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# BACKWARD BIFURCATION IN *SIR* ENDEMIC MODELS

THIS THESIS IS PRESENTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE  
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## Abstract

In the well known *SIR* endemic model, the infection-free steady state is globally stable for  $\mathcal{R}_0 < 1$  and unstable for  $\mathcal{R}_0 > 1$ . Hence, we have a forward bifurcation when  $\mathcal{R}_0 = 1$ . When  $\mathcal{R}_0 > 1$ , an asymptotically stable endemic steady state exists. The basic reproduction number  $\mathcal{R}_0$  is the main threshold bifurcation parameter used to determine the stability of steady states of *SIR* endemic models.

In this thesis we study extensions of the *SIR* endemic model for which a backward bifurcation may occur at  $\mathcal{R}_0 = 1$ . We investigate the biologically reasonable conditions for the change of stability. We also analyse the impact of different factors that lead to a backward bifurcation both numerically and analytically. A backward bifurcation leads to sub-critical endemic steady states and hysteresis.

We also provide a general classification of such models, using a small amplitude expansion near the bifurcation. Additionally, we present a procedure for projecting three dimensional models onto two dimensional models by applying some linear algebraic techniques. The four extensions examined are: the *SIR* model with a susceptible recovered class; nonlinear transmission; exogenous infection; and with a carrier class.

Numerous writers have mentioned that a nonlinear transmission function in relation to the infective class, can only lead to a system with an unstable endemic steady state. In spite of this we show that in a nonlinear transmission model, we have a function depending on the infectives and satisfying certain biological conditions, and leading to a sub-critical endemic equilibriums.

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