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Small rigid floating bodies under the influence of water waves

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The passive motion of a small body under the influence of small amplitude waves in deep water is considered. This motion is broken down into one component due to gravity and one due to the drag forces of the water. Two similar existing models describing this situation are discussed and the major differences between these models are outlined. A detailed investigation shows the role of the centripetal force and how it can be included in the models.

1 Introduction

The interaction of floating bodies and water waves is a much investigated subject. Large objects on the water surface reflect and scatter waves. If the wavelength is much longer than the dimension of the object, the wavefield is little modified by the body and wave diffraction is negligible. The object is then passively driven by the waves. Under the same condition viscous effects can be neglected and the problem can be linearised if the wave amplitude is small compared to the wavelength.

These assumptions apply to many applications such as the effect of water waves on small ice floes. Rumer, Crisman & Wake (4) derived a one-dimensional slope sliding model for ice transport in the Great Lakes. Their resulting system of equations formulated for the horizontal velocity of the floating body has been used by Shen & Ackley (6) and Frankenstein & Shen (1) to model the effect of waves on Pancake Ice Collisions. Based on this model, Meylan (3) studied the general behaviour of a single round ice floe affected by linear gravity waves and presented numerical solutions. Analytic approximate solutions for a few special cases of the problem were found by Shen & Zhong (7). All these considerations were only concerned with the velocity in horizontal direction. Marchenko (2) derived a vector-based model for the motion of small bodies under the influence of water waves. Using this model, he investigated the different behaviours of a body floating on a periodic wave: Periodic motion and the case where the body is captured by the wave.

In the following the models of Rumer *et al.* and Marchenko are presented and derived explicitly. The basic differences in both models are outlined and in a thorough comparison, which includes the derivation of the system of equations in a third way using Hamilton's principle, the major difference is discovered.

2 Mathematical formulation

The general problem of the interaction of water waves and small floating bodies is quite complicated and non-linear. However, under some assumptions concerning the water and the floating object the problem can be simplified. These assumptions are stated and the governing equations for the water are summarised making use of the linear water wave theory.

2.1 Assumptions concerning the water

It is assumed that the water is incompressible, so that its density is constant with respect to time. The analogous of friction in the water itself, viscosity, is neglected, all rotational motion is ignored and it is assumed that the water is deep. A Cartesian coordinate system is adopted where x is the dimension in the direction of the wave propagation. The z-axis is directed vertically upwards and $z = 0$ denotes the undisturbed water surface. The surface displacement of the water from this mean water level is given by $z = \eta(x,t)$. Only two-dimensional motions are considered, i.e. the dependence on y is omitted. In figure 71 the general setting is illustrated.

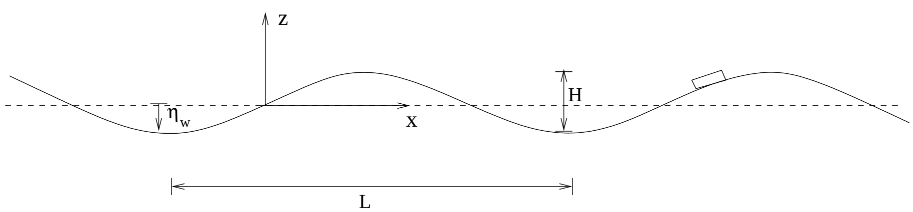

Figure 71: A floating body sliding down a linear gravity wave.

In the following linear gravity waves are considered. It is assumed that the amplitude of the wave is much smaller than its wavelength L. The height H of the wave is given by the distance from the trough to the crest, so the wave amplitude is given by $A = H/2$. The waves are supposed to be periodic such that the wave period T is the time required by one wave to pass a particular point. Therefore, the wave speed is given by $c = L/T$. The angular frequency $\omega = 2\pi/T$ and the wavenumber $k = 2\pi/L$ are used in the following, thus the speed of the wave is given by $c = \omega/k$. The dimension of the floating body is assumed to be much smaller than the wavelength and therefore the wave is not notably affected by its presence.

2.2 Governing equations

Since the water is assumed to be incompressible, the divergence of its velocity vector field V_w must be zero everywhere,

$$\nabla \bullet V_w = 0. \tag{2.88}$$

For irrotational motion, the velocity vector field of the water can be expressed as the gradient field of a scalar velocity potential $\Phi(x,z;t)$,

$$V_w = \nabla\Phi, \tag{2.89}$$

so the x- and the z-component of the water particle velocity can be calculated by $V_{wx} = \frac{\partial\Phi}{\partial x}$ and $V_{wz} = \frac{\partial\Phi}{\partial z}$ respectively. Substituting (2.89) in (2.88), the velocity potential Φ satisfies Laplace's equation,

$$\Delta\phi = 0. \tag{2.90}$$

At the free water surface $\eta(x,t)$ two boundary conditions must hold. One boundary condition is the kinematic condition

$$\frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial t} = \frac{\partial\Phi}{\partial z} \quad \text{on } z = \eta(x,t)$$

due to the fact that water particles cannot cross the air-water interface. The other boundary condition modelling the energy equilibrium at the water surface is described by Bernoulli's equation,

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left(\nabla\Phi\right)^2 + \frac{p}{\rho_w} + gz = 0,$$

where ρ_w is the density of the water, g is the acceleration due to gravity and p is the water pressure relative to the air pressure, which can be assumed constant at the water surface and therefore omitted. This non-linear Bernoulli equation can now be linearised under the assumption that the amplitude is small compared to the wavelength. Also substituting $z = \eta(x,t)$ leads to the dynamic boundary condition,

$$\frac{\partial \Phi}{\partial t} + g\eta = 0 \quad \text{on } z = 0. \tag{2.91}$$

The linearisation of the kinematic boundary condition gives

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z} \quad \text{on } z = 0. \tag{2.92}$$

The two boundary conditions are applied at $z = 0$. A perturbation analysis shows that this is consistent with the linearised theory (8).

Moreover, the waves are periodic and it is assumed that at time $t = 0$ the wave is at its steepest point with positive slope (compare figure 71). Using Laplace's equation and the linearised boundary conditions as well as a condition implying that the vertical water velocity vanishes as the vertical coordinate decreases, the velocity potential can be written as

$$\Phi(x,z;t) = -\frac{gA}{\omega}\cos(kx - \omega t)e^{kz}. \tag{2.93}$$

where the radian frequency ω is related to the wavenumber k by the dispersion relation $\omega^2 = gk$ (5). It follows from the dynamic boundary condition (2.91) that the surface displacement is given by

$$\eta(x;t) = A\sin(kx - \omega t). \tag{2.94}$$

3 Two models

When modelling the motion of a body under the influence of water waves, different forces have to be considered. Firstly, the floe slides down the surface of the wave due to gravity. Secondly, there is the drag force of the water. Generally, the drag force depends on the shape and size of the moving body in a complicated way. The main effects of the viscosity of the fluid can be modelled by a drag force proportional to the velocity of the body through the fluid, these dominate at low velocities. At higher speed the drag force becomes proportional to the square of the velocity, due to the onset of turbulence in the water.

To model the motion of the body, Newton's second law can be used. Since the body displaces water, the effective mass of the body (in contrast to the real mass) in the water needs to be considered. This effective mass in water is greater than the real mass, which can be modelled by introducing a so-called added mass term (5).

The interaction between waves and a single small body were modelled by Rumer *et al.* and by Marchenko Marchenko. In both models the force upon the body is broken down into two components: The gravity force and the drag force. At first the Rumer *et al.* model is introduced.

3.1 Derivation of the Rumer *et al.* model

Rumer *et al.* derived a system of equations formulated for the velocity of the body in the horizontal direction measured in a stationary coordinate system. The system is based on Newton's second law. The body's own inertia is balanced by the forces acting upon it,

$$m_i(1 + C_m)a = F_g + F_w, \tag{3.95}$$

where m_i is the mass of the floating body, a is the acceleration of the body, F_g is the force on the body due to gravity and F_w is the drag force of the water. The coefficient C_m is the added mass coefficient.

To find a system of equations for the velocity of the body in the horizontal direction, the acceleration a is split up into its vertical component, $\frac{d^2z}{dt^2}$, and its horizontal component, $\frac{d^2x}{dt^2}$. The force due to gravity and the vertical inertia are projected onto the tangential and the x-component is taken. This is illustrated in figure 72.

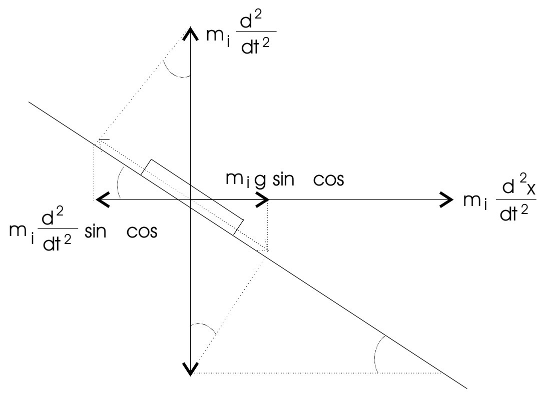

Figure 72: Forces on the body

The projection of the gravity force onto the tangential is given by $m_i g \sin\phi$, so the force in x-direction due to gravity is given by

$$F_{gx} = m_i g \cos\phi \sin\phi = \frac{m_i}{2} g \sin(2\phi). \tag{3.96}$$

where the angle ϕ is given by $-\arctan\frac{dz}{dx}$. The displacement from the horizontal equilibrium position is given by $\eta(x,t)$ so that $\frac{dz}{dx} = \frac{d\eta}{dx}$. To simplify the notation $\eta'(x,t) = \frac{\partial\eta}{\partial x}$ denotes the derivative of η with respect to x. F_{gx} can now be rewritten as

$$F_{gx} = m_i \, \frac{1}{2} \, g \, \sin\left(- \operatorname{arcsin}\left(\frac{2\,\eta'}{1+\eta'^2} \right)\right) \; = \; -m_i \, g \, \frac{\eta'}{1+\eta'^2}. \tag{3.97}$$

Analogously, the x-component of the inertia in vertical direction can be calculated as

$$-\frac{\partial^2\eta}{\partial t^2}\frac{\eta'}{1+\eta'^2} \tag{3.98}$$

pointing in the negative horizontal direction (compare figure 72).

The drag force of the water is dependent on the difference of the velocities of the body and the water. The x-component of the drag force can be modelled as

$$F_{wx} = \rho_w C_w A_i \, |V_{wx} - V_{ix}| \cdot (V_{wx} - V_{ix}). \tag{3.99}$$

where V_{wx} and $V_{ix} = \frac{dx}{dt}$ are the x-components of the surface water particle velocity and the body velocity respectively, ρ_w is the density of the water and A_i represents the wetted surface area of the floating body. The coefficient C_w is called the drag coefficient. Its value is dependent on the shape of the wetted part of the body and the water velocity field.

Substituting equations (3.97), (3.98) and (3.99) into Newton's law (3.95) and using the fact that the velocity of the body in the horizontal direction is given by $V_{ix} = \frac{dx}{dt}$, the final system of equations of Rumer *et al.* for the motion of a small rigid body is given by

$$\begin{aligned} m_i\,(1+C_m)\frac{dV_{ix}}{dt} &= -m_i\,(g+\frac{\partial^2\eta}{\partial t^2})\,\frac{\eta'}{1+\eta'^2} \\ &\quad + \rho_w C_w A_i\,|V_{wx}-V_{ix}|\,(V_{wx}-V_{ix}) \qquad (3.100)\\ \frac{dx}{dt} &= V_{ix}. \end{aligned}$$

3.2 Derivation of the Marchenko model

The system of equations derived by Marchenko is given in a coordinate system moving with the same speed as the wave. In the moving coordinate system the coordinate in the horizontal direction will be denoted by $\bar{x}$. Since the coordinate system is moving with speed c, the relation to the coordinate x in a stationary frame is given by $\bar{x} = x - ct$. In analogy, the velocities measured in the moving frame are denoted by $\bar{V}_i$ and $\bar{V}_w$.

Using vector notation in $(\bar{x}, z)$-coordinates, the gravity force on the body is given by

$$F_g = \begin{pmatrix} 0 \\ -m_i g \end{pmatrix}.$$

Marchenko decomposes the drag force F_w into the normal and tangential directions to the water surface, so that F_w can be written as

$$F_w = R_n n + R_\tau \tau, \qquad (3.101)$$

where R_n and R_τ are the drag forces and $n \in \mathbb{R}^2$ and $\tau \in \mathbb{R}^2$ are unit vectors normal and tangential to the water surface respectively, which can be calculated as

$$n = \frac{1}{\sqrt{1+\eta'^2}}\begin{pmatrix} -\eta' \\ 1 \end{pmatrix} \quad \text{and} \quad \tau = \frac{1}{\sqrt{1+\eta'^2}}\begin{pmatrix} 1 \\ \eta' \end{pmatrix}.$$

η' denotes the derivative of the water displacement with respect to $\bar{x}$, $\eta' = \frac{d\eta}{d\bar{x}}$. Although here η depends on $\bar{x}$ (and not x) no special notation is used since it will always be clear which η is meant. Since neither the floating body nor the water particle moves in this direction, the velocities in the normal direction are zero. Therefore the projections $(V_i \,|\, n)$ and $(V_w \,|\, n)$ are both zero where $(\cdot \,|\, \cdot)$ denotes the standard scalar product in two-dimensional Euclidean space. The inertial acceleration vector of the body in the tangential direction is given by

$$a_\tau = \frac{d\bar{V}_{i\tau}}{dt}\,\tau,$$

where $\bar{V}_{i\tau} = (\bar{V}_i \,|\, \tau) \in \mathbb{R}$ is the velocity of the body in tangential direction. The acceleration vector in direction n or $-n$ is given by the centripetal acceleration,

$$a_n = \frac{\eta''}{|\eta''|} \cdot \frac{\bar{V}_{i\tau}^2}{r}\,n,$$

where the radius r of the curvature circle is the reciprocal of the absolute value of the curvature, $r = |1/C|$ with $C = \eta''/\sqrt{(1+\eta'^2)^3}$. The fraction $\eta''/|\eta''|$ with $\eta'' = \frac{d^2\eta}{d\bar{x}^2}$ only determines the direction of the acceleration. Therefore the equation of motion for the body in a system of coordinates moving with the wave is given by Newton's Law,

$$m_i\Big(\frac{d\bar{V}_{i\tau}}{dt}\,\tau + \frac{\bar{V}_{i\tau}^2}{r}\,\frac{\eta''}{|\eta''|}\,n\Big) = F_g + F_w. \qquad (3.102)$$

Taking the scalar product with respect to τ yields

$$m_i \frac{d\bar{V}_{i\tau}}{dt} = -m_i g \frac{\eta'}{\sqrt{1+\eta'^2}} + R_\tau. \tag{3.103}$$

The velocity of the body tangential to the water surface is given by the vector

$$\bar{V}_{i\tau}\tau = \bar{V}_{i\tau} \cdot \begin{pmatrix} 1 \\ \eta' \end{pmatrix} \cdot \frac{1}{\sqrt{1+\eta'}},$$

therefore the velocity in the horizontal direction is given by

$$\frac{d\bar{x}}{dt} = \frac{\bar{V}_{i\tau}}{\sqrt{1+\eta'^2}}. \tag{3.104}$$

It is assumed that the drag force in tangential direction depends on the difference in the velocities of the body and the water surface. It is supposed that the drag force in the tangential direction is given by

$$R_\tau = \frac{1}{2}\rho_w C_w A_i \,|\bar{V}_{w\tau} - \bar{V}_{i\tau}| \cdot (\bar{V}_{w\tau} - \bar{V}_{i\tau}) \tag{3.105}$$

where $\bar{V}_{w\tau} = (\bar{V}_w \,|\, \tau)$ is the tangential velocity of a particle on the water surface, ρ_w is the water density, C_w is the drag coefficient and A_i is the wetted area of the body.

The system of differential equations for the tangential velocity of the body in a coordinate system moving with the wave as obtained by Marchenko is given by

$$\begin{aligned} m_i \frac{d\bar{V}_{i\tau}}{dt} &= -g m_i \frac{\eta'}{\sqrt{1+\eta'^2}} + \frac{1}{2}\rho_w C_w A_i |\bar{V}_{w\tau} - \bar{V}_{i\tau}|(\bar{V}_{w\tau} - \bar{V}_{i\tau}), \\ \frac{d\bar{x}}{dt} &= \frac{\bar{V}_{i\tau}}{\sqrt{1+\eta'^2}}. \end{aligned} \tag{3.106}$$

4 Comparison and discussion of the models

The two models considered in the previous section are based on the same basic ideas although these ideas are carried out quite differently. In both models Newton's second law is used and the forces on the body are broken down into one component due to gravity and one due to the drag force.

Small differences in the models are the use of a term which accounts for the effective mass of the body in the water, the added mass term, which is included in the Rumer *et al.* model but not in the Marchenko model, and the term 1/2 in the drag force. The main difference, however, lies in the coordinate system and the direction of the forces. In the model of Rumer *et al.* all motions are analysed using a stationary coordinate system and the problem is formulated for the horizontal velocitiy of the body. In Marchenko's model on the other hand, the coordinate system moves with the same speed as the wave and the problem is formulated for the tangential velocitiy of the body.

4.1 Comparison of the models

Because of the different coordinate systems and directions of the velocities it is not obvious if both systems are equivalent. In order to directly compare the systems of equations, it is useful to convert them such that they are formulated for both the tangential and the horizontal velocities using the same coordinate system. This is split into two parts, one for the inertia and the force due to gravity, the other one for the drag force.

4.1.1 A simplified situation without drag force

Both, Rumer *et al.* and Marchenko, split the external forces into one component due to drag and one due to gravity. For simplicity, the drag force will be ignored in the following comparison but will be investigated later on. Therefore the simplified situation where the wave is fixed in time and the drag forces and the added mass coefficient are zero is considered in the following.

This situation is equivalent to Marchenko's model without drag, where the coordinate system is moving with the same speed as the wave. Therefore Marchenko's model in this case is given by

$$\begin{aligned} m_i \frac{d\bar{V}_{i\tau}}{dt} &= -g m_i \frac{\eta'}{\sqrt{1+\eta'^2}} \\ \frac{d\bar{x}}{dt} &= \frac{\bar{V}_{i\tau}}{\sqrt{1+\eta'^2}}. \end{aligned} \tag{4.107}$$

In the Rumer *et al.* model the coordinate system is fixed and the wave moves with velocity c in horizontal direction. Omitting all drag forces, this is equivalent to a fixed wave and a coordinate system moving with speed $-c$. The variables measured in this moving frame can be related to variables measured in a stationary frame by $\bar{x} = x - ct$ and $\bar{V}_{ix} = V_{ix} - c$ while the accelerations are the same. The surface water displacement becomes $\eta(kx - \omega t) = \eta(k\bar{x})$ and the derivatives of η with respect to x become

$$\frac{\partial \eta(kx-\omega t)}{\partial x} = k\eta(kx-\omega t) = \frac{\partial \eta(k\bar{x})}{\partial \bar{x}}, \qquad \frac{\partial^2 \eta(kx-\omega t)}{\partial t^2} = \frac{\partial^2 \eta(k\bar{x})}{\partial t^2}.$$

Using these relations, the Rumer *et al.* model without drag and added mass for a non-moving frame and a fixed wave becomes

$$\begin{aligned} m_i \frac{d\bar{V}_{ix}}{dt} &= -m_i \big(g + \frac{\partial^2 \eta(k\bar{x})}{\partial t^2}\big) \frac{\eta'(k\bar{x})}{1+\eta'(k\bar{x})^2} \\ \frac{d\bar{x}}{dt} &= V_{ix} - c = \bar{V}_{ix}. \end{aligned}$$

Because in this simple situation the wave is not moving, η is not explicitly dependent on t but still implicitly depends on t since the position is given by $\bar{x}(t)$. So $\frac{d^2\eta}{dt^2}$ is given by

$$\frac{d^2\eta}{dt^2} = \frac{d}{dt}\left(\frac{d\eta}{d\bar{x}}\frac{d\bar{x}}{dt}\right) = \frac{d\eta}{d\bar{x}}\frac{d\bar{V}_{ix}}{dt} + \frac{d^2\eta}{d\bar{x}^2}\bar{V}_{ix}^2 = \eta'\frac{d\bar{V}_{ix}}{dt} + \eta''\bar{V}_{ix}^2 \tag{4.108}$$

Substituting this in the system of equations leads to the simplified version of the Rumer *et al.* model

$$\begin{aligned} m_i \frac{d\bar{V}_{ix}}{dt}\big(1 + \frac{\eta'^2}{1+\eta'^2}\big) &= m_i(-g - \bar{V}_{ix}^2\eta'')\frac{\eta'}{1+\eta'^2} \\ \frac{d\bar{x}}{dt} &= \bar{V}_{ix}. \end{aligned} \tag{4.109}$$

Solving Marchenko's system of equations, the velocity of the body in τ direction, $\bar{V}_{i\tau}$ is obtained, whereas the solution of system 4.109 gives the velocity in x-direction. These two velocities are related to each other by

$$\begin{aligned} \bar{V}_{i\tau} = \bar{V}_{ix}\frac{1}{\sqrt{1+\eta'^2}} + \bar{V}_{iz}\frac{\eta'}{\sqrt{1+\eta'^2}} &= \frac{d\bar{x}}{dt}\frac{1}{\sqrt{1+\eta'^2}} + \frac{d\eta}{dt}\frac{\eta'}{\sqrt{1+\eta'^2}} \\ = \frac{d\bar{x}}{dt}\frac{1}{\sqrt{1+\eta'^2}} + \eta'\frac{d\bar{x}}{dt}\frac{\eta'}{\sqrt{1+\eta'^2}} &= \bar{V}_{ix}\sqrt{1+\eta'^2} \end{aligned} \tag{4.110}$$

(this relation was already used by Marchenko in his system of equations). So the velocity in the horizontal direction can be calculated from the velocity in the tangential direction and the point on the wave and vice-versa. Relation (4.110) can now be used to transform Marchenko's system of equations such that it is formulated for the velocity of the body in horizontal direction, and also to convert the forces and velocities in the Rumer *et al.* model in τ-directional forces and velocities. Before the two models from section 3 are compared, the system of equations of motion for the simplified situation is derived in a third way. Using Hamilton's principle, at first a system of equations formulated for the horizontal velocity of the body is derived, which does not completely match the system of equations obtained by Rumer *et al.* In the second calculation, Hamilton's principle is used again, this time using τ-directional velocities. The resulting system of equations is the same as the one found by Marchenko.

Hamilton's principle

Hamilton's principle states that the integral over the difference of the kinetic energy K and the potential energy P, $\int(K-P)dt$, always becomes minimal. Considering a body at the water surface $\eta(\bar{x})$, the potential and the kinetic energy are given by

$$P = m_i\, g\, \eta(\bar{x}), \tag{4.111}$$

$$K = \frac{1}{2}\, m_i\, \bar{V}_{i\tau}^2, \tag{4.112}$$

respectively. To be able to apply the Euler differential equation with respect to x in order to solve the variational problem, the second equation should depend on $\bar{V}_{ix}$. Using relation (4.110) as well as the notation $\dot{\bar{x}} := \frac{d\bar{x}}{dt} = \bar{V}_{ix}$, the kinetic energy is given by

$$K = \frac{1}{2}\, m_i \dot{\bar{x}}^2 (1 + \eta'^2). \tag{4.113}$$

Applying Hamilton's principle, the integral $\int(K-P)dt$ must become minimal. The integrand therefore has to satisfy Euler's differential equation,

$$F_x = \frac{d}{dt}\left(F_{\dot{\bar{x}}}\right),$$
$$\text{where } F(x) = K - P = \frac{1}{2}\, m_i\, \dot{\bar{x}}^2 (1+\eta'^2) - m_i\, g\eta.$$

(in this case the subscripts $\bar{x}$ and $\dot{\bar{x}}$ denote the derivatives with respect to $\bar{x}$ and $\dot{\bar{x}}$). Therefore, the Euler differential equation becomes

$$m_i \dot{\bar{x}}^2 \eta' \eta'' - m_i g \eta' = \frac{d}{dt}\left(m_i \dot{\bar{x}} + m_i \eta'^2 \dot{\bar{x}}\right).$$

Calculating the derivative yields

$$m_i \dot{\bar{x}}^2 \eta' \eta'' - m_i g \eta' = m_i \frac{d}{dt}\dot{\bar{x}} + 2 m_i \eta' \eta'' \dot{\bar{x}}^2 + m_i \eta'^2 \frac{d}{dt}\dot{\bar{x}}$$

and the resulting system of equations is given by

$$\begin{aligned} m_i \frac{d\bar{V}_{ix}}{dt} &= m_i(-g - \bar{V}_{ix}^2 \eta'') \frac{\eta'}{1+\eta'^2} \\ \frac{d\bar{x}}{dt} &= \bar{V}_{ix}. \end{aligned} \tag{4.114}$$

This system is not the same as the one derived by Rumer *et al.*, system (4.109).

Now the system of equations for τ-directions is derived. Applying Hamilton's principle with equations (4.111) and (4.112) and using the Euler differential equation leads to

$$-\frac{\partial \eta}{\partial \tau} m_i g = \frac{d\bar{V}_\tau}{dt}.$$

Using

$$\frac{\partial \eta}{\partial \tau} = \frac{\partial \eta}{\partial(\bar{x}\sqrt{1+\eta'^2})} = \frac{1}{\sqrt{1+\eta'^2}} \frac{d\eta}{d\bar{x}}$$

leads to the system of equations

$$\begin{aligned} m_i \frac{d\bar{V}_{i\tau}}{dt} &= -g m_i \frac{\eta'}{\sqrt{1+\eta'^2}} \\ \frac{d\bar{x}}{dt} &= \frac{\bar{V}_{i\tau}}{\sqrt{1+\eta'^2}} \end{aligned} \tag{4.115}$$

which is exactly the same as the one derived by Marchenko, system (4.107).

Marchenko's model

Marchenko's model (4.107) for the simplified situation using tangential velocities is the same one as the model which was just derived using Hamilton's principle. It is now of interest how Marchenko's system of equations transforms to a system using velocities in horizontal-direction. As suggested earlier, this transformation can be performed easily using relation (4.110). Then the derivative $\frac{d\bar{V}_{i\tau}}{dt}$ can be written as

$$\frac{d\bar{V}_{i\tau}}{dt} = \frac{d}{dt}\left(\bar{V}_{ix}\sqrt{1+\eta'^2}\right) = \frac{d\bar{V}_{ix}}{dt}\sqrt{1+\eta'^2} + \bar{V}_{ix}\frac{1}{\sqrt{1+\eta'^2}}\eta'\eta''\frac{d\bar{x}}{dt}.$$

Substituting this in Marchenko's model, the transformed system of equations is

$$\begin{aligned} m_i \frac{d\bar{V}_{ix}}{dt} &= m_i(-g - \bar{V}_{ix}^2\eta'')\frac{\eta'}{1+\eta'^2}, \\ \frac{d\bar{x}}{dt} &= \bar{V}_{ix}, \end{aligned} \tag{4.116}$$

and this system is the same as the one derived using Hamilton's principle.

The Rumer *et al.* model

The system of equations for the motion of the body in the simplified situation was derived in four ways, using Marchenko's model and Hamilton's principle each in x- and τ-direction. For each direction the systems of equations are the same. However, the system formulated for the horizontal velocity, equations (4.114) or (4.116) is not the same as the one derived by Rumer *et al.*, system (4.109). It is possible to change $\bar{V}_{ix}$ to $\bar{V}_{i\tau}$ by substituting relation (4.110) in system (4.109), but since the systems formulated for horizontal velocities are already different, this substitution only leads to a system of equations different to the ones derived above. To find the source of this difference, the Rumer *et al.* model is derived again, but this time all forces are projected onto the tangential (instead of the horizontal). This is illustrated in figure 73.

The angle ϕ is given by $\phi = \arctan(-\eta') = -\arctan\eta'$. The projection of the force due to gravity is given by

$$F_g = m_i g \sin(-\arctan\eta') = m_i g \sin(-\arcsin\frac{\eta'}{\sqrt{1+\eta'^2}}) = -m_i g\frac{\eta'}{\sqrt{1+\eta'^2}}$$

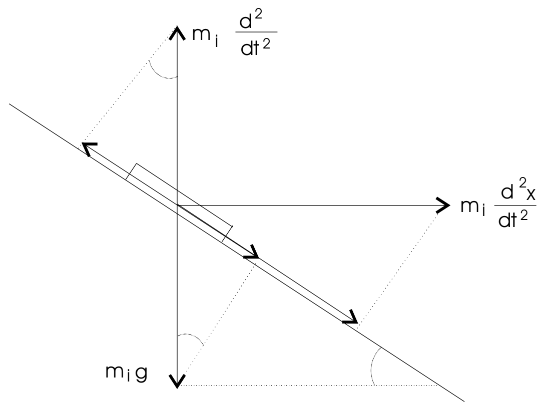

Figure 73: Projection of the forces onto the tangential

Also, the inertial forces are projected onto the tangential. The force in vertical direction is given by $m_i \frac{d^2 z}{dt^2} = m_i \frac{d^2 \eta}{dt^2}$, and its projection to the tangential of the water surface is given by

$$m_i \frac{d^2\eta}{dt^2} \sin(-\arctan \eta') = -m_i \frac{d^2 z}{dt^2} \frac{\eta'}{\sqrt{1+\eta'^2}},$$

where the vector points in negative tangential direction (compare figure 73). The projection of the horizontal inertial force, $m_i \frac{d^2 \bar{x}}{dt^2} = m_i \frac{d\bar{V}_{ix}}{dt}$, can be calculated by

$$m_i \frac{d^2\bar{x}}{dt^2} \cos(-\arctan \eta') = m_i \frac{d^2 \bar{x}}{dt^2} \frac{1}{\sqrt{1+\eta'^2}}.$$

Keeping in mind the direction of the forces and using Newton's second law leads to

$$m_i \frac{d^2\bar{x}}{dt^2} \frac{1}{\sqrt{1+\eta'^2}} - \big(- \frac{\partial^2 \eta}{\partial t^2} \frac{\eta'}{\sqrt{1+\eta'^2}} \big) = -m_i g \frac{\eta'}{\sqrt{1+\eta'^2}},$$

which can be rewritten using the relation (4.108) and $\frac{d\bar{x}}{dt} = \bar{V}_{ix}$ to obtain the system of equations

$$m_i \frac{d\bar{V}_{ix}}{dt} = m_i(-g - \bar{V}_{ix}^2 \eta'') \frac{\eta'}{1+\eta'^2}$$
$$\frac{d\bar{x}}{dt} = \bar{V}_{ix}.$$

This system of equations for the motion of a body is formulated for the velocity in the horizontal direction. It is the same as the ones derived starting from Marchenko's model and using Hamilton's principle. However, it is not the same as the original model by Rumer *et al.* In the derivation of their system of equations, Rumer *et al.* projected the inertia in the vertical direction onto the horizontal. But since it is the inertia in the vertical direction, it does not have a horizontal component. This is different for the gravity force which is an external force acting on the body. A body sliding down a slope due to gravity is accelerated in the horizontal direction. Therefore the gravity force has a portion in horizontal direction. On the other hand, the slope sliding model of Rumer *et al.* does not account for the centripetal force acting on the body which is necessary since the body moves on a curved path. This means it is also possible to derive the correct system of equations by projecting all forces onto the horizontal, but using the centripetal force instead of the inertia in the vertical direction. This situation is illustrated in figure 74.

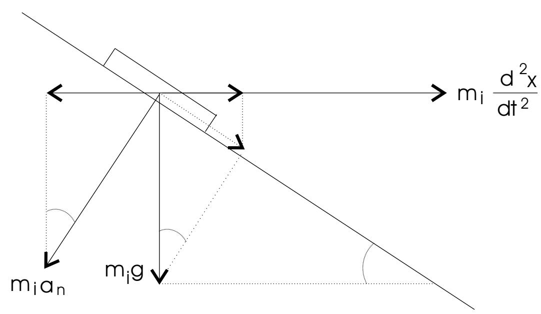

Figure 74: Projection of the forces onto the x-direction

The centripetal force a_n is given by

$$a_n = \frac{\eta''}{\sqrt{1+\eta'^2}^3}\bar{V}_{i\tau}^2 = \frac{\eta''}{\sqrt{1+\eta'^2}}\bar{V}_{ix}^2$$

and its projection onto the negative horizontal direction (compare figure 74) is given by $-a_n\eta'/\sqrt{1+\eta'^2}$. Therefore the system of equations becomes

$$m_i\frac{d\bar{V}_{ix}}{dt} + \eta''\bar{V}_{ix}^2\frac{\eta'}{1+\eta'^2} = -m_i g\frac{\eta'}{1+\eta'^2}$$
$$\frac{d\bar{x}}{dt} = \bar{V}_{ix}.$$

which is also the same one as the one derived by Hamilton's principle and from Marchenko's model, equations (4.114) and (4.116).

4.1.2 Discussion of the drag force

In both models not only a force due to gravity acts upon the body but there is also a force which accounts for the pull exerted on the body by the wave, the drag force. The drag force depends on the shape and size of the moving body and the fluid it is moving in in a complicated way. Moreover, in the situation of a floating body, the body does not move through one medium but moves along the boundary of two media. An approximation for the drag force is often given by

$$F_w = \frac{1}{2}\rho A_i C_w v|v|, \tag{4.117}$$

where ρ is the density of the medium which the body moves through, A_i is the maximum cross sectional area presented by the moving object, C_w is the dimensionless drag coefficient and v is the velocity of the body relative the medium (5).

Both, Marchenko and Rumer *et al.*, base their model of the drag force on equation (4.117) and use the water as the medium, which is reasonable because the drag exerted by the atmosphere is small compared to the drag exerted by the water. Marchenko models v as the difference in the velocity of the body and the water particle at the surface in tangential direction. Rumer *et al.*, however, approximates the velocity v in (4.117) by the difference of the horizontal velocities of the body and the surface water particle.

The drag force in Marchenko's model can be expressed using the x-directional velocities by using relation (4.110) and $V_{w\tau}$ in terms of the horizontal and vertical component of the water particle velocity to obtain

$$F_w = \frac{1}{2}\,\rho_w C_w A_i \left|\bar{V}_{wx}\frac{1}{\sqrt{1+\eta'^2}} + \bar{V}_{wz}\frac{\eta'}{\sqrt{1+\eta'^2}} - \bar{V}_{ix}\sqrt{1+\eta'^2}\right| \times \left(\bar{V}_{wx}\frac{1}{\sqrt{1+\eta'^2}} + \bar{V}_{wz}\frac{\eta'}{\sqrt{1+\eta'^2}} - \bar{V}_{ix}\sqrt{1+\eta'^2}\right).$$

Substituting the drag in the first equation of Marchenko's system of equations yields

$$\begin{aligned} m_i\frac{d\bar{V}_{ix}}{dt} &= m_i(-g - \bar{V}_{ix}^2\eta'')\frac{\eta'}{1+\eta'^2} + \rho_w C_w A_i\,\frac{\sqrt{1+\eta'^2}}{2} \\ &\quad\times\left|\frac{\bar{V}_{wx}+\bar{V}_{wz}\eta'}{1+\eta'^2} - \bar{V}_{ix}\right|\left(\frac{\bar{V}_{wx}+\bar{V}_{wz}\eta'}{1+\eta'^2} - \bar{V}_{ix}\right) \\ \frac{d\bar{x}}{dt} &= V_{ix}. \end{aligned} \tag{4.118}$$

Here it has to be noted that the water particle velocity in tangential direction cannot be further simplified since the water does not necessarily move in tangential direction. However, equation (4.117) is only an approximation of the real, very complicated drag force. Therefore it is reasonable to further approximate the water particle velocity in tangential direction, for example by the water particle velocity in the horizontal direction as done by Rumer *et al.*

4.2 Summary and conclusions

To model how a small body is affected by a periodic wave field, two similar slope sliding models were considered: The system of equations derived by Rumer *et al.* uses velocities in horizontal direction and a stationary coordinate system while the system of equations for the motion given by Marchenko is formulated for the velocities in the tangential direction measured in a coordinate system moving with the same speed as the wave. The velocities and displacements in these coordinate systems can be related to each other. In a thorough comparison, also including the derivation of a system of equations using Hamilton's principle, it was shown that the two systems are not equivalent since the Rumer *et al.* model does not include the centripetal force correctly.

This error occurred since in the system of equations a projection of the inertia in vertical direction onto the horizontal direction was used, but the centripetal force due to the curvature of the wave was ignored. To correct the system of equations, there are two options: Either all forces introduced by Rumer *et al.*, including the inertia in vertical and horizontal direction, are projected onto the tangential instead of the normal (in which case the centripetal force has no influence since it points perpendicular to the tangential), or the centripetal force is included and all forces are projected onto the horizontal direction (in which case the vertical inertia has no influence). Both options lead to the same system of equations as the one derived using Hamilton's principle or Marchenko's model.

The Rumer *et al.* model was used by Shen & Ackley, Meylan Shen & Zhong and others. However, they always omitted the term $\frac{\partial^2\eta}{\partial t^2}$ reasoning that it is very small due to the small amplitude of the wave. Compared to the correct system, they ommitted the term due to the centripetal force. This is reasonable for small amplitude waves and therefore their approximation can be regarded as valid.

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