The Jacobi triple product, quintuple product, Winquist and Macdonald identities

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Uros Abaz
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Abstract

This thesis consists of seven chapters.
Chapter 1 is an introduction to the infinite products. Here we provide a proof for representing sine function as an infinite product. This chapter also describes the notation used throughout the thesis as well as the method used to prove the identities. Each of the other chapters may be read independently, however some chapters assume familiarity with the Jacobi triple product identity.
Chapter 2 is about the Jacobi triple product identity as well as several implications of this identity.
In Chapter 3 the quintuple product identity and some of its special cases are derived. Even though there are many known proofs of this identity since 1916 when it was first discovered, the proof presented in this chapter is new. Some beautiful formulas in number theory are derived at the end of this chapter.
The simplest two dimensional example of the Macdonald identity, A2, is investigated in full detail in Chapter 4. Ian Macdonald first outlined the proof for this identity in 1972 but omitted many of the details hence making his work hard to follow.
In Chapters 5 and 6 we somewhat deviate from the method which uses the two specializations to evaluate the constant term and prove Winquist's identity and Macdonald's identity for G2. Some of the work involved in proving G2 identity is new.
Finally in Chapter 7 we discuss the work presented with some concluding remarks as well as underlining the possibilities for the future research.
Throughout the thesis we point to the relevant papers in this area which might provide different strategies for proving above identities.

Contents

1 Introduction

This thesis is about infinite products. We will state and prove some mathematical identities involving infinite products. The multivariate infinite products in this thesis were first studied by L. Winquist in 1969 [19]. In 1972, I. G. Macdonald [14] associated the infinite product identities with irreducible root systems and gave a very general method for proving such identities. He also classified such identities into infinite families and exceptional cases. Further, D. Stanton in 1989, explicitly [16], detailed the proofs for infinite families. In 1997, S. Cooper [8] provided a detailed proof for G_2 and $G_2^\vee$ identities, which belong to the exceptional cases. For other exceptional cases, namely $F_4, F_4^\vee, E_6, E_7, E_8$, only Macdonald's proofs exist.

1.1 Infinite product representation of the sine function

The simplest function which vanishes on a bilateral arithmetic progression is the sine function. This is illustrated by the following theorem.

Theorem 1.1 $\sin \pi x = \pi x_{n=1}^{\infty}(1-\frac{x^2}{n^2})$ *has simple zeros at the integers.*

Proof. Let,

$$U_n = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{1.1.1}$$

Using the trigonometric identity, $\sin(A+B) = \sin A\cos B + \cos A\sin B,$ we get that,

$$\begin{aligned}U_n + U_{n-2} &= \frac{\sin(n+1)\theta}{\sin\theta} + \frac{\sin(n-1)\theta}{\sin\theta}\\ &= \frac{\sin n\theta\cos\theta + \cos n\theta\sin\theta + \sin n\theta\cos\theta - \cos n\theta\sin\theta}{\sin\theta}\\ &= 2\cos\theta U_{n-1}.\end{aligned}$$

Similarly, using the same trigonometric identity we derive,

$$\begin{aligned}U_{2n+2} + U_{2n-2} &= \frac{\sin(2n+1)\theta\cos 2\theta + \cos(2n+1)\theta\sin 2\theta}{\sin\theta}\\ &+ \frac{\sin(2n+1)\theta\cos 2\theta - \cos(2n+1)\theta\sin 2\theta}{\sin\theta}\\ &= \frac{2\sin(2n+1)\theta\cos 2\theta}{\sin\theta}\\ &= 2\cos 2\theta U_{2n}\\ &= (2-4\sin^2\theta)U_{2n}.1.1.2\end{aligned} \tag{1}$$

Also, it can be easily checked, using (1.1.1) that,

$$U_0 = 1 \tag{1.1.3}$$

and

$$U_{-2} = -1. \tag{1.1.4}$$

Next, we let $n \in \mathrm{N}^*$, where $\mathrm{N}^* = \{0,1,2,3...\}$ and show that U_{2n} is a polynomial of degree n in $\sin^2\theta$. When $n = 0$, (1.1.2) becomes,

$$U_2 + U_{-2} = (2 - 4\sin^2\theta)U_0. \tag{1.1.5}$$

Then by (1.1.3), (1.1.4) and (1.1.5),

$$U_2 = 3 - 4\sin^2\theta$$

and hence U_2 is a polynomial of degree 1 in $\sin^2\theta$. Next, assume that U_{2k} is a polynomial of degree k in $\sin^2\theta$ for $0 \le k \le n$. Consider $U_{2(n+1)}$. By (1.1.2),

$$U_{2n+2} = (2 - 4\sin^2\theta)U_{2n} - U_{2n-2}.$$

It follows that U_{2n+2} is a polynomial of degree $n+1$ in $\sin^2\theta$.

We can write U_{2n} as

$$U_{2n} = c_0 + c_1\sin^2\theta + c_2\sin^4\theta + ... + c_n\sin^{2n}\theta,$$

with the factorization

$$U_{2n} = c_0(1 - r_1\sin^2\theta)(1 - r_2\sin^2\theta)...(1 - r_n\sin^2\theta).$$

Now, we seek the roots of this equation. From (1.1.1) we see that $U_{2n} = 0$ when $\theta = \pm j\pi/(2n+1), \quad j \in \mathrm{Z}$, which enables us to further write above factorization as

$$U_{2n} = c_0\left(1 - \frac{\sin^2\theta}{\sin^2(\pi/2n+1)}\right)\left(1 - \frac{\sin^2\theta}{\sin^2(2\pi/2n+1)}\right)...\left(1 - \frac{\sin^2\theta}{\sin^2(n\pi/2n+1)}\right)$$

In order to determine c_0, we take the limit as $\theta \to 0$.

$$\lim_{\theta\to 0}\frac{\sin(2n+1)\theta}{\sin\theta} = \lim_{\theta\to 0} c_0 {}^{n}_{j=1}\left(1 - \frac{\sin^2\theta}{\sin^2(j\pi/(2n+1)}\right),$$

Therefore,

$$c_0 = 2n+1.$$

Putting above results together,

$$\sin(2n+1)\theta = (2n+1)\sin\theta^{n}_{j=1}\left(1 - \frac{\sin^2\theta}{\sin^2(j\pi/(2n+1))}\right). \tag{1.1.6}$$

Further, we let $\theta = \pi x/(2n+1)$ in (1.1.6) and let n go to ∞.

$$\sin(\pi x) = \lim_{n\to\infty}(2n+1)\sin\frac{\pi x}{2n+1}{}^{n}_{j=1}\left(1 - \frac{\sin^2(\pi x/(2n+1))}{\sin^2(j\pi/(2n+1)}\right)$$

$$= \pi x^{\infty}_{j=1}\left(1 - \frac{\pi^2 x^2}{\pi^2 j^2}\right) = \pi x^{\infty}_{j=1}\left(1 - \frac{x^2}{j^2}\right).$$

This completes the proof of Theorem 1.1. There are other proofs of this theorem that have appeared in the literature. Refer to [2] for the proof involving the Gamma function and also [15] where the product for the sine function is derived by integrating a partial fraction for the cotangent.

1.2 Infinite product representation for other functions

We continue our investigation by trying to find a function that vanishes along the geometric progression. Let q be a complex parameter such that $q < |1|$. Then we can define the theta product analytic on $\mathbb{C} \setminus \{0\}$ to be

$$f(x;q) = \prod_{n=1}^{\infty} (1 - xq^{n-1})(1 - x^{-1}q^{n}).$$

It has simple zeros at $x = q^n$, $n = 0, \pm 1, \pm 2, \ldots$ and an essential singularity at $x = 0$. Since the theta product converges absolutely and uniformly on the compact subsets of a complex plane which do not contain the origin, it has a Laurent expansion in the powers of x, valid in the annulus $0 < |x| < \infty$.

In Chapter 2, we will show that the explicit form of the Laurent expansion is given by,

$$\prod_{n=1}^{\infty}(1 - xq^{n-1})(1 - x^{-1}q^{n}) = c(q)\sum_{n=-\infty}^{\infty} q^{(n^2-n)/2}(-x)^n,$$

where

$$c(q) = \prod_{j=1}^{\infty} (1 - q^{j})^{-1}.$$

This identity is more commonly written as,

$$\prod_{n=1}^{\infty}(1 - xq^{n-1})(1 - x^{-1}q^{n})(1 - q^{n}) = \sum_{n=-\infty}^{\infty} q^{(n^2-n)/2}(-x)^n, \tag{1.2.1}$$

and is called the Jacobi triple product identity.

The Macdonald identities, due to Ian Macdonald, are multivariate generalizations of the Jacobi triple product identity. When $q = 0$, the Macdonald identities reduce to the Weyl denominator formulas for the root systems. Macdonald [14] showed how to associate and convert each of the affine root systems to multivariate infinite products and also gave a general formula for the multivariate Laurent series expansion. We use Macdonald's classification to list the irreducible root systems below.

- Infinite families: $A_{n-1}, n \geq 2;\ B_n, n \geq 3;\ B_n^{\vee}, n \geq 3;\ C_n, n \geq 2;$
  $C_n^{\vee}, n \geq 2;\ BC_n, n \geq 1;\ D_n, n \geq 4.$

- Exceptional cases: $G_2, G_2^{\vee}, F_4, F_4^{\vee}, E_6, E_7, E_8.$

See Macdonald [14] for the definitions of an affine root system and also how to associate infinite products to affine root system. We will provide several explicit examples of the Macdonald identities in Section 1.4.

1.3 Notation

Before listing some of the examples let us introduce standard notation used to represent products. From now on, we will assume $|q| < 1$.

Definition 1.2 *Let* $n = 1,2,3...$ *then,*

1. $(a;q)_0 = 1;$

2. $(a;q)_n = {}_{j=0}^{n-1} (1-aq^j);$

3. $(a;q)_\infty = {}_{j=0}^{\infty} (1-aq^j);$

4. $(a_1, a_2, ..., a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty ... (a_n;q)_\infty .$

Throughout the thesis these will be refered to as *the products to the base of* q. Also note when writing $f(x;q)$ we will drop the dependence on q and just write $f(x)$.

1.4 Examples

Below we list several Macdonald identities and prove them in chapters to follow.

1. A_1 : This corresponds to the Jacobi triple product identity (1.2.1). Written as a product to the base of q,

$$(x, qx^{-1}; q)_\infty = {}_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} x^n. \tag{1.4.1}$$

2. BC_1 : Quintuple product identity

$$(qx, qx^{-1}, q; q)_\infty (x^2, q^2 x^{-2}; q^2)_\infty = {}_{n=-\infty}^{\infty} q^{n(3n-1)/2} (x^{3n} - x^{1-3n}). \tag{1.4.2}$$

3. A_2 :

$$\left(\frac{x_1}{x_2}, \frac{qx_2}{x_1}, \frac{x_1}{x_3}, \frac{qx_3}{x_1}, \frac{x_2}{x_3}, \frac{qx_3}{x_2}, q, q; q\right)_\infty$$

$$= {}_{m_1+m_2+m_3=0} q^{\frac{3}{2}(m_1^2+m_2^2+m_3^2)+m_1+2m_2+3m_3} x_1^{3m_1} x_2^{3m_2} x_3^{3m_3} {}_{1\le i\le j\le 3} \left(1 - \frac{x_i q^{m_i}}{x_j q^{m_j}}\right). \tag{1.4.3}$$

4. B_2 : Winquist's identity

$$(x, qx^{-1}, y, qy^{-1}, xy, qx^{-1}y^{-1}, xy^{-1}, qx^{-1}y, q, q; q)_\infty$$

$$= \qquad q^{\frac{1}{2}(3m^2+3n^2-3m-n)} x^{3m} y^{3n}$$

$$m+n \equiv 0\,(mod\ 2)$$

$$\times (1-xq^m)(1-yq^n)(1-xyq^{m+n})(1-xy^{-1}q^{m-n}), 1.4.4 \tag{2}$$

5. G_2 :

$$\left(x_1^2 x_2^{-1} x_3^{-1}, q x_1^{-2} x_2 x_3, x_1^{-1} x_2^2 x_3^{-1}, q x_1 x_2^{-2} x_3, x_1 x_2 x_3^{-2}, q x_1^{-1} x_2^{-1} x_3^2, q; q\right)_\infty$$

$$\times (x_1^3 x_3^{-3}, q x_1^{-3} x_3^3, x_1^{-3} x_2^3, q x_1^3 x_2^{-3}, x_2^3 x_3^{-3}, q x_2^{-3} x_3^3, q; q)_\infty$$

$$= \qquad q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2} x_1^{12m_1} x_2^{12m_2} x_3^{12m_3}$$

$$m_1+m_2+m_3=0$$

$$\times \left(1-\frac{x_1^2}{x_2 x_3}\right)\left(1-\frac{x_2^2}{x_1 x_3}\right)\left(1-\frac{x_1 x_2}{x_3^2}\right)\left(1-\frac{x_1^3}{x_3^3}\right)\left(1-\frac{x_2^3}{x_1^3}\right)\left(1-\frac{x_2^3}{x_3^3}\right).$$

$$1.4.5 \tag{3}$$

1.5 Method of proofs

Our proofs are based on four steps:

1. Define function f to be an infinite product. We seek a Laurent series expansion for the product.

2. Consider several functional equations involving f in order to derive the recurrence relations between the coefficients. Also, look at the symmetries of f and express all non-zero coefficients in terms of a single one, c_0.

3. Employ the specializations on f .

4. Last step involves comparison of the above specializations in order to evaluate c_0.

Using our method we are able to provide an ouline on how to go about proving Macdonald's identities. However, note that many other proofs exist and we will point to the papers containing them wherever known.

2 Jacobi triple product identity

2.1 Introduction

As the name suggests, this identity is due to Jacobi who discovered it in 1829. It is fundamental in the theory of theta functions, elliptic functions and analytic number theory. The substitution used to evaluate constant terms in our proof is due to Macdonald [14]. It was made explicit by Cooper [7] and rediscovered by S. Kongsiriwong and Z.-G. Liu [13].

2.2 Proof of the Jacobi triple product identity

We state the identity as a following theorem,

Theorem 2.1 *Let* $x \neq 0$ *and* $|q| < 1$. *Then,*

$$\prod_{n=1}^{\infty}(1-xq^{n-1})(1-x^{-1}q^{n})(1-q^{n}) = \sum_{n=-\infty}^{\infty} q^{(n^2-n)/2}(-x)^{n}.$$

Next we present the proof. Let,

$$f(x;q) = \prod_{n=1}^{\infty}(1-xq^{n-1})(1-x^{-1}q^{n}).$$

The infinite product in the definition of $f(x)$ converges on any compact subset of the complex plane that does not contain the origin. Therefore, f has a Laurent series expansion in the powers of x, valid in the annulus $0 < |x| < \infty$. Now we determine the coefficients in four steps.

2.2.1 Deriving recurrence relations

Let us consider,

$$\frac{f(x)}{f(xq)} = \prod_{n=1}^{\infty}\frac{(1-xq^{n-1})(1-x^{-1}q^{n})}{(1-xq^{n})(1-x^{-1}q^{n-1})}$$

$$= \frac{1-x}{1-x^{-1}} = -x.$$

Therefore,

$$f(x) = -xf(xq).$$

If we substitute the Laurent series into this functional equation we get,

$$\sum_{n=-\infty}^{\infty} c_n x^n = -\sum_{n=-\infty}^{\infty} c_n q^n x^{n+1}.$$

Now we equate coefficients of x^n on both sides to get

$$c_n = -q^{n-1}c_{n-1}.$$

We can iterate this recurrence relation to show that any coefficient c_n, for $n > 0$, can be expressed in terms of c_0, i.e,

$$\begin{aligned} c_n &= (-q^{n-1})(-q^{n-2})...(-q)(-1)c_0 \\ &= (-1)^n q^{n(n-1)/2} c_0. \end{aligned}$$

Further, if $n < 0$ we can rewrite the above recurrence relation to get,

$$c_n = -q^{-n} c_{n+1}.$$

Iterating this equation gives,

$$\begin{aligned} c_n &= (-q^{-n})(-q^{-n-1})...(-q)c_0 \\ &= (-1)^n q^{n(n-1)/2} c_{0.} \end{aligned}$$

We see that the same formula applies to the coefficients when $n < 0$. Therefore,

$$f(x) = {}_{n=1}^{\infty} (1 - xq^{n-1})(1 - x^{-1}q^n) = c_0{}_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} x^n \tag{2.2.1}$$

and all that is left is to determine c_0.

2.2.2 First specialization

Let $x = \omega$, where $\omega = \exp(2\pi i/3)$, a primitive third root of unity. From the infinite product defining f we have,

$$\begin{aligned} f(\omega) &= (1-\omega)_{n=1}^{\infty}(1-\omega q^n)(1-\omega^2 q^n) \\ &= (1-\omega)_{n=1}^{\infty}(1-\omega q^n)(1-\omega^2 q^n)\frac{(1-q^n)}{(1-q^n)} \\ &= (1-\omega)_{n=1}^{\infty}\frac{(1-q^{3n})}{(1-q^n)}.2.2.2 \end{aligned} \tag{4}$$

From the infinite series for f we have,

$$\begin{aligned} f(\omega) &= c_0{}_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}\omega^n \\ &= c_0\Big({}_{n=-\infty}^{\infty}(-1)^{3n} q^{3n(3n-1)/2} + \omega_{n=-\infty}^{\infty}(-1)^{3n+1} q^{3n(3n+1)/2} \\ &\quad + \omega^2{}_{n=-\infty}^{\infty}(-1)^{3n+2} q^{(3n+2)(3n+1)/2}\Big)2.2.3 \end{aligned} \tag{5}$$

By expanding the series associated with ω^2 we show that it equals zero. Split series into two,

$${}_{n=0}^{\infty}(-1)^n q^{(3n+2)(3n+1)/2} + {}_{n=-1}^{-\infty}(-1)^n q^{(3n+2)(3n+1)/2}.$$

Now replace n with $-n-1$ in the second series to get

$${}_{n=0}^{\infty}(-1)^n q^{(3n+2)(3n+1)/2} + {}_{n=0}^{\infty}(-1)^{-n-1} q^{(-3n-1)(-3n-2)/2} = 0.$$

Further replacing n with $-n$ in the series associated with ω, we get

$$f(\omega) = c_0(1-\omega)_{n=-\infty}^{\infty}(-1)^n q^{3n(3n-1)/2}. \tag{2.2.4}$$

Combining (2.2.2) and (2.2.4),

$$c_0{}_{n=-\infty}^{\infty}(-1)^n q^{3n(3n-1)/2} = {}_{n=1}^{\infty}\frac{(1-q^{3n})}{(1-q^n)}.$$

2.2.3 Second specialization

We will now replace q with q^9 and then let $x = q^3$. The infinite product gives,

$$f(q^3) =_{n=1}^{\infty} (1-q^3 q^{9n-9})(1-q^{-3}q^{9n})$$

$$=_{n=1}^{\infty} (1-q^{9n-6})(1-q^{9n-3})\frac{(1-q^{9n})}{(1-q^{9n})}$$

$$=_{n=1}^{\infty} \frac{(1-q^{3n})}{(1-q^{9n})}.2.2.5 \tag{6}$$

On the other hand the infinite series gives,

$$f(q^3) = c_0(q^9)_{n=-\infty}^{\infty}(-1)^n q^{9n(n-1)/2} q^{3n}$$

$$= c_0(q^9)_{n=-\infty}^{\infty}(-1)^n q^{3n(3n-1)/2}.2.2.6 \tag{7}$$

Consequently, equating (2.2.5) and (2.2.6),

$$c_0(q^9)_{n=-\infty}^{\infty}(-1)^n q^{3n(3n-1)/2} =_{n=1}^{\infty} \frac{(1-q^{3n})}{(1-q^{9n})}. \tag{2.2.7}$$

Note that in the equations (2.2.6) and (2.2.7), as well as in next chapter, we have emphasized the dependance on q by writing $c_0(q)$ and $c_0(q^9)$ since q has been replaced with q^9. This will also be done in any of the future chapters where necessary.

2.2.4 Evaluating constant term

Since the first and second specialization yield the results with equal series sides we can divide the two, getting

$$\frac{c_0(q)}{c_0(q^9)} =_{n=1}^{\infty} \frac{(1-q^{9n})}{(1-q^n)},$$

and so,

$$c_0(q)_{n=1}^{\infty}(1-q^n) = c_0(q^9)_{n=1}^{\infty}(1-q^{9n}).$$

Now, this expression can be iterated to get

$$c_0(q)_{n=1}^{\infty}(1-q^n) = c_0(q^9)_{n=1}^{\infty}(1-q^{9n}) = c_0(q^{81})_{n=1}^{\infty}(1-q^{81n})$$

$$= \ldots = c_0(q^{9^k})_{n=1}^{\infty}(1-q^{9^k n}).$$

By taking a limit as $k \to \infty$,

$$c_0(q)_{n=1}^{\infty}(1-q^n) = c_0(0) = 1.$$

Combining (2.2.1) and this result for c_0 completes the proof of Theorem 2.1.

2.3 Implications of the Jacobi triple product identity

In this chapter we will show several interesting results that can be derived by manipulating the Jacobi triple product identity. Corollary 2.2 is due to L. Euler and Corollary 2.3 is due to C. F. Gauss.

Corollary 2.2

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}.$$

Proof. We replace q with q^3 in Theorem 2.1 and then set $x=q$ to get

$$\prod_{n=1}^{\infty}(1-q\cdot q^{3n-3})(1-q^{3n}\cdot q^{-1})(1-q^{3n})=\sum_{n=-\infty}^{\infty}(-1)^n q^{3n(n-1)/2}\cdot q^n.$$

With further simplification we get,

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}.$$

Corollary 2.3

$$\prod_{n=1}^{\infty}(1-q^n)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(n^2+n)/2}.$$

Proof. We will prove this by observing what happens if we divide both sides of the identity by $1-x$ and let x go to $1.$ We get,

$$\lim_{x\to 1}\frac{1}{1-x}\prod_{n=1}^{\infty}(1-xq^{n-1})(1-x^{-1}q^n)(1-q^n)=\lim_{x\to 1}\frac{1}{1-x}\sum_{n=-\infty}^{\infty}q^{(n^2-n)/2}(-x)^n.$$

From the first factor on the product side we can pull out $1-x$ term and take the limit, whereas on the series side we need to do some manipulation before taking the limit using L'Hôpital's rule. It follows,

$$\prod_{n=1}^{\infty}(1-q^n)^3=\lim_{x\to 1}\frac{1}{1-x}\left(\sum_{n=1}^{\infty}(-1)^n q^{(n^2-n)/2}x^n+\sum_{n=0}^{-\infty}(-1)^n q^{(n^2-n)/2}x^n\right).$$

In the second series on the right hand side we replace n with $1-n$ to get,

$$\prod_{n=1}^{\infty}(1-q^n)^3=\lim_{x\to 1}\frac{1}{1-x}\sum_{n=1}^{\infty}(-1)^n q^{(n^2-n)/2}\left(x^n-x^{1-n}\right).$$

Next, we apply L'Hôpital's rule and take the limit to get,

$$\prod_{n=1}^{\infty}(1-q^n)^3=\sum_{n=1}^{\infty}(-1)^n q^{(n^2-n)/2}(-1)(2n-1).$$

Therefore,

$$\prod_{n=1}^{\infty}(1-q^n)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(n^2+n)/2}.$$

3 Quintuple product identity

3.1 Introduction

The quintuple product identity has received much attention and various proofs of it have appeared. It can be traced back to 1916, when it appeared in R. Fricke's book [11]. Around the same time S. Ramanujan stated the identity, but this was not discovered until about 70 years later. The identity was first studied, applied and written about by G. N. Watson [18] in 1929, while proving Ramanujan's theorems on continued fractions. He stated the identity in the following form,

$$
\begin{aligned}
&{}_{n=1}^{\infty}(1-q^{5n})(1+q^{5n-1}x^{-1})(1+q^{5n-4}x)(1-q^{10n-7}x^{-2})(1-q^{10n-3}x^{2}) \\
&= {}_{n=-\infty}^{\infty} (-1)^{n} q^{n(15n+1)/2} x^{3n} + {}_{n=-\infty}^{\infty} (-1)^{m} q^{(5n-2)(3n-1)/2} x^{1-3n} 3.1.1
\end{aligned} \tag{8}
$$

For the complete summary of known proofs for the quintuple product identity refer to Cooper [9]. The proof we are about to give is new. It is based on a proof by H. C. Chan [4].

3.2 Proof of the quintuple product identity

In this chapter a complete proof of the quintuple product identity is given.

Theorem 3.1 *For* $x \neq 0$ *and* $|q| < 1$,

$$
\begin{aligned}
&{}_{n=1}^{\infty}(1-q^{n})(1-xq^{n-1})(1-x^{-1}q^{n})(1-x^{2}q^{2n-1})(1-x^{-2}q^{2n-1}) \\
&= {}_{n=-\infty}^{\infty} q^{n(3n-1)/2}(x^{3n}-x^{1-3n}).3.2.1
\end{aligned} \tag{9}
$$

Proof. Let

$$f(x) = {}_{n=1}^{\infty} (1-xq^{n-1})(1-x^{-1}q^{n})(1-x^{2}q^{2n-1})(1-x^{-2}q^{2n-1}).$$

Then f has a Laurent series expansion

$$f(x) = {}_{n=-\infty}^{\infty} c_{n}(q)x^{n},$$

valid in the annulus $0 < |x| < \infty$.

3.2.1 Deriving recurrence relations

We begin by considering the first functional equation for f. Observe that

$$
\begin{aligned}
\frac{f(x)}{f(xq)} &= \prod_{n=1}^{\infty} \frac{(1-xq^{n-1})(1-x^{-1}q^{n})(1-x^{2}q^{2n-1})(1-x^{-2}q^{2n-1})}{(1-xq^{n})(1-x^{-1}q^{n-1})(1-x^{2}q^{2n+1})(1-x^{-2}q^{2n-3})} \\
&= \frac{(1-x)(1-x^{2}q)}{(1-x^{-1})(1-x^{-2}q^{-1})} \\
&= x^{3}q
\end{aligned}
$$

This functional equation implies a relationship between the coefficients c_n. We write,

$$\sum_{n=-\infty}^{\infty} c_n x^n = x^3 q \sum_{n=-\infty}^{\infty} c_n q^n x^n.$$

Equating the coefficients of x^n on both sides gives the following recurrence relation,

$$c_n = c_{n-3} q^{n-2}.$$

Therefore, any coefficient c_n can be expressed in terms of one of the three coefficients c_0, c_1 or c_2. More precisely,

$$
\begin{aligned}
c_{3n} &= (q^{3n-2})(q^{3n-5})...(q)c_0 = q^{n(3n-1)/2}c_0, \\
c_{3n+1} &= (q^{3n-1})(q^{3n-4})...(q^2)c_1 = q^{n(3n+1)/2}c_1, \\
c_{3n+2} &= (q^{3n})(q^{3n-3})...(q^3)c_2 = q^{3n(n+1)/2}c_2, 3.2.2
\end{aligned} \tag{10}
$$

for $n > 0$. Now, when $n < 0$ we replace n with $3-n$ and write the above recurrence relation as

$$c_n = q^{-n-1}c_{n+3}.$$

Further,

$$
\begin{aligned}
c_{3n} &= (q^{-3n-1})(q^{-3n-4})...(q^2)c_0 = q^{n(3n-1)/2}c_0, \\
c_{3n-1} &= (q^{-3n-2})(q^{3n-5})...(q^1)c_1 = q^{n(3n+1)/2}c_1, \\
c_{3n-2} &= (q^{-3n-3})(q^{-3n-6})...(q^0)c_1 = q^{3n(n+1)/2}c_2,
\end{aligned}
$$

Then we see that all of the negative coefficients can be written in terms of one of the three coefficients c_0, c_1 or c_2. Thus (3.2.2) holds for all n and we obtain the series expansion for f,

$$f(x) = c_0 \sum_{n=-\infty}^{\infty} q^{(3n^2-n)/2} x^{3n} + c_1 \sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2} x^{3n+1} + c_2 \sum_{n=-\infty}^{\infty} q^{(3n^2+3n)} x^{3n+2}. \tag{3.2.3}$$

This completes the first part of the calculation.

Next, we derive the second recurrence relation between the coefficients of f, that will enable us to reduce the above sum of the three infinite series to just one infinite series. Observe that,

$$
\begin{aligned}
\frac{f(x)}{f(x^{-1})} &= \prod_{n=1}^{\infty} \frac{(1-xq^{n-1})(1-x^{-1}q^{n})(1-x^{2}q^{2n-1})(1-x^{-2}q^{2n-1})}{(1-x^{-1}q^{n-1})(1-xq^{n})(1-x^{-2}q^{2n-1})(1-x^{2}q^{2n-1})} \\
&= \frac{(1-x)}{(1-x^{-1})} = -x.
\end{aligned}
$$

Then,

$$\sum_{n=-\infty}^{\infty} c_n x^n = -x \sum_{n=-\infty}^{\infty} c_n x^{-n}.$$

From the above functional equation we get the second recurrence relation. Equating the coefficients of x^n on both sides,

$$c_n = -c_{1-n}$$

When $n = 0$, $c_0 = -c_1$. Also when $n = 2$, $c_2 = -c_{-1}$. However, from (3.2.2) we see that by letting $n = -1$ we get $c_{-1} = c_2$. Further, this implies that $c_2 = -c_2$ and therefore $c_2 = 0$. Now (3.2.3) reduces to,

$$f(x) = c_0(q){}_{n=-\infty}^{\infty} q^{n(3n-1)/2}(x^{3n} - x^{1-3n}). \tag{3.2.4}$$

Next, we symmetrize f by writing it as a difference of the two series.

$$f(x) = c_0(q)\left[{}_{n=-\infty}^{\infty} q^{n(3n-1)/2} x^{3n} - {}_{n=-\infty}^{\infty} q^{n(3n-1)/2} x^{1-3n}\right].$$

By replacing n with $n+1$ in the second sum and only rearranging the terms in the first sum we get,

$$f(x) = {}_{n=1}^{\infty} (1 - xq^{n-1})(1 - x^{-1}q^n)(1 - x^2 q^{2n-1})(1 - x^{-2} q^{2n-1})$$

$$= c_0(q)[{}_{n=-\infty}^{\infty} q^{n(3n+2)/2}\left(\sqrt{q}\right)^{-3n} x^{3n} - {}_{n=-\infty}^{\infty} q^{n(3n+2)/2}\left(\sqrt{q}\right)^{3n+2} x^{-(3n+2)}]$$

$$= c_0(q){}_{n=-\infty}^{\infty} q^{n(3n+2)/2}\left[\left(\frac{x}{\sqrt{q}}\right)^{3n} - \left(\frac{\sqrt{q}}{x}\right)^{3n+2}\right] \tag{3.2.5}$$

This symmetric form of $f(x)$ will be needed later. We will now employ two specializations in order to determine c_0.

3.2.2 First specialization

Let $x = -\omega$, where $\omega = \exp(2\pi i/3)$ is a primitive third root of unity. Now,

$$f(-\omega) = (1+\omega){}_{n=1}^{\infty}(1+\omega q^n)(1+\omega^2 q^n){}_{n=1}^{\infty}(1-\omega^2 q^{2n-1})(1-\omega q^{2n-1})$$

$$= (1+\omega){}_{n=1}^{\infty} \frac{(1+q^n)(1+\omega q^n)(1+\omega^2 q^n)}{(1+q^n)}$$

$$\times{}_{n=1}^{\infty} \frac{(1-q^{2n-1})(1-\omega^2 q^{2n-1})(1-\omega q^{2n-1})}{(1-q^{2n-1})}$$

$$= (1+\omega){}_{n=1}^{\infty} \frac{(1+q^{3n})(1-q^{6n-3})}{(1+q^n)(1-q^{2n-1})}$$

$$= 1+\omega, 3.2.6 \tag{11}$$

because,

$${}_{n=1}^{\infty}(1+q^n)(1-q^{2n-1}) = {}_{n=1}^{\infty} (1+q^n)(1-q^{2n-1})\frac{(1-q^n)}{(1-q^n)}$$

$$= {}_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{2n-1})}{(1-q^n)}$$

$$= {}_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^n)} = 1,$$

and similarly,

$$ {}_{n=1}^{\infty}(1+q^{3n})(1-q^{6n-3}) = 1. $$

Also using (3.2.4),

$$ f(-\omega) = (1+\omega)c_0(q)_{n=-\infty}^{\infty} q^{n(3n-1)/2}(-1)^n. \tag{3.2.7} $$

Replacing q with q^4 and combining (3.2.6) and (3.2.7),

$$ 1 = c_0(q^4)_{n=-\infty}^{\infty} q^{2n(3n-1)}(-1)^n \tag{3.2.8} $$

3.2.3 Second specialization

Now we let $x = i\sqrt{q}$ in (3.2.5). The product side is,

$$ \begin{aligned} f(i\sqrt{q}) &= {}_{n=1}^{\infty}(1-i\sqrt{q}q^{n-1})(1+i\sqrt{q}q^{n-1})(1+q^{2n})(1+q^{2n-2}) \\ &= 2_{n=1}^{\infty}(1+q^{2n-1})(1+q^{2n})^2 \\ &= 2_{n=1}^{\infty}(1+q^n)(1+q^{2n})\frac{(1-q^n)}{(1-q^n)} \\ &= 2_{n=1}^{\infty}\frac{(1-q^{4n})}{(1-q^n)}.3.2.9 \end{aligned} \tag{12} $$

The series side becomes,

$$ \begin{aligned} f(i\sqrt{q}) &= c_0(q)_{n=-\infty}^{\infty} q^{n(3n+2)/2}\left(i^{3n} - i^{-3n-2}\right) \\ &= 2c_0(q)_{n=-\infty}^{\infty} q^{n(3n+2)/2}\left(\frac{i^{-4n}i^{3n} + i^{4n}i^{-3n}}{2}\right) \\ &= 2c_0(q)_{n=-\infty}^{\infty} q^{n(3n+2)/2}\left(\frac{e^{i\pi n/2} + e^{-i\pi n/2}}{2}\right) \\ &= 2c_0(q)_{n=-\infty}^{\infty} q^{n(3n+2)/2}\cos\left(\frac{n\pi}{2}\right)3.2.10 \end{aligned} \tag{13} $$

Since $\cos(\frac{n\pi}{2}) = 0$ for n odd, we replace n with $-2m$ in the above series expression and equate it to (3.2.9),

$$ {}_{n=1}^{\infty}\frac{(1-q^{4n})}{(1-q^n)} = c_0(q)_{m=-\infty}^{\infty} q^{2m(3m-1)}(-1)^m. \tag{3.2.11} $$

3.2.4 Evaluating constant term

Observe that the series sides in (3.2.8) and (3.2.11) are identical. Therefore, dividing (3.2.11) by (3.2.8) gives,

$$_{n=1}^{\infty}\frac{(1-q^{4n})}{(1-q^{n})}=\frac{c_0(q)}{c_0(q^4)}.$$

Then we rearrange the terms and iterate,

$$c_0(q)={}_{n=1}^{\infty}\frac{(1-q^{4n})}{(1-q^{n})}c_0(q^4)$$
$$={}_{n=1}^{\infty}\frac{(1-q^{16n})}{(1-q^{n})}c_0(q^{16})$$
$$={}_{n=1}^{\infty}\frac{(1-q^{(2^{2k})n})}{(1-q^{n})}c_0(q^{2^{2k}})$$

Taking the limit as $k\to\infty$ and using the fact that $c_0(0)=1$,

$$c_0(q)={}_{n=1}^{\infty}(1-q^n)^{-1}.$$

This completes the proof.

3.3 Implications of the quintuple product identity

Since the motivation for the above proof came from Chan's [4] proof in this chapter we first show that the form used here and in Chan's paper are equivalent. Then we state two corollaries which are proved using the quintuple product identity.

Observe that if we replace x with $x^{-1}q$ and q with q^2 in Theorem 3.1 we get,

$$_{n=1}^{\infty}(1-q^{2n})(1-x^{-1}q^{2n-1})(1-xq^{2n-1})(1-x^{-2}q^{4n})(1-x^2q^{4n-4})$$
$$={}_{n=-\infty}^{\infty}q^{n(3n-1)}(x^{-3n}q^{3n}-x^{3n-1}q^{1-3n}).$$

Next we shift n on the product side so that it ranges between 0 and ∞, and also replace n with $-n$ on the sum side. Now,

$$_{n=0}^{\infty}(1-q^{2n+2})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})(1-x^2q^{4n})(1-x^{-2}q^{4n+4})$$
$$={}_{n=-\infty}^{\infty}q^{3n^2+n}(x^{3n}q^{-3n}-x^{-3n-1}q^{3n+1}).3.3.1 \qquad (14)$$

Thus we get the quintuple product identity form stated by Chan [4]

Corollary 3.2

$$_{n=1}^{\infty}\frac{(1-q^n)^5}{(1-q^{2n})^2}={}_{n=-\infty}^{\infty}q^{n(3n+1)/2}(6n+1).$$

Proof. We will start with the equation (3.2.1) and replace x with x^{-1} and then replace n with $-n$ on the series side to get,

$$_{n=1}^{\infty}(1-q^n)(1-x^{-1}q^{n-1})(1-xq^n)(1-x^2q^{2n-1})(1-x^{-2}q^{2n-1})$$

$$={}_{n=-\infty}^{\infty} q^{n(3n+1)/2}(x^{3n}-x^{-3n-1}).$$

First, we will rewrite the product side as,

$$(1-\frac{1}{x})_{n=1}^{\infty}(1-q^n)(1-x^{-1}q^n)(1-xq^n)(1-x^2q^{2n-1})\frac{(1-x^2q^{2n})}{(1-x^2q^{2n})}$$

$$\times_{n=1}^{\infty}(1-x^{-2}q^{2n-1})\frac{(1-x^{-2}q^{2n})}{(1-x^{-2}q^{2n})}$$

$$={}_{n=-\infty}^{\infty} q^{n(3n+1)/2}(x^{3n}-x^{-3n-1}).3.3.2 \tag{15}$$

Next, we divide both sides of (3.3.2) by $(x-1)$ and take the limit as $x\to 1$ to get,

$${}_{n=1}^{\infty}\frac{(1-q^n)^5}{(1-q^{2n})^2}=\lim_{x\to 1}\frac{1}{(x-1)}{}_{n=-\infty}^{\infty} q^{n(3n+1)/2}(x^{3n}-x^{-3n-1}).$$

On the series side we apply L'Hôpital's rule once to get,

$${}_{n=1}^{\infty}\frac{(1-q^n)^5}{(1-q^{2n})^2}={}_{n=-\infty}^{\infty} q^{n(3n+1)/2}(6n+1).$$

This completes the proof of the first corollary.

Corollary 3.3

$${}_{n=1}^{\infty}\frac{(1-q^{2n})^5}{(1-q^n)^2}={}_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n}(3n+1).$$

Proof. We will start by replacing x by x^{-1} in (3.3.1) and shifting the infinite product so that it starts at $n=1$. This gives,

$${}_{n=1}^{\infty}(1-q^{2n})(1-xq^{2n-1})(1-x^{-1}q^{2n-1})(1-x^{-2}q^{4n-4})(1-x^2q^{4n})$$

$$={}_{n=-\infty}^{\infty} q^{3n^2+n}(x^{-3n}q^{-3n}-x^{3n+1}q^{3n+1})$$

$$={}_{n=-\infty}^{\infty} q^{3n^2-2n}x^{-3n}-{}_{n=-\infty}^{\infty} q^{3n^2+4n+1}x^{3n+1}.$$

We have now split the series into two and will replace n with $n-1$ in the second series, and then combine them back together to get

$${}_{n=1}^{\infty}(1-q^{2n})(1-xq^{2n-1})(1-x^{-1}q^{2n-1})(1-x^{-2}q^{4n-4})(1-x^2q^{4n})$$

$$={}_{n=-\infty}^{\infty} q^{3n^2-2n}x^{-3n}-{}_{n=-\infty}^{\infty} q^{3n^2-2n}x^{3n-2}$$

$$={}_{n=-\infty}^{\infty} q^{3n^2-2n}(x^{-3n}-x^{3n-2}).$$

Replacing n with $-n$ in the series,

$${}_{n=1}^{\infty}(1-q^{2n})(1-xq^{2n-1})(1-x^{-1}q^{2n-1})(1-x^2q^{4n})(1-x^{-2}q^{4n-4})$$

$$={}_{n=-\infty}^{\infty} q^{3n^2+2n}(x^{3n}-x^{-3n-2}).$$

Next we rearrange the product,

$$(1-x^{-2})_{n=1}^{\infty}(1-q^{2n})(1-xq^{2n-1})(1-x^{-1}q^{2n-1})(1-x^2q^{4n})(1-x^{-2}q^{4n})$$

$$= (1-x^{-2})_{n=1}^{\infty}(1-q^{2n})(1-xq^{2n-1})(1-x^{-1}q^{2n-1})(1-xq^{2n})(1+xq^{2n})$$
$$\times(1-x^{-1}q^{2n})(1+x^{-1}q^{2n})$$

Further,

$$= (1-x^{-2})_{n=1}^{\infty}(1-q^{2n})(1-xq^{n})(1-x^{-1}q^{n})(1+xq^{2n})(1+x^{-1}q^{2n})$$
$$= (1-x^{-2})_{n=1}^{\infty}(1-q^{2n})(1-xq^{n})\frac{(1+xq^{n})}{(1+xq^{n})}$$
$$\times(1-x^{-1}q^{n})\frac{(1+x^{-1}q^{n})}{(1+x^{-1}q^{n})}(1+xq^{2n})(1+x^{-1}q^{2n})$$
$$= (1-x^{-2})_{n=1}^{\infty}\frac{(1-q^{2n})(1-x^{2}q^{n})(1-x^{-2}q^{n})(1+xq^{2n})(1+x^{-1}q^{2n})}{(1+xq^{2n})(1+x^{-1}q^{2n})}$$
$$=_{n=-\infty}^{\infty} q^{3n^2+2n}(x^{3n}-x^{-3n-2}).$$

This is the final form in which we divide both sides by $1-x^{-2}$ and let $x\to -1$. Applying L'Hôpital's rule once on the series side we get,

$$_{n=1}^{\infty} \frac{(1-q^{2n})^{5}}{(1-q^{n})^{2}} =_{n=-\infty}^{\infty} (-1)^{n} q^{3n^2+2n}(3n+1).$$

For further discussion on the two corollaries refer to [9].

4 Macdonald identity for A_2

4.1 Introduction

As it can be seen from the list in Chapter 1, A_2 is the identity associated with one of the irreducible root systems. It is the simplest two dimensional example of Macdonald's identity. It is also important in the sense that the result of this identity has been used to prove other identities. For example, Cooper [8] used A_2 identity to prove the Macdonald identity for G_2. For a proof different to the one presented in the next section refer to Cooper [5].

4.2 Proof of A_2 identity

In this chapter we will provide a complete proof of the Macdonald identity for A_2. The identity is stated in the following theorem.

Theorem 4.1 *For* $x_1, x_2, x_3 \neq 0$ *and* $|q| < 1$,

$$\prod_{n=1}^{\infty}(1-q^{n-1}\frac{x_1}{x_2})(1-q^{n}\frac{x_2}{x_1})(1-q^{n-1}\frac{x_1}{x_3})(1-q^{n}\frac{x_3}{x_1})(1-q^{n-1}\frac{x_2}{x_3})(1-q^{n}\frac{x_3}{x_2})(1-q^{n})^2$$

$$= \sum_{m_1+m_2+m_3=0} q^{\frac{3}{2}(m_1^2+m_2^2+m_3^2)+m_1+2m_2+3m_3} x_1^{3m_1} x_2^{3m_2} x_3^{3m_3} \prod_{1\le i\le j\le 3}(1-\frac{x_i q^{m_i}}{x_j q^{m_j}}).$$

Proof. Let,

$$f(x_1,x_2,x_3) = \prod_{n=1}^{\infty}(1-q^{n-1}\frac{x_1}{x_2})(1-q^{n}\frac{x_2}{x_1})(1-q^{n-1}\frac{x_1}{x_3})(1-q^{n}\frac{x_3}{x_1})(1-q^{n-1}\frac{x_2}{x_3})(1-q^{n}\frac{x_3}{x_2}). \tag{4.2.1}$$

Since $f(x_1,x_2,x_3)$ is analytic in each of x_1, x_2, x_3 in the region $0<|x_1|,|x_2|,|x_3|<\infty$ and hence it has a Laurent expansion of the form,

$$f(x_1,x_2,x_3) = \sum_{m_1+m_2+m_3=0} c(m_1,m_2,m_3) x_1^{m_1} x_2^{m_2} x_3^{m_3}.$$

Note, the condition $m_1+m_2+m_3=0$ arises because f is homogeneous of order 0, i.e., $f(\lambda x_1,\lambda x_2,\lambda x_3) = f(x_1,x_2,x_3)$, for all $\lambda \in \mathbb{C}\setminus\{0\}$, where $\mathbb{C}$ is the set of complex numbers.

4.2.1 Deriving recurrence relations

Observe that in the following calculation all the terms which do not involve x_1 cancel out and hence we get,

$$\frac{f(qx_1,x_2,x_3)}{f(x_1,x_2,x_3)} =_{n=1}^{\infty} \frac{(1-q^n\frac{x_1}{x_2})(1-q^{n-1}\frac{x_2}{x_1})(1-q^n\frac{x_1}{x_3})(1-q^{n-1}\frac{x_3}{x_1})}{(1-q^{n-1}\frac{x_1}{x_2})(1-q^n\frac{x_2}{x_1})(1-q^{n-1}\frac{x_1}{x_3})(1-q^n\frac{x_3}{x_1})}$$

$$= \frac{(1-\frac{x_2}{x_1})(1-\frac{x_3}{x_1})}{(1-\frac{x_1}{x_2})(1-\frac{x_1}{x_3})} = \frac{x_2x_3}{x_1^2}.$$

Similarly it can be derived that,

$$\frac{f(x_1,qx_2,x_3)}{f(x_1,x_2,x_3)} = \frac{x_1x_3}{qx_2^2}$$

and

$$\frac{f(x_1,x_2,qx_3)}{f(x_1,x_2,x_3)} = \frac{x_1x_2}{q^2x_3^2}.$$

By equating the coefficients of $x_1^{m_1}x_2^{m_2}x_3^{m_3}$ in the above three equations we get the following three recurrence relations respectively,

$$c(m_1,m_2,m_3) = q^{m_1-2}c(m_1-2,m_2+1,m_3+1)4.2.2 \tag{16}$$

$$c(m_1,m_2,m_3) = q^{m_2-1}c(m_1+1,m_2-2,m_3+1)4.2.3 \tag{17}$$

$$c(m_1,m_2,m_3) = q^{m_3}c(m_1+1,m_2+1,m_3-2)4.2.4 \tag{18}$$

Next we combine (4.2.2) and (4.2.3) to get,

$$q^{m_1-2}c(m_1-2,m_2+1,m_3+1) = q^{m_2-1}c(m_1+1,m_2-2,m_3+1).$$

We use substitutions $i_1 = m_1-2, i_2 = m_2+1$ and $i_3 = m_3+1$. Then the above equation becomes,

$$c(i_1,i_2,i_3) = q^{i_2-i_1-2}c(i_1+3,i_2-3,i_3). \tag{4.2.5}$$

Similarly if we combine (4.2.2) and (4.2.4) and use the same subtitutions we derive,

$$c(i_1,i_2,i_3) = q^{i_3-i_1-1}c(i_1+3,i_2,i_3-3). \tag{4.2.6}$$

When combining (4.2.3) and (4.2.4) we use a different substituions, $i_1 = m_1+1, i_2 = m_2-2$ and $i_3 = m_3+1,$ to get,

$$c(i_1,i_2,i_3) = q^{i_3-i_2-2}c(i_1,i_2+3,i_3-3). \tag{4.2.7}$$

We have now converted a set of the three recurrence relations into a new set where each relation manipulates only two of the variables and fixes the third one. Then let $i_1 = 3k_1+e_1,$ $i_2 = 3k_2+e_2$ and $i_3 = -i_1-i_2 = -3(k_1+k_2)-(e_1+e_2),$ where $0 \le e_1 \le 2, -1 \le e_2 \le 1$ and $-2 \le e_3 \le 0.$

(4.2.5) becomes,

$$c(3k_1+e_1,3k_2+e_2,i_3) = q^{3(k_1-k_2)+(e_1-e_2)-4}c(3(k_1-1)+e_1,3(k_2+1)+e_2,i_3).$$

Iterating this equation k_1 times we get,

$$c(3k_1+e_1,3k_2+e_2,i_3)=q^{-3k_1k_2+k_1(e_1-e_2-1)}c(e_1,3(k_1+k_2)+e_2,i_3). \tag{4.2.8}$$

Next we let $k^{'}=k_1+k_2$ and use equation (4.2.7). Then,

$$c(e_1,3k^{'}+e_2,-3k^{'}-e_1-e_2)=q^{6k^{'}+e_1+2e_2-4}c(e_1,3(k^{'}-1)+e_2,-3(k^{'}-1)-e_1-e_2).$$

Iterate this equation $k^{'}$ times to get,

$$c(e_1,3k^{'}+e_2,-3k^{'}-e_1-e_2)=q^{3k^{'}(k^{'}+1)+k^{'}e_1+2k^{'}e_2-4k^{'}}c(e_1,e_2,-e_1-e_2). \tag{4.2.9}$$

Combining (4.2.8) and (4.2.9) we get,

$$\begin{aligned}&c(3k_1+e_1,3k_2+e_2,-3(k_1+k_2)-e_1-e_2)\\&=q^{3k_1^2+3k_2^2+3k_1k_2-2k_1-k_2+2k_1e_1+k_1e_2+k_2e_1+2k_2e_2}c(e_1,e_2,-e_1-e_2).\end{aligned}$$

If we define e_3 and k_3 by $e_1+e_2+e_3=0$ and $k_1+k_2+k_3=0$, then the above equation simplifies to

$$\begin{aligned}&c(3k_1+e_1,3k_2+e_2,3k_3+e_3)\\&=q^{\frac{3}{2}(k_1^2+k_2^2+k_3^3)+k_1+2k_2+3k_3+k_1e_1+k_2e_2+k_3e_3}c(e_1,e_2,e_3).\end{aligned}$$

Now, we have reduced our task of determining infinitely many coefficients to determining only 9, namely $c(e_1,e_2,e_3)$, with $0\le e_1\le 2$, $-1\le e_2\le 1$ and $e_1+e_2+e_3=0$. We now consider two new functional equations and show that 6 out of 9 listed coefficient can all be written as $\pm c(0,0,0)$ and that other 3 are equal to zero. Observe that,

$$\frac{f(x_1,x_2,x_3)}{f(x_2,x_1,x_3)}=\frac{(1-x_1x_2^{-1})}{(1-x_1^{-1}x_2)}=-\frac{x_1}{x_2}$$

and

$$\frac{f(x_1,x_2,x_3)}{f(x_1,x_3,x_2)}=\frac{(1-x_2x_3^{-1})}{(1-x_2^{-1}x_3)}=-\frac{x_2}{x_3}.$$

These two functional equations respectively imply the following relations between the coefficients,

$$c(k_1,k_2,k_3)=-c(k_2+1,k_1-1,k_3) \tag{4.2.10}$$

and

$$c(k_1,k_2,k_3)=-c(k_1,k_3+1,k_2-1). \tag{4.2.11}$$

Next, starting with $c(0,0,0)$ we apply (4.2.10) and (4.2.11) alternatively to get,

$$c(0,0,0)=-c(1,-1,0)=c(1,1,-2)=-c(2,0,-2)=c(2,-1,-1)=c(0,1,-1).$$

On the other hand using (4.2.10),

$$\begin{aligned}&c(0,-1,1)=-c(0,-1,1)=0,\\&c(1,0,-1)=-c(1,0,-1)=0\end{aligned}$$

and

$$c(2,1,-3)=-c(2,1-3)=0.$$

Using these results about the coefficients we write,

$$
\begin{aligned}
f(x_1,x_2,x_3) &= c_0(q) \sum_{k_1+k_2+k_3=0} q^{\frac{3}{2}(k_1^2+k_2^2+k_3^2)+k_1+2k_2+3k_3} \\
&\times (x_1^{3k_1}x_2^{3k_2}x_3^{3k_3} - x_1^{3k_1+1}x_2^{3k_2-1}x_3^{3k_3}q^{k_1-k_2} \\
&+ x_1^{3k_1+1}x_2^{3k_2+1}x_3^{3k_3-2}q^{k_1+k_2-2k_3} - x_1^{3k_1+2}x_2^{3k_2}x_3^{3k_3-2}q^{2k_1-2k_3} \\
&+ x_1^{3k_1+2}x_2^{3k_2-1}x_3^{3k_3-1}q^{2k_1-k_2-k_3} - x_1^{3k_1}x_2^{3k_2+1}x_3^{3k_3-1}q^{k_2-k_3}) \\
&= c_0(q) \sum_{k_1+k_2+k_3=0} q^{\frac{3}{2}(k_1^2+k_2^2+k_3^2)+k_1+2k_2+3k_3} x_1^{3k_1}x_2^{3k_2}x_3^{3k_3} \\
&\times (1 - x_1x_2^{-1}q^{k_1-k_2} + x_1x_2x_3^{-2}q^{k_1+k_2-2k_3} - x_1^2x_3^{-2}q^{2k_1-2k_3} \\
&+ x_1^2x_2^{-1}x_3^{-1}q^{2k_1-k_2-k_3} - x_2x_3^{-1}q^{k_2-k_3}) \\
&= c_0(q) \sum_{k_1+k_2+k_3=0} q^{\frac{3}{2}(k_1^2+k_2^2+k_3^2)+k_1+2k_2+3k_3} x_1^{3k_1}x_2^{3k_2}x_3^{3k_3} \\
&\times \prod_{1\le i<j\le 3} \left(1 - \frac{x_i q^{k_i}}{x_j q^{k_j}}\right).
\end{aligned}
$$

Using the Weyl denominator formula and writing $k_i^{'}s$ as $m_i^{'}s$ (to coincide with Theorem 4.1),

$$
\begin{aligned}
f(x_1,x_2,x_3) &= c_0(q) \sum_{m_1+m_2+m_3=0} q^{\frac{3}{2}(m_1^2+m_2^2+m_3^2)+m_1+2m_2+3m_3} x_1^{3m_1}x_2^{3m_2}x_3^{3m_3} \\
&\times \sum_{\sigma\in S_3} sgn\sigma (x_1q^{m_1})^{\sigma_1-1}(x_2q^{m_2})^{\sigma_2-2}(x_3q^{m_3})^{\sigma_3-3}. 4.2.12
\end{aligned}
\tag{19}
$$

4.2.2 First specialization

Next we employ the first of the two specializations on (4.2.12).

Replace x_1, x_2 and x_3 with it, i^2t and i^3t respectively, where $i = \exp(\pi i/2)$, the fourth root of unity. Also, let $Q(\underline{m}) = \frac{3}{2}(m_1^2+m_2^2+m_3^2) + m_1 + 2m_2 + 3m_3$ then,

$$
\begin{aligned}
f(it,i^2t,i^3t) &= c_0(q) \sum_{m_1+m_2+m_3=0} q^{Q(\underline{m})} i^{3m_1+6m_2+9m_3} \\
&\times \sum_{\sigma\in S_3} sgn\sigma q^{m_1\sigma_1+m_2\sigma_2+m_3\sigma_3-(m_1+2m_2+3m_3)} i^{\sigma_1+2\sigma_2+3\sigma_3-14}.
\end{aligned}
$$

The group S_3 has 6 members, which represent the 6 permutations of the 3 elements. These permutations are: $(\sigma_1,\sigma_2,\sigma_3) = (1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$ and $(3,2,1)$. Below we substitute these permutations into f and write,

$$
f(it,i^2t,i^3t) = f_{(1,2,3)} - f_{(1,3,2)} - f_{(2,1,3)}
$$

$$+ f_{(2,3,1)} + f_{(3,1,2)} - f_{(3,2,1)},$$

where

$$f_{(1,2,3)} = c_0(q) \sum_{m_1+m_2+m_3=0} q^{Q(m_1,m_2,m_3)} i^{3m_1+6m_2+9m_3},$$

$$f_{(1,3,2)} = -c_0(q) \sum_{m_1+m_2+m_3=0} q^{Q(m_1,m_2,m_3)} i^{3m_1+9m_2+6m_3-1},$$

$$f_{(2,1,3)} = -c_0(q) \sum_{m_1+m_2+m_3=0} q^{Q(m_1,m_2,m_3)} i^{6m_1+3m_2+9m_3-1},$$

$$f_{(2,3,1)} = c_0(q) \sum_{m_1+m_2+m_3=0} q^{Q(m_1,m_2,m_3)} i^{9m_1+3m_2+6m_3-3},$$

$$f_{(3,1,2)} = c_0(q) \sum_{m_1+m_2+m_3=0} q^{Q(m_1,m_2,m_3)} i^{6m_1+9m_2+3m_3-3},$$

$$f_{(3,2,1)} = -c_0(q) \sum_{m_1+m_2+m_3=0} q^{Q(m_1,m_2,m_3)} i^{9m_1+6m_2+3m_3}.$$

Further simplifying $f(it, i^2t, i^3t)$ we get,

$$f(it, i^2t, i^3t) = c_0(q) \sum_{m_1+m_2+m_3=0} q^{Q(m_1,m_2,m_3)} H(m_1, m_2, m_3)$$

where,

$$H(m_1,m_2,m_3) = (i^{3m_1+6m_2+9m_3} - i^{3m_1+9m_2+6m_3-1} - i^{6m_1+3m_2+9m_3-1}$$
$$+ i^{9m_1+3m_2+6m_3-3} + i^{6m_1+9m_2+3m_3-3} - i^{9m_1+6m_2+3m_3})$$

Since, $m_3 = -m_1 - m_2$, we need to consider 16 different cases for m_1 and m_2. The 16 cases correspond to $m_1, m_2 = 0,1,2,3\,(mod\ 4)$ and are presented in the table below

$$|c|c|c|c|c|012304i0-4i01004i-4i200003-4i004i$$

These results can be verified by sustituting appropriate values for m_1 and m_2 into $H(m_1,m_2,m_3)$, where $m_3 = -m_1 - m_2$. It can be observed that $m_1 \equiv 1\,(mod\ 4), m_2 \equiv 2\,(mod\ 4)$ and $m_3 \equiv 3\,(mod\ 4)$ all separately imply that $H(m_1,m_2,m_3) = 0$. Also when $m_1 \equiv 3$ and $m_2 \equiv 0$ and $m_3 \equiv 1\,(mod\ 4), H(m_1,m_2,m_3) = 0$. Substituting other values for m_1 and m_2 gives $H(m_1,m_2,m_3) = \pm 4i$. We therefore use this to write,

$$f(it, i^2t, i^3t) = 4ic_0(q) \sum_{m_1+m_2+m_3=0} [q^{Q(4m_1,4m_2,4m_3)} - q^{Q(4m_1+2,4m_2,4m_3-2)}$$
$$+ q^{Q(4m_1+2,4m_2+1,4m_3-3)} - q^{Q(4m_1+3,4m_2+1,4m_3-4)}$$
$$- q^{Q(4m_1,4m_2+3,4m_3-3)} + q^{Q(4m_1+3,4m_2+3,4m_3-6)}] \quad (4.2.13)$$

Body Math We return to the product representation of $f(x_1, x_2, x_3)$ and use the same substitution as for the series. Note that when writing out the product we use the compact notation described in Section 1.3.

$$f(it, i^2t, i^3t) = (-i, qi, -1, -q, -i, qi; q)_\infty$$
$$= (1+i)^2(1+1)(-qi, qi; q)_\infty^2(-q; q)_\infty^2$$
$$= 4i(-q^2; q^2)_\infty^2(-q; q)_\infty^2 \, 4.2.14 \qquad (20)$$

Equating the series and product shown in (4.2.13) and (4.2.14) respectively,

$$(-q^2; q^2)_\infty^2(-q; q)_\infty^2 = c_0(q) \underset{m_1+m_2+m_3=0}{\qquad} [q^{Q(4m_1, 4m_2, 4m_3)} - q^{Q(4m_1+2, 4m_2, 4m_3-2)}$$
$$+ q^{Q(4m_1+2, 4m_2+1, 4m_3-3)} - q^{Q(4m_1+3, 4m_2+1, 4m_3-4)}$$
$$- q^{Q(4m_1, 4m_2+3, 4m_3-3)} + q^{Q(4m_1+3, 4m_2+3, 4m_3-6)}], \qquad (4.2.15)$$

where,

$$Q(m_1, m_2, m_3) = \frac{3}{2}(m_1^2 + m_2^2 + m_3^2) + m_1 + 2m_2 + 3m_3.$$

4.2.3 Second specialization

In the second specialization we replace q, x_1, x_2 and x_3 with q^{16}, tq^8, tq^4 and t respectively. Same as in the first substitution we let $Q(\underline{m}) = \frac{3}{2}(m_1^2 + m_2^2 + m_3^2) + m_1 + 2m_2 + 3m_3$. Then the product side becomes,

$$f(tq^8, tq^4, t) = (q^4, q^{12}, q^8, q^8, q^4, q^{12}; q^{16})_\infty$$
$$= \frac{(q^4, q^8, q^{12}; q^{16})_\infty^2(q^{16}; q^{16})_\infty^2}{(q^{16}; q^{16})_\infty^2} = \frac{(q^4; q^4)_\infty^2}{(q^{16}; q^{16})_\infty^2} . 4.2.16 \qquad (21)$$

With the above substitution the series side is,

$$f(tq^8, tq^4, t) = c_0(q^{16}) \underset{m_1+m_2+m_3=0}{\qquad} q^{24(m_1^2+m_2^2+m_3^2)+16m_1+32m_2+48m_3}$$
$$\times q^{24m_1+12m_2} \underset{\sigma\in S_3}{} sgn\sigma (q^{16m_1+8})^{\sigma_1-1}(q^{16m_2+4})^{\sigma_2-2}(q^{16m_3})^{\sigma_3-3}$$

We let $P(\underline{m}) = sgn\sigma (q^{16m_1+8})^{\sigma_1-1}(q^{16m_2+4})^{\sigma_2-2}(q^{16m_3})^{\sigma_3-3}$ and observe what happens when we substitute 6 elements of S_3 separately.

$$(\sigma_1, \sigma_2, \sigma_3) = (1,2,3) \Rightarrow P(\underline{m}) = q^{24m_1+12m_2}$$
$$(\sigma_1, \sigma_2, \sigma_3) = (1,3,2) \Rightarrow P(\underline{m}) = -q^{24m_1+28m_2-16m_3+4}$$
$$(\sigma_1, \sigma_2, \sigma_3) = (2,1,3) \Rightarrow P(\underline{m}) = -q^{40m_1-4m_2+4}$$
$$(\sigma_1, \sigma_2, \sigma_3) = (2,3,1) \Rightarrow P(\underline{m}) = q^{40m_1+28m_2-32m_3+12}$$
$$(\sigma_1, \sigma_2, \sigma_3) = (3,1,2) \Rightarrow P(\underline{m}) = q^{56m_1-4m_2-16m_3+12}$$
$$(\sigma_1, \sigma_2, \sigma_3) = (3,2,1) \Rightarrow P(\underline{m}) = -q^{56m_1+12m_2-32m_3+16}.$$

Combining the 6 expressions for $P(\underline{m})$ together and multiplying through by $q^{24m_1+12m_2}$ gives,

$$
\begin{aligned}
f(tq^8, tq^4, t) = c_0(q^{16}) & \mathop{}_{m_1+m_2+m_3=0} q^{24(m_1^2+m_2^2+m_3^2)}[q^{40m_1+44m_2+48m_3} \\
& - q^{40m_1+60m_2+32m_3+4} - q^{56m_1+28m_2+48m_3+4} + q^{56m_1+60m_2+16m_3+12} \\
& + q^{72m_1+28m_2+32m_3+12} - q^{72m_1+44m_2+16m_3+16}]
\end{aligned}
\tag{4.2.17}
$$

Combining the equations (4.2.16) and (4.2.17),

$$
\begin{aligned}
\frac{(q^4;q^4)_\infty^2}{(q^{16};q^{16})_\infty^2} = c_0(q^{16}) & {}_{m_1+m_2+m_3=0} q^{24(m_1^2+m_2^2+m_3^2)}[q^{40m_1+44m_2+48m_3} \\
& - q^{40m_1+60m_2+32m_3+4} - q^{56m_1+28m_2+48m_3+4} + q^{56m_1+60m_2+16m_3+12} \\
& + q^{72m_1+28m_2+32m_3+12} - q^{72m_1+44m_2+16m_3+16}]
\end{aligned}
\tag{4.2.18}
$$

Interestingly, we can write (4.2.18) as,

$$
\begin{aligned}
\frac{(q^4;q^4)_\infty^2}{(q^{16};q^{16})_\infty^2} = c_0(q^{16}) & {}_{m_1+m_2+m_3=0}[q^{Q(4k_1,4k_2,4k_3)} + q^{Q(4k_1+3,4k_2+3,4k_3-6)} \\
& - q^{Q(4k_1+3,4k_2+1,4k_3-4)} + q^{Q(4k_1+2,4k_2+1,4k_3-3)} \\
& - q^{Q(4k_1+2,4k_2,4k_3-2)} - q^{Q(4k_1,4k_2+3,4k_3-3)}]
\end{aligned}
\tag{4.2.19}
$$

4.2.4 Evaluating constant term

Observe that the series in (4.2.15) and (4.2.19) are equal. Therefore, we can divide the equation (4.2.15) by the equation (4.2.19) to get,

$$\frac{(-q^2;q^2)_\infty^2(-q;q)_\infty^2}{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^{-2}} = \frac{c_0(q)}{c_0(q^{16})}.$$

Then

$$
\begin{aligned}
c_0(q) &= \frac{(q^{16};q^{16})_\infty^2(-q^2;q^2)_\infty^2(-q;q)_\infty^2 \dfrac{(q;q)_\infty^2}{(q;q)_\infty^2}}{(q^4;q^4)_\infty^2} c_0(q^{16}) \\
&= \frac{(q^{16};q^{16})_\infty^2}{(q;q)_\infty^2} c_0(q^{16}).
\end{aligned}
$$

Iterating this equation,

$$c_0(q) = \frac{(q^{256};q^{256})_\infty^2}{(q;q)_\infty^2} c_0(q^{256}) = \frac{(q^{2^{4k}};q^{2^{4k}})_\infty^2}{(q;q)_\infty^2} c_0(q^{2^{4k}}) \text{for} k = 0,1,2...$$

Now, we take the limit as $k \to \infty$ and take into account that $c_0(0) = 1$ to get,

$$c_0(q) = (q;q)_\infty^{-2}.$$

Combining (4.2.1) with the above equation for c_0,

$$\left(\frac{x_1}{x_2}, \frac{qx_2}{x_1}, \frac{x_1}{x_3}, \frac{qx_3}{x_1}, \frac{x_2}{x_3}, \frac{qx_3}{x_2}, q, q; q\right)_\infty$$

$$=_{m_1+m_2+m_3=0} q^{\frac{3}{2}(m_1^2+m_2^2+m_3^2)+m_1+2m_2+3m_3} x_1^{3m_1} x_2^{3m_2} x_3^{3m_3} \times_{1\le i\le j\le 3} \left(1-\frac{x_i q^{m_i}}{x_j q^{m_j}}\right)$$

which completes the proof of theorem 4.1.

5 Winquist's identity

5.1 Introduction

This identity is due to Winquist [19], who discovered it in 1969. He used it to prove Ramanujan's partition congruence $p(11n+6) \equiv 0\,(mod 11)$, where $p(n)$ is the partition function which gives a number of ways of writing the integer as a sum of positive integers, where the order of addends is not considered significant.

Example 5.1 *Since* 4 *can be written as*

$$\begin{aligned} 4 &= 4 \\ &= 3+1 = 2+2 \\ &= 2+1+1 = 1+1+1+1 \end{aligned}$$

it follows that $p(4) = 5$.

Example 5.2 *In this example we look at the above mentioned Ramanujan's partition congruence when* $n = 0$*, i.e. we seek* $p(6)$. 6 *can be written as*

$$\begin{aligned} 6 &= 6 \\ &= 5+1 = 4+2 \\ &= 4+1+1 = 3+3 \\ &= 3+2+1 = 3+1+1 \\ &= 2+2+2 = 2+2+1+1 \\ &= 2+1+1+1+1 = 1+1+1+1+1+1. \end{aligned}$$

Therefore it follow that $p(6) = 11 \equiv 0\ (mod\ 11)$.

5.2 Proof of Winquist's identity

We state Winquist's identity in the following theorem.

Theorem 5.3 *For* $x, y \neq 0$ *and* $|q| < 1$,

$$\begin{aligned} &{}_{n=1}^{\infty}(1-q^n)^2(1-xq^{n-1})(1-x^{-1}q^n)(1-yq^{n-1})(1-y^{-1}q^n) \\ &\times(1-xyq^{n-1})(1-x^{-1}y^{-1}q^n)(1-xy^{-1}q^{n-1})(1-x^{-1}yq^n) \\ &=_{m\,n}\, q^{\frac{1}{2}(3m^2+3n^2-3m-n)} x^{3m} y^{3n} \\ &\times(1-xq^m)(1-yq^n)(1-xyq^{m+n})(1-xy^{-1}q^{m-n}), 5.2.1 \end{aligned} \tag{22}$$

where m and n on the series side range over all integers satisfying $m+n \equiv 0\ (mod\ 2)$.

Proof. Let

$$f(x,y) = {}_{n=1}^{\infty} (1-xq^{n-1})(1-x^{-1}q^{n})(1-yq^{n-1})(1-y^{-1}q^{n})$$
$$\times(1-xyq^{n-1})(1-x^{-1}y^{-1}q^{n})(1-xy^{-1}q^{n-1})(1-x^{-1}yq^{n}).$$
$$5.2.2 \qquad (23)$$

Then f has a Laurent series expansion

$$f(x,y) = {}_{m=-\infty}^{\infty}{}_{n=-\infty}^{\infty}\, c_{m,n}(q)x^{m}y^{n},$$

valid in the annulus $0<|x|,|y|<\infty$.

5.2.1 Deriving recurrence relations

As it can be seen f has been defined as a function of two variables. Similar to the proof of the Jacobi triple product identity we look at functional equations involving f but here we consider symmetry in both variable x and y. Below is the first set of functional equations.

$$\frac{f(x,y)}{f(xq,y)} = {}_{n=1}^{\infty} \frac{(1-xq^{n-1})(1-x^{-1}q^{n})(1-yq^{n-1})(1-y^{-1}q^{n})}{(1-xq^{n})(1-x^{-1}q^{n-1})(1-yq^{n-1})(1-y^{-1}q^{n})}$$
$$\times\frac{(1-xyq^{n-1})(1-x^{-1}y^{-1}q^{n})(1-xy^{-1}q^{n-1})(1-x^{-1}yq^{n})}{(1-xyq^{n})(1-x^{-1}y^{-1}q^{n-1})(1-xy^{-1}q^{n})(1-x^{-1}yq^{n-1})}$$
$$=\frac{(1-x)(1-xy)(1-xy^{-1})}{(1-x^{-1})(1-x^{-1}y^{-1})(1-x^{-1}y)}$$
$$=-x^{3}5.2.3 \qquad (24)$$

Similarly,

$$\frac{f(x,y)}{f(x,yq)} = -y^{3}q. \qquad (5.2.4)$$

These functional equations respectively imply the following two relations between the coefficients $c_{m,n}$. We write,

$${}_{m=-\infty}^{\infty}{}_{n=-\infty}^{\infty}c_{m,n}x^{m}y^{n} = -x^{3}{}_{m=-\infty}^{\infty}{}_{n=-\infty}^{\infty}c_{m,n}x^{m}y^{n}q^{m}$$

and

$${}_{m=-\infty}^{\infty}{}_{n=-\infty}^{\infty}c_{m,n}x^{m}y^{n} = -y^{3}q\,{}_{m=-\infty}^{\infty}{}_{n=-\infty}^{\infty}c_{m,n}x^{m}y^{n}q^{n}.$$

Equating the coefficients of x^{n} on both sides gives,

$$c_{m,n} = -c_{m-3,n}q^{m-3}$$

and

$$c_{m,n} = -c_{m,n-3}q^{n-2}.$$

Consequently, by combining these two relations we reduce the number of unknown coefficient to 9.

$$c_{3m+A,3n+B} = (-1)^{m+n}q^{3m(m-1)/2+3n(n-1)/2+n+mA+nB}c_{A,B}, \qquad (5.2.5)$$

where $0\le A,B\le 2$.

Let us consider the second set of functional equations that will further reduce our task of finding the 9 coefficient to just determining $c_{0,0}$.

$$\frac{f(x,y)}{f(y,x)} = {}_{n=1}^{\infty} \frac{(1-xq^{n-1})(1-x^{-1}q^n)(1-yq^{n-1})(1-y^{-1}q^n)}{(1-yq^{n-1})(1-y^{-1}q^n)(1-xq^{n-1})(1-x^{-1}q^n)}$$
$$\times \frac{(1-xyq^{n-1})(1-x^{-1}y^{-1}q^n)(1-xy^{-1}q^{n-1})(1-x^{-1}yq^n)}{(1-xyq^{n-1})(1-x^{-1}y^{-1}q^n)(1-x^{-1}yq^{n-1})(1-xy^{-1}q^n)}$$
$$= \frac{(1-xy^{-1})}{(1-x^{-1}y)}$$
$$= -xy^{-1}.$$

Also,

$$\frac{f(x,y)}{f(x,y^{-1})} = {}_{n=1}^{\infty} \frac{(1-xq^{n-1})(1-x^{-1}q^n)(1-yq^{n-1})(1-y^{-1}q^n)}{(1-xq^{n-1})(1-x^{-1}q^n)(1-y^{-1}q^{n-1})(1-yq^n)}$$
$$\times \frac{(1-xyq^{n-1})(1-x^{-1}y^{-1}q^n)(1-xy^{-1}q^{n-1})(1-x^{-1}yq^n)}{(1-xy^{-1}q^{n-1})(1-x^{-1}yq^n)(1-xyq^{n-1})(1-x^{-1}y^{-1}q^n)}$$
$$= \frac{(1-y)}{(1-y^{-1})}$$
$$= -y.$$

The two functional equations give,

$$ {}_{m=-\infty}^{\infty} {}_{n=-\infty}^{\infty} c_{m,n} x^m y^n = - {}_{m=-\infty}^{\infty} {}_{n=-\infty}^{\infty} c_{m,n} x^{n+1} y^{m-1} and$$
$$ {}_{m=-\infty}^{\infty} {}_{n=-\infty}^{\infty} c_{m,n} x^m y^n = - {}_{m=-\infty}^{\infty} {}_{n=-\infty}^{\infty} c_{m,n} x^m y^{1-n}.$$

Further we get,

$$c_{m,n} = -c_{n+1,m-1} \tag{5.2.6}$$

and

$$c_{m,n} = -c_{m,1-n}. \tag{5.2.7}$$

Next we obtain the orbit of the constant term $c_{0,0}$ by applying (5.2.6) and (5.2.7) alternatively. I.e.,

$$c_{0,0} = -c_{1,-1} = c_{1,2} = -c_{3,0} = c_{3,1} = -c_{2,2} = c_{2,-1} = -c_{0,1} = c_{0,0}.$$

This shows that 4 of the 9 coefficients sought by (5.2.5) are equal to $\pm c_{0,0}$. We also show the remaining 5 coefficient are equal to zero. Observe that by applying (5.2.7) and (5.2.6) alternatively on $c_{1,1}, c_{2,0}$ and $c_{0,2}$ we get,

$$c_{1,1} = -c_{1,0} = c_{1,0} = 0,$$
$$c_{2,0} = -c_{2,1} = c_{2,1} = 0 \text{and}$$
$$c_{0,2} = -c_{0,-1} = c_{0,-1} = 0.$$

Using the Laurent expansion we summarize,

$$f(x,y) = {}_{m=-\infty}^{\infty} {}_{n=-\infty}^{\infty} (c_{3m,3n} x^{3m} y^{3n} + c_{3m,3n+1} x^{3m} y^{3n+1}$$
$$+ c_{3m+1,3n+2} x^{3m+1} y^{3n+2} + c_{3m+2,3n+2} x^{3m+2} y^{3n+2}).$$

Since each of the 4 non-zero coefficients belongs to two orbits of constant terms we reduce the double sum to range over all integers satisfying $m+n \equiv 0\,(mod\ 2)$. We write this as,

$$f(x,y) = {}_{mn}\,(c_{3m,3n}x^{3m}y^{3n} + c_{3m,3n+1}x^{3m}y^{3n+1} + c_{3m+1,3n+2}x^{3m+1}y^{3n+2}$$
$$+ c_{3m+2,3n+2}x^{3m+2}y^{3n+2} + c_{3m+3,3n+1}x^{3m+3}y^{3n+1} + c_{3m+3,3n}x^{3m+3}y^{3n}$$
$$+ c_{3m+2,3n-1}x^{3m+2}y^{3n-1} + c_{3m+1,3n-1}x^{3m+1}y^{3n-1})$$
$$= c_{0,0}{}_{mn}\,q^{3m(m-1)/2+3n(n-1)/2+n}x^{3m}y^{3n}(1 - yq^{n} + xy^{2}q^{m+2n}$$
$$- x^{2}y^{2}q^{2m+2n} + x^{3}yq^{3m+n} - x^{3}q^{3m} + x^{2}y^{-1}q^{2m-n} - xy^{-1}q^{m-n}),$$

where $m+n \equiv 0\,(mod\ 2)$. Factorizing,

$$f(x,y) = c_{0,0}{}_{mn}\,q^{3m(m-1)/2+3n(n-1)/2+n}x^{3m}y^{3n}$$
$$\times (1 - xq^{m})(1 - yq^{n})(1 - xyq^{m+n})(1 - xy^{-1}q^{m-n}),5.2.8 \qquad (25)$$

where $m+n \equiv 0\,(mod\ 2)$.

5.2.2 Specialization

Let $x = -\omega^{j}$ and $y = -\omega^{k}$, where $\omega = \exp(2\pi i/3)$, a primitive third root of unity and sum $f(x,y)$ over $0 \le j,k \le 2$. Note that, $f(-\omega^{j},-\omega^{k}) = 0$, for $(j,k) \in \{(0,0),(1,1),(1,2),(2,1),(2,2)\}$. Therefore,

$${}^{2}_{j=0}{}^{2}_{k=0}\, f(-\omega^{j},-\omega^{k}) = f(-1,-\omega) + f(-1,-\omega^{2}) + f(-\omega,-1) + f(-\omega^{2},-1).$$

Using (5.2.2),

$${}^{2}_{j=0}{}^{2}_{k=0}\, f(-\omega^{j},-\omega^{k}) = {}^{\infty}_{n=1}\,(1+q^{n})^{2}(1+\omega q^{n})(1+\omega^{2}q^{n})(1-\omega q^{n})^{2}(1-\omega^{2}q^{n})^{2}$$
$$\times 2(1+\omega)(1-\omega)(1-\omega^{2}) + 2(1+\omega^{2})(1-\omega^{2})(1-\omega)$$
$$+ 2(1+\omega)(1-\omega)(1-\omega) + 2(1+\omega^{2})(1-\omega^{2})(1-\omega^{2})]$$
$$= 18{}^{\infty}_{n=1}(1+q^{n})(1+q^{3n})(1-q^{3n})^{2}(1-q^{n})^{-2}.5.2.9 \qquad (26)$$

Now, we will use the same substitution and summation on (5.2.8). This gives,

$${}^{2}_{j=0}{}^{2}_{k=0}\, f(-\omega^{j},-\omega^{k}) = c_{0,0}{}_{mn}\,q^{3m(m-1)/2+3n(n-1)/2+n}(-1)^{m+n}$$
$$\times {}^{2}_{j=0}{}^{2}_{k=0}\,(1 + \omega^{k}q^{n} - \omega^{j+2k}q^{m+2n} - \omega^{2j+2k}q^{2m+2n}$$
$$+ \omega^{3j+k}q^{3m+n} + q^{3m} - \omega^{2j-k}q^{2m-n} + \omega^{j-k}q^{m-n})$$
$$= 9c_{0,0}{}_{mn}\,q^{3m(m-1)/2+3n(n-1)/2+n}(1+q^{3m}),$$

where $m+n \equiv 0\,(mod\ 2)$. The above double sum can be split into two sums by multiplying it out by $(1+q^{3m})$,

$${}^{2}_{j=0}{}^{2}_{k=0}\, f(-\omega^{j},-\omega^{k}) = 9c_{0,0}{}_{mn}\,q^{3m(m-1)/2+3n(n-1)/2+n}$$
$$+ 9c_{0,0}{}_{mn}\,q^{3m(m-1)/2+3n(n-1)/2+n+3m}$$

where $m+n \equiv 0\,(mod\ 2)$ still holds. In the second sum we let $m \to m-1$ and make it equal to the first one, but with the condition that $m+n \equiv 1\,(mod\ 2)$. Now we combine the two sums again and now m and n are ranging over all integers.

$$\sum_{j=0}^{2}\sum_{k=0}^{2} f(-\omega^j,-\omega^k) = 9c_{0,0}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{3m(m-1)/2+3n(n-1)/2+n}$$
$$= 9c_{0,0}\sum_{m=-\infty}^{\infty} q^{3m(m-1)/2} \times \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}.$$

Using the Jacobi triple product identity (Theorem 2.1) twice, with (q, x) replaced with $(q^3,-1)$ and $(q^3,-q)$, we get

$$\sum_{j=0}^{2}\sum_{k=0}^{2} f(-\omega^j,-\omega^k) = 9c_{0,0}\prod_{n=1}^{\infty}(1+q^{3n-3})(1+q^{3n})(1-q^{3n})^2(1+q^{3n-2})(1+q^{3n-1})$$
$$= 18c_{0,0}\prod_{n=1}^{\infty}(1+q^n)(1+q^{3n})(1-q^{3n})^2.$$
$$5.2.10 \qquad (27)$$

5.2.3 Evaluating constant term

Next, we can combine (5.2.9) and (5.2.10) to determine $c_{0,0}$.

$$\prod_{n=1}^{\infty}(1+q^n)(1+q^{3n})(1-q^{3n})^2(1-q^n)^{-2} = c_{0,0}\prod_{n=1}^{\infty}(1+q^n)(1+q^{3n})(1-q^{3n})^2.$$

Therefore,

$$c_{0,0}(q) = \prod_{n=1}^{\infty}(1-q^n)^{-2}.$$

This completes the proof of Theorem 5.3.

5.3 Implications of Winquist's identity

Corollary 5.4

$$\prod_{n=1}^{\infty}(1-q^n)^{10} = -\frac{1}{2}\sum_{m+n\equiv 0\,(mod\ 2)} q^{\frac{1}{2}(3m^2+3n^2-3m-n)}$$
$$\times(2m-1)(6n-1)(3n+3m-2)(3n-3m+1).$$

Proof. We divide both sides of Winquist's identity by $(1-x)(1-y)(1-xy)(1-xy^{-1})$ and take the limit as $x, y \to 1$ to get,

$$\lim_{x,y\to 1}\frac{1}{(1-x)(1-y)(1-xy)(1-xy^{-1})}\prod_{n=1}^{\infty}(1-xq^{n-1})(1-x^{-1}q^n)(1-yq^{n-1})$$
$$\times(1-y^{-1}q^n)(1-xyq^{n-1})(1-x^{-1}y^{-1}q^n)(1-xy^{-1}q^{n-1})(1-x^{-1}yq^n)(1-q^n)^2$$
$$= \lim_{x,y\to 1}\frac{1}{(1-x)(1-y)(1-xy)(1-xy^{-1})}\sum_{m+n\equiv 0\,(mod\ 2)} q^{\frac{1}{2}(3m^2+3n^2-3m-n)}x^{3m}y^{3n}$$
$$\times(1-xq^m)(1-yq^n)(1-xyq^{m+n})(1-xy^{-1}q^{m-n}).$$

Taking the limit on the left hand side reduces the product to the desired form. On the right hand side we expand the double sum to get,

$$
{}_{n=1}^{\infty}(1-q^n)^{10} = \lim_{x,y\to 1}\frac{1}{(1-x)(1-y)(1-xy)(1-xy^{-1})}
$$

$$
\times \quad {}_{m+n=0\,(mod\ 2)} \quad q^{\frac{1}{2}(3m^2+3n^2-3m-n)}
$$

$$
\times(x^{3m}y^{3n}-x^{3m+1}y^{3n-1}q^{m-n}-x^{3m}y^{3n+1}q^{n}
$$

$$
+x^{3m+1}y^{3n+2}q^{m+2n}+x^{3m+2}y^{3n-1}q^{2m-n}-x^{3m+3}y^{3n}q^{3m}
$$

$$
-x^{3m+2}y^{3n+2}q^{2m+2n}+x^{3m+3}y^{3n+1}q^{3m+n}).
$$

In the 8 expanded sums, replace (m,n) with (m,n), (n,m), $(m,-n)$, $(n,-m)$, $(-n,m)$, $(-m,n)$, $(-n,-m)$, *and* $(-m,-n)$, respectively to get,

$$
{}_{n=1}^{\infty}(1-q^n)^{10} = \lim_{x,y\to 1}\frac{1}{(1-x)(1-y)(1-xy)(1-xy^{-1})}
$$

$$
\times \quad {}_{m+n=0\,(mod\ 2)} \quad q^{\frac{1}{2}(3m^2+3n^2-3m-n)}H_{m,n}(x,y),
$$

where

$$
H_{m,n}(x,y)=x^{3m}y^{3n}-x^{3m+1}y^{3n-1}q^{m-n}-x^{3m}y^{3n+1}q^{n}+x^{3m+1}y^{3n+2}q^{m+2n}
$$

$$
+x^{3m+2}y^{3n-1}q^{2m-n}-x^{3m+3}y^{3n}q^{3m}-x^{3m+2}y^{3n+2}q^{2m+2n}
$$

$$
+x^{3m+3}y^{3n+1}q^{3m+n}.
$$

L'Hôpital's rule was applied 4 times and using computer algebra package to calculate the limit, $\frac{\partial^4}{\partial x^3 \partial y}H_{m,n}(x,y)$, we get,

$$
{}_{n=1}^{\infty}(1-q^n)^{10} = \quad {}_{m+n=0\,(mod\ 2)} \quad q^{\frac{1}{2}(3m^2+3n^2-3m-n)}\frac{1}{6}\frac{\partial^4}{\partial x^3 \partial y}H_{m,n}(x,y)
$$

$$
=-\frac{1}{2} \quad {}_{m+n=0\,(mod\ 2)} \quad q^{\frac{1}{2}(3m^2+3n^2-3m-n)}
$$

$$
\times(2m-1)(6n-1)(3n+3m-2)(3n-3m+1).
$$

This completes the proof of the corollary.

6 Macdonald identity for G_2

6.1 Introduction

The Macdonald identities for the G_2 family of root systems can be generalized with the function,

$$f_{s,t}(x,y;q)=(x,q^s x^{-1},xy,q^s x^{-1}y^{-1},x^2y,q^s x^{-2}y^{-1};q^s)_\infty$$
$$\times(x^3y,q^t x^{-3}y^{-1},y,q^t y^{-1},x^3y^2,q^t x^{-3}y^{-2};q^t)_\infty,$$

where s and t are positive integers. We denote the Laurent expansion of $f_{s,t}$ by

$$f_{s,t}(x,y;q)={}_{mn}\ c_{s,t}(m,n;q)x^m y^m.$$

Note that we have switched to using a notation described in Section 1.3, as the product involved in the calculation is too long to write out in full.

We will assume $s=1$ and $t=1$ and reduce the above function $f_{s,t}$ to the two variable equivalent of the equation (1.4.5). A proof of the two variable case has been presented previously. Cooper [8] showed that coefficients of $x^m y^n$ in the expansion of $f_{s,t}$ can be obtained by multiplying two Macdonald's identities for A_2 together. Macdonald [14] and Stanton [16] prove the two variable case using the method described in Section 1.5. The method we use to determine the constant term is new.

In our proof we use substitutions $x=x_1^2x_2^{-1}x_3^{-1}$ and $y=x_2^3x_1^{-3}$ to convert $f_{s,t}$ into a function of three variables. This also gives a product with a high degree of symmetry.

$$f(x_1,x_2,x_3;q)$$
$$=\left(x_1^2x_2^{-1}x_3^{-1},qx_1^{-2}x_2x_3,x_1^{-1}x_2^2x_3^{-1},qx_1x_2^{-2}x_3,x_1x_2x_3^{-2},qx_1^{-1}x_2^{-1}x_3^2;q\right)_\infty$$
$$\times(x_1^3x_3^{-3},qx_1^{-3}x_3^3,x_1^{-3}x_2^3,qx_1^3x_2^{-3},x_2^3x_3^{-3},qx_2^{-3}x_3^3;q)_\infty,6.1.1 \tag{28}$$

with the Laurent expansion,

$$f(x_1,x_2,x_3;q)=\sum_{m_1+m_2+m_3=0}c(m_1,m_2,m_3;q)x_1^{m_1}x_2^{m_2}x_3^{m_3},$$

valid in the annulus $0<|x_1|,|x_2|,|x_3|<\infty$. Because of homogeneity of f we place $m_1+m_2+m_3=0$ restriction on the triple sum.

6.2 Proof of G_2 identity

We state the identity in the following theorem.

Theorem 6.1 *For* $x\neq 0$ *and* $|q|<1$,

$$\left(x_1^2x_2^{-1}x_3^{-1},qx_1^{-2}x_2x_3,x_1^{-1}x_2^2x_3^{-1},qx_1x_2^{-2}x_3,x_1x_2x_3^{-2},qx_1^{-1}x_2^{-1}x_3^2,q;q\right)_\infty$$

$$\times (x_1^3x_3^{-3}, qx_1^{-3}x_3^3, x_1^{-3}x_2^3, qx_1^3x_2^{-3}, x_2^3x_3^{-3}, qx_2^{-3}x_3^3, q;q)_\infty$$

$$= \sum_{m_1+m_2+m_3=0} q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2} x_1^{12m_1} x_2^{12m_2} x_3^{12m_3}$$

$$\times H(q^{m_1/3}x_1, q^{m_2/3}x_2, q^{m_3/3}x_3), 6.2.1 \tag{29}$$

where

$$H(x_1,x_2,x_3) = \left(1-\frac{x_1^2}{x_2x_3}\right)\left(1-\frac{x_2^2}{x_1x_3}\right)\left(1-\frac{x_1x_2}{x_3^2}\right)\left(1-\frac{x_1^3}{x_3^3}\right)\left(1-\frac{x_2^3}{x_1^3}\right)\left(1-\frac{x_2^3}{x_3^3}\right).$$

Proof. To prove this theorem we let $f(x_1,x_2,x_3)$ as in (6.1.1).

6.2.1 Deriving recurrence relations

As with the previous identities we start the proof by considering recurrence relations involving f. Observe that,

$$\frac{f(x_1, q^{\frac{1}{3}}x_2, q^{-\frac{1}{3}}x_3)}{f(x_1,x_2,x_3)}$$

$$= \frac{\left(x_1^2x_2^{-1}x_3^{-1}, qx_1^{-2}x_2x_3, qx_1^{-1}x_2^2x_3^{-1}, x_1x_2^{-2}x_3, qx_1x_2x_3^{-2}, x_1^{-1}x_2^{-1}x_3^2;q\right)_\infty}{\left(x_1^2x_2^{-1}x_3^{-1}, qx_1^{-2}x_2x_3, x_1^{-1}x_2^2x_3^{-1}, qx_1x_2^{-2}x_3, x_1x_2x_3^{-2}, qx_1^{-1}x_2^{-1}x_3^2;q\right)_\infty}$$

$$\times \frac{(qx_1^3x_3^{-3}, x_1^{-3}x_3^3, qx_1^{-3}x_2^3, x_1^3x_2^{-3}, q^2x_2^3x_3^{-3}, q^{-1}x_2^{-3}x_3^3;q)_\infty}{(x_1^3x_3^{-3}, qx_1^{-3}x_3^3, x_1^{-3}x_2^3, qx_1^3x_2^{-3}, x_2^3x_3^{-3}, qx_2^{-3}x_3^3;q)_\infty}$$

$$= \frac{(1-x_1x_2^{-2}x_3)(1-x_1^{-1}x_2^{-1}x_3^2)(1-x_1^{-3}x_3^3)(1-x_1^3x_2^{-3})}{(1-x_1^{-1}x_2^2x_3^{-1})(1-x_1x_2x_3^{-2})(1-x_1^3x_3^{-3})(1-x_1^{-3}x_2^3)}$$

$$\times \frac{(1-q^{-1}x_2^{-3}x_3^3)(1-x_2^{-3}x_3^3)}{(1-x_2^3x_3^{-3})(1-qx_2^3x_3^{-3})} = \frac{x_3^{12}}{qx_2^{12}}. 6.2.2 \tag{30}$$

Similarly we can show that,

$$\frac{f(q^{\frac{1}{3}}x_1, x_2, q^{-\frac{1}{3}}x_3)}{f(x_1,x_2,x_3)} = \frac{x_3^{12}}{q^2x_1^{12}} 6.2.3 \tag{31}$$

$$\frac{f(q^{-\frac{1}{3}}x_1, q^{\frac{1}{3}}x_2, x_3)}{f(x_1,x_2,x_3)} = \frac{x_1^{12}}{q^3x_2^{12}} 6.2.4 \tag{32}$$

From (6.2.2) we can obtain the following relation,

$$\sum_{m_1+m_2+m_3=0} c(m_1,m_2,m_3)x_1^{m_1}x_2^{m_2}x_3^{m_3}$$

$$= \frac{qx_2^{12}}{x_3^{12}} \sum_{m_1+m_2+m_3=0} c(m_1,m_2,m_3)x_1^{m_1}q^{\frac{m_2}{3}}x_2^{m_2}q^{-\frac{m_3}{3}}x_3^{m_3}$$

$$= \sum_{m_1+m_2+m_3=0} c(m_1,m_2,m_3)x_1^{m_1}x_2^{m_2+12}x_3^{m_3-12}q^{\frac{1}{3}(m_2-m_3)+1}.$$

Equating the powers of x on the two sides we get,

$$c(m_1,m_2,m_3) = q^{\frac{1}{3}(m_2-12-m_3-12)+1}c(m_1,m_2-12,m_3+12)$$

$$= q^{\frac{1}{3}(m_2-m_3)-7}c(m_1,m_2-12,m_3+12).$$

Similarly from (6.2.3) and (6.2.4) respectively we derive,

$$c(m_1,m_2,m_3) = q^{\frac{1}{3}(m_1-m_3)-6}c(m_1-12,m_2,m_3+12)$$

and

$$c(m_1,m_2,m_3) = q^{\frac{1}{3}(m_2-m_1)-5}c(m_1+12,m_2-12,m_3).$$

Now, from the product representation of $f(x_1,x_2,x_3)$ we see that only integer power of q should be present in the series expansion. Therefore, from the last recurrence relation we derive,

$$\frac{m_2-m_1}{3} = z \qquad where\ z \in \mathrm{Z}.$$

Further we conclude that,

$$m_2 - m_1 \equiv 0\ (mod\ 3).$$

Hence,

$$m_2 \equiv m_1\ (mod\ 3).$$

We will use the above three recurrence relations to show that every coefficient of the form $c(m_1,m_2,m_3)$ can be expressed in terms of $c(j_1,j_2,j_3)$ where, $0 \le j_1, j_2, j_3 \le 11$.

Initially we will use the first of the recurrence relations. We replace m_i with $12m_i + j_i$ for $i \in \{1,2,3\}$.

$$c(12m_1+j_1,12m_2+j_2,12m_3+j_3)$$

$$= q^{4(m_2-m_3)+\frac{1}{3}(j_2-j_3)-7}c(12m_1+j_1,12(m_2-1)+j_2,12(m_3+1)+j_3).$$

We can iterate the above equation. After the first iteration,

$$c(12m_1+j_1,12m_2+j_2,12m_3+j_3)$$

$$= q^{4(m_2-m_3)+\frac{1}{3}(j_2-j_3)-7}q^{4(m_2-1-m_3-1)+\frac{1}{3}(j_2-j_3)-7}$$

$$\times c(12m_1+j_1,12(m_2-2)+j_2,12(m_3+2)+j_3).$$

After m_2 iterations this becomes,

$$c(12m_1+j_1,12m_2+j_2,12m_3+j_3)$$

$$= q^{4(m_2-m_3)+4(m_2-m_3-2)+...+4(m_2-m_3)-2(m_2-1)+\frac{1}{3}m_2(j_2-j_3)-7m_2}$$

$$\times c(12m_1 + j_1, j_2, 12(m_3 + m_2) + j_3).$$

We can sum up the power of q to write,

$$c(12m_1 + j_1, 12m_2 + j_2, 12m_3 + j_3)$$
$$= q^{4m_2(m_2-m_3)-4m_2(m_2-1)+\frac{1}{3}m_2(j_2-j_3)-7m_2} c(12m_1 + j_1, j_2, 12(m_3 + m_2) + j_3).$$
$$6.2.5 \qquad (33)$$

Now, we will use the second recurrence relation in order to fix m_2 and to iterate m_1 and m_3. We let $m_1 \to 12m_1 + j_1$, $m_2 \to j_2$ and $m_3 \to 12(m_3 + m_2) + j_3$ where $m_1 + m_2 + m_3 = 0$. Then,

$$c(12m_1 + j_1, j_2, 12(m_3 + m_2) + j_3) = q^{4(m_1-m_3-m_2)+\frac{1}{3}(j_1-j_3)-6}$$
$$\times c(12(m_1 - 1) + j_1, j_2, 12(m_3 + m_2 + 1) + j_3.$$

Simplifying we get,

$$c(12m_1 + j_1, j_2, 12(m_3 + m_2) + j_3) = q^{8m_1+\frac{1}{3}(j_1-j_3)-6} c(12(m_1 - 1) + j_1, j_2, -12(m_1 - 1) + j_3.$$

After m_1 iterations,

$$c(12m_1 + j_1, j_2, 12(m_3 + m_2) + j_3)$$
$$= q^{8m_1+8(m_1-1)+\ldots 8(m_1-(m_1-1))+\frac{1}{3}m_1(j_1-j_3)-6m_1} c(j_1, j_2, j_3)$$
$$= q^{4m_1(m_1+1)+\frac{1}{3}m_1(j_1-j_3)-6m_1} c(j_1, j_2, j_3).6.2.6 \qquad (34)$$

Combining (6.2.5) and (6.2.6),

$$c(12m_1 + j_1, 12m_2 + j_2, 12m_3 + j_3)$$
$$= q^{4m_2(m_2-m_3)-4m_2(m_2-1)+\frac{1}{3}m_2(j_2-j_3)-7m_2+4m_1(m_1+1)+\frac{1}{3}m_1(j_1-j_3)-6m_1} c(j_1, j_2, j_3).$$

The power of q can be reduced to give,

$$c(12m_1 + j_1, 12m_2 + j_2, 12m_3 + j_3) = q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2+\frac{1}{3}(m_1j_1+m_2j_2+m_3j_3)}$$
$$\times c(j_1, j_2, j_3).6.2.7 \qquad (35)$$

Hence we have shown that every coefficient can be expressed in terms of one of the coefficients $c(j_1, j_2, j_3)$, where $0 \le j_1, j_2, j_3 \le 11$.

Now, we let S be the set of all such coefficients which are non-zero, i.e.

$$S = \{(0,0,0), (-3,3,0), (-4,5,-1), (-4,8,-4), (-3,9,-6), (0,9,-9),$$
$$(2,-1,-1), (5,-1,-4), (6,0,-6), (6,3,-9), (5,5,-10), (2,8,-10)\}$$

The set S can be derived in similar way used at the end of Section 4.2.1. Elements of the set correspond to the powers of x_1, x_2 and x_3 in the expansion of $H(x_1, x_2, x_3)$. Using (6.2.7) and summing over S, the Laurent expansion for f becomes,

$$f(x_1, x_2, x_3) = \sum_{m_1+m_2+m_3=0} c(m_1, m_2, m_3) x_1^{m_1} x_2^{m_2} x_3^{m_3}$$

$$
\begin{aligned}
&= \sum_{m_1+m_2+m_3=0} \sum_{(j_1,j_2,j_3)\in S} c(12m_1+j_1, 12m_2+j_2, 12m_3+j_3) \\
&\times x_1^{12m_1+j_1} x_2^{12m_2+j_2} x_3^{12m_3+j_3} \\
&= \sum_{m_1+m_2+m_3=0} \sum_{(j_1,j_2,j_3)\in S} c(j_1, j_2, j_3) \\
&\times q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2+\frac{1}{3}(m_1 j_1+m_2 j_2+m_3 j_3)} x_1^{12m_1+j_1} x_2^{12m_2+j_2} x_3^{12m_3+j_3} \\
&= \sum_{m_1+m_2+m_3=0} q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2} x_1^{12m_1} x_2^{12m_2} x_3^{12m_3} \\
&\times \sum_{(j_1,j_2,j_3)\in S} c(j_1, j_2, j_3) q^{\frac{1}{3}(m_1 j_1+m_2 j_2+m_3 j_3)} x_1^{j_1} x_2^{j_2} x_3^{j_3} \\
&= \sum_{m_1+m_2+m_3=0} q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2} x_1^{12m_1} x_2^{12m_2} x_3^{12m_3} \\
&\times \sum_{(j_1,j_2,j_3)\in S} c(j_1, j_2, j_3) (q^{\frac{m_1}{3}} x_1)^{j_1} (q^{\frac{m_2}{3}} x_2)^{j_2} (q^{\frac{m_3}{3}} x_3)^{j_3} \\
&= c_0(q) \sum_{m_1+m_2+m_3=0} q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2} x_1^{12m_1} x_2^{12m_2} x_3^{12m_3} \\
&\times H(q^{\frac{m_1}{3}} x_1, q^{\frac{m_2}{3}} x_2, q^{\frac{m_3}{3}} x_3)
\end{aligned}
$$

where,

$$
H(x_1, x_2, x_3) = \left(1-\frac{x_1^2}{x_2 x_3}\right)\left(1-\frac{x_2^2}{x_1 x_3}\right)\left(1-\frac{x_1 x_2}{x_3^2}\right)\left(1-\frac{x_1^3}{x_3^3}\right)\left(1-\frac{x_2^3}{x_1^3}\right)\left(1-\frac{x_2^3}{x_3^3}\right)
$$

Therefore, our problem has been reduced to determining c_0.

6.2.2 Specialization

This proof for G_2 is similar to the method used to prove Winquist's identity in the sense that we employ only one specialization on both sum and product representations of f. Let $x_1 = 1, x_2 = -1$ and $x_3 = \mu^j$, where $\mu = \exp(\pi i/6)$, the twelwth root of unity. After the substitution we sum $f(x_1, x_2, x_3)$ over j for $0 \le j < 12$.

Firstly, we will look at what happens to the series side under this substitution. Observe that,

$$
(1)^{12m_1}(-1)^{12m_2}\exp(2\pi i j/12)^{12m_3} = 1.
$$

Then we get,

$$
\begin{aligned}
\sum_{j=0}^{11} f(1,-1,\mu^j) &= \sum_{j=0}^{11} c_0(q) \sum_{m_1+m_2+m_3=0} q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2} \\
&\times H(q^{m_1/3}, -q^{m_2/3}, q^{m_3/3}\mu^j).
\end{aligned}
$$

Expanding H we get that,

$$H(q^{m_1/3},-q^{m_2/3},q^{m_3/3}\mu^j)=1+\frac{q^{m_2}}{q^{m_1}}-\frac{q^{5m_2/3}}{q^{4m_1/3}q^{m_3/3}\mu^j}-\frac{q^{8m_2/3}}{q^{4m_1/3}q^{4m_3/3}\mu^{4j}}$$
$$-\frac{q^{3m_2}}{q^{2m_3}q^{m_1}\mu^{6j}}+\frac{q^{3m_2}}{q^{3m_3}\mu^{9j}}+\frac{q^{2m_1/3}}{q^{m_2/3}q^{m_3/3}\mu^j}-\frac{q^{5m_1/3}}{q^{m_2/3}q^{4m_3/3}\mu^{4j}}$$
$$-\frac{q^{2m_1}}{q^{2m_3}\mu^{6j}}-\frac{q^{2m_1}q^{m_2}}{q^{3m_3}\mu^{9j}}+\frac{q^{5m_1/3}q^{5m_2/3}}{q^{10m_3/3}\mu^{10j}}+\frac{q^{2m_1/3}q^{8m_2/3}}{q^{10m_3/3}\mu^{10j}}$$

It can be checked that,

$$\sum_{j=0}^{11}\frac{1}{\mu^{Aj}}=0\text{unless}A\equiv 0\ (mod 12). \tag{6.2.8}$$

Using this we can see that all but the first two terms of H sum up to zero. Therefore, we compute that,

$$\sum_{j=0}^{11}H(q^{m_1/3},-q^{m_2/3},q^{m_3/3}\mu^j)=12(1+q^{m_2-m_1}).$$

Now, the Laurent series for $f(x_1,x_2,x_3)$ is,

$$\sum_{j=0}^{11}f(1,-1,\mu^j)=12c_0(q)\sum_{m_1+m_2+m_3=0}q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2}(1+q^{m_2-m_1}).$$
$$=12c_0(q)\sum_{m_1+m_2+m_3=0}q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2}$$
$$+12c_0(q)\sum_{m_1+m_2+m_3=0}q^{2(m_1^2+m_2^2+m_3^2)-3m_1-2m_2}.$$

By interchanging m_1 and m_2 in the first series the two series become equal,

$$\sum_{j=0}^{11}f(1,-1,\mu^j)=24c_0(q)\sum_{m_1+m_2+m_3=0}q^{2(m_1^2+m_2^2+m_3^2)-3m_1-2m_2}.$$

Since $m_3=-m_1-m_2$, we rewrite this result as a product of the three infinite separate sums, where we are only interested in the coefficient of x^0 and we emphasise this by including $[x^0]$ in our notation, i.e. if $f(x)=\sum_{n=-\infty}^{\infty}c_nx^n$, then $[x^n]f(x)=c_n$.

$$\sum_{j=0}^{11}f(1,-1,\mu^j)=24c_0(q)[x^0]\sum_{m_1=-\infty}^{\infty}q^{2m_1^2-2m_1}x^{m_1}\sum_{m_2=-\infty}^{\infty}q^{2m_2^2-m_2}x^{m_2}$$
$$\times\sum_{m_3=-\infty}^{\infty}q^{2m_3^2+m_3}x^{m_3}.$$

Using the Jacobi triple product identity we write the three sums as three infinite products,

$$\sum_{j=0}^{11}f(1,-1,\mu^j)=24c_0(q)[x^0]\left(-x,-\frac{q^4}{x},q^4;q^4\right)_\infty\left(-xq,-\frac{q^3}{x},q^4;q^4\right)_\infty$$
$$\times\left(-xq^3,-\frac{q}{x},q^4;q^4\right)_\infty$$

$$= 24c_0(q)\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}[x^0]\left(-x,-\frac{q^4}{x},q^4;q^4\right)_\infty\left(-qx,-\frac{q}{x},q^2;q^2\right)_\infty.$$

Once again we will use the Jacobi triple product identity, but this time to convert the two infinite products involving x into their sum equivalences.

$${}_{j=0}^{11} f(x_1,x_2,x_3) = 24c_0(q)\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}[x^0]{}_{m=-\infty}^{\infty} q^{2m^2-2m}x^m{}_{n=-\infty}^{\infty}q^{n^2}x^n.$$

In order to pull out only the coefficient of x^0 we need to have $m+n=0$. In other words, we can write $n=-m$ and combine the two sums.

$${}_{j=0}^{11} f(x_1,x_2,x_3) = 24c_0(q)\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}{}_{m+n=0} q^{3n^2+2n}.$$

By the Jacobi triple product identity,

$${}_{j=0}^{11} f(x_1,x_2,x_3) = 24c_0(q)\frac{(q^4;q^4)_\infty^2\left(-q;q^2\right)_\infty(q^6;q^6)_\infty}{(q^2;q^2)_\infty(-q^3;q^6)_\infty}$$

Secondly, we can turn our attention to the product representation of $f(x_1,x_2,x_3)$. We will use the same substitution and summation as for the Laurent expansion. Before we continue with the substitution we rewrite f as,

$$f(x_1,x_2,x_3) = \left(\frac{x_1^2}{x_2x_3},\frac{qx_2x_3}{x_1^2},\frac{x_2^2}{x_1x_3},\frac{qx_1x_3}{x_2^2},\frac{x_1x_2}{x_3^2},\frac{qx_3^2}{x_1x_2};q\right)_\infty$$

$$\times\left(\frac{x_1^3}{x_3^3},\frac{qx_3^3}{x_1^3},\frac{x_2^3}{x_1^3},\frac{qx_1^3}{x_2^3},\frac{x_2^3}{x_3^3},\frac{qx_3^3}{x_2^3};q\right)_\infty.$$

$$= H(x_1,x_2,x_3)\left(\frac{qx_1^2}{x_2x_3},\frac{qx_2x_3}{x_1^2},\frac{qx_2^2}{x_1x_3},\frac{qx_1x_3}{x_2^2},\frac{qx_1x_2}{x_3^2},\frac{qx_3^2}{x_1x_2};q\right)_\infty$$

$$\times\left(\frac{qx_1^3}{x_3^3},\frac{qx_3^3}{x_1^3},\frac{qx_2^3}{x_1^3},\frac{qx_1^3}{x_2^3},\frac{qx_2^3}{x_3^3},\frac{qx_3^3}{x_2^3};q\right)_\infty,$$

where $H(x_1,x_2,x_3)$ was defined in Section 6.2.1. Then we let $x_1=1, x_2=-1$ and $x_3=\mu^j$ and observe how $f(1,-1,\mu^j)$ behaves for different values of $0\le j<11$.

$$f(1,-1,\mu^j) = H(1,-1,\mu^j)$$

$$\times(q\mu^{6-j},q\mu^{6+j},q\mu^{12-j},q\mu^j,q\mu^{6-2j},q\mu^{6+2j},q\mu^{-3j},q\mu^{3j},-q,-q,q\mu^{6-3j},q\mu^{6+3j};q)_\infty$$

It can be checked by hand or by computer that $H(1,-1,\mu^j)$ is non-zero only for $j=1,5,7,11.$ Therefore,

$${}_{j=0}^{11} f(1,-1,\mu^j) = \left[H(1,-1,\mu)+H(1,-1,\mu^5)+H(1,-1,\mu^7)+H(1,-1,\mu^{11})\right]$$

$$\times(q\mu,q\mu^3,q\mu^3,q\mu^4,q\mu^5,q\mu^6,q\mu^6,q\mu^7,q\mu^8,q\mu^9,q\mu^9,q\mu^{11};q)_\infty.$$

We simplify the infinite product and since the missing eight terms are all zero we also sum $H(1,-1,\mu^j)$ over all $j's,$ to get,

$$\sum_{j=0}^{11} f(1,-1,\mu^j) = \frac{(q^{12};q^{12})_\infty (q^4;q^4)_\infty (q^3;q^3)_\infty (-q;q)_\infty}{(q^6;q^6)_\infty (q;q)_\infty^2} \sum_{j=0}^{11} H(1,-1,\mu^j)$$

Using (6.2.8) we can deduce that the sum on the right-hand side is equal to 24. Hence,

$$\sum_{j=0}^{11} f(1,-1,\mu^j) = 24\frac{(q^{12};q^{12})_\infty (q^4;q^4)_\infty (q^3;q^3)_\infty (-q;q)_\infty}{(q^6;q^6)_\infty (q;q)_\infty^2}$$

6.2.3 Evaluating constant term

One thing left to do is equate the product and the series substitutions to get,

$$c_0(q)\frac{(q^4;q^4)_\infty^2 \left(-q;q^2\right)_\infty (q^6;q^6)_\infty}{(q^2;q^2)_\infty (-q^3;q^6)_\infty} = \frac{(q^{12};q^{12})_\infty (q^4;q^4)_\infty (q^3;q^3)_\infty (-q;q)_\infty}{(q^6;q^6)_\infty (q;q)_\infty^2}.$$

Further,

$$\begin{aligned}
c_0(q) &= \frac{1}{(q;q)_\infty^2}\frac{(q^{12};q^{12})_\infty (q^3;q^3)_\infty (-q;q)_\infty (q^2;q^2)_\infty (-q^3;q^6)_\infty}{(q^6;q^6)_\infty (q^4;q^4)_\infty \left(-q;q^2\right)_\infty (q^6;q^6)_\infty}. \\
&= \frac{1}{(q;q)_\infty^2}\frac{(q^{12};q^{12})_\infty (q^3;q^3)_\infty (-q;q^2)_\infty (q^4;q^4)_\infty (-q^3;q^6)_\infty}{(q^3;q^3)_\infty (-q^3;q^3)_\infty (q^4;q^4)_\infty (-q;q^2)_\infty (q^6;q^6)_\infty} \\
&= \frac{1}{(q;q)_\infty^2}\frac{(q^{12};q^{12})_\infty (-q^3;q^6)_\infty}{(-q^3;q^3)_\infty (q^6;q^6)_\infty} \\
&= \frac{1}{(q;q)_\infty^2}.
\end{aligned}$$

Summarizing the result,

$$\begin{aligned}
&\left(\frac{x_1^2}{x_2x_3}, \frac{qx_2x_3}{x_1^2}, \frac{x_2^2}{x_1x_3}, \frac{qx_1x_3}{x_2^2}, \frac{x_1x_2}{x_3^2}, \frac{qx_3^2}{x_1x_2}, \frac{x_1^3}{x_3^3}, \frac{qx_3^3}{x_1^3}, \frac{x_2^3}{x_1^3}, \frac{qx_1^3}{x_2^3}, \frac{x_2^3}{x_3^3}, \frac{qx_3^3}{x_2^3}, q, q; q\right)_\infty \\
&= \sum_{m_1+m_2+m_3=0} q^{2(m_1^2+m_2^2+m_3^2)-2m_1-3m_2} x_1^{12m_1} x_2^{12m_2} x_3^{12m_3} H(q^{m_1/3}x_1, q^{m_2/3}x_2, q^{m_3/3}x_3),
\end{aligned}$$

where

$$H(x_1,x_2,x_3) = \left(1-\frac{x_1^2}{x_2x_3}\right)\left(1-\frac{x_2^2}{x_1x_3}\right)\left(1-\frac{x_1x_2}{x_3^2}\right)\left(1-\frac{x_1^3}{x_3^3}\right)\left(1-\frac{x_2^3}{x_1^3}\right)\left(1-\frac{x_2^3}{x_3^3}\right).$$

This completes the proof of Theorem 6.1.

7 Conclusion

In this thesis several of the Macdonald identities were studied and proved.

We supplied the details for Macdonald's proof of A_2 as the one provided by Macdonald in [14] can be somewhat difficult to follow. Also a proof of Winquist's identity was provided in the thesis. This identity was proved in a similar way to A_2 identity. The only fundamental difference in the proof of Winquist's identity was that we employed only one specialization and summed such specialization over some integers. The proof of the Macdonald identity for G_2 in Section 6.2 follows the same procedure as that of Winquist's identity. In this proof for the Macdonald identity for G_2, a method used to evaluate the constant term is new.

Following the four steps described in Section 1.5 we gave a new proof for the quintuple product identity, in Section 3.2.

Chapters containing the Jacobi triple product identity, the quintuple product identity and Winquist's identity contain some further implications of the identities. We showed how to derive several interesting formulas in number theory using these identities.

Further work in this area could involve similar analysis for the exceptional root systems: F_4, E_6, E_7 and E_8. These have not been studied in detail but are known to also give way to some beautiful identities in number theory.

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