $u(x,T)$ is bounded as $T \to \infty$, for all $x \in \overline{V}$. Here again we are using the Lebesgue dominated convergence theorem.

So, taking limits as $T \to \infty$, we obtain finally:

$$(\hat{u},L_1^*\xi) + (f_1[\hat{u}],\xi) = 0.$$

Now L_1 is invertible, since for $g \in L_2(V)$, the system $L_1 w = g$, $\hat{B}_{lin} w = 0$

has a unique solution, by a result of Agmon, Douglis and Nirenberg[1]. The same applies to L_1^*, since the order of the adjoint boundary condition is 0 or 1 by Theorem 2.1(b) on p.115 of the book by Lions and Magenes[24], and so the uniqueness theorem of Agmon, Douglis and Nirenberg applies. Let G_1 be the inverse of L_1. Then by a result of Riesz and Nagy[30, p.304], G_1^* is the inverse of L_1^*. Put $w = -G_1 f_1[\hat{u}]$. Then:

$$
\begin{aligned}
(w, L_1^*\xi) &= -(G_1 f_1[\hat{u}], L_1^*\xi)\\
&= -(f_1[\hat{u}], G_1^* L_1^*\xi)\\
&= -(f_1[\hat{u}], \xi).
\end{aligned}
$$

Hence, from above, $(\hat{u}, L_1^*\xi) = -(f_1[\hat{u}], \xi) = (w, L_1^*\xi)$. Therefore $(\hat{u}-w, L_1^*\xi) = 0$ for all ξ in the domain of L_1^*. But the invertibility of L_1^* implies that the range of L_1^* is all of $L_2(V)$. Hence $(\hat{u}-w, \eta) = 0$ for all $\eta \in L_2(V)$. Thus $\hat{u} = w$ a.e., i.e. $\hat{u} = -G_1 f_1[\hat{u}]$ a.e. So $\hat{u}$ is a weak solution of the elliptic problem (7). By Theorems 2.2.1 and 2.2.2 of Sattinger[32], we conclude that $\hat{u}$ is equal a.e. to a classical solution of (7), as required.

Applying Theorems 8 and 9 to problems (13) and (14), we obtain the following theorem.

THEOREM 10: *Suppose that*

(i) *the coefficient* $\hat{d}_0(x)$ *in* $\hat{B}_{lin}$ *is strictly positive for all* $x \in \partial V$, *and* $\lambda\hat{f}(x,0) \geq 0$ *for all* $x \in V$;

(ii) $\hat{f}$ *satisfies a uniform Lipschitz condition in* u *on any finite* u-*interval, and has partial derivative* $\hat{f}_u$ *continuous for all* $x \in \overline{V}$ *and all real* u;

(iii) *there exists a solution* $u(x,t)$ *of* (13) *such that, for some constant* K, $u(x,t) \leq K$ *for all* $(x,t) \in \overline{D}$.

Then there exists a non-negative solution of (14) *which is equal a.e. to the limit as* $t \to \infty$ *of the solution of* (13).

If in addition

(iv) *the coefficients* $\hat{c}$ *in* $\hat{L}$ *and* $\hat{d}_1$ *in* $\hat{B}_{lin}$ *satisfy* $\hat{c}(x) \leq 0$ *for all* $x \in V$ *and* $\hat{d}_1(x) > 0$ *for all* $x \in \partial V$;

(v) $\lambda\hat{f}(x,u) > 0$ *for all* $x \in V$ *and* $u \geq 0$

then the solution of (14) *is strictly positive on* $\overline{V}$.

Proof: Since 0 is obviously a lower solution for both (13) and (14), we can immediately apply Theorems 8 and 9 to deduce the existence of a solution $\mu(x)$ of (14) such that $\lim_{t\to\infty} u(x,t) = \mu(x)$ a.e. Since $u(x,t) \geq 0$ for all $(x,t) \in \overline{D}$ by Theorem 2, and μ is continuous, it follows that $\mu(x) \geq 0$ for all $x \in \overline{V}$, thus proving the first part.

The second part follows from a form of the maximum principle. Clearly μ is not identically zero, by hypothesis (v). Suppose $\mu(x_0) = 0$ for some $x_0 \in \overline{V}$. If $x_0 \in V$, we obtain a contradiction by using Theorem 6 of Ch.2 of Protter and Weinberger[28]; if $x_0 \in \partial V$, we obtain a contradiction by using Theorem 8 of the same reference. So $\mu(x) > 0$ for all $x \in \overline{V}$, completing the proof.

We have now shown that, under quite wide conditions, the existence of a bounded solution of (13) implies the existence of a positive solution of (14) which is the limit a.e. of the solution of (13) as $t \to \infty$. Conversely, we showed earlier that, again under quite wide conditions, the existence of a (minimal) positive solution of (14) implies the existence of a bounded solution of (13) which tends to the (minimal) positive solution of (14) as $t \to \infty$.

Suppose we adopt the alternative approach to the study of thermal explosions, whereby one equates the boundedness over all positive time of the solution of the time-dependent problem with the absence of a thermal explosion. This would lead us to define the critical parameter λ^* for the pair of problems (13), (14) to be the least upper bound of the set of positive values of λ for which the solution of (13) is bounded. The results of this chapter establish fairly general conditions under which the two approaches to the problem and the two definitions of the critical parameter are equivalent. Certainly this is so for the cases discussed in Ch.4 and also for most forms of the original heat-generation problem as discussed by Boddington, Gray and Harvey[4]. It seems reasonable to suggest that the two approaches are in fact equivalent under much wider conditions than those given in this chapter.

For the remainder of this thesis, we shall treat the second approach to the thermal explosion problem as the fundamental one, and concentrate on describing the behaviour of the solution of the time-dependent problem under various assumptions. The information obtained will be compared with that obtained by studying the steady-state problem.

7 CONSTRUCTION OF UPPER AND LOWER SOLUTIONS FOR THE TIME-DEPENDENT PROBLEM

In this chapter we shall examine the behaviour of the solution of the time-dependent problem under various assumptions as to the nature of the function f(x,t,u), in particular the rate of growth of f(x,t,u) as a function of u. We shall do this by constructing upper and lower solutions for various cases, and then applying a suitable comparison theorem. In all the theorems of this chapter, the existence of a solution of the time-dependent problem will be taken as a hypothesis. However, in many cases we will construct both an upper and a lower solution, whereupon the existence of a solution will follow from Theorem 3 if the conditions of that theorem are satisfied. In any event, it is quite sufficient for our purposes in this chapter to show that *if* a solution exists, it must behave in such-and-such a way.

We shall begin by working with a specific domain V_m (described below) for which calculations are relatively simple, and later deal with the problem of extending our theory to general domains. We shall then discuss the results obtained in this chapter and the relationship between them and the results already known for the steady-state problem. The chapter will conclude with an examination of some theorems concerning the effect of reactant consumption.

<u>Notation:</u>

We write $V_m = \{x: \sum_{i=1}^{n} x_i^{2m_i} < 1\}$ where the $m_i (i = 1,2,\dots n)$ are arbitrarily chosen positive integers. Lu and $B_{lin}u$ will be as defined in Ch.3. In the directional derivative $\frac{\partial u}{\partial n}$ which appears in $B_{lin}u$, we shall take the unit vector field n to be such that, for each $i = 1,2,\dots n$, $n_i(x) = \alpha_i(x)x_i^{2m_i-1}$ where α_i is of class $C^{1+\alpha}(\partial V_m)$.

Since the outward unit normal vector field ν to $\partial V_m = \{x: \sum_{i=1}^{n} x_i^{2m_i} = 1\}$ is given by

$$\nu_i(x) = \frac{2m_i x_i^{2m_i-1}}{\sqrt{\sum_{i=1}^{n} 4m_i^2 x_i^{4m_i-2}}} \quad \text{for all } x \in \partial V_m \text{ and } i = 1,2,\dots n$$

the vector field n will be outwardly directed and nowhere tangential to ∂V_m provided that, for each $x \in \partial V_m$,

$$n(x).\nu(x) = \frac{\sum_{i=1}^{n} 2m_i\alpha_i(x)x_i^{4m_i-2}}{\sqrt{\sum_{i=1}^{n} 4m_i^2 x_i^{4m_i-2}}}$$

is positive. We assume that this is the case. We then have that

$$\frac{\partial u}{\partial n} = \sum_{i=1}^{n} \alpha_i(x)x_i^{2m_i-1}\frac{\partial u}{\partial x_i} .$$

We suppose further that the quantity $\sum_{i=1}^{n} \alpha_i(x)x_i^{2m_i}$ is bounded below and above on ∂V_m by positive numbers $\theta(\alpha_1,\dots \alpha_n,m_1,\dots m_n)$ and $\Theta(\alpha_1,\dots \alpha_n,m_1,\dots m_n)$ respectively, and that the quantity $\sum_{i=1}^{n} x_i^2$ is bounded below and above on ∂V_m by positive numbers $\psi(m_1,\dots m_n)$ and $\Psi(m_1,\dots m_n)$ respectively.

We shall be concerned with the parabolic initial-boundary problem

$$\left.\begin{array}{l} Lu - \frac{\partial u}{\partial t} + \lambda f(x,t,u) = 0 \quad \text{for } x \in V_m,\ 0 < t \le T \\ B_{lin}u = 0 \quad \text{for } x \in \partial V_m,\ 0 < t \le T \\ u(x,0) = u_0(x) \quad \text{for } x \in \overline{V}_m \end{array}\right\} \quad \dots(15)$$

where f is continuous for $x \in \overline{V}_m$, $0 \le t \le T$ and all u, $u_0 \in C^{2+\alpha}(\overline{V}_m)$ and the parameter λ is assumed to be positive.

Construction of Lower Solutions on V_m:

THEOREM 11: *Suppose that*

(a) *For all* $t > 0$ *and* $x \in V_m$, $f(x,t,0) \ge 0$; *furthermore,* f *satisfies a uniform Lipschitz condition in* u *on any finite* u-*interval.*

(b) $u_0(x) \ge 0$ *for all* $x \in \overline{V}_m$.

(c) *The coefficient* $d_0(x,t)$ *of* u *in* $B_{lin}u$ *is strictly positive for all* $t > 0$ *and* $x \in \partial V_m$.

Then: (i) *For any* $T > 0$, *a lower solution for* (15) *is given by*

$$w(x,t) = 0 \text{ for all } x \in \overline{V}_m,\ 0 \le t \le T.$$

(ii) *For any* $T > 0$, *if* $u(x,t)$ *is a solution of* (15), $u(x,t) \ge 0$ *for all* $x \in \overline{V}_m$ *and* $0 \le t \le T$.

Proof: (I) $w(x,0) = 0 \le u_0(x)$ for all $x \in \overline{V}_m$.

(II) $B_{lin}w = 0$ for $t > 0$ and $x \in \partial V_m$.

(III) $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w) = \lambda f(x,t,0) \geq 0$ for $t > 0$ and $x \in V_m$.

This proves part (i) of the theorem. If $u(x,t)$ is a solution of (15), it follows by Theorem 2 that $u(x,t) \geq w(x,t) = 0$ for all $x \in \overline{V}_m$ and $0 \leq t \leq T$, thus proving part (ii).

THEOREM 12: *Suppose that*

(a) *There exist constants* $A_i > 0 (i = 1,\ldots n)$, $B_i \geq 0 (i = 1,\ldots n)$ *and* $C \geq 0$ *such that, for all* $x \in V_m$ *and* $t > 0$, $0 < a_{ii}(x,t) \leq A_i (i = 1,\ldots n)$, $|b_i(x,t)| \leq B_i (i = 1,\ldots n)$ *and* $c(x,t) \geq -C$.

(b) *There exists a constant* $M > 0$ *such that, for all* $t > 0$, $x \in V_m$ *and* $u \geq 0$, $f(x,t,u) \geq M$. *Furthermore,* f *satisfies a uniform Lipschitz condition in* u *on any finite* u-*interval.*

(c) *There exist constants* $D_0 > 0$ *and* $\delta_1 > 0$ *such that, for all* $x \in \partial V_m$ *and* $t > 0$, $0 < d_0(x,t) \leq D_0$ *and* $\delta_1 \leq d_1(x,t)$; *we require also that these constants be such that* $\Psi < \psi + \frac{2\delta_1\theta}{D_0}$.

(d) $u_0(x) \geq 0$ *for all* $x \in \overline{V}_m$.

Then: (i) *For any* $T > 0$, *a lower solution for* (15) *is given by*

$$w(x,t) = \lambda K(A - \sum_{i=1}^{n} x_i^2)(1 - e^{-t}) \text{ for all } x \in \overline{V}_m,\ 0 \leq t \leq T$$

where A *and* K *are constants chosen so as to satisfy*

$$\Psi < A < \psi + \frac{2\delta_1\theta}{D_0} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

$$0 < K < \frac{M}{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + (C+1)A} \qquad \ldots\ldots\ldots\ldots\ldots(17)$$

(ii) *If* $u(x,t)$ *is a solution of* (15), *then for any* $T > 0$, $u(x,t) > 0$ *for* $0 < t \leq T$, *and if* $\lim_{T\to\infty} u(x,T) = \hat{u}(x)$ *exists, then for all* $x \in \overline{V}_m$,

$$\hat{u}(x) \geq \frac{\lambda M(\psi + \frac{2\delta_1\theta}{D_0} - \Psi)}{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + (C+1)(\psi + \frac{2\delta_1\theta}{D_0})} > 0.$$

Proof: (I) $w(x,0) = 0 \leq u_0(x)$ for all $x \in \overline{V}_m$, by hypothesis (d).

(II) $B_{lin}w = d_0(x,t)w + d_1(x,t)\sum_{i=1}^{n} \alpha_i(x)x_i^{2m_i-1}\frac{\partial w}{\partial x_i}$

$$= d_0(x,t)\lambda K(A - \sum_{i=1}^{n} x_i^2)(1 - e^{-t}) + d_1(x,t)\sum_{i=1}^{n} \alpha_i(x)x_i^{2m_i-1}\{\lambda K(-2x_i)(1-e^{-t})\}$$

$$= \lambda K(1 - e^{-t})\{d_0(x,t)(A - \sum_{i=1}^{n} x_i^2) - 2d_1(x,t)\sum_{i=1}^{n} \alpha_i(x)x_i^{2m_i}\}$$

$$\leq \lambda K(1 - e^{-t})\{D_0(A-\psi) - 2\delta_1\theta\} \quad \text{for } t > 0 \text{ and } x \in \partial V_m$$

$$< 0 \quad \text{for all } t > 0 \text{ by (16).}$$

(III) $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$= \sum_{i=1}^{n} a_{ii}(x,t)\lambda K(-2)(1-e^{-t}) + \sum_{i=1}^{n} b_i(x,t)\lambda K(-2x_i)(1-e^{-t})$$

$$+ c(x,t)\lambda K(A- \sum_{i=1}^{n} x_i^2)(1-e^{-t}) - \lambda K(A- \sum_{i=1}^{n} x_i^2)e^{-t} + \lambda f(x,t,w)$$

$$\geq -2\lambda K \sum_{i=1}^{n} A_i - 2\lambda K \sum_{i=1}^{n} B_i - C\lambda KA - \lambda KA + \lambda M \quad \text{for } t > 0 \text{ and } x \in V_m$$

$$= \lambda\{-K(2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA + A) + M\}$$

$$> 0 \quad \text{for all } t > 0 \text{ by (17).}$$

Thus part (i) of the theorem is proved. By Theorem 2, it follows that if $u(x,t)$ is a solution of (15), then $u(x,t) \geq w(x,t)$ for $x \in \overline{V}_m$ and $0 \leq t \leq T$. It follows that $u(x,t) > 0$ for $x \in \overline{V}_m$, $0 < t \leq T$. Furthermore, $w(x,t) \to \lambda K(A - \sum_{i=1}^{n} x_i^2)$ as $t \to \infty$, so if $\lim_{T\to\infty} u(x,T) = \hat{u}(x)$ exists, then for all $x \in \overline{V}_m$, $\hat{u}(x) \geq \lambda K(A- \sum_{i=1}^{n} x_i^2) \geq \lambda K(A-\Psi)$. Since A may be chosen arbitrarily close to $\psi + \frac{2\delta_1\theta}{D_0}$ and K may be chosen arbitrarily close to $\dfrac{M}{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + (C+1)A}$, part (ii) of the theorem follows at once.

Note: The condition that f should satisfy a uniform Lipschitz condition in u on any finite u-interval may be removed if we alter hypothesis (d) to read "$u_0(x) > 0$ for all $x \in \overline{V}_m$." The only change in the proof is that Theorem 1 rather than Theorem 2 is used in proving (ii).

THEOREM 13: *Hypotheses as for Theorem 12 except that in hypothesis* (b),

we suppose that there exist constants $M_1 > 0$, $M_2 > 0$ *such that, for all* $t > 0$, $x \in V_m$ *and* $u \geq 0$, $f(x,t,u) \geq M_1 u + M_2$. *We still assume that* f *satisfies a uniform Lipschitz condition in* u *on any finite* u*-interval.*
Then: (i) *For any* $T > 0$, *a lower solution for* (15) *is given by*

$$w(x,t) = t(A - \sum_{i=1}^{n} x_i^2) \text{ for all } x \in \overline{V}_m,\ 0 \leq t \leq T$$

if A *is a constant chosen so as to satisfy*

$$\Psi < A < \psi + \frac{2\delta_1\theta}{D_0}$$

and if
$$\lambda > \max\left\{ \frac{2(\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i) + CA}{M_1(A - \Psi)},\ \frac{A}{M_2} \right\} \quad \ldots\ldots\ldots\ldots\ldots(18)$$

(ii) *If*
$$\lambda > \max\left\{ \frac{2(\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i) + C(\psi + \frac{2\delta_1\theta}{D_0})}{M_1(\psi + \frac{2\delta_1\theta}{D_0} - \Psi)},\ \frac{\psi + \frac{2\delta_1\theta}{D_0}}{M_2} \right\},$$

and if $u(x,t)$ *is a solution of* (15), *then* $u(x,T) \to \infty$ *as* $T \to \infty$, *uniformly for* $x \in \overline{V}_m$.

Proof: (I) and (II) are similar to the proof of Theorem 12.

(III) $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$= \sum_{i=1}^{n} a_{ii}(x,t)(-2t) + \sum_{i=1}^{n} b_i(x,t)(-2tx_i) + c(x,t)t(A - \sum_{i=1}^{n} x_i^2) - (A - \sum_{i=1}^{n} x_i^2) + \lambda f(x,t,w)$$

$$\geq -2t\sum_{i=1}^{n} A_i - 2t\sum_{i=1}^{n} B_i - CtA - A + \lambda[M_1 t(A-\Psi) + M_2] \quad \text{for } t > 0 \text{ and } x \in V_m$$

$$= t[\lambda M_1(A-\Psi) - 2\{\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i\} - CA] + \lambda M_2 - A$$

> 0 for all $t > 0$ by (18).

Thus part (i) of the theorem is proved. Now if λ satisfies the condition of part (ii), then we can choose A sufficiently close to $\psi + \frac{2\delta_1\theta}{D_0}$ so that (18) is satisfied and therefore part (i) will hold. It follows by Theorem 2 that if $u(x,t)$ is a solution of (15), then $u(x,t) \geq w(x,t)$ for all $x \in \overline{V}_m$ and $0 \leq t \leq T$. Now $w(x,t) \geq t(A-\Psi)$ for all $x \in \overline{V}_m$, so $w(x,t) \to \infty$ as $t \to \infty$, uniformly for $x \in \overline{V}_m$, which proves part (ii).

Note: As in the case of Theorem 12, the requirement that f should satisfy a uniform Lipschitz condition in u on any finite u-interval may be removed if we alter hypothesis (d) to read "$u_0(x) > 0$ for all $x \in \overline{V}_m$."

THEOREM 14: *Suppose that*

(a) *As for Theorem 12.*

(b) *There exists a constant* $M > 0$ *such that, for all* $t > 0$, $x \in V_m$ *and* $u \geq 0$, $f(x,t,u) \geq Mu$.

(c) *There exist constants* $D_0 \geq 0$ *and* $\delta_1 > 0$ *such that, for all* $x \in \partial V_m$ *and* $t > 0$, $0 \leq d_0(x,t) \leq D_0$ *and* $\delta_1 \leq d_1(x,t)$; *if* $D_0 > 0$, *then we require also that* $\Psi < \psi + \dfrac{2\delta_1\theta}{D_0}$.

(d) *There exists a constant* $\epsilon > 0$ *such that* $u_0(x) > \epsilon$ *for all* $x \in \overline{V}_m$.

Then: (i) *For any* $T > 0$, *a strict lower solution for* (15) *is given by*

$$w(x,t) = \frac{\epsilon e^t}{A}\left(A - \sum_{i=1}^{n} x_i^2\right) \text{ for all } x \in \overline{V}_m,\ 0 \leq t \leq T$$

if A *is a constant chosen so as to satisfy*

$$\Psi < A < \psi + \frac{2\delta_1\theta}{D_0}$$

(*if* $D_0 > 0$; *if* $D_0 = 0$ *then* A *is chosen to satisfy* $A > \Psi$)

and if
$$\lambda > \frac{2\left\{\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i\right\} + (C+1)A}{M(A - \Psi)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(19)$$

(ii) *If* $\lambda > \dfrac{2\left\{\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i\right\} + (C+1)\left(\psi + \frac{2\delta_1\theta}{D_0}\right)}{M\left(\psi + \frac{2\delta_1\theta}{D_0} - \Psi\right)}$ (*if* $D_0 > 0$; *in the case where* $D_0 = 0$ *we require* $\lambda > \frac{C+1}{M}$) *and if* $u(x,t)$ *is a solution of* (15), *then* $u(x,T) \to \infty$ *as* $T \to \infty$, *uniformly for* $x \in \overline{V}_m$.

Proof: (I) $w(x,0) = \dfrac{\epsilon}{A}\left(A - \sum_{i=1}^{n} x_i^2\right) \leq \epsilon$ for all $x \in \overline{V}_m$

$< u_0(x)$ for all $x \in \overline{V}_m$, by hypothesis (d)

(II) is similar to the proof of Theorem 12.

(III) $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$= \sum_{i=1}^{n} a_{ii}(x,t)\{\frac{-2\varepsilon e^t}{A}\} + \sum_{i=1}^{n} b_i(x,t)\{\frac{-2x_i\varepsilon e^t}{A}\} + c(x,t)\frac{\varepsilon}{A} e^t(A - \sum_{i=1}^{n} x_i^2)$$

$$- \frac{\varepsilon e^t}{A}(A - \sum_{i=1}^{n} x_i^2) + \lambda f(x,t,w)$$

$$\geq \{\frac{-2\varepsilon e^t}{A}\}\sum_{i=1}^{n} A_i + \{\frac{-2\varepsilon e^t}{A}\}\sum_{i=1}^{n} B_i - \varepsilon e^t C - \varepsilon e^t + \frac{\lambda M \varepsilon e^t}{A}(A-\Psi)$$

for $t > 0$ and $x \in V_m$

$$= \frac{\varepsilon e^t}{A}\{-2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - (C+1)A + \lambda M(A-\Psi)\}$$

> 0 for all $t > 0$ by (19).

Thus part (i) of the theorem is proved. Now if λ satisfies the condition of part (ii), then we can choose A sufficiently close to $\psi + \frac{2\delta_1\theta}{D_0}$ (if $D_0 > 0$) or sufficiently large (if $D_0 = 0$) so that (19) is satisfied and therefore part (i) will hold. It follows by Theorem 1 that if $u(x,t)$ is a solution of (15), then $u(x,t) > w(x,t)$ for all $x \in \overline{V}_m$ and $0 \leq t \leq T$. Now $w(x,t) \geq \frac{\varepsilon e^t}{A}(A-\Psi)$ for all $x \in \overline{V}_m$, so $w(x,t) \to \infty$ as $t \to \infty$, uniformly for $x \in \overline{V}_m$, which proves part (ii).

THEOREM 15: *Suppose that*

(a) *As for Theorem 12.*

(b) *There exist constants* $M > 0$ *and* $\alpha > 0$ *such that, for all* $t > 0$, $x \in V_m$ *and* $u \geq 0$, $f(x,t,u) \geq Mu^{1+\frac{1}{\alpha}}$.

(c) *As for Theorem 14.*

(d) *As for Theorem 14.*

Then: (i) *For any* T *such that* $0 < T < t^*$, *a strict lower solution for* (15) *is given by*

$$w(x,t) = \frac{(t^*)^{\alpha}\varepsilon}{A(t^*-t)^{\alpha}}(A - \sum_{i=1}^{n} x_i^2) \text{ for all } x \in \overline{V}_m,\ 0 \leq t \leq T$$

if A *is a constant chosen so as to satisfy*

$$\Psi < A < \psi + \frac{2\delta_1\theta}{D_0}$$

(*if* $D_0 > 0$; *if* $D_0 = 0$ *then* A *is chosen to satisfy* $A > \Psi$)

and if $\lambda > \dfrac{A^{1/\alpha}}{M\epsilon^{1/\alpha}(A-\Psi)^{1+\frac{1}{\alpha}}}\{\frac{\alpha A}{t^*} + 2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA\}$ ....(20)

(ii) *If* $\lambda > \dfrac{(\psi + \frac{2\delta_1\theta}{D_0})^{1/\alpha}}{M\epsilon^{1/\alpha}(\psi + \frac{2\delta_1\theta}{D_0} - \Psi)^{1+\frac{1}{\alpha}}}\{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + C(\psi + \frac{2\delta_1\theta}{D_0})\}$

(*if* $D_0 > 0$; *in the case where* $D_0 = 0$ *we require* $\lambda > \frac{C}{M\epsilon^{1/\alpha}}$) *and if* $u(x,t)$ *is a solution of* (15), *then there exists a finite number* $\tau > 0$ *such that* $u(x,T) \to \infty$ *as* $T \to \tau-$, *uniformly for* $x \in \overline{V}_m$. *In particular, given any positive* ϵ, *this will be the case for all sufficiently large* λ; *and given any positive* λ, *this will be the case for all sufficiently large* ϵ.

Proof: (I) is similar to the proof of Theorem 14.

(II) is similar to the proof of Theorem 12.

(III) $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$= \sum_{i=1}^{n} a_{ii}(x,t)\frac{-2(t^*)^{\alpha}\epsilon}{A(t^*-t)^{\alpha}} + \sum_{i=1}^{n} b_i(x,t)\frac{-2x_i(t^*)^{\alpha}\epsilon}{A(t^*-t)^{\alpha}}$$

$$+ c(x,t)\frac{(t^*)^{\alpha}\epsilon}{A(t^*-t)^{\alpha}}(A - \sum_{i=1}^{n} x_i^2) - \frac{\alpha(t^*)^{\alpha}\epsilon}{A(t^*-t)^{\alpha+1}}(A - \sum_{i=1}^{n} x_i^2) + \lambda f(x,t,w)$$

$$\geq \{\frac{-2(t^*)^{\alpha}\epsilon}{A(t^*-t)^{\alpha}}\}\sum_{i=1}^{n} A_i + \{\frac{-2(t^*)^{\alpha}\epsilon}{A(t^*-t)^{\alpha}}\}\sum_{i=1}^{n} B_i - \frac{(t^*)^{\alpha}\epsilon C}{(t^*-t)^{\alpha}} - \frac{\alpha(t^*)^{\alpha}\epsilon}{(t^*-t)^{\alpha+1}}$$

$$+ \frac{\lambda M(t^*)^{\alpha+1}\epsilon^{1+\frac{1}{\alpha}}(A-\Psi)^{1+\frac{1}{\alpha}}}{A^{1+\frac{1}{\alpha}}(t^*-t)^{\alpha+1}} \quad \text{for } x \in V_m \text{ and } 0 < t < t^*$$

$$= \frac{(t^*)^{\alpha}\epsilon}{A(t^*-t)^{\alpha+1}}[-(2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA)(t^*-t) - \alpha A + \frac{\lambda M t^* \epsilon^{1/\alpha}(A-\Psi)^{1+\frac{1}{\alpha}}}{A^{1/\alpha}}]$$

$$\geq \frac{(t^*)^{\alpha}\epsilon}{A(t^*-t)^{\alpha+1}}[-(2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA)t^* - \alpha A + \frac{\lambda M t^* \epsilon^{1/\alpha}(A-\Psi)^{1+\frac{1}{\alpha}}}{A^{1/\alpha}}]$$

for $0 < t < t^*$

> 0 by (20).

Thus part (i) of the theorem is proved. Now if λ satisfies the condition of part (ii), then we can choose A sufficiently close to $\psi + \frac{2\delta_1\theta}{D_0}$ (if $D_0 > 0$) or sufficiently large (if $D_0 = 0$) so that

$$\lambda > \frac{A^{1/\alpha}}{M\epsilon^{1/\alpha}(A-\Psi)^{1+\frac{1}{\alpha}}} \{2 \sum_{i=1}^{n} A_i + 2 \sum_{i=1}^{n} B_i + CA\}$$

and we may then choose t^* large enough so that (20) is satisfied and therefore part (i) will hold. It follows by Theorem 1 that if $u(x,t)$ is a solution of (15), then $u(x,t) > w(x,t)$ for all $x \in \overline{V}_m$ and $0 \leq t \leq T$. But $w(x,t) \geq \frac{(t^*)^\alpha \epsilon}{A(t^*-t)^\alpha}(A-\Psi)$ for all $x \in \overline{V}_m$, so $w(x,t) \to \infty$ as $t \to t^*-$, uniformly for $x \in \overline{V}_m$. It follows that there must exist a τ with $0 < \tau \leq t^*$, such that $u(x,T) \to \infty$ as $T \to \tau-$; that this limit is uniform for $x \in \overline{V}_m$ can be seen at once by redefining $w(x,t)$ with $t^* = \tau$. Thus part (ii) is proved.

THEOREM 16: *Suppose that*

(a) *As for Theorem 12.*

(b) *There exist constants* $M_1 > 0$, $M_2 > 0$, $\alpha > 0$ *such that, for all* $t > 0$, $x \in V_m$ *and* $u \geq 0$, $f(x,t,u) \geq M_1 u^{1+\frac{1}{\alpha}} + M_2$. *Furthermore,* f *satisfies a uniform Lipschitz condition in* u *on any bounded* u*-interval.*

(c) *As for Theorem 12.*

(d) *As for Theorem 12.*

Then: (i) *For any* T *such that* $0 < T < t^*$, *a lower solution for* (15) *is given by*

$$w(x,t) = \frac{t}{(t^*-t)^\alpha}(A - \sum_{i=1}^{n} x_i^2) \text{ for all } x \in \overline{V}_m,\ 0 \leq t \leq T$$

if A *is a constant chosen so as to satisfy*

$$\Psi < A < \psi + \frac{2\delta_1\theta}{D_0}$$

and if λ *is chosen sufficiently large, depending on the constants* A_i, B_i $(i = 1,\dots n)$, C, M_1, M_2, α, A, Ψ *and* t^*.

(ii) *If* A *and* λ *are chosen as in* (i), *and if* $u(x,t)$ *is a solution of* (15), *then there exists a number* τ *with* $0 < \tau < t^*$ *such that* $u(x,T) \to \infty$ *as* $T \to \tau-$, *uniformly for* $x \in \overline{V}_m$.

Proof: (I) and (II) are similar to the proof of Theorem 12.

(III) $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$= \sum_{i=1}^{n} a_{ii}(x,t)\,\frac{-2t}{(t^*-t)^{\alpha}} + \sum_{i=1}^{n} b_i(x,t)\,\frac{-2x_i t}{(t^*-t)^{\alpha}}$$

$$+ \quad c(x,t)\,\frac{t}{(t^*-t)^{\alpha}}(A - \sum_{i=1}^{n} x_i^2) - \frac{t^* + (\alpha-1)t}{(t^*-t)^{\alpha+1}}(A - \sum_{i=1}^{n} x_i^2) + \lambda f(x,t,w)$$

$$\geq \{\frac{-2t}{(t^*-t)^{\alpha}}\}\sum_{i=1}^{n} A_i + \{\frac{-2t}{(t^*-t)^{\alpha}}\}\sum_{i=1}^{n} B_i - \frac{tCA}{(t^*-t)^{\alpha}} - \frac{A[t^* + (\alpha-1)t]}{(t^*-t)^{\alpha+1}}$$

$$+ \frac{\lambda M_1 t^{1+\frac{1}{\alpha}}(A-\Psi)^{1+\frac{1}{\alpha}}}{(t^*-t)^{\alpha+1}} + \lambda M_2 \quad \text{for } x \in V_m \text{ and } 0 < t < t^*$$

$$= \frac{1}{(t^*-t)^{\alpha+1}}\left[\begin{array}{l} -(2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA)t(t^*-t) - A[t^* + (\alpha-1)t] \\ \qquad + \lambda\{M_1 t^{1+\frac{1}{\alpha}}(A-\Psi)^{1+\frac{1}{\alpha}} + M_2(t^*-t)^{\alpha+1}\} \end{array}\right].$$

Now put $$K = 2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA$$

and $$g(t) = -Kt(t^*-t) - A[t^* + (\alpha-1)t]$$
$$= Kt^2 + t\{-Kt^* - A(\alpha-1)\} - At^*.$$

Then the minimum value over all real t of the quadratic g(t) is

$$\frac{-4KAt^* - \{Kt^* + A(\alpha-1)\}^2}{4K} = -At^* - \frac{\{Kt^* + A(\alpha-1)\}^2}{4K}$$

attained when $t = \frac{Kt^* + A(\alpha-1)}{2K}$. Thus, for $0 < t < t^*$ and $x \in V_m$, we have:

$$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$$

$$\geq \frac{1}{(t^*-t)^{\alpha+1}}[-At^* - \frac{\{Kt^* + A(\alpha-1)\}^2}{4K} + \lambda\{M_1 t^{1+\frac{1}{\alpha}}(A-\Psi)^{1+\frac{1}{\alpha}} + M_2(t^*-t)^{\alpha+1}\}].$$

The expression $M_1 t^{1+\frac{1}{\alpha}}(A-\Psi)^{1+\frac{1}{\alpha}} + M_2(t^*-t)^{\alpha+1}$ is obviously continuous and strictly positive for $0 \leq t \leq t^*$, so there exists $\delta > 0$ such that

$$M_1 t^{1+\frac{1}{\alpha}}(A-\Psi)^{1+\frac{1}{\alpha}} + M_2(t^*-t)^{\alpha+1} \geq \delta \quad \text{for } 0 \leq t \leq t^*.$$

If we then choose $\lambda \geq \frac{1}{\delta}[At^* + \frac{\{Kt^* + A(\alpha-1)\}^2}{4K}]$, then

$$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w) \geq 0 \quad \text{for } 0 < t < t^* \text{ and } x \in V_m.$$

Thus part (i) of the theorem is proved. It follows by Theorem 2 that if $u(x,t)$ is a solution of (15), then $u(x,t) \geq w(x,t)$ for all $x \in \overline{V}_m$ and $0 \leq t \leq T$. The rest of the argument parallels the proof of Theorem 15.

Note: In the case of Theorem 16 where $\alpha = 1$, δ is the minimum value on $[0,t^*]$ of the expression

$$M_1 t^2 (A-\Psi)^2 + M_2(t^*-t)^2$$

$$= t^2[M_1(A-\Psi)^2 + M_2] - 2M_2 t^* t + M_2(t^*)^2.$$

This quadratic attains its overall minimum value at

$$t = \frac{2M_2 t^*}{2[M_1(A-\Psi)^2 + M_2]}$$

which is a value between 0 and t^*. Hence δ is the overall minimum of the quadratic.

$$\text{Thus} \quad \delta = \frac{4M_2(t^*)^2[M_1(A-\Psi)^2 + M_2] - 4M_2^2(t^*)^2}{4[M_1(A-\Psi)^2 + M_2]}$$

$$= \frac{M_1 M_2 (t^*)^2 (A-\Psi)^2}{M_1(A-\Psi)^2 + M_2}.$$

Using this we can prove the following:

COROLLARY: *Suppose that*

(a) *As for Theorem 12.*

(b) *There exist constants* $M_1 > 0$, $M_2 > 0$ *such that, for all* $t > 0$, $x \in V_m$ *and* $u \geq 0$, $f(x,t,u) \geq M_1 u^2 + M_2$. *Furthermore,* f *satisfies a uniform Lipschitz condition in* u *on any bounded* u*-interval.*

(c) *As for Theorem 12.*

(d) *As for Theorem 12.*

Then: (i) *For any* T *such that* $0 < T < t^*$, *a lower solution for* (15) *is given by*

$$w(x,t) = \frac{t}{t^*-t}\left(A - \sum_{i=1}^{n} x_i^2\right) \text{ for all } x \in \overline{V}_m,\ 0 \leq t \leq T$$

if A *is a constant chosen so as to satisfy*

$$\Psi < A < \psi + \frac{2\delta_1 \theta}{D_0}$$

$$\text{and if } \lambda \geq \frac{[M_1(A-\Psi)^2 + M_2][\frac{A}{t^*} + \frac{1}{4}(2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA)]}{M_1M_2(A-\Psi)^2} \quad \ldots\ldots(21)$$

(ii) *If*

$$\lambda > \frac{[2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + C(\psi + \frac{2\delta_1\theta}{D_0})][M_1(\psi + \frac{2\delta_1\theta}{D_0} - \Psi)^2 + M_2]}{4M_1M_2(\psi + \frac{2\delta_1\theta}{D_0} - \Psi)^2}$$

and if $u(x,t)$ *is a solution of* (15), *then there exists a finite number* $\tau > 0$ *such that* $u(x,T) \to \infty$ *as* $T \to \tau-$, *uniformly for* $x \in \overline{V}_m$.

Proof: (i) follows from Theorem 16 with $\alpha = 1$, using the value of δ obtained in the note above.

If λ satisfies the condition of part (ii), then we can choose A sufficiently close to $\psi + \frac{2\delta_1\theta}{D_0}$ so that

$$\lambda > \frac{[2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA][M_1(A-\Psi)^2 + M_2]}{4M_1M_2(A-\Psi)^2}$$

and we may then choose t^* large enough so that (21) is satisfied and therefore part (i) will hold. The rest of the argument parallels the proof of Theorem 15.

<u>Construction of Upper Solutions on V_m:</u>

THEOREM 17: *Suppose that*

(a) *There exist constants* $A_i > 0 (i = 1,\ldots n)$, $B_i \geq 0 (i = 1,\ldots n)$ *and* C *such that, for all* $x \in V_m$ *and* $t > 0$, $0 < A_i \leq a_{ii}(x,t)$ $(i = 1,\ldots n)$, $|b_i(x,t)| \leq B_i$ $(i = 1,\ldots n)$ *and* $c(x,t) \leq C$.

(b) *For any bounded positive* u-*interval* I, *there exists a corresponding positive number* M, *depending only on* I, *such that* $f(x,t,u) \leq M$ *for all* $x \in V_m$, $t > 0$ *and* $u \in I$.

(c) *There exist constants* $\delta_0 > 0$ *and* $D_1 \geq 0$ *such that, for all* $x \in \partial V_m$ *and* $t > 0$, $d_0(x,t) \geq \delta_0$ *and* $0 \leq d_1(x,t) \leq D_1$.

(d) *We require in addition that, if* $C \geq 0$, *then*

$$2\sum_{i=1}^{n} B_i + C(\Psi + \frac{2D_1\Theta}{\delta_0}) < 2\sum_{i=1}^{n} A_i.$$

(e) *There exists a constant* $\epsilon > 0$ *such that* $u_0(x) < \epsilon$ *for all* $x \in \overline{V}_m$. *Then*: (i) *For any* $T > 0$, *a strict upper solution for* (15) *is given by*

$$w(x,t) = \frac{\epsilon}{A-\Psi}\left(A - \sum_{i=1}^{n} x_i^2\right) \text{ for all } x \in \overline{V}_m,\ 0 \le t \le T$$

(so that $w(x,t)$ *is actually independent of* t*) if* A *is a constant chosen so as to satisfy*

$$\left.\begin{array}{ll} \Psi + \dfrac{2D_1\Theta}{\delta_0} < A < \dfrac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i}{C} & \text{if } C > 0 \\[2ex] A > \Psi + \dfrac{2D_1\Theta}{\delta_0} & \text{if } C = 0 \\[2ex] A > \max\left\{\Psi + \dfrac{2D_1\Theta}{\delta_0},\ \Psi + \dfrac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i}{C}\right\} & \text{if } C < 0 \end{array}\right\} \quad \ldots\ldots(22)$$

and if $\qquad 0 < \lambda < m_0(\epsilon, A) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(23)$

where $m_0(\epsilon,A)$, *apart from depending on* ϵ *and* A, *depends on the coefficients in the operators* L *and* B_{lin}, *the quantities* Ψ *and* Θ, *and the nature of the function* f, *but not on* T.

(ii) *If* $0 < \lambda < m_0(\epsilon,A)$, *and if* $u(x,t)$ *is a solution of* (15), *then for all* $T > 0$ *and* $x \in \overline{V}_m$, $u(x,T) \le \frac{\epsilon A}{A-\Psi}$.

Proof: (I) $w(x,0) = \frac{\epsilon}{A-\Psi}\left(A - \sum_{i=1}^{n} x_i^2\right) \ge \epsilon > u_0(x)$ for all $x \in \overline{V}_m$, by hypothesis (e).

(II)
$$B_{lin}w = d_0(x,t)w + d_1(x,t)\sum_{i=1}^{n} \alpha_i(x) x_i^{2m_i-1}\frac{\partial w}{\partial x_i}$$

$$= d_0(x,t)\frac{\epsilon}{A-\Psi}\left(A - \sum_{i=1}^{n} x_i^2\right) + d_1(x,t)\sum_{i=1}^{n} \alpha_i(x) x_i^{2m_i-1}\left(\frac{-2\epsilon x_i}{A-\Psi}\right)$$

$$= \frac{\epsilon}{A-\Psi}\left\{d_0(x,t)\left(A - \sum_{i=1}^{n} x_i^2\right) - 2d_1(x,t)\sum_{i=1}^{n} \alpha_i(x) x_i^{2m_i}\right\}$$

$$\ge \frac{\epsilon}{A-\Psi}\{\delta_0(A-\Psi) - 2D_1\Theta\} \text{ for } t > 0 \text{ and } x \in \partial V_m$$

$$> 0 \quad \text{for all } t > 0, \text{ since } A > \Psi + \frac{2D_1\Theta}{\delta_0} \text{ by (22).}$$

(III) $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$= \sum_{i=1}^{n} a_{ii}(x,t)(-2)\left(\frac{\epsilon}{A-\Psi}\right) + \sum_{i=1}^{n} b_i(x,t)(-2x_i)\left(\frac{\epsilon}{A-\Psi}\right) + c(x,t)w + \lambda f(x,t,w).$$

Now for $t > 0$ and $x \in V_m$, $\varepsilon \le w(x,t) \le \frac{\varepsilon A}{A-\Psi}$, so by hypothesis (b) we have that for all $t > 0$ and $x \in V_m$, $f(x,t,w) \le M(\varepsilon,A,\Psi)$. If $C \ge 0$, then by (22) and hypothesis (d) we have $2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - CA > 0$, while if $C < 0$, then by (22) we have $2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(A-\Psi) > 0$.

Thus we define the positive number $m_0(\varepsilon,A)$ as follows:

$$\left.\begin{aligned}
&\text{If } C \ge 0,\ m_0(\varepsilon,A) = \frac{\varepsilon\{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - CA\}}{(A-\Psi)M(\varepsilon,A,\Psi)} \\
&\text{If } C < 0,\ m_0(\varepsilon,A) = \frac{\varepsilon\{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(A-\Psi)\}}{(A-\Psi)M(\varepsilon,A,\Psi)}
\end{aligned}\right\} \quad \ldots\ldots(24)$$

Now if $C \ge 0$, then we have that for all $t > 0$ and $x \in V_m$:

$$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w) \le \left(\frac{-2\varepsilon}{A-\Psi}\right)\sum_{i=1}^{n} A_i + \left(\frac{2\varepsilon}{A-\Psi}\right)\sum_{i=1}^{n} B_i + \frac{C\varepsilon A}{A-\Psi} + \lambda M(\varepsilon,A,\Psi).$$

If $C < 0$, then we have that for all $t > 0$ and $x \in V_m$:

$$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w) \le \left(\frac{-2\varepsilon}{A-\Psi}\right)\sum_{i=1}^{n} A_i + \left(\frac{2\varepsilon}{A-\Psi}\right)\sum_{i=1}^{n} B_i + \left(\frac{\varepsilon}{A-\Psi}\right)C(A-\Psi) + \lambda M(\varepsilon,A,\Psi).$$

In either case, it follows by (23) and (24) that $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w) < 0$ for all $t > 0$ and $x \in V_m$. Thus part (i) of the theorem is proved.

Now if λ satisfies the condition of part (ii), then part (i) holds, and it follows by Theorem 1 that if $u(x,t)$ is a solution of (15), then $u(x,t) < w(x,t)$ for all $x \in \overline{V}_m$ and $0 \le t \le T$. It follows that for all $T > 0$ and $x \in \overline{V}_m$, $u(x,T) < w(x,T) \le \frac{\varepsilon A}{A-\Psi}$. Thus part (ii) is proved.

Note: That some such condition as hypothesis (d), which places a definite upper bound on $c(x,t)$, is required in Theorem 17, is shown by the example given in Ch.2.

In that example, we have $c(x,t) = k$ for all x and t, and $f(x,t,u) = 1$ for all x, t and u. It is easily checked that hypotheses (a), (b), (c) and (e) of Theorem 17 are satisfied, with $C = k$. As explained in Ch.2, if $C = k \ge \frac{\pi^2}{4}$, the solution of the time-dependent problem is unbounded as $t \to \infty$, for any $\lambda > 0$. Thus Theorem 17 does not hold in this case unless we require $C < \frac{\pi^2}{4}$, i.e. unless we put an upper bound on $c(x,t)$ as in hypothesis (d) of Theorem 17.

We may also observe in passing that, for the example of Ch.2, we have $n = 1$, $B_1 = 0$, $A_1 = 1$, $\Psi = 1$ and $D_1 = 0$, so that hypothesis (d) of Theorem 17 reduces to $C < 2$, which is not much stronger a condition than the weakest possible condition $C < \frac{\pi^2}{4} \approx 2.47$.

THEOREM 18: *Suppose that*

(a) *As for Theorem 17.*

(b) *There exist constants* $M_1 > 0$, $M_2 > 0$ *such that, for all* $t > 0$, $x \in V_m$ *and* $u \geq 0$, $f(x,t,u) \leq M_1 u + M_2$.

(c) *As for Theorem 17.*

(d) *As for Theorem 17.*

(e) $u_0(x)$ *is bounded above for* $x \in \overline{V}_m$.

Then: If $u(x,t)$ *is a solution of* (15), *and if* λ *satisfies*

$$\left.\begin{aligned} & 0 < \lambda < \frac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(\Psi + \frac{2D_1\Theta}{\delta_0})}{M_1(\Psi + \frac{2D_1\Theta}{\delta_0})} \quad (\textit{if } C \geq 0) \\ & \textit{or} \quad 0 < \lambda < \frac{-C}{M_1} \quad (\textit{if } C < 0) \end{aligned}\right] \tag{25}$$

then there exists a constant $K > 0$ *such that, for all* $T > 0$ *and* $x \in \overline{V}_m$, $u(x,T) \leq K$, *where* K *depends on* λ.

Proof: We may apply Theorem 17 in this case. In the notation of Theorem 17, if $\epsilon < w < \frac{\epsilon A}{A-\Psi}$, then $f(x,t,w) \leq M_1(\frac{\epsilon A}{A-\Psi}) + M_2$, so in the proof of Theorem 17 we may take $M(\epsilon,A,\Psi) = M_1(\frac{\epsilon A}{A-\Psi}) + M_2$. Thus, if $C \geq 0$, then

$$m_0(\epsilon,A) = \frac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - CA}{M_1 A + M_2(\frac{A-\Psi}{\epsilon})},$$ while if $C < 0$, then we have

$$m_0(\epsilon,A) = \frac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(A-\Psi)}{M_1 A + M_2(\frac{A-\Psi}{\epsilon})}.$$

Now if $C \geq 0$ and λ satisfies (25), then by choosing A sufficiently close to $\Psi + \frac{2D_1\Theta}{\delta_0}$ and ϵ sufficiently large (as we may do, since ϵ may be chosen as large as we please) we can ensure that

$$0 < \lambda < m_0(\epsilon,A) < \frac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(\Psi + \frac{2D_1\Theta}{\delta_0})}{M_1(\Psi + \frac{2D_1\Theta}{\delta_0})}$$

and so part (ii) of Theorem 17 will hold, which proves Theorem 18 if we take $K = \frac{\epsilon A}{A-\Psi}$.

If $C < 0$ and λ satisfies (25), then by choosing A sufficiently large and ϵ equal to, say, A^2, we can ensure that $m_0(\epsilon,A)$ is as close as we please to $\frac{-C}{M_1}$, and so we can certainly ensure that

$$0 < \lambda < m_0(\epsilon,A).$$

So again part (ii) of Theorem 17 will hold, and Theorem 18 is proved with $K = \frac{\epsilon A}{A-\Psi}$.

COROLLARY: *Suppose we modify the hypotheses of Theorem 18 by requiring that* $f(x,t,u) \leq M_2$ *for all* $t > 0$, $x \in V_m$ *and* $u \geq 0$. *Then if* $u(x,t)$ *is a solution of* (15), *we have that for any* $\lambda > 0$ *there exists a constant* $K > 0$, *depending on* λ, *such that* $u(x,T) \leq K$ *for all* $T > 0$ *and* $x \in \overline{V}_m$.

Proof: For any $\lambda > 0$, we may choose M_1 sufficiently small so that (25) holds, and the required result follows at once by Theorem 18.

Notes: (i) For the example discussed in Ch.2, with $k = \lambda$, we have $n = 1$, $B_1 = 0$, $A_1 = 1$, $\Psi = 1$, $D_1 = 0$, $C = 0$; also $f(x,t,u) = u+1$ for all x, t and u, so that $M_1 = M_2 = 1$. Thus in this case Theorem 18 tells us that the solution to the time-dependent problem will be bounded above as $t \to \infty$ if $0 < \lambda < 2$; in fact, as explained in Ch.2, the solution will be bounded above as $t \to \infty$ if $0 < \lambda < \frac{\pi^2}{4} \approx 2.47$.

(ii) Theorem 18 and its Corollary relate well to results (iv) and (v) of Keller and Cohen (see pp.15,16). Keller and Cohen showed (result (v)) that if $\hat{f}(x,u) < F(x) + \rho(x)u$ for $x \in V$, $u > 0$, then steady-state solutions exist if $0 < \lambda < \mu_1\{\rho\}$; Theorem 18 shows that if $f(x,t,u) \leq M_1u + M_2$ for $x \in \overline{V}_m$, $t > 0$, $u \geq 0$ then time-dependent solutions are bounded as $t \to \infty$ if $0 < \lambda < \mu$(say) where μ depends on M_1 but not on M_2. Keller and Cohen also show (result (iv)) that if $\hat{f}(x,u) < F(x)$ for $x \in V$, $u > 0$, then steady-state solutions exist for all $\lambda > 0$; the Corollary to Theorem 18 shows that if $f(x,t,u) \leq M_2$ for all $x \in \overline{V}_m$, $t > 0$, $u \geq 0$ then time-dependent solutions are bounded as

$t \to \infty$ for all $\lambda > 0$.

THEOREM 19: *Suppose that*

(a) *As for Theorem 17.*

(b) *There exists a constant* $M > 0$ *such that, for all* $x \in V_m$, $t > 0$ *and* $u \geq 0$, $f(x,t,u) \leq Mu$.

(c), (d), (e) *as for Theorem 17.*

(f) *For all* $t > 0$ *and* $x \in V_m$, $f(x,t,0) \geq 0$; *furthermore,* f *satisfies a uniform Lipschitz condition in* u *on any finite* u*-interval.*

(g) $u_0(x) \geq 0$ *for all* $x \in V_m$.

Then: (i) *For any* $T > 0$, *a strict upper solution for* (15) *is given by*

$$w(x,t) = \frac{\varepsilon}{A-\Psi} e^{-\lambda t}\left(A - \sum_{i=1}^{n} x_i^2\right) \text{ for all } x \in \overline{V}_m,\ 0 \leq t \leq T$$

if A *is a constant chosen so as to satisfy*

$$\left.\begin{array}{ll} \Psi + \dfrac{2D_1\Theta}{\delta_0} < A < \dfrac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i}{C} & \text{if } C > 0 \\[2ex] A > \Psi + \dfrac{2D_1\Theta}{\delta_0} & \text{if } C = 0 \\[2ex] A > \max\left\{\Psi + \dfrac{2D_1\Theta}{\delta_0},\ \Psi + \dfrac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i}{C}\right\} & \text{if } C < 0 \end{array}\right\}$$

$$\text{and if}\quad \left.\begin{array}{ll} 0 < \lambda < \dfrac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - CA}{M(A+1)} & \text{if } C \geq 0 \\[2ex] 0 < \lambda < \dfrac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(A-\Psi)}{M(A+1)} & \text{if } C < 0 \end{array}\right\} \quad \ldots.(26)$$

(ii) *If* $u(x,t)$ *is a solution of* (15), *and if* λ *satisfies*

$$\left.\begin{array}{l} 0 < \lambda < \dfrac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C\left(\Psi + \dfrac{2D_1\Theta}{\delta_0}\right)}{M_1\left(\Psi + \dfrac{2D_1\Theta}{\delta_0} + 1\right)} \quad (\text{if } C \geq 0) \\[3ex] \text{or}\quad 0 < \lambda < \dfrac{-C}{M} \quad (\text{if } C < 0) \end{array}\right\} \quad (27)$$

then $u(x,T) \to 0$ *as* $T \to \infty$, *uniformly for* $x \in \overline{V}_m$.

Proof: (I) and (II) are similar to the proof of Theorem 17.

(III) $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$= \sum_{i=1}^{n} a_{ii}(x,t) \frac{-2\epsilon e^{-\lambda t}}{A-\Psi} + \sum_{i=1}^{n} b_i(x,t) \frac{-2x_i \epsilon e^{-\lambda t}}{A-\Psi} + c(x,t) \frac{\epsilon e^{-\lambda t}}{A-\Psi}(A-\sum_{i=1}^{n} x_i^2)$$

$$+ \frac{\lambda\epsilon e^{-\lambda t}}{A-\Psi}(A-\sum_{i=1}^{n} x_i^2) + \lambda f(x,t,w)$$

$$\le \left(\frac{-2\epsilon e^{-\lambda t}}{A-\Psi}\right)\sum_{i=1}^{n} A_i + \left(\frac{2\epsilon e^{-\lambda t}}{A-\Psi}\right)\sum_{i=1}^{n} B_i + c(x,t)\frac{\epsilon e^{-\lambda t}}{A-\Psi}(A-\sum_{i=1}^{n} x_i^2)$$

$$+ \frac{\lambda\epsilon e^{-\lambda t}A}{A-\Psi} + \lambda M \frac{\epsilon e^{-\lambda t}}{A-\Psi}(A-\sum_{i=1}^{n} x_i^2)$$

$$\le \frac{\epsilon e^{-\lambda t}}{A-\Psi}\{-2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + c(x,t)(A-\sum_{i=1}^{n} x_i^2) + \lambda A + \lambda MA\}$$

for all $t > 0$, $x \in V_m$.

Now if $C \ge 0$, then we have that for all $t > 0$ and $x \in V_m$:

$$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w) \le \frac{\epsilon e^{-\lambda t}}{A-\Psi}\{-2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA + \lambda M(A+1)\}$$

< 0 for all $t > 0$ by (26).

If $C < 0$, then we have that for all $t > 0$ and $x \in V_m$:

$$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w) \le \frac{\epsilon e^{-\lambda t}}{A-\Psi}\{-2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + C(A-\Psi) + \lambda M(A+1)\}$$

< 0 for all $t > 0$ by (26).

Thus part (i) of the theorem is proved.

Now if λ satisfies (27), then by choosing A sufficiently close to $\Psi + \frac{2D_1\Theta}{\delta_0}$ (if $C \ge 0$), or sufficiently large (if $C < 0$), we may ensure that (26), and therefore part (i) of the theorem, will hold. It follows by Theorem 1 that if $u(x,t)$ is a solution of (15), then $u(x,t) < w(x,t)$ for all $x \in \overline{V}_m$ and $0 \le t \le T$. Hence, for all $T > 0$ and $x \in \overline{V}_m$:

$$u(x,T) < w(x,T) = \frac{\epsilon e^{-\lambda T}}{A-\Psi}(A-\sum_{i=1}^{n} x_i^2) \le \frac{A\epsilon e^{-\lambda T}}{A-\Psi}.$$

Also, by hypotheses (f) and (g), it follows from Theorem 11 that $u(x,T) \ge 0$ for all $T > 0$ and $x \in \overline{V}_m$. Since $\lambda > 0$, part (ii) of the theorem follows at once.

Note: Theorem 19 relates well to result (vi) of Keller and Cohen (see p.16). Keller and Cohen showed that if there exists a positive $\rho(x)$

such that $\hat{f}(x,u) < \rho(x)u$ for $x \in V$, $u > 0$, then no positive steady-state solutions exist if $0 < \lambda < \mu_1\{\rho\}$. Theorem 19 shows that, if $f(x,t,u) \le Mu$ for all $x \in V_m$, $t > 0$, $u \ge 0$, and if $0 < \lambda < \mu$(say) where μ depends on M, then all solutions of the time-dependent problem (provided the initial function is bounded) tend to zero as $t \to \infty$, so that no positive steady-state solutions will exist if $0 < \lambda < \mu$.

THEOREM 20: *Hypotheses as for Theorem 17 except that, in hypothesis* (b), *we suppose that there exist constants* $M_1 \ge 0$, $M_2 > 0$ *and* α *such that* $0 < \alpha \le 1$, *such that, for all* $x \in V_m$, $t > 0$ *and* $u \ge 0$,

$f(x,t,u) \le M_1 + M_2 u^{1-\alpha}$.

Then: (i) *For any* $T > 0$, *a strict upper solution for* (15) *is given by*

$$w(x,t) = K\lambda^{1/\alpha}M_2^{1/\alpha}(A - \sum_{i=1}^{n} x_i^2) \text{ for all } x \in \overline{V}_m,\ 0 \le t \le T$$

(so that $w(x,t)$ *is actually independent of* t*) if* $\lambda > 0$, A *is a constant chosen so as to satisfy*

$$\left.\begin{aligned} &\Psi + \frac{2D_1\Theta}{\delta_0} < A < \frac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i}{C} && \text{if } C > 0 \\ &A > \Psi + \frac{2D_1\Theta}{\delta_0} && \text{if } C = 0 \\ &A > \max\left\{\Psi + \frac{2D_1\Theta}{\delta_0},\ \Psi + \frac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i}{C}\right\} && \text{if } C < 0 \end{aligned}\right\}$$

and K *is a constant chosen so as to satisfy*

$$\left.\begin{aligned} &K > \max\left\{1, \left[\frac{\lambda^{1-\frac{1}{\alpha}}M_2^{-1/\alpha}M_1 + A^{1-\alpha}}{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - CA}\right]^{1/\alpha}, \frac{\epsilon}{\lambda^{1/\alpha}M_2^{1/\alpha}(A-\Psi)}\right\} && \text{if } C \ge 0 \\ &K > \max\left\{1, \left[\frac{\lambda^{1-\frac{1}{\alpha}}M_2^{-1/\alpha}M_1 + A^{1-\alpha}}{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(A-\Psi)}\right]^{1/\alpha}, \frac{\epsilon}{\lambda^{1/\alpha}M_2^{1/\alpha}(A-\Psi)}\right\} && \text{if } C < 0 \end{aligned}\right\} \quad (28)$$

(ii) *For any* $\lambda > 0$, *if* $u(x,t)$ *is a solution of* (15), *and* K *and* A *are as above, then* $u(x,T) \le K\lambda^{1/\alpha}M_2^{1/\alpha}A$ *for all* $x \in \overline{V}_m$ *and* $T > 0$.

Proof: (I) $w(x,0) \ge K\lambda^{1/\alpha}M_2^{1/\alpha}(A-\Psi) > \epsilon > u_0(x)$ for all $x \in \overline{V}_m$, using

(28) and hypothesis (e).

(II) is similar to the proof of Theorem 17.

(III) $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$= \sum_{i=1}^{n} a_{ii}(x,t)(-2K\lambda^{1/\alpha}M_2^{1/\alpha}) + \sum_{i=1}^{n} b_i(x,t)(-2x_i K\lambda^{1/\alpha}M_2^{1/\alpha}) + c(x,t)w + \lambda f(x,t,w)$$

$$\le -2K\lambda^{1/\alpha}M_2^{1/\alpha}\sum_{i=1}^{n} A_i + 2K\lambda^{1/\alpha}M_2^{1/\alpha}\sum_{i=1}^{n} B_i + c(x,t)K\lambda^{1/\alpha}M_2^{1/\alpha}(A-\sum_{i=1}^{n} x_i^2) + \lambda\{M_1 + M_2K^{1-\alpha}\lambda^{\frac{1}{\alpha}-1}M_2^{\frac{1}{\alpha}-1}(A-\sum_{i=1}^{n} x_i^2)^{1-\alpha}\}.$$

Thus, for all $t > 0$ and $x \in V_m$:

$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$\le K^{1-\alpha}\lambda^{1/\alpha}M_2^{1/\alpha}\{-2K^{\alpha}\sum_{i=1}^{n} A_i + 2K^{\alpha}\sum_{i=1}^{n} B_i + c(x,t)K^{\alpha}(A-\sum_{i=1}^{n} x_i^2) + \lambda^{1-\frac{1}{\alpha}}M_2^{-1/\alpha}M_1K^{\alpha-1} + A^{1-\alpha}\}$$

$$\le K^{1-\alpha}\lambda^{1/\alpha}M_2^{1/\alpha}\{K^{\alpha}[-2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + c(x,t)(A-\sum_{i=1}^{n} x_i^2)] + \lambda^{1-\frac{1}{\alpha}}M_2^{-1/\alpha}M_1 + A^{1-\alpha}\}$$

since $K > 1$ (by (28)) and $\alpha-1 \le 0$, so $K^{\alpha-1} \le 1$.

Now if $C \ge 0$, then we have that for all $t > 0$ and $x \in V_m$:

$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$\le K^{1-\alpha}\lambda^{1/\alpha}M_2^{1/\alpha}\{K^{\alpha}[-2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA] + \lambda^{1-\frac{1}{\alpha}}M_2^{-1/\alpha}M_1 + A^{1-\alpha}\}$$

< 0 for all $t > 0$ by (28).

If $C < 0$, then we have that for all $t > 0$ and $x \in V_m$:

$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$\le K^{1-\alpha}\lambda^{1/\alpha}M_2^{1/\alpha}\{K^{\alpha}[-2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + C(A-\Psi)] + \lambda^{1-\frac{1}{\alpha}}M_2^{-1/\alpha}M_1 + A^{1-\alpha}\}$$

< 0 for all $t > 0$ by (28).

Thus part (i) of the theorem is proved.

Now if $\lambda > 0$, and $u(x,t)$ is a solution of (15), and K and A are as above, then it follows by Theorem 1 that $u(x,t) < w(x,t)$ for all $x \in \overline{V}_m$ and $0 \le t \le T$. Hence, for all $T > 0$ and $x \in \overline{V}_m$, $u(x,T) < w(x,T) \le K\lambda^{1/\alpha}M_2^{1/\alpha}A$. This proves part (ii) of the theorem.

Extension to Other Domains:

Theorems 12 to 20 apply only to the domain $V_m = \{x: \sum_{i=1}^{n} x_i^{2m_i} < 1\}$ (the m_i being arbitrary positive integers); Theorem 11, though stated for the domain V_m, will obviously hold for any domain V. If one wishes to extend Theorems 12 to 20 in a constructive way to some other specific domain V^*, this may be possible if one can explicitly construct a diffeomorphism from $\overline{V}^*$ to $\overline{V}_m$, i.e. if one can find open sets $O_1 \supset \overline{V}^*$, $O_2 \supset \overline{V}_m$ and a homeomorphism of $\overline{V}^*$ onto $\overline{V}_m$ which can be extended to a differentiable function $g: O_1 \to O_2$ with differentiable inverse. It is necessary also that the second partial derivatives of g should exist on O_1. The construction of such a diffeomorphism is only possible in certain simple cases.

Then if $u(x,t)$ satisfies (15) on the region $\{(x,t): x \in \overline{V}^*, t \ge 0\}$, we can define $v(x,t) = u(g^{-1}(x),t)$ on the region $\{(x,t): x \in \overline{V}_m, t \ge 0\}$, and use standard calculus techniques to transform (15) into the corresponding initial-boundary value problem satisfied by v. It may then be possible to apply Theorems 12 to 20 to this problem. The next chapter includes one simple example of this technique.

However, if one is prepared to abandon to some extent the explicitly constructive approach used in Theorems 12 to 20, one may prove a collection of theorems similar to Theorems 12 to 20 but applying to an arbitrary domain V. The method of proof that will be used here requires that we restrict ourselves to a time-independent differential operator. We shall denote by $\hat{L}_1 u$ the expression

$$\sum_{i,j=1}^{n} \hat{a}_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \hat{b}_i(x) \frac{\partial u}{\partial x_i}$$

where $\hat{a}_{ij} = \hat{a}_{ji} \in C^{\alpha}(\overline{V})$ for $i,j = 1,2,\ldots n$, and $\hat{b}_i \in C^{\alpha}(\overline{V})$ for $i = 1,2,\ldots n$. The differential operator $\hat{L}_1$ is assumed to be uniformly elliptic. We shall be concerned with the parabolic initial-boundary value problem

$$\left.\begin{aligned} \hat{L}_1 u + c(x,t)u - \frac{\partial u}{\partial t} + \lambda f(x,t,u) = 0 \quad \text{for } (x,t) \in D_T \\ B_{1in} u = 0 \quad \text{for } (x,t) \in S_T \\ u(x,0) = u_0(x) \quad \text{for } x \in \overline{V} \end{aligned}\right\} \quad ..(29)$$

where f is continuous for $x \in \overline{V}$, $0 \le t \le T$ and all u, $u_0 \in C^{2+\alpha}(\overline{V})$ and the parameter λ is assumed to be positive.

Now consider the eigenvalue problem

$$\left.\begin{aligned} \hat{L}_1\varphi + \mu\varphi = 0 \quad \text{for } x \in V \\ a\varphi + b\frac{\partial\varphi}{\partial n} = 0 \quad \text{for } x \in \partial V \end{aligned}\right\} \quad ....(30)$$

where a and b are positive constants. If A denotes the inverse of the operator $\hat{L}_1$ with boundary condition as in (30), then A is a compact (i.e. completely continuous) operator on the Banach space $C(\overline{V})$ of continuous functions defined on $\overline{V}$, with the supremum norm (see Browder[7]). Further:

$Ah = g$ for $x \in \overline{V}$ $\Leftrightarrow$ $\hat{L}_1 g = h$ for $x \in V$ and $ag + b\frac{\partial g}{\partial n} = 0$ for $x \in \partial V$.

Suppose now that $Ah = g$ for $x \in \overline{V}$, where $h(x) \ge 0$ for all $x \in \overline{V}$. If g attains its maximum M in V, then it follows by Theorem 5 of Ch.2 of Protter and Weinberger[28] that $g(x) = M$ for all $x \in \overline{V}$. But then $\frac{\partial g}{\partial n} = 0$ for all $x \in \partial V$, and so from $ag + b\frac{\partial g}{\partial n} = 0$ for all $x \in \partial V$, it follows that $g(x) = 0$ for all $x \in \partial V$. Thus $M = 0$, and g is identically zero on $\overline{V}$. So if g is not identically zero on $\overline{V}$ (equivalently, if h is not identically zero on $\overline{V}$), then the maximum of g on $\overline{V}$ is attained at a point P on ∂V. Then, by Theorem 7 of Ch.2 of Protter and Weinberger[28], $\frac{\partial g}{\partial n} > 0$ at P. Since $ag + b\frac{\partial g}{\partial n} = 0$ at P, it follows that $g(x) < 0$ at P, so $g(x) < 0$ for all $x \in \overline{V}$.

Thus we have shown that if $h(x) \ge 0$ for all $x \in \overline{V}$, and h is not identically zero on $\overline{V}$, then $-Ah > 0$ for all $x \in \overline{V}$.

Now the set of non-negative functions on $\overline{V}$ is a cone in the Banach space $C(\overline{V})$, with interior the set of strictly positive functions on $\overline{V}$. It follows from the above discussion that the operator $-A$ is strongly positive with $n = 1$, with respect to this cone (using the terminology of Krein and Rutman[20, p.266]). It follows by Theorem 6.3 of Krein and Rutman[20, p.267] that $-A$ has a unique normalised eigenfunction which is strictly positive on $\overline{V}$, and the corresponding eigenvalue is real, positive and simple. Since (30) may be written

$(-A)\varphi = \frac{1}{\mu}\varphi$, it follows that (30) has a unique normalised eigenfunction $\varphi_1(x;a,b)$ which is strictly positive on $\overline{V}$, and the corresponding eigenvalue $\mu_1(a,b)$ is real, positive and simple. Obviously, there exist positive constants $\alpha_1(a,b)$, $\alpha_2(a,b)$ such that

$$\alpha_1(a,b) \le \varphi_1(x;a,b) \le \alpha_2(a,b) \quad \text{for all } x \in \overline{V}.$$

Since $a\varphi_1 + b\,\frac{\partial\varphi_1}{\partial n} = 0$ on ∂V and $\varphi_1 > 0$ on ∂V, it follows also that $\frac{\partial\varphi_1}{\partial n} < 0$ on ∂V.

We are now in a position to state and prove a theorem analogous to our earlier Theorem 12.

THEOREM 12^A: *Suppose that*

(a) *There exists a constant* $C \ge 0$ *such that* $c(x,t) \ge -C$ *for all* $x \in V$ *and* $t > 0$.

(b) *There exists a constant* $M > 0$ *such that, for all* $t > 0$, $x \in V$ *and* $u \ge 0$, $f(x,t,u) \ge M$. *Furthermore,* f *satisfies a uniform Lipschitz condition in* u *on any finite* u*-interval.*

(c) *There exist constants* $D_0 > 0$ *and* $\delta_1 > 0$ *such that, for all* $x \in \partial V$ *and* $t > 0$, $0 < d_0(x,t) \le D_0$ *and* $\delta_1 \le d_1(x,t)$.

(d) $u_0(x) \ge 0$ *for all* $x \in \overline{V}$.

Then: (i) *For any* $T > 0$, *a lower solution for* (29) *is given by*

$$w(x,t) = \lambda K\varphi_1(x;D_0,\tfrac{1}{2}\delta_1)(1-e^{-t}) \text{ for all } x \in \overline{D}_T$$

where K *is a constant chosen so as to satisfy*

$$0 < K < \frac{M}{\alpha_2(D_0,\frac{1}{2}\delta_1)\{C + \mu_1(D_0,\frac{1}{2}\delta_1) + 1\}} \qquad \text{..........(31)}$$

(ii) *If* $u(x,t)$ *is any solution of* (29), *then for any* $T > 0$, $u(x,t) > 0$ *for* $0 < t \le T$, *and if* $\lim_{T\to\infty} u(x,T) = \hat{u}(x)$ *exists, then for all* $x \in \overline{V}$,

$$\hat{u}(x) \ge \frac{\lambda M\alpha_1(D_0,\frac{1}{2}\delta_1)}{\alpha_2(D_0,\frac{1}{2}\delta_1)\{C + \mu_1(D_0,\frac{1}{2}\delta_1) + 1\}} > 0.$$

Proof: (I) $w(x,0) = 0 \le u_0(x)$ for all $x \in \overline{V}$, by hypothesis (d).

(II)
$$\begin{aligned} B_{lin}w &= d_0(x,t)w + d_1(x,t)\frac{\partial w}{\partial n} \\ &= \lambda K(1-e^{-t})\{d_0(x,t)\varphi_1(x;D_0,\tfrac{1}{2}\delta_1) + d_1(x,t)\frac{\partial\varphi_1}{\partial n}\} \\ &\le \lambda K(1-e^{-t})\{D_0\varphi_1(x;D_0,\tfrac{1}{2}\delta_1) + d_1(x,t)\frac{\partial\varphi_1}{\partial n}\} \text{ for } t > 0,\ x \in \partial V \end{aligned}$$

$$= \lambda K(1-e^{-t})\{-\tfrac{1}{2}\delta_1 + d_1(x,t)\}\frac{\partial\varphi_1}{\partial n}$$

< 0 for all $t > 0$ since $d_1(x,t) \geq \delta_1 > \frac{1}{2}\delta_1$ for $t > 0$, $x \in \partial V$, and also $\frac{\partial\varphi_1}{\partial n} < 0$ for $x \in \partial V$.

(III) $\hat{L}_1 w + c(x,t)w - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$= \lambda K(1-e^{-t})\{-\mu_1(D_0,\tfrac{1}{2}\delta_1) + c(x,t)\}\varphi_1 - \lambda K\varphi_1 e^{-t} + \lambda f(x,t,w)$$

$$\geq \lambda K\{-\mu_1(D_0,\tfrac{1}{2}\delta_1) - C\}\alpha_2(D_0,\tfrac{1}{2}\delta_1) - \lambda K\alpha_2(D_0,\tfrac{1}{2}\delta_1) + \lambda M$$

> 0 for $t > 0$ and $x \in V$, by (31).

Thus part (i) of the theorem is proved. By Theorem 2, it follows that if $u(x,t)$ is a solution of (29), then $u(x,t) \geq w(x,t)$ for all $(x,t) \in \overline{D}_T$. It follows that $u(x,t) > 0$ for $x \in \overline{V}$, $0 < t \leq T$. Furthermore, $w(x,t) \to \lambda K\varphi_1$ as $t \to \infty$, so if $\lim_{T\to\infty} u(x,T) = \hat{u}(x)$ exists, then for all $x \in \overline{V}$, $\hat{u}(x) \geq \lambda K\varphi_1 \geq \lambda K\alpha_1(D_0,\frac{1}{2}\delta_1)$. Since K may be chosen arbitrarily close to $\dfrac{M}{\alpha_2(D_0,\frac{1}{2}\delta_1)\{C + \mu_1(D_0,\frac{1}{2}\delta_1) + 1\}}$, part (ii) of the theorem follows at once.

Note: As for Theorem 12, the condition that f should satisfy a uniform Lipschitz condition in u on any finite u-interval may be removed if we assume that $u_0(x) > 0$ for all $x \in \overline{V}$.

In a similar fashion, one can state and prove Theorems 13^A to 16^A analogous to Theorems 13 to 16. The remaining "lower solution" theorem, Theorem 11, holds for arbitrary domains V in any case, as already remarked.

Next we state and prove a theorem analogous to our earlier Theorem 17.

THEOREM 17^A: *Suppose that*

(a) *There exist constants* $\delta_0 > 0$ *and* $D_1 > 0$ *such that, for all* $x \in \partial V$ *and* $t > 0$, $d_0(x,t) \geq \delta_0$ *and* $0 \leq d_1(x,t) \leq D_1$.

(b) *There exists a constant* $C < \mu_1(\frac{1}{2}\delta_0, D_1)$ *such that* $c(x,t) \leq C$ *for all* $x \in V$ *and* $t > 0$.

(c) *For any bounded positive u-interval* I, *there exists a corresponding positive number* M, *depending only on* I, *such that* $f(x,t,u) \leq M$ *for all* $x \in V$, $t > 0$ *and* $u \in I$.

(d) *There exists a constant* $\epsilon > 0$ *such that* $u_0(x) < \epsilon$ *for all* $x \in \overline{V}$.

Then: (i) *For any* $T > 0$, *a strict upper solution for* (29) *is given by*

$$w(x,t) = \frac{\epsilon}{\alpha_1(\frac{1}{2}\delta_0,D_1)}\,\varphi_1(x;\tfrac{1}{2}\delta_0,D_1) \text{ for all } x \in \overline{D}_T$$

(so that $w(x,t)$ *is actually independent of* t*) if*

$$0 < \lambda < \frac{\epsilon[\mu_1(\frac{1}{2}\delta_0,D_1) - C]}{M[\epsilon,\alpha_1(\frac{1}{2}\delta_0,D_1),\alpha_2(\frac{1}{2}\delta_0,D_1)]} \qquad \text{............(32)}$$

where $M(\epsilon,\alpha_1,\alpha_2)$ *is defined in the proof below, and is independent of* T.

(ii) *If* $0 < \lambda < \dfrac{\epsilon[\mu_1(\frac{1}{2}\delta_0,D_1) - C]}{M[\epsilon,\alpha_1(\frac{1}{2}\delta_0,D_1),\alpha_2(\frac{1}{2}\delta_0,D_1)]}$, *and if* $u(x,t)$ *is a solution of* (29), *then for all* $T > 0$ *and* $x \in \overline{V}$, $u(x,T) < \dfrac{\epsilon\alpha_2(\frac{1}{2}\delta_0,D_1)}{\alpha_1(\frac{1}{2}\delta_0,D_1)}$.

Proof: (I) $w(x,0) \geq \epsilon > u_0(x)$ for all $x \in \overline{V}$, by hypothesis (d).

(II) $B_{lin}w = d_0(x,t)w + d_1(x,t)\dfrac{\partial w}{\partial n}$

$$= \frac{\epsilon}{\alpha_1(\frac{1}{2}\delta_0,D_1)}\left\{d_0(x,t)\varphi_1(x;\tfrac{1}{2}\delta_0,D_1) + d_1(x,t)\frac{\partial\varphi_1}{\partial n}\right\}$$

$$\geq \frac{\epsilon}{\alpha_1(\frac{1}{2}\delta_0,D_1)}\left\{d_0(x,t)\varphi_1(x;\tfrac{1}{2}\delta_0,D_1) + D_1\frac{\partial\varphi_1}{\partial n}\right\} \text{ for } t > 0,$$

$x \in \partial V$, using the fact that $\dfrac{\partial\varphi_1}{\partial n} < 0$ for $t > 0$, $x \in \partial V$

$$= \frac{\epsilon}{\alpha_1(\frac{1}{2}\delta_0,D_1)}\{d_0(x,t) - \tfrac{1}{2}\delta_0\}\,\varphi_1(x;\tfrac{1}{2}\delta_0,D_1)$$

> 0 for all $t > 0$ and $x \in \partial V$ since $d_0(x,t) \geq \delta_0 > \frac{1}{2}\delta_0$ for all $t > 0$ and $x \in \partial V$.

(III) $\hat{L}_1 w + c(x,t)w - \dfrac{\partial w}{\partial t} + \lambda f(x,t,w)$

$$= \frac{\epsilon}{\alpha_1(\frac{1}{2}\delta_0,D_1)}\{-\mu_1(\tfrac{1}{2}\delta_0,D_1) + c(x,t)\}\,\varphi_1(x;\tfrac{1}{2}\delta_0,D_1) + \lambda f(x,t,w)$$

$$\leq \epsilon\{-\mu_1(\tfrac{1}{2}\delta_0,D_1) + C\} + \lambda f(x,t,w) \quad \text{for all } t > 0 \text{ and } x \in V,$$

using the fact that $C - \mu_1(\frac{1}{2}\delta_0,D_1) < 0$, and so $c(x,t) - \mu_1(\frac{1}{2}\delta_0,D_1) < 0$ for all $t > 0$ and $x \in V$.

Now for $t > 0$ and $x \in V$, $\epsilon \leq w(x,t) \leq \dfrac{\epsilon\alpha_2(\frac{1}{2}\delta_0,D_1)}{\alpha_1(\frac{1}{2}\delta_0,D_1)}$, so by hypothesis (c) we have that for all $t > 0$ and $x \in V$,

$$f(x,t,w) \leq M[\epsilon,\alpha_1(\tfrac{1}{2}\delta_0,D_1),\alpha_2(\tfrac{1}{2}\delta_0,D_1)].$$

Hence, for all $t > 0$ and $x \in V$:

$$\hat{L}_1 w + c(x,t)w - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$$
$$\leq \epsilon\{-\mu_1(\tfrac{1}{2}\delta_0,D_1) + C\} + \lambda M[\epsilon,\alpha_1(\tfrac{1}{2}\delta_0,D_1),\alpha_2(\tfrac{1}{2}\delta_0,D_1)]$$
$$< 0 \quad \text{for all } t > 0 \text{ and } x \in V \text{ by (32).}$$

Thus part (i) of the theorem is proved.

Now if λ satisfies the condition of part (ii), then part (i) holds, and it follows by Theorem 1 that if $u(x,t)$ is a solution of (29), then $u(x,t) < w(x,t)$ for all $(x,t) \in \overline{D}_T$. It follows that for all $T > 0$ and $x \in \overline{V}$, $u(x,T) < w(x,T) \leq \dfrac{\epsilon\alpha_2(\frac{1}{2}\delta_0,D_1)}{\alpha_1(\frac{1}{2}\delta_0,D_1)}$. Thus part (ii) is proved.

In a similar fashion, one can state and prove Theorems 18^A to 20^A analogous to Theorems 18 to 20. Thus the picture built up in Theorems 12 to 20 of the behaviour of the solution of the time-dependent problem for different classes of functions f holds not only for the special domain V_m but for any domain V. We shall be discussing that picture shortly. However, it should be pointed out that both sets of theorems, Theorems 12 to 20 and Theorems 12^A to 20^A, are of interest. Theorems 12^A to 20^A are indeed more general as regards the domain V, but the constructive nature of Theorems 12 to 20 allows us to use these theorems to obtain, for example, easily calculated bounds for the critical parameter λ^*, as well as other quantitative information should we need it. This aspect of the matter will be examined in the next chapter.

Discussion:

We shall now summarise and examine one of the most important aspects of Theorems 11 to 20 and 12^A to 20^A from our point of view, namely the information they give concerning the relationship between the nature of the function f and the behaviour of the solution u of the time-dependent problem as $t \to \infty$. To avoid undue complication, we shall here assume that we are dealing with the time-dependent problem

$$\left.\begin{aligned} \hat{L}u - \frac{\partial u}{\partial t} + \lambda\hat{f}(x,u) &= 0 \quad \text{for } (x,t) \in D \\ \hat{B}_{lin}u &= 0 \quad \text{for } (x,t) \in S \\ u(x,0) &= u_0(x) \quad \text{for } x \in \overline{V} \end{aligned}\right\} \quad \ldots\ldots\ldots(33)$$

and its related steady-state problem

$$\left.\begin{aligned} \hat{L}u + \lambda\hat{f}(x,u) &= 0 \quad \text{for } x \in V \\ \hat{B}_{lin}u &= 0 \quad \text{for } x \in \partial V \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(34)$$

We shall suppose that (33) and (34) are such that the discussion at the end of Ch.6 applies, so that the existence of a positive solution of (34) is equivalent to the boundedness over all positive time of the solution of (33) in the case $u_0 = 0$. This means, in particular, that $\hat{f}(x,u) > 0$ for all $x \in V$ and $u \geq 0$, and $u_0(x) \geq 0$ for all $x \in \overline{V}$.

First let us review what is known about (34) from the results of Keller and Cohen, and Amann (see Ch.4). We suppose here that (34) is such that the theory discussed in Ch.4 applies; this requires in particular that (34) be self-adjoint. It is convenient to consider three categories of functions $\hat{f}$.

(1) $\hat{f}$ monotone increasing and concave in u (but not asymptotically linear):

In this case, Keller and Cohen show (result (viii)) that either there exists $\lambda^* > 0$ such that a positive solution of (34) exists for $0 < \lambda < \lambda^*$ but not for $\lambda \geq \lambda^*$, or else a positive solution of (34) exists for all $\lambda > 0$.

(2) $\hat{f}$ monotone increasing and asymptotically linear in u:

In this case, Amann has shown that there exists a finite $\lambda^* > 0$ such that positive solutions of (34) exist for $0 < \lambda < \lambda^*$ but not for $\lambda > \lambda^*$; a positive solution of (34) may or may not exist for $\lambda = \lambda^*$.

(3) $\hat{f}$ monotone increasing and convex in u (but not asymptotically linear):

In this case, Keller and Cohen show (result (vii)) that there exists a finite $\lambda^* \geq 0$ such that positive solutions of (34) exist for $0 < \lambda < \lambda^*$ but not for $\lambda > \lambda^*$; a positive solution of (34) may or may not exist for $\lambda = \lambda^*$. Note that, as far as is known from the results of Keller and Cohen, it is possible that $\lambda^* = 0$, i.e. that for certain $\hat{f}$ there are no positive solutions of (34).

Now suppose that (33) is such that the theorems of the present chapter apply. In considering the relation between the nature of the function $\hat{f}$ and the behaviour of the solution of (33) as $t \to \infty$, we again find three categories of functions $\hat{f}$ appearing, which while not identical to the above, clearly correspond closely to them.

$(1)^A$ $\hat{f}(x,u) \leq M_1 + M_2 u^{1-\alpha}$ for some $M_1 > 0$, $M_2 > 0$, $0 < \alpha \leq 1$:

For any $\lambda > 0$, the solution of (33) is in this case bounded above as $t \to \infty$ (Theorems 20, 20^A). In the light of the discussion at the end of Ch.6, this means that positive solutions of (34) exist for all $\lambda > 0$. Thus the possibility of a bounded spectrum, which was left open by Keller and Cohen, can be ruled out in this case.

A partial result in this direction was in fact obtained by Keller and Cohen[19, Theorem 4.2], who showed that if $\lim_{u \to \infty} \hat{f}_u(x,u) = 0$ then positive solutions of (34) exist for all $\lambda > 0$. An earlier paper by Hudjaev[16], which has not yet been mentioned, deals more fully with this point. Hudjaev's paper has much in common with the paper of Keller and Cohen, but he requires that the coefficient $\hat{a}_0(x)$ of u be zero (using the notation of Ch.4), and also that $\hat{f}(x,u)$ can be written in the form $\alpha(x)F(u)$. With these restrictions, he proves[16, Theorem 2] that a necessary condition for positive solutions of (34) to exist for all $\lambda > 0$ is that $\liminf_{u \to \infty} \frac{F(u)}{u} = 0$, while a sufficient condition is $\lim_{u \to \infty} \frac{F(u)}{u} = 0$. These conditions are obviously closely related to our condition $\hat{f}(x,u) \le M_1 + M_2 u^{1-\alpha}$.

$(2)^A$ <u>$M_1 u + M_2 \le \hat{f}(x,u) \le M_1^* u + M_2^*$ for some positive M_1, M_2, M_1^*, M_2^*; and $\hat{f}$ satisfies a uniform Lipschitz condition in u on any bounded u-interval</u>:

(i) If λ is sufficiently small, then the solution of (33) is bounded above as $t \to \infty$ (Theorems 18, 18^A), and so (34) will have a positive solution.

(ii) If λ is sufficiently large, then the solution of (33) tends to ∞ either as $t \to \infty$ or as t tends to some finite value (Theorems 13, 13^A), and so (34) will have no positive solution.

These facts are in general accord with the results proved by Amann for asymptotically linear $\hat{f}$, but deal with a much wider class of functions.

$(3)^A$ <u>$\hat{f}(x,u) \ge Mu^{1+\alpha}$ for some $M > 0$, $\alpha > 0$; and $\hat{f}$ satisfies a uniform Lipschitz condition in u on any bounded u-interval</u>:

(i) If λ is sufficiently small (depending on the initial function u_0), then the solution of (33) is bounded above as $t \to \infty$ (Theorems 17, 17^A), and so (34) will have a positive solution. This suggests that in the case of convex $\hat{f}$ discussed by Keller and Cohen, positive solutions of (34) will exist for all sufficiently small λ, so that the spectrum is always non-empty.

(ii) For any $\lambda > 0$, if u_0 is sufficiently large (depending on λ), then the solution of (33) tends to ∞ as t tends to some finite value (Theorems 15, 15^A).

(iii) For any positive u_0, if λ is sufficiently large (depending on u_0), then the solution of (33) tends to ∞ as t tends to some finite value (Theorems 15, 15^A).

(iv) If we impose the stronger condition that $\hat{f}(x,u) \geq M_1 u^{1+\alpha} + M_2$ for some $M_1 > 0$, $M_2 > 0$, $\alpha > 0$ (as well as satisfying the Lipschitz condition), then for any non-negative u_0, if λ is sufficiently large, the solution of (33) tends to ∞ as t tends to some finite value (Theorems 16, 16^A). Thus, for sufficiently large λ, (34) has no positive solution.

Thus the information obtained by studying the time-dependent problem parallels in many respects that obtained by studying the steady-state problem; however, the study of the time-dependent problem does seem to have some advantages.

(a) The conditions which need to be imposed on the function $\hat{f}$ are much less stringent; there are no requirements involving differentiability, monotonicity, concavity or the like, only fairly crude inequalities.

(b) There is no requirement that the problem be self-adjoint.

(c) Taking the theorems of this chapter in their full generality, one can deal with time-dependent differential and boundary operators as well as a time-dependent f; indeed, one can handle oscillating systems where no related steady-state problem exists.

(d) As already mentioned, it is possible to extract interesting quantitative data fairly easily from Theorems 12 to 20 of this chapter; Ch.8 deals with this point.

The Effect of Reactant Consumption:

In discussing the heat-generation problem in Ch.1, we assumed that there was no consumption of reactant. Since this hardly seems a realistic assumption, it is time we considered the effect of reactant consumption. One way of doing this is to suppose that, for any fixed x and u, the heat generation function f(x,t,u) decays to zero in a suitably well-behaved manner as $t \to \infty$. We obtain the following theorem:

THEOREM 21: *Suppose that*

(a) *As for Theorem 17.*

(b) *For all* $x \in V_m$, $t \geq 0$ *and* $u \geq 0$, $f(x,t,u) \leq M(u)F(t)$, *where*:

(1) $M(u)$ *is bounded and positive on any finite positive* u-*interval*;

(2) $F(t) \to 0$ *as* $t \to \infty$, $F(t) > 0$ *and bounded above on* $\{t: t \geq 0\}$;

(3) F *is differentiable for all* $t \geq 0$, *and there exist positive constants* Γ_1, Γ_2, γ *such that* $\frac{|F'(t)|}{F(t)} \leq \Gamma_1$ *for all* $t \geq 0$, *and* $\frac{F(t)}{F(\lambda t)} \leq \Gamma_2$ *for all* $t \geq 0$ *if* $0 < \lambda < \gamma$. *Note that these are not severe restrictions, since they are satisfied by, for example*

$$F(t) = Ae^{-kt} \quad (A > 0,\ k > 0)$$

and by
$$F(t) = \begin{cases} Ae^{-k(t-1)} & (0 \leq t \leq 1) \\ At^{-k} & (t > 1) \end{cases} \qquad (A > 0,\ k > 0).$$

(c), (d) *and* (e) *as for Theorem 17.*

(f) *and* (g) *as for Theorem 19.*

Then: (i) *For any* $T > 0$, *a strict upper solution for* (15) *is given by*

$$w(x,t) = \frac{\epsilon}{F(0)(A-\Psi)}\left(A - \sum_{i=1}^{n} x_i^2\right)F(\lambda t) \text{ for all } x \in \overline{V}_m,\ 0 \leq t \leq T$$

if A *is a constant chosen so as to satisfy*

$$\left.\begin{array}{ll} \Psi + \dfrac{2D_1\Theta}{\delta_0} < A < \dfrac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i}{C} & \text{if } C > 0 \\ A > \Psi + \dfrac{2D_1\Theta}{\delta_0} & \text{if } C = 0 \\ A > \max\left\{\Psi + \dfrac{2D_1\Theta}{\delta_0},\ \Psi + \dfrac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i}{C}\right\} & \text{if } C < 0 \end{array}\right\}$$

and if $\quad 0 < \lambda < n_0(\epsilon,A) \quad$ ....(35)

where $n_0(\epsilon,A)$, *apart from depending on* ϵ *and* A, *depends on the coefficients in the operators* L *and* B_{lin}, *the quantities* Ψ *and* Θ, *and the nature of the functions* M *and* F, *but not on* T.

(ii) *If* $0 < \lambda < n_0(\epsilon,A)$, *and if* $u(x,t)$ *is a solution of* (15), *then* $u(x,T) \to 0$ *as* $T \to \infty$, *uniformly for* $x \in \overline{V}_m$.

Proof: (I) and (II) are similar to the proof of Theorem 17.

(III) By hypothesis (b)(2), $F(\lambda t)$ is bounded above on $\{t: t \geq 0\}$, so $w(x,t)$ is bounded above on $\{(x,t): x \in \overline{V}_m,\ t \geq 0\}$. By hypothesis (b)(1), there exists a positive number $N(\epsilon,A)$ such that $M(w) \leq N(\epsilon,A)$ for all $x \in \overline{V}_m$, $t \geq 0$.

$$\therefore \quad Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$$

$$= \sum_{i=1}^{n} a_{ii}(x,t) \frac{-2\epsilon}{F(0)(A-\Psi)} F(\lambda t) + \sum_{i=1}^{n} b_i(x,t) \frac{-2x_i\epsilon}{F(0)(A-\Psi)} F(\lambda t)$$

$$+ \; c(x,t) \frac{\epsilon F(\lambda t)}{F(0)(A-\Psi)} (A - \sum_{i=1}^{n} x_i^2) - \frac{\lambda\epsilon}{F(0)(A-\Psi)} (A - \sum_{i=1}^{n} x_i^2) F'(\lambda t)$$

$$+ \; \lambda f(x,t,w)$$

$$\leq \left(\frac{-2\epsilon F(\lambda t)}{F(0)(A-\Psi)}\right) \sum_{i=1}^{n} A_i + \left(\frac{2\epsilon F(\lambda t)}{F(0)(A-\Psi)}\right) \sum_{i=1}^{n} B_i + c(x,t) \frac{\epsilon F(\lambda t)}{F(0)(A-\Psi)} (A - \sum_{i=1}^{n} x_i^2)$$

$$+ \; \frac{\lambda \epsilon \Gamma_1 F(\lambda t) A}{F(0)(A-\Psi)} + \lambda N(\epsilon,A) F(t) \quad \text{for } t > 0 \text{ and } x \in V_m,$$

using hypothesis (b)

$$\leq F(\lambda t)\Big[\frac{\epsilon}{F(0)(A-\Psi)}\{- 2 \sum_{i=1}^{n} A_i + 2 \sum_{i=1}^{n} B_i + c(x,t)(A - \sum_{i=1}^{n} x_i^2)\}$$

$$+ \; \lambda\{\frac{\epsilon\Gamma_1 A}{F(0)(A-\Psi)} + N(\epsilon,A)\Gamma_2\}\Big]$$

if $0 < \lambda < \gamma$, by hypothesis (b)(3).

We now define the positive number $n_0(\epsilon,A)$ as follows:

$$\left.\begin{aligned} &\text{If } C \geq 0,\ n_0(\epsilon,A) = \min\left\{\gamma,\ \frac{\epsilon[2 \sum_{i=1}^{n} A_1 - 2 \sum_{i=1}^{n} B_i - CA]}{\epsilon\Gamma_1 A + F(0)(A-\Psi)N(\epsilon,A)\Gamma_2}\right\} \\ &\text{If } C < 0,\ n_0(\epsilon,A) = \min\left\{\gamma,\ \frac{\epsilon[2 \sum_{i=1}^{n} A_i - 2 \sum_{i=1}^{n} B_i - C(A-\Psi)]}{\epsilon\Gamma_1 A + F(C)(A-\Psi)N(\epsilon,A)\Gamma_2}\right\} \end{aligned}\right\} \quad (36)$$

Now if $C \geq 0$, then we have that for all $t > 0$ and $x \in V_m$, if $0 < \lambda < \gamma$:

$$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$$

$$\leq F(\lambda t)\Big[\frac{\epsilon}{F(0)(A-\Psi)}\{- 2 \sum_{i=1}^{n} A_i + 2 \sum_{i=1}^{n} B_i + CA\} + \lambda\{\frac{\epsilon\Gamma_1 A}{F(0)(A-\Psi)} + N(\epsilon,A)\Gamma_2\}\Big].$$

If $C < 0$, then we have that for all $t > 0$ and $x \in V_m$, if $0 < \lambda < \gamma$:

$$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$$

$$\leq F(\lambda t)\Big[\frac{\epsilon}{F(0)(A-\Psi)}\{- 2 \sum_{i=1}^{n} A_i + 2 \sum_{i=1}^{n} B_i + C(A-\Psi)\} + \lambda\{\frac{\epsilon\Gamma_1 A}{F(0)(A-\Psi)} + N(\epsilon,A)\Gamma_2\}\Big].$$

In either case, it follows by (35) and (36) that $Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w) < 0$ for all $t > 0$ and $x \in V_m$. Thus part (i) of the theorem is proved.

Now if λ satisfies the condition of part (ii), then part (i) holds, and it follows by Theorem 1 that if $u(x,t)$ is a solution of (15), then $u(x,t) < w(x,t)$ for all $x \in \overline{V}_m$ and $0 \leq t \leq T$. It follows that for all $T > 0$ and $x \in \overline{V}_m$, $u(x,T) < w(x,T) \leq \frac{\epsilon AF(\lambda T)}{F(0)(A-\gamma)}$. Also, by hypotheses (f) and (g), it follows from Theorem 11 that $u(x,T) \geq 0$ for all $T > 0$ and $x \in \overline{V}_m$. Since, by hypothesis (b)(2), $F(\lambda T) \to 0$ as $T \to \infty$, part (ii) is proved.

Note: As in the case of Theorems 12 to 20, one can state and prove Theorem 21^A, analogous to Theorem 21 but applying to an arbitrary domain V.

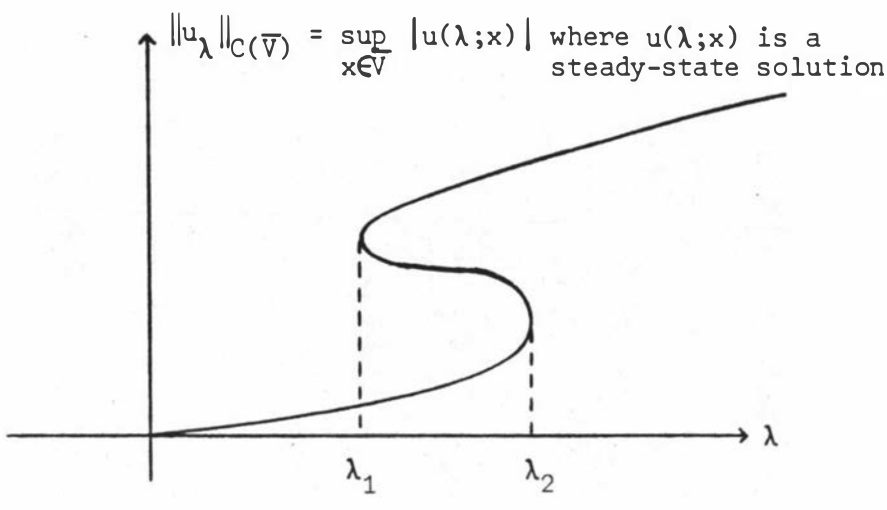
Thus, if we allow for reactant consumption, then part (ii) of the theorem shows that, in terms of the criterion for thermal explosion that we are working with at present, thermal explosion cannot occur. Since thermal explosions undoubtedly do occur, there appears to be something wrong with our criterion. However, a recent paper by Sattinger[33] sheds some light on this matter.

Sattinger discusses a model of combustion with reactant consumption which involves two simultaneous partial differential equations, and so is rather more complex than the one we have used. However, he reaches the same conclusion - that the solution $u(x,t)$ tends to zero as $t \to \infty$, regardless of the initial condition, at least for the particular system he is dealing with. He then points out that whether or not thermal explosion takes place depends, not on the final state reached by the system, but on the manner in which that state is attained. If the system is initially in what he calls a "subcritical" state, the combustion proceeds very slowly, with the temperature reaching an almost steady value, which it holds for a considerable time before ultimately falling to the ambient value. Alternatively, the reaction may after a certain time begin to proceed very quickly, with a rapid rise to a very high temperature, and this is what in practice constitutes an explosion. The fact that, in this case also, the temperature will in theory ultimately fall to the ambient value, is irrelevant, since the explosion will already have taken place at some finite time.

It therefore appears that, if reactant consumption is to be taken into account, new criteria for thermal explosion must be used, relating to the early behaviour of a reacting system rather than to its final state. However, as Sattinger shows for the particular problem discussed in his paper, the early behaviour of the system when reactant consumption is taken into account is related to the existence of steady-state solutions for the equation in the form we have treated it, ignoring reactant consumption. Thus it may well be that, in many cases at least, there are two equivalent ways of deciding whether or not explosion takes place - one may ignore reactant consumption and adopt one or the other of the (equivalent) criteria used in this thesis, or one may take reactant consumption into account and adopt Sattinger's criterion. The two approaches may give the same conclusion even though the latter is based on a model much closer to the actual situation than is the former.

It must be said at once that there is a further complication. The problem dealt with by Sattinger corresponds to the problem without reactant consumption in which $\hat{f}(x,u) = C_1\exp(-\frac{C_2}{u+T_a})$, the Arrhenius formula (see Ch.1). Our Theorems 20, 20^A apply to this function, and show that the solution u(x,t) of problem (15) is in this case always bounded as $t \to \infty$, so that, in terms of our criteria, explosion will never take place. But Sattinger shows that it can take place in terms of his criterion, if λ is sufficiently large, and it is apparent from his discussion that, at least for this particular $\hat{f}$, neither of the criteria adopted in this thesis is appropriate (this possibility was also suggested in a private communication by Dr. G.C.Wake).

For the case where $\hat{f}(x,u) = C_1\exp(-\frac{C_2}{u+T_a})$, there are positive steady-state solutions for all $\lambda > 0$, as is shown by our Theorems 20, 20^A and 10. If one analyses the situation more deeply, as has been done, for example, by Parter[27], one finds that there exist two finite values $\lambda_1,\lambda_2 > 0$ such that the steady-state problem has one positive solution for $0 < \lambda < \lambda_1$, two for $\lambda = \lambda_1$, three for $\lambda_1 < \lambda < \lambda_2$, two for $\lambda = \lambda_2$ and one for $\lambda > \lambda_2$, as illustrated:



As λ passes through the value λ_2, the number of positive steady-state solutions changes from three to one, and (more important, perhaps) the size of the minimal positive solution increases by an abrupt jump. Sattinger shows that it is the value λ_2 which is critical in the sense that for $\lambda > \lambda_2$, thermal explosion takes place in terms of his criterion, taking reactant consumption into account.

Thus the orthodox criteria for thermal explosion, which we use in this thesis and which are used by many other authors, seem not to apply to the Arrhenius function, at any rate, though they appear to work well enough if one uses instead the Frank-Kamenetskii approximation for which $\hat{f}(x,u) = e^u$, as is done, for example, by Boddington, Gray and Harvey[4] (the use of the Frank-Kamenetskii approximation is commented on in Ch.1 of this thesis).

There is evidently plenty of scope for more detailed investigation of the steady-state problem for different functions $\hat{f}$, looking not merely at whether or not positive solutions exist but at the number and size of such solutions for different values of λ. The problems involved seem likely to be difficult - so far only some simple special cases have been studied. In this connection, the theorems we have proved for the time-dependent problem, giving constructive bounds for the solution, may prove helpful in determining the size of possible steady-state solutions, though it is equally likely that these bounds will turn out to be too crude to be useful for this purpose.

8 BOUNDS FOR THE CRITICAL PARAMETER

In this chapter, we shall apply the results obtained in the previous chapter to the problem of finding bounds for the critical parameter λ^* (as defined at the end of Ch.6). The steady-state theory reviewed in Ch.4 gives us upper and lower bounds for λ^* in certain cases, these bounds generally involving the principal eigenvalue of some related linear problem. Also, Wake and Rayner[36] have recently developed a variational method for estimating λ^*, again working with the steady-state problem. The theorems we have proved in Ch.7 provide another means of obtaining rigorous bounds for λ^*. As compared to the steady-state methods, our method has the advantage that it gives bounds for λ^* which are easily computable by elementary methods. We shall shortly illustrate this by calculating these bounds in the cases of two important functions $\hat{f}$. However, these bounds have no pretensions to being highly accurate estimates; some idea of their closeness to the exact value of λ^* will be obtained later in this chapter by comparing them with the results obtained by Boddington, Gray and Harvey[4] using an empirical formula for λ^*.

Preliminaries:

Consider first the original heat-generation problem described in Ch.1. Using the notation defined in that chapter, this problem is of the form

$$\left.\begin{aligned} K\sum_{i=1}^{3}\frac{\partial^2 T}{\partial x_i^2} - \rho c\frac{\partial T}{\partial t} + g_1(T) &= 0 \quad \text{for } (x,t)\in D \\ K\frac{\partial T}{\partial \nu} + Hg_2(T) &= 0 \quad \text{for } (x,t)\in S \\ T(x,0) &= T_a \quad \text{for } x\in \overline{V}. \end{aligned}\right\}$$

We shall assume that the boundary condition is linear, corresponding to heat loss following Newton's law of cooling, so that $g_2(T) = T-T_a$. If we divide the differential equation by K and change to a new time scale, which we may do without loss of generality, the problem reduces to

$$\left.\begin{aligned} \sum_{i=1}^{3}\frac{\partial^2 T}{\partial x_i^2} - \frac{\partial T}{\partial t} + \frac{1}{K}g_1(T) &= 0 \quad \text{for } (x,t)\in D \\ K\frac{\partial T}{\partial \nu} + H(T-T_a) &= 0 \quad \text{for } (x,t)\in S \\ T(x,0) &= T_a \quad \text{for } x\in \overline{V}. \end{aligned}\right\}$$

We shall examine two important possibilities for the function g_1.

1) The Arrhenius formula $g_1(T) = q\rho A \exp(-\frac{E}{RT})$, where q is the exothermicity per unit mass of the reactant, ρ is again the density of the reactant, A is a proportionality constant, E is the activation energy of the reaction and R is the universal gas constant. We follow here the study by Boddington, Gray and Harvey[4]; they use the common procedure of replacing the Arrhenius function by the Frank-Kamenetskii approximation, and so we shall do the same. The difficulties associated with the use of the Arrhenius function itself were discussed at the end of the previous chapter.

Accordingly we make the change of variable $u = \frac{E(T-T_a)}{RT_a^2}$, so that

$$\begin{aligned}\exp(-\frac{E}{RT}) &= \exp(-\frac{E}{R}\{\frac{1}{T_a + RT_a^2E^{-1}u}\})\\ &\approx \exp(-\frac{E}{RT_a}\{1 - \frac{RT_au}{E}\}) \quad \text{if } \frac{RT_au}{E} \text{ is small}\\ &= \exp(-\frac{E}{RT_a})e^u.\end{aligned}$$

This is the Frank-Kamenetskii approximation, and using this we obtain finally the initial-boundary value problem

$$\left.\begin{aligned}\sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} + \frac{E}{KRT_a^2} q\rho A \exp(-\frac{E}{RT_a})e^u = 0 \quad &\text{for } (x,t) \in D\\ Hu + K\frac{\partial u}{\partial \nu} = 0 \quad &\text{for } (x,t) \in S\\ u(x,0) = 0 \quad &\text{for } x \in \overline{V}.\end{aligned}\right\}$$

We then write $\lambda = \frac{q\rho AE}{KRT_a^2}\exp(-\frac{E}{RT_a})$, this being the same as the parameter γ in the notation of Boddington, Gray and Harvey[4].

2) The modified Arrhenius formula $g_1(T) = q\rho AT \exp(-\frac{E}{RT})$, the constant A being not necessarily the same as in the previous case. As mentioned in Ch.1, recent theory suggests that reaction rates may well be governed by formulae of this kind rather than by the original Arrhenius formula. Making the same change of variable as before gives:

$$T\exp(-\frac{E}{RT}) = (T_a + \frac{RT_a^2u}{E})\exp(\frac{-E}{R\{T_a + RT_a^2E^{-1}u\}})$$

$$= \frac{RT_a^2}{E}\left(u + \frac{E}{RT_a}\right)\exp\left[-\left(\frac{E}{RT_a}\right)^2\left(\frac{1}{u + \frac{E}{RT_a}}\right)\right]$$

$$= \frac{T_a}{\xi}(u + \xi)\exp\left(\frac{-\xi^2}{u + \xi}\right), \text{ writing } \xi = \frac{E}{RT_a}.$$

This leads to the initial-boundary value problem

$$\left.\begin{aligned} \sum_{i=1}^{3}\frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} + \frac{E}{KRT_a^2}\,q\rho A\left(\frac{T_a}{\xi}\right)(u + \xi)\exp\left(\frac{-\xi^2}{u + \xi}\right) &= 0 \quad \text{for } (x,t) \in D \\ Hu + K\frac{\partial u}{\partial \nu} &= 0 \quad \text{for } (x,t) \in S \\ u(x,0) &= 0 \quad \text{for } x \in \overline{V}. \end{aligned}\right\}$$

If we again write $\lambda = \frac{q\rho AE}{KRT_a^2}\exp\left(-\frac{E}{RT_a}\right)$, then the non-linear term in the above differential equation becomes

$$\lambda \exp\left(\frac{E}{RT_a}\right)\left(\frac{T_a}{\xi}\right)(u + \xi)\exp\left(\frac{-\xi^2}{u + \xi}\right) = \lambda\left(\frac{T_a e^{\xi}}{\xi}\right)(u + \xi)\exp\left(\frac{-\xi^2}{u + \xi}\right).$$

We need to consider a slightly modified version of the problem, to allow bounds for λ^* to be calculated for certain specific domains V, so we shall suppose henceforth that we are dealing with the problem

$$\left.\begin{aligned} \sum_{i=1}^{3} A_i\frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} + \lambda f(u) &= 0 \quad \text{for } x \in V_m,\ t > 0 \\ Hu + K\sum_{i=1}^{3}\alpha_i(x)x_i^{2m_i-1}\frac{\partial u}{\partial x_i} &= 0 \quad \text{for } x \in \partial V_m,\ t > 0 \\ u(x,0) &= 0 \quad \text{for } x \in \overline{V}_m \end{aligned}\right\} \tag{37}$$

where A_1, A_2, A_3, H and K are positive constants, and in all other respects we follow the notation of Ch.7. We shall, in the light of the above discussion, consider two possibilities for the function f:

(a) $f_1(u) = e^u$, the Frank-Kamenetskii version.

(b) $$\left.\begin{aligned} f_2(u) &= \left(\frac{T_a e^{\xi}}{\xi}\right)(u + \xi)\exp\left(\frac{-\xi^2}{u + \xi}\right) && (u > -\xi) \\ &= 0 && (u \le -\xi) \end{aligned}\right\}$$, the modified Arrhenius form. Note that this function is non-decreasing and asymptotically linear in u, and also satisfies a uniform Lipschitz condition for all u, since its derivative is bounded. Since the values of u that we work with are always positive, we may define $f_2(u)$ as we please for $u \le -\xi$, so we define it in such a way as to make f_2 suitably well-behaved.

Bounds for the Critical Value λ^*:

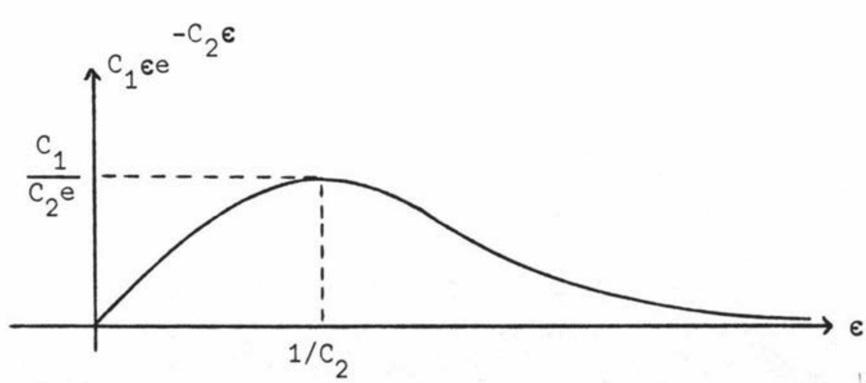
Let u(x,t) be the solution of (37). For our present purposes, we shall define the critical value λ^* to be the value of λ below which u(x,t) is bounded as $t \to \infty$, and above which u(x,t) is unbounded as $t \to \infty$ or as t tends to some finite value. As shown in Ch.6, this is the same as the steady-state critical value above which no positive steady-state solutions exist.

(a) $f(u) \equiv f_1(u)$: For the special problem (37), Theorem 17 tells us that the solution u(x,t) is bounded as $t \to \infty$ if

$$0 < \lambda < \frac{2\varepsilon \sum_{i=1}^{3} A_i}{(A-\Psi)M(\varepsilon,A,\Psi)} .$$

Thus $\dfrac{2\varepsilon \sum_{i=1}^{3} A_i}{(A-\Psi)M(\varepsilon,A,\Psi)}$ is a lower bound for λ^*, for any choice of $\varepsilon > 0$. Now $M(\varepsilon,A,\Psi)$ may be taken as the least upper bound of f(u) on the interval $\varepsilon \le u \le \frac{\varepsilon A}{A-\Psi}$. Thus, for $f(u) \equiv f_1(u)$, we have $M(\varepsilon,A,\Psi) = e^{\frac{\varepsilon A}{A-\Psi}}$. Thus we have that for any choice of $\varepsilon > 0$, the following is a lower bound for λ^*:

$$\frac{2\varepsilon \sum_{i=1}^{3} A_i}{A-\Psi} e^{-\frac{\varepsilon A}{A-\Psi}} = C_1 \varepsilon e^{-C_2\varepsilon}, \text{ say.}$$



From the graph, it is clear that the best lower bound for λ^*, namely $\frac{C_1}{C_2 e}$, will be obtained by choosing $\varepsilon = \frac{1}{C_2}$, which we may do. We obtain therefore as a lower bound for λ^* the value $\frac{C_1}{C_2 e} = \frac{2}{Ae} \sum_{i=1}^{3} A_i$.

Now (i) of Theorem 17 tells us that A may be arbitrarily chosen greater than $\Psi + \frac{2K\Theta}{H} = \Psi + \frac{2\Theta}{h}$ where $h = \frac{H}{K}$. Since A may be chosen as close to this value as we please, it follows that a lower bound for λ^* is given by:

$$\underline{\lambda}^* = \frac{2\sum_{i=1}^{3} A_i}{e(\Psi + \frac{2\Theta}{h})} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38)$$

Further, since $f_1(u) = e^u \geq 1 + \frac{u^2}{2}$ for all $u \geq 0$, we can apply the Corollary of Theorem 16 to (37) in the case $f(u) \equiv f_1(u)$. It follows from this Corollary that, provided λ is sufficiently large, u(x,t) is unbounded as t tends to some finite value. This allows us to obtain an upper bound for λ^* in the case $f(u) \equiv f_1(u)$. In fact, the Corollary tells us that, as long as $\psi + \frac{2\theta}{h} > \Psi$, an upper bound for λ^* is given by

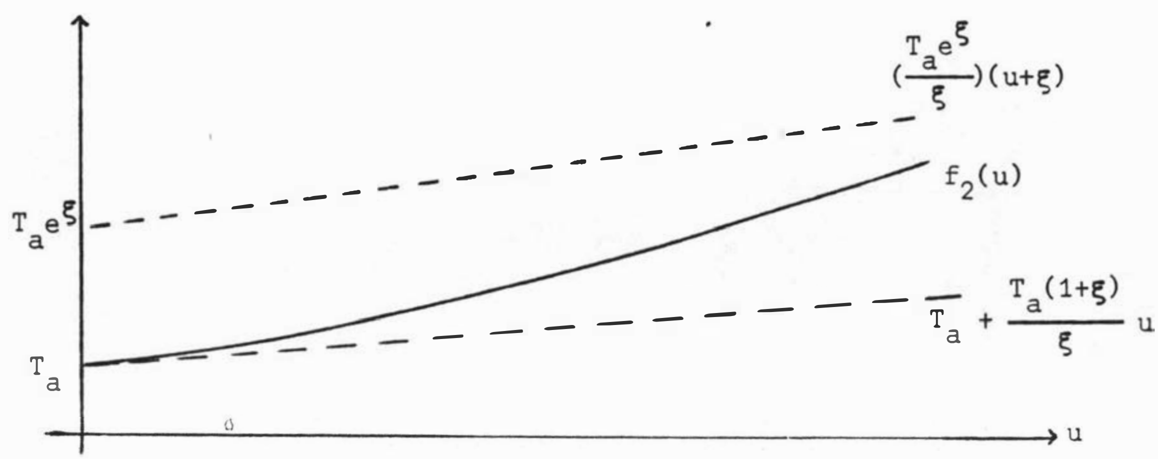
$$\overline{\lambda}^* = \frac{\sum_{i=1}^{3} A_i[\frac{1}{2}(\psi + \frac{2\theta}{h} - \Psi)^2 + 1]}{(\psi + \frac{2\theta}{h} - \Psi)^2} \qquad \ldots\ldots\ldots\ldots\ldots(39)$$

(b) $f(u) \equiv f_2(u)$: We have that $f_2(0) = T_a$. As $u \to \infty$, $f_2(u)$ is asymptotic to $(\frac{T_a e^{\xi}}{\xi})(u+\xi)$, and $f_2(u) < (\frac{T_a e^{\xi}}{\xi})(u+\xi)$ for all $u \geq 0$.

Further, for $u \geq 0$, $f_2'(u) = (\frac{T_a e^{\xi}}{\xi})\left\{\frac{\xi^2}{u+\xi} e^{-\frac{\xi^2}{u+\xi}} + e^{-\frac{\xi^2}{u+\xi}}\right\}$.

So $f_2'(0) = \frac{T_a(1+\xi)}{\xi} < \frac{T_a e^{\xi}}{\xi}$, and $f_2'(u) \to \frac{T_a e^{\xi}}{\xi}$ as $u \to \infty$. Also, for $u \geq 0$,

$$f_2''(u) = (\frac{T_a e^{\xi}}{\xi})\frac{\xi^4}{(u+\xi)^3} e^{-\frac{\xi^2}{u+\xi}} > 0.$$



Thus, for all $u \geq 0$, $T_a + \frac{T_a(1+\xi)}{\xi} u \leq f_2(u) \leq (\frac{T_a e^{\xi}}{\xi})(u+\xi)$. It follows from Theorem 18 that a lower bound for λ^* is given in this case by

$$\underline{\lambda}^* = \frac{2\xi \sum_{i=1}^{3} A_i}{T_a e^{\xi}(\Psi + \frac{2\Theta}{h})} .$$

It also follows from Theorem 13 that, as long as $\psi + \frac{2\theta}{h} > \Psi$, an upper bound for λ^* is given in this case by

$$\overline{\lambda}^* = \max\left\{\frac{2\xi \sum_{i=1}^{3} A_i}{T_a(1+\xi)(\psi + \frac{2\theta}{h} - \Psi)} , \frac{\psi + \frac{2\theta}{h}}{T_a}\right\}.$$

It will be seen from these illustrations that the calculations involved in determining $\underline{\lambda}^*$ and $\overline{\lambda}^*$ are quite simple. One could readily perform similar calculations for other forms of the function f(u) if desired.

Comparison with Known Values:

While we do not know in general how close the bounds $\underline{\lambda}^*$ and $\overline{\lambda}^*$ are to the true value of λ^*, it is possible to get some feeling for this by comparing these bounds with the known value of λ^* in certain special cases. Using the Frank-Kamenetskii approximation in the original heat-generation problem, Boddington, Gray and Harvey[4] have obtained an empirical formula for λ^* which appears to agree well with all known information; we shall denote the value obtained using their formula by λ^*_{est}. Values of λ^*_{est} for various special regions V are given in Table 1 on p.92, using the notation of Boddington, Gray and Harvey. We shall first compare, for each of the special regions in Table 1, our lower bound $\underline{\lambda}^*$ with λ^*_{est}; the size of the upper bound $\overline{\lambda}^*$ relative to λ^*_{est} will be investigated at the end of the chapter. Apart from giving some feel for the size of $\underline{\lambda}^*$ and $\overline{\lambda}^*$, our calculations will also serve to illustrate the technique mentioned in Ch.7 (p.72) of transforming from a region V* to the region V_m for which Theorems 12 to 20 hold, and for which formulas (38) and (39) for $\underline{\lambda}^*$ and $\overline{\lambda}^*$ were calculated.

(continued on p.93)

REGION	j	$F(j)$	SEMENOV RADIUS R_S	RECIPROCAL SQUARE MEAN RADIUS R_0^{-2}	$\frac{1}{\lambda^*_{est}} = R_0^2\left[\frac{1}{3F(j)} + \frac{e}{j+1}\left(\frac{1}{Bi}\right)\right]$ where $Bi = hR_S$ is the Biot number
Sphere (radius = a)	2	1.111	a	$\frac{1}{a^2}$	$a^2\left[\frac{1}{3.333} + \frac{e}{3}\left(\frac{1}{Bi}\right)\right] = a^2\left[\frac{0.9061 + (0.3000)Bi}{Bi}\right]$
Infinite cylinder (radius = a)	1	1.000	$\frac{3a}{2}$	$\frac{2}{3a^2}$	$\frac{3a^2}{2}\left[\frac{1}{3} + \frac{e}{2}\left(\frac{1}{Bi}\right)\right] = a^2\left[\frac{2.0387 + (0.5000)Bi}{Bi}\right]$
Infinite slab (thickness = 2a)	0	0.857	3a	$\frac{1}{3a^2}$	$3a^2\left[\frac{1}{2.571} + \frac{e}{1}\left(\frac{1}{Bi}\right)\right] = a^2\left[\frac{8.1548 + (1.1669)Bi}{Bi}\right]$
Equicylinder (height = 2a, radius = a)	2.728	1.178	a	$\frac{1+\sqrt{2}}{3a^2} = \frac{2.4142}{3a^2}$	$\frac{3a^2}{2.4142}\left[\frac{1}{3.534} + \frac{e}{3.728}\left(\frac{1}{Bi}\right)\right] = a^2\left[\frac{0.9061 + (0.3516)Bi}{Bi}\right]$
Thin circular disc (thickness = 2a, radius = 10a)	0.437	0.9243	$\frac{5a}{2}$	$\frac{1}{3a^2}\left[1 + \frac{0.02}{\sqrt{101}}\right] = \frac{1.0020}{3a^2}$	$\frac{3a^2}{1.0020}\left[\frac{1}{2.773} + \frac{e}{1.437}\left(\frac{1}{Bi}\right)\right] = a^2\left[\frac{5.6636 + (1.0797)Bi}{Bi}\right]$
Long circular cylinder (height = 10a, radius = a)	1.418	1.050	$\frac{15a}{11}$	$\frac{1}{3a^2}\left[\frac{1}{25} + 2\sqrt{\frac{25}{26}}\right] = \frac{2.0012}{3a^2}$	$\frac{3a^2}{2.0012}\left[\frac{1}{3.150} + \frac{e}{2.418}\left(\frac{1}{Bi}\right)\right] = a^2\left[\frac{1.6853 + (0.4759)Bi}{Bi}\right]$
Cube (side = 2a)	3.280	1.222	a	$\frac{1 + \frac{2\sqrt{3}}{\pi}}{3a^2} = \frac{2.1027}{3a^2}$	$\frac{3a^2}{2.1027}\left[\frac{1}{3.666} + \frac{e}{4.280}\left(\frac{1}{Bi}\right)\right] = a^2\left[\frac{0.9061 + (0.3892)Bi}{Bi}\right]$
Infinite square rod (side = 2a)	1.443	1.051	$\frac{3a}{2}$	$\frac{1 + \frac{2}{\pi}}{3a^2} = \frac{1.6366}{3a^2}$	$\frac{3a^2}{1.6636}\left[\frac{1}{3.153} + \frac{e}{2.443}\left(\frac{1}{Bi}\right)\right] = a^2\left[\frac{2.0396 + (0.5814)Bi}{Bi}\right]$

TABLE 1: Values of λ^*_{est} for various regions.

(continued from p.91)

We begin by working with the heat-generation problem

$$\left.\begin{aligned} \sum_{i=1}^{3} \frac{\partial^2 u}{\partial y_i^2} - \frac{\partial u}{\partial t} + \lambda e^u &= 0 \quad \text{for } y \in V^*,\ t > 0 \\ Hu + K\frac{\partial u}{\partial \nu} &= 0 \quad \text{for } y \in \partial V^*,\ t > 0 \\ u(y,0) &= 0 \quad \text{for } y \in \overline{V^*} \end{aligned}\right\} \qquad \ldots.(40)$$

where $V^* = \{y: (\frac{y_1}{a_1})^{2m_1} + (\frac{y_2}{a_2})^{2m_2} + (\frac{y_3}{a_3})^{2m_3} < 1\}$; this is the time-dependent version of the problem considered by Boddington, Gray and Harvey, apart from our choice of the region V^*. Following the remarks on p.72, we make the change of coordinates $x_i = \frac{y_i}{a_i}$ $(i = 1,2,3)$. This transforms V^* into the region V_m with dimension $n = 3$. We then have:

$$\frac{\partial u}{\partial y_i} = \frac{1}{a_i}\frac{\partial u}{\partial x_i}, \quad \frac{\partial^2 u}{\partial y_i^2} = \frac{1}{a_i^2}\frac{\partial^2 u}{\partial x_i^2} \quad (i = 1,2,3).$$

Also, for the region V^*, $\frac{\partial u}{\partial \nu} = \sum_{i=1}^{3} \nu_i(y)\frac{\partial u}{\partial y_i}$, where:

$\nu(y)$ = outward unit normal to V^*

$$= \frac{1}{\sqrt{\sum_{i=1}^{3} \frac{4m_i^2}{a_i^2}(\frac{y_i}{a_i})^{4m_i-2}}}\left(\frac{2m_1}{a_1}(\frac{y_1}{a_1})^{2m_1-1}, \frac{2m_2}{a_2}(\frac{y_2}{a_2})^{2m_2-1}, \frac{2m_3}{a_3}(\frac{y_3}{a_3})^{2m_3-1}\right)$$

$$= \frac{1}{\sqrt{\sum_{i=1}^{3} \frac{4m_i^2}{a_i^2} x_i^{4m_i-2}}}\left(\frac{2m_1 x_1^{2m_1-1}}{a_1}, \frac{2m_2 x_2^{2m_2-1}}{a_2}, \frac{2m_3 x_3^{2m_3-1}}{a_3}\right).$$

Thus, changing coordinates, $\frac{\partial u}{\partial \nu}$ transforms into

$$\sum_{i=1}^{3}\left\{\frac{\frac{2m_i x_i^{2m_i-1}}{a_i}}{\sqrt{\sum_{i=1}^{3} \frac{4m_i^2}{a_i^2} x_i^{4m_i-2}}}\right\}\frac{1}{a_i}\frac{\partial u}{\partial x_i} = \sum_{i=1}^{3} \alpha_i(x) x_i^{2m_i-1}\frac{\partial u}{\partial x_i}$$

where $\alpha_i(x) = \dfrac{m_i}{a_i^2\sqrt{\sum_{i=1}^{3} \frac{m_i^2}{a_i^2} x_i^{4m_i-2}}}$.

Results concerning $\underline{\lambda}^*$ for the problem (40) will therefore be the same as for the problem (37) with $A_i = \frac{1}{a_i^2}$ $(i = 1,2,3)$, $\alpha_i(x)$ as above and $f(u) \equiv e^u$.

The easiest cases to deal with are those for which $m_i = 1$ $(i = 1,2,3)$. In that case V_m is the spherical region $\{x: \sum_{i=1}^{3} x_i^2 < 1\}$, and so, in the notation of Ch.7, $\psi = \Psi = 1$. Further, θ and Θ are the extreme values of $\sum_{i=1}^{3} \alpha_i(x) x_i^2 = \sqrt{\sum_{i=1}^{3} \frac{x_i^2}{a_i^2}}$ on ∂V_m. It is easily shown, using Lagrange multiplier techniques, that $\theta = \frac{1}{\max\{a_i\}}$, $\Theta = \frac{1}{\min\{a_i\}}$. Thus, using formula (38), we obtain:

$$\underline{\lambda}^* = \frac{2\sum_{i=1}^{3} \frac{1}{a_i^2}}{e\left(1 + \frac{2}{\min\{a_i\}h}\right)} .$$

We now apply this to cases where V^* is one of the first three regions listed in Table 1.

1). Sphere, radius a:

Here $a_1 = a_2 = a_3 = a$, and so

$$\underline{\lambda}^* = \frac{\frac{6}{a^2}}{e\left(1 + \frac{2}{ah}\right)}$$

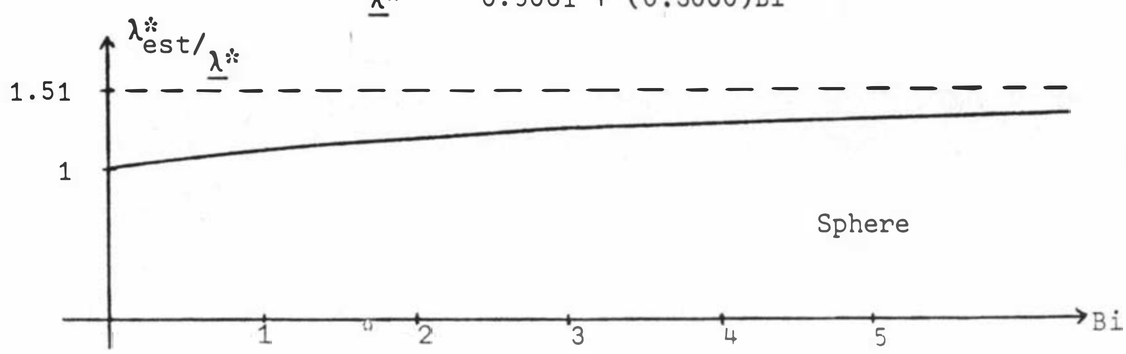
$$= \frac{6\,Bi}{a^2 e(2 + Bi)} \quad \text{where } Bi = ah \text{ is the Biot}$$

number for the sphere.

$$\text{i.e.}\quad \underline{\lambda}^* = \frac{1}{a^2}\left[\frac{Bi}{0.9061 + (0.4530)Bi}\right].$$

From Table 1:
$$\lambda^*_{est} = \frac{1}{a^2}\left[\frac{Bi}{0.9061 + (0.3000)Bi}\right].$$

$$\text{Thus}\quad \frac{\lambda^*_{est}}{\underline{\lambda}^*} = \frac{0.9061 + (0.4530)Bi}{0.9061 + (0.3000)Bi} .$$



2). <u>Infinite cylinder, radius a:</u>

Here we take $a_1 = a_2 = a$ and let $a_3 \to \infty$, and so

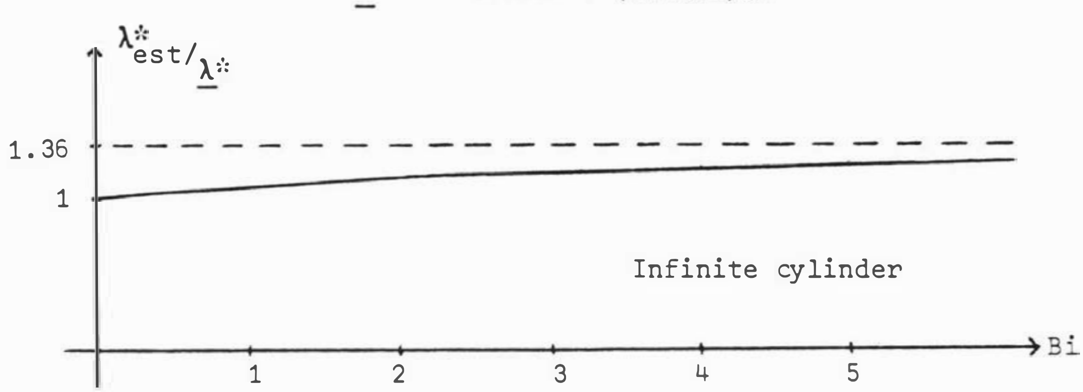
$$\underline{\lambda}^* = \frac{\frac{4}{a^2}}{e(1 + \frac{2}{ah})}$$

$$= \frac{4\,Bi}{a^2 e(3 + Bi)}$$ where $Bi = \frac{3ah}{2}$ is the Biot number for the infinite cylinder.

i.e. $$\underline{\lambda}^* = \frac{1}{a^2}\left[\frac{Bi}{2.0387 + (0.6796)Bi}\right].$$

From Table 1: $$\lambda^*_{est} = \frac{1}{a^2}\left[\frac{Bi}{2.0387 + (0.5000)Bi}\right].$$

Thus $$\frac{\lambda^*_{est}}{\underline{\lambda}^*} = \frac{2.0387 + (0.6796)Bi}{2.0387 + (0.5000)Bi}.$$



3). <u>Infinite slab, thickness 2a:</u>

Here we take $a_1 = a$ and let $a_2 \to \infty$, $a_3 \to \infty$, and so

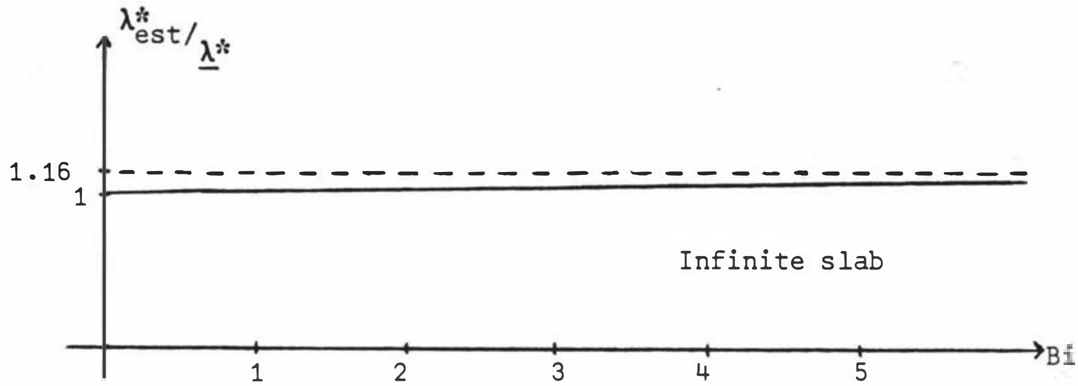
$$\underline{\lambda}^* = \frac{\frac{2}{a^2}}{e(1 + \frac{2}{ah})}$$

$$= \frac{2\,Bi}{a^2 e(6 + Bi)}$$ where $Bi = 3ah$ is the Biot number for the infinite slab.

i.e. $$\underline{\lambda}^* = \frac{1}{a^2}\left[\frac{Bi}{8.1548 + (1.3591)Bi}\right].$$

From Table 1: $$\lambda^*_{est} = \frac{1}{a^2}\left[\frac{Bi}{8.1548 + (1.1669)Bi}\right].$$

Thus $$\frac{\lambda^*_{est}}{\underline{\lambda}^*} = \frac{8.1548 + (1.3591)Bi}{8.1548 + (1.1669)Bi}.$$



The next cases we wish to consider are those where V^* is one of the three finite cylindrical regions. To obtain these we take $a_2 = a_1$, $m_1 = m_2 = 1$ and let $m_3 \to \infty$. In the limit, as $m_3 \to \infty$, V_m becomes the right circular cylindrical region $\{x: x_1^2 + x_2^2 < 1, |x_3| < 1\}$. Thus $\psi = 1$, $\Psi = 2$. Further, on ∂V_m:

$$\sum_{i=1}^{3} \alpha_i(x) x_i^{2m_i} = \frac{\dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_1^2} + \dfrac{m_3\, x_3^{2m_3}}{a_3^2}}{\sqrt{\dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_1^2} + \dfrac{m_3^2\, x_3^{4m_3-2}}{a_3^2}}} .$$

So in the limit, as $m_3 \to \infty$, we have:

$$\sum_{i=1}^{3} \alpha_i(x) x_i^{2m_i} = \frac{1}{a_1} \quad \text{where } x_1^2 + x_2^2 = 1,\ |x_3| < 1$$

$$= \frac{1}{a_3} \quad \text{where } |x_3| = 1.$$

Thus $\theta = \min\{\frac{1}{a_1}, \frac{1}{a_3}\} = \frac{1}{\max\{a_1, a_3\}}$ and similarly $\Theta = \frac{1}{\min\{a_1, a_3\}}$.

So, using formula (38), we obtain:

$$\underline{\lambda}^* = \frac{2\left(\dfrac{2}{a_1^2} + \dfrac{1}{a_3^2}\right)}{e\left(2 + \dfrac{2}{\min\{a_1, a_3\} h}\right)} .$$

We now apply this to the three finite cylindrical regions listed in Table 1.

4). Equicylinder, height 2a, radius a:

Here $a_3 = a_1 = a$, and so

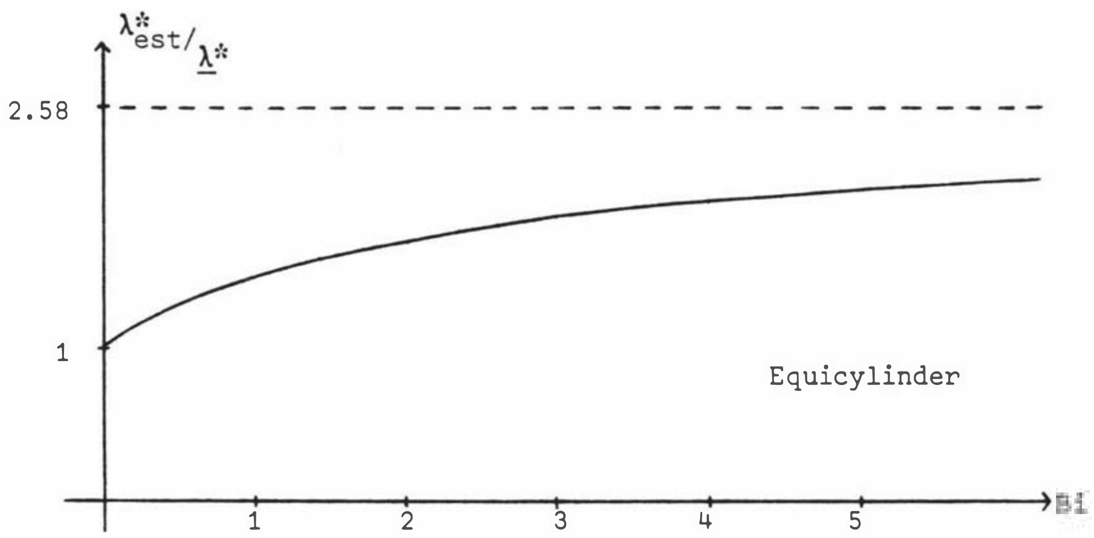
$$\underline{\lambda}^* = \frac{\frac{6}{a^2}}{e(2 + \frac{2}{ah})}$$

$$= \frac{3\,Bi}{a^2 e(1 + Bi)}$$ where $Bi = ah$ is the Biot number for the equicylinder.

$$\text{i.e.} \quad \underline{\lambda}^* = \frac{1}{a^2}\left[\frac{Bi}{0.9061 + (0.9061)Bi}\right].$$

From Table 1: $$\lambda^*_{est} = \frac{1}{a^2}\left[\frac{Bi}{0.9061 + (0.3516)Bi}\right].$$

$$\text{Thus} \quad \frac{\lambda^*_{est}}{\underline{\lambda}^*} = \frac{0.9061 + (0.9061)Bi}{0.9061 + (0.3516)Bi}.$$



5). <u>Thin circular disc, thickness 2a, radius 10a</u>:

Here $a_1 = 10a$, $a_3 = a$, and so

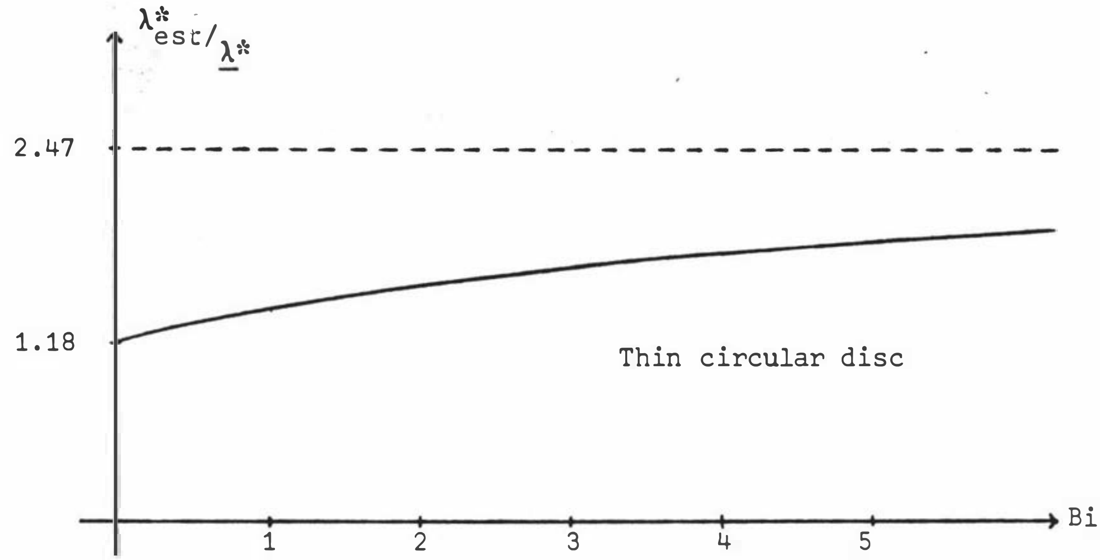
$$\underline{\lambda}^* = \frac{2(\frac{2}{100a^2} + \frac{1}{a^2})}{e(2 + \frac{2}{ah})}$$

$$= \frac{(2.04)Bi}{a^2 e(5 + 2\,Bi)}$$ where $Bi = \frac{5ah}{2}$ is the Biot number for the thin circular disc.

$$\text{i.e.} \quad \underline{\lambda}^* = \frac{1}{a^2}\left[\frac{Bi}{6.6625 + (2.6650)Bi}\right].$$

From Table 1: $$\lambda^*_{est} = \frac{1}{a^2}\left[\frac{Bi}{5.6636 + (1.0797)Bi}\right].$$

$$\text{Thus} \quad \frac{\lambda^*_{est}}{\underline{\lambda}^*} = \frac{6.6625 + (2.6650)Bi}{5.6636 + (1.0797)Bi}.$$



6). Long circular cylinder, height 10a, radius a:

Here $a_1 = a$, $a_3 = 5a$, and so

$$\underline{\lambda}^* = \frac{2(\frac{2}{a^2} + \frac{1}{25a^2})}{e(2 + \frac{2}{ah})}$$

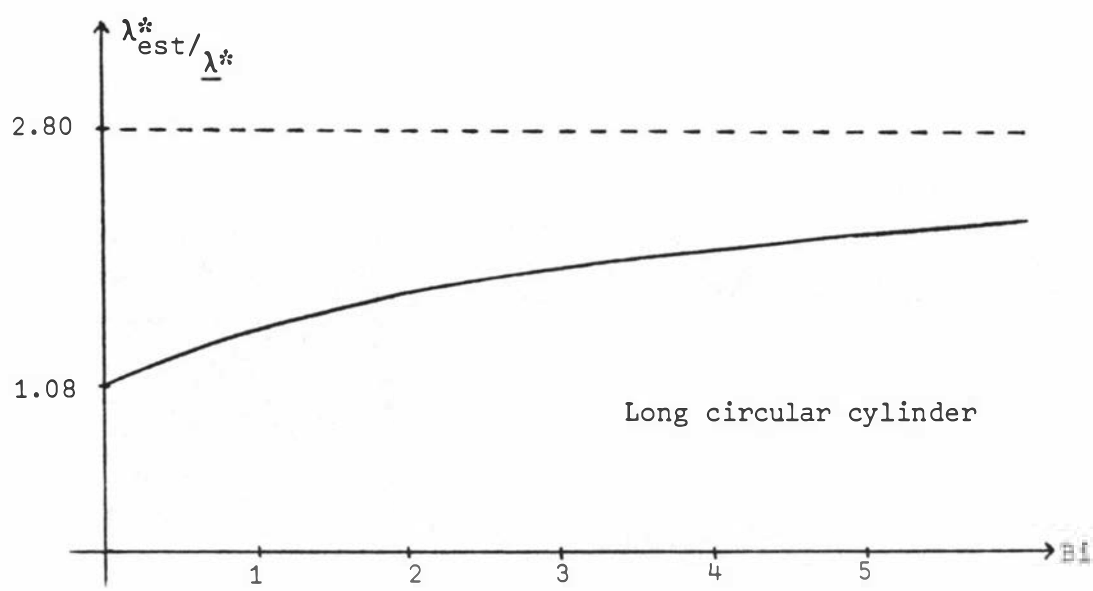
$$= \frac{(22.44)Bi}{a^2 e(15 + 11\,Bi)} \quad \text{where } Bi = \frac{15ah}{11} \text{ is the}$$

Biot number for the long circular cylinder.

$$\text{i.e.} \quad \underline{\lambda}^* = \frac{1}{a^2}\left[\frac{Bi}{1.8170 + (1.3325)Bi}\right].$$

From Table 1: $$\lambda^*_{est} = \frac{1}{a^2}\left[\frac{Bi}{1.6853 + (0.4759)Bi}\right].$$

$$\text{Thus} \quad \frac{\lambda^*_{est}}{\underline{\lambda}^*} = \frac{1.8170 + (1.3325)Bi}{1.6853 + (0.4759)Bi}.$$



Finally, we wish to consider the cases where V^* is one of the last two regions listed in Table 1, namely the cube and the infinite square rod. To obtain these we take $m_1 = m_2 = m_3 = m$ and let $m \to \infty$. The region V^* then tends to the rectangular prism with dimensions $2a_1 \times 2a_2 \times 2a_3$, and V_m tends to the cubical region $\{x: |x_i| < 1,\ i = 1,2,3\}$. Thus $\psi = 1$, $\Psi = 3$. Further, on ∂V_m:

$$\sum_{i=1}^{3} \alpha_i(x) x_i^{2m_i} = \frac{\displaystyle\sum_{i=1}^{3} \frac{x_i^{2m}}{a_i^2}}{\sqrt{\displaystyle\sum_{i=1}^{3} \frac{x_i^{4m-2}}{a_i^2}}}\ .$$

So in the limit, as $m \to \infty$, we have:

$$\sum_{i=1}^{3} \alpha_i(x) x_i^{2m_i} = \sqrt{\sum_{\substack{i \text{ such that} \\ |x_i| = 1}} \left(\frac{1}{a_i^2}\right)}\ .$$

Thus $\theta = \dfrac{1}{\max\{a_i\}}$ and $\Theta = \sqrt{\displaystyle\sum_{i=1}^{3} \frac{1}{a_i^2}}$. So, using formula (38), we obtain:

$$\underline{\lambda}^* = \frac{2\displaystyle\sum_{i=1}^{3} \frac{1}{a_i^2}}{e\left(3 + \dfrac{2}{h}\sqrt{\displaystyle\sum_{i=1}^{3} \frac{1}{a_i^2}}\right)}\ .$$

We now apply this to the last two regions listed in Table 1.

7). <u>Cube, side 2a</u>:

Here $a_1 = a_2 = a_3 = a$, and so

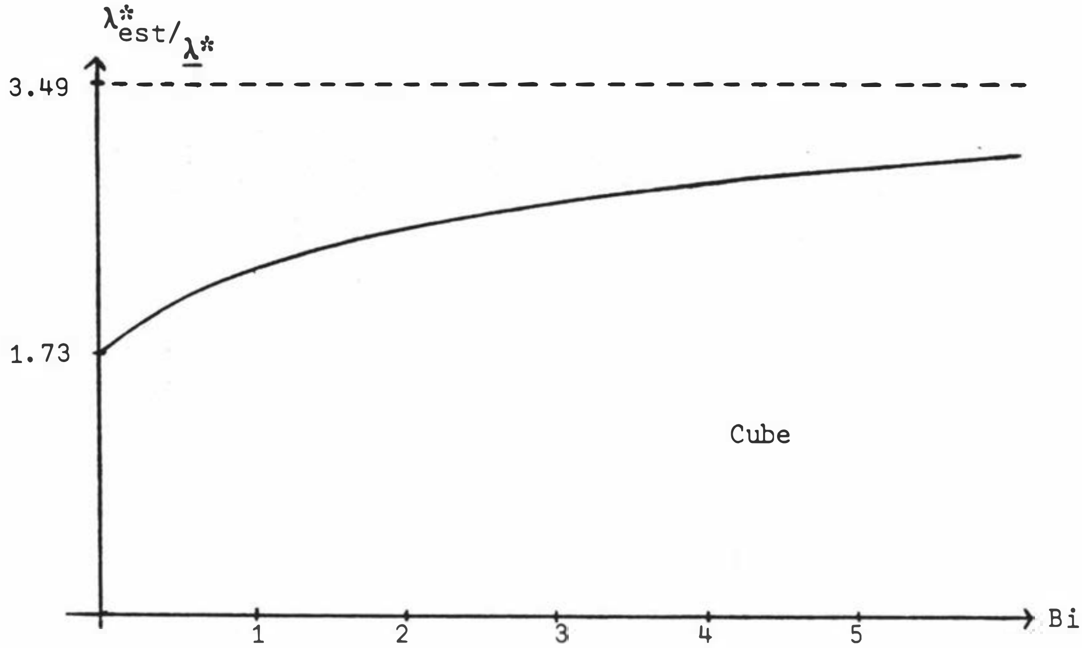
$$\underline{\lambda}^* = \frac{\dfrac{6}{a^2}}{e\left(3 + \dfrac{2\sqrt{3}}{ah}\right)}$$

$$= \frac{6\,Bi}{a^2 e(2\sqrt{3} + 3\,Bi)}$$ where $Bi = ah$ is the Biot number for the cube.

$$\text{i.e. } \underline{\lambda}^* = \frac{1}{a^2}\left[\frac{Bi}{1.5694 + (1.3591)Bi}\right].$$

From Table 1: $$\lambda^*_{est} = \frac{1}{a^2}\left[\frac{Bi}{0.9061 + (0.3892)Bi}\right].$$

$$\text{Thus } \frac{\lambda^*_{est}}{\underline{\lambda}^*} = \frac{1.5694 + (1.3591)Bi}{0.9061 + (0.3892)Bi}\ .$$



8). <u>Infinite square rod, side 2a:</u>

Here we take $a_1 = a_2 = a$ and let $a_3 \to \infty$, and so

$$\underline{\lambda^*} = \frac{\frac{4}{a^2}}{e(3 + \frac{2\sqrt{2}}{ah})}$$

$$= \frac{4\,Bi}{3a^2 e(\sqrt{2} + Bi)} \quad \text{where } Bi = \frac{3ah}{2} \text{ is}$$

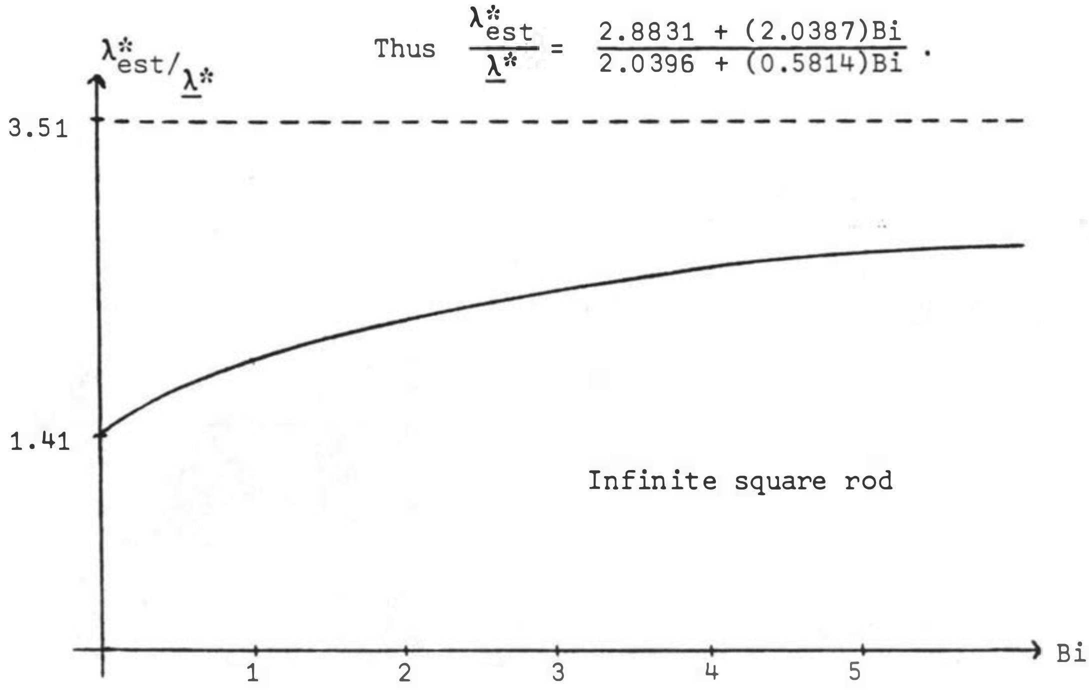
the Biot number for the infinite square rod.

$$\text{i.e.} \quad \underline{\lambda^*} = \frac{1}{a^2}\left[\frac{Bi}{2.8831 + (2.0387)Bi}\right].$$

From Table 1:

$$\lambda^*_{est} = \frac{1}{a^2}\left[\frac{Bi}{2.0396 + (0.5814)Bi}\right].$$

$$\text{Thus} \quad \frac{\lambda^*_{est}}{\underline{\lambda^*}} = \frac{2.8831 + (2.0387)Bi}{2.0396 + (0.5814)Bi}\ .$$



From the foregoing calculations, it can be seen that in the cases considered, the ratio $\frac{\lambda^*_{est}}{\underline{\lambda}^*}$ lies between 1 and about $\frac{7}{2}$ in all cases, and is much less than this in some cases. It is clear that in general $\underline{\lambda}^*$ cannot be regarded as in any sense an approximation to λ^*_{est}; nevertheless it may be of interest as being a rigorous lower bound for λ^* which is not too remote from the true value, particularly for small values of the Biot number and certain types of region.

The upper bound given by (39), though of theoretical interest since it shows that λ^* is finite, is not as useful for estimation purposes as the lower bound $\underline{\lambda}^*$ already discussed. We can illustrate this by considering the case where V^* is a sphere of radius a (see p.94). In this case $\psi = \Psi = 1$, $\theta = \frac{1}{a}$ and $A_1 = A_2 = A_3 = \frac{1}{a^2}$, and so we obtain:

$$\overline{\lambda}^* = \frac{\frac{3}{a^2}\left[\frac{2}{a^2h^2} + 1\right]}{\frac{4}{a^2h^2}}$$

$$= \frac{3}{4a^2}\left[2 + (Bi)^2\right] \quad \text{where } Bi = ah \text{ is the Biot}$$

number for the sphere. Thus, using the appropriate value of λ^*_{est} from Table 1, we obtain:

$$\frac{\overline{\lambda}^*}{\lambda^*_{est}} = \frac{3[2 + (Bi)^2][0.9061 + (0.3)Bi]}{4\ Bi}$$

$$= \frac{3}{4}\left[\frac{1.8122}{Bi} + 0.6 + (0.9061)Bi + (0.3)(Bi)^2\right]$$

$$= \frac{3}{4}\, g(Bi), \text{ say.}$$

Evidently $g(Bi) \to \infty$ as $Bi \to 0+$ or as $Bi \to \infty$. Further:

$$g'(Bi) = -\frac{1.8122}{(Bi)^2} + 0.9061 + (0.6)Bi$$

$= 0$ when $Bi = 1.0799$ (to four places), the solution being obtained using the Newton-Raphson method. Thus $g(Bi)$ attains its minimum for $Bi > 0$ when $Bi = 1.0799$, whence $g(Bi) \geq 3.6$ for all $Bi > 0$. Hence $\frac{\overline{\lambda}^*}{\lambda^*_{est}} \geq 2.7$ for all $Bi > 0$. Thus the upper bound $\overline{\lambda}^*$ is typically very much larger than λ^*, which is why it is of theoretical rather than practical interest.

9 NON-LINEAR BOUNDARY CONDITIONS

In this final chapter, we shall examine to what extent the methods of Chs.7 and 8 can be adapted to non-linear boundary conditions. We shall begin by working with the initial-boundary value problem

$$\left.\begin{aligned} Lu - \frac{\partial u}{\partial t} + \lambda f(x,t,u) = 0 \quad & \text{for } x \in V_m,\ 0 < t \le T \\ B_{gen}u = 0 \quad & \text{for } x \in \partial V_m,\ 0 < t \le T \\ u(x,0) = u_0(x) \quad & \text{for } x \in \overline{V}_m \end{aligned}\right\} \qquad (41)$$

where, as in the problem (15) discussed in Ch.7, f is continuous for $x \in \overline{V}_m$, $0 \le t \le T$ and all u, $u_0 \in C^{2+\alpha}(\overline{V}_m)$ and λ is taken to be positive. Recall that $B_{gen}u = d_0(x,t)g(u) + d_1(x,t)\,\frac{\partial u}{\partial n}$, where g(u) is strictly increasing for all u (for those functions g which occur in physical problems, we are only concerned with $u \ge 0$, and we may extend the definition of g(u) to negative u so as to satisfy this condition without loss of applicability). As in the case of problem (15) in Ch.7, we shall suppose that the derivative $\frac{\partial u}{\partial n}$ appearing in $B_{gen}u$ is of the form $\frac{\partial u}{\partial n} = \sum_{i=1}^{n} \alpha_i(x)x_i^{2m_i-1}\,\frac{\partial u}{\partial x_i}$, and shall likewise follow in other respects the notation used in studying problem (15) (see p.53).

If we assume further that $g(0) \le 0$ (which is certainly true in physical applications) then Theorem 11 extends at once to problem (41). Theorems 17 to 20, on the construction of upper solutions, and also Theorem 21 on reactant consumption, also extend at once to problem (41) if we assume that $g(u) \ge u$ for all $u \ge 0$; however, this is certainly not true of the function $g(u) = u^{5/4}$ which occurs when cooling at the boundary is by natural convection. However, a slightly different condition on g takes account of the case $g(u) = u^{5/4}$ and still allows us to extend the important Theorems 17, 18 and 20, where the upper solution constructed is independent of t. Theorems 19 and 21 have so far proved impossible to extend using this condition on g, because the fact that the upper solution tends to zero as $t \to \infty$ creates technical difficulties of an apparently insuperable nature in the construction of the proof. However, extending Theorems 17, 18 and 20 leads to the following theorems.

THEOREM 17^B: *Suppose that*

(a) *As for Theorem 17 except that we suppose* $c(x,t) \le 0$ *for all* $x \in V_m$, $t > 0$.

(b),(c) *As for Theorem 17.*

(d) $\sum_{i=1}^{n} B_i < \sum_{i=1}^{n} A_i$.

(e) *As for Theorem 17.*

(f) *There exist constants* $N > 0$ *and* $p > 1$ *such that* $g(u) \ge Nu^p$ *for all* $u \ge 0$.

Then: (i) *For any* $T > 0$, *a strict upper solution for* (41) *is given by*

$$w(x,t) = \frac{\epsilon}{A-\Psi}\left(A - \sum_{i=1}^{n} x_i^2\right) \text{ for all } x \in \overline{V}_m,\ 0 \le t \le T$$

if A *is a constant chosen so as to satisfy*

$$A > \Psi + \frac{2D_1\Theta}{N\delta_0\epsilon^{p-1}} \qquad \text{....................(42)}$$

and if $\qquad 0 < \lambda < m_0(\epsilon,A)$

where $m_0(\epsilon,A)$ *does not depend on* T.

(ii) *If* $0 < \lambda < m_0(\epsilon,A)$, *and if* $u(x,t)$ *is a solution of* (41), *then for all* $T > 0$ *and* $x \in \overline{V}_m$, $u(x,T) \le \frac{\epsilon A}{A-\Psi}$.

Proof: (I) As for Theorem 17.

(II) $$B_{gen}w = d_0(x,t)g(w) + d_1(x,t)\sum_{i=1}^{n}\alpha_i(x)x_i^{2m_i-1}\frac{\partial w}{\partial x_i}$$

$$\ge d_0(x,t)N\left(\frac{\epsilon}{A-\Psi}\right)^p\left(A-\sum_{i=1}^{n}x_i^2\right)^p + d_1(x,t)\sum_{i=1}^{n}\alpha_i(x)x_i^{2m_i-1}\left(\frac{-2\epsilon x_i}{A-\Psi}\right)$$

$$\ge \delta_0 N\epsilon^p - \frac{2D_1\epsilon\Theta}{A-\Psi} \quad \text{for } t > 0 \text{ and } x \in \partial V_m$$

> 0 for all $t > 0$ by (42).

(III) As for the case of Theorem 17 where $C = 0$.

THEOREM 18^B: *This is to Theorem* 17^B *as Theorem 18 is to Theorem 17. The conclusion of the theorem is:*

If $u(x,t)$ *is a solution of* (41), *and if* λ *satisfies*

$$0 < \lambda < \frac{2\sum_{i=1}^{n}A_i - 2\sum_{i=1}^{n}B_i}{M_1\Psi}$$

then there exists a constant $K > 0$ *such that, for all* $T > 0$ *and* $x \in \bar{V}_m$, $u(x,t) \le K$, *where* K *depends on* λ.
COROLLARY: *Goes through exactly as it does for Theorem 18.*

THEOREM 20^B: *This is to Theorem* 17^B *as Theorem 20 is to Theorem 17. The conclusion of the theorem is analogous to that of Theorem 20 in the case* $C = 0$, *the only change being that the condition that* A *must satisfy becomes*

$$A > \Psi + \left[\frac{2D_1\Theta}{\delta_0 N K^{p-1}\lambda^{\frac{p-1}{\alpha}} M_2^{\frac{p-1}{\alpha}}}\right]^{1/p} \qquad \ldots\ldots\ldots\ldots.(43)$$

Proof: Similar to the case $C = 0$ of Theorem 20. (II) is slightly modified:

$$B_{gen}w = d_0(x,t)g(w) + d_1(x,t)\sum_{i=1}^{n}\alpha_i(x)x_i^{2m_i-1}\frac{\partial w}{\partial x_i}$$

$$\ge \delta_0 N K^p \lambda^{p/\alpha} M_2^{p/\alpha}(A-\Psi)^p - 2D_1K\lambda^{1/\alpha}M_2^{1/\alpha}\Theta \quad \text{for } t > 0 \text{ and } x \in \partial V_m$$

$$> 0 \quad \text{by (43).}$$

Extending the "lower solution" theorems, Theorems 12 to 16, to problem (41) poses far more problems. Certainly the extension is immediate if we assume $g(u) \le u$ for all $u \ge 0$, but unfortunately this condition is not satisfied by the non-linear functions g which arise in applications. If we require g to satisfy some more realistic condition, then we immediately run into technical difficulties unless $\psi = \Psi$, i.e. unless the region V_m is spherical. So let us take V_m to be the sphere $\{x: \sum_{i=1}^{n} x_i^2 < 1\}$, so that $m_i = 1$ for $i = 1,2,\ldots n$, and $\psi = \Psi = 1$. In that case we obtain the following extension of Theorem 12.

THEOREM 12^B: *Hypotheses* (a), (b), (c) *and* (d) *are as for Theorem 12, except that the condition* $\Psi < \psi + \frac{2\delta_1\theta}{D_0}$ *is omitted in* (c). *In addition, we suppose that*

(e) V_m *is the sphere* $\{x: \sum_{i=1}^{n} x_i^2 < 1\}$.

(f) *There exists a constant* $Q > 0$ *such that* $g(u) \le Que^u$ *for all* $u \ge 0$ *(this condition is satisfied by the functions* g *which arise in applications).*

Then: (i) *For any* $T > 0$, *a lower solution for* (41) *is given by*

$$w(x,t) = \lambda K(A - \sum_{i=1}^{n} x_i^2)(1 - e^{-t}) \text{ for all } x \in \overline{V}_m,\ 0 \le t \le T$$

where A *and* K *are constants chosen so as to satisfy*

$$A > 1$$

$$(A-1)e^{\lambda K(A-1)} < \frac{2\delta_1\theta}{QD_0} \qquad \dots\dots\dots\dots\dots\dots\dots(44)$$

$$0 < K < \frac{M}{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + (C+1)A}$$

(note that by choosing A *sufficiently close to* 1, (44) *can always be satisfied; the choice of* A *will depend upon the value of* λ*, but this is of no consequence for this theorem).*

(ii) *If* $u(x,t)$ *is a solution of* (41), *then for any* $T > 0$, $u(x,t) > 0$ *for* $0 < t \le T$, *and if* $\lim_{T\to\infty} u(x,T) = \hat{u}(x)$ *exists, then for all* $x \in \overline{V}_m$,

$$\hat{u}(x) \ge \frac{\lambda M\alpha}{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + (C+1)(1+\alpha)}$$ *where* α *is the solution of the equation* $\alpha e^{\lambda K\alpha} = \dfrac{2\delta_1\theta}{QD_0}$.

Proof: (I) As for Theorem 12.

(II) $$B_{gen}w = d_0(x,t)g(w) + d_1(x,t)\sum_{i=1}^{n}\alpha_i(x)x_i\frac{\partial w}{\partial x_i}$$

$$\le d_0(x,t)Q\lambda K(A-\sum_{i=1}^{n}x_i^2)(1-e^{-t})\exp\{\lambda K(A-\sum_{i=1}^{n}x_i^2)(1-e^{-t})\}$$

$$+ d_1(x,t)\sum_{i=1}^{n}\alpha_i(x)x_i\{\lambda K(-2x_i)(1-e^{-t})\}$$

$$\le \lambda K(1-e^{-t})\{D_0Q(A-1)e^{\lambda K(A-1)} - 2\delta_1\theta\} \quad \text{for } t > 0 \text{ and } x \in \partial V_m$$

< 0 for all $t > 0$ by (44).

(III) As for Theorem 12.

This proves part (i). Part (ii) follows as in the proof of Theorem 12, if we observe that A-1 may be chosen arbitrarily close to α.

Unfortunately, theorems parallel to Theorems 13 to 16 cannot be obtained by this method, because in each case a condition on λ is obtained which involves the value of A, which in turn depends on λ, and so a vicious circle results.

While the preceding theorems are rather incomplete, they do at least show that certain qualitative aspects of the behaviour of solutions of (15) still hold for (41). It may well be that in other respects there are qualitative changes in behaviour when we change from a linear to a non-linear boundary condition.

The extension to more general domains V which was carried out, in the case of the linear boundary condition, in Theorems 12^A to 20^A, has not proved possible so far in the case of the non-linear boundary condition. The difficulty is the unavailability of a theory for non-linear operators comparable to the theory for linear operators which was used in proving Theorems 12^A to 20^A.

In Ch.8 we used Theorem 17 to obtain a lower bound $\underline{\lambda}^*$ for the critical value λ^* in the heat-generation problem, which turned out to be quite close to λ^* in certain cases where a good approximation to the value of λ^* was known. It is of interest to see whether we can similarly use Theorem 17^B to obtain a lower bound for λ^* in the case of certain non-linear boundary conditions, and if so, what information we can deduce about the size of λ^* in the case of the non-linear boundary conditions, compared to its size in the case of the linear boundary condition.

We shall consider a modification of problem (37) on p.88, the modifications consisting of the introduction of a non-linear boundary condition and the assumption that $f(u) \equiv e^u$, giving the following problem:

$$\left.\begin{aligned} \sum_{i=1}^{3} A_i \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} + \lambda e^u = 0 \quad & \text{for } x \in V_m,\ t > 0 \\ g(u) + K \sum_{i=1}^{3} \alpha_i(x) x_i^{2m_i - 1} \frac{\partial u}{\partial x_i} = 0 \quad & \text{for } x \in \partial V_m,\ t > 0 \\ u(x,0) = 0 \quad & \text{for } x \in \overline{V}_m. \end{aligned}\right\} \qquad (45)$$

We shall consider particularly two possibilities for g(u):

(a) <u>The natural convection boundary condition</u>:

In this case $g(u) = Hu^{5/4}$, so in the notation of Theorem 17^B we have $N = H$, $p = \frac{5}{4}$, $\delta_0 = 1$.

(b) <u>The thermal radiation boundary condition</u>:

In this case $g(u) = \sigma\varepsilon[(u+T_a)^4 - T_a^4] \geq \sigma\varepsilon u^4$ for all $u \geq 0$, so we may take $N = \sigma\varepsilon$, $p = 4$, $\delta_0 = 1$ (where here and here only, ε denotes the emissivity of the surface).

Reasoning exactly as for the linear boundary condition on p.89, we obtain from Theorem 17[B] that a lower bound for λ^* (more precisely, a number such that for all λ smaller than this, all solutions of (45) are bounded as $T \to \infty$) is given by $\dfrac{2\epsilon \sum_{i=1}^{3} A_i}{A-\Psi} e^{-\frac{\epsilon A}{A-\Psi}}$.

Now A may be arbitrarily chosen greater than $\Psi + \dfrac{2K\Theta}{N\epsilon^{p-1}}$, so if we put $\eta = A - (\Psi + \dfrac{2K\Theta}{N\epsilon^{p-1}})$, then both ϵ and η are arbitrary positive numbers. The expression for the lower bound on λ^* now becomes:

$$\frac{2\epsilon \sum_{i=1}^{3} A_i}{\eta + \frac{2K\Theta}{N\epsilon^{p-1}}} \exp\left\{- \frac{\epsilon(\eta + \Psi + \frac{2K\Theta}{N\epsilon^{p-1}})}{\eta + \frac{2K\Theta}{N\epsilon^{p-1}}}\right\} = \frac{K_1\epsilon}{\eta + K_2\epsilon^{1-p}} \exp\left\{-\epsilon\left(1 + \frac{\Psi}{\eta + K_2\epsilon^{1-p}}\right)\right\}$$

where $K_1 = 2\sum_{i=1}^{3} A_i$, $K_2 = \dfrac{2K\Theta}{N}$

$$= \beta(\epsilon,\eta), \text{ say.}$$

We define $\beta(\epsilon,\eta)$ for $\epsilon = 0$ or $\eta = 0$ so that the function β is continuous on the set $\{(\epsilon,\eta): \epsilon,\eta \geq 0\}$; it is easily verified that this is possible.

We wish to find the best possible lower bound for λ^* that Theorem 17[B] will give us, so the next step is to find, if possible, a point (ϵ,η) with $\epsilon,\eta \geq 0$ which maximises $\beta(\epsilon,\eta)$.

$$\text{Now} \quad \beta_\epsilon(\epsilon,\eta) = \frac{K_1\epsilon}{\eta + K_2\epsilon^{1-p}}\left\{- 1 - \frac{\Psi\eta + p\Psi K_2\epsilon^{1-p}}{(\eta + K_2\epsilon^{1-p})^2}\right\} \exp\left\{-\epsilon\left(1 + \frac{\Psi}{\eta + K_2\epsilon^{1-p}}\right)\right\}$$

$$+ \frac{K_1\eta + pK_1K_2\epsilon^{1-p}}{(\eta + K_2\epsilon^{1-p})^2} \exp\left\{-\epsilon\left(1 + \frac{\Psi}{\eta + K_2\epsilon^{1-p}}\right)\right\}$$

$$= \frac{\exp\left\{-\epsilon\left(1 + \frac{\Psi}{\eta + K_2\epsilon^{1-p}}\right)\right\}}{(\eta + K_2\epsilon^{1-p})^3} \{K_1\epsilon[-(\eta + K_2\epsilon^{1-p})^2 - (\Psi\eta + p\Psi K_2\epsilon^{1-p})] + (\eta + K_2\epsilon^{1-p})(K_1\eta + pK_1K_2\epsilon^{1-p})\}$$

= 0 if and only if

$$\epsilon\{-K_1\eta^2 - K_1\Psi\eta\} + \epsilon^{2-p}\{-2\eta K_1K_2 - pK_1K_2\Psi\} + \epsilon^{3-2p}\{-K_1K_2^2\} + \epsilon^{1-p}\{p\eta K_1K_2 + \eta K_1K_2\} + \epsilon^{2-2p}\{pK_1K_2^2\} + K_1\eta^2 = 0 \qquad \text{.....(46)}$$

$$\text{Also, } \beta_\eta(\epsilon,\eta) = \frac{-K_1\epsilon}{(\eta + K_2\epsilon^{1-p})^2} \exp\left\{-\epsilon\left(1 + \frac{\Psi}{\eta + K_2\epsilon^{1-p}}\right)\right\}$$

$$+ \frac{K_1\epsilon}{\eta + K_2\epsilon^{1-p}} \left\{\frac{\epsilon\Psi}{(\eta + K_2\epsilon^{1-p})^2}\right\} \exp\left\{-\epsilon\left(1 + \frac{\Psi}{\eta + K_2\epsilon^{1-p}}\right)\right\}$$

$$= \frac{K_1\epsilon\exp\left\{-\epsilon\left(1 + \frac{\Psi}{\eta + K_2\epsilon^{1-p}}\right)\right\}}{(\eta + K_2\epsilon^{1-p})^3} \{-\eta - K_2\epsilon^{1-p} + \epsilon\Psi\}$$

$$= 0 \text{ if and only if } \eta = \epsilon\Psi - K_2\epsilon^{1-p} \qquad \ldots\ldots\ldots\ldots(47)$$

If (46) and (47) hold simultaneously, then substitution from (47) into (46) gives the equation $K_1\epsilon^3\Psi^2 = 0$, which yields $\epsilon = 0$. Now $\beta(\epsilon,\eta) \to 0$ as $\epsilon \to 0$, regardless of the value of η, so $\epsilon = 0$ clearly does not give the desired maximum for $\beta(\epsilon,\eta)$. For $\epsilon,\eta > 0$, we now know that the graph of $\beta(\epsilon,\eta)$ has no points where the tangent plane is horizontal. Further, it is easy to see that as $\sqrt{\epsilon^2 + \eta^2} \to \infty$ in the first quadrant, $\beta(\epsilon,\eta) \to 0$. Hence $\beta(\epsilon,\eta)$ must attain its maximum value in the first quadrant on the $\eta = 0$ axis. So our problem reduces to finding a positive value of ϵ which maximises the value of $\beta(\epsilon,0)$. Now, putting $\eta = 0$ in (46), we obtain:

$\beta_\epsilon(\epsilon,0) = 0$ if and only if

$$\epsilon^{2-p}(-pK_1K_2\Psi) + \epsilon^{3-2p}(-K_1K_2^2) + \epsilon^{2-2p}(pK_1K_2^2) = 0$$

$$\text{i.e. } -p\Psi\epsilon^p - K_2\epsilon + pK_2 = 0 \qquad \ldots\ldots\ldots\ldots\ldots(48)$$

If we write $h(\epsilon) = -p\Psi\epsilon^p - K_2\epsilon + pK_2$, then $h(0) = pK_2 > 0$ and $h'(\epsilon) = -p^2\Psi\epsilon^{p-1} - K_2 < 0$ for all $\epsilon > 0$, so h is strictly decreasing for $\epsilon > 0$. Since $h(\epsilon) < 0$ for ϵ sufficiently large, it follows that (48) has exactly one positive solution. We denote the unique positive solution of (48) by ϵ_p^*.

Since $\beta(\epsilon,0) \to 0$ as $\epsilon \to 0$ and as $\epsilon \to \infty$, it follows that the maximum value of $\beta(\epsilon,0)$ for $\epsilon > 0$ is attained when $\epsilon = \epsilon_p^*$; so we have finally that the desired first quadrant maximum value of $\beta(\epsilon,\eta)$ is

$$\beta(\epsilon_p^*,0) = \frac{K_1}{K_2}(\epsilon_p^*)^p \, e^{-\epsilon_p^*\{1 + \frac{\Psi}{K_2}(\epsilon_p^*)^{p-1}\}} \qquad \ldots\ldots\ldots(49)$$

Since η may be chosen arbitrarily close to 0, it follows that $\beta(\epsilon_p^*,0)$ is a lower bound for λ^* which is the largest that can be obtained from the data of Theorem 17^B.

It is of interest to compare $\beta(\epsilon_p^*,0)$ with the lower bound $\underline{\lambda}^*$ that was obtained in the case where the boundary condition was linear, and also with the empirical estimate λ^*_{est} of Boddington, Gray and Harvey[4], to see whether we can establish a difference between the value of λ^* for the linear boundary condition and the value of λ^* for the non-linear boundary condition. Before doing this, we discuss a special case which we shall use for illustrative purposes.

<u>Example: Sphere of radius 1:</u>

In order to give some feel for the numbers involved in the problem under discussion, we shall illustrate the discussion at each stage by taking the special case where V_m is a three-dimensional sphere of radius 1; we shall consider the two values of p that are of special practical significance, namely $p = \frac{5}{4}$ and $p = 4$.

As discussed on p.94, for a spherical region of radius 1, we have $\sum_{i=1}^{3} A_i = 3$, so $K_1 = 6$. Also $\Upsilon = \Theta = 1$, and so $K_2 = \frac{2K}{N}$. If we write $h = \frac{N}{K}$ by analogy with the linear case, then $K_2 = \frac{2}{h}$ and (48) reduces to:

$$- ph\epsilon^p - 2\epsilon + 2p = 0.$$

For the special values $p = \frac{5}{4}$ and $p = 4$, this becomes:

$$p = \frac{5}{4}: \qquad - 5h\epsilon^{5/4} - 8\epsilon + 10 = 0 \quad \ldots\ldots\ldots\ldots\ldots(50)$$

$$p = 4: \qquad - 2h\epsilon^4 - \epsilon + 4 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(51)$$

Also, in this case:

$$\beta(\epsilon_p^*,0) = 3h(\epsilon_p^*)^p \, e^{-\epsilon_p^*\{1 + \frac{h}{2}(\epsilon_p^*)^{p-1}\}}.$$

From p.94 we have finally that in this special case:

$$\underline{\lambda}^* = \frac{h}{0.9061 + (0.4530)h}, \quad \lambda^*_{est} = \frac{h}{0.9061 + (0.3000)h}.$$

<u>Relationship between $\beta(\epsilon_p^*,0)$ and $\underline{\lambda}^*$:</u>

We have from (38) on p.90 that $\underline{\lambda}^*$ for the linear boundary condition (if we take H = N so as to relate it to the non-linear boundary condition) is given by $\underline{\lambda}^* = \frac{K_1}{e(\Psi+K_2)}$. We now consider the ratio

$$r(K_2,p) = \frac{\beta(\epsilon_p^*,0)}{\underline{\lambda}^*} = \frac{\Psi + K_2}{K_2}(\epsilon_p^*)^p \, e^{-\epsilon_p^*\{1 + \frac{\Psi}{K_2}(\epsilon_p^*)^{p-1}\} + 1}.$$

Since ϵ_p^* satisfies (48), we have:

$$\frac{\Psi(\epsilon_p^*)^p}{K_2} = 1 - \frac{\epsilon_p^*}{p} \qquad \text{........................(52)}$$

$$\therefore \quad r(K_2,p) = \frac{\Psi + K_2}{K_2}(\epsilon_p^*)^p \, e^{-\epsilon_p^*(1 - \frac{1}{p})}$$

$$= \frac{\Psi + K_2}{\Psi}(1 - \frac{\epsilon_p^*}{p})e^{-\epsilon_p^*(1 - \frac{1}{p})}.$$

Now it is easily seen from (48) that as $K_2 \to 0$ (i.e. as $Bi \to \infty$ where Bi is the Biot number, since Bi is proportional to $\frac{1}{K_2}$), $\epsilon_p^* \to 0$, while as $K_2 \to \infty$ (i.e. as $Bi \to 0$), $\epsilon_p^* \to p$. We illustrate this in the case of the spherical region of radius 1 by giving the values of $\epsilon_{5/4}^*$ and ϵ_4^* for various values of h, where in this case $h = Bi = \frac{2}{K_2}$. The equations involved were solved using the Newton-Raphson method.

h(= Bi)	$\epsilon_{5/4}^*$ (equation (50))	ϵ_4^* (equation (51))
0.00001	not calculated	3.994906
0.0001	1.249917	3.951251
0.001	1.249175	3.646415
0.01	1.241807	2.789326
0.1	1.173651	1.817523
1	0.786836	1.097572
10	0.233816	0.640207
100	0.0425397	0.367118
1,000	0.00690082	0.208661
10,000	0.00109779	0.118034
100,000	0.000174091	0.0665939

It follows in general that, as $K_2 \to 0$, $r(K_2,p) \to r(0+,p) = 1$, while as

$K_2 \to \infty$, $r(K_2,p) \to r(\infty,p) = p^p\, e^{1-p}$. The value of $r(\infty,p)$ increases very rapidly with increasing p, as the following table illustrates:

p	$r(\infty,p) = p^p\, e^{1-p}$
5/4	1.029
2	1.472
4	12.745
8	1.530×10^4
16	5.643×10^{12}

We can thus assert that for large values of K_2, i.e. for small Biot number, the value of $\beta(\epsilon_p^*,0)$ is greater then the value of $\underline{\lambda}^*$, and that the ratio between them for small Biot number increases very rapidly as p increases, i.e. as the non-linearity becomes more pronounced.

On pp.94-100, we discussed in detail the relation between $\underline{\lambda}^*$ and the empirical λ^*_{est} for various regions, and determined in particular the limit of the ratio $\dfrac{\lambda^*_{est}}{\underline{\lambda}^*}$ as $Bi \to 0$, as shown in the next table.

Region	$\lim\limits_{Bi\to 0} \dfrac{\lambda^*_{est}}{\underline{\lambda}^*}$
(1) Sphere, radius a	1
(2) Infinite cylinder, radius a	1
(3) Infinite slab, thickness 2a	1
(4) Equicylinder, height 2a, radius a	1
(5) Thin circular disc, thickness 2a, radius 10a	1.18
(6) Long circular cylinder, height 10a, radius a	1.08
(7) Cube, side 2a	1.73
(8) Infinite square rod, side 2a	1.41

If we compare the value of $\lim\limits_{Bi\to 0} \dfrac{\lambda^*_{est}}{\underline{\lambda}^*}$ with that of the quantity $r(\infty,p) = \lim\limits_{Bi\to 0} \dfrac{\beta(\epsilon_p^*,0)}{\underline{\lambda}^*}$, we see that for regions 1, 2, 3 and 4, the value of $\beta(\epsilon_p^*,0)$ exceeds λ^*_{est} when the Biot number is small, both for $p = \frac{5}{4}$ and for $p = 4$, and when $p = 4$ the excess is quite large. This shows that, for small Biot number, the true value of λ^* in the case of a non-linear boundary condition exceeds the approximate estimate λ^*_{est} obtained for a linear boundary condition. The excess is certainly large when $p = 4$, but may be rather small when $p = \frac{5}{4}$.

For regions 5, 6, 7 and 8, the above remarks apply only to the case $p = 4$. When $p = \frac{5}{4}$, $\beta(\epsilon_p^*,0)$ does not exceed λ^*_{est} and so no firm conclusion can be drawn.

We refer several times above to the condition that the Biot number be "small". To give an idea of the size of Biot number for which the above remarks apply, we again illustrate by considering the case of the spherical region of radius 1, where Bi = h in our notation. In Tables 2 and 3 on p.113, we tabulate, for both $p = \frac{5}{4}$ and $p = 4$, values of $\beta(\epsilon_p^*,0)$, $\underline{\lambda}^*$, λ^*_{est}, $\dfrac{\beta(\epsilon_p^*,0)}{\underline{\lambda}^*}$ and $\dfrac{\beta(\epsilon_p^*,0)}{\lambda^*_{est}}$ for various values of h. From these tables we have:

(a) $\underline{p = \frac{5}{4}}$: In this case, the value of the Biot number h at which $\beta(\epsilon_p^*,0)$ starts to exceed λ^*_{est} is slightly less than 0.1.

(b) $\underline{p = 4}$: In this case, the value of the Biot number h at which $\beta(\epsilon_p^*,0)$ starts to exceed λ^*_{est} is slightly less than 1.

It appears from Tables 2 and 3 that as h decreases (i.e. as K_2 increases) the ratio $r(K_2,p) = \dfrac{\beta(\epsilon_p^*,0)}{\underline{\lambda}^*}$ decreases at first to a value rather less than 1, and then increases to its limiting value $r(\infty,p) = p^p e^{1-p}$ as $h \to 0$ (i.e. $K_2 \to \infty$). We shall show that this is the case in general.

We have that ϵ_p^* is defined implicitly as a function of K_2 by equation (48); if we differentiate (48) with respect to K_2, we obtain

$$\frac{d\epsilon_p^*}{dK_2} = \frac{p - \epsilon_p^*}{K_2 + p^2\Psi(\epsilon_p^*)^{p-1}} \,.$$

(continued on p.114)

h	$\underline{\lambda}^*$	λ^*_{est}	$\beta(\epsilon^*_p,0)$	$\frac{\beta(\epsilon^*_p,0)}{\underline{\lambda}^*}$	$\frac{\beta(\epsilon^*_p,0)}{\lambda^*_{est}}$
100,000	2.2075	3.3332	2.2074	1.000	0.662
10,000	2.2071	3.3323	2.2048	0.999	0.662
1,000	2.2031	3.3233	2.1922	0.995	0.660
100	2.1642	3.2356	2.1143	0.977	0.653
10	1.8396	2.5601	1.7126	0.931	0.669
1	0.7358	0.8291	0.6987	0.950	0.843
0.1	0.1051	0.1068	0.1066	1.014	0.998
0.01	0.01098	0.01100	0.01128	1.027	1.026
0.001	0.001103	0.001103	0.001135	1.029	1.029
0.0001	0.0001104	0.0001104	0.0001136	1.029	1.029

TABLE 2: Values of parameters for different values of h, with $p=\frac{5}{4}$.

h	$\underline{\lambda}^*$	λ^*_{est}	$\beta(\epsilon^*_p,0)$	$\frac{\beta(\epsilon^*_p,0)}{\underline{\lambda}^*}$	$\frac{\beta(\epsilon^*_p,0)}{\lambda^*_{est}}$
100,000	2.2075	3.3332	2.0648	0.935	0.620
10,000	2.2071	3.3323	1.9606	0.888	0.588
1,000	2.2031	3.3233	1.7891	0.812	0.538
100	2.1642	3.2356	1.5220	0.703	0.470
10	1.8396	2.5601	1.1470	0.624	0.448
1 .	0.7358	0.8291	0.7031	0.956	0.848
0.1	0.1051	0.1068	0.3081	2.932	2.885
0.01	0.01098	0.01100	0.08245	7.509	7.496
0.001	0.001103	0.001103	0.01268	11.496	11.496
0.0001	0.0001104	0.0001104	0.001389	12.582	12.582
0.00001	0.00001104	0.00001104	0.0001406	12.726	12.726

TABLE 3: Values of parameters for different values of h, with p=4.

(continued from p.112)

Since, by (52), ϵ_p^* cannot exceed p, it follows that $\frac{d\epsilon_p^*}{dK_2} > 0$ for all $K_2 > 0$, so ϵ_p^* increases steadily from 0 to p as K_2 goes from 0 to ∞.

Also, $r(K_2,p) = \frac{\Psi + K_2}{\Psi}(1 - \frac{\epsilon_p^*}{p})e^{-\epsilon_p^*(1 - \frac{1}{p})}$, and so:

$$r_{K_2}(K_2,p) = (1 + \frac{K_2}{\Psi})(1 - \frac{\epsilon_p^*}{p})(\frac{\epsilon_p^* - p}{K_2 + p^2\Psi(\epsilon_p^*)^{p-1}})(1 - \frac{1}{p})e^{-\epsilon_p^*(1 - \frac{1}{p})}$$

$$+ (1 + \frac{K_2}{\Psi})(\frac{\epsilon_p^* - p}{K_2 + p^2\Psi(\epsilon_p^*)^{p-1}})\frac{1}{p}e^{-\epsilon_p^*(1 - \frac{1}{p})} + \frac{1}{\Psi}(1 - \frac{\epsilon_p^*}{p})e^{-\epsilon_p^*(1 - \frac{1}{p})}$$

$$= \frac{(p - \epsilon_p^*)e^{-\epsilon_p^*(1 - \frac{1}{p})}}{(K_2 + p^2\Psi(\epsilon_p^*)^{p-1})p\Psi}\{(\Psi + K_2)(\epsilon_p^* - p)(1 - \frac{1}{p}) - (\Psi + K_2) + (K_2 + p^2\Psi(\epsilon_p^*)^{p-1})\}$$

$= \rho(K_2)A(K_2)$ where $\rho(K_2)$ is positive for all $K_2 > 0$, and:

$$A(K_2) = \Psi(\epsilon_p^* - p)(1 - \frac{1}{p}) + K_2(1 - p)(\frac{p - \epsilon_p^*}{p}) - \Psi + p^2\Psi(\epsilon_p^*)^{p-1}$$

$$= \Psi(\epsilon_p^* - p - \frac{\epsilon_p^*}{p}) + \Psi(1 - p)(\epsilon_p^*)^p + p^2\Psi(\epsilon_p^*)^{p-1} \quad \text{by (52).}$$

Now as $K_2 \to 0$, $\epsilon_p^* \to 0$, so $A(0+) = -\Psi p < 0$. Also, as $K_2 \to \infty$, $\epsilon_p^* \to p$, so $A(\infty) = -\Psi + \Psi(1-p)p^p + p^2\Psi p^{p-1} = \Psi(p^p-1) > 0$ since $p > 1$.

Further, $A'(K_2) = [\Psi(1 - \frac{1}{p}) + \Psi p(1-p)(\epsilon_p^*)^{p-1} + p^2(p-1)\Psi(\epsilon_p^*)^{p-2}]\frac{d\epsilon_p^*}{dK_2}$

$$= [\Psi(1 - \frac{1}{p}) + \Psi p(p-1)(\epsilon_p^*)^{p-2}(p-\epsilon_p^*)]\frac{d\epsilon_p^*}{dK_2}$$

> 0 for all $K_2 > 0$ since $p > 1$ and, for all $K_2 > 0$, $\epsilon_p^* < p$ and $\frac{d\epsilon_p^*}{dK_2} > 0$.

Thus, as K_2 goes from 0 to ∞, $A(K_2)$ increases steadily from $-\Psi p$ to $\Psi(p^p-1)$, so $r_{K_2}(K_2,p)$ increases steadily from a negative value to a positive value. Hence, as required, we have shown that as K_2 goes from

0 to ∞, $r(K_2,p) = \dfrac{\beta(\varepsilon_p^*,0)}{\lambda^*}$ initially decreases and then increases. Since the Biot number is proportional to $\frac{1}{K_2}$, it follows that as the Biot number increases from 0 to ∞, $r(K_2,p)$ initially decreases and then increases.

We have now shown that if the Biot number is small, the non-linear boundary condition gives a value of λ^* which is higher than the estimate λ^*_{est} of Boddington, Gray and Harvey[4] for the linear boundary condition, at least when $p = 4$. When $p = \frac{5}{4}$, this conclusion has been proved only for some of the regions considered by Boddington, Gray and Harvey. For larger Biot number, no firm conclusion can be drawn, since we have no means of knowing how close our lower bound $\beta(\varepsilon_p^*,0)$ is to the true value of λ^*.

However, one may speculate that the behaviour of λ^* for the non-linear boundary condition may perhaps correspond in a qualitative sense to the behaviour of $\beta(\varepsilon_p^*,0)$, i.e. if we write λ_p^* for the critical value of λ in the case of the non-linear boundary condition ($p > 1$) and λ_1^* for the critical value in the case of the linear boundary condition, then the ratio $\frac{\lambda_p^*}{\lambda_1^*}$ may decrease initially as the Biot number increases from near zero, then increase to a finite limiting value as the Biot number tends to infinity. One may also ask whether, for all $p > 1$, $\frac{\lambda_p^*}{\lambda_1^*} > 1$ for sufficiently small Biot number. From the evidence given in this chapter, this seems quite likely, but there is evidently plenty of scope for further research on the case of a non-linear boundary condition.

APPENDIX

We give here the details of the example discussed in Ch.2. First we need to calculate several Fourier series and inverse Laplace transforms.

LEMMA 1: *If* $f(x) = -f(2-x)$ *for* $1 < x \le 2$, *then*

$$\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \begin{cases} 2\int_0^1 f(x)\cos\frac{n\pi x}{2}\,dx & (n \text{ odd}) \\ 0 & (n \text{ even}). \end{cases}$$

Proof: $$\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 f(x)\cos\frac{n\pi x}{2}\,dx + \int_1^2 \{-f(2-x)\}\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 f(x)\cos\frac{n\pi x}{2}\,dx - \int_1^0 f(u)\cos\frac{n\pi(2-u)}{2}(-1)du$$

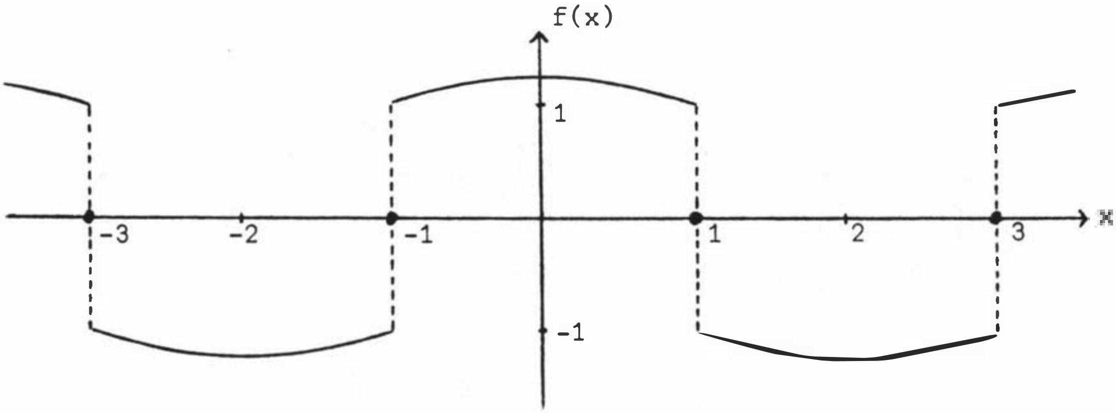
$$= \int_0^1 f(x)\cos\frac{n\pi x}{2}\,dx - \int_0^1 f(u)\cos(n\pi - \frac{n\pi u}{2})du.$$

The lemma now follows since $\cos(n\pi - \frac{n\pi u}{2}) = \cos\frac{n\pi u}{2}$ if n is even, and $\cos(n\pi - \frac{n\pi u}{2}) = -\cos\frac{n\pi u}{2}$ if n is odd.

LEMMA 2: *Suppose that* $k \ne \frac{(2n+1)^2\pi^2}{4}$ *for any integer* n. *Let* f *be defined by*

$$f(x) = \begin{cases} \frac{\cos\sqrt{k}x}{\cos\sqrt{k}} & (0 \le x < 1) \\ 0 & (x = 1) \\ -f(2-x) & (1 < x \le 2) \end{cases}$$

and let f *be an even function, and periodic of period* 4. *Thus* f *is discontinuous at* ±1, ±3, ±5, ... *as shown*:



Then the Fourier series of $f(x)$ *is* $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)\pi}{k - \frac{(2n+1)^2\pi^2}{4}} \cos\frac{(2n+1)\pi x}{2},$$

and this series converges to f(x) *for all real* x.

Proof: Since f is even and has period 4, it has a Fourier series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$ where:

$$a_n = \int_0^2 f(x)\cos \frac{n\pi x}{2}\, dx = \begin{cases} 2\int_0^1 f(x)\cos \frac{n\pi x}{2}\, dx & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases} \quad \text{by Lemma 1.}$$

Hence the Fourier series of f(x) is of the form $\sum_{n=0}^{\infty} a_{2n+1}\cos \frac{(2n+1)\pi x}{2}$,

$$\text{where:}\quad a_{2n+1} = 2\int_0^1 \frac{\cos\sqrt{k}x}{\cos\sqrt{k}} \cos \frac{(2n+1)\pi x}{2}\, dx$$

$$= \frac{2}{\cos\sqrt{k}} \int_0^1 \tfrac{1}{2}\{\cos[\sqrt{k} + \tfrac{(2n+1)\pi}{2}]x + \cos[\sqrt{k} - \tfrac{(2n+1)\pi}{2}]x\}dx$$

$$= \frac{1}{\cos\sqrt{k}} \left[\frac{\sin\{\sqrt{k} + \frac{(2n+1)\pi}{2}\}x}{\sqrt{k} + \frac{(2n+1)\pi}{2}} + \frac{\sin\{\sqrt{k} - \frac{(2n+1)\pi}{2}\}x}{\sqrt{k} - \frac{(2n+1)\pi}{2}} \right]_0^1$$

$$= \frac{1}{\cos\sqrt{k}} \left\{ \frac{\cos\sqrt{k}\,\sin\frac{(2n+1)\pi}{2}}{\sqrt{k} + \frac{(2n+1)\pi}{2}} + \frac{-\cos\sqrt{k}\,\sin\frac{(2n+1)\pi}{2}}{\sqrt{k} - \frac{(2n+1)\pi}{2}} \right\}$$

$$= \frac{-\frac{(2n+1)\pi}{2}\sin\frac{(2n+1)\pi}{2} - \frac{(2n+1)\pi}{2}\sin\frac{(2n+1)\pi}{2}}{k - \frac{(2n+1)^2\pi^2}{4}}$$

$$= \frac{(-1)^{n+1}(2n+1)\pi}{k - \frac{(2n+1)^2\pi^2}{4}} \quad \text{for } n = 0,1,2,\ldots\ , \text{ as required.}$$

The convergence of the Fourier series of f(x) to f(x) for all real x follows from a standard theorem on Fourier series.

LEMMA 3: *The hypotheses are as for Lemma 2 except that*

$$f(x) = \frac{\cos\sqrt{k}x}{\cos\sqrt{k}} - 1 \quad \text{for } 0 \le x < 1.$$

Then f *is continuous for all real* x, *the Fourier series of* f(x) *is*

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}4k}{\{k - \frac{(2n+1)^2\pi^2}{4}\}(2n+1)\pi} \cos \frac{(2n+1)\pi x}{2}, \text{ and this series converges}$$

to f(x) *for all real* x.

Proof: As in the proof of Lemma 2, the Fourier series of f(x) converges to f(x) for all real x, and has the form $\sum_{n=0}^{\infty} a_{2n+1}\cos\frac{(2n+1)\pi x}{2}$, where:

$$a_{2n+1} = 2\int_0^1 f(x)\cos\frac{(2n+1)\pi x}{2}\,dx$$

$$= 2\int_0^1 \frac{\cos\sqrt{k}x}{\cos\sqrt{k}}\cos\frac{(2n+1)\pi x}{2}\,dx - 2\int_0^1 \cos\frac{(2n+1)\pi x}{2}\,dx$$

$$= \frac{(-1)^{n+1}(2n+1)\pi}{k - \frac{(2n+1)^2\pi^2}{4}} - 2\left[\frac{\sin\frac{(2n+1)\pi x}{2}}{\frac{(2n+1)\pi}{2}}\right]_0^1 \quad \text{as in the proof of Lemma 2}$$

$$= \frac{(-1)^{n+1}(2n+1)\pi}{k - \frac{(2n+1)^2\pi^2}{4}} + \frac{(-1)^{n+1}4}{(2n+1)\pi}$$

$$= \frac{(-1)^{n+1}4k}{\{k - \frac{(2n+1)^2\pi^2}{4}\}(2n+1)\pi} \quad \text{for } n = 0,1,2,\ldots\text{, as required.}$$

LEMMA 4: *Let f be defined as in Lemma 2 except that*

$$f(x) = (-1)^N x\sin\frac{(2N+1)\pi x}{2} - \frac{(-1)^N}{(2N+1)\pi}\cos\frac{(2N+1)\pi x}{2} \quad \text{for } 0 \le x < 1$$

where N *is an arbitrary non-negative integer. Like the function discussed in Lemma 2,* f *is discontinuous at* ±1, ±3, ±5, ... *and its graph has a similar appearance to the graph of the function discussed in Lemma 2. Then the Fourier series of* f(x) *is*

$$\sum_{n\neq N} \frac{(-1)^{n+1}(2n+1)\pi}{\frac{(2N+1)^2\pi^2}{4} - \frac{(2n+1)^2\pi^2}{4}}\cos\frac{(2n+1)\pi x}{2}$$

and this series converges to f(x) *for all real* x.

Proof: As in the proof of Lemma 2, the Fourier series of f(x) converges to f(x) for all real x, and has the form $\sum_{n=0}^{\infty} a_{2n+1}\cos\frac{(2n+1)\pi x}{2}$, where:

$$a_{2n+1} = 2\int_0^1 f(x)\cos\frac{(2n+1)\pi x}{2}\,dx$$

$$= 2\int_0^1 (-1)^N x\sin\frac{(2N+1)\pi x}{2}\cos\frac{(2n+1)\pi x}{2}\,dx$$

$$- 2\int_0^1 \frac{(-1)^N}{(2N+1)\pi}\cos\frac{(2N+1)\pi x}{2}\cos\frac{(2n+1)\pi x}{2}\,dx.$$

For $n \neq N$, $\sin\frac{(2N+1)\pi x}{2}\cos\frac{(2n+1)\pi x}{2}$

$$= \frac{1}{2}\left[\sin\left\{\frac{(2N+1)\pi}{2} + \frac{(2n+1)\pi}{2}\right\}x + \sin\left\{\frac{(2N+1)\pi}{2} - \frac{(2n+1)\pi}{2}\right\}x\right].$$

Thus $2\int_0^1 x \sin\frac{(2N+1)\pi x}{2}\cos\frac{(2n+1)\pi x}{2}\,dx$

$$= \left[-\frac{x\cos\{\frac{(2N+1)\pi}{2}+\frac{(2n+1)\pi}{2}\}x}{\frac{(2N+1)\pi}{2}+\frac{(2n+1)\pi}{2}} + \frac{\sin\{\frac{(2N+1)\pi}{2}+\frac{(2n+1)\pi}{2}\}x}{\{\frac{(2N+1)\pi}{2}+\frac{(2n+1)\pi}{2}\}^2}\right.$$

$$\left. - \frac{x\cos\{\frac{(2N+1)\pi}{2}-\frac{(2n+1)\pi}{2}\}x}{\frac{(2N+1)\pi}{2}-\frac{(2n+1)\pi}{2}} + \frac{\sin\{\frac{(2N+1)\pi}{2}-\frac{(2n+1)\pi}{2}\}x}{\{\frac{(2N+1)\pi}{2}-\frac{(2n+1)\pi}{2}\}^2}\right]_0^1 .$$

Now $\frac{(2N+1)\pi}{2}+\frac{(2n+1)\pi}{2} = (n+N+1)\pi$ and $\frac{(2N+1)\pi}{2}-\frac{(2n+1)\pi}{2} = (N-n)\pi$, so:

$2\int_0^1 x \sin\frac{(2N+1)\pi x}{2}\cos\frac{(2n+1)\pi x}{2}\,dx$

$$= -\frac{(-1)^{n+N+1}}{\frac{(2N+1)\pi}{2}+\frac{(2n+1)\pi}{2}} - \frac{(-1)^{N-n}}{\frac{(2N+1)\pi}{2}-\frac{(2n+1)\pi}{2}}$$

$$= \frac{(-1)^{n+1+N}(2n+1)\pi}{\frac{(2N+1)^2\pi^2}{4}-\frac{(2n+1)^2\pi^2}{4}} .$$

Also $2\int_0^1 \cos\frac{(2N+1)\pi x}{2}\cos\frac{(2n+1)\pi x}{2}\,dx = 0.$

Hence, for $n \neq N$, $a_{2n+1} = \frac{(-1)^{n+1}(2n+1)\pi}{\frac{(2N+1)^2\pi^2}{4}-\frac{(2n+1)^2\pi^2}{4}}$, as required.

Also, $\sin\frac{(2N+1)\pi x}{2}\cos\frac{(2N+1)\pi x}{2} = \frac{1}{2}\sin(2N+1)\pi x$ and

$$\cos\frac{(2N+1)\pi x}{2}\cos\frac{(2N+1)\pi x}{2} = \frac{1}{2}\{1+\cos(2N+1)\pi x\}.$$

Thus $2\int_0^1 x \sin\frac{(2N+1)\pi x}{2}\cos\frac{(2N+1)\pi x}{2}\,dx$

$$= \left[-\frac{x\cos(2N+1)\pi x}{(2N+1)\pi} + \frac{\sin(2N+1)\pi x}{(2N+1)^2\pi^2}\right]_0^1 = \frac{1}{(2N+1)\pi} .$$

Also $2\int_0^1 \cos\frac{(2N+1)\pi x}{2}\cos\frac{(2N+1)\pi x}{2}\,dx = \left[x + \frac{\sin(2N+1)\pi x}{(2N+1)\pi}\right]_0^1 = 1.$

Hence $a_{2N+1} = \frac{(-1)^N}{(2N+1)\pi} - \frac{(-1)^N}{(2N+1)\pi} = 0$ as required.

LEMMA 5: *Let f be defined as in Lemma 2 except that*

$$f(x) = (-1)^N x\sin\frac{(2N+1)\pi x}{2} + \frac{3(-1)^N}{(2N+1)\pi}\cos\frac{(2N+1)\pi x}{2} - 1 \quad \text{for } 0 \le x < 1.$$

Then f is continuous for all real x, the Fourier series of f(x) is

$$\sum_{n\neq N} \frac{(-1)^{n+1}(2N+1)^2\pi}{\left\{\frac{(2N+1)^2\pi^2}{4} - \frac{(2n+1)^2\pi^2}{4}\right\}(2n+1)} \cos \frac{(2n+1)\pi x}{2}$$

and this series converges to f(x) *for all real* x.

Proof: As in the proof of Lemma 2, the Fourier series of f(x) converges to f(x) for all real x, and has the form $\sum_{n=0}^{\infty} a_{2n+1}\cos \frac{(2n+1)\pi x}{2}$, where:

$$a_{2n+1} = 2\int_0^1 f(x)\cos \frac{(2n+1)\pi x}{2}\,dx$$

$$= 2\int_0^1 (-1)^N x \sin \frac{(2N+1)\pi x}{2} \cos \frac{(2n+1)\pi x}{2}\,dx$$

$$+ \; 2\int_0^1 \frac{3(-1)^N}{(2N+1)\pi} \cos \frac{(2N+1)\pi x}{2} \cos \frac{(2n+1)\pi x}{2}\,dx \; - \; 2\int_0^1 \cos \frac{(2n+1)\pi x}{2}\,dx.$$

Using the integrals evaluated in the proof of Lemma 4, we have:

For $n \neq N$, $$a_{2n+1} = \frac{(-1)^{n+1}(2n+1)\pi}{\frac{(2N+1)^2\pi^2}{4} - \frac{(2n+1)^2\pi^2}{4}} - \frac{4(-1)^n}{(2n+1)\pi}$$

$$= \frac{(-1)^{n+1}(2N+1)^2\pi}{\left\{\frac{(2N+1)^2\pi^2}{4} - \frac{(2n+1)^2\pi^2}{4}\right\}(2n+1)} \quad \text{as required.}$$

Also, $a_{2N+1} = \frac{(-1)^N}{(2N+1)\pi} + \frac{3(-1)^N}{(2N+1)\pi} - \frac{4(-1)^N}{(2N+1)\pi} = 0$ as required.

LEMMA 6: *If* $k \neq \frac{(2n+1)^2\pi^2}{4}$ *for any* n = 0,1,2,.... , *then*

$$L^{-1}\left[\frac{\cosh x\sqrt{s-k}}{s \cosh\sqrt{s-k}}\right] = \pi \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)\cos \frac{(2n+1)\pi x}{2}}{k - \frac{(2n+1)^2\pi^2}{4}}\left\{e^{\left(k - \frac{(2n+1)^2\pi^2}{4}\right)t} - 1\right\}.$$

If $k = \frac{(2N+1)^2\pi^2}{4}$ *for some* N = 0,1,2,.... , *then*

$$L^{-1}\left[\frac{\cosh x\sqrt{s-k}}{s \cosh\sqrt{s-k}}\right] = \pi \sum_{n\neq N} \frac{(-1)^n(2n+1)\cos \frac{(2n+1)\pi x}{2}}{k - \frac{(2n+1)^2\pi^2}{4}}\left\{e^{\left(k - \frac{(2n+1)^2\pi^2}{4}\right)t} - 1\right\}$$

$$+ \pi(-1)^N(2N+1)t \cos \frac{(2N+1)\pi x}{2}.$$

Proof: From Spiegel's tables[34, p.252, entry 125],

$$L^{-1}\left[\frac{\cosh x\sqrt{s}}{\cosh \sqrt{s}}\right] = \pi \sum_{n=0}^{\infty} (-1)^n(2n+1)e^{-\frac{(2n+1)^2\pi^2 t}{4}} \cos \frac{(2n+1)\pi x}{2}.$$

$$\therefore \quad L^{-1}\left[\frac{\cosh x\sqrt{s-k}}{\cosh\sqrt{s-k}}\right] = \pi\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{kt - \frac{(2n+1)^2\pi^2 t}{4}}\cos\frac{(2n+1)\pi x}{2}.$$

$$\therefore \quad L^{-1}\left[\frac{\cosh x\sqrt{s-k}}{s\cosh\sqrt{s-k}}\right] = \pi\sum_{n=0}^{\infty}(-1)^n(2n+1)\cos\frac{(2n+1)\pi x}{2}\int_0^t e^{(k - \frac{(2n+1)^2\pi^2}{4})u}\,du.$$

Now if $k \neq \frac{(2n+1)^2\pi^2}{4}$, then $\int_0^t e^{(k - \frac{(2n+1)^2\pi^2}{4})u}\,du = \dfrac{e^{(k - \frac{(2n+1)^2\pi^2}{4})t} - 1}{k - \frac{(2n+1)^2\pi^2}{4}}$

while if $k = \frac{(2n+1)^2\pi^2}{4}$ then $\int_0^t e^{(k - \frac{(2n+1)^2\pi^2}{4})u}\,du = t$. The lemma follows.

LEMMA 7:

$$L^{-1}\left[\frac{\cosh x\sqrt{s-k}}{(s-k)\cosh\sqrt{s-k}}\right] = e^{kt} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}\cos\frac{(2n+1)\pi x}{2}\,e^{(k - \frac{(2n+1)^2\pi^2}{4})t}.$$

Proof: From Spiegel's tables[34, p.252, entry 129],

$$L^{-1}\left[\frac{\cosh x\sqrt{s}}{s\cosh\sqrt{s}}\right] = 1 + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}e^{-\frac{(2n+1)^2\pi^2 t}{4}}\cos\frac{(2n+1)\pi x}{2}.$$

The lemma follows from a standard theorem on Laplace transforms.

<u>Example of a Time-dependent Problem:</u>

Consider the equation $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + ku + \lambda = 0$ $(-1 < x < 1,\ t > 0)$ where we assume $k > 0$, $\lambda > 0$; further, $u(x,t)$ satisfies the initial-boundary conditions

$$u(x,0) = 0 \text{ for } -1 \le x \le 1$$
$$u(-1,t) = 0 \text{ for } t \ge 0$$
$$u(1,t) = 0 \text{ for } t \ge 0.$$

Let $y(s,x)$ be the Laplace transform with respect to t of u. Taking Laplace transforms, the problem becomes:

$$\frac{d^2y}{dx^2} - sy + ky + \frac{\lambda}{s} = 0, \quad \text{with } y(s,-1) = y(s,1) = 0.$$

The equation is $D^2y - (s-k)y = -\frac{\lambda}{s}$. We may assume $s > k$, so that $s-k > 0$. A particular integral is

$$\frac{1}{D^2 - (s-k)}\left(-\frac{\lambda}{s}\right) = \frac{\lambda}{s(s-k)} = \frac{\lambda}{k(s-k)} - \frac{\lambda}{ks}.$$

Thus the general solution is

$$y(s,x) = Ae^{\sqrt{s-k}\,x} + Be^{-\sqrt{s-k}\,x} + \frac{\lambda}{k(s-k)} - \frac{\lambda}{ks}.$$

$$\therefore\ y(s,-1) = 0 = Ae^{-\sqrt{s-k}} + Be^{\sqrt{s-k}} + \frac{\lambda}{k(s-k)} - \frac{\lambda}{ks} \quad \ldots\ldots\ldots\ldots(1)$$

$$y(s,1) = 0 = Ae^{\sqrt{s-k}} + Be^{-\sqrt{s-k}} + \frac{\lambda}{k(s-k)} - \frac{\lambda}{ks} \quad \ldots\ldots\ldots\ldots(2)$$

(1) - (2): $\qquad (A - B)(e^{-\sqrt{s-k}} - e^{\sqrt{s-k}}) = 0.$

Hence A = B, and substituting back into (1) gives at once:

$$A = B = \frac{\dfrac{\lambda}{ks} - \dfrac{\lambda}{k(s-k)}}{e^{\sqrt{s-k}} + e^{-\sqrt{s-k}}}.$$

Thus the required solution is

$$y(s,x) = \left\{\frac{\lambda}{ks} - \frac{\lambda}{k(s-k)}\right\}\frac{\cosh x\sqrt{s-k}}{\cosh\sqrt{s-k}} + \frac{\lambda}{k(s-k)} - \frac{\lambda}{ks}.$$

Now $L^{-1}[\frac{1}{s-k}] = e^{kt}$ and $L^{-1}[\frac{1}{s}] = 1$. Using these and Lemmas 6 and 7, we may take inverse Laplace transforms to obtain a formal solution u(x,t).

<u>Case 1: $k \neq \frac{(2n+1)^2\pi^2}{4}$ for any $n = 0,1,2,\ldots$:</u>

$$u(x,t) = \frac{\lambda\pi}{k}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)\cos\frac{(2n+1)\pi x}{2}}{k - \frac{(2n+1)^2\pi^2}{4}}\left\{e^{(k - \frac{(2n+1)^2\pi^2}{4})t} - 1\right\}$$

$$- \frac{4\lambda}{\pi k}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}\cos\frac{(2n+1)\pi x}{2}\,e^{(k - \frac{(2n+1)^2\pi^2}{4})t} - \frac{\lambda}{k}$$

$$= \frac{\lambda}{k\pi}\sum_{n=0}^{\infty}(-1)^n\cos\frac{(2n+1)\pi x}{2}\,e^{(k - \frac{(2n+1)^2\pi^2}{4})t}\left\{\frac{(2n+1)\pi^2}{k - \frac{(2n+1)^2\pi^2}{4}} + \frac{4}{2n+1}\right\}$$

$$+ \frac{\lambda\pi}{k}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)}{k - \frac{(2n+1)^2\pi^2}{4}}\cos\frac{(2n+1)\pi x}{2} - \frac{\lambda}{k}$$

$$= \frac{4\lambda}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\{k - \frac{(2n+1)^2\pi^2}{4}\}(2n+1)}\cos\frac{(2n+1)\pi x}{2}\,e^{(k - \frac{(2n+1)^2\pi^2}{4})t}$$

$$+ \frac{\lambda\pi}{k}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)}{k - \frac{(2n+1)^2\pi^2}{4}}\cos\frac{(2n+1)\pi x}{2} - \frac{\lambda}{k}.$$

Now from Lemma 2 we can see that the second of these sums is discontinuous at $x = \pm 1$. We must therefore redefine it at $x = \pm 1$ to make it continuous; again using Lemma 2 and remembering that in this problem we are only concerned with the interval $-1 \le x \le 1$, we can see that the second sum must be replaced by $\frac{\lambda \cos\sqrt{k}x}{k\cos\sqrt{k}}$. This gives the formal solution:

$$u(x,t) = \frac{4\lambda}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{\{k - \frac{(2n+1)^2\pi^2}{4}\}(2n+1)} \cos\frac{(2n+1)\pi x}{2}\, e^{(k - \frac{(2n+1)^2\pi^2}{4})t} + \frac{\lambda\cos\sqrt{k}x}{k\cos\sqrt{k}} - \frac{\lambda}{k} \quad \ldots\ldots\ldots\ldots(S_1)$$

It is necessary to verify that this is indeed a solution of the problem. Clearly $u(-1,t) = u(1,t) = 0$ for $t \ge 0$.

Further,
$$\begin{aligned} u(x,0) &= \frac{4\lambda}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{\{k - \frac{(2n+1)^2\pi^2}{4}\}(2n+1)} \cos\frac{(2n+1)\pi x}{2} + \frac{\lambda\cos\sqrt{k}x}{k\cos\sqrt{k}} - \frac{\lambda}{k} \\ &= -\frac{\lambda}{k}\left(\frac{\cos\sqrt{k}x}{\cos\sqrt{k}} - 1\right) + \frac{\lambda\cos\sqrt{k}x}{k\cos\sqrt{k}} - \frac{\lambda}{k} \quad \text{for } -1 \le x \le 1, \text{ by Lemma 3} \\ &= 0. \end{aligned}$$

Thus all the boundary conditions are satisfied. Checking the differential equation, we have:

$$\frac{\partial^2 u}{\partial x^2} = \frac{4\lambda}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)\pi^2}{\{k - \frac{(2n+1)^2\pi^2}{4}\}4} \cos\frac{(2n+1)\pi x}{2}\, e^{(k - \frac{(2n+1)^2\pi^2}{4})t} - \frac{\lambda\cos\sqrt{k}x}{\cos\sqrt{k}}.$$

$$\frac{\partial u}{\partial t} = \frac{4\lambda}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos\frac{(2n+1)\pi x}{2}\, e^{(k - \frac{(2n+1)^2\pi^2}{4})t}.$$

Note that, here and in Case 2 below, the uniform convergence of the differentiated series follows easily from the fact that, for $t \ge t_0 > 0$ and n sufficiently large, the general term of each series is smaller in absolute value than $e^{(k - \frac{(2n+1)^2\pi^2}{4})t_0}$.

$$\therefore\ \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + ku + \lambda = \frac{4\lambda}{\pi}\sum_{n=0}^{\infty} (-1)^n \cos\frac{(2n+1)\pi x}{2}\, e^{(k - \frac{(2n+1)^2\pi^2}{4})t} \times\left\{\frac{-(2n+1)\pi^2}{4\{k - \frac{(2n+1)^2\pi^2}{4}\}} - \frac{1}{2n+1} + \frac{k}{(2n+1)\{k - \frac{(2n+1)^2\pi^2}{4}\}}\right\}$$

$= 0$ as required.

Thus (S_1) is an actual solution of the problem. It is clear from the form of (S_1) that if $0 < k < \frac{\pi^2}{4}$ then $u(x,t)$ is bounded as $t \to \infty$, and in fact $u(x,t) \to \frac{\lambda}{k}\{\frac{\cos\sqrt{k}x}{\cos\sqrt{k}} - 1\}$ as $t \to \infty$. If $k > \frac{\pi^2}{4}$ then $u(x,t)$ is unbounded as $t \to \infty$.

<u>Case 2: $k = \frac{(2N+1)^2\pi^2}{4}$ for some $N = 0,1,2,\ldots$:</u>

$$u(x,t) = \frac{\lambda\pi}{k}\sum_{n\neq N}\frac{(-1)^n(2n+1)\cos\frac{(2n+1)\pi x}{2}}{k-\frac{(2n+1)^2\pi^2}{4}}\{e^{(k-\frac{(2n+1)^2\pi^2}{4})t}-1\}$$
$$+\frac{\lambda\pi}{k}(-1)^N(2N+1)t\cos\frac{(2N+1)\pi x}{2}$$
$$-\frac{4\lambda}{\pi k}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}\cos\frac{(2n+1)\pi x}{2}e^{(k-\frac{(2n+1)^2\pi^2}{4})t}-\frac{\lambda}{k}.$$

Calculating as in the previous case, this gives:

$$u(x,t) = \frac{4\lambda}{\pi}\sum_{n\neq N}\frac{(-1)^n}{\{k-\frac{(2n+1)^2\pi^2}{4}\}(2n+1)}\cos\frac{(2n+1)\pi x}{2}e^{(k-\frac{(2n+1)^2\pi^2}{4})t}$$
$$+\frac{4\lambda(-1)^N}{\pi k(2N+1)}\cos\frac{(2N+1)\pi x}{2}+\frac{\lambda\pi}{k}\sum_{n\neq N}\frac{(-1)^{n+1}(2n+1)}{k-\frac{(2n+1)^2\pi^2}{4}}\cos\frac{(2n+1)\pi x}{2}$$
$$+\frac{\lambda\pi}{k}(-1)^N(2N+1)t\cos\frac{(2N+1)\pi x}{2}-\frac{\lambda}{k}.$$

From Lemma 4 we see that, in order to make the second sum in this expression continuous at $x = \pm 1$, we must replace it by

$$\frac{\lambda}{k}[(-1)^N x\sin\frac{(2N+1)\pi x}{2}-\frac{(-1)^N}{(2N+1)\pi}\cos\frac{(2N+1)\pi x}{2}]$$

leading to the formal solution

$$u(x,t) = \frac{4\lambda}{\pi}\sum_{n\neq N}\frac{(-1)^n}{\{k-\frac{(2n+1)^2\pi^2}{4}\}(2n+1)}\cos\frac{(2n+1)\pi x}{2}e^{(k-\frac{(2n+1)^2\pi^2}{4})t}$$
$$+\frac{3\lambda(-1)^N}{\pi k(2N+1)}\cos\frac{(2N+1)\pi x}{2}+\frac{\lambda(-1)^N x}{k}\sin\frac{(2N+1)\pi x}{2}$$
$$+\frac{\lambda\pi}{k}(-1)^N(2N+1)t\cos\frac{(2N+1)\pi x}{2}-\frac{\lambda}{k}\quad\ldots\ldots\ldots\ldots\ldots\ldots(S_2)$$

As before, it is necessary to verify that this is indeed a solution of the problem. Clearly $u(-1,t) = u(1,t) = 0$ for $t \geq 0$.

Further, $u(x,0) = \frac{4\lambda}{\pi} \sum_{n\neq N} \frac{(-1)^n}{\{k - \frac{(2n+1)^2\pi^2}{4}\}(2n+1)} \cos \frac{(2n+1)\pi x}{2}$

$$+ \frac{3\lambda(-1)^N}{\pi k(2N+1)} \cos \frac{(2N+1)\pi x}{2} + \frac{\lambda(-1)^N x}{k} \sin \frac{(2N+1)\pi x}{2} - \frac{\lambda}{k}$$

$$= -\frac{\lambda}{k}\left[\sum_{n\neq N} \frac{(-1)^{n+1}(2N+1)^2\pi}{\{\frac{(2N+1)^2\pi^2}{4} - \frac{(2n+1)^2\pi^2}{4}\}(2n+1)} \cos \frac{(2n+1)\pi x}{2}\right]$$

$$+ \frac{3\lambda(-1)^N}{\pi k(2N+1)} \cos \frac{(2N+1)\pi x}{2} + \frac{\lambda(-1)^N x}{k} \sin \frac{(2N+1)\pi x}{2} - \frac{\lambda}{k}$$

$= 0$ for $-1 \leq x \leq 1$, by Lemma 5.

Thus the boundary conditions are satisfied. Checking the differential equation, we have:

$$\frac{\partial^2 u}{\partial x^2} = \frac{4\lambda}{\pi} \sum_{n\neq N} \frac{(-1)^{n+1}(2n+1)\pi^2}{\{k - \frac{(2n+1)^2\pi^2}{4}\}4} \cos \frac{(2n+1)\pi x}{2}\, e^{(k - \frac{(2n+1)^2\pi^2}{4})t}$$

$$+ \frac{3\lambda(-1)^{N+1}(2N+1)\pi}{4k} \cos \frac{(2N+1)\pi x}{2} + \frac{\lambda(-1)^N(2N+1)\pi}{k} \cos \frac{(2N+1)\pi x}{2}$$

$$+ \frac{\lambda(-1)^{N+1}(2N+1)^2\pi^2 x}{4k} \sin \frac{(2N+1)\pi x}{2} + \frac{\lambda(-1)^{N+1}(2N+1)^3\pi^3 t}{4k} \cos \frac{(2N+1)\pi x}{2}.$$

$$\frac{\partial u}{\partial t} = \frac{4\lambda}{\pi} \sum_{n\neq N} \frac{(-1)^n}{2n+1} \cos \frac{(2n+1)\pi x}{2}\, e^{(k - \frac{(2n+1)^2\pi^2}{4})t}$$

$$+ \frac{\lambda\pi}{k}(-1)^N(2N+1)\cos \frac{(2N+1)\pi x}{2}.$$

$\therefore \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + ku + \lambda = 0$, as is readily checked. Thus (S_2) is an actual solution in this case; evidently, we have in this case that $u(x,t)$ is always unbounded as $t \to \infty$.

<u>The Corresponding Steady-state Problem:</u>

This is the problem $\frac{d^2u}{dx^2} + ku + \lambda = 0$ $(-1 < x < 1)$, where $k > 0$, $\lambda > 0$, and the boundary conditions are $u(-1) = u(1) = 0$. The general solution of the equation is $u(x) = A \cos \sqrt{k}x + B \sin \sqrt{k}x - \frac{\lambda}{k}$.

$$\therefore\ u(-1) = 0 = A \cos \sqrt{k} - B \sin \sqrt{k} - \frac{\lambda}{k}.$$

$$u(1) = 0 = A \cos \sqrt{k} + B \sin \sqrt{k} - \frac{\lambda}{k}.$$

Adding: $\qquad A \cos\sqrt{k} = \frac{\lambda}{k}$.

Subtracting: $\qquad B \sin\sqrt{k} = 0$.

Now if $\sqrt{k} \neq n\pi$, i.e. $k \neq n^2\pi^2$ for $n = 1,2,3,\ldots.$, we must have $B = 0$. If $k = n^2\pi^2$ for some $n = 1,2,3,\ldots.$, then B is arbitrary. If $\sqrt{k} \neq \frac{(2n+1)\pi}{2}$, i.e. $k \neq \frac{(2n+1)^2\pi^2}{4}$ for $n = 0,1,2,\ldots.$, then $A = \frac{\lambda}{k\cos\sqrt{k}}$. If $k = \frac{(2n+1)^2\pi^2}{4}$ for some $n = 0,1,2,\ldots.$, no solution is possible. So we have the following cases:

If $k \neq \frac{m^2\pi^2}{4}$ for any $m = 1,2,3,\ldots.$, the solution is

$$u(x) = \frac{\lambda}{k}\left[\frac{\cos\sqrt{k}x}{\cos\sqrt{k}} - 1\right].$$

If $k = n^2\pi^2$ for some $n = 1,2,3,\ldots.$, the solution is

$$u(x) = \frac{\lambda}{k}\left[\frac{\cos\sqrt{k}x}{\cos\sqrt{k}} - 1\right] + B\sin\sqrt{k}x$$

where B is arbitrary.

If $k = \frac{(2n+1)^2\pi^2}{4}$ for some $n = 0,1,2,\ldots.$, no solution exists.

In particular, if $0 < k < \frac{\pi^2}{4}$ then the steady-state problem has the positive solution $u(x) = \frac{\lambda}{k}\left[\frac{\cos\sqrt{k}x}{\cos\sqrt{k}} - 1\right]$, which is also the limit of the solution $u(x,t)$ of the time-dependent problem as $t \to \infty$. For larger values of k, positive solutions of the steady-state problem do not exist.

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