Mathematics of Cell Growth

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Abstract

We present a model that describes growth, division and death of cells structured by size. Here, size can be interpreted as DNA content or physical size. The model is an extension of that studied by Hall and Wake [24] and incorporates the symmetric as well as the asymmetric division of cells.

We first consider the case of symmetric cell division. This leads to an initial boundary value problem that involves a first-order linear PDE with a functional term. We study the separable solution to this problem which plays an important role in the long term behaviour of solutions. We also derive a solution to the problem for arbitrary initial cell distributions. The method employed exploits the hyperbolic character of the underlying differential operator, and the advanced nature of the functional argument to reduce the problem to a sequence of simple Cauchy problems. The existence of solutions for arbitrary initial distributions is established along with uniqueness. The asymptotic relationship with the separable solution is established, and because the solution is known explicitly, higher order terms in the asymptotics can be obtained. Adding variability to the growth rate of cells leads to a modified Fokker-Planck equation with a functional term. We find the steady size distribution solution to this equation. We also obtain a constructive existence and uniqueness theorem for this equation with an arbitrary initial size-distribution and with a no-flux condition.

We then proceed to study the binary asymmetric division of cells. This leads to an initial boundary value problem that involves a first-order linear PDE with two functional terms. We find and prove the unimodality of the steady size distribution solution to this equation. The existence of higher eigenfunctions is also discussed. Adding stochasticity to the growth rate of cells yields a second-order functional differential equation with two non-local terms.

These problems, being a particular kind of functional differential equations exhibit unusual characteristics. Although the associated boundary value problems are well-posed, the spectral problems that arise by separating the variables, cannot be easily shown to have a complete set of eigenfunctions or the usual orthogonality properties.

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Published work

The following publications are based on the research work done during the course of this thesis:

1. **Ali A. Zaidi**, B. van-Brunt, G. C. Wake, A model for asymmetrical cell division, *Mathematical Biosciences and Engineering*, accepted November 2014.

2. Graeme Wake, **Ali A. Zaidi** and Bruce van-Brunt, Tumour Cell Biology and some New Non-local Calculus. *The Impact of Applications on Mathematics -Proceedings of Forum "Math-for-Industry" 2013*, Springer 2014.

3. **Ali A. Zaidi**, B. van-Brunt, G. C. Wake, Solutions to an advanced functional partial differential equation of the pantograph-type, submitted December 2014.

Oral Presentations

The following oral presentations and talks were given, based on the research work done during the course of this thesis:

1. Presented and won **first** prize for my talk on "Solutions to an advanced functional partial differential equation of the pantograph-type" at the second INMS Postgradaute Student Conference 2014.

2. Presented and won **second** prize for my talk on "A size structured cell growth model" at the first INMS Postgradaute Student Conference 2013.

3. Presented a talk on "Asymmetric cell division arising in stem cells and cancer" at the ANZIAM Conference 2013.

4. Presented a talk on "Solutions to an advanced functional partial differential equation of the pantograph-type" at the NZ Math and Stat Postgraduate Student Conference 2014.

5. Presented a talk on "A size structured cell growth model" at the NZ Math Colloquium 2013.

6. Presented a talk on "A size structured cell growth model" at the NZ Math and Stat Postgraduate Student Conference 2013.

7. Presented a talk on "Multiple delay differential equations and a nonlinear eigenvalue problem" at the NZ Math Colloquium 2012.

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