A CLASS OF ABSOLUTE RETRACTS

A Thesis presented in partial fulfilment
of the requirements for the degree of
Doctor of Philosophy in Mathematics
at Massey University.

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1973

MASSEY UNIVERSITY

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ABSTRACT

A restricted version of the Tietze Theorem is that a continuous mapping of a closed subspace of a metric space ranging in a closed interval may be extended to a continuous function defined upon the whole metric space. This may be viewed as a property of the closed interval and is expressed by saying that the interval is an absolute extensor. Thus, absolute extensors may be viewed as a generalisation of real intervals, and many of the desirable properties of intervals have been generalised to the class of absolute extensors.

In 1951, Dugundji showed that every convex subset of a locally convex linear topological space is an absolute extensor, thus dramatically extending the Tietze theorem.

In this thesis, a class of subsets of a normed linear space is defined. This new class of sets includes the convex sets and it is shown that these new sets are also absolute extensors.

ACKNOWLEDGEMENT

I wish to express my gratitude to Dr. Ivan L. Reilly for his guidance during the preparation of this thesis.

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Introduction

The first theorem concerning absolute extensors was undoubtedly the Tietze extension theorem, a theorem which is found in virtually every textbook of general topology, and which is one of the fundamental theorems of modern analysis. Yet, it was not until 1931 that Borsuk [1] introduced the concept of absolute retracts and not until 1951 that Dugundji [4] proved the first significant generalization of the Tietze theorem. The first textbook of the subject was Hu's book 'Theory of Retracts', Hu [7], in 1965. Until that time, the only widely read texts to have mentioned such spaces were Lefschetz, 'Topics in Topology', and Kuratowski, 'Topologie I - II', each devoting a relatively small space to this important class of spaces.

It is difficult to understand why this should be so. The theory is satisfying and elegant, the problems difficult and challenging, and, as Hu [7] observes, 'the theory of retracts serves as the natural link connecting combinatorial topology and set-theoretic topology'.

A possible explanation for the slow development of the basic concepts is that of the emphasis in the usual statement of the Tietze theorem. The theorem is ordinarily viewed as a statement concerning the class of all normal spaces, which of course it is. But, and here the emphasis changes completely, it is also a statement of a very important property of the unit interval, and it is with this emphasis that the theorem is in the spirit of the theory of absolute extensors. The Dugundji theorem (Th. 2.17) is an elegant and far reaching generalization of the Tietze theorem.

The problem of extending continuous functions is closely related to a problem which superficially appears to be of quite a different nature. If $\{U_a : a \in A\}$ is an indexed family of non-empty subsets of a set Y, then the axiom of choice guarantees the existence of a function $f : A \to Y$ such that

$f(a) \in U_a$ for each $a \in A$. Suppose that A and Y are topological spaces. May f be chosen so as to be continuous? Chapter III is concerned with this problem and with relating this problem to the problem of continuous extensions.

Due to the lack of general familiarity with the theory of absolute extensors, Chapter I is devoted to the basic definitions and the fundamental properties of these spaces.

Indexed coverings of a space are special cases of indexed families of subsets, and certain results concerning the properties of such collections are necessary for the proofs of the fundamental theorems of the theory. Chapter II is devoted to the systematic treatment of certain types of coverings and, since this provides all of the machinery necessary for the proof of the Dugundji theorem, the chapter concludes with a proof of this fundamental and important theorem (Th. 2.17).

Chapter III investigates the connexion between the problem of finding continuous extensions and that of obtaining a continuous choice function for an indexed family of sets.

Chapter IV is concerned with proving the main theorem concerning continuous choice functions. A new class of sets is defined and it is then shown that the basic theorem of Chapter III may be extended to include these sets. The class thus provides a wide new class of metrizable absolute extensors.

Another point deserves comment. Most authors define an indexed collection of sets as a function having the index set as domain and ranging in the set of all (non-empty) subsets of some set. They then proceed to ignore (at least notationally, and frequently philosophically as well) their own definition. In this paper the definition is used systematically in all statements and, more importantly, in all proofs. I believe that the proofs become more precise and 'cleaner' as a result. Agreement on this may depend upon the reader's background and temperament.

Finally, as pointed out in the text, the Urysohn Lemma remains the fundamental and perhaps the only tool for constructing continuous functions 'from nothing'. For this reason, and because of its fundamental importance in the theory of absolute extensors, I have included a proof of the theorem in the appendix, along with several definitions and small theorems which do not seem to be a part of every general topologist's vocabulary.

CHAPTER I: ABSOLUTE EXTENSORS

There are two problems of topology, closely related, which have stimulated considerable research in the subject: When can a given function be extended? When is a given space a retract of some larger space? The definitions which follow make precise these questions.

1.1 Definition: Suppose that X is a topological space, A a subspace of X, and $f : A \to Y$ is a function. Then $F : X \to Y$ is called an extension of f if and only if $F(a) = f(a)$ for all $a \in A$.

1.2 The Extension Problem: Given A, X, Y and f as in definition 1.1, under what conditions does there exist a continuous extension of f?

1.3 Definition: Suppose that A is a subspace of X. A is called a retract of X if and only if there is a continuous function $r : X \to A$ such that $r(a) = a$ for every $a \in A$. The function r is called a retraction of X onto A.

1.4 The Retraction Problem: Given X and A, is there a retraction of X onto A?

1.5 Remark: It is clear that the retraction problem is a special case of the extension problem. Let Y = A and f be the identity map $i_A : A \to A$. Then there is a retraction of X onto A if and only if there is a continuous extension of f. In fact, the problems are 'equivalent', in a sense made precise by lemma 1.7.

1.6 Remark: Pursuing the extension problem in one direction leads to the discovery that in many important cases the answer depends only upon the homotopy class of f. This is the direction followed, for example, in Hu [6]. We shall pursue a different approach here, leading to quite a different theory for which we need only several of the basic properties concerning the two problems. The following lemma shows that the two problems are very closely related.

1.7 Lemma: Suppose that $f : A \to Y$ is continuous where $A \subset X$. Then f has an extension $F : X \to Y$ if and only if Y is a retract of $\mathrm{adj}(X,Y,f)$.

1.8 Remark: The space $\mathrm{adj}(X,Y,f)$ is described in the Appendix I.

Proof of 1.7: Let $W = \mathrm{adj}(X,Y,f)$ and suppose that $r : W \to Y$ is a retraction onto Y. Let $p : X \cup Y \to W$ be the natural projection and define, for each $x \in X$, $F(x) = r(p(x))$. F is clearly continuous and, for $x \in A$, $F(x) = r(p(x)) = r(f(x)) = f(x)$. Thus F is a continuous extension of f.

For the converse, suppose that F is a continuous extension of f, and let $w \in W$. Define $r : W \to Y$ as follows: if $w \in Y$, then $r(w) = w$. If $w \notin Y$, then there is a unique $x \in X-A$ for which $p(x) = w$. Let $r(w) = F(x)$. Then certainly $r : W \to Y$ and $r|Y$ is the identity on Y. It remains to show that r is continuous. Consider the composition $r \circ p : X \cup Y \to Y$. Then $(r \circ p)|X = F$ and $(r \circ p)|Y = i_Y$ and thus $r \circ p$ is continuous. Since p is a quotient mapping, it follows that r is continuous.

1.8.1 Lemma: If X is a Hausdorff space and if $r : X \to A$ is a retraction of X onto a subspace A, then A is closed in X.

Proof: Assume that $\{x_\nu\}$ is a net in A for which $\lim x_\nu = x$. Since r is continuous and since limits are unique, $r(x) = r(\lim x_\nu) = \lim r(x_\nu) = \lim x_\nu = x$. Thus $x \in A$, and A is closed in X.

1.9 Remark: Let P be a property of topological spaces. P is called a hereditary property if $P(A)$ is always implied by $P(X)$ and $A \subset X$. P is weakly hereditary if $P(A)$ is always implied by P(X) and A closed in X. If we restrict ourselves to the class of Hausdorff spaces, then the preceeding lemma shows that a retract of X shares all of the weakly hereditary properties possessed by X itself. In particular, if X is normal, then so is every retract of X. Retracts, as one would expect, inherit many properties from X other than the weakly hereditary ones. Hu [7] is an excellent catalogue of such properties. Only one of these is of interest to our work.

1.10 Definition: A space X is said to have the fixed point property if and only if for each continuous function $f : X \to X$ there is a point $x \in X$ for which $f(x) = x$.

1.11 Lemma: Suppose that X has the fixed point property and that A is a retract of X. Then A has the fixed point property.

Proof: Let $r : X \to A$ be a retraction of X onto A and let $f : A \to A$ be continuous. Since $(f \circ r)$ is a continuous map of X into itself, it follows that there is some $x \in X$ for which $f(r(x)) = x$. Since x is in the range of f, it must be that $x \in A$, and so $r(x) = x$. Thus $f(x) = f(r(x)) = x$, and so A has the fixed point property.

1.12 Remark: The definition 1.1 of extension essentially involves three objects: the spaces X and Y and the function f. As previously remarked, concentrating attention on the function f leads naturally into homotopy theory. In the following, attention is focused on the spaces X and Y, leading to quite a different theory.

1.13 Definition: Y is an extensor for X if and only if for each closed subspace A of X and each continuous $f : A \to Y$, there is a continuous $F : X \to Y$ which is an extension of f.

<u>1.14 Definition</u>: Y is a retractor for X if and only if e(Y) is a retract of X whenever $e : Y \to X$ is a homeomorphism of Y onto a closed subspace of X.

<u>1.15 Remark</u>: Suppose that some embedding of Y into X is a retract of X. Is Y a retractor for X? The following example shows that the answer is generally 'no'.

<u>1.16 Example</u>: Let $X = [0,1] \cup [2,3]$ with the topology inherited from the real line and let $Y = \{y_1, y_2\}$ with the discrete topology. Define embeddings e_1 and e_2 as follows:

$$e_1(y_1) = 0 \qquad e_2(y_1) = 0$$
$$e_1(y_2) = 2 \qquad e_2(y_2) = 1$$

Then $e_1(Y)$ and $e_2(Y)$ are homeomorphs of Y which are closed in X. $e_1(Y)$ is easily seen to be a retract of X, but $e_2(Y)$ cannot be since any retraction $r : X \to e_2(Y)$ must have $r(0) = 0$ and $r(1) = 1$; It follows that $r|[0,1] : [0,1] \to \{0,1\}$ is a retraction of the closed unit interval onto its end points, which is impossible.

<u>1.17 Lemma</u>: If Y is an extensor for X, then Y is a retractor for X.

<u>Proof</u>: Suppose $e : Y \to X$ is a homeomorphism of Y onto a closed subset e(Y) of X. Then $e^{-1} : e(Y) \to Y$ is continuous and so has a continuous extension $f : X \to Y$. Let $r = (e \circ f) : X \to e(Y)$. For $x \in e(Y)$, $r(x) = (e \circ f)(x) = (e \circ e^{-1})(x) = x$, and so r is a retraction.

The following theorem shows that a non-normal space has no interesting Hausdorff extensors.

<u>1.18 Theorem</u>: Suppose that Y is a Hausdorff space and that X is a space which is not normal. Then Y is an extensor for X if and only if Y consists of at most one point.

<u>Proof</u>: If Y has at most one point, then every function to Y is continuous and so Y is an extensor for every space.

Conversely, assume that y_1 and y_2 are distinct points of Y_1 and that Y is an extensor for X. Since X is not normal, there are disjoint closed sets A and B which do not have disjoint neighborhoods in X. Define $f : A \cup B \to Y$ by $f(x) = y_1$ if $x \in A$ and $f(x) = y_2$ if $x \in B$. Let $F : X \to Y$ be a continuous extension of f, and let N_1 and N_2 be disjoint neighborhoods of y_1 and y_2 respectively in Y. Then $F^{-1}(N_1)$ and $F^{-1}(N_2)$ are disjoint neighborhoods of A and B respectively, contrary to the choice of A and B. Thus Y is not an extensor for X.

1.19 Remark: If, in the definition 1.13 of extensor, attention is focused upon the space Y, then it becomes natural to ask if Y may be an extensor for all spaces X. The preceeding Theorem 1.18 shows that the theory arising from such questions would be uninteresting. Some restriction is necessary.

1.20 Definition: Y is an absolute extensor (AE) if and only if Y is an extensor for every metrizable space.

Similarly,

1.21 Definition: Y is an absolute retract (AR) if and only if Y is a retractor for every metrizable space.

1.22 Remark: The restriction to metrizable spaces in these definitions requires a certain amount of comment. In view of Theorem 1.18, the more natural restriction would be to normal spaces. However, the theory for metrizable spaces develops in a much more satisfying fashion and, in addition, there are certain cases in which AE's and AR's as defined above operate 'properly' for a larger class of spaces. The best results in this direction are due to Michael [10] and are summarized in theorems 1.24 and 1.25 below.

Also, notice that every non-metrizable space Y is an AR according to definition 1.21. As will be seen, there are also many non-metrizable AE's.

<u>1.23 Lemma</u>: Every AE is also an AR.

<u>Proof</u>: Follows immediately from lemma 1.17.

The following theorems of E. Michael help to clarify the situation when the spaces are other than metrizable. Proofs of the theorems may be found in Michael [10] or in Hu [7]. Definitions of possibly unfamiliar terms may be found in the appendix II.

<u>1.24 Theorem</u>: Suppose that Y is a metrizable AE. Then

a) Y is an extensor for every space which is both fully normal and perfectly normal.

b) Y is an extensor for every fully normal space if and only if Y is metrically complete.

c) Y is an extensor for every perfectly normal space if and only if Y is separable.

d) Y is an extensor for every normal space if and only if Y is both separable and metrically complete.

<u>1.25 Theorem</u>: Suppose that Y is a metrizable AR. Then

a) Y is a retractor for every space which is both fully normal and perfectly normal.

b) Y is a retractor for every fully normal space if and only if Y is metrically complete.

c) Y is a retractor for every perfectly normal space if and only if Y is separable.

d) Y is a retractor for every normal space if and only if Y is both separable and metrically complete.

e) Y is a retractor for every Tychonoff space if and only if Y is compact.

Finally, the next two lemmas show that products and retracts of AE's are again AE's.

<u>1.26 Lemma</u>: A topological product of AE's is an AE.

<u>Proof</u>: Let Y_α be an AE for each $\alpha \in I$, and let X be a metrizable space with a closed subspace A. If $f : A \to \Pi\{Y_\alpha : \alpha \in I\}$ is continuous, then $(\pi_\alpha \circ f)$ is continuous for each $\alpha \in I$, where π_α is the natural projection onto Y_α. Since Y_α is an AE, there is a continuous map $F_\alpha : X \to Y_\alpha$ such that $F_\alpha(x) = \pi_\alpha(f(x))$ for each $x \in A$. Define $F : X \to \Pi\{Y_\alpha : \alpha \in I\}$ by $[F(x)](\alpha) = F_\alpha(x)$. Since $(\pi_\alpha \circ F) = F_\alpha$ is continuous for each $\alpha \in I$, it follows that F is continuous. For $x \in A$, $\alpha \in I$, $[F(x)](\alpha) = F_\alpha(x) = \pi_\alpha(f(x)) = [f(x)](\alpha)$, and so F is an extension of f.

<u>1.27 Lemma</u>: A retract of an AE is an AE.

<u>Proof</u>: Assume that Y is an AE and that $r : Y \to Y_1$ is a retraction of Y onto a subspace of Y_1. Let X be a metrizable space and A a closed subspace of X with $f : A \to Y_1$ a continuous function. Then f has an extension $F : X \to Y$. Let $F_1 = (r \circ F) : X \to Y_1$. For $x \in A$, we have $F_1(x) = r(F(x)) = r(f(x)) = f(x)$, and so F_1 is a continuous extension of f.

<u>1.28 Remark</u>: The classical Tietze extension theorem states that $I = [0,1]$ is an extensor for every normal space, and so, in particular, I is an AE. The standard proof is based upon the Urysohn lemma (see appendix III), and is probably the first theorem which is in the spirit of the theory of absolute extensors. We shall exhibit the Tietze theorem as a corollary of theorem 2.17, but it is of considerable interest to note that the Urysohn lemma is still required in the proof of the Dugundji theorem. The Urysohn lemma remains the fundamental, and possibly the only, tool for constructing continuous functions from 'nothing'.

Finally, we note that AE's are contractible and locally contractible. Further, if a contractible space is locally an AE, then the space itself is an AE. These results were first proved by Hanner [5], but are more readily found in Hu [7].

CHAPTER II: INDEXED COVERINGS and the DUGUNDJI THEOREM

In this Chapter, certain properties of indexed coverings are developed. The emphasis throughout is upon the view that these coverings are certain types of functions. Throughout this Chapter, Y will denote a topological space, P(Y) the set of all subsets of Y, and A a non-empty set. In Chapter III, we shall consider the case when A is also topological.

2.1 Definition: An indexed collection of subsets of Y is a function G from a set A, to be called the indexing set, to the set of all subsets of Y, P(Y). The collection is a covering of Y if and only if $\cup\{G(a) : a \in A\} = Y$. The collection is open if G(a) is open in Y for each $a \in A$, finite if the set A is finite, and closed if G(a) is closed for each $a \in A$. The closure, boundary, and interior operators of Y are used to define the indexed collections KG, BG, and IG by means of the equations

$$(KG)(a) = K(G(a))$$
$$(BG)(a) = B(G(a))$$
$$(IG)(a) = I(G(a))$$

for each $a \in A$.

2.2 Definition: Suppose that $G : A \to P(Y)$ and $H : B \to P(Y)$ are two indexed collections of subsets of Y. Then H is a refinement of G if and only if there is a function $\lambda : B \to A$ such that $H(b) \subset G(\lambda(b))$ for each $b \in B$.

2.3 Definition: Suppose $G : A \to P(Y)$ is an indexed collection of subsets of Y. G is said to be point-finite if and only if $\{a : x \in G(a)\}$ is a finite set for each $x \in Y$. G is locally finite if and only if each point of Y has a neighborhood N for which $\{a : G(a) \cap N \neq \emptyset\}$ is a finite set. Notice that KG is locally finite if G is.

2.4 Remark: The more usual definitions of the above concepts refer to coverings as collections of sets, and the situation is at times confused. For example, suppose that for each positive integer m, $G(m) = Y$. Is $\{G(m) : m \in N\}$ a finite 'cover' of Y? With our definition 2.1, G is clearly not a finite cover, but using the 'set of sets' definition it is not always clear how the author intends such a collection to be considered.

Also, it should be noted that the unfortunate situation concerning the use of adjectives is well established.

<u>2.5 Definition</u>: A normal topological space is paracompact if and only if each open covering has a locally finite open refinement.

Without question, the single most important topological property of metrizable spaces, at least in the theory of absolute extensors, is the following theorem of A.H. Stone.

<u>2.6 Theorem</u>: A metrizable space is paracompact.

<u>Proof</u>: See A.H. Stone [11].

The following theorem is due to J. Dieudonne [3].

<u>2.7 Theorem</u>: Suppose that Y is a normal space and that $U : A \to P(Y)$ is a point-finite open covering of Y. Then there is an open covering $G : A \to P(Y)$ of Y such that

$$KG(a) \subset U(a) \text{ for each } a \in A.$$

<u>Proof</u>: First suppose that A consists of only two points. With no loss of generality, we may suppose that $A = \{1,2\}$. Then $Y - U(1)$ and $Y - U(2)$ are closed sets and disjoint since U is a covering. Thus, $U(1)$ is an open neighborhood of $Y - U(2)$. Since Y is normal, there is a neighborhood V of $Y - U(2)$ such that $KV \subset U(1)$. Let $G(1) = IV$, $G(2) = Y - KV$. Then G is clearly an open collection. Also,

$$KG(1) = KIV \subset KV \subset U(1), \text{ and}$$

$$KG(2) = K(Y-KV) = Y-IKV \subset Y - (Y-U(2)) = U(2).$$

Finally, $G(1) \cup G(2) = IV \cup (Y-KV) \subset IV \cup (Y-IV) = Y$ so that G is indeed a covering of Y.

The case when A is a finite set follows easily by induction from the above.

Now suppose that A is infinite. Let $\mathcal{F}$ be the set of all indexed open covers F of Y such that

i) $F : A \to P(Y)$

ii) for each $a \in A$, $KF(a) \subset U(a)$ or $F(a) = U(a)$

Define a partial ordering on $\mathcal{F}$ by

$$F_1 \ \alpha \ F_2 \text{ if and only if } F_1(a) \supset F_2(a) \text{ and}$$
$$F_1(a) = F_2(a) \text{ whenever } KF_1(a) \subset U(a).$$

It is immediate that the relation so defined upon $\mathcal{F}$ is reflexive and transitive. We shall show that $\mathcal{F}$ has maximal elements relative to this partial order, and that any such maximal element satisfies the conclusions of the theorem.

Let $\mathcal{D}$ be a monotone set in $\mathcal{F}$, that is, if E and F are any two covers in $\mathcal{D}$, then either $E \ \alpha \ F$ or $F \ \alpha \ E$.

We show that $\mathcal{D}$ has an upper bound in $\mathcal{F}$. For each $a \in A$, define $G(a) = \bigcap\{F(a) : F \in \mathcal{D}\}$. It will be shown that G is the required upper bound.

i) G is a covering of Y: Assume the contrary and let $x \in Y - \bigcup\{G(a) : a \in A\}$. Since the covering U is point finite, there is a finite set $B \subset A$ such that

$$x \notin U(a) \text{ for } a \in A-B.$$

For each $b \in B$, let $F_b \in \mathcal{D}$ such that $x \notin F_b(b)$. This is possible since, if $x \in F(b)$ for all $F \in \mathcal{D}$ then $x \in G(b)$, contrary to the choice of x. Since $\{F_b : b \in B\}$ is a finite set of coverings, there is a covering $F \in \mathcal{D}$ such that $F_b \ \alpha \ F$ for each $b \in B$.

Thus, since $F(b) \subset F_b(b)$ for each $b \in B$, it follows that $x \notin F(b)$ for $b \in B$. On the other hand, for $a \in A-B$, we have $F(a) \subset U(a)$ and so, again, $x \notin F(a)$ by the choice of the set B. Thus, $x \notin F(a)$ for any $a \in A$ and this contradicts the fact that F is a covering of Y.

Thus, G is a covering of Y.

ii) G is open: If $F(a) = U(a)$ for all $F \in \mathcal{D}$, then $G(a) = U(a)$ which is open.

Suppose that there is some $F \in \mathcal{D}$ for which $KF(a) \subset U(a)$. For all $E \in \mathcal{D}$, either $E \,\alpha\, F$ or $F \,\alpha\, E$. If $E \,\alpha\, F$, then $E(a) \supset F(a)$, and so $F(a) \subset \cap\{E(a) : E \,\alpha\, F\}$. If $F \,\alpha\, E$, then $F(a) = E(a)$, and so $F(a) = \cap\{E(a) : F \,\alpha\, E\}$. Consequently,

$$F(a) = \cap\{E(a) : E \,\alpha\, F\} \cap \cap\{E(a) : F \,\alpha\, E\} = \cap\{E(a) : E \in \mathcal{D}\} = G(a).$$

Thus G(a) is open.

It also follows that G is in the set $\mathcal{F}$. We now show that G is an upper bound for $\mathcal{D}$. Let $F \in \mathcal{D}$. Then $G(a) = \cap\{E(a) : E \in \mathcal{D}\} \subset F(a)$. Also, if $KF(a) \subset U(a)$, then the argument in (ii) above shows that $G(a) = F(a)$. It follows that $F \,\alpha\, G$ and so that G is an upper bound for $\mathcal{D}$.

By Zorn's Lemma, $\mathcal{F}$ has maximal elements. Let F be any maximal element of $\mathcal{F}$. We show that $KF(a) \subset U(a)$ for each $a \in A$.

Indeed, suppose that $KF(b) \cap (Y-U(b)) \neq \emptyset$ for some $b \in A$. Since $F \in \mathcal{F}$, it must then be the case that $F(b) = U(b)$. Let $G(1) = \cup\{F(a) : a \neq b\}$ and $G(2) = F(b)$. Then G is surely an open covering of Y and so, by the finite case, there is an open covering $E : \{1,2\} \to P(Y)$ such that $KE(1) \subset G(1)$ and $KE(2) \subset G(2)$. Let $D : A \to P(Y)$ be defined by $D(a) = F(a)$ if $a \neq b$, and $D(b) = E(2)$.

Then F α D, but it is not the case that D α F, thus contradicting the maximality of F. Hence, for all $a \in A$, $KF(a) \subset U(a)$, thus completing the proof of the theorem.

<u>2.8 Theorem</u>: Suppose that Y is paracompact and Hausdorff and that $U : A \to P(Y)$ is an open covering of Y. Then there is a locally finite open covering $G : A \to P(Y)$ such that $KG(a) \subset U(a)$ for every $a \in A$.

<u>Proof</u>: Since Y is paracompact, U has an open locally finite refinement. Let $V : B \to P(Y)$ be a locally finite open covering of Y with $\lambda : B \to A$ such that $V(b) \subset U(\lambda(b))$ for each $b \in B$. By theorem 2.7 there is an open covering $W : B \to P(Y)$ such that $KW(b) \subset V(b)$ for each $b \in B$. Notice that, for any $N \subset Y$, $\{b : W(b) \cap N \neq \emptyset\} \subset \{b : V(b) \cap N \neq \emptyset\}$, and so W is also locally finite.

For each $b \in B$, $W(b) \subset KW(b) \subset V(b) \subset U(\lambda(b))$. Define G by $G(a) = \cup\{W(b) : b \in \lambda^{-1}(a)\}$. It is true that G(a) may be empty. In any case, $KG(a) = K\cup\{W(b) : b \in \lambda^{-1}(a)\}$

$$= \cup\{KW(b) : b \in \lambda^{-1}(a)\} \subset U(a).$$

G is clearly an open covering of Y. To see that G is also locally finite, let $x \in Y$ and N be the neighborhood of x for which $\{b \in B : W(b) \cap N \neq \emptyset\}$ is a finite set. But $G(a) \cap N \neq \emptyset$ if and only if there is some $b \in B$ with $\lambda(b) = a$ and $W(b) \cap N \neq \emptyset$. Thus, $\{a \in A : G(a) \cap N \neq \emptyset\}$ has no more elements than $\{b \in B : W(b) \cap N \neq \emptyset\}$, and this is finite. Thus G is locally finite.

<u>2.9 Corollary</u>: Suppose that Y is paracompact and Hausdorff. Let N denote the positive integers and suppose that $V : N \to P(Y)$ is an open covering of Y for which $V(n) \subset V(n+1)$ for all $n \in N$. Then there is a closed locally finite covering $F : N \to P(Y)$ of Y such that, for all $n \in N$, $F(n) \subset V(n)$ and $F(n) \subset F(n+1)$.

<u>Proof</u>: From the theorem 2.8 there is an open locally

finite covering $G : N \to P(Y)$ for which $KG(n) \subset V(n)$ for each $n \in N$. Define $F(n) = \bigcup_{i=1}^{n} KG(i)$. Clearly F is closed, $F(n) \subset V(n)$, $F(n) \subset F(n+1)$ for each n, and F is a covering of Y.

Let $x \in Y$ and W be a neighborhood of x for which $\{n : KG(n) \cap W \neq \emptyset\}$ is finite. But $F(n) \cap W \neq \emptyset$ if and only if there is some $i \leq n$ for which $KG(i) \cap W \neq \emptyset$. Thus the number of elements in $\{n : F(n) \cap W \neq \emptyset\}$ does not exceed $\{n : KG(n) \cap W \neq \emptyset\}$ which is finite. Thus F is locally finite.

It will be convenient to construct continuous functions 'in pieces'. To this end, it is required to know certain conditions under which the resulting function is continuous. The follwoing two theorems are perhaps better known than the other results in this Chapter, but we shall use them frequently and so include them here for completeness.

2.10 Lemma: If X and Y are topological spaces, $F : A \to P(X)$ is a closed locally finte cover of X and $f : X \to Y$ is a function such that $f|F(a)$ is continuous for each $a \in A$, then f is continuous.

Proof: Suppose that M is closed in Y. Then $f^{-1}(M) \cap F(a)$ is closed in F(a) and hence in X. Thus,

$$\begin{aligned} Kf^{-1}(M) &= K\bigcup\{f^{-1}(M) \cap F(a) : a \in A\} \\ &= \bigcup\{K(f^{-1}(M) \cap F(a)) : a \in A\} \\ &= \bigcup\{f^{-1}(M) \cap F(a) : a \in A\} = f^{-1}(M) \end{aligned}$$

Hence, f is continuous.

2.11 Lemma: Suppose that $F : A \to P(X)$ is an open cover of X and that $f : X \to Y$ is such that $f|F(a)$ is continuous for each $a \in A$. Then f is continuous.

Proof is immediate.

The most efficient method of 'piecing together' results obtained locally is by means of partitions of unity. In the following, $C^*(X)$ denotes the bounded continuous real valued functions having domain X.

<u>2.12 Definition</u>: An indexed partition of unity on X is a function $P : A \to C^*(X)$ where

i) A is a non-empty set known as the indexing set

ii) $0 \leq (P(a))(x) \leq 1$ for each $a \in A$ and $x \in X$.

iii) For each $x \in X$, $\Sigma\{(P(a))(x) : a \in A\} = 1$.

Notice that P(a) is required to be continuous for each $a \in A$.

<u>2.13 Definition</u>: If $V : A \to P(X)$ is an indexed collection of subsets of X and if $p : A \to C^*(X)$ is an indexed partition of unity, then p is said to be subordinate to V if and only if $p(a)|(X-V(a)) \equiv 0$ for all $a \in A$.

The adjective 'indexed' will be frequently omitted in the following.

The best result concerning the existence of partitions of unity is the following theorem.

<u>2.14 Theorem</u>: Suppose that X is normal and that $V : A \to P(X)$ is an open locally finite cover of X. Then there is a partition of unity subordinate to V.

<u>Proof</u>: By theorem 2.7, there is an open covering $G : A \to P(X)$ such that $KG(a) \subset V(a)$ for each $a \in A$. By the Urysohn Lemma (see appendix III), there is a continuous function $q(a) : X \to [0,1]$ such that

$$q(a)|KG(a) \equiv 1 \text{ and } q(a)|(X-V(a)) \equiv 0.$$

Note that, since V is locally finite, each point $x \in X$ has a neighborhood N such that $\{a \in A : q(a)|N \not\equiv 0\}$ is finite. In particular $q(a)(x) \neq 0$ for only finitely many $a \in A$. Define $(p(a))(x) = \dfrac{(q(a))(x)}{\Sigma\{(q(b))(x) : b \in A\}}$

It is clear that $\Sigma\{(p(a))(x) : a \in A\} = 1$ for each $x \in X$. By the above remarks, it follows that each point $x \in X$ has a neighborhood $N(x)$ such that $p(a)|N(x)$ is continuous, and so $p(a)$ is continuous by lemma 2.11.

2.15 Corollary: If X is paracompact and Hausdorff and $G : A \to P(X)$ is an open covering of X, then there is a partition of unity subordinate to G.

Proof: By theorem 2.8, there is a locally finite open covering $V : A \to P(X)$ such that $V(a) \subset G(a)$ for each $a \in A$. There is a partition of unity subordinate to V, and hence subordinate to G.

2.16 Theorem: Let X be a metrizable space and A a closed subspace of X. Then there is an indexed open cover of X-A such that
i) $U : X-A \to P(X-A)$
ii) U is locally finite
iii) if $a \in B(A)$ and N is any neighborhood of a, then $\{x : U(x) \subset N\}$ is infinite.
iv) If $a \in A$ and if N is a neighborhood of a, then there is a neighborhood N_1 of a such that $U(x) \subset N$ whenever $U(x) \cap N_1 \neq \emptyset$.

Proof: Let d be a metric for the topology on X. For $x \in X-A$, let $V(x) = S(x, \frac{d(x,A)}{2})$. Then $V : (X-A) \to P(X-A)$ is an open collection and is a covering since X-A is open in X. By theorems 2.8 and 2.6, there is an open locally finite covering $U : (X-A) \to P(X-A)$ such that $U(x) \subset V(x)$ for each $x \in (X-A)$. It remains to verify that U also satisfies iii and iv.

Let $a \in B(A)$ and let N be any neighborhood of a. Let $r>0$ be such that $S(a,r) \subset N$. Since a is a limit point of X-A, there are infinitely many points x in X-A such that $d(x,a) < r/4$. Since $d(x,A) \leq d(x,a)$, it follows that, for these points, $V(x) \subset S(a,r)$ and, hence, $U(x) \subset V(x) \subset S(a,r) \subset N$. Thus $\{x : U(x) \subset N\}$ is infinite and this verifies iii.

Suppose that $a \in A$ and that N is a neighborhood of a. Let $r>0$ be such that $S(a,r) \subset N$ and let $N_1 = S(a, r/4)$. If $U(x) \cap N_1 \neq \emptyset$, then $V(x) \cap N_1 \neq \emptyset$ and consequently, $d(x,a) \leq \frac{d(x,A)}{2} + \frac{r}{4} \leq \frac{d(x,a)}{2} + \frac{r}{4}$.

Thus, $d(x,a) \leq r/2$ and hence,

$U(x) \subset V(x) = S(x, \frac{d(x,A)}{2}) \subset S(x, \frac{d(x,a)}{2}) \subset S(x, r/4) \subset S(a,r) \subset N$, thus verifying iv, and completing the proof of the theorem.

The following theorem was the first to provide a really large class of AE's. It provides a striking generalization of the Tietze extension theorem and, together with the Wojdyslawski embedding (theorem 2.21), provides an interesting characterization of metrizable AE's. The theorem is due to Dugundji [7].

<u>2.17 Theorem</u>: A convex subset of a locally convex linear topological space is an AE.

<u>Proof</u>: Let Y be a convex subset of the locally convex linear topological space L. Let X be a metrizable space, A a closed subspace, and $f : A \to Y$ a continuous function. Further, let $U:(X-A) \to P(X-A)$ be the covering of theorem 2.16. By theorem 2.14, there is a partition of unity p subordinate to U, $p : (X-A) \to C^*(X-A)$. Let $\lambda_1 : (X-A) \to (X-A)$ be a choice function for U, that is, $\lambda_1(x) \in U(x)$ if $U(x) \neq \emptyset$ and $\lambda_1(x)$ is an arbitrary element of $X-A$ if $U(x) = \emptyset$.

Let $\lambda_2 : (X-A) \to A$ be such that $d(\lambda_1(x),\lambda_2(x)) < 2d(\lambda_1(x),A)$. Notice that $p(x) \equiv 0$ if $U(x) = \emptyset$. The extension F of f is defined immediately by

$$F(x) = \begin{cases} \Sigma\{p(y)(x)f(\lambda_2(y)) : y \in (X-A)\} & \text{if } x \in (X-A) \\ f(x) & \text{if } x \in A. \end{cases}$$

Notice that the sum appearing in the definition of F is always a finite sum since p is subordinate to the locally finite covering U, and that F is trivially an extension of f. It remains to show that F is continuous.

Suppose first that $x \in (X-A)$. There is a neighborhood N of x for which $\{y : U(y) \cap N \neq \emptyset\}$ is finite. Thus $F|N$ is just a finite sum with continuous coefficients and so is continuous on N and hence at x.

Let $a \in A$ and let V be a convex neighborhood of $f(a) = F(a)$. Since f is continuous on A, there is a $\delta > 0$ such that $f(y) \in V$ whenever $y \in A$ and $d(a,y) < \delta$. Let W be a neighborhood of a such that $U(x) \subset S(a,\delta/3)$ whenever $U(x) \cap W \neq \emptyset$. W may clearly be chosen so that $W \subset S(a,\delta)$.

We now claim that if $\lambda_1(x) \in W$ and $U(x) \neq \emptyset$, then $F(\lambda_2(x)) = f(\lambda_2(x)) \in V$. Indeed, since $U(x) \neq \emptyset$, it follows that $\lambda_1(x) \in U(x)$ and so $U(x) \cap W \neq \emptyset$. Thus, $U(x) \subset S(a,\delta/3)$, and, in particular, $d(a,\lambda_1(x)) < \delta/3$. Finally, $d(\lambda_2(x),a) \leq d(\lambda_2(x),\lambda_1(x)) + d(\lambda_1(x),a) < \delta$, and so $f(\lambda_2(x)) \in V$.

Now let W_1 be a neighborhood of a such that $W_1 \subset W$ and $U(x) \subset W$ for any x such that $U(x) \cap W_1 \neq \emptyset$. It will be shown that $F(W_1) \subset V$.

Suppose that $x \in W_1 \cap (X-A)$, so that $F(x) = \Sigma\{p(y)(x)f(\lambda_2(y)) : y \in X-A\}$. Consider any y for which $(p(y))(x) \neq 0$. Then $x \in U(y)$ and hence $U(y) \cap W_1 \neq \emptyset$ and consequently $U(y) \subset W$. Thus, since $\lambda_1(y) \in U(y)$, we have $\lambda_1(y) \in W$ and hence $f(\lambda_2(y)) \in V$. Since V is convex, it follows immediately that $F(x) \in V$.

If $x \in W_1 \cap A$, then $d(x,a) < \delta$ since $W_1 \subset W \subset S(a,\delta)$, and so, again, $F(x) \in V$. Hence, $F(W_1) \subset V$ and so F is continuous.

2.18 Corollary: With the notation of 2.17, if $f : A \to L$, then $F(X) \subset co(f(A))$

Proof: is immediate from the definition of F.

2.19 Corollary: $[0,1]$ is an extensor for every normal space.

Proof: By the theorem, $[0,1]$ is an AE. Since, in addition, $[0,1]$ is metrizable, separable, and metrically complete, the corollary follows from theorem 1.24.

2.20 Remark: Several points concerning the proof of theorem 2.17 should be noted. First is the use of the partition of unity, the existence of which depended upon the Urysohn Lemma. Secondly, the reader should note the heavy use of the axiom of choice and its equivalents. The function λ_1 is directly guaranteed by the axiom of choice, while Zorn's Lemma was used in the proof of 2.7, upon which the covering U ultimately depends.

The Dugundji theorem revived interest in a theorem which had been known for some time. Kuratowski had noticed that a bounded metric space may be isometrically embedded in the Banach space of all continuous bounded real valued functions defined on the space. In 1939, Wojdyslawski [13] showed that the embedding is a closed subset of its convex hull. The theorem is given here.

2.21 Theorem: Let X be a metrizable space and d a bounded metric for the topology on X. There is an isometry $F : X \to C^*(X)$ such that $F(X)$ is closed in $co(F(X))$. In addition, if X is separable, then so is $co(F(X))$.

Proof: Define $F : X \to C^*(X)$ by $F(x)(y) = d(x,y)$. Then,

$$\begin{aligned} ||F(x) - F(y)|| &= \sup\{|F(x)(z)-F(y)(z)| : z \in X\} \\ &= \sup\{|d(x,z)-d(y,z)| : z \in X\} \\ &\leq d(x,y) \end{aligned}$$

But $d(x,y) = |F(x)(y)-F(y)(z)|$ and so $\|F(x)-F(y)\| = d(x,y)$ and F is an isometry.

To show that F(X) is closed in co(F(X)), let $q \in (co(F(X))-F(X))$. Since $q \notin F(X)$, there are a finite number of points $x_i \in X$ and reals $r_i \geq 0$ such that $q \neq F(x_i)$, $\Sigma r_i = 1$ and $q = \Sigma r_i F(x_i)$. Let $\delta > 0$ be such that $\delta < \frac{1}{2}\min\|q-F(x_i)\|$ and let $N = S(q,\delta) \cap co(F(X))$. It will be shown that $N \subset (co(F(X))-F(X))$.

To this end, assume that $F(x) \in N$ for some $x \in X$. Then $\|F(x_i)-F(x)\| > \delta$ for each i and so $d(x_i,x) = F(x_i)(x) > \delta$ for each i. Thus.

$$\|q-F(X)\| \geq |q(x)-F(x)(x)| = |q(x)| = |\Sigma r_i F(x_i)(x)|$$
$$= \Sigma r_i F(x_i)(x) > \Sigma r_i \delta = \delta \text{ and so } F(x) \notin N.$$

Hence, $N \subset (co(F(X))-F(X))$ and so F(X) is closed in co(F(X)). As the assertion concerning separability is not required in the following, the proof is omitted.

There are two important results which follow from the theorem.

<u>2.22 Theorem</u>: A metrizable space is an AE if and only if it is a retract of a convex set of a Banach space.

<u>Proof</u>: If X is an AE, then X is an AR by Lemma 1.23, and so there is a retraction $r : co(F(X)) \to F(X)$.

If X is a retract of a convex set, then X is an AE by lemma 1.27.

<u>2.23 Theorem</u>: A metrizable AR is an AE.

<u>Proof</u>: If X is a metrizable AR then X is (homeomorphic to) a retract of a convex set by Theorem 2.21.

<u>2.24 Theorem</u>: If X is a compact metrizable AE, then X has the fixed point property.

<u>Proof</u>: Since X is compact, F(X) is closed in K(co(F(X)))

which, by the theorem of Mazur [9], is itself compact. Thus, there is a retraction $r : K(co(F(X))) \to F(X)$. Since $K(co(F(X)))$ has the fixed point property by a Theorem of Tychonoff [12], it follows from lemma 1.11 that $F(X)$, and hence X, has the fixed point property.

CHAPTER III: CONTINUITY OF INDEXED COLLECTIONS. THE BASIC THEOREM.

3.1 Remark: We now turn to a problem which, at first sight, seems quite removed from the problem of continuous extensions. One formulation of the axiom of choice is as follows: If $C : X \to P(Y)$ is an indexed collection of subsets of Y such that $C(x) \neq \emptyset$ for each $x \in X$, then there is a function $f : X \to Y$ such that $f(x) \in C(x)$ for each $x \in X$.

In case X and Y are both topological spaces, it is natural to ask if f might not be continuous. There is an intimate connection between this problem and the extension problem which is clarified by theorem 3.12 below. In the following $P'(X)$ denotes the set of all non-empty subsets of X.

3.2 Definition: For $V \subset X$, $st(V) = \{Z \in P'(X) : Z \cap V \neq \emptyset\}$. 'st(V)' is read 'star of V'.

3.3 Definition: If X is a topological space, then $\{st(V) : V \text{ open in } X\}$ is a sub-base for a topology on $P'(X)$. Unless the contrary is explicitly stated, $P'(X)$ will be assumed to be equipped with this topology which will be known as the usual topology on $P'(X)$.

3.4 Remark: If X and Y are topological spaces and if $C : X \to P'(Y)$ is an indexed collection of non-empty subsets of Y, then C may or may not be continuous. We will show that continuity of C is related to the question of the existence of a continuous choice function. Also, it should be noted that the topology which we are using is not the only useful topology. That it is the appropriate topology for our purposes follows from Theorem 3.12. The interested reader should consult Čech [2, Chapter VI]. In the following, X and Y are topological spaces, and $P'(X)$, $P'(Y)$ are given the usual topology.

3.5 Lemma: Let $C : X \to P'(Y)$. Then C is continuous if and only if $\{x \in X : C(x) \cap G \neq \emptyset\}$ is open in X for each G which is open in Y.

Proof: Assume that C is continuous, and let G be open in Y. Then $st(G)$ is open in $P'(Y)$ and, since C is continuous, $C^{-1}(st(G))$ is open in X. But,

$$C^{-1}(st(G)) = \{x \in X : C(x) \in st(G)\}$$
$$= \{x \in X : C(x) \cap G \neq \emptyset\}.$$

Conversely, if $\{x \in X : C(x) \cap G \neq \emptyset\}$ is open in X, then the set equalities above show that $C^{-1}(st(G))$ is open in X. Since $\{st(G) : G \text{ open in } Y\}$ is a subbase for the topology of $P'(Y)$, it follows that C is continuous.

3.6 Lemma: Suppose that Y is a locally convex linear topological space. Then $\{st(V) : V \text{ open and convex in } Y\}$ is a subbase for the usual topology on $P'(Y)$.

Proof: Let T denote the topology on $P'(Y)$ which is generated by $\{st(V) : V \text{ open and convex in } Y\}$. Since each of the sets in this collection is open in the usual topology of $P'(Y)$, it follows that T is weaker than the usual topology. It will now be shown that, if G is open in Y, then $st(G)$ is in the topology T.

Let $Z_o \in st(G)$, so that $Z_o \cap G \neq \emptyset$, and let $y_o \in Z_o \cap G$. Since Y is locally convex, there is an open convex set U such that $y_o \in U \subset G$. Since $y_o \in U \cap Z_o$, it follows that $Z_o \in st(U)$, and that $st(U)$ is a subbasic open set of T. Further, if $Z \in st(U)$, then $Z \cap U \neq \emptyset$ and so $Z \cap G \neq \emptyset$ and hence $Z \in st(G)$. Thus, $st(U) \subset st(G)$. Thus, $st(G)$ is open in the T topology. Since the usual topology is the weakest topology which contains $\{st(G) : G \text{ open in } Y\}$, it follows that T coincides with the usual topology.

3.7 Remark: If Y is a linear space, then the operator co may be considered as a function co : P'(Y) → P'(Y).

<u>3.8 Theorem</u>: Suppose that Y is a locally convex linear topological space. Then co : P'(Y) → P'(Y) is continuous.

<u>Proof</u>: Let U be an open set in Y so that st(U) is a subbasic open set in P'(Y). Let $Z_o \in co^{-1}(st(U))$. It will be shown that there is a neighborhood N of Z_o which lies in $co^{-1}(st(V))$.

Since $Z_o \in co^{-1}(st(U))$, we have $co(Z_o) \in st(U)$ and so $co(Z_o) \cap U \neq \emptyset$. Let $y_o \in co(Z_o) \cap U$. There are thus a finite number of points z_1, z_2 ..., z_n, and non-negative reals $r_1, \ldots, r_n$ such that $z_i \in Z_o$ for i = 1, ..., n, $\Sigma r_i = 1$, and $y_o = \Sigma r_i z_i$.

Further, since Y is locally convex, there is an open convex neighborhood V of 0 such that $(y_o + V) \subset U$. Let $V_i = z_i + V$ for each i = 1, ..., n, and let $N = st(V_1) \cap \ldots \cap st(V_n)$. Note that N is a basic open set in P'(Y) and that $z_i \in Z_o \cap V_i$ for each i = 1, ..., n. It follows immediately that $Z_o \in st(V_i)$ for each i = 1, ..., n and so $Z_o \in N$. Thus N is a neighborhood of Z_o in P'(Y).

Now, let $Z \in N$. Then $Z \cap V_i \neq \emptyset$ for each i =1, ...,n. Let $v_i \in Z \cap V_i$ for each i = 1, ..., n, so that $v_i = z_i + w_i$, where $w_i \in V$. Then $\Sigma r_i v_i = \Sigma r_i z_i + \Sigma r_i w_i = y_o$ + (some point of V).

Hence, $\Sigma r_i v_i \in U$ and so $co(Z) \cap U \neq \emptyset$. Thus $co(Z) \in st(V)$, and this shows that $N \subset co^{-1}(st(U))$. Hence co is continuous.

<u>3.9 Remark</u>: If Z is convex, then co(Z) = Z. Hence the set of convex subsets of Y is a retract of P'(Y). Since P'(Y) is not generally a Hausdorff space, it cannot be immediately concluded that the class of convex subsets of Y is closed

in P'(Y) and, in fact, this is not generally so, as may be seen from the following example.

3.10 Example: Let $Y = R^1$, $Z_o = \{r : r \text{ is a rational}, 0 \leq r \leq 1\}$. Then $co(Z_o) = [0,1] = K(Z_o)$, but since $K(Z_o) \cap U \neq \emptyset$ if and only if $Z_o \cap U \neq \emptyset$ when U is open, it follows that Z_o is in the $P'(R^1)$ closure of the class of convex subsets of R^1, although Z_o is not convex.

3.11 Remark: The argument used in example 3.10 may be used to show that Z and K(Z) never have disjoint neighborhoods in P'(Y). Thus, if Y contains any set which is not closed, then P'(Y) is not Hausdorff.

What has all of this to do with continuous extensions?

3.12 Theorem: Suppose that A is a closed subspace of X, and that $f : A \to Y$. Define $C(x) = \begin{cases} \{f(x)\} & \text{if } x \in A \\ Y & \text{if } x \notin A \end{cases}$

then $C : X \to P'(Y)$ is such that

i) The map $F : X \to Y$ is an extension of f if and only if F is a choice function for C.

ii) C is continuous if and only if f is continuous.

Proof:

i) is clear

ii) Suppose f is continuous. Let G be open in Y and suppose that x_o is such that $C(x_o) \cap G \neq \emptyset$. If $x_o \in (X-A)$, then, since A is closed, there is a neighborhood N of x_o such that $N \subset (X-A)$. For each $x \in N$, $C(x) = Y$ and so $C(x) \cap G \neq \emptyset$, i.e., $N \subset \{x : G \cap C(x) \neq \emptyset\}$. On the other hand, if $x_o \in A$, then, since f is continuous, there is a neighborhood N of x_o such that $f(N \cap A) \subset G$. Again, for $x \in (N-A)$, $C(x) = Y$, and so $N \subset \{x : C(x) \cap G \neq \emptyset\}$.

Thus $\{x : C(x) \cap G \neq \emptyset\}$ must be open, and so C is continuous by Lemma 3.5.

Conversely, let C be continuous and let $x_o \in A$. Let G be a neighborhood of $f(x_o)$ in Y. Since $C(x_o) = \{f(x_o)\}$, it follows that $x_o \in \{x : C(x) \cap G \neq \emptyset\}$. Since C is continuous, there is a neighborhood N of x_o in X such that $N \subset \{x : C(x) \cap G \neq \emptyset\}$. Thus, for $x \in (N \cap A)$, $C(x) = \{f(x)\}$ and $C(x) \cap G \neq \emptyset$ and so $f(x) \in G$. Thus $f(N \cap A) \subset G$ and so f is continuous on A.

<u>3.13 Remark</u>: Consequently, it is now seen that the problem of finding continuous extensions is a special case of finding continuous choice functions for continuous indexed collections. The remainder of this chapter is devoted to proving the basic theorem 3.20 on continuous choices, along with certain lemmas necessary to prove the main theorem 4.3 which extends the basic theorem 3.20.

<u>3.14 Lemma</u>: Suppose that $C : X \to P'(Y)$ is continuous. Let $D(x) = K(C(x))$ for each $x \in X$. Then $D : X \to P'(Y)$ is continuous.

<u>Proof</u>: Let G be open in Y. Then $K(C(x)) \cap G \neq \emptyset$ if and only if $C(x) \cap G \neq \emptyset$. Thus $\{x : D(x) \cap G \neq \emptyset\} = \{x : C(x) \cap G \neq \emptyset\}$ and so D is continuous.

<u>3.15 Lemma</u>: Suppose that $C : X \to P'(Y)$ is continuous and that A is a closed subset of X. Further suppose that $f : A \to Y$ is continuous. Let $D : X \to P'(Y)$ be defined by

$$D(x) = \begin{cases} \{f(x)\} & \text{if } x \in A \\ C(x) & \text{if } x \in (X-A) \end{cases}$$

Then D is continuous.

<u>Proof</u>: Let $E(x) = \begin{cases} \{f(x)\} & \text{if } x \in A \\ Y & \text{if } x \in (X-A) \end{cases}$ Then E is continuous by theorem 3.12. Also, $\{x : D(x) \cap G \neq \emptyset\} = \{x : C(x) \cap G \neq \emptyset\} \cap \{x : E(x) \cap G \neq \emptyset\}$ and so D is continuous.

<u>3.16 Lemma</u>: Let $C : X \to P'(Y)$ be continuous and let G be open in Y. If $D(x) = C(x) \cap G$ for all $x \in X$, then

$D : X \to P'(Y)$ is continuous.

Proof: Let H be open in Y. Then $\{x : D(x) \cap H \neq \emptyset\} = \{x : C(x) \cap G \cap H \neq \emptyset\}$ and so D is also continuous.

3.17 Lemma: Suppose that Y is a metric space, that $C : X \to P'(Y)$ is continuous, and that $r > 0$. Let $f : X \to Y$ be a continuous function such that $C(x) \cap S(f(x),r) \neq \emptyset$ for each $x \in X$. Define $D : X \to P'(Y)$ by $D(x) = C(x) \cap S(f(a),r)$. Then D is continuous.

Proof: Let G be open in Y and let $x_o \in \{x : D(x) \cap G \neq \emptyset\}$. Since $C(x_o) \cap S(f(x_o),r) \cap G \neq \emptyset$, there is a real $s > 0$ such that $s < r$ and $C(x_o) \cap S(f(x_o),s) \cap G \neq \emptyset$. Let $y_o \in C(x_o) \cap S(f(x_o),s) \cap G$. Let

$$W_1 = \{x : C(x) \cap S(f(x_o),s) \cap G \neq \emptyset\}$$
$$W_2 = \{x : f(x) \in S(f(x_o),r-s)\}.$$

Note that W_1 is open, since C is continuous, and W_2 is open since f is continuous. Thus $W_1 \cap W_2$ is a neighborhood of x_o. Finally, let $x \in W_1 \cap W_2$. Then, for all $y \in S(f(x_o),s)$, we have

$$d(y,f(x)) \leq d(y,f(x_o)) + d(f(x_o),f(x)) < s + r - s = r.$$

Thus, $S(f(x_o),s) \subset S(f(x),r)$ and consequently, $W_1 \cap W_2 \subset \{x : C(x) \cap S(f(x),r) \cap G \neq \emptyset\}$. Hence, D is continuous.

3.18 Lemma: Suppose that Y is a normal linear space and that X is metrizable. Suppose that $C : X \to P'(Y)$ is continuous, and that $C(x)$ is convex for each $x \in X$. If $r > 0$, then there is a continuous function $f : X \to Y$ such that, for each $x \in X$, $f(x) \in S(C(x),r)$.

Proof: Let $U : Y \to P(X)$ be defined by

$$U(y) = \{x : y \in S(C(x),r)\} = \{x : C(x) \cap S(y,r) \neq \emptyset\}.$$

It is clear, since C is continuous, that U is an open covering of X. By Theorem 2.8, there is a locally finite open covering $V : Y \to P(x)$ such that $V(y) \subset U(y)$ for each $y \in Y$. By Theorem 2.14, there is a partition of unity p subordinate to V. Define f(x) = $\Sigma\{p(y)(x)\cdot y : y \in Y\}$ for each $x \in X$. Notice that the sum is finite since $p(y)(x) \neq \emptyset$ for only finitely many y. In fact, each $x \in X$ has a neighborhood N such that $f|N$ is defined by a finite sum, and so f is continuous on X.

Finally, if $p(y)(x) \neq 0$, then $x \in V(y) \subset U(y)$ and so $y \in S(C(x),r)$. Since $S(C(x),r)$ is a convex set, it follows that $f(x) \in S(C(x),r)$.

3.19 Remark: Notice here again the fundamental use of partitions of unity which in turn depend upon the Urysohn Lemma. The following is due to Michael [14].

3.20 The Basic Theorem of Continuous Choices: Suppose that X is metrizable and that Y is a banach space. Let $C : X \to P'(Y)$ be continuous and such that C(x) is a closed convex subset of Y for each $x \in X$. Then there is a continuous function $f : X \to Y$ such that $f(x) \in C(x)$ for each $x \in X$.

Proof: It is proposed to inductively construct a sequence of functions $f_i : X \to Y$ such that

a) f_i is continuous for i = 1, 2, ...

b) $\|f_i(x) - f_{i-1}(x)\| < 1/2^{i-2}$ for i = 2, 3, ...

c) $\|C(x) - f_i(x)\| < 1/2^i$ for i = 1, 2, ...

Construction of f_1: In Lemma 3.18, choose r = ½. The conclusion of the lemma yields f_1 satisfying a and c.

Inductive Hypothesis: $f_1, \ldots, f_n$ have been constructed which satisfy a, b, and c above, for each i = 1, ..., n.

Construction of f_{n+1}: Define $B(x) = C(x) \cap S(f_n(x), 1/2^n)$. Then $B(x) \neq \emptyset$ for all $x \in X$ by part c of the inductive hypothesis, and $B : X \to P'(Y)$ is continuous by lemma

3.17, since both f_n and C are continuous. Also, B(x) is convex for each $x \in X$ since it is the interaction of two convex sets. Thus, by lemma 3.18, there is a continuous function $f_{n+1} : X \to Y$ such that $f_{n+1}(x) \in S(B(x), {}^1/_{2^{n+1}})$ for each $x \in X$. Thus,

$$\|f_{n+1}(x)-f_n(x)\| \leq \frac{1}{2^{n+1}} + \frac{1}{2^n} = \frac{3}{2^{n+1}} = \frac{3}{4} \cdot \frac{1}{2^{n-1}} < \frac{1}{2^{n-1}}$$

and $\|C(x)-f_{n+1}(x)\| < {}^1/_{2^{n+1}}$. Hence f_{n+1} satisfies a, b, and c above.

Hence there is a sequence of functions $\{f_n\}_{n=1}^{\infty}$ satisfying a, b, and c. By b, the sequence is uniformly Canchy. Since Y is a Banach space, there is a continuous $f : X \to Y$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$. By c, and since C(x) is closed in Y for each $x \in X$, it follows that $f(x) \in C(x)$ for each $x \in X$, thus completing the theorem.

<u>3.21 Corollary</u>: If X and Y are as in theorem 3.20 and if $D : X \to P'(Y)$ is continuous, then there is a continuous function $f : X \to Y$ such that $f(x) \in K\ co(D(x))$ for each $x \in X$.

<u>Proof</u>: Let $C(x) = K\ co(D(x))$ for each $x \in X$. By lemmas 3.14 and 3.8, it follows that C is continuous.

<u>3.22 Corollary</u>: A closed convex subset of a Banach space is an AE.

<u>Proof</u>: Let Z be a closed convex subset of a Banach space Y and A a closed subset of a metrizable space X. If $f : A \to Z$ is continuous, let $C(x) = \begin{cases} \{f(x)\} & \text{if } x \in A \\ Z & \text{if } x \in (X-A) \end{cases}$

C is continuous by Theorem 3.12 and so there is a continuous $F : X \to Z$ such that $F(x) \in C(x)$ for each $x \in X$.

<u>3.23 Remark</u>: It should be noted that 3.22 is weaker than the Dugundji theorem (Th. 2.17).

CHAPTER IV: THE MAIN THEOREM

This Chapter is devoted to extending the basic theorem on continuous choices. We define a new class of sets, which includes the class of convex sets, and show that the basic choice theorem still holds for this extended class. As a consequence, we have a new class of AE's.

Throughout this section, Y will denote an arbitrary Banach space.

4.1 Definition: Let $E \subset Y$ and $t \geq 0$. The class of sets $\mathfrak{C}(t,Y)$ is defined by $E \in \mathfrak{C}(t,Y)$ if and only if for each finite subset $\{x_i\}_{i=1}^n \subset E$ if $x \in co(\{x_i\}_{i=1}^n)$ then $\|x - E\| \leq t \max\{\|x_i - x_j\| : 1 \leq i, j \leq n\}$.

4.2 Remark: The definition indicates that convex combinations of points may 'escape' from E, but not too far. If E is closed, then $E \in \mathfrak{C}(0,Y)$ if and only if E is convex. Consider the two plane sets V and U. The first is in $\mathfrak{C}(t,R^2)$ for some $t < \frac{1}{2}$, the exact values of t depending upon the size of the included angle. The second figure is not in $\mathfrak{C}(t,R^2)$ for $t < \frac{1}{2}$. The main theorem applies only to sets in $\mathfrak{C}(t,Y)$ for $t < \frac{1}{2}$. It should be noted that the two figures are compact one-dimensional homeomorphic AE's. Although the figure U is locally $\mathfrak{C}(t,R^2)$ for $t < \frac{1}{2}$, a later example will provide a compact one-dimensional AE which is not locally $\mathfrak{C}(t,Y)$ for any $t < \frac{1}{2}$.

Further, it should be noted that a set consisting of two points in the plane is in $\mathfrak{C}(\frac{1}{2},R^2)$ but not in $\mathfrak{C}(t,R^2)$ for any $t < \frac{1}{2}$, and so, in that sense at least, the main theorem cannot be improved.

4.3 The Main Theorem: Suppose that X is a metrizable space and that Y is a Banach space. Let $0 \leq t < \frac{1}{2}$ and let $A : X \to P'(Y)$ be continuous. If A(x) is closed for each $x \in X$ and if $A(x) \in \mathfrak{C}(t,Y)$ for each $x \in X$, then there is a continuous function $f : X \to Y$ such that $f(x) \in A(x)$ for all $x \in X$.

Proof: Let N denote the natural numbers and let s be a real number such that $2t < s < 1$. Let a be a real number such that

4.4) $a > \sum_{k=0}^{\infty} s^k > 1$

and 4.5) $A(x) \cap S(0,a) \neq \emptyset$ for at least one $x \in X$.

Define $V : N \to P(X)$ by $V(n) = \{x \in X : A(x) \cap S(0,a^n) \neq \emptyset\}$. Then V is open since A is continuous, and V is a covering of X since $\cup\{S(0,a^n) : n \in N\} = Y$. Also, since $a^{n+1} > a^n$, it follows that $V(n) \subset V(n+1)$ for all $n \in N$. By Corollary 2.9, there is a covering $F : N \to P(X)$ such that

4.6) $F(n) \subset V(n)$ for all $n \in N$

4.7) $F(n) \subset F(n+1)$ for all $n \in N$

4.8) $F(n)$ is closed for each $n \in N$

4.9) F is locally finite

It is proposed to construct a sequence of functions $\{f_n\}_{n=1}^{\infty}$ with the following properties:

4.10) $f_n : F(n) \to Y$ is continuous for all $n \in N$

4.11) $f_n(x) \in A(x)$ for all $x \in F(n)$ and for all $n \in N$

4.12) $\|f_n(x)\| < a^{n+1}$ for all $x \in F(n)$ and for all $n \in N$

4.13) $f_{n+1}|F(n) = f_n$ for all $n \in N$.

The construction of such a sequence will proceed by induction.

Construction of f_1: This will also be an inductive construction. Define $g_o(x) = 0$ for all $x \in F(1)$. Note that $\|g_o(x)-A(x)\| = \inf\{\|y\| : y \in A(x)\} < a$ for all $x \in F(1)$, since $F(1) \subset V(1) = \{x \in X : A(x) \cap S(0,a) \neq \emptyset\}$

Assume that the functions $g_0, \ldots, g_n$ have been defined and that the following properties are satisfied:

4.14) $g_k : F(1) \to Y$ is continuous for $k = 0, 1, .., n$

4.15) $\|g_k(x) - A(x)\| < s^k a$ for each $x \in F(1)$ and $k = 0, 1, .., n$

4.16) $\|g_k(x) - g_{k+1}(x)\| \leq s^k a$ for each $x \in F(1)$ and $k = 0, 1, .., n-1$

Define $B : F(1) \to P'(Y)$ by $B(x) = A(x) \cap S(g_n(x), s^n a)$ for all $x \in F(1)$. Note that $B(x) \neq \emptyset$ by 4.15, and that B is continuous by Lemma 3.17. Hence, by corollary 3.21, there is a continuous function $g_{n+1} : F(1) \to Y$ such that $g_{n+1}(x) \in K(co(B(x)))$ for each $x \in F(1)$. Since $g_{n+1} \in KS(g_n(x), s^n a)$, property 4.16 follows immediately. It remains to verify 4.15 for the function g_{n+1}.

Let $\epsilon > 0$. Then there is a $y \in co(B(x))$ for which $\|g_{n+1}(x) - y\| < \epsilon$. Since $y \in co(B(x))$, there are a finite number of points $\{y_i\}_{i=1}^p \subset B(x)$ and such that $y \in co(\{y_i\}_{i=1}^p)$. Notice that if $1 \leq i, j \leq p$, then $\|y_i - y_j\| < 2s^n a$ since $B(x) \subset S(g_n(x), s^n a)$. Thus, we have

$$\begin{aligned} \|g_{n+1}(x) - A(x)\| &\leq \|g_{n+1}(x) - y\| + \|y - A(x)\| \\ &< \epsilon + t \max\{\|y_i - y_j\| : 1 \leq i, j \leq p\} \\ &\leq \epsilon + t \cdot 2s^n \cdot a, \text{ since } A(x) \in \mathfrak{C}(t, Y). \end{aligned}$$

Thus, $\|g_{n+1}(x) - A(x)\| \leq t \cdot 2s^n a < s^{n+1} a$ since $2t < s$ by the choice of s, and this verifies 4.15.

Hence, there is a sequence of functions $\{g_i\}_{i=1}^{\infty}$ which satisfy properties 4.14-4.16. The sequence is uniformly Cauchy by 4.16 and hence $f_1(x) = \lim_{i \to \infty} g_i(x)$ defines a continuous function $f_1 : F(1) \to Y$. We must verify that f_1 satisfies 4.11 and 4.12.

<u>Verification of 4.11:</u> Let $\epsilon > 0$ and let M be such that $\|f_1(x) - g_n(x)\| < \epsilon$ for all $n > M$ and all $x \in F(1)$.

Then, $\|f_1(x)-A(x)\| \leq \|f_1(x)-g_n(x)\| + \|g_n(x)-A(x)\|$
$< \epsilon + s^n \cdot a$ for all $n > M$ and all $x \in F(1)$

Thus, $\|f_1(x)-A(x)\| = 0$ and, since $A(x)$ is assumed closed in Y, it follows that $f_1(x) \in A(x)$ for each $x \in F(1)$.

Verification of 4.12: for all $x \in F(1)$, we have

$$\|f_1(x)\| = \|f_1(x)-g_o(x)\| \leq \|f_1(x)-g_n(x)\| + \sum_{k=0}^{n-1}\|g_{k+1}(x)-g_k(x)\|$$

$$\leq \|f_1(x)-g_n(x)\| + \sum_{k=0}^{\infty}\|g_{k+1}(x)-g_k(x)\| \text{ for all } n \in N.$$

Thus, $\|f_1(x)\| \leq \sum_{k=0}^{\infty}\|g_{k+1}(x)-g_k(x)\| \leq a\sum_{k=0}^{\infty} s^k < a^2$.

This completes the construction of f_1.

Inductive Hypothesis: Assume that functions $f_1, \ldots, f_{n_o}$ have been constructed which satisfy properties 4.10-4.13 for all $n \leq n_o$.

Construction of f_{n_o+1}: Define $C : F(n_o+1) \to P'(Y)$

$$\text{by } C(x) = \begin{cases} \{f_{n_o}(x)\} & \text{if } x \in F(n_o) \\ A(x) & \text{if } x \in F(n_o+1)-F(n_o) \end{cases}$$

Since $F(n_o)$ is closed in $F(n_o+1)$ and since f_{n_o} is continuous, it follows from Theorem 3.12 that C is continuous. Also note that $C(x)$ is closed and $C(x) \in \mathcal{A}(t,Y)$ for each $x \in F(n_o+1)$.

The construction of f_{n_o+1} is by induction.

Let $h_o : F(n_o+1) \to Y$ be defined by $h_o(x) = 0$ for $x \in F(n_o+1)$. If $x \in F(n_o)$ then $\|h_o(x)-C(x)\| = \|f_{n_o}(x)\| < a^{n_o+1}$ by 4.12.

If $x \in F(n_o+1)-F(n_o)$, then $\|h_o(x)-C(x)\| = \|h_o(x)-A(x)\| < a^{n_o+1}$, since $F(n_o+1) \subset V(n_o+1) = \{x \in X : A(x) \cap S(0,a^{n_o+1}) \neq \emptyset\}$. Thus for all $x \in F(n_o+1)$, we have $\|h_o(x) - C(x)\| < a^{n_o+1}$.

Assume now that functions $h_o, h_1, \ldots, h_p$ have been defined which satisfy the following conditions:

4.17) $h_k : F(n_o+1) \to Y$ is continuous for $k = 1, \ldots, p$

4.18) $\|h_k(x)-C(x)\| < s^k a^{n_o+1}$ for all $x \in F(n_o+1)$ and $k = 1, \ldots, p$

4.19) $\|h_{k+1}(x)-h_k(x)\| \leq s^k a^{n_o+1}$ for all $x \in F(n_o+1)$ and $k = 1, \ldots, p-1$

Define $D : F(n_o+1) \to P'(Y)$ by $D(x) = C(x) \cap S(h_p(x),s^p a^{n_o+1})$. Note that $D(x) \neq \emptyset$ by 4.18 and that D is continuous by lemma 3.17. Thus, by corollary 3.21, there is a continuous function $h_{p+1} : F(n_o+1) \to Y$ such that $h_{p+1}(x) \in K(co(D(x)))$ for each $x \in F(n_o+1)$. Thus we need to verify that h_{p+1} satisfies 4.18 and 4.19.

<u>Verification of 4.18</u>: Let $x \in F(n_o+1)$ and $\epsilon > 0$. Then there is a $y \in co(D(x))$ for which $\|h_{p+1}(x)-y\| < \epsilon$. Also, there are a finite number of points $\{y_i\}_{i=1}^q \subset D(x)$ such that $y \in co(\{y_i\}_{i=1}^q)$. Notice that if $1 \leq i, j \leq q$ then $\|y_i-y_j\| < 2s^p a^{n_o+1}$, since $D(x) \subset S(h_p(x),s^p a^{n_o+1})$.

$$\begin{aligned} \text{Hence, } \|h_{p+1}(x)-C(x)\| &\leq \|h_{p+1}(x)-y\| + \|y-C(x)\| \\ &< \epsilon + t \max\{\|y_i-y_j\| : 1\leq i, j\leq q\} \\ &\leq \epsilon + t.2s^p a^{n_o+1}, \text{ since } C(x) \in \mathcal{C}(t,Y). \end{aligned}$$

Thus, $\|h_{p+1}(x)-C(x)\| \leq t.2s^p a^{n_o+1} < s^{p+1}a^{n_o+1}$, since $2t < s$ by the choice of s.

Verification of 4.19: Since $h_{p+1}(x) \in KS(h_p(x), s^p a^{n_o+1})$, it follows immediately that $\|h_{p+1}(x)-h_p(x)\| \le s^p a^{n_o+1}$.

Thus, there is a sequence of functions $\{h_k\}_{k=1}^{\infty}$ which satisfy 4.17-4.19. By 4.19, the sequence converges uniformly, and thus $f_{n_o+1}(x) = \lim_{k\to\infty} h_k(x)$ defines a continuous function $f_{n_o+1} : F(n_o+1) \to Y$.

We now show that f_{n_o+1} satisfies properties 4.11-4.13.

Verification of 4.11 and 4.13: Observe that

$$\begin{aligned}\|f_{n_o+1}(x)-C(x)\| &\le \|f_{n_o+1}(x)-h_k(x)\| + \|h_k(x)-C(x)\| \\ &< \|f_{n_o+1}(x)-h_k(x)\| + s^k a^{n_o+1} \text{ for all}\end{aligned}$$

$k \in N$ and $x \in F(n_o+1)$. Thus, $\|f_{n_o+1}(x)-C(x)\| = 0$ for all $x \in F(n_o+1)$. Since $C(x)$ is closed for all x, it follows that $f_{n_o+1}(x) \in C(x)$. For $x \in F(n_o)$, then, $f_{n_o+1}(x) = f_{n_o}(x)$ and $f_{n_o}(x) \in A(x)$ by the induction hypothesis. For $x \in F(n_o+1)-F(n_o)$, we have $C(x) = A(x)$, and so again $f_{n_o+1}(x) \in A(x)$.

Verification of 4.12: We have

$$\begin{aligned}\|f_{n_o+1}(x)\| = \|f_{n_o+1}(x)-h_o(x)\| &\le \|f_{n_o+1}(x)-h_k(x)\| + \sum_{i=0}^{k-1}\|h_{i+1}(x)-h_i(x)\| \\ &\le \|f_{n_o+1}(x)-h_k(x)\| + \sum_{i=0}^{\infty}\|h_{i+1}(x)-h_i(x)\| \text{ for all } k.\end{aligned}$$

Thus, $\|f_{n_o+1}(x)\| \le \sum_{i=0}^{\infty}\|h_{i+1}(x)-h_i(x)\| \le a^{n_o+1}\sum_{i=0}^{\infty} s^i < a^{n_o+2}$.

This completes the construction of the sequence $\{f_i\}_{i=1}^{\infty}$. Define $f : X \to Y$ by $f(x) = f_n(x)$ for $x \in F(n)$. The definition is unambiguous by 4.13, and f is continuous since F is a closed locally finite covering of X. Finally $f(x) \in A(x)$ by 4.11. This completes the proof of the main theorem.

<u>4.20 Corollary</u>: If K is a closed subset of a Banach space Y, if $0 \leq t < \frac{1}{2}$, and if $K \in \mathcal{C}(t,Y)$ then K is an absolute extensor.

<u>Proof</u>: Let $f : A \to K$ be continuous, where A is a closed subspace of a metrizable space X. Define $B : X \to P'(Y)$ by

$$B(x) = \begin{cases} \{f(x)\} & \text{if } x \in A \\ K & \text{if } x \in X-A. \end{cases}$$

Then B is continuous by lemma 3.12 and B has a continuous choice function by the main theorem.

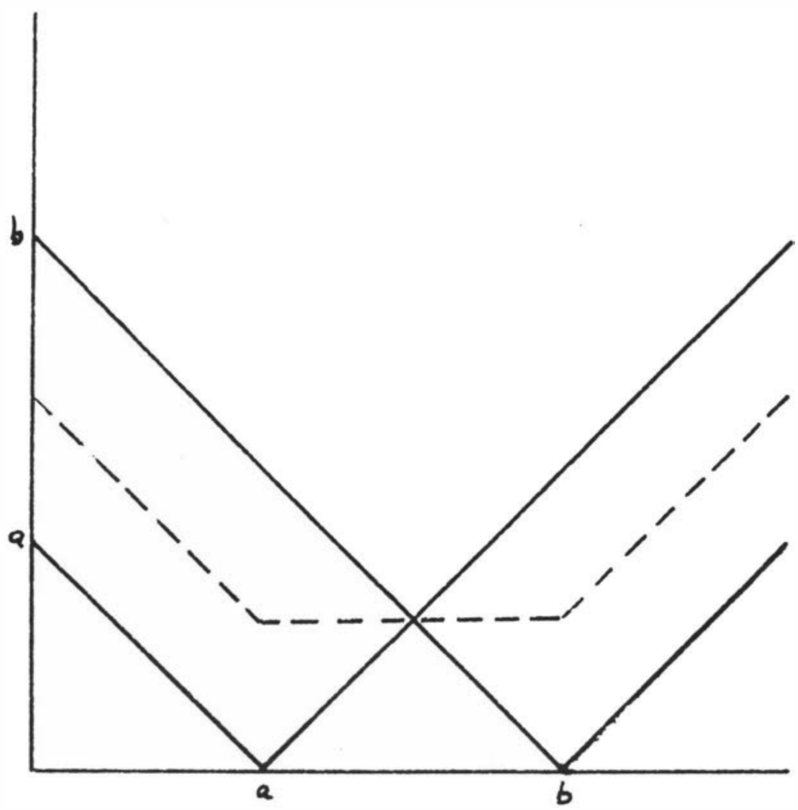
<u>4.21 Corollary</u>: If $K \in \mathcal{C}(t,Y)$ and if K is closed, then K is contractible and locally contractible.

<u>4.22 Corollary</u>: If $K \in \mathcal{C}(t,Y)$, $t < \frac{1}{2}$, and if K is compact, then K has the fixed point property.

<u>Proof</u>: Follows from theorem 2.24, since K is a compact metrizable AE.

<u>4.23 Example</u>: A compact one dimensional metrizable AE which is not even locally $\mathcal{C}(t,Y)$ for $t < \frac{1}{2}$: Let $I=[0,1]$ and $F : I \to C(I)$ be the Wojdyslawski embedding, i.e., $F(x)(r) = |x-r|$. Then certainly $F(I)$ is an AE, but $F(I) \notin \mathcal{C}(t,C(I))$ for $0 \leq t < \frac{1}{2}$, as may easily be seen from the graph on page 40.

The dotted graph is $\frac{1}{2}F(a) + \frac{1}{2}F(b)$, and the distance from this to $F(I)$ is easily seen to be $\frac{1}{2}|a-b|$.



Example 4.23: The solid graphs represent F(a) and F(b). The dashed graph is ½ F(a) + ½ F(b). The distance from ½ F(a) + ½ F(b) to any other point in F(I) may not be less than ½||F(a) - F(b)||. Thus, F(I) $\notin \mathcal{Q}(t, C(I))$ for any t < ½, even though F(I) is a compact one dimensional absolute retract.

APPENDIX I: THE SPACE ADJ(X,Y,f)

Definition: The topological sum of two topological spaces X and Y is the topological space X + Y given as follows: The underlying set is the disjoint union of the sets X and Y. The topology is the set $\{V \cup W : V \text{ open in } X \text{ and } W \text{ open in } Y\}$.

In particular, note that X and Y are both embedded as closed subsets of X + Y, and are considered in the natural way as subsets of X + Y.

Definition: Suppose that $A \subset X$ and that $f : A \to Y$ is continuous. The space Adj(X,Y,f) is the quotient space of X + Y under the identification of x and f(x), i.e., $\mathrm{Adj}(X,Y,f) = \frac{X + Y}{E}$, where E is the equivalence relation given by $\langle x_1, x_2\rangle \in E \Leftrightarrow x_1 = x_2$ or $fx_1 = fx_2$ or $fx_1 = x_2$ or $fx_2 = x_1$

Denote by p the quotient mapping from X + Y to Adj(X,Y,f).

Lemma: If A is closed, then $p|Y$ is a homeomorphic embedding of Y as a closed subspace of Adj(X,Y,f).

Proof: $p|Y$ is easily seen to be injective. To see that it is closed, let F be a closed subset of Y. Then $p^{-1}(p(F)) = f^{-1}(F) \cup F$ which is closed in X + Y. Since p is a quotient map, it follows that p(F) is closed in Adj(X,Y,f). Consequently, $p|Y$ is a homeomorphism and $(p|Y)(Y)$ is closed in Adj(X,Y,f).

As usual, Y is identified with p(Y) and so considered to be a subset of Adj(X,Y,f).

APPENDIX II: SOME STANDARD DEFINITIONS

Definition: A subset K of a topological space X is a G_δ if and only if K is the intersection of a countable number of sets, each open in X.

Definition: A topological space X is perfectly normal if and only if each closed set of X is a G_δ.

Definition: If $V : A \to P(X)$ is an indexed collection of subsets of X, $st_V : X \to P(X)$ is defined by

$$st_V(x) = \cup\{V(a) : x \in V(a),\ a \in A\}.$$

Definition: V is a star refinement of $W : A \to P(X)$ if and only if $st_V : X \to P(X)$ is a refinement of W.

Definition: X is fully normal if and only if each open cover W of X has a star-refinement which is also an open cover.

Theorem (Stone [11]) X is fully normal if and only if X is regular and each open cover has a locally finite refinement which is also an open cover.

Definition: X is metrically complete if and only if there is a metric for the topology of X which is a complete metric.

APPENDIX III: THE URYSOHN LEMMA

Theorem: Let X_o and X_1 be closed subsets of a normal space X. There is a continuous function $f : X \to [0,1]$ such that $f|X_o = 0$ and $f|X_1 = 1$.

Proof: Let D be the set of dyadic rationals between 0 and 1, i.e. $D = \{\frac{p}{2^q} : p, q$ positive integers with $p < 2^q\}$. An indexed collection of open sets $A : D \to P(X)$ is defined inductively on q as follows:

q=1: Normality of X permits the choice of an open set $A(\frac{1}{2})$ such that

$$X_o \subset A(\tfrac{1}{2}) \subset KA(\tfrac{1}{2}) \subset X-X_1$$

q=n+1: Choose, using normality, for each $p=2k+1$, $k=1,2,\ldots,2^n-2$ an open set $A(\frac{2k+1}{2^{n+1}})$ such that

$$KA\left(\frac{k}{2^n}\right) \subset A\left(\frac{2k+1}{2^{n+1}}\right) \subset KA\left(\frac{2k+1}{2^{n+1}}\right) \subset A\left(\frac{k+1}{2^n}\right)$$

and, for $k=0$ and $k=2^n-1$,

$$X_o \subset A\left(\frac{1}{2^{n+1}}\right) \subset KA\left(\frac{1}{2^{n+1}}\right) \subset A\left(\frac{1}{2^n}\right)$$

$$KA\left(\frac{2^n-1}{2^n}\right) \subset A\left(\frac{2^{n+1}-1}{2^{n+1}}\right) \subset KA\left(\frac{2^{n+1}-1}{2^{n+1}}\right) \subset X-X_1.$$

Observe that the collection A satisfies

a) if $r \in D$, $s \in D$ and $r < s$, then $KA(r) \subset A(s)$

b) for all $r \in D$, $X_o \subset A(r) \subset KA(r) \subset X-X_1$.

Define $f : X \to [0,1]$ by

$$f(x) = \begin{cases} 0 \text{ if } x \in A(r) \text{ for all } r \in D \\ \sup\{r \in D : x \notin A(r)\} \text{ otherwise.} \end{cases}$$

Note that $f|X_o = 0$ and $f|X_1 = 1$ follow immediately from (b). It remains to show that f is continuous. Let $x \in X$.

Case $0 < f(x) < 1$: Let $\epsilon > 0$ be given. It may be assumed that $0 < f(x)-\epsilon$ and $f(x)+\epsilon < 1$. Choose $\eta_1 < \epsilon$ such that $f(x)+\eta_1 \in D$ and $\eta_2 < \epsilon$ such that $f(x)-\eta_2 \in D$.

Since $f(x) = \sup\{r \in D : x \notin A(r)\}$, it follows that $x \in A(f(x)+\eta_1)$. Further, by replacing η_2 with $\eta_2/2$ if necessary, it may be supposed that $x \notin KA(f(x)-\eta_2)$. Consequently $A(f(x)+\eta_1) \cap (X-KA(f(x)-\eta_2))$ is an open neighborhood of x. If y is any element of this neighborhood, then $f(y) = \sup\{r \in D : y \notin A(r)\}$ and so

$$f(x)-\epsilon < f(y) < f(x)+\epsilon$$

and so f is continuous at x.

<u>Case $f(x)=0$</u>: Let $\epsilon > 0$ be given and choose $\eta \in D$ such that $0 < \eta < \min(\epsilon,1)$. For $y \in A(\eta)$, $f(y) = \sup\{r \in D : y \notin A(r)\} \leq \eta \leq \epsilon$, or $f(y) = 0$ and consequently f is continuous at x.

<u>Case $f(x)=1$</u>: Let $\epsilon > 0$ be given and choose $\eta \in D$ such that $o < \eta < \min(\epsilon,1)$. We may assume that $x \notin KA(1-\eta)$. Thus, for $y \in X-KA(1-\eta)$, we have $f(y) = \sup\{r \in D : y \notin A(r)\} \geq 1-\eta > 1-\epsilon$ and so, again, f is continuous at x.

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NOTES ON NOTATION

The symbols K, B, and I denote the closure, boundary, and interior operators in a fixed topological space. The complement of A in X is written as X - A, and parentheses are frequently omitted where not necessary. Thus, for example, KIV-A means K(I(V))-A.

In a metric space, S(a,r) denotes the set of points whose distance from the point a is less than r. Slightly less standard is the use of S(A,r), where A is a non-empty set to denote the set of points whose distance from the set A is less than r. Note that S(KA,r) = S(A,r). Even less standard, but very useful, is the notation $\|A\|$ to denote $\inf\{\|a\| : a \in A\}$ when A is a subset of a normed linear space.

Other notation conforms to standard mathematical usage and should require no explanation.