Backward Bifurcation in *SIR* endemic models

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Abstract

In the well known *SIR* endemic model, the infection-free steady state is globally stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$. Hence, we have a forward bifurcation when $\mathcal{R}_0 = 1$. When $\mathcal{R}_0 > 1$, an asymptotically stable endemic steady state exists. The basic reproduction number $\mathcal{R}_0$ is the main threshold bifurcation parameter used to determine the stability of steady states of *SIR* endemic models.

In this thesis we study extensions of the *SIR* endemic model for which a backward bifurcation may occur at $\mathcal{R}_0 = 1$. We investigate the biologically reasonable conditions for the change of stability. We also analyse the impact of different factors that lead to a backward bifurcation both numerically and analytically. A backward bifurcation leads to sub-critical endemic steady states and hysteresis.

We also provide a general classification of such models, using a small amplitude expansion near the bifurcation. Additionally, we present a procedure for projecting three dimensional models onto two dimensional models by applying some linear algebraic techniques. The four extensions examined are: the *SIR* model with a susceptible recovered class; nonlinear transmission; exogenous infection; and with a carrier class.

Numerous writers have mentioned that a nonlinear transmission function in relation to the infective class, can only lead to a system with an unstable endemic steady state. In spite of this we show that in a nonlinear transmission model, we have a function depending on the infectives and satisfying certain biological conditions, and leading to a sub-critical endemic equilibriums.

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