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Generalized Inverses, Stationary Distributions and Mean First Passage Times with applications to Perturbed Markov Chains

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Abstract
In an earlier paper (Hunter, 2002) it was shown that mean first passage times play an important role in determining bounds on the relative and absolute differences between the stationary probabilities in perturbed finite irreducible discrete time Markov chains. Further when two perturbations of the transition probabilities in a single row are carried out the differences between the stationary probabilities in the unperturbed and perturbed situations are easily expressed in terms of a reduced number of mean first passage times. Using this procedure we provide an updating procedure for mean first passage times to determine changes in the stationary distributions under successive perturbations. Simple procedures for determining both stationary distributions and mean first passage times in a finite irreducible Markov chain are also given. The techniques used in the paper are based upon the application of generalized matrix inverses.

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1. Introduction

Cho and Meyer [1] and Hunter [8] have shown that there is a strong interconnection between the mean first passage times and stationary distributions in examining the effects of small changes, or perturbations, on the transition probabilities in finite irreducible discrete time Markov chains. We explore these interconnections in more detail, especially in relation to updating Markov chains through successive two-element perturbations. Since multiple-element perturbations do not have the nice properties in respect to the observed effects upon the stationary distributions evident in the two-element perturbation situation, we consider decomposing such perturbations into a string of two-element perturbations. This requires the updating of the mean first passage times, in conjunction with the stationary distributions, at each perturbation.

Our techniques involve the use of generalized matrix inverses. Following a summary of the relevant properties of such matrices for Markovian kernels we look at procedures for obtaining stationary distributions and mean first passage times. We then summarize the results for the changes in these brought about by general perturbations but more specifically by two-element perturbations. This leads to a general procedure for updating both stationary distributions and mean first passage times under successive two-element perturbations.

2. Generalized inverses of Markovian kernels

Let $P = [p_{ij}]$ be the transition matrix of a finite irreducible, m-state Markov chain with state space $S = \{1, 2, \dots, m\}$ and stationary probability vector $\boldsymbol{\pi}' = (\pi_1, \pi_2, \dots, \pi_m)$.

The following results summarise the key features of generalized inverses of the Markovian kernel $I - P$ that we shall make use of in developing our results.

2.1 G is a generalized inverse (g-inverse), or a 1-condition g-inverse, of $I - P$ if and only if

$$(I - P)G(I - P) = I - P. \tag{2.1}$$

2.2 (Hunter, [2]). If G is any g-inverse of I – P then there exists vectors **f, g**, **t** and **u** with $\mathbf{u}'\mathbf{e} \neq 0$ and $\boldsymbol{\pi}'\mathbf{t} \neq 0$ where $\mathbf{e}' = (1, 1, \ldots, 1)$ such that

$$G = [I - P + \mathbf{t}\mathbf{u}']^{-1} + \mathbf{e}\mathbf{f}' + \mathbf{g}\boldsymbol{\pi}' \tag{2.2}$$

2.2 (Hunter, [6]) If G is any g-inverse of I – P, define $A \equiv I - (I - P)G$ and $B \equiv I - G(I - P)$,

then $$G = [I - P + \boldsymbol{\alpha}\boldsymbol{\beta}']^{-1} + \gamma\mathbf{e}\boldsymbol{\pi}', \tag{2.3}$$

where $$\boldsymbol{\alpha} = A\mathbf{e},\ \boldsymbol{\beta}' = \boldsymbol{\pi}'B,\ \gamma + 1 = \boldsymbol{\pi}'G\boldsymbol{\alpha} = \boldsymbol{\beta}'G\mathbf{e} = \boldsymbol{\beta}'G\boldsymbol{\alpha}, \tag{2.4}$$

and $$\boldsymbol{\pi}'\boldsymbol{\alpha} = 1,\ \boldsymbol{\beta}'\mathbf{e} = 1. \tag{2.5}$$

Further $$A = \boldsymbol{\alpha}\boldsymbol{\pi}' \tag{2.6}$$

and $$B = \mathbf{e}\boldsymbol{\beta}'. \tag{2.7}$$

2.2 The parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and γ uniquely specify and characterize the g-inverse so that we can denote such a g-inverse as $G(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$.

2.2 Generalized inverses may satisfy additional conditions. In particular, we have the following conditions:

Condition 1: $(I - P)G(I - P) = I - P$,
Condition 2: $G(I - P)G = G$,
Condition 3: $[(I - P)G]' = (I - P)G$,
Condition 4: $[G(I - P)]' = G(I - P)$,
Condition 5: $(I - P)G = (I - P)G$.

In Hunter [6] it is shown that

$G(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$ satisfies condition 2 if and only if $\gamma = -1$,
$G(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$ satisfies condition 3 if and only if $\boldsymbol{\alpha} = \boldsymbol{\pi}/\boldsymbol{\pi}'\boldsymbol{\pi}$,
$G(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$ satisfies condition 4 if and only if $\boldsymbol{\beta} = \mathbf{e}/\mathbf{e}'\mathbf{e}$,
$G(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$ satisfies condition 5 if and only if $\boldsymbol{\alpha} = \mathbf{e}$ and $\boldsymbol{\beta} = \boldsymbol{\pi}$.

Special unique generalized inverses are the *Moore-Penrose* g-inverse of I – P which satisfies conditions 1, 2, 3 and 4 with $G = G(\boldsymbol{\pi}/\boldsymbol{\pi}'\boldsymbol{\pi}, \mathbf{e}/\mathbf{e}'\mathbf{e}, -1)$ and the *group inverse* which satisfies conditions 1, 2 and 5 with $G = G(\mathbf{e}, \boldsymbol{\pi}, -1) = A^{\#}$ as originally derived by Paige, Styan and Wachter [12] and Meyer [10], respectively.

2.6 The following results are easily established (see Section 3.3, [2])

(a) $$\mathbf{u}'[I - P + \mathbf{t}\mathbf{u}']^{-1} = \boldsymbol{\pi}'/(\boldsymbol{\pi}'\mathbf{t}). \tag{2.8}$$
(b) $$[I - P + \mathbf{t}\mathbf{u}']^{-1}\mathbf{t} = \mathbf{e}/(\mathbf{u}'\mathbf{e}). \tag{2.9}$$
(c) $$\mathbf{u}'[I - P + \mathbf{t}\mathbf{u}']^{-1}\mathbf{t} = 1. \tag{2.10}$$

2. Stationary distributions

There are a variety of techniques that can be used for the computation of stationary distributions involving the solution of the singular system of linear equations, $\boldsymbol{\pi}'(I - P) = \mathbf{0}'$, subject to the boundary condition $\boldsymbol{\pi}'\mathbf{e} = 1$.

Since, as we shall see later, the derivation of mean first passage times involves either the computation of a matrix inverse or a matrix generalized inverse, we consider those techniques for solving the stationary distributions using generalized inverses. This will enable us later to consider the joint computation of the stationary distributions and mean first passage times with a minimal set of computations.

Theorem 3.1.: (Hunter[2]) If G is any g-inverse of I – P, A = I – (I – P)G, and $\mathbf{v}$ *is any vector such that* $\mathbf{v}'A\boldsymbol{e} \neq 0$ *then*

$$\boldsymbol{\pi}' = \frac{v'A}{v'Ae}\ . \tag{3.1}$$

❑

Numerous special cases follow from the above Theorem. See Hunter [2], [3], [7].

The key observation however is that A has a very special structure as exhibited by (2.6) viz. $A \equiv I - (I - P)G = \boldsymbol{\alpha\pi}'$ where $\boldsymbol{\alpha}'$ is a subset of the parameters that specify the g-inverse of I – P. Since, from (2.5), $\boldsymbol{\pi}'\boldsymbol{\alpha} = 1$ it is clear that $\boldsymbol{\alpha} \neq 0$ (for otherwise $\boldsymbol{\pi}'\boldsymbol{\alpha}$ would be zero) and consequently $A\mathbf{e} = \boldsymbol{\alpha} \neq 0$ and a suitable choice of $\mathbf{v}'$ for Theorem 3.1 is always possible to obtain.

Suppose we let $\mathbf{v}' = \mathbf{e}_i' = (0, \ldots, 0, 1, 0, \ldots, 0)$, the elementary vector with 1 in the ith position and zero elsewhere. Then $\mathbf{e}_i'A\mathbf{e} = \mathbf{e}_i'\boldsymbol{\alpha} = \boldsymbol{\alpha}_i'$ which must be non-zero for at least one such i. Since $\mathbf{e}_i'A$ is the vector consisting of the elements of the ith row of A, the implications of the above observation is that we can always find at least one row of A that does not contain a non-zero element. Furthermore, if there is at least one element non-zero in that row all the elements in that row must be non-zero since the rows of A are scaled versions of $\boldsymbol{\pi}'$. Thus if $A = [a_{ij}]$ then there is at least one i such $a_{i1} \neq 0$ in which case $a_{ij} \neq 0$ for $j = 1, \ldots, m$.

This leads to following result, obtained earlier by Hunter, [7].

Theorem 3.2: Let G *be any g-inverse of* I – P. *Let* $A = I - (I - P)G \equiv [a_{ij}]$.
Let r be the smallest integer i $(1 \leq i \leq m)$ *such that* $\sum_{k=1}^{m} a_{ik} \neq 0,$ *then*

$$\pi_j = \frac{a_{rj}}{\sum_{k=1}^{m} a_{rk}}, \quad j = 1, 2, \ldots, m. \tag{3.2}$$

❑

In applying Theorem 3.2 one typically needs to first find a_{11}. If $a_{11} \neq 0$ then the first row of A will suffice to find the stationary probabilities. If not find $a_{21}, a_{31}, \ldots$ and stop at the first non-zero a_{r1}. If we know something of the structure of the g-inverse package being used typically we need only find the first row of A. For example MATLAB uses the pseudo inverse routine pinv(I – P) which satisfies conditions 1, 2, 3 and 4 of §2.5.

Corollary 3.2.1: If G *is any g-inverse of* I – P *that satisfies either condition 3 or condition 5 of §2.5, and if* $A = I - (I - P)G \equiv [a_{ij}]$ *then*

$$\pi_j = \frac{a_{1j}}{\sum_{k=1}^{m} a_{1k}}, \quad j = 1, 2, \ldots, m. \tag{3.3}$$

Proof: G satisfies condition 3 if $\boldsymbol{\alpha} = \boldsymbol{\pi}/\boldsymbol{\pi}'\boldsymbol{\pi}$, in which case $\alpha_1 \neq 0$. Similarly G satisfies condition 5 if $\boldsymbol{\alpha} = \mathbf{e}$ in which case $\alpha_1 = 1$. The non-zero form of α_1 ensures $a_{11} \neq 0$.

❑

Conditions 2 or 4 of §2.4 do not place any restrictions upon $\boldsymbol{\alpha}$ and consequently the non-zero nature of a_{11} cannot be guaranteed in these situations.

While (3.1), (3.2) and (3.3) are useful expressions for obtaining the stationary probabilities, the added computation of A following the derivation of a g-inverse G is typically unnecessary, especially when additional special properties of G are given. Typically one can take G as special matrix inverse.

Theorem 3.3: If $G = [I - P + \mathbf{tu}']^{-1}$ *where* $\boldsymbol{u}$ *and* $\boldsymbol{t}$ *are any vectors such that* $\mathbf{u}'\mathbf{e} \neq 0$ *and* $\boldsymbol{\pi}'\mathbf{t} \neq 0$*, then*

$$\boldsymbol{\pi}' = \frac{\mathbf{u}'G}{\mathbf{u}'G\mathbf{e}}. \tag{3.4}$$

and hence, if $\mathbf{u}' = (u_1, u_2, \ldots, u_m)$,

$$\pi_j = \frac{\sum_{k=1}^{m} u_k g_{kj}}{\sum_{r=1}^{m} u_r \sum_{s=1}^{m} g_{rs}} = \frac{\sum_{k=1}^{m} u_k g_{kj}}{\sum_{r=1}^{m} u_r g_{r.}},\ j = 1,\ 2,\ ...,\ m. \tag{3.5}$$

Proof: Using (2.8) it is easily seen that $\mathbf{u}'[I - P + \mathbf{tu}']^{-1}\mathbf{e} = \boldsymbol{\pi}'\mathbf{e}/(\boldsymbol{\pi}'\mathbf{t}) = 1/(\boldsymbol{\pi}'\mathbf{t})$ and (3.4) follows. Elemental expression (3.5) follows.

❑

In this paper we consider the use of a variety of special g-inverses, which we enumerate below with their specific parameters (cf. §2.3). Let $\mathbf{p}_r^{(r)\prime} = \mathbf{e}_r'P$ denote the rth row of the transition matrix P and $\mathbf{p}_r^{(c)} = P\mathbf{e}_r$ denote the rth column of P.

Table 1: Special g-inverses

Identifier	g-inverse $[I - P + \mathbf{tu}']^{-1}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}'$	γ
$G_{rr}^{(r)}$	$[I - P + \mathbf{e}_r\mathbf{p}_r^{(r)\prime}]^{-1}$	$\mathbf{e}_r/\pi_r$	$\mathbf{p}_r^{(r)\prime}$	$(1/\pi_r) - 1$
$G_{rr}^{(c)}$	$[I - P + \mathbf{p}_r^{(c)}\mathbf{e}_r']^{-1}$	$\mathbf{p}_r^{(c)}/\pi_r$	$\mathbf{e}_r'$	$(1/\pi_r) - 1$
G_{rr}	$[I - P + \mathbf{e}_r\mathbf{e}_r']^{-1}$	$\mathbf{e}_r/\pi_r$	$\mathbf{e}_r'$	$(1/\pi_r) - 1$
$G_{er}^{(r)}$	$[I - P + \mathbf{e}\mathbf{p}_r^{(r)\prime}]^{-1}$	$\mathbf{e}$	$\mathbf{p}_r^{(r)\prime}$	0
$G_{re}^{(c)}$	$[I - P + \mathbf{p}_r^{(c)}\mathbf{e}']^{-1}$	$\mathbf{p}_r^{(c)}/\pi_r$	$\mathbf{e}'/m$	$(1/m\pi_r) - 1$
G_{ee}	$[I - P + \mathbf{ee}']^{-1}$	$\mathbf{e}$	$\mathbf{e}'/m$	$(1/m) - 1$
G_{er}	$[I - P + \mathbf{e}\mathbf{e}_r']^{-1}$	$\mathbf{e}$	$\mathbf{e}_r'$	0
G_{re}	$[I - P + \mathbf{e}_r\mathbf{e}']^{-1}$	$\mathbf{e}_r/\pi_r$	$\mathbf{e}'/m$	$(1/m\pi_r) - 1$

All these results follow from the observation that if $G = [I - P + \mathbf{tu}']^{-1}$ then the parameters are given by $\boldsymbol{\alpha} = \mathbf{t}/\boldsymbol{\pi}'\mathbf{t}$, $\boldsymbol{\beta}' = \mathbf{u}'/\mathbf{u}'\mathbf{e}$ and $\gamma + 1 = 1/\{(\boldsymbol{\pi}'\mathbf{t})(\mathbf{u}'\mathbf{e})\}$.

The special structure of the g-inverses given in Table 1 lead, in most cases, to very simple forms for the stationary probabilities.

In applying Theorem 3.3, observe that $\boldsymbol{\pi}' = \mathbf{u}'G$ if and only if $\mathbf{u}'G\mathbf{e} = 1$ if and only if $\boldsymbol{\pi}'\mathbf{t} = 1$.

Simple sufficient conditions for $\boldsymbol{\pi}'\mathbf{t} = 1$ are $\mathbf{t} = \mathbf{e}$ or $\mathbf{t} = \boldsymbol{\alpha}$ (cf. (2.5)). (This later condition is of use only if $\boldsymbol{\alpha}$ does not explicitly involve any of the stationary probabilities.)

Corollary 3.3.1: If $G = [I - P + \mathbf{eu}']^{-1}$ *where* $\mathbf{u}'\mathbf{e} \neq 0$,

$$\boldsymbol{\pi}' = \mathbf{u}'G\,. \tag{3.6}$$

and hence if $\mathbf{u}' = (u_1, u_2, \ldots, u_m)$ *and* $G = [g_{ij}]$ *then*

$$\pi_j = \sum_{k=1}^{m} u_k g_{kj},\ j = 1,\ 2,\ ...,\ m. \tag{3.7}$$

❑

In particular, we have the following special cases:

(i) If $\mathbf{u}' = \mathbf{e}'$ then $G \equiv G_{ee} = [I - P + \mathbf{ee}']^{-1} = [g_{ij}]$ and

$$\pi_j = \sum_{k=1}^{m} g_{kj} \equiv g_{.j}. \quad (3.8)$$

(ii) If $\mathbf{u}' = \mathbf{p}_r^{(r)\prime}$ then $G \equiv G_{er}^{(r)} = [I - P + \mathbf{ep}_r^{(r)\prime}]^{-1} = [g_{ij}]$ and

$$\pi_j = \sum_{k=1}^{m} p_{rk} g_{kj}. \quad (3.9)$$

(iii) If $\mathbf{u}' = \mathbf{e}_r'$ then $G \equiv G_{er} = [I - P + \mathbf{ee}_r']^{-1} = [g_{ij}]$ and

$$\pi_j = g_{rj}. \quad (3.10)$$

Corollary 3.3.2: If $G = [I - P + \mathbf{te}']^{-1}$ *where* $\boldsymbol{\pi}'\mathbf{t} \neq 0$,

$$\boldsymbol{\pi}' = \frac{\mathbf{e}'G}{\mathbf{e}'G\mathbf{e}}, \quad (3.11)$$

and hence, if $G = [g_{ij}]$, *then*

$$\pi_j = \frac{\sum_{k=1}^{m} g_{kj}}{\sum_{r=1}^{m}\sum_{s=1}^{m} g_{rs}} = \frac{g_{.j}}{g_{..}}, \quad j = 1, 2, ..., m. \quad (3.12)$$

❑

In particular, results (3.12) hold for $G = G_{re}^{(c)}$, G_{ee} and G_{re}.

In the special case of G_{ee}, using (2.10) it follows that $g_{..} = 1$, and (3.12) reduces to (3.8).

Corollary 3.3.3: If $G = [I - P + \mathbf{te}_r']^{-1}$ *where* $\boldsymbol{\pi}'\mathbf{t} \neq 0$,

$$\boldsymbol{\pi}' = \frac{\mathbf{e}_r'G}{\mathbf{e}_r'G\mathbf{e}}, \quad (3.13)$$

and hence, if $G = [g_{ij}]$, *then*

$$\pi_j = \frac{g_{rj}}{\sum_{s=1}^{m} g_{rs}} = \frac{g_{rj}}{g_{r.}}, \quad j = 1, 2, ..., m. \quad (3.14)$$

❑

In particular, results (3.14) hold for $G = G_{rr}^{(c)}$, G_{rr} and G_{er}.

In the special case of G_{er}, $g_{r.} = 1$ since $\mathbf{u}'[I - P + \mathbf{tu}']^{-1}\mathbf{t} = 1$ and (3.14) reduces to (3.10).

Corollary 3.3.3: If $G = [I - P + \mathbf{tp}_r^{(r)\prime}]^{-1}$ *where* $\boldsymbol{\pi}'\mathbf{t} \neq 0$,

$$\boldsymbol{\pi}' = \frac{\mathbf{p}_r^{(r)\prime}G}{\mathbf{p}_r^{(r)\prime}G\mathbf{e}}, \quad (3.15)$$

and hence, if $G = [g_{ij}]$, *then*

$$\pi_j = \frac{\sum_{k=1}^{m} p_{rk} g_{kj}}{\sum_{i=1}^{m}\sum_{s=1}^{m} p_{ri} g_{is}}, \quad j = 1, 2, ..., m. \quad (3.16)$$

❑

In particular, results (3.16) hold for $G = G_{rr}^{(r)}$ and $G_{er}^{(r)}$.

In the special case of $G_{er}^{(r)}$, the denominator of (3.16) is 1 and (3.16) reduces to (3.9).

Thus we have been able to find simple elemental expressions for the stationary probabilities using any of the generalized inverses in Table 1. In the special cases of G_{ee}, $G_{er}^{(r)}$ and G_{er} the denominator of the expression given by equations (3.5) is always 1. (In each other case, observe that denominator of the expression, $\mathbf{u}'G\mathbf{e}$, is in fact $1/\pi_r$ and that $\mathbf{u}'G = \boldsymbol{\pi}'/\pi_r$.)

We explore inter-relationships between some of the g-inverses in Table 1 by utilizing the following result given by Theorem 3.3 of Hunter [5].

Theorem 3.4: Let P *be the transition matrix of a finite irreducible transition matrix of a Markov chain with stationary probability vector* $\boldsymbol{\pi}'$. *Suppose that for* i = 1, 2, $\boldsymbol{\pi}'\mathbf{t}_i \neq 0$ *and* $\mathbf{u}_i'\mathbf{e} \neq 0$. *Then*

$$[I - P + \mathbf{t}_2\mathbf{u}_2']^{-1} = [I - \frac{\mathbf{e}\mathbf{u}_2'}{\mathbf{u}_2'\mathbf{e}}][I - P + \mathbf{t}_1\mathbf{u}_1']^{-1}[I - \frac{\mathbf{t}_2\boldsymbol{\pi}'}{\boldsymbol{\pi}'\mathbf{t}_2}] + \frac{\mathbf{e}\boldsymbol{\pi}'}{(\boldsymbol{\pi}'\mathbf{t}_2)(\mathbf{u}_2'\mathbf{e})}. \tag{3.17}$$

and hence that

$$[I - P + \mathbf{t}_2\mathbf{u}_2']^{-1} - [I - P + \mathbf{t}_1\mathbf{u}_1']^{-1}$$

$$= \frac{\mathbf{e}\mathbf{u}_2'}{\mathbf{u}_2'\mathbf{e}}[I - P + \mathbf{t}_1\mathbf{u}_1']^{-1}\frac{\mathbf{t}_2\boldsymbol{\pi}'}{\boldsymbol{\pi}'\mathbf{t}_2} - \frac{\mathbf{e}\mathbf{u}_2'}{\mathbf{u}_2'\mathbf{e}}[I - P + \mathbf{t}_1\mathbf{u}_1']^{-1} - [I - P + \mathbf{t}_1\mathbf{u}_1']^{-1}\frac{\mathbf{t}_2\boldsymbol{\pi}'}{\boldsymbol{\pi}'\mathbf{t}_2} + \frac{\mathbf{e}\boldsymbol{\pi}'}{(\boldsymbol{\pi}'\mathbf{t}_2)(\mathbf{u}_2'\mathbf{e})}. \tag{3.18}$$

❑

In particular, we wish to focus on the difference between $G_{rr}^{(c)} = [I - P + \mathbf{p}_r^{(c)}\mathbf{e}_r']^{-1}$ and $G_{rr}^{(r)} = [I - P + \mathbf{e}_r\mathbf{p}_r^{(r)\prime}]^{-1}$.

We first need to establish some preliminary results:

Lemma 3.5:

(a) The rth row of $G_{rr}^{(c)}$ is given by $\mathbf{e}_r'G_{rr}^{(c)} = \dfrac{\boldsymbol{\pi}'}{\pi_r}$. (3.19)

(b) The rth column of $G_{rr}^{(c)}$ is given by $G_{rr}^{(c)}\mathbf{e}_r = \mathbf{e}_r$. (3.20)

(c) The rth row of $G_{rr}^{(r)}$ is given by $\mathbf{e}_r'G_{rr}^{(r)} = \mathbf{e}_r'$. (3.21)

(d) The rth column of $G_{rr}^{(r)}$ is given by $G_{rr}^{(r)}\mathbf{e}_r = \mathbf{e}$. (3.22)

Proof: (a) Follows from (2.8) and the fact that $\boldsymbol{\pi}'\mathbf{p}_r^{(c)} = \boldsymbol{\pi}'P\mathbf{e}_r = \boldsymbol{\pi}'\mathbf{e}_r = \pi_r$.

(b) First observe that, by (2.9) $G_{rr}^{(c)}\mathbf{p}_r^{(c)} = \mathbf{e}$, so that by the definition of $G_{rr}^{(c)}$,

$$I = G_{rr}^{(c)} - G_{rr}^{(c)}P + G_{rr}^{(c)}\mathbf{p}_r^{(c)}\mathbf{e}_r'. \tag{3.23}$$

Now post-multiplication of (3.23) by $\mathbf{e}_r$ yields

$$\mathbf{e}_r = G_{rr}^{(c)}\mathbf{e}_r - G_{rr}^{(c)}P\mathbf{e}_r + G_{rr}^{(c)}\mathbf{p}_r^{(c)}\mathbf{e}_r'\mathbf{e}_r = G_{rr}^{(c)}\mathbf{e}_r - G_{rr}^{(c)}\mathbf{p}_r^{(c)} + G_{rr}^{(c)}\mathbf{p}_r^{(c)} = G_{rr}^{(c)}\mathbf{e}_r, \tag{3.24}$$

and (3.20) follows.

(c) By the definition of $G_{rr}^{(r)}$,

$$I = G_{rr}^{(r)} - PG_{rr}^{(r)} + \mathbf{e}_r\mathbf{p}_r^{(r)\prime}G_{rr}^{(r)}. \tag{3.25}$$

Now pre-multiplication of (3.25) by $\mathbf{e}_r'$ yields

$$\mathbf{e}_r' = \mathbf{e}_r'G_{rr}^{(r)} - \mathbf{e}_r'PG_{rr}^{(r)} + \mathbf{e}_r'\mathbf{e}_r\mathbf{p}_r^{(r)\prime}G_{rr}^{(r)} = \mathbf{e}_r'G_{rr}^{(r)} - \mathbf{p}_r^{(r)\prime}G_{rr}^{(r)} + \mathbf{p}_r^{(r)\prime}G_{rr}^{(r)} = \mathbf{e}_r'G_{rr}^{(r)}, \tag{3.26}$$

and (3.21) follows.

(d) Follows from (2.9) and the fact that $\mathbf{p}_r^{(r)\prime}\mathbf{e} = 1$.

❑

Theorem 3.6:

$$G_{rr}^{(c)} - G_{rr}^{(r)} = \frac{\mathbf{e}_r\boldsymbol{\pi}'}{\pi_r} - \mathbf{e}\mathbf{e}_r'. \tag{3.27}$$

Proof: From equation (3.18) it is easily seen that

$$G_{rr}^{(c)} - G_{rr}^{(r)} = \frac{\mathbf{e}\mathbf{e}_r'}{\mathbf{e}_r'\mathbf{e}}G_{rr}^{(r)}\frac{\mathbf{p}_r^{(c)}\boldsymbol{\pi}'}{\boldsymbol{\pi}'\mathbf{p}_r^{(c)}} - \frac{\mathbf{e}\mathbf{e}_r'}{\mathbf{e}_r'\mathbf{e}}G_{rr}^{(r)} - G_{rr}^{(r)}\frac{\mathbf{p}_r^{(c)}\boldsymbol{\pi}'}{\boldsymbol{\pi}'\mathbf{p}_r^{(c)}} + \frac{\mathbf{e}\boldsymbol{\pi}'}{(\mathbf{e}_r'\mathbf{e})(\boldsymbol{\pi}'\mathbf{p}_r^{(c)})}. \tag{3.28}$$

Using the observations that $\mathbf{e}_r'\mathbf{e} = 1$, $\mathbf{e}_r'G_{rr}^{(r)} = \mathbf{e}_r'$, $\mathbf{p}_r^{(c)} = P\mathbf{e}_r$, $\boldsymbol{\pi}'\mathbf{p}_r^{(c)} = \boldsymbol{\pi}'\mathbf{e}_r = \pi_r$, $\mathbf{e}_r'P\mathbf{e}_r = p_{rr}$, equation (3.28) simplifies to

$$G_{rr}^{(c)} - G_{rr}^{(r)} = \frac{p_{rr}\mathbf{e}\boldsymbol{\pi}'}{\pi_r} - \mathbf{e}\mathbf{e}_r' - \frac{G_{rr}^{(r)}\mathbf{P}\mathbf{e}_r\boldsymbol{\pi}'}{\pi_r} + \frac{\mathbf{e}\boldsymbol{\pi}'}{\pi_r}. \tag{3.29}$$

Now observe that, by the definition of $G_{rr}^{(r)}$,

$$I = G_{rr}^{(r)} - G_{rr}^{(r)}P + G_{rr}^{(r)}\mathbf{e}_r\mathbf{p}_r^{(r)\prime}. \tag{3.30}$$

Post-multiplying (3.30) by $\mathbf{e}_r$ and using the results of (3.22) yields

$$\mathbf{e}_r = G_{rr}^{(r)}\mathbf{e}_r - G_{rr}^{(r)}P\mathbf{e}_r + G_{rr}^{(r)}\mathbf{e}_r\mathbf{e}_r'P\mathbf{e}_r = \mathbf{e} - G_{rr}^{(r)}P\mathbf{e}_r + \mathbf{e}p_{rr}. \tag{3.31}$$

Substitution of the expression for $G_{rr}^{(r)}P\mathbf{e}_r$ from (3.31) into (3.29) yields (3.27).

❑

A close study of equation (3.27) shows that $G_{rr}^{(c)}$ and $G_{rr}^{(r)}$ differ only in the rth row and rth column, with specific elements in the rth row and column in each matrix given by Lemma 3.5, and with all the other elements identical. A formal proof follows from (3.27), since for i ≠ r and j ≠ r, the (i, j)th element of $G_{rr}^{(c)} - G_{rr}^{(r)}$ is given by

$$\mathbf{e}_i'(G_{rr}^{(c)} - G_{rr}^{(r)})\mathbf{e}_j = (\mathbf{e}_i'\mathbf{e}_r)(\frac{\boldsymbol{\pi}'\mathbf{e}_j}{\pi_r}) - (\mathbf{e}_i'\mathbf{e})(\mathbf{e}_r'\mathbf{e}_j) = 0.$$

(A proof can be constructed via determinants and cofactors defining the inverses $G_{rr}^{(c)}$ and $G_{rr}^{(r)}$ upon noting that in constructing $I - P + \mathbf{e}_r\mathbf{p}_r^{(r)\prime}$ the only elements of I – P that are changed are in the rth *row* where each element is zero apart from the (r,r)th element which is 1. Similarly that in constructing $I - P + \mathbf{p}_r^{(c)}\mathbf{e}_r'$ the only elements of I – P that are changed are in the rth *column* where each element is zero apart from the (r,r)th element which is 1.)

This is of interest later in the computation of mean first passages times and useful in tying together some different results.

Similar connections can be developed regarding relationships between other g-inverses.

Corollary 3.6.1:

(i) $$G_{rr} - G_{rr}^{(r)} = \frac{\mathbf{e}\boldsymbol{\pi}'}{\pi_r} - \mathbf{e}\mathbf{e}_r' = \mathbf{e}(\frac{\boldsymbol{\pi}'}{\pi_r} - \mathbf{e}_r'). \tag{3.32}$$

(ii) $$G_{rr} - G_{rr}^{(c)} = \frac{\mathbf{e}\boldsymbol{\pi}'}{\pi_r} - \frac{\mathbf{e}_r\boldsymbol{\pi}'}{\pi_r} = (\mathbf{e} - \mathbf{e}_r)\frac{\boldsymbol{\pi}'}{\pi_r}. \tag{3.33}$$

Proof: These results follow directly from Theorem 3.4 and the properties of Lemma 3.5.

❑

Note also the following extension to Lemma 3.5 which follows directly using (2.8) and (2.9).

Corollary 3.5.1:

(a) The rth row of G_{rr} is given by $\mathbf{e}_r' G_{rr} = \frac{\boldsymbol{\pi}'}{\pi_r}$. (3.34)

(b) The rth column of G_{rr} is given by $G_{rr}\mathbf{e}_r = \mathbf{e}$. (3.35)

❑

4. Mean first passage times

All known general procedures for finding mean first passage times involve the determination of either matrix inverses or generalised inverses. The following theorem summarises the general determination of $M = [m_{ij}]$, the mean first passage time matrix of a finite irreducible, Markov chain with transition matrix P. Let $E = \mathbf{ee}' = [1]$ and $D = M_d = (\Pi_d)^{-1}$ where $\Pi = \mathbf{e}\boldsymbol{\pi}'$.

Theorem 4.1: *(i) Let* G *be any g-inverse of* I – P*, then*

$$M = [G\Pi - E(G\Pi)_d + I - G + EG_d]D. \tag{4.1}$$

(ii) Let H = G(I – Π)*, then*

$$M = [EH_d - H + I]D. \tag{4.2}$$

(iii) Let C = I – H*, then*

$$M = [C - EC_d + E]D. \tag{4.3}$$

Proof: (i) Expression (4.1) appears in Hunter [3] as Theorem 7.3.6 having initially appeared in the literature in Hunter [2].
(ii) Expression (4.2) follows from (4.1) upon substitution. The technique was also used in a disguised form in Corollary 3.1.1 of Hunter [6].
(iii) Expression (4.3) follows form (4.2). It was first derived in Hunter [7].

❑

The advantages of expressions (4.2) and (4.3) is that we can deduce simple elemental forms of m_{ij} direct from these results.

Corollary 4.1.1: *(i) If* $C = [c_{ij}]$ *then*

$$m_{ij} = [c_{ij} - c_{jj} + 1]\frac{1}{\pi_j}, \textit{ for all } i, j. \tag{4.4}$$

(ii) If $H = [h_{ij}]$ *then*

$$m_{ij} = [h_{jj} - h_{ij} + \delta_{ij}]\frac{1}{\pi_j} = \begin{cases} \frac{1}{\pi_j}, & i = j, \\ [h_{jj} - h_{ij}]\frac{1}{\pi_j}, & i \neq j. \end{cases} \tag{4.5}$$

(iii) If G = [g_{ij}] *then*

$$m_{ij} = [g_{jj} - g_{ij} + \delta_{ij}]\frac{1}{\pi_j} + [g_{i.} - g_{j.}], \text{ for all } i, j. \quad (4.6)$$

Proof: (i) Result (4.4) follows directly from equation (4.3) (correcting the results given in Hunter [7]).
(ii) Result (4.5) follows either from (4.2) or (4.4) since $h_{ij} = \delta_{ij} - c_{ij}$.
(iii) Since H = G – GΠ,

$$h_{ij} = g_{ij} - \sum_{k=1}^{m} g_{ik}\pi_j = g_{ij} - g_{i.}\pi_j, \text{ for all i, j.} \quad (4.7)$$

and results (4.6) follow from (4.5). Note also that since C = I – H

$$c_{ij} = \delta_{ij} - g_{ij} + \sum_{k=1}^{m} g_{ik}\pi_j = \delta_{ij} - g_{ij} + g_{i.}\pi_j, \text{ for all i, j.} \quad (4.8)$$

and hence results (4.6) follow alternatively from (4.4).

❑

Note that expression (4.4) has the advantage that no special treatment of the i = j case is required.

In our earlier study of perturbations (Hunter [8]) it was seen that expressions for the changes in the stationary probabilities were more conveniently expressed in terms of $N = [n_{ij}] = [(1 - \delta_{ij})m_{ij}\pi_j]$ so that $N = (M - M_d)(M_d)^{-1}$. The following follows directly from (4.2) and (4.6).

Theorem 4.2:

$$N = [n_{ij}] = EH_d - H \text{ where } H = G(I - \Pi),$$
$$\text{so that } n_{ij} = (g_{jj} - g_{ij}) + (g_{i.} - g_{j.})\pi_j, \text{ for all } i, j. \quad (4.9)$$

❑

Note that $n_{jj} = 0$ for all j.

The following joint computation procedure for π_j and m_{ij} was given in Hunter [7], based upon Theorem 3.2 and Corollary 4.1.1 (iii) above. (The version below corrects some minor errors given in the initial derivation.)

Theorem 4.3:

1. *Compute* G = [g_{ij}], *be any g-inverse of* I – P.
1. *Compute sequentially rows* 1, 2, …r (≤ m) *of* A = I – (I – P)G ≡ [a_{ij}]

   until $\sum_{k=1}^{m} a_{rk}$ *(1 ≤ r ≤ m) is the first non-zero sum.*

3. *Compute* $\pi_j = \dfrac{a_{rj}}{\sum_{k=1}^{m} a_{rk}}, \quad j = 1, 2, ..., m.$

4. *Compute* $m_{ij} = \begin{cases} \dfrac{\sum_{k=1}^{m} a_{rk}}{a_{rj}}, & i = j, \\ \dfrac{[g_{jj} - g_{ij}]\sum_{k=1}^{m} a_{rk}}{a_{rj}} + \sum_{k=1}^{m}[g_{ik} - g_{jk}], & i \neq j. \end{cases}$

❑

While this theorem outlines a procedure for the joint computation of all the π_j and m_{ij}, following the computation of any g-inverse, the procedure contains the unnecessary additional computation of the elements of A.

Instead we can use one of the special g-inverses given in Table 1 to find all the π_j and m_{ij} and/or n_{ij}. The results are summarised in Table 2.

Table 2: Joint computation of $\{\pi_j\}$and $[n_{ij}]$ using special g-inverses

g-inverse	π_j	n_{ij}
$G_{rr}^{(r)} = [I - P + \mathbf{e}_r\mathbf{p}_r^{(r)\prime}]^{-1}$	$\Sigma_k p_{rk} g_{kj}/\Sigma_i\Sigma_s p_{ri} g_{is}$	$(g_{jj} - g_{ij}) + (g_{i.} - g_{.j})\Sigma_k p_{rk} g_{kj}/\Sigma_i\Sigma_s p_{ri} g_{is}$
$G_{rr}^{(c)} = [I - P + \mathbf{p}_r^{(c)}\mathbf{e}_r']^{-1}$	$g_{rj}/g_{r.}$	$(g_{jj} - g_{ij}) + (g_{i.} - g_{.j})g_{rj}/g_{r.}$
$G_{rr} = [I - P + \mathbf{e}_r\mathbf{e}_r']^{-1}$	$g_{rj}/g_{r.}$	$(g_{jj} - g_{ij}) + (g_{i.} - g_{.j})g_{rj}/g_{r.}$
$G_{er}^{(r)} = [I - P + \mathbf{e}\mathbf{p}_r^{(r)\prime}]^{-1}$	$\Sigma_k p_{rk} g_{kj}$	$(g_{jj} - g_{ij}) + (g_{i.} - g_{.j})\Sigma_k p_{rk} g_{kj}$
$G_{re}^{(c)} = [I - P + \mathbf{p}_r^{(c)}\mathbf{e}']^{-1}$	$g_{.j}/g_{..}$	$(g_{jj} - g_{ij}) + (g_{i.} - g_{.j})g_{.j}/g_{..}$
$G_{ee} = [I - P + \mathbf{e}\mathbf{e}']^{-1}$	$g_{.j}$	$(g_{jj} - g_{ij}) + (g_{i.} - g_{.j})g_{.j}$
$G_{er} = [I - P + \mathbf{e}\mathbf{e}_r']^{-1}$	g_{rj}	$(g_{jj} - g_{ij}) + (g_{i.} - g_{.j})g_{rj}$
$G_{re} = [I - P + \mathbf{e}_r\mathbf{e}']^{-1}$	$g_{.j}/g_{..}$	$(g_{jj} - g_{ij}) + (g_{i.} - g_{.j})g_{.j}/g_{..}$

Before we complete this section we wish to remark on other alternative procedures for finding the mean first passage times.

If the stationary probability vector has already been computed then the standard procedure is to compute either Kemeny and Snell's '*fundamental matrix*', ([9]), $Z \equiv [I - P + \Pi]^{-1}$, where $\Pi = \mathbf{e}\boldsymbol{\pi}'$, or Meyer's '*group inverse*', ([10]), $A^{\#} \equiv Z - \Pi$. Both of these matrices are in fact g-inverses of $I - P$. The relevant results, which follow from Corollary 4.1.1 (iii) are as follows.

Corollary 4.1.2:

(i) If $Z = [I - P + \mathbf{e}\boldsymbol{\pi}']^{-1} = [z_{ij}]$ *then*

$$M = [m_{ij}] = [I - Z + EZ_d]D, \tag{4.10}$$

and

$$m_{ij} = \begin{cases} \dfrac{1}{\pi_j}, & i = j; \\ \dfrac{(z_{jj} - z_{ij})}{\pi_j}, & i \neq j. \end{cases} \tag{4.11}$$

(ii) If $A^{\#} = [I - P + \mathbf{e}\boldsymbol{\pi}']^{-1} - \mathbf{e}\boldsymbol{\pi}' = [a_{ij}^{\#}]$ *then*

$$M = [m_{ij}] = [I - A^{\#} + EA_d^{\#}]D, \tag{4.12}$$

and

$$m_{ij} = \begin{cases} \dfrac{1}{\pi_j}, & i = j; \\ \dfrac{(a_{jj}^{\#} - a_{ij}^{\#})}{\pi_j}, & i \neq j. \end{cases} \tag{4.13}$$

Proof: See Hunter, [3], Corollary 7.3.6C. These are also special cases of (4.6) since $Z\mathbf{e} = \mathbf{e}$ and $A^{\#}\mathbf{e} = \mathbf{0}$, so that $\Sigma_j z_{ij} = z_{i.} = 1$ for all i and $\Sigma_j a_{ij}^{\#} = a_{i.}^{\#} = 0$ for all i.

❑

Further, in deriving the mean first passage times one is in effect solving the set of equations

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}. \tag{4.14}$$

If we hold j fixed, (j = 1, 2, …, m), and let $\mathbf{m}_j' = (m_{1j}, m_{2j}, \ldots, m_{mj})$ then equation (4.14) yields

$$\mathbf{m}_j = [I - P + \mathbf{p}_j^{(c)}\mathbf{e}_j]^{-1}\mathbf{e} \ = \ G_{jj}^{(c)}\mathbf{e}. \quad (4.15)$$

(This results appears in Hunter [3], as Corollary 7.3.3A). Note the appearance of one of the special g-inverses considered in this paper of the form of $G_{rr}^{(c)}$ with r = j.

Thus an explicit evaluation of m_{ij} (for fixed j, $1 \le i \le m$) can be obtained from (4.15). Further we can deduce expressions for m_{ij} direct from $G_{jj}^{(r)}$ or G_{jj} using the inter-relationships derived in Theorem 3.6 and Corollary 3.6.1. The following Theorem summarises the results.

Theorem 4.4: For fixed j, $1 \le i \le m$,

(i) $$m_{ij} = \mathbf{e}_i' G_{jj}^{(c)}\mathbf{e}. \quad (4.16)$$

Further, if $G_{jj}^{(c)} = [g_{rs}]$, *then* $m_{ij} = g_{i.}$

(ii) $$m_{ij} = \mathbf{e}_i' G_{jj}^{(r)}\mathbf{e} + \frac{\delta_{ij}}{\pi_j} - 1. \quad (4.17)$$

Further, if $G_{jj}^{(r)} = [g_{rs}]$, *then* $m_{ij} = g_{i.} + \dfrac{\delta_{ij}}{\pi_j} - 1 = \begin{cases} \dfrac{1}{\pi_j}, & i = j, \\ g_{i.} - 1, & i \ne j. \end{cases}$

(iii) $$m_{ij} = \mathbf{e}_i' G_{jj}\mathbf{e} + \frac{\delta_{ij} - 1}{\pi_j}. \quad (4.18)$$

Further, if $G_{jj} = [g_{rs}]$ *then* $m_{ij} = \begin{cases} g_{j.} & i = j, \\ g_{i.} - g_{j.} & i \ne j. \end{cases}$

❑

All of these results are consistent with equation (4.6).

For example, for (4.16), with $G_{jj}^{(c)} = [g_{rs}]$ from equation (3.14), $\pi_i = g_{ji}/g_{j.}$ for all i. Observe that from (3.19) and (3.20) that the jth row and column of $G_{jj}^{(c)}$ are, respectively, $\boldsymbol{\pi}'/\pi_j$ and $\mathbf{e}_j$, so that for fixed j, $g_{jj} = 1$, and for $i \ne j$, $g_{ij} = 0$ and $g_{ji} = \pi_i/\pi_j$ with $g_{j.} = 1/\pi_j$. Substitution in (4.6), for fixed j, yields $m_{jj} = 1/\pi_j = g_{j.}$ and for $i \ne j$, $m_{ij} = [g_{jj} - g_{ij}]\, g_{j.} + [g_{i.} - g_{j.}] = g_{j.} + g_{i.} - g_{j.} = g_{i.}$, as given by (4.16). Similarly one can examine (4.17) and (4.18) to establish the required equivalence.

Note that using $G_{jj}^{(c)}$ and G_{jj} an expression for m_{jj} (and hence also an expression for π_j) is found directly from the elements of the g-inverse as $g_{j.}$. In the case of $G_{jj}^{(r)}$ a subsidiary computation for m_{jj} (or alternatively π_j) is required. From (3.16) it is easy to see that $\mathbf{p}_j^{(r)'} G_{jj}^{(r)}\mathbf{e} = 1/\pi_j = m_{jj}$.

5. Perturbations and stationary distributions

We use the same terminology used in Hunter [8]. Let $P^{(1)} = [p_{ij}^{(1)}]$ be the transition matrix of a finite irreducible, m-state Markov chain. Let $P^{(2)} = [p_{ij}^{(2)}] = P^{(1)} + \boldsymbol{E}$ be the transition matrix of the perturbed Markov chain where $\boldsymbol{E} = [\varepsilon_{ij}]$ is the matrix of perturbations. We assume that the perturbed Markov chain is also irreducible with the same state space $S = \{1, 2, \ldots, m\}$. For i = 1, 2, let $\boldsymbol{\pi}^{(i)'} = (\pi_1^{(i)}, \pi_2^{(i)}, \ldots, \pi_m^{(i)})$ be the stationary probability vectors for the respective Markov chains.

In Hunter [8] we established the following key results for the differences in the stationary probability vectors $\boldsymbol{\pi}^{(2)'} - \boldsymbol{\pi}^{(1)'}$ using arbitrary g-inverses of $I - P^{(1)}$:

Theorem 5.1: If G *is any g-inverse of* $I - P^{(1)}$ *then, for any general perturbation* $\boldsymbol{E}$,

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,\mathrm{G}(\mathrm{I} - \Pi^{(1)}). \tag{5.1}$$

In particular

(i) *If* $\mathrm{G} = [\mathrm{I} - \mathrm{P}^{(1)} + \mathbf{tu}']^{-1} + \mathbf{ef}' + \mathbf{g}\boldsymbol{\pi}^{(1)\prime}$ *with* $\boldsymbol{\pi}^{(1)\prime}\mathbf{t} \neq 0$, $\mathbf{u}'\mathbf{e} \neq 0$, $\mathbf{f}'$ *and* $\mathbf{g}$ *arbitrary vectors, then*

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,[\mathrm{I} - \mathrm{P}^{(1)} + \mathbf{tu}']^{-1}(\mathrm{I} - \Pi^{(1)}). \tag{5.2}$$

(ii) *If* $G = [\mathrm{I} - \mathrm{P}^{(1)} + \mathbf{eu}']^{-1} + \mathbf{ef}' + \mathbf{g}\boldsymbol{\pi}^{(1)\prime}$ *with* $\boldsymbol{\pi}^{(1)\prime}\mathbf{t} \neq 0$, $\mathbf{u}'\mathbf{e} \neq 0$, $\mathbf{f}'$ *and* $\mathbf{g}$ *arbitrary vectors, then*

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,[\mathrm{I} - \mathrm{P}^{(1)} + \mathbf{eu}']^{-1}. \tag{5.3}$$

(ii) *If* $G = [\mathrm{I} - \mathrm{P}^{(1)} + \mathbf{eu}']^{-1} + \mathbf{ef}'$ *with* $\mathbf{u}'\mathbf{e} \neq 0$, *and* $\mathbf{f}'$ *an arbitrary vector, then*

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,\mathrm{G}. \tag{5.4}$$

❑

We then showed that mean first passage times have important connections to these results by establishing the following results.

Theorem 5.2: *If* $\mathrm{M}^{(1)}$ *is the mean first passage time matrix of the finite irreducible, Markov chain with transition matrix* $\mathrm{P}^{(1)}$, *then for any general perturbation* $\boldsymbol{E}$ *of* $\mathrm{P}^{(1)}$,

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = -\,\boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,(\mathrm{M}^{(1)} - \mathrm{M}_d^{(1)})(\mathrm{M}_d^{(1)})^{-1}. \tag{5.5}$$

Further, if $\mathrm{N}^{(1)} = [n_{ij}^{(1)}] = [(1 - \delta_{ij})\, m_{ij}^{(1)}\pi_j^{(1)}]$, *then*

$$\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} = -\,\boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,\mathrm{N}^{(1)}. \tag{5.6}$$

❑

By focussing on perturbations in a single row of the transition matrix we were able to establish some interesting results. Let $\mathbf{p}_r^{(i)\prime} = \mathbf{e}_r\mathrm{P}^{(i)}$ so that $\mathbf{p}_r^{(i)\prime}$ is the r[th] row of the transition matrix $\mathrm{P}^{(i)}$. Now suppose $\boldsymbol{E} = \mathbf{e}_r\boldsymbol{\varepsilon}_r'$ where $\boldsymbol{\varepsilon}_r' = \mathbf{p}_r^{(2)\prime} - \mathbf{p}_r^{(1)\prime}$, so that the perturbation replaces the rth row of the transition matrix $\mathrm{P}^{(1)}$ by the rth row of the transition matrix $\mathrm{P}^{(2)}$.

Suppose that $\boldsymbol{\varepsilon}_r' = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ where $\boldsymbol{\varepsilon}_r'\mathrm{e} = 0$. Substitution in equation (5.6) yields

$$\boldsymbol{\pi}^{(1)\prime} - \boldsymbol{\pi}^{(2)\prime} = \boldsymbol{\pi}^{(2)\prime}\mathbf{e}_r\boldsymbol{\varepsilon}_r'\mathrm{N}^{(1)} = \pi_r^{(2)}\boldsymbol{\varepsilon}_r'\mathrm{N}^{(1)}$$

so that in elemental form, for j = 1, 2, …, m,

$$\pi_j^{(1)} - \pi_j^{(2)} \;=\; \pi_r^{(2)}\sum_{i \neq j}\varepsilon_i n_{ij}^{(1)} \;=\; \pi_j^{(1)}\pi_r^{(2)}\sum_{i \neq j}\varepsilon_i m_{ij}^{(1)}. \tag{5.7}$$

Now restrict attention to the simplest perturbation of decreasing the (r,a)[th] element of $\mathrm{P}^{(1)}$ by an amount ε and increasing the (r,b)[th] element of $\mathrm{P}^{(1)}$ by the same amount to obtain the new transition matrix $\mathrm{P}^{(2)}$. Thus $p_{ra}^{(2)} = p_{ra}^{(1)} - \varepsilon < p_{ra}^{(1)}$ and $p_{rb}^{(2)} = p_{rb}^{(1)} + \varepsilon > p_{rb}^{(1)}$. We assume that the stochastic and irreducible nature of both $\mathrm{P}^{(1)}$ and $\mathrm{P}^{(2)}$ is preserved. For this special case we obtained the following results.

Theorem 5.3: Suppose that the transition probability $p_{ra}^{(1)}$ *in an irreducible chain is decreased by an amount* ε *while* $p_{rb}^{(1)}$ *is increased by an amount* ε. *If the resulting chain is irreducible then expressions for difference in the stationary probabilities* $\pi_j^{(1)} - \pi_j^{(2)}$ *are given by*

$$\pi_j^{(1)} - \pi_j^{(2)} \;=\; \begin{cases} \varepsilon\pi_a^{(1)}\pi_r^{(2)}m_{ba}^{(1)} & j = a, \\ -\,\varepsilon\pi_b^{(1)}\pi_r^{(2)}m_{ab}^{(1)} & j = b, \\ \varepsilon\pi_j^{(1)}\pi_r^{(2)}(m_{bj}^{(1)} - m_{aj}^{(1)}) & j \neq a,\, b. \end{cases} \tag{5.8}$$

❑

This resulting perturbation has some nice properties that are not necessarily replicated in other more general perturbations. In particular

(i)
$$-\varepsilon\pi_r^{(2)}m_{ab}^{(1)} = \frac{\pi_b^{(1)}-\pi_b^{(2)}}{\pi_b^{(1)}} \le \frac{\pi_j^{(1)}-\pi_j^{(2)}}{\pi_j^{(1)}} \le \frac{\pi_a^{(1)}-\pi_a^{(2)}}{\pi_a^{(1)}} = \varepsilon\pi_r^{(2)}m_{ba}^{(1)},\ 1 \le j \le m. \tag{5.9}$$

(i)
$$1-\varepsilon\pi_r^{(2)}m_{ba}^{(1)} = \frac{\pi_a^{(2)}}{\pi_a^{(1)}} \le \frac{\pi_j^{(2)}}{\pi_j^{(1)}} \le \frac{\pi_b^{(2)}}{\pi_b^{(1)}} = 1+\varepsilon\pi_r^{(2)}m_{ab}^{(1)},\ 1 \le j \le m. \tag{5.10}$$

(i)
$$\frac{\pi_a^{(2)}}{\pi_a^{(1)}} < 1 < \frac{\pi_b^{(2)}}{\pi_b^{(1)}},\ \textit{so that}\ \pi_a^{(2)} < \pi_a^{(1)}\ \textit{and}\ \pi_b^{(1)} < \pi_b^{(2)}. \tag{5.11}$$

The approach that we now explore for general perturbations is to regard the changes in stationary probabilities as being carried out by a succession of 2-element perturbations. This permits at each stage consideration of the effect that perturbation will have on the stationary probabilities.

The requirements we now seek are simple procedures to first update the stationary probability vector $\boldsymbol{\pi}^{(1)\prime}$ to $\boldsymbol{\pi}^{(2)\prime}$ and then the mean first passage matrix $M^{(1)}$ to $M^{(2)}$; alternatively, $N^{(1)}$ to $N^{(2)}$. Where possible we strive to use the same g-inverse or a slight modification in order to minimise the required computations.

We first observe that every expression for $\boldsymbol{\pi}^{(2)\prime}-\boldsymbol{\pi}^{(1)\prime}$ (cf (5.1) to (5.7)) is of the form

$$\boldsymbol{\pi}^{(2)\prime}-\boldsymbol{\pi}^{(1)\prime} = \boldsymbol{\pi}^{(2)\prime}\boldsymbol{E}\,\mathrm{K} \tag{5.12}$$

where typically K is a g-inverse, or involves $M^{(1)}$ or $N^{(1)}$.

For a general perturbation, involving just the rth row, we can express $\boldsymbol{E} = \mathbf{e}_r\boldsymbol{\varepsilon}'$. Define $\mathbf{k}' = \boldsymbol{\varepsilon}'\mathrm{K}$ so that $\boldsymbol{E}\mathrm{K} = \mathbf{e}_r\boldsymbol{\varepsilon}'\mathrm{K} = \mathbf{e}_r\mathbf{k}'$. For such expressions we have the following inter-relationships between $\boldsymbol{\pi}^{(1)\prime}$ and $\boldsymbol{\pi}^{(2)\prime}$.

Theorem 5.4: If $\boldsymbol{E} = \mathbf{e}_r\boldsymbol{\varepsilon}'$ and $\mathbf{k}' = \boldsymbol{\varepsilon}'\mathrm{K}$,

$$\boldsymbol{\pi}^{(1)\prime} = \boldsymbol{\pi}^{(2)\prime}(I-\mathbf{e}_r\mathbf{k}'), \tag{5.13}$$

$$\boldsymbol{\pi}^{(2)\prime} = \boldsymbol{\pi}^{(1)\prime} + \frac{\pi_r^{(1)}\mathbf{k}'}{1-\mathbf{k}'\mathbf{e}_r}, \tag{5.14}$$

Note also that $\pi_r^{(1)} = \pi_r^{(2)}(1-\mathbf{k}'\mathbf{e}_r)$. (5.15)

Proof: Since equation (5.12) can be reexpressed as $\boldsymbol{\pi}^{(2)\prime}-\boldsymbol{\pi}^{(1)\prime} = \boldsymbol{\pi}^{(2)\prime}\mathbf{e}_r\mathbf{k}'$, (5.13) immediately follows. Now it is easily established by matrix multiplication that

$$[I-\mathbf{e}_r\mathbf{k}']^{-1} = I + \frac{\mathbf{e}_r\mathbf{k}'}{1-\mathbf{k}'\mathbf{e}_r}$$

and equation (5.14) follows from (5.13). Further, (5.15) follows from (5.14) by noting that for i = 1, 2, $\boldsymbol{\pi}^{(i)\prime}\mathbf{e}_r = \pi_r^{(i)}$.

❑

In order to apply the results of Theorem 5.4 observe that we do not need to find all the elements of K. Consider a two element perturbation acting on the elements of the rth row of the transition matrix with $\boldsymbol{\varepsilon}' = \varepsilon(\mathbf{e}_b'-\mathbf{e}_a')$ corresponding to an increase of an amount ε to the the transition probability at position (r, b) and a decrease of an amount ε to the transition probability at position (r, b). Then $\mathbf{k}' = \boldsymbol{\varepsilon}'\mathrm{K} = \varepsilon(\mathbf{e}_b'-\mathbf{e}_a')\mathrm{K} = \varepsilon(\mathbf{k}_b'-\mathbf{k}_a')$ where $\mathbf{k}_i'$ is the ith row of K. Thus we need only determine the elements of the ath and bth rows of K, be it a g-inverse, or the N-matrix with elements involving the stationary probabilities combined with mean first passage times.

6. Updating procedures

The general approach that we take is the following:

Step 1: Select a suitable g-inverse $G^{(1)}$. (We have a variety of possible candidates given in Table 1.)
Step 2: Solve for the stationary probability vector $\boldsymbol{\pi}^{(1)\prime}$ using the g-inverse selected.
Step 3: Solve for the mean first passage time matrix $M^{(1)} = [m_{ij}^{(1)}]$ (or the matrix $N^{(1)}$) using the same g-inverse $G^{(1)}$. This permits us to determine the changes in $\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime}$.
Step 4: Solve for the stationary probability vector $\boldsymbol{\pi}^{(2)\prime}$ using e.g. Theorem 5.4 or $G^{(1)}$ or a suitable variant.
Step 5: Solve for the mean first passage time matrix $M^{(2)} = [m_{ij}^{(2)}]$ (or the matrix $N^{(2)}$) using either the same g-inverse $G^{(1)}$ or modification, $G^{(2)}$, which hopefully can be obtained by the updating procedure using a variant of Theorem 3.4.

Let us suppose that we make a perturbation on the rth row of $P^{(1)}$ of the form $\boldsymbol{E} = \mathbf{e}_r\boldsymbol{\varepsilon}'$ where $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}_r' = \mathbf{p}_r^{(2)\prime} - \mathbf{p}_r^{(1)\prime}$, so that the perturbation replaces the rth row of the transition matrix $P^{(1)}$ by the rth row of the transition matrix $P^{(2)}$.

Step 1: Consider using the g-inverse $G^{(1)} = [g_{ij}^{(1)}] = G_{rr}^{(r,1)} = [I - P^{(1)} + \mathbf{e}_r\mathbf{p}_r^{(1)\prime}]^{-1}$ as this is directly based upon the row selected for the perturbation.

Step 2: An expression for the stationary probabilities $\pi_j^{(1)}$ are given by (3.16):

$$\pi_j^{(1)} = \frac{\sum_{k=1}^{m} p_{rk}^{(1)} g_{kj}^{(1)}}{\sum_{i=1}^{m} \sum_{s=1}^{m} p_{ri}^{(1)} g_{is}^{(1)}} \quad , \ j = 1,\ 2,\ ...,\ m. \tag{6.1}$$

Step 3: Expressions for the $n_{ij}^{(1)}$ are given in Table 2 from Theorem 4.2, i.e.

$$n_{ij}^{(1)} = (g_{jj}^{(1)} - g_{ij}^{(1)}) + (g_{i.}^{(1)} - g_{j.}^{(1)}) \frac{\sum_{k=1}^{m} p_{rk}^{(1)} g_{kj}^{(1)}}{\sum_{i=1}^{m} \sum_{s=1}^{m} p_{ri}^{(1)} g_{is}^{(1)}}. \tag{6.2}$$

Step 4: Under the perturbation, with the new rth row as $\mathbf{p}_r^{(2)\prime}$, a suitable g-inverse would be $G_{rr}^{(r,2)} = [I - P^{(2)} + \mathbf{e}_r\mathbf{p}_r^{(2)\prime}]^{-1}$. Note however that since $P^{(2)} = P^{(1)} + \mathbf{e}_r(\mathbf{p}_r^{(2)\prime} - \mathbf{p}_r^{(1)\prime})$, $G_{rr}^{(r,2)} = G_{rr}^{(r,1)} = G^{(1)} = [g_{ij}^{(1)}]$ so that the same g-inverse can be used for finding the properties of the Markov chain with transition matrix $P^{(2)}$. Thus an expression for the stationary probabilities $\pi_j^{(2)}$ are given by (3.16):

$$\pi_j^{(2)} = \frac{\sum_{k=1}^{m} p_{rk}^{(2)} g_{kj}^{(1)}}{\sum_{i=1}^{m} \sum_{s=1}^{m} p_{ri}^{(2)} g_{is}^{(1)}}, \ j = 1,\ 2,\ ...,\ m. \tag{6.3}$$

Step 5: Expressions for the $n_{ij}^{(2)}$ are given, as in Step 3, but with $p_{ri}^{(1)}$ replaced by $p_{ri}^{(2)}$, as

$$n_{ij}^{(2)} = (g_{jj}^{(1)} - g_{ij}^{(1)}) + (g_{i.}^{(1)} - g_{j.}^{(1)}) \frac{\sum_{k=1}^{m} p_{rk}^{(2)} g_{kj}^{(1)}}{\sum_{i=1}^{m} \sum_{s=1}^{m} p_{ri}^{(2)} g_{is}^{(1)}}. \tag{6.4}$$

Thus, as long as perturbations are carried out in the same row, we have a very simple procedure for updating the stationary probabilities and the mean first passage times (via the n_{ij}.)

In matrix-vector form the above procedure is as follows:

(1) Let $G = [I - P^{(1)} + \mathbf{e}_r\mathbf{p}_r^{(1)\prime}]^{-1} = [I - P^{(2)} + \mathbf{e}_r\mathbf{p}_r^{(2)\prime}]^{-1}$.

(2) $\boldsymbol{\pi}^{(1)\prime} = \dfrac{\mathbf{p}_r^{(1)\prime}G}{\mathbf{p}_r^{(1)\prime}G\mathbf{e}}$.

(3) $N^{(1)} = E(H^{(1)})_d - H^{(1)}$ *where* $H^{(1)} = G(I - e\boldsymbol{\pi}^{(1)\prime})$.

(4) $\boldsymbol{\pi}^{(2)\prime} = \dfrac{\mathbf{p}_r^{(2)\prime}G}{\mathbf{p}_r^{(2)\prime}G\mathbf{e}}$.

(5) $N^{(2)} = E(H^{(2)})_d - H^{(2)}$, where $H^{(2)} = G(I - e\boldsymbol{\pi}^{(2)\prime})$.
Further $H^{(2)} - H^{(1)} = \mathbf{g}(\boldsymbol{\pi}^{(1)\prime} - \boldsymbol{\pi}^{(2)\prime})$ where $\mathbf{g} = G\mathbf{e}$. This implies that

$$N^{(2)} - N^{(1)} = E(\mathbf{g}(\boldsymbol{\pi}^{(1)\prime} - \boldsymbol{\pi}^{(2)\prime}))_d - \mathbf{g}(\boldsymbol{\pi}^{(1)\prime} - \boldsymbol{\pi}^{(2)\prime}). \tag{6.5}$$

Note also, since from (3.22) $G\mathbf{e}_r = \mathbf{e}$, that for $i = 1, 2$, $\pi_r^{(i)} = 1/\mathbf{p}_r^{(i)\prime}G\mathbf{e}$.

This leads to the following relationships between the original and updated mean first passage times.

Theorem 6.1: If the same g-inverse $G = [g_{ij}]$ *is used in the updating of a transition matrix of a finite irreducible Markov chain, the following relationships hold between the stationary probabilities and the mean first passage times for the two respective Markov chains:*

$$m_{ij}^{(2)}\pi_j^{(2)} - m_{ij}^{(1)}\pi_j^{(1)} = (g_{i.} - g_{j.})(\pi_j^{(2)} - \pi_j^{(1)}). \tag{6.6}$$

Proof: Extraction of the (i,j)th element of (6.5) leads to (6.6), since $\mathbf{g}' = (g_{1.}, g_{2.}, \ldots g_{m.})$.

❑

A more general result where the updating is carried using two different g-inverses is the following:

Theorem 6.2: If $G^{(1)} = [g_{ij}^{(1)}]$ *and* $G^{(2)} = [g_{ij}^{(2)}]$ *are two different g-inverses with* $G^{(i)}$ *(i= 1, 2) used to determine expressions for the stationary probabilities and mean first passage times in for the ith finite irreducible Markov chain then the following relationships hold between their stationary probabilities and the mean first passage times:*

$$m_{ij}^{(2)}\pi_j^{(2)} - m_{ij}^{(1)}\pi_j^{(1)} = (g_{jj}^{(2)} - g_{ij}^{(2)} - g_{jj}^{(1)} + g_{ij}^{(1)}) + (g_{i.}^{(2)} - g_{j.}^{(2)})\pi_j^{(2)} - (g_{i.}^{(1)} - g_{j.}^{(1)})\pi_j^{(1)}. \tag{6.7}$$

Proof: Observe that from Theorem 4.2 for k = 1, 2,
$N^{(k)} = [n_{ij}^{(k)}] = E(H^{(k)})_d - H^{(k)}$ where $H^{(k)} = G^{(k)}(I - \Pi^{(k)})$ with $\Pi^{(k)} = \mathbf{e}\pi^{(k)}$, so that

$$n_{ij}^{(k)} = [m_{ij}^{(k)}\pi_j^{(k)}] = (g_{jj}^{(k)} - g_{ij}^{(k)}) + (g_{i.}^{(k)} - g_{j.}^{(k)})\pi_j^{(k)}, \text{ for all i, j.}$$

Extraction of $[n_{ij}^{(2)}] - [n_{ij}^{(1)}]$ yields equation (6.7).

❑

Note that the relationships do not depend upon the nature of the updating nor on any special interrelationship between the stationary probabilities. Note that (6.6) also follows as special case of (6.7) when the two g-inverses are same i.e. $[g_{ij}^{(1)}] = [g_{ij}^{(2)}] = [g_{ij}]$. Further, in this case we can update the mean first passage times directly as follows:

Corollary 6.2.1: If $[g_{ij}^{(1)}] = [g_{ij}^{(2)}] = [g_{ij}]$ *then, for* $i \neq j$,

$$m_{ij}^{(2)} = m_{ij}^{(1)} + (g_{ij} - g_{jj})\left(\frac{1}{\pi_j^{(1)}} - \frac{1}{\pi_j^{(2)}}\right), \tag{6.8}$$

$$\text{with } m_{jj}^{(1)} = \frac{1}{\pi_j^{(1)}} \text{ and } m_{jj}^{(2)} = \frac{1}{\pi_j^{(2)}}.$$

Proof: Equation (6.8) follows directly from equation (4.6).

❑

Let us now focus on the main application case of two element perturbations. We start with the g-inverse $G = [I - P^{(1)} + \mathbf{e}_r\mathbf{p}_r^{(1)\prime}]^{-1} = [g_{ij}]$ with $\mathbf{p}_r^{(1)\prime}$ the rth row of $P^{(1)}$. Now suppose $\boldsymbol{E} = \mathbf{e}_r\boldsymbol{\varepsilon}_r'$ where $\boldsymbol{\varepsilon}_r' = \mathbf{p}_r^{(2)\prime} - \mathbf{p}_r^{(1)\prime}$, so that the perturbation replaces the rth row of the transition matrix $P^{(1)}$ by the rth row of the transition matrix $P^{(2)}$.

Theorem 6.3: *For the two element perturbation case, if the initial Markov chain with transition matrix* $P^{(1)}$ *has stationary probabilities* $\pi_j^{(1)}$ *and mean first passage times* $m_{ij}^{(1)}$ *then the perturbed Markov chain with transition matrix* $P^{(2)} = P^{(1)} + \boldsymbol{E}$ *where* $\boldsymbol{E} = \mathbf{e}_r(\mathbf{p}_r^{(2)\prime} - \mathbf{p}_r^{(1)\prime}) = \varepsilon\mathbf{e}_r(\mathbf{e}_b' - \mathbf{e}_a')$ *has stationary probabilities* $\pi_j^{(2)}$ *and mean first passage times* $m_{ij}^{(2)}$ *given by*

$$\pi_j^{(1)} = \pi_r^{(1)}\sum_{k=1}^{m} p_{rk}^{(1)} g_{kj},\ j = 1, 2, \ldots, m,\ \textit{where}\ \pi_r^{(1)} = 1\Big/\sum_{k=1}^{m} p_{ri}^{(1)} g_{i.},$$

$$\pi_j^{(2)} = \pi_r^{(2)}[\pi_j^{(1)} - \varepsilon(g_{aj} - g_{bj})],\ \mathrm{j} = 1, 2, \ldots, \mathrm{m},\ \textit{where}\ \pi_r^{(2)} = \pi_r^{(1)}\big/[1 - \varepsilon\,\pi_r^{(1)}(g_{a.} - g_{b.})],$$

so that

$$\pi_j^{(2)} - \pi_j^{(1)} = \varepsilon\pi_r^{(2)}[(g_{bj} - g_{aj}) - (g_{b.} - g_{a.})\pi_j^{(1)}],\ j = 1, 2, \ldots, m;$$

$$m_{ij}^{(1)} = [\mathrm{g_{jj}} - \mathrm{g_{ij}}]m_{jj}^{(1)} + [\mathrm{g_{i.}} - \mathrm{g_{j.}}],\ \textit{for all}\ i \neq j,\ \textit{where}\ m_{jj}^{(1)} = 1\big/\pi_j^{(1)},$$

$$m_{ij}^{(2)} = [\mathrm{g_{jj}} - \mathrm{g_{ij}}]m_{jj}^{(2)} + [\mathrm{g_{i.}} - \mathrm{g_{j.}}],\ \textit{for all}\ i \neq j,\ \textit{where}\ m_{jj}^{(2)} = 1\big/\pi_j^{(2)},$$

so that

$$m_{ij}^{(2)} - m_{ij}^{(1)} = [\mathrm{g_{jj}} - \mathrm{g_{ij}}]\left(\frac{1}{\pi_j^{(2)}} - \frac{1}{\pi_j^{(1)}}\right)$$

where $G = [g_{ij}] = [I - P^{(1)} + \mathbf{e}_r\mathbf{p}_r^{(1)\prime}]^{-1}$.

Proof: The expressions for $\pi_j^{(1)}$ and $\pi_j^{(2)}$ follow as special cases of (6.1) and (6.3). For the difference between $\pi_j^{(2)}$ and $\pi_j^{(1)}$ we can use one or more of the various forms for $\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime}$. In particular**,** from (5.2)

$$\begin{aligned}\boldsymbol{\pi}^{(2)\prime} - \boldsymbol{\pi}^{(1)\prime} &= \pi_r^{(2)}\boldsymbol{\varepsilon}_r'G(I - e\boldsymbol{\pi}^{(1)\prime}) = \varepsilon\pi_r^{(2)}(\mathbf{e}_b' - \mathbf{e}_a')G(I - e\boldsymbol{\pi}^{(1)\prime})\\ &= \varepsilon\pi_r^{(2)}(\mathbf{g}_b' - \mathbf{g}_a')(I - e\boldsymbol{\pi}^{(1)\prime})\\ &= -\varepsilon\pi_r^{(2)}(\mathbf{h}_b' - \mathbf{h}_a')\\ &= -\varepsilon\pi_r^{(2)}(\mathbf{n}_b' - \mathbf{n}_a'),\end{aligned}$$

where $\mathbf{g}_i'$, $\mathbf{h}_i'$, and $\mathbf{n}_i'$ represent the ith row of G, $H^{(1)} = G(I - \Pi^{(1)})$ and $N^{(1)}$ respectively.

Thus in element form

$$\pi_j^{(2)} - \pi_j^{(1)} = \varepsilon\pi_r^{(2)}(n_{aj} - n_{bj}) = \varepsilon\pi_r^{(2)}(h_{aj} - h_{bj}) = \varepsilon\pi_r^{(2)}[(g_{bj} - g_{aj}) - (g_{b.} - g_{a.})\pi_j^{(1)}].$$

(Note that for j = a, $n_{aa} = 0$ and for j = b, $n_{bb} = 0$. However this does not imply similar values for h_{aa} and h_{bb} since $n_{ij} = h_{jj} - h_{ij}$ where $h_{ij} = g_{ij} - g_{i.}\pi_j$.)

Expressions for $m_{ij}^{(1)}$ and $m_{ij}^{(2)}$ follow from (4.6) and the difference between $m_{ij}^{(2)}$ and $m_{ij}^{(1)}$ from (6.8).

❑

The matrix $G_{rr}^{(r)} = [I - P^{(1)} + \mathbf{e}_r\mathbf{p}_r^{(1)\prime}]^{-1}$ has special properties that can be utilised. In particular note that from (3.21) and (3.22) the rth row is $\mathbf{e}_r'$ and the rth column is $\mathbf{e}$, so that $g_{ir} = 1$ for all i while $g_{rj} = 0$ for all $j \neq r$, with $g_{rr} = g_{r.} = 1$. A simple consequence of this is that

For i ≠ r, $m_{ir}^{(1)} = m_{ir}^{(2)} = g_{i.} - 1,$ (6.9)

and, for j ≠ r, $m_{rj}^{(1)} = \dfrac{g_{jj}}{\pi_j^{(1)}} + 1 - g_{j.}$ and $m_{rj}^{(2)} = \dfrac{g_{jj}}{\pi_j^{(2)}} + 1 - g_{j.}.$ (6.10)

Further, $m_{rr}^{(1)} - m_{rr}^{(2)} = \dfrac{1}{\pi_j^{(1)}} - \dfrac{1}{\pi_j^{(1)}} = \varepsilon(g_{b.} - g_{a.}).$ (6.11)

Further special results can be deduced by making use of the Theorem 4.4.

In general, a single row perturbation affects all the stationary probabilities. However, from (6.9), it is clear that the mean first passage times from any state (≠ r) to state r do not change when a perturbation is carried out in the rth row of the transition matrix, as to be intuitively expected.

In conclusion note that when subsequent perturbations are made to another row, say the sth row, the procedures outlined above will still hold but with $G^{(1)} = [I - P^{(1)} + \mathbf{e}_r\mathbf{p}_r^{(1)\prime}]^{-1} = [I - P^{(2)} + \mathbf{e}_r\mathbf{p}_r^{(2)\prime}]^{-1}$ replaced by $G^{(2)} = [I - P^{(2)} + \mathbf{e}_s\mathbf{p}_s^{(2)\prime}]^{-1}$. This change can be effected by updating $G^{(1)}$ to $G^{(2)}$, using (3.17):

$$G^{(2)} = [I - P^{(2)} + \mathbf{e}_s\mathbf{p}_s^{(2)\prime}]^{-1} = [I - \mathbf{e}\mathbf{p}_s^{(2)\prime}]G^{(1)}[I - \frac{\mathbf{e}_s\boldsymbol{\pi}'}{\boldsymbol{\pi}'\mathbf{e}_s}] + \frac{\mathbf{e}\boldsymbol{\pi}'}{\boldsymbol{\pi}'\mathbf{e}_s}.$$

The computation of the mean first passages times in the updated chain can also be carried out using either an updated fundamental matrix or group inverse, using the formulae of Corollary 4.1.2. These approaches have been considered, respectively, by Hunter [4] for the case of a rank one update of the form $\boldsymbol{E} = \mathbf{a}\mathbf{b}'$ where it can be deduced that

$$Z^{(2)} = [I - \mathbf{e}\boldsymbol{\pi}^{(2)\prime} + \mathbf{e}\boldsymbol{\pi}^{(1)\prime}]Z^{(1)}[I + \frac{\mathbf{a}\mathbf{b}'Z^{(1)}}{1 - \mathbf{b}'Z^{(1)}\mathbf{a}}],$$

and Meyer and Shoaf [11] ,for the case of changing the ith row of $P^{(1)}$, $\mathbf{p}_i^{(1)\prime}$, to $\mathbf{p}_i^{(2)\prime}$, where it is shown, in the terminology of this paper, that

$$A^{\#(2)} = A^{\#(1)} + \pi_i^{(1)}\mathbf{e}\mathbf{b}_i'[A^{\#(1)} - \mathbf{b}_i'A^{\#(1)}\mathbf{e}_i I] - A^{\#(1)}\mathbf{e}_i\mathbf{b}_i' \text{ where } \mathbf{b}_i' = \frac{[\mathbf{p}_i^{(1)\prime} - \mathbf{p}_i^{(2)\prime}]A^{\#(1)}}{1 + [\mathbf{p}_i^{(1)\prime} - \mathbf{p}_i^{(2)\prime}]A^{\#(1)}\mathbf{e}_i}.$$

The utilisation of g-inverses into the joint computation of stationary distribtiions and mean first passage times leads to a significant simplification in that at most a single matrix inverse needs to be computed and often this involves a row and/or a column with very simple form further reducing the necessary computations. While no computational examples have been included in this paper, a variety of new procedures have been presented that warrant further examination from a computational efficiency perspective.

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