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BACKWARD BIFURCATION IN SIR ENDEMIC MODELS

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Abstract

In the well known SIR endemic model, the infection-free steady state is globally stable for $R_0 < 1$ and unstable for $R_0 > 1$. Hence, we have a forward bifurcation when $R_0 = 1$. When $R_0 > 1$, an asymptotically stable endemic steady state exists. The basic reproduction number $R_0$ is the main threshold bifurcation parameter used to determine the stability of steady states of SIR endemic models.

In this thesis we study extensions of the SIR endemic model for which a backward bifurcation may occur at $R_0 = 1$. We investigate the biologically reasonable conditions for the change of stability. We also analyse the impact of different factors that lead to a backward bifurcation both numerically and analytically. A backward bifurcation leads to sub-critical endemic steady states and hysteresis.

We also provide a general classification of such models, using a small amplitude expansion near the bifurcation. Additionally, we present a procedure for projecting three dimensional models onto two dimensional models by applying some linear algebraic techniques. The four extensions examined are: the SIR model with a susceptible recovered class; nonlinear transmission; exogenous infection; and with a carrier class.

Numerous writers have mentioned that a nonlinear transmission function in relation to the infective class, can only lead to a system with an unstable endemic steady state. In spite of this we show that in a nonlinear transmission model, we have a function depending on the infectives and satisfying certain biological conditions, and leading to a sub-critical endemic equilibriums.
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## Contents

1 **Introduction**  
1.1 The Basic *SIR* Model ............................... 1
1.2 The *SIR* Epidemic Model .......................... 2
1.3 The *SIR* Endemic Model ........................... 5  
  1.3.1 Steady State Solutions ........................... 6
  1.3.2 Stability ....................................... 7
  1.3.3 Bifurcation Analysis ............................ 7
  1.3.4 Phase-Plane Analysis ........................... 8
  1.3.5 Summary ....................................... 8

2 **2D Extensions** ................................... 10  
2.1 The *SIR* model with Susceptible *R* Class .......... 10  
  2.1.1 Steady State Solutions .......................... 11
  2.1.2 Saddle Node Equation .......................... 11
  2.1.3 Stability ....................................... 12
  2.1.4 Bifurcation Analysis ............................ 14
  2.1.5 Phase-Plane Analysis ........................... 15
2.2 The *SIR* model with Nonlinear Transmission Class .... 18  
  2.2.1 Steady State Solutions .......................... 19
  2.2.2 Saddle Node Equation .......................... 20
  2.2.3 Stability ....................................... 21
  2.2.4 Bifurcation Analysis ............................ 22
  2.2.5 Phase-Plane Analysis ........................... 22
2.3 The *SIR* model with Exogenous Infection Class ........ 26  
  2.3.1 Steady State Solutions .......................... 27
  2.3.2 Saddle Node Equation .......................... 28
  2.3.3 Stability ....................................... 29
  2.3.4 Bifurcation Analysis ............................ 31
  2.3.5 Phase-Plane Analysis ........................... 31
2.4 Summary ....................................... 34

3 **General Analysis of 2D Models** .................. 36  
3.1 Susceptible *R* Class ............................... 36
3.2 Nonlinear Transmission Class ........................ 39
3.3 Exogenous Infection Class ........................... 42
4 3D Extensions

4.1 SEIR model
   4.1.1 Steady State Solutions
   4.1.2 Saddle Node Equation
   4.1.3 Stability
   4.1.4 Bifurcation Analysis
   4.1.5 Phase-Plane Analysis

4.2 The SEIR model with Partial Recovery
   4.2.1 Steady State Solutions
   4.2.2 Saddle Node Equation
   4.2.3 Stability
   4.2.4 Bifurcation Analysis
   4.2.5 Phase-Plane Analysis

4.3 SEIR model with Full Recovery
   4.3.1 Steady State Solutions
   4.3.2 Saddle Node Equation
   4.3.3 Stability
   4.3.4 Bifurcation Analysis
   4.3.5 Phase-Plane Analysis

4.4 The SIR Model with Carrier Class
   4.4.1 Steady State Solutions
   4.4.2 Saddle Node Equation
   4.4.3 Stability
   4.4.4 Examples 1, 2

4.5 Summary

5 General Analysis of 3D Models

5.1 The SEIR model
5.2 SEIR model with Partial Recovery
5.3 SEIR model with Full Recovery
5.4 The SIR Model with Carrier Class

6 Discussion
List of Tables

1  Summary of the notations used in the SIR Model 2
2  Parameter values for susceptible $R$ class. 12
3  Parameter values for nonlinear transmission class. 21
4  Parameter values for exogenous infection class. 28
5  Frequently Used Notation in Chapters 3 and 5 37
6  Parameter values for the SEIR model. 47
7  Parameter values for the SEIR model with partial recovery. 55
8  Parameter values for the SEIR model with full recovery. 63
9  Parameter values for the SIR model with carrier class. 72
List of Figures

1. Phase-Plane for SIR epidemic model when $R_0 = 5$. ........................................... 4
2. Triangle Invariance of SIR endemic model. ................................................................. 6
3. Bifurcation analysis for SIR endemic model. Stable infection-free steady state for $R_0 < 1$; unstable infection-free $i$ and stable endemic steady state $i^*$ for $R_0 > 1$. .......................................................... 8
4. Phase-Plane for SIR endemic model when: (a) $R_0 = 0.5 < 1$; (b) $R_0 = 2 > 1$. Other parameter values are $\mu = 0.02$ and $\gamma = 0.05$. ......................................................... 9
5. Bifurcation diagram for the SIR model with susceptible $R$ class giving curves ($R_0, s^*$) and ($R_0, i^*$). Broken lines signify unstable steady state while unbroken & dotted lines are stable ones. Light dark arrow points downward at $R_0 = R_{\text{saddle}}$ where two endemic equilibriums coincide. The ($R_0, s^*$) and ($R_0, i^*$) curves have backward bifurcations when $P > P_{\text{crit}}$ where $P_{\text{crit}} = 1 + \frac{\mu}{\gamma}$. ............................................................... 14
6. Phase-Plane for the SIR model with susceptible $R$ class for $P = 0 < P_{\text{crit}}$ when: (a) $R_0 = 0.8 < 1$; (b) $R_0 = 1.2 > 1$. ................................................................. 15
7. Phase-Planes for susceptible $R$ class for $P = \frac{P_{\text{crit}}}{2}$ when: (a) $R_0 = 0.8 < 1$; (b) $R_0 = 1.2 > 1$. ................................................................. 16
8. Phase-Planes for susceptible $R$ class for $P = P_{\text{crit}}$ when: (a) $R_0 = 0.8 < 1$; (b) $R_0 = 2 > 1$. ................................................................. 16
9. Phase-Planes for susceptible $R$ class for $P = 2.8 > P_{\text{crit}}$ when: (a) $R_0 = 0.5 < R_{\text{saddle}}$; (b) $R_0 = R_{\text{saddle}}$. ................................................................. 17
10. Phase-Planes for susceptible $R$ class for $P = 2.8 > P_{\text{crit}}$ when: (a) $R_{\text{saddle}} < R_0 = 0.92 < 1$; (b) $R_0 = 2$. ................................................................. 17
11. Bifurcation diagram for the SIR model with nonlinear transmission class. Labels are as in Fig. 5. For this class, $P_{\text{crit}} = 1 + \frac{2}{\mu}$. ................................................................. 23
12. Phase-Planes for the SIR model with nonlinear transmission class for $P = 0 < P_{\text{crit}}$ when: (a) $R_0 = 0.8 < 1$; (b) $R_0 = 1.2 > 1$. ................................................................. 24
13. Phase-Planes for nonlinear transmission class for $P = \frac{P_{\text{crit}}}{2}$ when: (a) $R_0 = 0.8 < 1$; (b) $R_0 = 1.2 > 1$. ................................................................. 24
14. Phase-Planes for nonlinear transmission class for $P = 3.5 = P_{\text{crit}}$ when: (a) $R_0 = 0.8 < 1$; (b) $R_0 = 1.2 > 1$. ................................................................. 25
15. Phase-Plane for nonlinear transmission class for $P > P_{\text{crit}}$ when: (a) $R_0 = 0.5$; (b) $R_0 = R_{\text{saddle}} = 0.951919$. ................................................................. 25
16. Phase-Plane for nonlinear transmission class for $P > P_{\text{crit}}$ when: (a) $R_{\text{saddle}} < R_0 = 0.96 < 1$; (b) $R_0 = 1.2 > 1$. ................................................................. 26
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>Bifurcation diagram for the SIR model with exogenous infection class. Labels are as in Fig. 5. The critical value of $P$ for this class is $\frac{\nu (\nu + \mu)}{\mu^2}$.</td>
<td>30</td>
</tr>
<tr>
<td>18</td>
<td>Phase-Planes for the SIR model with exogenous infection class for $P = 0 &lt; P_{\text{crit}}$ when: (a) $R_0 = 0.6 &lt; R_{\text{saddle}}$; (b) $R_0 = 2$ within triangle $s + i = 1$.</td>
<td>31</td>
</tr>
<tr>
<td>19</td>
<td>Phase-Planes for exogenous infection class for $P = 5 &lt; P_{\text{crit}}$ when: (a) $R_0 = 0.6 &lt; R_{\text{saddle}}$; (b) $R_0 = 1.2$.</td>
<td>32</td>
</tr>
<tr>
<td>20</td>
<td>Phase-Planes for exogenous infection class for $P = P_{\text{crit}} = 8.75$ when: (a) $R_0 = 0.6 &lt; R_{\text{saddle}}$; (b) $R_0 = 2$.</td>
<td>32</td>
</tr>
<tr>
<td>21</td>
<td>Phase-Planes for exogenous infection class for $P = 14 &gt; P_{\text{crit}}$ when: (a) $R_0 = 0.6 &lt; R_{\text{saddle}}$; (b) $R_0 = 0.9838 = R_{\text{saddle}}$.</td>
<td>33</td>
</tr>
<tr>
<td>22</td>
<td>Phase-Planes for exogenous infection class for $P = 14 &gt; P_{\text{crit}}$ when: (a) $R_{\text{saddle}} &lt; R_0 = 0.99 &lt; 1$; (b) $R_0 = 1.2 &gt; 1$ is of interest.</td>
<td>33</td>
</tr>
<tr>
<td>23</td>
<td>Enlarged top-center part of Figure 5. A clearer view for $s^<em><em>1 &lt; 0$, and the values for $R</em>{01}$ as we perturb the variable $i^</em>$. Unbroken lines show stable steady states while broken lines signify unstable. The curve $R_{01} &lt; 0$ when $P &gt; P_{\text{crit}}$ and curve $R_{01} &gt; 0$ when $P &lt; P_{\text{crit}}$.</td>
<td>39</td>
</tr>
<tr>
<td>24</td>
<td>Enlarged diagram taken from Fig. 11. Labels are as in Fig. 23.</td>
<td>41</td>
</tr>
<tr>
<td>25</td>
<td>An enlarged top-center portion of the bifurcation fig. (17). Labels are as in Fig. 23.</td>
<td>44</td>
</tr>
<tr>
<td>26</td>
<td>Bifurcation Diagram for the SEIR model. Curves are $s^<em>$ and $i^</em>$ as the functions of $R_0$. Continuous lines show stable steady state and broken lines are unstable steady states. There are curves for $P &lt; P_{\text{crit}}; P = P_{\text{crit}}$ and $P &gt; P_{\text{crit}}$. For this model, $P_{\text{crit}} = \frac{\nu (\nu + \mu)}{\mu^2}$.</td>
<td>50</td>
</tr>
<tr>
<td>27</td>
<td>Phase-Planes for the SEIR Model for $P = 0$: when (a) $R_0 = 0.6 &lt; 1$; (b) $R_0 = 1.2 &gt; 1$.</td>
<td>51</td>
</tr>
<tr>
<td>28</td>
<td>Phase-Planes for the SEIR Model for $P = 5 &lt; P_{\text{crit}}$ when: (a) $R_0 = 0.6 &lt; 1$; (b) $R_0 = 1.2 &gt; 1$.</td>
<td>51</td>
</tr>
<tr>
<td>29</td>
<td>Phase-Planes for the SEIR Model for $P = 8.75 = P_{\text{crit}}$ when: (a) $R_0 = 0.6 &lt; 1$; (b) $R_0 = 1.2 &gt; 1$.</td>
<td>52</td>
</tr>
<tr>
<td>30</td>
<td>Phase-Planes for the SEIR Model for $P &gt; P_{\text{crit}}$ when: (a) $R_0 = 0.8 &lt; R_{\text{saddle}}$; (b) $R_{\text{saddle}} = 0.977 &lt; 1$.</td>
<td>52</td>
</tr>
<tr>
<td>31</td>
<td>Phase-Planes for the SEIR Model for $P &gt; P_{\text{crit}}$ when: (a) $R_0 = 0.987 &lt; 1$; (b) $R_0 = 1.2 &gt; 1$.</td>
<td>53</td>
</tr>
<tr>
<td>32</td>
<td>Bifurcation Diagram for the SEIR model with partial recovery. Labels are as in Figure 26. This model have $P_{\text{crit}} = \frac{(\mu + \gamma)(\mu + \nu) - \kappa \gamma}{\gamma \nu (1 - \epsilon)}$.</td>
<td>58</td>
</tr>
<tr>
<td>33</td>
<td>Phase-Plane for the SEIR Model with partial recovery for $P = 0$: (a) $R_0 = 0.7 &lt; 1$; (b) $R_0 = 1.2 &gt; 1$.</td>
<td>59</td>
</tr>
</tbody>
</table>
Phase-Plane for the SEIR Model with partial recovery for $P = 1.3 = \frac{P_{\text{crit}}}{2}$ when: (a) $R_0 = 0.7 < 1$; (b) $R_0 = 1.2 > 1$. .............................. 60

Phase-Plane for the SEIR Model with partial recovery for $P = 2.6 = P_{\text{crit}}$ when: (a) $R_0 = 0.7 < 1$; (b) $R_0 = 1.2 > 1$. .............................. 60

Phase-Plane for the SEIR Model with partial recovery for $P > P_{\text{crit}}$ when: (a) $R_0 < R_{\text{saddle}}$; (b) $R_{\text{saddle}} = 0.9545 < 1$. .............................. 61

Phase-Plane for the SEIR Model with partial recovery for $P > P_{\text{crit}}$ when: (a) $R_{\text{saddle}} < R_0 = 0.98 < 1$; (b) $R_0 = 1.2 > 1$. .............................. 61

Bifurcation Diagram for the SEIR model with full recovery. The critical value of $P_{\text{crit}} = \frac{(\mu_+\gamma)(\mu_+\nu)}{\gamma}$. .......................................................... 66

Phase-Planes for the SEIR Model with full recovery for $P = 0$ when: (a) $R_0 = 0.7 < 1$; (b) $R_0 > 1$. .......................................................... 66

Phase-Planes for the SEIR Model with full recovery for $P = 0.98 = \frac{P_{\text{crit}}}{2}$ when: (a) $R_0 = 0.7 < 1$; (b) $R_0 = 2 > 1$. .......................................................... 67

Phase-Plane for the SEIR Model with full recovery for $P = 1.96 = P_{\text{crit}}$ when: (a) $R_0 = 0.7 < 1$; (b) $R_0 = 2 > 1$. .......................................................... 67

Phase-Plane for the SEIR Model with full recovery for $P > P_{\text{crit}}$ when: (a) $R_0 < R_{\text{saddle}}$; (b) $R_{\text{saddle}} = 0.9308 < 1$. .......................................................... 68

Phase-Plane for the SEIR Model with full recovery for $P > P_{\text{crit}}$ when: (a) $R_{\text{saddle}} < R_0 = 0.98 < 1$; (b) $R_0 = 2 > 1$. .......................................................... 68

Bifurcation diagram for the SIR model with carrier class (Example 1) for the function $g(x^*) = 1 - e^{-0.9x^*}$. We show $1 - s^*$ as a function of $R_0$. Backward bifurcation occurs when $P > P_{\text{crit}} = \frac{\mu_+\delta}{\gamma^s(0)}$ at $(R_0, 1 - s^*) = (1, 0)$. .......................................................... 75

Phase-Plane for the Example 1 for $P = 0 < P_{\text{crit}}$ when: (a) $R_0 = 0.7$; (b) $R_0 = 1.1$. .......................................................... 76

Phase-Plane for the Example 1 for $P = 0.8333 = P_{\text{crit}}$ when: (a) $R_0 = 0.7$; (b) $R_0 = 1.1$. .......................................................... 77

Phase-Plane for the Example 1 for $P = 1.6 > P_{\text{crit}}$ when: (a) $R_0 = 0.7 < 1$; (b) $R_0 = R_{\text{saddle}} = 0.8299 < 1$. .......................................................... 77

Phase-Plane for the Example 1 for $P = 1.6 > P_{\text{crit}}$ when: (a) $R_{\text{saddle}} < R_0 = 0.95 < 1$; (b) $R_0 = 1.1 > 1$. .......................................................... 78

Time-series plots for the carrier class Example 1. For $P = 0 < P_{\text{crit}}$: (a) $R_0 = 0.7 < 1$; (b) $R_0 = 1.1 > 1$ and For $P = 0.8333 = P_{\text{crit}}$: (c) $R_0 = 0.7 < 1$; (d) $R_0 = 1.1 > 1$. .......................................................... 78
Time-series for $q(x) = 1 - e^{-0.9x}$ the carrier class Example 1. For $P = 1.6 > P_{crit}$: (a) $R_0 = 0.6 < 1$; (b) $R_{saddle} = R_0 = 0.8299 < 1$; (c) $R_{saddle} < R_0 = 0.95 < 1$; (d) $R_0 = 1.1 > 1$.  

Bifurcation Diagram for the carrier class for the function $q(x^*) = \frac{x^2}{x^2+2}$. In this diagram, 1 - $s^*$ is plotted against $R_0$. Backward bifurcation occurs when $P > P_{crit} = 0.9$ for $q'(0) = 0$. Multiple endemic steady states exist in the region of $R_{saddle} < R_0 < 1.19$.  

Phase-Planes for the Example 2 for $P = 0 < P_{crit}$ when: (a) $R_0 = 0.7 < 1.19$; (b) $R_0 = 1.2 > 1.19$.  

Phase-Planes for the Example 2 for $P = 0.9 = P_{crit}$ when: (a) $R_0 = 0.7 < 1.19$; (b) $R_0 = 1.2 > 1.19$.  

Phase-Planes for the Example 2 for $P = 1.8 > P_{crit}$ when: (a) $R_0 = 0.7 < 1.19$; (b) $R_{saddle} = R_0 = 1.09 < 1.19$.  

Phase-Planes for the Example 2 for $P = 1.8 > P_{crit}$ when: (a) $R_{saddle} < R_0 = 1.17 < 1.19$; (b) $R_0 = 1.2 > 1.19$.  

Time-series plots for the carrier class Example 2. For $P = 0 < P_{crit}$: (a) $R_0 = 0.7 < 1$; (b) $R_0 = 1.2 > 1$ and For $P = 0.9 = P_{crit}$: (c) $R_0 = 0.7 < 1$; (d) $R_0 = 1.2 > 1$.  

Time-series plots for the carrier class Example 2 for $P = 1.8 > P_{crit}$: (a) $R_0 = 0.7 < 1$; (b) $R_0 = R_{saddle} = 1.09 < 1.19$; (c) $R_{saddle} < R_0 = 1.16 < 1.19$; (d) $R_0 = 1.2 > 1$.  

An enlarged top-center portion of Fig. 26. In this sketch, it is clearly shown that $s_1^* < 0$, and $R_{01} < 0$ give backward bifurcation, while $R_{01} > 0$ give forward bifurcation.  

Enlarged top-center portion of Bifurcation diagram 32. The critical value $P_{crit} = (\mu + \gamma)(\mu + \nu) - \kappa \gamma \nu$.  

Blow up of the bifurcation diagram 38. Labels are as in Fig. 26. The critical value of $P$ for this class is $P_{crit} = (\mu + \gamma)(\mu + \nu)$.  

A top-center blow up of Bifurcation diagram 44. Labels are as in Fig. 26. The critical value for $P$ is $P_{crit} = \frac{\mu + \delta}{\gamma q'(x^*)}$.