Discrete Groups and Computational Geometry

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ABSTRACT

Let f and g be Möbius transformations with finite-orders p and q respectively. Further, let $\gamma = tr[f,g] - 2$, where $tr[f,g]$ is the trace of the commutator of f and g in the standard $SL(2,\mathbb{C})$ representation of Möbius transformations.

The group $G = \langle f, g \rangle$ is then defined, up to conjugacy, by the parameter set (p, q, γ), whenever $\gamma \neq 0$. If the group G is discrete and non-elementary, then it is a Kleinian group. Kleinian groups are intimately related to hyperbolic 3-orbifolds.

Here we develop a computer program that constructs a fundamental domain for such Kleinian groups. These constructions are undertaken directly from the parameters given above. We use this program to investigate, and add to, recent work on the classification of arithmetic Kleinian groups generated by two (finite-order) elliptic transformations.

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