An initial-boundary value problem arising in cell population growth modelling

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Adel Almalki

Supervisor: Assoc. Pro. Bruce van Brunt
Co-supervisor: Dr. Tammy Lynch

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Abstract

A partial differential equation modelling cell populations undergoing growth and division characterised by size is studied. This equation is a special case of the fragmentation equation studied by Michel *et al.* [5] with no dispersion term, and the problem is of the initial-boundary value type.

Eigenfunction solutions are derived for separable solutions and related properties such as spectrum, uniqueness and unimodality are investigated. We show that the spectrum is continuous and that the decay of the eigenfunctions is exponential at a critical eigenvalue and algebraic otherwise.

The existence of a fast decay general solution $n(x, t)$ is then established. The problem can be solved analytically, and it is shown that the solution is unique and smooth. The solution properties are illustrated with some numerical simulations.

Finally, the role of exponential decaying eigenfunction solutions is interpreted from the standpoint of the general solution. The asymptotic behaviour as $t \to \infty$ of the general solution is examined. Slow decay eigenfunction solutions are briefly discussed, but their mathematical role remains to be explored.

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Abbreviations

SSD	**S**teady **S**ize **D**istribution
PDE	**P**artial **D**ifferential **E**quation
PDF	**P**robability **D**ensity **F**unction
ODE	**O**rdinary **D**ifferential **E**quation

Symbols

$n(x,t)$	Number density	$\frac{1}{[l]}$
x	Cell size	$[l]$
t	Time	$[s]$
g	Cell growth rate	$\frac{[l]}{[s]}$
μ	Cell death rate	$\frac{1}{[s]}$
$W(\xi,x)$	Division kernel	$\frac{1}{[ls]}$

The notations l and s refer to size unit and second, respectively.

This thesis is dedicated to my parents with all of my love.

Chapter 1

Introduction

1.1 Preliminary

Modelling populations characterised by some property dates back to the eighteenth century, on human populations, when researchers began to take into account the characteristics of individuals unlike early models that were dependent only on the total population. Euler was one of the main researchers who studied structured population, in which he considered age as a property for modelling (cf. Hall [2]. Another property that has been given prominence for modelling populations is the size of individuals. These population models have been extended to cell growth (e.g. [1–7, 9]). In particular, models structured by size have attracted the interests of researchers since the 1960's [2]. This is due in part to the fact that cell size can now be easily and precisely measured owing to advances in instruments. These measurements may track DNA content, weight, radius or mass among other quantities.

In this thesis, we study a special case of the fragmentation equation to model cell population growth characterised by a single variable "size". Here, size may be any of the above quantities. The case of study that we examine corresponds to an initial-boundary value problem involving a hyperbolic PDE. Although this is perhaps a simple case, it nonetheless displays the analytical character found in more complicated models and it has the decisive merit that the equation can be solved explicitly and the solution properties studied directly. To set the stage we first give an outline of the model.

1.2 Developing the model

Let $n(x,t)$ denote the number density of cells of size x at time t. If l denotes the unit of size, then n has units $[\frac{1}{l}]$. The total number of cells between sizes a and b, $0 < a < b$, at time t is given by

$$\int_a^b n(x,t)dx.$$

The number of cells of size x may be increased by smaller cells growing to that size or by the division of larger cells to form cells of size x. The number of cells may decrease owing to division to smaller sized cells or through mortality. In brief, we have

$$\frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}\Big(g(x,t)n(x,t)\Big) = C(x,t) - D(x,t) - E(x,t), \tag{1.1}$$

where $g(x,t)$ is the rate of growth $[\frac{l}{s}]$, $C(x,t)$ is the rate at which the number of cells of size x is increasing from the division of larger cells $[\frac{1}{ls}]$, $D(x,t)$ is the rate of loss of individuals from cells of size x $[\frac{1}{s}]$, and $E(x,t)$ is the rate at which the number of cells of size x die $[\frac{1}{s}]$. These functions, in general, can be regarded as stochastic; however, we shall model the process using a cell division kernel that is deterministic and independent of time.

Following the structure of the fragmentation equation (cf. Michel *et al.* [5]) we introduce a division kernel $T(x,\xi)$. This kernel represents the rate at which cells of size $\xi > x$ divides to form cells of size x, i.e. the rate of change for the number density $n(x,t)$ as a result of cells of size $\xi > x$ dividing to form x sized cells. The units are $[\frac{1}{ls}]$. Since cells do not divide to form larger cells, it is clear that $T(x,\xi) = 0$ if $\xi < x$. The function $C(x,t)$ is thus modelled by

$$C(x,t) = \int_x^\infty T(x,\xi)n(\xi,t)d\xi.$$

The loss of cells owing the division of x-sized cells to smaller sized cells is clearly dependent on $T(x,\xi)$. To ascertain the form of $D(x,t)$ we consider the process in the absence of growth and death (conservation of biomass). Suppose that the cells are neither growing nor dying. Then $g(x,t) = 0$, $E(x,t) = 0$, and equation (1.1) becomes

$$n_t(x,t) = \int_x^\infty T(x,\xi)n(\xi,t)d\xi - D(x,t). \tag{1.2}$$

The total biomass at time t is given by

$$\mu_1(t) = \int_0^\infty xn(x,t)dx, \tag{1.3}$$

and under the restrictions of no growth or death, the biomass should be constant, i.e.

$$\mu_1'(t) = \Big(\int_0^\infty xn(x,t)dx\Big)_t = \int_0^\infty xn_t(x,t)dx = 0. \tag{1.4}$$

Multiplying equation (1.2) by x and integrating with respect to x from 0 to ∞ gives

$$\frac{\partial}{\partial t}\int_0^\infty xn(x,t)dx = \int_0^\infty \Big(x\int_x^\infty T(x,\xi)n(\xi,t)d\xi - xD(x,t)\Big)dx, \tag{1.5}$$

and condition (1.4) implies

$$\int_0^\infty \Big(x\int_x^\infty T(x,\xi)n(\xi,t)d\xi - xD(x,t)\Big)dx = 0. \tag{1.6}$$

Interchanging the order of the integration for the first term yields

$$\begin{aligned}\int_0^\infty x\int_x^\infty T(x,\xi)n(\xi,t)d\xi dx &= \int_0^\infty n(\xi,t)\int_0^\xi xT(x,\xi)dxd\xi \\ &= \int_0^\infty n(x,t)\int_0^x \tau T(\tau,x)d\tau dx, \end{aligned}\tag{1.7}$$

where we have changed the dummy variables of integration, $\xi \to x$ and $x \to \tau$. Equations (1.6) and (1.7) imply that

$$D(x,t) = n(x,t)\int_0^x \frac{\tau}{x}T(\tau,x)d\tau.$$

In a biological sense, the above expression reflects that the total biomass produced by x-sized cells dividing to cells of size $\tau < x$ is $\int_0^x \tau T(\tau,x)d\tau$, so that the total biomass per x cell left in the division process for the whole system is then defined by the above relationship for $D(x,t)$.

Assuming the mortality rate is proportional to the population, we have

$$E(x,t) = \mu(x,t)n(x,t),$$

where μ is the mortality rate $[\frac{1}{s}]$. Equation (1.1) thus becomes

$$n_t(x,t) + \Big(g(x,t)n(x,t)\Big)_x = \int_x^\infty T(x,\xi)n(\xi,t)d\xi - n(x,t)\int_0^x \frac{\tau}{x}T(\tau,x)d\tau - \mu(x,t)n(x,t). \quad (1.8)$$

We will assume that μ is constant. In this case the mortality can be transformed out of equation (1.8). Let

$$n(x,t) = e^{-\mu t}\tilde{n}(x,t),$$

then equation (1.8) becomes

$$\tilde{n}_t(x,t) + \Big(g(x,t)\tilde{n}(x,t)\Big)_x = \int_x^\infty T(x,\xi)\tilde{n}(\xi,t)d\xi - \tilde{n}(x,t)\int_0^x \frac{\tau}{x}T(\tau,x)d\tau. \quad (1.9)$$

We henceforth drop the tilde in equation (1.9) unless there is scope for confusion. The above equation is a special case of a fragmentation equation (cf. [5]), in which a general relative entropy inequality was extended using a "fragmentation operator" to model the cell division process with a drift term. As we do not expect solutions to blow up spatially, but maybe timewise, and not accept cells of negative sizes, we supplement the above equation with boundary data

$$\lim_{x\to\infty} n(x,t) = 0, \quad (1.10)$$

$$\lim_{x\to\infty} \frac{\partial}{\partial x}n(x,t) = 0, \quad (1.11)$$

$$n(0,t) = 0, \quad (1.12)$$

as well as given initial cell population

$$n(x,0) = n_0(x). \quad (1.13)$$

1.2.1 The separation of variables

Biologists study cell division by studying specimens and developing cell distribution curves based on cell size. The distribution curve is a plot of $n(x,t)$ for a fixed time. They noted that after a certain time the shape of one curve is virtually the same as the shape of the next one in time but perhaps scaled (cf. Hall [2], Begg [4]). In other words, for t_1, t_2 sufficiently large, the graph of $n(x, t_1)$ is essentially the same as $n(x, t_2)$ apart from a scaling. This indicates that the overall shape of the population remains steady while the number of cells may increase or decrease. In this case n is termed a steady size distribution, or SSD. Mathematically, an SSD corresponds to the separable solution,

$$n(x,t) = N(t)y(x), \tag{1.14}$$

where $N(t)$ is the population at time $t \in [0, \infty)$ and $y(x)$ is a probability density function (PDF) that models the shape of the size distribution. Since y is a PDF, it is required that $y(x) \geq 0$ for all $x \geq 0$ and that

$$\int_0^\infty y(x)dx = 1. \tag{1.15}$$

For the remainder of this thesis y will denote the x dependent function of the separable solution. We will also assume that g is independent of time. Substituting (1.14) in equation (1.9) gives

$$\frac{N_t(t)}{N(t)} = -\frac{1}{y(x)}\Big(g(x)y(x)\Big)_x + \frac{1}{y(x)}\int_x^\infty T(x,\xi)y(\xi)d\xi - \int_0^x \frac{\tau}{x}T(\tau,x)d\tau \equiv \lambda, \tag{1.16}$$

where λ is an eigenvalue. Evidently,

$$N(t) = N_0 e^{\lambda t}, \tag{1.17}$$

where N_0 is a constant.

The above procedure precipitates a number of mathematical questions foremost of which concerns the class of functions T for which an eigenvalue exists. In addition, the model requires y to be non-negative and thus T must produce at least one eigenvalue that gives a non-negative eigenfunction. These questions are substantial and require tools such as the Krein-Rutman theorem. The reader is directed to da Casta *et al*. [6] and Begg [4] for applications of this theorem to the special case detailed in the next section. Of independent interest is the paper by Perthame and Ryzhik [7], in which the problem is attacked by studying a sequence of solutions to problems on compact interval using the Perron-Frobenius theorem to assert the existence of a positive eigenfunction. For the choice of T in this thesis it is possible to establish directly the existence of positive eigenfunctions.

1.3 Special case

Consider the division kernel modelled by the Dirac delta function δ,

$$T(x,\xi) = \alpha b(\xi)\delta(\frac{\xi}{\alpha} - x),$$

where α is a constant and $b(\xi) > 0$ is a division rate. Symmetric cell division occurs when the mother cell is divided into a number of $\alpha > 1$ cells of equal sizes. If $\alpha = 2$ then the reproduction process divides each cell exactly into two halves, which is often the biological case. This assumption was first introduced in modelling cell growth by Koch and Schaechter according to Heijmans [9]. This choice of T in equation (1.9) gives

$$n_t(x,t) + \Big(g(x,t)n(x,t)\Big)_x = \alpha^2 b(\alpha x)n(\alpha x,t) - b(x)n(x,t). \tag{1.18}$$

The above equation is a modified Fokker-Planck equation (no dispersion), which has been studied with and without dispersion by various researchers (e.g. [2–4, 6, 7]). Following the separation of variables in the previous section with the assumption that the growth rate g and the division rate b are constants we get

$$gy'(x) - \alpha^2 by(\alpha x) + by(x) + \lambda y(x) = 0. \tag{1.19}$$

The above ODE is supplemented with the boundary conditions derived from (1.10)-(1.12),

$$\lim_{x\to\infty} y(x) = 0, \tag{1.20}$$

$$\lim_{x\to\infty} \frac{d}{dx} y(x) = 0, \tag{1.21}$$

and

$$y(0) = 0. \tag{1.22}$$

The value of λ can be deduced by integrating (1.19) from 0 to ∞ and applying the normalizing condition (1.15). This yields

$$\lambda = (\alpha - 1)b,$$

and equation (1.19) reduces to

$$y'(x) = \alpha^2 a y(\alpha x) - \alpha a y(x), \tag{1.23}$$

where $a = \frac{b}{g}$. Equation (1.23) was investigated in a model developed by Hall and Wake [1]. This equation is a special case of the so-called pantograph equation which has been studied extensively by Kato and McLeod [8] among others.

1.4 The case of study

Perhaps the simplest division kernel is

$$T(x, \xi) = H(\xi - x),$$

where H is the Heaviside function. Such a choice may prove unrealistic for biological applications. However, it does capture some of the features shared by more complicated choices of T. For this choice, equation (1.9) becomes

$$n_t(x,t) + g n_x(x,t) = \int_x^\infty n(\xi, t) d\xi - \frac{x}{2} n(x,t), \tag{1.24}$$

where the growth rate g in this model is assumed to be a positive constant. We study solutions to this equation, subject to conditions (1.10)-(1.13), throughout this thesis. Note that the conditions (1.12) and (1.13) are both defined at $(x,t) = (0,0)$. This, in turn, places restrictions on the initial data at the origin. Equation (1.24) implies

$$n_t(0,0) + g n_x(0,0) = \int_0^\infty n(\xi, 0) d\xi, \tag{1.25}$$

and equation (1.12) indicates that

$$n_t(0,t) = 0. \tag{1.26}$$

This means n_0 must satisfy

$$n_0'(0) = \frac{1}{g} \int_0^\infty n_0(\xi) d\xi. \tag{1.27}$$

The initial data are assumed to be a PDF so that

$$\int_0^\infty n_0(\xi)d\xi = 1.$$

We thus see that

$$n_0'(0) = \frac{1}{g}.$$

Note that the special case considered in the last section also leads to a condition on $n_0'(x)$ at $x = 0$. For that case equation (1.18) implies $n_0'(0) = 0$.

1.5 Outline of the thesis

1.5.1 Chapter 2

In chapter 2, we seek solutions of the form $n(x,t) = N(t)y(x)$ to equation (1.24). Specifically, we seek solutions $y(x)$ to an eigenvalue problem derived from (1.24) that satisfy conditions (1.20)-(1.22). Here $g > 0$ is a constant and $y(x) \geq 0$ is required to be a PDF. We then investigate the spectrum and a number of important properties of these solutions including uniqueness and unimodality. We show that the spectrum is of the form $[\lambda_0, \infty)$ and that the solution decays exponentially for $\lambda = \lambda_0$ as $x \to \infty$. If $\lambda > \lambda_0$ we show that $y \sim O(\frac{1}{x^3})$ as $x \to \infty$. For any fixed $\lambda \in [\lambda_0, \infty)$, we establish that the separable solution is unique and unimodal.

1.5.2 Chapter 3

The separable solutions of chapter 2 require that $n_0(x) = y(x)$. In chapter 3, we solve equation (1.24) subject to the initial data (1.12) and (1.13), where n_0 is an arbitrary PDF that satisfies the compatibility conditions

$$n_0(0) = 0,$$

and

$$n_0'(0) = \frac{1}{g}.$$

We convert equation (1.24) to a second order PDE and solve it by the method of characteristics with characteristic projections u and v given by (3.5) rather than pursue solutions to the singular Fredholm equation (3.2). We find that the general solution given by (3.18) contains two unknown functions $F(u)$ and $G(v)$, and that condition (1.13) defines $F(u)$ only for $u < 0$. In order to determine F for $u \geq 0$ another condition must be determined. We derive this condition in section 3.3 from a pair of ordinary differential equations involving the total cell number $\int_0^\infty n(x,t)dx$ and the total biomass $\int_0^\infty xn(x,t)dx$ at time t.

The initial-boundary value problem posed by equation (1.24) and conditions (1.12) and (1.13) can be decomposed into two problems: a Cauchy problem in a region R_2 when $u > 0$ and a Goursat problem in a region R_1 when $u < 0$. The two solutions are linked along the characteristic $u = 0$, on which a continuity condition is imposed. The union of the solution to the Cauchy problem in R_2, and the solution to the Goursat problem in R_1 gives the general solution to this problem (see figure 3.1). The solution strategy is to first solve the Cauchy problem for equation (1.24) in R_2 subject to condition (1.12) and the flux condition derived in section 3.3. Continuity is then used to define n along the characteristic $u = 0$. Once $n(0, v)$ is determined, then n_0, which is a condition along the $v = 0$ axis (i.e. the x-axis), can be used to form the Goursat problem.

As noted the flux condition $n_u(gv, v)$ comes from differential equations involving the total biomass and the total number density. The crucial assumptions are that the solution decays faster than $\frac{1}{x^3}$ as $x \to \infty$ for all $t \geq 0$ and the initial data also decay this fast (cf. (3.24) and (3.25)). The flux condition along with the initial-boundary value data allow us to determine the unknown functions G and F under the assumption that $n \sim o(\frac{1}{x^3})$ as $x \to \infty$. It is not clear, however, what the flux condition becomes if n decays slowly. The separable solutions with $\lambda > \lambda_0$ provide explicit evidence that such solutions are possible. We discuss this further in chapter 4.

In section 3.5.1, we show that the solutions to the Cauchy problem and the Goursat problem can be joined along the characteristic projection $u = 0$ so that the solution is smooth, i.e. n_u and n_v exist and are continuous functions on $u = 0$. We then address the question of uniqueness in section 3.5.2, where standard results such as Holmgren's theorem are used to establish the global uniqueness of the Cauchy problem. Once this is established, we show that the Goursat problem is also unique.

We end this chapter by giving two examples to illustrate the theory. We choose two types of initial data n_0: one is discontinuous; one is smooth. The examples illustrate the smoothness of the solution on $u = 0$ and the former initial data show discontinuities are propagated along characteristics as expected from a hyperbolic PDE. Both solutions show that the Cauchy problem solution, in time, dominates the solution. This motivates the work in chapter 4.

1.5.3 Chapter 4

We note in chapter three that the solution in R_2 dominates the long term solution, and that this solution depends indirectly on the initial data n_0 only through the first moment, which appears in the condition for $n_x(0,t)$ that is derived in section 3.3. We show that there is a relationship between this condition and the separable solution. In certain examples on the fragmentation equation, it is known that there is an associated eigenvalue problem and that, for arbitrary initial data, the solution is asymptotic to the eigenfunction as $t \to \infty$ (cf. [4, 5, 7]). The eigenvalue problem corresponds to separable solutions and this prompts us to examine the role of the separable solutions, particularly the one that decays exponentially. We show that this solution is special in that, as expected, there is no need to blend the solutions from R_1 and R_2 (the F is the same), but more importantly, for any PDF initial data, the general solution is asymptotic to this separable solution as $t \to \infty$.

We show in chapter 2, that for higher eigenvalues λ, $y(x;\lambda) \sim \frac{2g}{x^3}$ as $x \to \infty$. These solutions do not satisfy condition (3.24), and this gives rise to questions concerning solutions that decay no faster than $\frac{1}{x^3}$. This type of solution may be of limited interest biologically, but it indicates that there is another class of solutions for the general problem. We look briefly at these solutions and rewrite them in terms of the general solution. The mathematical role of these solutions remains to be explored.

Chapter 2

The separable solution

In the first section of this chapter we seek solutions $y(x)$ to an eigenvalue problem derived from equation (1.24) by the separation of variables $n(x,t) = N(t)y(x)$, introduced in chapter 1, subject to conditions (1.20)-(1.22) as well as the requirement that y be a PDF. In the next section, we investigate the spectrum, uniqueness and unimodality of these solutions.

2.1 Derivation of the separable solution

We seek solutions of the form

$$n(x,t) = N(t)y(x), \tag{2.1}$$

where N(t) is given by (1.17) and $y(x)$ is a PDF. Substituting relation (2.1) into (1.24) gives

$$-gy'(x) + \int_x^\infty y(\xi)d\xi - \frac{x}{2}y(x) - \lambda y(x) = 0, \tag{2.2}$$

where λ and g are constants, and the dash denotes differentiation with respect to the indicated argument. Let

$$\psi(x) = \int_x^\infty y(\xi)d\xi,$$

then equation (2.2) becomes

$$\psi''(x) + \Big(\frac{x+2\lambda}{2g}\Big)\psi'(x) + \frac{1}{g}\psi(x) = 0. \tag{2.3}$$

Let

$$\frac{x+2\lambda}{2g} = s,$$

and $\psi(x) = \theta(s)$. Then equation (2.3) can be expressed, after multiplying by $4g^2 s$ and integrating with respect to s, as

$$\theta'(s) + \Big(2gs - \frac{1}{s}\Big)\theta(s) = \frac{K_1}{s}, \tag{2.4}$$

where K_1 is an arbitrary constant of integration. Let

$$2gs = -f'(s).$$

Substituting this into (2.4) and using the integrating factor method gives

$$\frac{d}{ds}\Big(\frac{1}{s}e^{-f(s)}\theta(s)\Big) = \frac{K_1}{s^2}e^{-f(s)}. \tag{2.5}$$

Integrating (2.5) with respect to s from $\frac{\lambda}{g}$ to s yields

$$\theta(s) = K_1 s e^{f(s)} \int_{\frac{\lambda}{g}}^{s} \frac{1}{\xi^2} e^{-f(\xi)} d\xi + K_2 s e^{f(s)}, \tag{2.6}$$

where K_2 is another arbitrary constant. Replacing $\theta(s)$ by $\psi(x)$ gives

$$\psi(x) = K_1\Big(\frac{x+2\lambda}{2g}\Big)e^{-g\left(\frac{x^2+4\lambda x+4\lambda^2}{4g^2}\right)} \int_0^x \frac{e^{g\left(\frac{\xi^2+4\lambda\xi+4\lambda^2}{4g^2}\right)}}{(\frac{\xi+2\lambda}{2g})^2}\frac{1}{2g}d\xi + K_2\Big(\frac{x+2\lambda}{2g}\Big)e^{-g\left(\frac{x^2+4\lambda x+4\lambda^2}{4g^2}\right)}.$$

Combining and rearranging constants we thus see that $\psi(x)$ is of the form

$$\psi(x) = K_1(x+2\lambda)e^{\frac{-x^2-4\lambda x}{4g}} + K_2(x+2\lambda)e^{\frac{-x^2-4\lambda x}{4g}} \int_0^x \frac{e^{\frac{\xi^2+4\lambda\xi}{4g}}}{(\xi+2\lambda)^2}d\xi. \tag{2.7}$$

Let

$$\psi_1(x) = (x+2\lambda)e^{\frac{-x^2-4\lambda x}{4g}}, \tag{2.8}$$

and

$$\psi_2(x) = \psi_1(x) \int_0^x \frac{e^{\frac{\xi^2+4\lambda\xi}{4g}}}{(\xi+2\lambda)^2} d\xi. \tag{2.9}$$

Then

$$\psi(x) = K_1\psi_1(x) + K_2\psi_2(x). \tag{2.10}$$

Now,

$$y(x) = -\psi'(x); \tag{2.11}$$

consequently, equations (1.20)-(1.22) imply that $\psi(x)$ must satisfy

$$\lim_{x\to\infty} \psi(x) = 0, \tag{2.12}$$

$$\psi(0) = 1, \tag{2.13}$$

$$\frac{d}{dx}\psi(0) = 0, \tag{2.14}$$

and, in addition, since $y(x)$ is PDF, we require

$$\psi(x) \geq 0, \tag{2.15}$$

for all $x > 0$.

2.2 Properties of the separable solution

In this section, we show that the spectrum to the eigenvalue problem posed by the separation of variables is continuous. Then, we show that the eigenfunction solution $y(x;\lambda)$ is unique for fixed λ and unimodal.

2.2.1 The spectrum

We first examine the asymptotic behaviour of $\psi_1(x)$ and $\psi_2(x)$ as $x \to \infty$. Evidently,

$$\lim_{x\to\infty} \psi_1(x) = \lim_{x\to\infty} (x+2\lambda)e^{\frac{-x^2-4\lambda x}{4g}} = 0, \tag{2.16}$$

and it is obvious that ψ_1 decays exponentially as $x \to \infty$. L'Hopital's rule can be used to determine the limit of $\psi_2(x)$ as $x \to \infty$, *viz.*,

$$\lim_{x\to\infty} \psi_2(x) = \lim_{x\to\infty} \frac{\int_0^x \frac{e^{\frac{\xi^2+4\lambda\xi}{4g}}}{(\xi+2\lambda)^2} d\xi}{\frac{e^{\frac{x^2+4\lambda x}{4g}}}{(x+2\lambda)}}$$

$$= \lim_{x\to\infty} \frac{2g}{x^2 + 4\lambda x + 4\lambda^2 - 2g} = 0. \tag{2.17}$$

In fact, the above calculation shows that

$$\psi_2(x) \sim \frac{2g}{x^2},$$

as $x \to \infty$. A plot of $\psi_1(x)$ and $\psi_2(x)$ at certain growth rate over time is illustrated in figure 2.1.

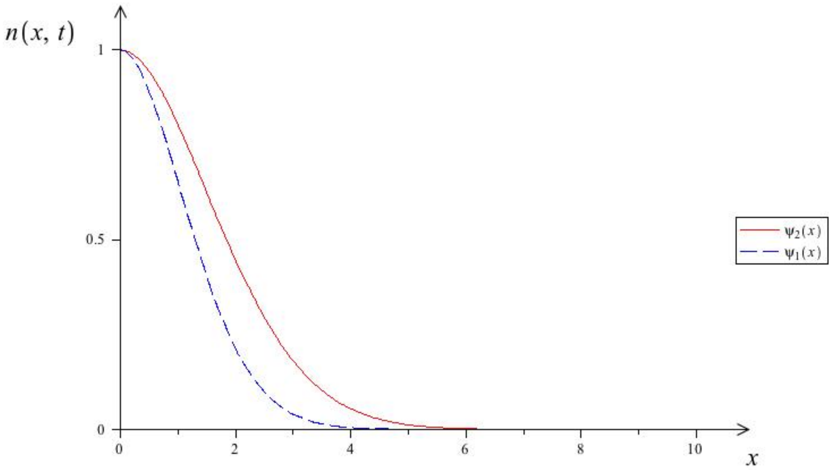

FIGURE 2.1: The asymptotic behaviour of ψ_1 and ψ_2

Equations (2.16) and (2.17) show that condition (2.12) is satisfied for any choice of K_1 and K_2. Condition (2.13) implies

$$\psi(0) = 2\lambda K_1 = 1. \tag{2.18}$$

Hence, since λ and K_1 can not be zero,

$$K_1 = \frac{1}{2\lambda}. \tag{2.19}$$

Note that λ and K_1 must be both negative or both positive. Suppose $\lambda < 0$, then

$$K_2 = 0,$$

because the integral defining ψ_2 in equation (2.9) contains a singularity when $\lambda < 0$ that causes the integral to diverge. Thus, if $\lambda < 0$, then

$$\psi(x) = K_1\psi_1(x) < 0,$$

for all $x > -2\lambda > 0$. But y is a PDF and hence $\psi(x) \geq 0$ for all $x > 0$. We thus conclude that $\lambda > 0$. Let

$$\int_0^x \frac{e^{\frac{\xi^2+4\lambda\xi}{4g}}}{(\xi+2\lambda)^2}d\xi = U(x).$$

Then

$$\begin{aligned}\psi'(x) &= K_1\psi_1'(x) + K_2\Big(\psi_1'(x)U(x) + \psi_1(x)U'(x)\Big) \\ &= \psi_1'(x)\Big(K_1 + K_2U(x)\Big) + K_2\psi_1(x)U'(x), \qquad (2.20)\end{aligned}$$

and applying condition (2.14) yields

$$K_2 = \frac{2\lambda^2}{g} - 1. \qquad (2.21)$$

We now show that for any non-negative solution ψ

$$\lambda \geq \sqrt{\frac{g}{2}}. \qquad (2.22)$$

Suppose (2.22) is false. Then equation (2.21) indicates that $K_2 < 0$. Since $K_2 \neq 0$,

$$\psi(x) \sim K_2\frac{2g}{x^2}, \qquad (2.23)$$

as $x \to \infty$. In particular, $\psi(x) < 0$ for all x sufficiently large, and this contradicts the condition that $\psi(x) \geq 0$ for all $x \geq 0$. We thus conclude that λ satisfies the inequality (2.22).

In summary, for all eigenvalues that satisfy (2.22), equations (2.10), (2.19) and (2.21) can be used to construct a solution to equation (2.3) that satisfy conditions (2.12)-(2.15).

The spectrum is thus the interval $[\lambda_0, \infty)$, where

$$\lambda_0 = \sqrt{\frac{g}{2}}.$$

If $\lambda = \lambda_0$ then $K_2 = 0$ and the solution decays exponentially as $x \to \infty$; if $\lambda > \lambda_0$, then relation (2.23) is valid for $K_2 > 0$. For the remainder of this thesis we use $y(x;\lambda)$ to denote the separable solution associated with the eigenvalue λ.

2.2.2 Uniqueness and unimodality

Suppose that $\lambda \in [\lambda_0, \infty)$ is fixed. Then Picard's theorem shows that the initial value problem consisting of the ordinary differential equation (2.3) and the initial conditions (2.13), (2.14) has a unique solution. In this sense, the eigenvalues are simple.

We now show that for any $\lambda \in [\lambda_0, \infty)$ the PDF y must be unimodal, i.e. y has precisely one non-zero local maximum in $[0, \infty)$. First note that y is continuous on $[0, \infty)$, $y(0)=0$, and $y(x) \to 0$ as $x \to \infty$. Since y must be positive somewhere in $[0, \infty)$ to be a PDF, it is clear that y must have at least one local maximum in $[0, \infty)$. Suppose there exist two points x_0 and x_2 at which y has a local maximum. Then there must exist a point x_1 between these points at which y has a local minimum. It is clear from the solution ψ that y has derivatives of all orders for $x \in (0, \infty)$ and thus at x_1,

$$y'(x_1) = 0,$$

and

$$y''(x_1) \geq 0.$$

Differentiating equation (2.2) with respect to x gives

$$gy''(x) = -\frac{3}{2}y(x) - \frac{x}{2}y'(x) - \lambda y'(x), \tag{2.24}$$

so that at x_1,

$$gy''(x_1) = -\frac{3}{2}y(x_1). \tag{2.25}$$

Now, $y(x_1) \geq 0$, and this means

$$y(x_1) = y'(x_1) = y''(x_1) = 0. \tag{2.26}$$

Equation (2.2) thus implies

$$\int_{x_1}^{\infty} y(\xi)d\xi = 0, \tag{2.27}$$

and since $y(x) \geq 0$ for all $x \geq 0$ this indicates that $y(x) = 0$ for all $x \geq x_1$. In this case we see that y cannot have a non-zero local maximum beyond x_1, which contradicts our assumption. We conclude that y has only one non-zero local maximum for $x \in [0, \infty)$. (In fact, the solution (2.10) shows that there cannot be an interval of the form $[x, \infty)$ wherein y is zero, since $\psi'(x)$ is not zero over such interval). Figure 2.2 illustrates unimodality of the separable solution over the eigenvalues at a growth rate of $g = 2$.

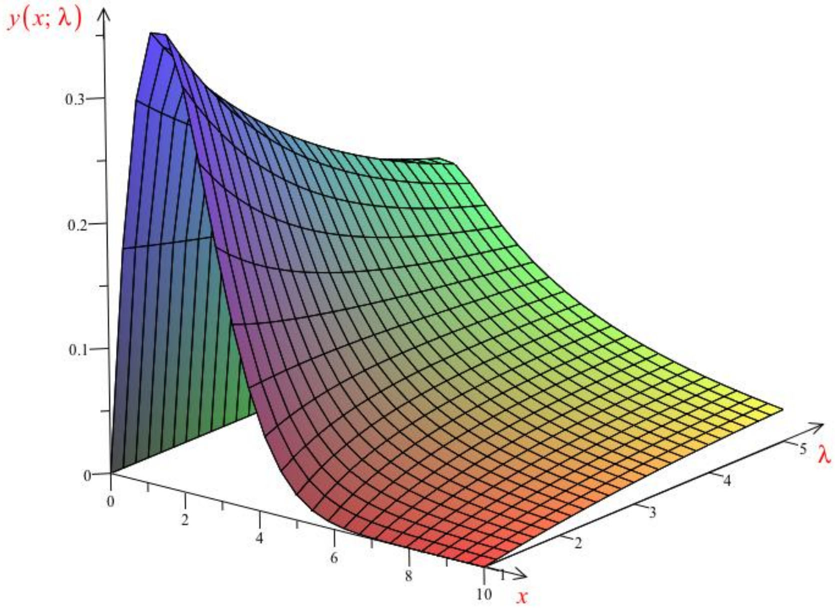

FIGURE 2.2: Unimodal character of $y(x; \lambda)$

Chapter 3

The general solution

3.1 Introduction

In this chapter, we seek a classical solution $n(x, t)$ to equation (1.24) subject to the conditions (1.12) and (1.13), where $n_0(x)$ is an arbitrary function that must satisfy the compatibility conditions (1.26) and (1.27).

In the previous chapter we looked for solutions of the form

$$n(x, t) = N(t)y(x).$$

This solution form allows one to "split" the PDE into an ODE for y and an ODE for N. The solution of the PDE is then formed as the product of the solutions of the ODE's. We showed in chapter 2 that the spectrum to the eigenvalue problem posed by the separation of variables is continuous. A general class of solutions can constructed of the form

$$n(x, t) = \int_{\lambda_0}^{\infty} C(\lambda)e^{\lambda t}y(x; \lambda)d\lambda, \tag{3.1}$$

where C is any function for which the integral is well defined. If we impose the initial condition (1.13), then we have the singular Fredholm equation

$$n_0(x) = \int_{\lambda_0}^{\infty} C(\lambda)y(x; \lambda)d\lambda, \tag{3.2}$$

for the function C. It was shown in chapter 2 that the decay of the eigenfunctions $y(x;\lambda)$ as $x \to \infty$ depends on whether $\lambda = \lambda_0$ or $\lambda > \lambda_0$. In the former case, the solution decays exponentially; in the latter case $y(x;\lambda) \sim O(\frac{1}{x^3})$ as $x \to \infty$. For the remainder of this thesis we will concentrate on solutions such that $x^3 n(x,t) \to 0$ as $x \to \infty$. Rather than pursue solutions to the singular Fredholm equation (3.2) and construct solutions from (3.1), we will attack the problem directly by first converting it to a second order PDE and use the method of characteristics. Recall the integrodifferential equation (1.24),

$$n_t(x,t) + g n_x(x,t) = \int_x^{\infty} n(\xi,t)d\xi - \frac{x}{2}n(x,t). \tag{3.3}$$

Conditions (1.12) and (1.13) indicate that the problem is an "initial-boundary value problem". Solutions for these problems can be elusive. We derive the general solution form in the next section. In section 3.3 a flux condition is derived that, along with the initial-boundary value conditions, will allow us to determine the unknown functions in section 3.4. We discuss certain properties for solutions in section 3.5 involving continuity and a proof of uniqueness. In the last section we present some examples for different initial conditions $n_0(x)$.

3.2 Derivation of the general solution

Differentiating (3.3) with respect to x gives

$$g n_{xx}(x,t) + n_{tx}(x,t) + \frac{x}{2}n_x(x,t) + \frac{3}{2}n(x,t) = 0, \tag{3.4}$$

which is a homogenous second-order hyperbolic PDE with the characteristic projections

$$u = gt - x, \quad v = t. \tag{3.5}$$

These characteristics are the two natural directions at a point in which information flows. Let

$$n(x,t) = n(x(u,v), t(u,v)) = \hat{n}(u,v). \tag{3.6}$$

Now,

$$n_x(x,t) = -\hat{n}_u(u,v), \tag{3.7}$$

$$n_t(x,t) = g\hat{n}_u(u,v) + \hat{n}_v(u,v), \tag{3.8}$$

$$n_{tx}(x,t) = -g\hat{n}_{uu}(u,v) - \hat{n}_{vu}(u,v) = n_{xt}(x,t), \tag{3.9}$$

and

$$n_{xx}(x,t) = \hat{n}_{uu}(u,v); \tag{3.10}$$

hence, (3.4) transforms to

$$\hat{n}_{uv}(u,v) + \Big(\frac{gv-u}{2}\Big)\hat{n}_u(u,v) - \frac{3}{2}\hat{n}(u,v) = 0. \tag{3.11}$$

For the remainder of this thesis we will drop the circumflexes unless there is a danger of confusion. Let

$$n(u,v) = e^{\frac{-gv^2}{4}+\frac{uv}{2}}W(u,v). \tag{3.12}$$

Then equation (3.11) yields

$$W_{uv}(u,v) + \frac{v}{2}W_v(u,v) - W(u,v) = 0. \tag{3.13}$$

Differentiating (3.13) with respect to v gives

$$W_{uvv}(u,v) + \frac{v}{2}W_{vv}(u,v) - \frac{1}{2}W_v(u,v) = 0, \tag{3.14}$$

and differentiating again leads to the equation

$$W_{uvvv}(u,v) + \frac{v}{2}W_{vvv}(u,v) = 0. \tag{3.15}$$

Equation (3.15) has solutions W_{vvv} of the form

$$W_{vvv}(u,v) = G(v)e^{\frac{-uv}{2}}. \tag{3.16}$$

Integrating (3.16) three times from zero to v leads to the following expression

$$W(u,v) = W(u,0) + vW_v(u,0) + \frac{v^2}{2}W_{vv}(u,0) + \frac{1}{2}\int_0^v G(\tau)(\tau-v)^2 e^{-\frac{\tau u}{2}}d\tau, \tag{3.17}$$

where G is an arbitrary function. The unknown functions $W(u,0), W_v(u,0)$ and $W_{vv}(u,0)$ can be reduced to a single unknown function F, say, by use of equations (3.13) and (3.14).

Specifically,

$$W_{uv}(u,0) = W(u,0),$$

and

$$W_{uvv}(u,0) = \frac{1}{2}W_v(u,0).$$

Let

$$2W_{vv}(u,0) = F(u).$$

Then

$$W_v(u,0) = F'(u),$$

and

$$W(u,0) = F''(u).$$

The solution (3.17) can therefore be written as

$$W(u,v) = F''(u) + vF'(u) + \frac{v^2}{4}F(u) + \frac{1}{2}\int_0^v G(\tau)(\tau - v)^2 e^{-\frac{\tau u}{2}} d\tau. \tag{3.18}$$

Note that condition (1.13) implies

$$F''(-x) = n_0(x),$$

but that n_0 is defined only for $x \geq 0$. The above differential equation thus determines $F(u)$ for $u \leq 0$. The function F is not determined for $u > 0$ by the initial condition. We thus have two unknown functions F (for $u > 0$) and G after condition (1.13) is imposed, so that another condition must be gleaned. We return to the above differential equation in section 3.4.

A solution to the initial-boundary value problem can be viewed as the combination of a solution to a Cauchy problem in one region and the solution of a Goursat problem in another region. In particular, the characteristic $u = 0$ divides the positive quadrant of the xt-plane into two regions. Let R_1 denote the region where $u < 0$ and R_2 denote the region where $u > 0$ (see figure 3.1). The solution strategy is to solve a Cauchy problem for equation (3.4) in region R_2 and use the continuity of solutions to define n along the characteristic $u = 0$. Once $n(0,v)$ is determined, then with the condition along the $v = 0$ axis (i.e. the x-axis) a Goursat problem can be formulated. The immediate problem is that we have only the condition $n(gt,t) = 0$ on the non-characteristic curve formed by

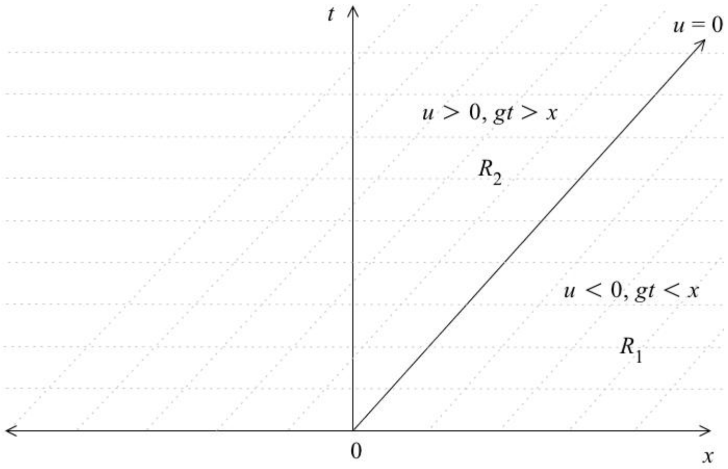

FIGURE 3.1: Characteristic coordinates

the t-axis. Another condition such as $n_u(gt,t) = m(t)$ is needed for the Cauchy problem, and it is this sort of condition that we seek in the next section.

3.3 Derivation of a compatibility condition

The solution (3.18) can be written as

$$n(u,v) = \begin{cases} e^{\frac{-gv^2}{4}+\frac{uv}{2}} W_1(u,v), & \text{if } (u,v) \in R_1, \\ e^{\frac{-gv^2}{4}+\frac{uv}{2}} W_2(u,v), & \text{if } (u,v) \in R_2, \end{cases} \tag{3.19}$$

where, for $i = 1, 2$,

$$W_i(u,v) = F_i''(u) + vF_i'(u) + \frac{v^2}{4}F_i(u) + \frac{1}{2}\int_0^v G(\tau)(\tau - v)^2 e^{-\frac{\tau u}{2}} d\tau.$$

The functions F_1 and F_2 will be determined in section 3.4.2. A typical Cauchy problem for R_2 is to solve PDE (3.4) subject to condition (1.12) and a condition of the form

$$n_x(0,t) = m(t),$$

where m is some given function. As noted earlier, the function $n_x(0,t)$ is not determined directly by $n(x,0) = n_0(x)$, $n(0,t) = 0$ and the PDE. Nonetheless, under the assumption that $n(x,t)$ decays faster than $\frac{1}{x^3}$ as $x \to \infty$ for all $t \geq 0$, we can derive this function.

There is a relationship between the total biomass and the total cell number, and this can be used to get a flux condition. Multiplying equation (3.3) by x and then integrating with respect to x from 0 to ∞ gives

$$\frac{\partial}{\partial t}\int_0^\infty xn(x,t)dx = \frac{1}{2}L(t) + g\int_0^\infty n(x,t)dx, \tag{3.20}$$

where

$$L(t) = \lim_{x\to\infty} x^2 \int_x^\infty n(\xi,t)d\xi.$$

Another relationship can be derived by integrating (3.3) with respect to x from 0 to ∞, *viz.*,

$$\frac{\partial}{\partial t}\int_0^\infty n(x,t)dx = \frac{1}{2}\int_0^\infty xn(x,t)dx. \tag{3.21}$$

Differentiating (3.21) with respect to t and using (3.20) yields

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int_0^\infty n(x,t)dx &= \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty xn(x,t)dx \\ &= \frac{g}{2}\int_0^\infty n(x,t)dx + \frac{1}{4}L(t). \end{aligned}\tag{3.22}$$

Let

$$\int_0^\infty n(x,t)dx = m(t),$$

then equation (3.22) becomes

$$m(t)'' - \frac{g}{2}m(t) = \frac{1}{4}L(t). \tag{3.23}$$

We know from the analysis of the separable solution that $L(t) \neq 0$ if $\lambda > \lambda_0$, and that $L(t) = 0$ for $\lambda = \lambda_0$. Although, it is clear from the separable case that solutions exist for which $L(t) \neq 0$, we restrict our attention to solutions $n(x,t)$ that decay more rapidly than $\frac{1}{x^3}$ as $x \to \infty$, for all $t \geq 0$. We thus assume that

$$\lim_{x\to\infty} x^3 n(x,t) = 0, \tag{3.24}$$

for all $t \geq 0$. This places the additional restriction on the initial condition,

$$\lim_{x\to\infty} x^3 n_0(x) = 0, \tag{3.25}$$

but it ensures that

$$L(t) = 0. \tag{3.26}$$

In applications, condition (3.25) is not that restrictive. Recall that $n_0(x)$ is the initial number density of cells of size x, which generally has compact support. Relation (3.26) and differential equation (3.23) imply

$$m(t) = k_1 e^{\sqrt{\frac{g}{2}}t} + k_2 e^{-\sqrt{\frac{g}{2}}t}, \tag{3.27}$$

where k_1 and k_2 are constants. Now,

$$m(0) = \int_0^\infty n_0(x)dx = 1,$$

and

$$m'(0) = \frac{1}{2}\int_0^\infty x n_0(x)dx \equiv \frac{1}{2}\mu_1;$$

hence, k_1 and k_2 satisfy

$$k_1 + k_2 = 1,$$

$$\sqrt{\frac{g}{2}}(k_1 - k_2) = \frac{1}{2}\mu_1.$$

We thus have

$$k_1 = \frac{1}{\sqrt{2g}}\Big[\frac{\sqrt{2g} + \mu_1}{2}\Big], \tag{3.28}$$

and

$$k_2 = -\frac{1}{\sqrt{2g}}\Big[\frac{\mu_1 - \sqrt{2g}}{2}\Big]. \tag{3.29}$$

Note that PDE (3.3) shows that

$$n_x(0,t) = \frac{1}{g}\int_0^\infty n(\xi,t)d\xi = \frac{1}{g}m(t). \tag{3.30}$$

3.4 Application of the initial-boundary conditions

3.4.1 The boundary conditions

The function G in the solution (3.19) can be determined if W_{vvv} can be determined (equation (3.16)). In this section we use the condition (3.30) to derive an expression for

W_{vvv} and show that under assumption (3.24), $G(v) \equiv 0$.

In terms of the characteristic coordinates, conditions (1.12), (1.26) and (3.30) are

$$n(0,t) = \hat{n}(gv, v) = 0, \tag{3.31}$$

$$n_t(0,t) = g\hat{n}_u(gv, v) + \hat{n}_v(gv, v) = 0, \tag{3.32}$$

and

$$n_x(0,t) = -\hat{n}_u(gv, v) = \frac{1}{g}m(v). \tag{3.33}$$

Note that since $t = v$ we write $m(t)$ as $m(v)$ for the convenience of the argument and vice versa. For the fast decay solution we have

$$\frac{\partial}{\partial t}\int_0^\infty xn(x,t)dx = g\int_0^\infty n(x,t)dx.$$

This says that the rate of change of the total biomass with respect to time is constant when there is no growth (i.e. $g = 0$), other than that things may gain mass. Let

$$\frac{gv^2}{4} - \frac{uv}{2} = b(u,v),$$

then,

$$\begin{aligned} W_{vvv}(u,v) = b'(u,v)e^{b(u,v)}\Big[(b'(u,v))^2\hat{n}(u,v) + 2b'(u,v)\hat{n}_v(u,v) + b''(u,v)\hat{n}(u,v) + \hat{n}_{vv}(u,v)\Big] \\ + e^{b(u,v)}\Big[2b'(u,v)b''(u,v)\hat{n}(u,v) + (b'(u,v))^2\hat{n}_v(u,v) + 2b''(u,v)\hat{n}_v(u,v) \\ + 2b'(u,v)\hat{n}_{vv}(u,v) + b'''(u,v)\hat{n}(u,v) + b''(u,v)\hat{n}_v(u,v) + \hat{n}_{vvv}(u,v)\Big], \end{aligned}$$

where the dash here denotes the derivative with respect to v. From now on we drop the circumflexes unless there is a danger of confusion. At $x = 0$ we have

$$b(gv,v) = -\frac{gv^2}{4}, \qquad b'(gv,v) = 0, \qquad b''(gv,v) = \frac{g}{2}, \qquad b'''(gv,v) = 0,$$

which implies

$$W_{vvv}(gv,v) = e^{-\frac{gv^2}{4}}\Big[\frac{3g}{2}n_v(gv,v) + n_{vvv}(gv,v)\Big]. \tag{3.34}$$

Equations (3.32) and (3.33) imply

$$n_v(gv,v) = m(v). \tag{3.35}$$

The next move is to work out $n_{vvv}(gv, v)$, which can be deduced by differentiating equation (3.32) three times with respect to v to get

$$n_{vvv}(gv, v) = -g^3 n_{uuu}(gv, v) - 3g^2 n_{uuv}(gv, v) - 3g n_{uvv}(gv, v). \tag{3.36}$$

In order to find $n_{uuu}(gv, v)$, $n_{uuv}(gv, v)$ and $n_{uvv}(gv, v)$ we need to determine $n_{uu}(gv, v)$, $n_{vv}(gv, v)$ and $n_{uv}(gv, v)$. We can find $n_{uv}(gv, v)$ simply by applying condition (3.31) on the characteristic equation (3.11), which yields

$$n_{uv}(gv, v) = 0. \tag{3.37}$$

To find $n_{uu}(gv, v)$ and $n_{vv}(gv, v)$ we differentiate (3.32) with respect to v to get

$$g^2 n_{uu}(gv, v) + 2g n_{uv}(gv, v) + n_{vv}(gv, v) = 0,$$

and (3.37) implies

$$n_{uu}(gv, v) = -\frac{1}{g^2} n_{vv}(gv, v).$$

Now, differentiating (3.33) with respect to v with the use of (3.37) one finds

$$n_{uu}(gv, v) = -\frac{1}{g^2} m'(v), \tag{3.38}$$

which implies

$$n_{vv}(gv, v) = m'(v). \tag{3.39}$$

To determine $n_{uuv}(gv, v)$ and $n_{uvv}(gv, v)$ we differentiate (3.11) once with respect to u and then with respect to v. At $x = 0$, equations (3.33) and (3.35) show that

$$n_{uuv}(gv, v) = -\frac{2}{g} m(v), \tag{3.40}$$

and

$$n_{uvv}(gv, v) = 2m(v). \tag{3.41}$$

Now, differentiating (3.33) twice with respect to v gives

$$g^2 n_{uuu}(gv, v) + 2g n_{uuv}(gv, v) + n_{uvv}(gv, v) = -\frac{1}{g} m''(v).$$

Equations (3.40) and (3.41) imply

$$n_{uuu}(gv, v) = -\frac{1}{g^3} m''(v) + \frac{2}{g^2} m(v). \tag{3.42}$$

Substituting (3.42), (3.41) and (3.40) into (3.36) yields

$$n_{vvv}(gv, v) = m''(v) - 2gm(v).$$

Equation (3.34) thus implies

$$W_{vvv}(gv, v) = e^{-\frac{gv^2}{4}} \Big[m''(v) - \frac{g}{2} m(v) \Big]. \tag{3.43}$$

Equation (3.23) implies that

$$W_{vvv}(gv, v) = \frac{1}{4} e^{-\frac{gv^2}{4}} L(v), \tag{3.44}$$

but under the condition (3.24) we know that $L(v) = 0$, and this gives

$$G(v) = 0.$$

The general form of the solution is thus

$$n(u, v) = \begin{cases} e^{\frac{-gv^2}{4} + \frac{uv}{2}} \Big[F_1''(u) + vF_1'(u) + \frac{v^2}{4} F_1(u) \Big] & \text{if } u \leq 0, \\ e^{\frac{-gv^2}{4} + \frac{uv}{2}} \Big[F_2''(u) + vF_2'(u) + \frac{v^2}{4} F_2(u) \Big] & \text{if } u > 0. \end{cases} \tag{3.45}$$

3.4.2 The initial condition

The application of the initial condition at $t = 0$, or equivalently $v = 0$, gives

$$F''(u) = n_0(-u), \tag{3.46}$$

for $u < 0$. This equation thus defines F_1. Integrating (3.46) from $-u$ to ∞ gives

$$F_1'(u) = \int_{-u}^{\infty} n_0(\tau) d\tau + q_1, \tag{3.47}$$

and integrating again yields

$$F_1(u) = \int_{-u}^{\infty} \int_{\xi}^{\infty} n_0(\tau) d\tau d\xi + u q_1 + q_2. \tag{3.48}$$

Here q_1 and q_2 are constants. Note that the function defined by equation (3.19) is a solution to equation (3.11), which was derived by differentiating equation (3.3) with respect to x and then using characteristic coordinates. It is clear that any smooth solution to equation (3.3) is a solution to (3.4) and hence defines a solution to (3.11), but the converse is not true as some information is lost in the differentiation. In other words, the class of solutions to (3.4) is larger than that of equation (3.3). This observation allows us to determine the constants q_1 and q_2. Rather than use the second order equation (3.11), we could have transformed equation (3.3) directly without differentiation. This leads to the following integrodifferential equation for W

$$W_v(u,v) = e^{-\frac{uv}{2}} \int_{-\infty}^{u} e^{\frac{\xi v}{2}} W(\xi, v) d\xi. \tag{3.49}$$

Let

$$M = F_1''(u) + vF_1'(u) + \frac{v^2}{4} F_1(u),$$

and

$$H = \frac{1}{2} \int_0^v G(\tau)(\tau - v)^2 e^{-\frac{\tau u}{2}} d\tau.$$

Then

$$\Big(M + H\Big)_v = e^{-\frac{uv}{2}} \int_{-\infty}^{u} e^{\frac{\xi v}{2}} M d\xi + e^{-\frac{uv}{2}} \int_{-\infty}^{u} e^{\frac{\xi v}{2}} H d\xi.$$

For any choice of G, it can be shown by direct calculation that

$$H_v = e^{\frac{-uv}{2}} \int_{-\infty}^{u} e^{\frac{\xi v}{2}} H d\xi;$$

hence, equation (3.49) places a restriction only on F, *viz.*,

$$\Big[F_1'(u) + \frac{v}{2} F_1(u)\Big] e^{\frac{uv}{2}} = \int_{-\infty}^{u} e^{\frac{\xi v}{2}} \Big[F_1''(\xi) + vF_1'(\xi) + \frac{v^2}{4} F_1(\xi)\Big] d\xi. \tag{3.50}$$

Now,

$$\int_{-\infty}^{u} e^{\frac{\xi v}{2}} F_1''(\xi)d\xi = F_1'(u)e^{\frac{uv}{2}} - \frac{v}{2}F_1(u)e^{\frac{uv}{2}} - \lim_{\xi\to-\infty} F_1'(\xi)e^{\frac{\xi v}{2}} + \frac{v}{2}\lim_{\xi\to-\infty} F_1(\xi)e^{\frac{\xi v}{2}} + \frac{v^2}{4}\int_{-\infty}^{u} F_1(\xi)e^{\frac{\xi v}{2}}d\xi, \quad (3.51)$$

and

$$v\int_{-\infty}^{u} e^{\frac{\xi v}{2}} F_1'(\xi)d\xi = vF_1(u)e^{\frac{uv}{2}} - v\lim_{\xi\to-\infty} F_1(\xi)e^{\frac{\xi v}{2}} - \frac{v^2}{2}\int_{-\infty}^{u} F_1(\xi)e^{\frac{\xi v}{2}}d\xi, \quad (3.52)$$

so that substituting (3.52) and (3.51) into (3.50) yields

$$\lim_{\xi\to-\infty} F_1'(\xi)e^{\frac{\xi v}{2}} + \frac{v}{2}\lim_{\xi\to-\infty} F_1(\xi)e^{\frac{\xi v}{2}} = 0. \quad (3.53)$$

Equation (3.53) must hold for all $v \geq 0$. In particular, for $v = 0$ we have

$$\lim_{\xi\to-\infty} F_1'(\xi) = 0, \quad (3.54)$$

and equation (3.53) reduces to

$$\frac{v}{2}\lim_{\xi\to-\infty} F_1(\xi)e^{\frac{\xi v}{2}} = 0,$$

which must be satisfied for all $v \geq 0$. Differentiating the above equation with respect to v gives

$$\lim_{\xi\to-\infty}\left(\frac{1}{2}F_1(\xi)e^{\frac{\xi v}{2}} + \frac{\xi v}{4}F_1(\xi)e^{\frac{\xi v}{2}}\right) = 0,$$

so that for $v = 0$, we have

$$\lim_{\xi\to-\infty} F_1(\xi) = 0. \quad (3.55)$$

Equations (3.54) and (3.47) imply that $q_1 = 0$; equations (3.55) and (3.48) imply that $q_2 = 0$. We thus have

$$F_1(u) = \int_{-u}^{\infty}\int_{\xi}^{\infty} n_0(\tau)d\tau d\xi. \quad (3.56)$$

It remains to determine F for $u > 0$, i.e. F_2. Condition (1.12) (or (3.31)) implies that

$$F_2''(gt) + tF_2'(gt) + \frac{t^2}{4}F_2(gt) = 0. \quad (3.57)$$

The general solution to equation (3.57) is

$$F_2(u) = c_1 e^{\frac{-u^2+2\sqrt{2g}u}{4g}} + c_2 e^{\frac{-u^2-2\sqrt{2g}u}{4g}}, \tag{3.58}$$

where c_1 and c_2 are constants. The constants c_i are determined by imposing a continuity condition on n across the characteristic projection $u = 0$, which divides the positive quadrant into two regions R_1 and R_2 (see figure 3.1). In the next section we determine these constants and establish the smoothness of n across the characteristic projection $u = 0$.

3.5 Properties of the solution

In this section we show that there are unique constants c_1 and c_2 such that F_2 defined by equation (3.58) leads to a solution n continuous across the characteristic projection $u = 0$. Once this is established we show that the solution constructed by "patching" the solution in R_1 with that in R_2 is unique.

3.5.1 Smoothness across $u = 0$

Recall that a solution to equation (3.11) is given by equation (3.45), where F_1 is defined by equation (3.56) and F_2 is given by equation (3.58). For $i = 1, 2$, let

$$n_i(u, v) = e^{\frac{-gv^2}{4}+\frac{uv}{2}} \left[F_i''(u) + vF_i'(u) + \frac{v^2}{4} F_i(u) \right]. \tag{3.59}$$

We require that n be continuous across the projection $u = 0$, i.e.

$$\lim_{u \to 0^-} n_1(u, v) = \lim_{u \to 0^+} n_2(u, v), \tag{3.60}$$

for all $v \geq 0$. Condition (3.60) implies

$$\lim_{u \to 0^-} \left(F_1''(u) + vF_1'(u) + \frac{v^2}{4} F_1(u) \right) = \lim_{u \to 0^+} \left(F_2''(u) + vF_2'(u) + \frac{v^2}{4} F_2(u) \right). \tag{3.61}$$

Now,

$$\lim_{u \to 0^-} F_1''(u) = n_0(0) = 0,$$

$$\lim_{u\to 0^-} F_1'(u) = \int_0^\infty n_0(\xi)d\xi = 1,$$

and

$$\begin{aligned}\lim_{u\to 0^-} F_1(u) &= \Big[\xi \int_\xi^\infty n_0(\tau)d\tau\Big]_0^\infty + \int_0^\infty \xi n_0(\xi)d\xi \\ &= \int_0^\infty \xi n_0(\xi)d\xi \\ &= \mu_1.\end{aligned}$$

The right hand side of equation (3.61) is

$$\lim_{u\to 0^+} \Big(F_2''(u) + vF_2'(u) + \frac{v^2}{4}F_2(u)\Big) = v\Big(\frac{c_1 - c_2}{\sqrt{2g}}\Big) + \frac{v^2}{4}(c_1 + c_2); \tag{3.62}$$

hence, equation (3.61) yields

$$v + \frac{v^2}{4}\mu_1 = v\Big(\frac{c_1 - c_2}{\sqrt{2g}}\Big) + \frac{v^2}{4}(c_1 + c_2),$$

which must be satisfied for all $v \geq 0$. The above equation implies

$$c_1 - c_2 = \sqrt{2g},$$

$$c_1 + c_2 = \mu_1,$$

so that

$$c_1 = \frac{\sqrt{2g} + \mu_1}{2} \equiv \sqrt{2g}k_1, \tag{3.63}$$

$$c_2 = \frac{\mu_1 - \sqrt{2g}}{2} \equiv -\sqrt{2g}k_2, \tag{3.64}$$

where k_1 and k_2 are given by (3.28) and (3.29), respectively. The differentiability of the solution across the projection curve $u = 0$ requires the first derivatives of n (n_u and n_v) to exist there, namely,

$$\lim_{u\to 0^-} (n_1(u,v))_u = \lim_{u\to 0^+} (n_2(u,v))_u, \tag{3.65}$$

$$\lim_{u\to 0^-} (n_1(u,v))_v = \lim_{u\to 0^+} (n_2(u,v))_v, \tag{3.66}$$

for all $v \geq 0$. For condition (3.65) we differentiate (3.59) with respect to u, which gives

$$(n_i(u,v))_u = \frac{v}{2}n_i(u,v) + e^{\frac{-gv^2}{4}+\frac{uv}{2}}\Big[F_i'''(u) + vF_i''(u) + \frac{v^2}{4}F_i'(u)\Big].$$

Then condition (3.65) implies

$$\lim_{u\to 0^-}\left(F_1'''(u)+vF_1''(u)+\frac{v^2}{4}F_1'(u)\right)=\lim_{u\to 0^+}\left(F_2'''(u)+vF_2''(u)+\frac{v^2}{4}F_2'(u)\right), \tag{3.67}$$

where

$$\lim_{u\to 0^-}F_1'''(u)=-\frac{1}{g}=\lim_{u\to 0^+}F_2'''(u);$$

hence, equation (3.67) yields

$$-\frac{1}{g}+\frac{v^2}{4}=-\frac{1}{g}+\frac{v^2}{4}.$$

Similarly for n_v we find that condition (3.66) implies

$$0+1+\mu_1=0+1+\mu_1.$$

Therefore, the solution $n(u,v)$ is continuous on the characteristic projection $u=0$ and it is also differentiable.

3.5.2 Uniqueness of the solution

In this section we establish the uniqueness of the solution by showing that the Cauchy problem in R_2 and the Goursat problem in R_1 (see figure 3.1) have unique solutions. Uniqueness is established once it is shown that equation (3.13) has a unique solution W for the Cauchy problem and a unique solution for the Goursat problem. Let

$$L[W]=W_{uv}(u,v)+\frac{v}{2}W_v(u,v)-W(u,v).$$

The Cauchy problem consists of solving $L(W)=0$ in R_2 subject to the initial conditions

$$W(gv,v)=0,$$

$$W_u(gv,v)=-\frac{1}{g}m(v)e^{-\frac{gv^2}{4}}.$$

It is clear that the operator L has analytic coefficients, the initial data are non-characteristic, and the initial data are analytic. The solution obtained (3.45) with F_2 given by (3.58) is the unique analytic solution guaranteed by the Cauchy-Kowalevski theorem. The result can be sharpened by Holmgren's theorem [10], which shows that this solution is unique among functions in C^2. Although, these results are local in character, we can appeal to the global uniqueness result for Cauchy problems since L is a linear operator (cf. Garabedian [11], pg. 110).

The Goursat problem consists of solving $L(W) = 0$ in R_1 subject to the conditions

$$W(u,0) = F_1''(u),$$

$$W(0,v) = w(v),$$

along the characteristics $v = 0$ and $u = 0$ respectively (cf. figure 3.1). Note that since the Cauchy problem has a unique solution that is in C^2, the function $w(v)$ is uniquely determined. The uniqueness of the solution in R_1 is shown by first transforming the linear second order differential equation $L(W) = 0$ into an integrodifferential equation and then exploiting a Lipschitz condition. To do so, we first integrate $L(W) = 0$ over a region A (see figure 3.2), which gives

$$W(u,v) = W_0(u,v) + \int_0^u \frac{-v}{2} W(\xi, v) d\xi + \frac{3}{2} \int_0^u \int_0^v W(\xi, \eta) d\eta d\xi, \qquad (3.68)$$

where

$$W_0(u,v) = W(u,0) + W(0,v) - W(0,0).$$

The above equation shows that $W(u,0)$, $W(0,v)$ and $W(0,0)$ depend on the data along

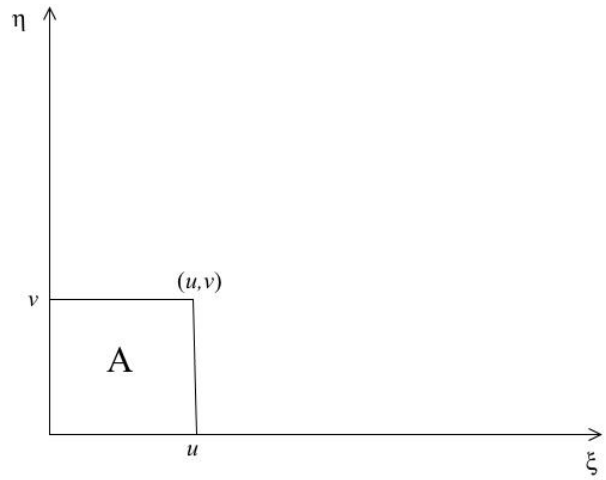

FIGURE 3.2: Goursat problem

the characteristic curves $u = 0$ and $v = 0$. Let

$$\| W(u,v) \| = \sup_{(u,v)\in A} | W(u,v) | .$$

Suppose that W and Z are two solutions of (3.68). Then

$$| W(u,v) - Z(u,v) | \leq \frac{v}{2}\int_0^u | W(\xi,v) - Z(\xi,v) | d\xi + \frac{3}{2}\int_0^u\int_0^v | W(\xi,\eta) - Z(\xi,\eta) | d\eta d\xi;$$

hence,

$$\| W(u,v) - Z(u,v) \| \leq \frac{v}{2}\int_0^u \| W(\xi,v) - Z(\xi,v) \| d\xi + \frac{3}{2}\int_0^u\int_0^v \| W(\xi,\eta) - Z(\xi,\eta) \| d\eta d\xi$$

$$\leq \| W(u,v) - Z(u,v) \| [2uv].$$

The above inequality shows that

$$\| W(u,v) - Z(u,v) \| = 0,$$

whenever,

$$uv < \frac{1}{2}.$$

We thus see that in any rectangular region $A_0 = [(u,v) : 0 \leq u \leq u_0,\ 0 \leq v \leq v_0]$ such that $u_0 v_0 < \frac{1}{2}$, the solution is unique. To extend the region of uniqueness, consider the shift (cf. figure 3.3)

$$u = u_0 + \hat{u},$$

and let

$$W(u,v) = W(u_0 + \hat{u}, v) = \hat{W}(\hat{u}, v).$$

Under this transformation equation (3.68) becomes

$$\hat{W}(\hat{u},v) = \hat{W}_0(\hat{u},v) + \int_0^{\hat{u}} \frac{-v}{2}\hat{W}(\xi,v)d\xi + \frac{3}{2}\int_0^{\hat{u}}\int_0^v \hat{W}(\xi,\eta)d\eta d\xi, \qquad (3.69)$$

where

$$\hat{W}_0(\hat{u},v) = \hat{W}(\hat{u},0) + \hat{W}(0,v) - \hat{W}(0,0).$$

Note that the characteristic data $\hat{W}(\hat{u},0) = W(u_0 + \hat{u}, 0)$ are given and that the characteristic data $\hat{W}(0,v) = W(u_0,v)$ are uniquely determined for $0 < v < v_0$, $u_0 v_0 < \frac{1}{2}$.

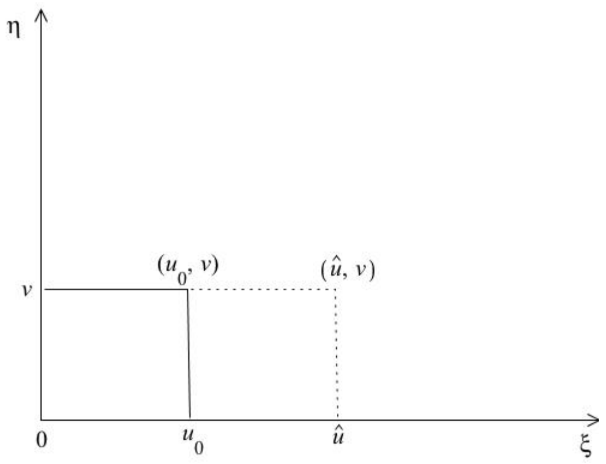

FIGURE 3.3: Goursat problem

Equation (3.69) is the same as (3.68), so the same argument can be used to show that the solution is unique for $0 \le u \le 2u_0$, $0 \le v \le v_0$, where $u_0 v_0 < \frac{1}{2}$. We can repeat this argument to establish uniqueness in the rectangle $0 \le u \le 3u_0$, $0 \le v \le v_0$, and, in general, for $0 \le u \le ku_0$, $0 \le v \le v_0$, where k is positive integer (see figure 3.4). In this manner we see that the solution must be unique in the strip $0 \le u < \infty$, $0 \le v \le v_0$.

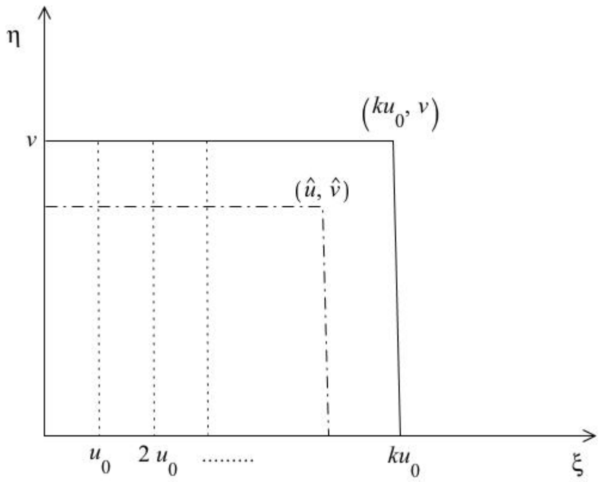

FIGURE 3.4: Goursat problem

To establish uniqueness in the quadrant $0 \le u < \infty$, $0 \le v < \infty$, suppose that there are two distinct solutions W and Z. Then there must be a point $(\hat{u}, \hat{v})$ such that $W(\hat{u}, \hat{v}) \ne Z(\hat{u}, \hat{v})$ for $\hat{u}, \hat{v} > 0$. Choose v_0 such that $v_0 > \hat{v}$ and choose u_0 such that $u_0 v_0 < \frac{1}{2}$. We can thus apply the arguments above to establish the solution must be unique in the strip $0 \le u < \infty$, $0 \le v \le v_0$. This strip must include the point $(\hat{u}, \hat{v})$ and hence $W(\hat{u}, \hat{v}) = Z(\hat{u}, \hat{v})$, which contradicts our assumption. The solution to the Goursat problem is therefore unique.

3.6 Examples of $n_0(x)$

In this section we give examples for two different choices of the initial data n_0. Recall that $n_0(x)$ is an initial probability distribution; hence, any choice must satisfy the conditions

$$n_0(x) \geq 0,$$

for $x \geq 0$, and

$$\int_0^\infty n_0(x)dx = 1.$$

In addition n_0 must satisfy

$$n_0(0) = 0,$$

owing to the condition $n(0, t) = 0$, $t \geq 0$; and n_0 must also satisfy the compatibility condition

$$n_0'(0) = \frac{1}{g}.$$

3.6.1 Example 1

The first example is the discontinuous function

$$n_0(x) = \begin{cases} \frac{x}{g} & \text{if } 0 \leq x \leq \sqrt{2g}, \\ 0 & \text{if } x > \sqrt{2g}. \end{cases}$$

Now,

$$n_0(x) = F_1''(-x),$$

so that in the (u, v) coordinates we have

$$F_1''(u) = \begin{cases} \frac{-u}{g} & \text{if } 0 \leq u \leq gv + \sqrt{2g}, \\ 0 & \text{if } u > gv + \sqrt{2g}, \end{cases}$$

which gives

$$F_1'(u) = \begin{cases} \frac{-u^2}{2g} & \text{if } 0 \le u \le gv + \sqrt{2g}, \\ 0 & \text{if } u > gv + \sqrt{2g}, \end{cases}$$

and

$$F_1(u) = \begin{cases} \frac{-u^3}{6g} & \text{if } 0 \le u \le gv + \sqrt{2g}, \\ 0 & \text{if } u > gv + \sqrt{2g}. \end{cases} \tag{3.70}$$

The general solution thus is constructed by patching the solution in R_1, with the solution in R_2 across the characteristic corresponding to $u = 0$ such that F_1 is defined by (3.70) and F_2 given by (3.58); thus,

$$n(u,v) = \begin{cases} e^{\frac{-gv^2}{4}+\frac{uv}{2}}\left[F_2''(u) + vF_2'(u) + \frac{v^2}{4}F_2(u)\right] & \text{if } 0 \le u \le gv, \\ e^{\frac{-gv^2}{4}+\frac{uv}{2}}\left[F_1''(u) + vF_1'(u) + \frac{v^2}{4}F_1(u)\right] & \text{if } gv \le u \le gv + \sqrt{2g}, \\ 0 & \text{if } u > gv + \sqrt{2g}. \end{cases}$$

Figure 3.5 depicts the behaviour of the above solution for $g = 2$ over time. Note that the discontinuity in n_0 at $x = 2$ propagates along the characteristic $u = 2t - x$ for $t > 0$.

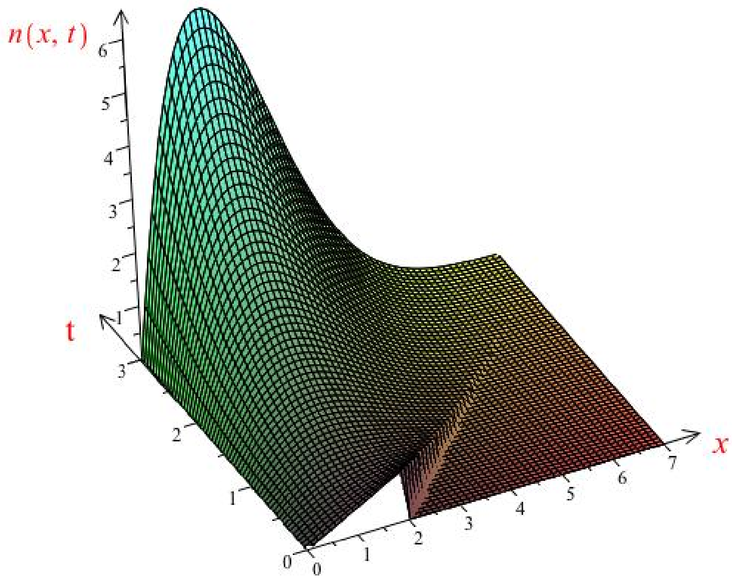

FIGURE 3.5: Discontinuous example of $n_0(x)$

3.6.2 Example 2

The next example uses a gamma type distribution for n_0. Let

$$n_0(x) = \frac{1}{g} x e^{-\frac{x}{\sqrt{g}}}. \tag{3.71}$$

Then

$$F_1''(u) = \frac{-u}{g} e^{\frac{u}{\sqrt{g}}},$$

$$F_1'(u) = e^{\frac{u}{\sqrt{g}}} - \frac{u}{g} e^{\frac{u}{\sqrt{g}}},$$

and

$$F_1(u) = 2g e^{\frac{u}{\sqrt{g}}} - u e^{\frac{u}{\sqrt{g}}}.$$

The general solution is constructed much like the previous example with F_1 defined above and F_2 given by (3.58); thus,

$$n(u,v) = \begin{cases} e^{\frac{-gv^2}{4}+\frac{uv}{2}} \left[F_2''(u) + vF_2'(u) + \frac{v^2}{4} F_2(u) \right] & \text{if } 0 \le u \le gv, \\ e^{\frac{-gv^2}{4}+\frac{uv}{2}} \left[F_1''(u) + vF_1'(u) + \frac{v^2}{4} F_1(u) \right] & \text{if } gv \le u < \infty. \end{cases}$$

Figure (3.6) illustrates the general solution for n_0 given by (3.71) with $g = 2$ over time.

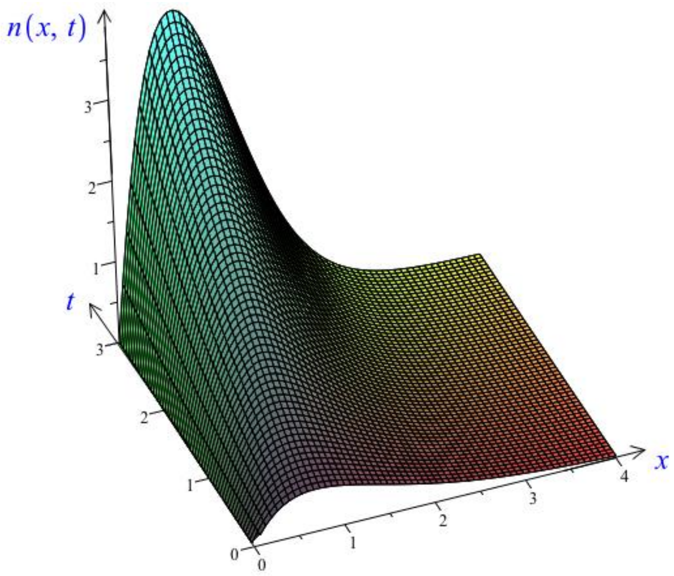

FIGURE 3.6: An exponential example of $n_0(x)$

Note that for both examples the solution surface is smooth along the characteristic corresponding to $u = 0$ as predicted in section 3.5. Note also that the solution in the "upper triangle" R_2 dominates the long term shape of the surface. This solution does not depend directly on n_0 except through the first moment μ_1 that appears in the initial condition for n_x along the $x = 0$ axis (cf. equations (3.58), (3.63) and (3.64)). In the next chapter we look closer at the role of the solution in R_2 and its relation to the separable solution.

Chapter 4

The role of the separable solution

In this chapter, we examine the role of the separable solution in relation to the general solution. In the previous chapter, we noted that the long term solution is dominated by the solution in the region R_2, and this solution depends on n_0 only through the first moment which appears in the condition $n_x(0,t)$. We first look at this condition and its relationship to the separable solution. We look at the separable solutions, particularly the one that decays faster than $\frac{1}{x^3}$. Then, we examine the asymptotic behaviour of the fast decay general solution as $t \to \infty$ for any PDF initial data. In the last section, we briefly discuss the other separable solutions $y(x;\lambda)$ corresponding to higher eigenvalues from the perspective of the general solution.

4.1 The critical eigenvalue solution

In this section we look at the separable solution corresponding to $\lambda = \lambda_0$ from the standpoint of the general solution. If $\lambda = \lambda_0$, then the separable solution is given by

$$n(x,t) = e^{\lambda_0 t} y(x;\lambda_0),$$

where

$$y(x;\lambda_0) = -K_1 \psi_1'(x),$$

ψ_1 is given by (2.8), and $K_1 = \frac{1}{\sqrt{2g}}$. In terms of the general solution, this solution corresponds to initial data choice

$$\begin{aligned} n_0(x) &= -K_1\psi_1'(x) \\ &= \frac{1}{2g\sqrt{2g}}\Big[e^{\frac{-x^2-2\sqrt{2g}x}{4g}}(x^2+2\sqrt{2g}x)\Big]. \end{aligned} \tag{4.1}$$

We first look at the compatibility condition derived in section 3.3. Consider the derivative of the general solution (3.59) with respect to u,

$$n_u(u,v) = \frac{v}{2}n(u,v) + e^{\frac{-gv^2}{4}+\frac{uv}{2}}\Big[F'''(u) + vF''(u) + \frac{v^2}{4}F'(u)\Big].$$

When $x = 0$, $u = gv$, and the derivative can be expressed as

$$n_u(gv,v) = e^{\frac{u^2}{4g}}\Big[F_2'''(u) + \frac{u}{g}F_2''(u) + \frac{u^2}{4g^2}F_2'(u)\Big],$$

where $F_2(u)$ is given by (3.58); thus,

$$n_u(gv,v) = -\frac{1}{g}\Big[\frac{1}{\sqrt{2g}}c_1e^{\sqrt{\frac{g}{2}}v} - \frac{1}{\sqrt{2g}}c_2e^{-\sqrt{\frac{g}{2}}v}\Big],$$

where c_1 and c_2 are given by (3.63) and (3.64), respectively. Recall that F_2 depends "weakly" on n_0 through the constants c_1 and c_2. These constants depend on the first moment of the distribution n_0, i.e.,

$$\mu_1 = \int_0^\infty xn_0(x)dx. \tag{4.2}$$

Substituting (4.1) into (4.2) gives

$$\begin{aligned} \mu_1 &= \int_0^\infty x\frac{1}{2g\sqrt{2g}}\Big[e^{\frac{-x^2-2\sqrt{2g}x}{4g}}(x^2+2\sqrt{2g}x)\Big]dx \\ &= \sqrt{2g}, \end{aligned}$$

which yields

$$c_1 = 1,$$

$$c_2 = 0.$$

We thus have

$$n_u(gv, v) = -\frac{1}{g\sqrt{2g}} e^{\sqrt{\frac{g}{2}}v},$$

and

$$F_2(u) = \sqrt{2g} e^{\frac{-u^2+2\sqrt{2g}u}{4g}},$$

for this separable solution.

In contrast, the function F_1 in the general solution depends strongly on the choice of n_0, *viz.*,

$$\begin{aligned} n_0(x) &= -K_1\psi_1'(x) \\ &= F_1''(-x). \end{aligned}$$

The above expression implies

$$F_1'(u) = -K_1 \int_{-u}^{\infty} \psi_1(\xi) d\xi,$$

and here

$$F_1(u) = \sqrt{2g} e^{\frac{-u^2+2\sqrt{2g}u}{4g}}.$$

The separable solution with $\lambda = \lambda_0$ is thus a special case when $F_1(u) = F_2(u)$.

The separable solution is not the only solution with this property. It is evident that any suitable initial data of the form

$$n_0(x) = F_2''(-x),$$

will also lead to the $F_1(u) = F_2(u)$. The separable solution, however, is special because of its asymptotic relationship to the general solution as $t \to \infty$.

It is known that there is an associated eigenvalue problem in certain examples on the fragmentation equation and that, for arbitrary initial data, the solution is asymptotic to the eigenfunction as $t \to \infty$ (cf. [4, 5, 7]). If $x > 0$ is fixed then, for t sufficiently large, the point (x, t) will lie in R_2 (figure 3.1). This means that for any fixed $x > 0$, the solution at (x, t) will be $n_2(x, t)$ for t large. Recall that

$$\begin{aligned} n_2(x,t) &= \frac{c_1}{4g^2} e^{\frac{-x^2-2\sqrt{2g}x}{4g}} (x^2 + 2\sqrt{2g}x) e^{\sqrt{\frac{g}{2}}t} + \frac{c_2}{4g^2} e^{\frac{-x^2+2\sqrt{2g}x}{4g}} (x^2 - 2\sqrt{2g}x) e^{-\lambda_0 t} \\ &= K_1 c_1 y(x, \lambda_0) e^{\lambda_0 t} + \frac{c_2}{4g^2} e^{\frac{-x^2+2\sqrt{2g}x}{4g}} (x^2 - 2\sqrt{2g}x) e^{-\lambda_0 t}. \end{aligned}$$

Evidently

$$\lim_{t\to\infty} \frac{n_2(x,t)}{K_1 y(x,\lambda_0)e^{\lambda_0 t}} = c_1,$$

and that means

$$n(x,t) \sim K_1 c_1 y(x,\lambda_0)e^{\lambda_0 t}.$$

We thus see that for any choice of initial data n_0 the long term behaviour of the solution is a multiple of the separable solution with $\lambda = \lambda_0$. The only trace of the initial data appears in the constants c_i and k_i, where μ_1 is the first moment of the initial distribution. If the initial data distributions n_0 and $\tilde{n}_0$ have the same first moment, then

$$\tilde{n}(x,t) \sim n(x,t),$$

as $t \to \infty$, where n and $\tilde{n}$ denote the solutions corresponding to n_0 and $\tilde{n}_0$ initial data respectively.

4.2 Solution for higher eigenvalues

If $\lambda > \lambda_0$, then it has been shown that $y(x;\lambda) \sim \frac{2g}{x^3}$ as $x \to \infty$. Although these slow decay solutions might be of limited interest in the biological model, they nonetheless signal another class of solutions for the general problem.

In section 3.3 a decay condition was placed on the general solution and an initial data (cf. equations (3.24), (3.25)). Under these conditions it follows that the function G that appears in the general solution (3.18) is zero (section 3.4.1). The separable solutions with $\lambda > \lambda_0$; however, do not satisfy condition (3.24), and this prompts queries about solutions that decay no faster than $\frac{1}{x^3}$.

Recall that $G(t)$ is gleaned from the solution of the differential equation (3.23), i.e.,

$$m''(t) - \frac{g}{2}m(t) = \frac{1}{4}L(t), \tag{4.3}$$

where $L(t)$ is given by

$$L(t) = \lim_{x\to\infty} x^2 \int_x^\infty n(\xi,t)d\xi.$$

Equations (3.16), (3.44) and (4.3) show that

$$G(t) = \frac{1}{4} e^{\frac{gt^2}{4}} L(t),$$

so that in general G is a non zero function.

To illustrate the above comments consider a separable solution

$$n(x,t) = -e^{\lambda t} \psi'(x;\lambda),$$

where $\psi(x;\lambda)$ is given by (2.7) and $\lambda > \lambda_0$. Then

$$\begin{aligned} L(t) &= \lim_{x\to\infty} x^2 \int_x^\infty n(\xi,t) d\xi \\ &= (4\lambda^2 - 2g)e^{\lambda t} \\ &\neq 0. \end{aligned}$$

For this choice

$$G(t) = \frac{1}{2}(2\lambda^2 - g)e^{\frac{gt^2}{4} - \lambda t}.$$

In terms of the general solution, the separable solution is given by

$$\begin{aligned} n(x,t) = e^{\frac{gt^2}{4} - \frac{tx}{2}} \Big[F''(gt-x) + tF'(gt-x) + \frac{t^2}{4} F(gt-x) \\ + \frac{1}{4}(2\lambda^2 - g) \int_0^t e^{\frac{g\tau^2}{4} - \lambda\tau - \frac{\tau(gt-x)}{2}} (\tau - t)^2 d\tau \Big]. \end{aligned}$$

Here,

$$n_0(x) = -\psi'(x;\lambda),$$

and F is given by

$$F(-x) = \int_x^\infty \int_\xi^\infty n_0(\tau) d\tau d\xi.$$

Chapter 5

Conclusion and future work

The general solution form to

$$n_t(x,t) + gn_x(x,t) = \int_x^\infty n(\xi,t)d\xi - \frac{x}{2}n(x,t),$$

is given by

$$n(x,t) = e^{\frac{gt^2}{4}-\frac{tx}{2}}\Big[F''(gt-x) + tF'(gt-x) + \frac{t^2}{4}F(gt-x) + \frac{1}{2}\int_0^t G(\tau)(\tau-t)^2 e^{-\frac{\tau(gt-x)}{2}}d\tau\Big].$$

This general form for the fast decay solutions is

$$n(x,t) = e^{\frac{gt^2}{4}-\frac{tx}{2}}\Big[F''(gt-x) + tF'(gt-x) + \frac{t^2}{4}F(gt-x)\Big].$$

The problem is well-posed for the rapid decay solutions. This type of solution reaches a steady size distribution that depends on the initial distribution n_0 only through the first moment and is asymptotic to the eigenfunction $y(x;\lambda_0)$ as $t \to \infty$. Essentially, the general solution is dominated by the solution in the region R_2 (see figure 3.1) for t large.

We showed in chapter 2 that there are eigenfunction solutions $y(x;\lambda)$ with continuous spectrum of the form $[\lambda_0, \infty)$, and that this solution is unique for fixed λ and unimodal. We also showed that $y(x;\lambda)$ decays exponentially when $\lambda = \lambda_0$ as $x \to \infty$, and when $\lambda > \lambda_0$, $y(x;\lambda) \sim O(\frac{1}{x^3})$ as $x \to \infty$. In chapter 3, we solved the initial-boundary value problem for solutions that decay faster than $\frac{1}{x^3}$ as $x \to \infty$, and showed that the general solution across u_0 is smooth and is unique. Furthermore, we illustrated the theory by giving discontinuous and smooth examples of n_0. These examples show that the solution is dominated by Cauchy problem solution in time. In chapter four, we examined the role of the separable solution. We found that the separable solution corresponding to λ_0 is special in that the solutions in R_1 and R_2 are the same. More significantly, we found that the general solution is asymptotic to this separable solution as $t \to \infty$ for any PDF initial data.

The separable solution with $\lambda > \lambda_0$ show that there are slow decay solutions to the problem. Such solutions may be of limited interest biologically, but they illustrate that another family of solutions exit. For slow decay solutions it is not clear what $L(t)$ (or equivalently $G(t)$) is in the general solution. The determination of G for the general solutions remains an open problem.

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