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Geodesics on Lie Groups: Euler Equations and Totally Geodesic Subgroups

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The geodesic motion on a Lie group equipped with a left or right invariant Riemannian metric is governed by the Euler-Arnold equation. This paper investigates conditions on the metric in order for a given subgroup to be totally geodesic. Results on the construction and characterisation of such metrics are given. The setting works both in the classical finite dimensional case, and in the category of infinite dimensional Fréchet Lie groups, in which diffeomorphism groups are included. Using the framework we give new examples of both finite and infinite dimensional totally geodesic subgroups. In particular, based on the cross helicity, we construct right invariant metrics such that a given subgroup of exact volume preserving diffeomorphisms is totally geodesic.

The paper also gives a general framework for the representation of Euler-Arnold equations in arbitrary choice of dual pairing.

Keywords: Euler equations, Totally geodesic subgroups, Diffeomorphism groups

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1 Introduction

In 1966 Vladimir Arnold demonstrated that Euler's equation for an ideal fluid is the geodesic equation on the group of volume preserving diffeomorphisms with respect to the right invariant L^2 metric [2]. Since then there has been a lot of interest in generalised Euler equations (also known as *Euler-Arnold equations*), i.e., geodesic equations on a Lie group equipped with an invariant metric. Examples of such equations include Burgers' equation ($\mathrm{Diff}(S^1)$ with a right-invariant L^2 metric), Korteweg-de Vries (Virasoro-Bott group with a right invariant L^2 metric), and Camassa-Holm ($\mathrm{Diff}(S^1)$ with a right invariant H^1 metric); these and other examples are surveyed in [21].

A Lie subgroup $H \subset G$ is called *totally geodesic* in G if geodesics in H are also geodesics in G. Motivated by the physical applications it is common to ask which subgroups (of a given group) are totally geodesic with respect to a given metric. In this paper we investigate a different question: we do not fix the metric, but ask if it is possible to choose one so that a given subgroup $H \subset G$ is totally geodesic. A motivation for this study comes from diffeomorphic image matching, where one may want to require a certain class of transformations (e.g. affine transformations) to be totally geodesic within a larger class.

There has been little systematic study of totally geodesic subgroups. However, in the case of diffeomorphisms, the following results are known:

1. The exact volume preserving diffeomorphisms of a flat compact Riemannian manifold without boundary is totally geodesic in the volume preserving diffeomorphisms, with respect to the right invariant L^2 metric. (Recall that the exact volume preserving diffeomorphisms are generated by vector fields that have a vector potential in terms of the curl operator.) This result is given in [4], and refined in [8].

2. The Hamiltonian diffeomorphisms of a closed Kähler manifold with flat metric is totally geodesic in the symplectic diffeomorphisms, with respect to the right invariant L^2 metric. This result is given in [4], and refined in [8].

3. Let G be a compact Lie group that acts on on a Riemannian manifold M by isometries. Let Φ_g denote the action. The subgroup of equivariant diffeomorphisms $\mathrm{Diff}_{\Phi_g}(M)$ is totally geodesic in $\mathrm{Diff}(M)$ and $\mathrm{Diff}_{\mathrm{vol}}(M)$, with respect to the right invariant L^2 metric. This result is given in [20].

4. The subgroup of diffeomorphisms on a Riemannian manifold M that leaves the point of a submanifold $N \subset M$ fixed is totally geodesic with respect to the right invariant L^2 metric. This result is given in [20].

5. The subgroup of diffeomorphisms of the cylindrical surface $S^1 \times [0,1]$ that rotates each horizontal circle rigidly by an angle is totally geodesic in the group of volume preserving diffeomorphisms of $S^1 \times [0,1]$. This result is given in [7].

The main contribution of this paper is a framework for the construction of a family of left or right invariant metrics on a Lie algebra G such that a given subgroup $H \subset G$ is totally geodesic with respect to each metric in the family. The requirement is that there is a bi-linear symmetric form on the Lie algebra $\mathfrak{g}$ of G with certain bi-invariance and non-degeneracy properties. The construction works both in the finite and infinite dimensional case (as in [8], we work in the category of Fréchet Lie groups [12]). In the finite dimensional case, using the Killing form as bi-linear symmetric form, the requirement is that $\mathfrak{h}$ is semisimple.

Using this technique, we can extend the list of totally geodesic examples above:

6. Let G be an n dimensional Lie group, and let $H \subset G$ be an m dimensional semisimple Lie subgroup of G. We construct a $(n+1)n/2 - (n-m)m$ dimensional manifold of left (or right) invariant metrics on G, for which H is totally geodesic in G. In particular, we give an example of a left invariant metric such that $\mathrm{SO}(3)$ is totally geodesic in $\mathrm{GL}(3)$.

In the infinite dimensional case of diffeomorphism groups, we are lead to bi-invariant forms. For exact divergence free and Hamiltonian vector fields respectively, a bi-invariant non-degenerate bi-linear symmetric form has been given by Smolentsev [16, 17, 18]. We give a generalisation of Smolentsev's result, extending it to include manifolds with boundaries. Using our framework, we then give the following new examples of totally geodesic subgroups of diffeomorphisms:

7. Let (M, g) be a compact Riemannian n–manifold with boundary. The (finite dimensional) group of isometries $\mathrm{Diff}_{\mathrm{iso}}(M)$ is totally geodesic in $\mathrm{Diff}(M)$ with respect to the right invariant H^1_α metric.

8. Let (M, g) be a compact Riemannian n–manifold with boundary. Then we give a strong condition for $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ to be totally geodesic in $\mathrm{Diff}_{\mathrm{vol}}(M)$ with respect to the right invariant H^1_α metric. This is an extension of a result in [8].

9. Let (M, g) be a compact contact 3–manifold with boundary. Then the exact contact diffeomorphisms are totally geodesic in the exact volume preserving diffeomorphisms, with respect to the right invariant L^2 metric.

The paper is organised as follows. Section 2 begins with a brief presentation of geodesic flow on groups and the Euler-Arnold equation. In Section 2.1 we state the infinite dimensional setting, which is based on Fréchet Lie groups. In particular, this setting allows groups of diffeomorphisms. As a subsidiary objective of the paper, we give in Section 2.2 a fairly detailed framework of how to represent the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ by a choice of pairing. It is our experience that, in the current literature, the choice of pairing used to represent a particular Euler-Arnold equation is often implicit, and varies from equation to equation and from research group to research group. As examples, we give two different representations of the rigid body equation, and five different representations of the ideal fluid equation, using various choices of pairing (all of them occur in literature).

In Section 3 we derive a characterisation on the algebra level, for a subgroup to be totally geodesic (Theorem 2). We point out that part of this result appears already as a main tool in the paper [8]. To gain geometric insight we also derive, in Section 3.1, the condition for a subgroup to be totally geodesic by the standard technique using the second fundamental form. Furthermore, in Section 3.2 we derive the correspondence of Theorem 2 in terms of Lie algebra structure coefficients and metric tensor elements, i.e., from the coordinate point view.

Section 4 presents a framework for constructing totally geodesic metrics. To some extent, this construction characterises all metrics which makes a subgroup *easy totally geodesic*, meaning that the orthogonal complement of the subalgebra is invariant under the adjoint action. In Section 4.2 we investigate the special case of semi-direct products.

Finally, the examples of totally geodesic diffeomorphism subgroups given in the list above are derived in Section 5.

2 Geodesic Flow and Euler-Arnold Equations

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We denote by $[\cdot,\cdot]$ the Lie algebra bracket on $\mathfrak{g}$, and the identity element in G is denoted e. For each $g \in G$ we denote by L_g and R_g the left and right translation maps on G, and by $T\mathsf{L}_g$ and $T\mathsf{R}_g$ their corresponding tangent maps (derivatives). To simplify the development we mainly work with left translation in our derivation. Notice however that all results also carry over to the setting of right translation.

Consider a real inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ on $\mathfrak{g}$. It is implicitly associated with an inertia operator $\mathcal{A} : \mathfrak{g} \to \mathfrak{g}^*$ such that $\langle\cdot,\cdot\rangle_{\mathcal{A}} = \langle \mathcal{A}\,\cdot,\cdot\rangle$. The tensor field over G given by

$$T_gG \times T_gG \ni (v_g, w_g) \longmapsto \langle T_g\mathsf{L}_{g^{-1}}v_g, T_g\mathsf{L}_{g^{-1}}w_g\rangle_{\mathcal{A}} =: \langle\!\langle v_g, w_g\rangle\!\rangle_{\mathcal{A},g},$$

defines a Riemannian metric $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathcal{A}}$ on G. The geodesic flow $\gamma : [0,1] \to G$ between two points $g_0, g_1 \in G$ fulfils (by definition) the variational problem

$$\delta \int_0^1 \frac{1}{2}\langle\!\langle\dot\gamma(t),\dot\gamma(t)\rangle\!\rangle_{\mathcal{A},\gamma(t)}\,\mathrm{d}t = 0, \quad \gamma(0) = g_0, \quad \gamma(1) = g_1. \tag{1}$$

This can be viewed as a Lagrangian problem, with a quadratic Lagrangian function $L : TG \to \mathbb{R}$ given by $L(v_g) = \frac{1}{2}\langle\!\langle v_g, v_g\rangle\!\rangle_{\mathcal{A},g}$.

From the construction of the metric on G it is straightforward to check that L is left invariant, i.e., $L(v_g) = L(T_g\mathsf{L}_h v_g)$ for each $h \in G$. In particular, it means that the Lagrangian L is fully determined by the reduced quadratic Lagrangian $l : \mathfrak{g} \to \mathbb{R}$ defined by restriction of L to $T_eG \equiv \mathfrak{g}$, i.e., $l(\xi) = \frac{1}{2}\langle\xi,\xi\rangle_{\mathcal{A}}$. By Euler-Poincaré reduction (cf. Marsden and Ratiu [15, Ch. 13]), the second order differential equation for geodesic motion, i.e., the Euler-Lagrange equation for L, can be reduced to a first order differential equation on the Lie algebra $\mathfrak{g}$ called the *Euler-Arnold equation*. In weak form it is given by

$$\langle\dot\xi,\eta\rangle_{\mathcal{A}} = \langle\xi, \mathrm{ad}_\xi(\eta)\rangle_{\mathcal{A}}, \qquad \forall\,\eta \in \mathfrak{g}, \tag{2a}$$

where $\mathrm{ad}_\xi := [\xi, \cdot]$. The corresponding strong form of the Euler-Arnold equation is

$$\dot{\xi} = \mathrm{ad}_\xi^{\top_\mathcal{A}}(\xi) \tag{2b}$$

where $\mathrm{ad}_\xi^{\top_\mathcal{A}}$ is the transpose of the map ad_ξ with respect to the inner product $\langle\cdot,\cdot\rangle_\mathcal{A}$, i.e., $\langle \mathrm{ad}_\xi^{\top_\mathcal{A}}(\psi), \eta\rangle_\mathcal{A} = \langle \psi, \mathrm{ad}_\xi(\eta)\rangle_\mathcal{A}$ for all $\xi, \psi, \eta \in \mathfrak{g}$. Throughout the rest of this paper we assume that $\mathrm{ad}_\xi^{\top_\mathcal{A}}$ is well defined for every $\xi \in \mathfrak{g}$, and that the strong form of the Euler-Arnold equation is locally well posed for every choice of initial data. In finite dimensions this is always the case since the inner product $\langle\cdot,\cdot\rangle_\mathcal{A}$ is non-degenerate, so $\mathcal{A}$ is an isomorphism. For infinite dimensional groups (see Section 2.1 below) the assumption is non-trivial.

Given a solution curve $\xi(t) \in \mathfrak{g}$ to the Euler-Arnold equation (2), the corresponding solution curve $\gamma(t) \in G$ is recovered by the reconstruction equation $\dot{\gamma}(t) = T_e\mathsf{L}_{\gamma(t)}\xi(t)$.

The Euler-Arnold equation (2b) is described in the Lagrangian framework of mechanics. It is also possible to obtain a Hamiltonian description. Indeed, by the Legendre transformation we can change to the momentum variable $\mu := \mathrm{d}l/\,\mathrm{d}\xi(\xi) = \mathcal{A}\xi$. In this variable the Euler-Arnold equation takes the form

$$\dot{\mu} = \mathrm{ad}_\xi^*(\mu), \quad \xi = \mathcal{A}^{-1}\mu, \tag{2c}$$

where $\mathrm{ad}_\xi^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by $\langle \mu, \mathrm{ad}_\xi(\eta)\rangle = \langle \mathrm{ad}_\xi^*(\mu), \eta\rangle$ for all $\eta \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. This is a Hamiltonian system with respect to the canonical Lie-Poisson bracket (cf. Marsden and Ratiu [15, Ch. 13]), for the reduced quadratic Hamiltonian function $h(\mu) = \frac{1}{2}\langle \mu, \mathcal{A}^{-1}\mu\rangle$.

Remark 2.1. If right invariance instead of left invariance is considered, the framework is almost identical, with the two deviations that the right hand side of the Euler-Arnold equation (2) switches sign, and right instead of left reconstruction should be used. Typically, finite dimensional examples are left invariant, and infinite dimensional examples are right invariant.

Remark 2.2. In this paper, we mean with "Lagrangian form" of an Euler-Arnold equation that it corresponds to a reduced Lagrangian function as described above. We do *not* mean the fluid particle representation of the equation.

2.1 Infinite Dimensional Setting

In addition to the classical setting of finite dimensional Lie groups, the framework described above is also valid for infinite dimensional *Fréchet Lie groups* and corresponding *Fréchet Lie algebras*. The prime examples, and the only ones we consider in this paper, are subgroups (including the full group itself) of the group $\mathrm{Diff}(M)$ of diffeomorphisms on an n–dimensional compact manifold M, with composition as group operation. If M has no boundary, the corresponding Fréchet Lie algebra is the space $\mathfrak{X}(M)$ of smooth vector field on M. If M has a boundary, it is the vector fields $\mathfrak{X}_t(M)$ in $\mathfrak{X}(M)$ that are tangent to the boundary. The Lie algebra bracket on $\mathfrak{X}(M)$ is minus the Jacobi-Lie commutator bracket, i.e., $\mathrm{ad}_\xi(\eta) \equiv [\xi, \eta] = -[\xi, \eta]_\mathfrak{X}$. The topology on $\mathfrak{X}(M)$, making it a Fréchet space, is given by the sequence of semi norms

$$\|\xi\|_0, \|\xi\|_1, \|\xi\|_2, \ldots \qquad \text{where} \qquad \|\xi\|_m := \sum_{k=0}^{m} \sup_{x\in M} \sum_{i=1}^{n} |\xi_i^{(k)}(x)|, \quad \xi = \sum_{i=1}^{n} \xi_i \partial_i \tag{3}$$

The Jacobi-Lie bracket $[\cdot,\cdot]_\mathfrak{X} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a smooth map with this topology, as is every differential operator $\mathfrak{X}(M) \to \mathfrak{X}(M)$. See Hamilton [9, Sect. I.4] for details on the category of Fréchet Lie algebras and Fréchet Lie groups.

Remark 2.3. Notice that the Fréchet topology defined in equation (3) is independent of the choice of coordinates on M, since all norms on M are equivalent.

The dual space of a Fréchet space is not itself a Fréchet space (see [9, Sect. I.1]). Thus, the Hamiltonian view-point, established by equation (2c) above, does not make sense, since it assumes that $\mathfrak{g}$ and $\mathfrak{g}^*$ are isomorphic. As a remedy, it is customary to introduce the so called *regular dual.* It is a subspace of the full dual which is isomorphic to $\mathfrak{g}$. Indeed, the regular dual is given by the image of the inertia operator $\mathfrak{g}^*_{\mathrm{reg}} = \mathcal{A}\mathfrak{g}$. Throughout this paper we will only work with the regular dual, so the subscript is omitted: $\mathfrak{g}^* := \mathfrak{g}^*_{\mathrm{reg}}$. Furthermore, if several inertia operators $\mathcal{A}_1, \mathcal{A}_2$ are considered for the same Fréchet Lie algebra, we assume that the regular part of the dual is invariant, i.e., that $\mathcal{A}_1\mathfrak{g} = \mathcal{A}_2\mathfrak{g}$.

2.2 Choice of Pairing

The most straightforward way to get a coordinate representation of the Euler-Arnold equation (2) is to introduce coordinates in $\mathfrak{g}$, and then compute the transpose of ad_ξ in these coordinates with respect to the given inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$. However, with this approach the inner product is implicitly entangled in the equation. Furthermore, in the infinite dimensional case of diffeomorphism groups it is only possible to explicitly compute the transpose map $\mathrm{ad}_\xi^{\top_{\mathcal{A}}}$ in a few special cases. Instead, a common approach is to "decouple" the dependence on the choice of inner product in the coordinate representation. That is, to have a representation of the Euler-Arnold equation similar to the Hamiltonian form (2c). In order to do so one needs to introduce a correspondence between elements in $\mathfrak{g}^*$ and elements in $\mathfrak{g}$ without reference to the given inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$. In finite dimensions one may choose any basis in $\mathfrak{g}$ and use the corresponding dual basis in $\mathfrak{g}^*$. In general, we identify every element in $\mathfrak{g}^*$ with a corresponding element in a space $\mathfrak{g}^\bullet$, isomorphic to $\mathfrak{g}^*$, via an isomorphism $\mathcal{L} : \mathfrak{g}^\bullet \to \mathfrak{g}^*$ which we call the *pairing operator.* That is, $\mu \in \mathfrak{g}^*$ is represented by $\bar{\mu} = \mathcal{L}^{-1}\mu \in \mathfrak{g}^\bullet$. In infinite dimensions the trick is to find an isomorphic space $\mathfrak{g}^\bullet$ and a suitable pairing operator in which ad_ξ^* is nicely represented, i.e., in which $\mathrm{ad}_\xi^\bullet := \mathcal{L}^{-1} \circ \mathrm{ad}_\xi^* \circ \mathcal{L}$ is simple to write down. It is our experience that the relation between the choice of inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$, the choice of dual pairing space $\mathfrak{g}^\bullet$, and the choice of pairing operator $\mathcal{L}$, has caused confusion in the current literature, especially when comparing different Euler-Arnold equations with different traditions.

In the Hamiltonian view-point, the Euler-Arnold equation (2) can be written in terms of $\bar{\mu}$ as

$$\begin{cases} \dfrac{\mathrm{d}\bar{\mu}}{\mathrm{d}t} = \mathrm{ad}_\xi^\bullet(\bar{\mu}) \\ \bar{\mu} = \mathcal{J}\xi \end{cases} \quad \text{where} \quad \mathcal{J} := \mathcal{L}^{-1}\mathcal{A}. \tag{4a}$$

One may also take the Lagrangian view-point, in which case the equation is written

$$\mathcal{J}\dot{\xi} = \mathrm{ad}_\xi^\bullet(\mathcal{J}\xi). \tag{4b}$$

Notice, in both cases, that the map $\mathrm{ad}_\xi^\bullet$ is used, not $\mathrm{ad}_\xi^{\top_{\mathcal{A}}}$ or ad_ξ^*. The dependence on the choice of inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ is captured through $\mathcal{J}$ alone.

The reconstructed variables $\mu(t) = \mathcal{L}\bar{\mu}(t)$ and $\xi(t) = \mathcal{J}^{-1}\bar{\mu}(t)$ are of course independent of the choice of pairing space and choice of hat map. For the choice $\mathfrak{g}^\bullet = \mathfrak{g}$ and $\mathcal{L} = \mathcal{A}$ both equation (4a) and equation (4b) exactly recover the original form (2b) of the Euler-Arnold equation. We call this choice the *inertia pairing.* Next we continue with two original examples by Arnold. Using our framework, we give a list of various choices of pairings, all occurring in the literature.

Example 2.1 (Rigid body)**.** This is the first example of Arnold [2]. The group is the set of rotation matrices SO(3). Its Lie algebra is the space of skew-symmetric matrices $\mathfrak{so}(3)$. A left-invariant metric on SO(3) is obtained from the inertia operator $\mathcal{A} : \mathfrak{so}(3) \to \mathfrak{so}(3)^*$, corresponding to moments of inertia for the rigid body. Next, we specify a choice of pairing. We give two commonly used examples.

(a) Chose $\mathfrak{g}^\bullet = \mathfrak{so}(3)$, and the map $\mathcal{L} : \mathfrak{g}^\bullet \to \mathfrak{so}(3)^*$ defined by the Frobenius inner product: $\langle \mathcal{L}\bar{\mu}, \xi\rangle = \frac{1}{2}\,\mathrm{tr}(\bar{\mu}^\top \xi)$. With this pairing operator it holds that $\mathrm{ad}_\xi^\bullet(\bar{\mu}) = -\mathrm{ad}_\xi(\bar{\mu}) = -\xi\bar{\mu} + \bar{\mu}\xi$, which follows from the fact that the Frobenius inner product on $\mathfrak{so}(3)$ is the negative of the bi-invariant Killing form on $\mathfrak{so}(3)$ (see Section 4 below). Thus, the rigid body equation in Hamiltonian form (4a) is given by the *Lax pair formulation*

$$\frac{\mathrm{d}\bar{\mu}}{\mathrm{d}t} = -[\xi, \bar{\mu}] = \bar{\mu}\xi - \xi\bar{\mu}, \quad \bar{\mu} = \mathcal{J}\xi,$$

and in the Lagrangian form (4b) it is

$$\mathcal{J}\dot{\xi} = -[\xi, \mathcal{J}\xi] = \mathcal{J}\xi\,\xi - \xi\,\mathcal{J}\xi,$$

where $\mathcal{J} = \mathcal{L}^{-1}\mathcal{A}$ is a linear map $\mathfrak{so}(3) \to \mathfrak{so}(3)$, self-adjoint with respect to the Frobenius inner product, giving the moments of inertia.

(b) This second choice of pairing is most frequently used. Let e_1, e_2, e_3 denote the basis

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $\mathfrak{so}(3)$. We chose $\mathfrak{g}^{\bullet} = \mathbb{R}^3$. Further, we use the traditional notation for the coordinate vector $\bar{\mu} = \boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$, corresponding to *angular momentum*. The pairing operator is given by $\mathcal{L}\boldsymbol{\pi} = \sum_{i=1}^{3} \pi_i e_i^*$, where e_1^*, e_2^*, e_3^* is the dual basis of $\mathfrak{so}(3)^*$. With this pairing operator it holds that $\mathrm{ad}_\xi^{\bullet}(\boldsymbol{\pi}) = \boldsymbol{\pi} \times \boldsymbol{\omega}$, where $\boldsymbol{\omega} \in \mathbb{R}^3$ is the coordinate vector of ξ in the basis e_1, e_2, e_3, corresponding to *angular velocity*. Thus, we recover the classical version of the rigid body equation, in Hamiltonian form

$$\dot{\boldsymbol{\pi}} = \boldsymbol{\pi} \times \boldsymbol{\omega}, \quad \boldsymbol{\pi} = \mathcal{J}\xi = \boldsymbol{J}\boldsymbol{\omega}$$

or in Lagrangian form

$$\boldsymbol{J}\dot{\boldsymbol{\omega}} = \boldsymbol{J}\boldsymbol{\omega} \times \boldsymbol{\omega}$$

where $\boldsymbol{J}$ is the symmetric 3×3 *inertia matrix*, defined by $\boldsymbol{J}\boldsymbol{e}_i = \mathcal{J}e_i = \mathcal{L}^{-1}\mathcal{A}e_i$, with $\{\boldsymbol{e}_i\}_{i=1}^3$ being the canonical basis in $\mathbb{R}^3$.

Example 2.2 (Ideal hydrodynamics)**.** This is the second example of Arnold [2]. Let (M, g) be a compact Riemannian manifold of dimension n, possibly with boundary. The group we consider is first $\mathrm{Diff}_{\mathrm{vol}}(M)$, i.e., the set of volume preserving diffeomorphism, and later also the subgroup $\mathrm{Diff}_{\mathrm{vol}}^{\mathrm{ex}}(M) \subset \mathrm{Diff}_{\mathrm{vol}}(M)$ of exact volume preserving diffeomorphisms.

It holds that $\mathrm{Diff}_{\mathrm{vol}}(M)$ is a Fréchet Lie subgroup of $\mathrm{Diff}(M)$; see [12]. Its Fréchet Lie algebra is $\mathfrak{X}_{\mathrm{vol},t}(M) = \mathfrak{X}_{\mathrm{vol}}(M) \cap \mathfrak{X}_t(M)$, i.e., the set of divergence free vector fields on M, tangent to the boundary ∂M. The metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathcal{A}}$ on $\mathrm{Diff}_{\mathrm{vol}}(M)$ is right translation of the L^2 inner product, i.e.,

$$\langle \xi, \eta \rangle_{\mathcal{A}} = \langle \xi, \eta \rangle_{L^2} := \int_M \mathsf{g}(\xi, \eta)\mathrm{vol},$$

where vol is the volume form associated with the Riemannian metric g. We now present various choices of pairings which renders different representations of the Euler-Arnold equation for ideal hydrodynamics.

(a) The classical choice is the inertia pairing, i.e., $\mathfrak{g}^{\bullet} = \mathfrak{X}_{\mathrm{vol},t}(M)$ and $\mathcal{L} = \mathcal{A}$ (the L^2 inner product is used for the pairing). Straightforward calculations yield (see e.g. any of [4, 18, 21])

$$\mathrm{ad}_\xi^{\bullet}(\xi) = P(\nabla_\xi \xi)$$

where ∇ denotes the Levi-Civita connection and $P : \mathfrak{X}(M) \to \mathfrak{X}_{\mathrm{vol},t}(M)$ is the L^2 projection onto $\mathfrak{X}_{\mathrm{vol},t}(M)$, i.e., projection along $\mathfrak{X}_{\mathrm{vol},t}^{\perp_{\mathcal{A}}}$. From the Hodge decomposition for manifold with boundary (see [1, Sect. 7.5]), it follows that $\mathfrak{X}_{\mathrm{vol},t}^{\perp_{\mathcal{A}}} = \mathrm{grad}(\mathcal{F}(M))$. We now recover the well known Euler equation of an ideal incompressible fluid:

$$\dot{\xi} = -\nabla_\xi \xi - \mathrm{grad}\, p, \quad \mathrm{div}\, \xi = 0. \tag{5}$$

Notice that the additional pressure function p (corresponding to Lagrangian multiplier) must be used in this representation, due to the projection operator P occurring in the expression for $\mathrm{ad}^\bullet_\xi$.

(b) There is another commonly used choice of pairing, in which the Euler fluid equation takes a simpler form (see e.g. [4, Sect. 7.B]). The space of k–forms on M is denoted $\Omega^k(M)$. Every vector field ξ on M corresponds to a 1-form $\xi^\flat$, by the flat operator $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ defined by contraction with the metric. Its inverse is given by the sharp operator $\sharp : \Omega^1(M) \to \mathfrak{X}(M)$. Consider the map $T : \Omega^1(M) \to \mathfrak{X}_{\mathrm{vol},t}(M)$ defined by $T\alpha = P\alpha^\sharp$, were P is the projection operator as above. Clearly, the kernel is given by $\ker T = \mathrm{grad}(\mathcal{F}(M))^\flat = \mathrm{d}\Omega^0(M)$, i.e., the exact 1–forms. Thus, we have a corresponding isomorphism $\mathcal{T} : \Omega^1(M)/\,\mathrm{d}\Omega^0(M) \to \mathrm{im}\, T = \mathfrak{X}_{\mathrm{vol},t}(M)$, so we may chose $\mathfrak{g}^\bullet = \Omega^1(M)/\,\mathrm{d}\Omega^0(M)$, with pairing operator defined by

$$\langle \mathcal{L}\bar{\mu}, \xi\rangle = \langle \mathcal{T}\bar{\mu}, \xi\rangle_{L^2} \qquad \forall \xi \in \mathfrak{X}_{\mathrm{vol},t}(M).$$

Since $\mathrm{Diff}_{\mathrm{vol}}(M)$ acts on $\mathfrak{X}_{\mathrm{vol},t}(M)$ by coordinate changes, and due to preservation of the volume form, it holds that $\mathrm{ad}^\bullet_\xi(\bar{\mu}) = \pounds_\xi \bar{\mu}$, where the Lie derivative is well defined on $\Omega^1(M)/\,\mathrm{d}\Omega^0(M)$ since it maps exact forms to exact forms. For details see [11, Chap. 3]. Thus, the Hamiltonian form of the Euler-Arnold equation with this pairing is

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mu} = -\pounds_\xi \bar{\mu}, \quad \xi = \mathcal{T}\bar{\mu}, \tag{6a}$$

and the corresponding Lagrangian form of the equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}[\xi^\flat] = -\pounds_\xi [\xi^\flat]. \tag{6b}$$

(c) There is another choice of pairing, "in between" the choices **(a)** and **(b)**, sometimes used in the literature (see e.g. [15, Sect. 14.1]). Recall that the metric on M induces the Hodge star operator $\star : \Omega^k(M) \to \Omega^{n-k}(M)$. The L^2 inner product on $\Omega^k(M)$ is given by

$$\langle \alpha, \beta\rangle_{L^2} := \int_M \alpha \wedge \star\beta. \tag{7}$$

From the Hodge decomposition theorem for manifolds with boundary (see [1, Sect. 7.5]) we get an L^2 orthogonal decomposition $\Omega^k(M) = \mathrm{d}\Omega^{k-1}(M) \oplus \mathsf{D}^k_t(M)$, where $\mathsf{D}^k_t(M) = \{\alpha \in \Omega^k(M); \delta\alpha = 0, i^*(\star\alpha) = 0\}$ are the co-closed tangential k–forms ($i : \partial M \to M$ is the natural inclusion and $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is the co-differential). It follows from the de Rham complex (see Figure 1) that $\mathfrak{X}_{\mathrm{vol},t}(M) \simeq \mathsf{D}^1_t(M)$, with an isomorphism given by $\xi \mapsto \xi^\flat$. Indeed, if $\xi \in \mathfrak{X}_{\mathrm{vol},t}(M)$ then $\mathrm{i}_\xi\mathrm{vol}$ is a closed normal $(n-1)$–form, since the isomorphism $\xi \mapsto \mathrm{i}_\xi\mathrm{vol}$ maps tangential vector fields to normal $(n-1)$–forms, and since div on $\mathfrak{X}(M)$ corresponds to d on $\Omega^{n-1}_n(M)$. Further, the Hodge star maps tangential k–forms to normal $(n-k)$–forms, and co-closed k–forms to closed $(n-k)$–forms, so

$$\mathfrak{X}_{\mathrm{vol},t}(M) \ni \xi \mapsto \star^{-1}\mathrm{i}_\xi\mathrm{vol} = \star^{-1} \star \xi^\flat = \xi^\flat \in \mathsf{D}^1_t(M)$$

is an isomorphism. Thus, we may chose $\mathfrak{g}^\bullet = \mathsf{D}^1_t(M)$, with pairing operator defined by

$$\langle \mathcal{L}\bar{\mu}, \xi\rangle = \langle \bar{\mu}, \xi^\flat\rangle_{L^2}.$$

The $\mathrm{ad}^\bullet_\xi$ operator is obtained by direct calculations:

$$\begin{aligned}
\langle \mathrm{ad}^*_\xi(\mu), \eta\rangle &= \langle \mathcal{L}\mathrm{ad}^\bullet_\xi(\mathcal{L}^{-1}\mu), \eta\rangle = \langle \mathcal{L}\mathrm{ad}^\bullet_\xi(\bar\mu), \eta\rangle \\
&= \langle \mathrm{ad}^\bullet_\xi(\bar\mu), \eta^\flat\rangle_{L^2} = \langle \bar\mu, (\mathrm{ad}_\xi(\eta))^\flat\rangle_{L^2} = \langle \bar\mu, -[\xi,\eta]^\flat\rangle_{L^2} \\
&= -\int_M \bar\mu \wedge \star[\xi,\eta]^\flat = -\int_M \bar\mu \wedge \mathrm{i}_{[\xi,\eta]}\mathrm{vol} \\
&= -\int_M \bar\mu \wedge \pounds_\xi \mathrm{i}_\eta \mathrm{vol} + \int_M \bar\mu \wedge \mathrm{i}_\eta \underbrace{\pounds_\xi \mathrm{vol}}_{0} \\
&= \int_M \pounds_\xi\bar\mu \wedge \mathrm{i}_\eta\mathrm{vol} - \int_M \pounds_\xi(\bar\mu \wedge \mathrm{i}_\eta\mathrm{vol}) \\
&= \int_M \pounds_\xi\bar\mu \wedge \star\eta^\flat - \int_{\partial M} i^*(\mathrm{i}_\xi(\bar\mu\wedge\star\eta^\flat)) \\
&= \int_M \pounds_\xi\bar\mu \wedge \star\eta^\flat - \int_{\partial M} \mathrm{i}_{i^*\xi}(i^*(\bar\mu) \wedge \underbrace{i^*(\star\eta^\flat)}_{0}) \\
&= \langle \pounds_\xi\bar\mu, \eta^\flat\rangle_{L^2}.
\end{aligned}$$

Thus, $\mathrm{ad}^\bullet_\xi(\bar\mu) = P(\pounds_\xi\bar\mu)$, where $P : \Omega^1(M) \to \mathsf{D}^1_t(M)$ is the L^2 orthogonal projection. Since the orthogonal complement of $\mathsf{D}^1_t(M)$ is $\mathrm{d}\Omega^0(M)$ (by the Hodge decomposition theorem), we get $\mathrm{ad}^\bullet_\xi(\bar\mu) = \pounds_\xi\bar\mu + \mathrm{d}p$, for some $p \in \Omega^0(M)$. Thus, the Hamiltonian form of the Euler-Arnold equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar\mu = -\pounds_\xi\bar\mu - \mathrm{d}p, \quad \delta\bar\mu = 0, \quad \bar\mu = \xi^\flat, \tag{8a}$$

and the Lagrangian form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi^\flat = -\pounds_\xi\xi^\flat - \mathrm{d}p, \quad \delta\xi^\flat = 0. \tag{8b}$$

Notice the resemblance with both the form (5) and the form (6). Indeed, applying the Riemannian lift yields (5), and applying the quotient map yields (6).

(d) Now let M be a 3–manifold, and consider the subgroup $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M) \subset \mathrm{Diff}_{\mathrm{vol}}(M)$ consisting of exact volume preserving diffeomorphisms (see Section 5.2 below). For this setting, one may use the *vorticity formulation* (see e.g. [16]). The space of exact divergence free tangential vector fields $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ is the Lie algebra of $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M)$. Furthermore, $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ is isomorphic to the space of normal exact 2–forms $\mathrm{d}\Omega^1_n(M)$, with isomorphism given by $\xi \mapsto \mathrm{i}_\xi\mathrm{vol}$. Also, the map $(\mathrm{d}\star)^{-1}$ is well defined on $\mathrm{d}\Omega^1_n(M)$, where it is non-degenerate and L^2 self-adjoint; see Lemma 19 below. Now, we chose $\mathfrak{g}^\bullet = \mathrm{d}\Omega^1_n(M)$ with pairing operator defined by

$$\langle \mathcal{L}\bar\mu, \xi\rangle = \langle \bar\mu, (\mathrm{d}\star)^{-1}\mathrm{i}_\xi\mathrm{vol}\rangle_{L^2}.$$

For $\xi, \eta \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ we now get

$$\begin{aligned}
\langle \mathrm{ad}^*_\xi(\bar\mu), \eta\rangle &= \langle \mathrm{ad}^\bullet_\xi\bar\mu, (\mathrm{d}\star)^{-1}\mathrm{i}_\eta\mathrm{vol}\rangle_{L^2} = \langle \bar\mu, (\mathrm{d}\star)^{-1}\mathrm{i}_{\mathrm{ad}_\xi(\eta)}\mathrm{vol}\rangle_{L^2} \\
&= -\langle \bar\mu, (\mathrm{d}\star)^{-1}\mathrm{i}_{[\xi,\eta]_{\mathfrak{X}}}\mathrm{vol}\rangle_{L^2} \\
&= -\langle \bar\mu, (\mathrm{d}\star)^{-1}\pounds_\xi\mathrm{i}_\eta\mathrm{vol}\rangle_{L^2} + \langle \bar\mu, (\mathrm{d}\star)^{-1}\mathrm{i}_\eta \underbrace{\pounds_\xi\mathrm{vol}}_{0}\rangle_{L^2} \\
&= \langle \pounds_\xi\bar\mu, (\mathrm{d}\star)^{-1}\mathrm{i}_\eta\mathrm{vol}\rangle_{L^2}
\end{aligned}$$

where the last equality follows from the bi-invariant property of cross helicity (see Section 5.4 below). Notice that $\pounds_\xi\bar\mu$ is exact since d commutes with the Lie derivative. Thus, $\mathrm{ad}^\bullet_\xi = \pounds_\xi$.

$$
\begin{tikzcd}
& \mathcal{F}(M) \arrow[r, "\mathrm{grad}"] \arrow[d, leftrightarrow, "\mathrm{Id}"'] \arrow[rrrr, bend left, "\mathrm{Id}"] & \mathfrak{X}(M) \arrow[rr, bend left, "\mathrm{Id}"] \arrow[d, shift left, "\flat"] & & \mathfrak{X}(M) \arrow[r, "\mathrm{div}"] \arrow[d, "\xi\mapsto \mathrm{i}_\xi\,\mathrm{vol}"] & \mathcal{F}(M) \arrow[d, "f\mapsto f\,\mathrm{vol}"] & \\
0 \arrow[r] & \Omega^0(M) \arrow[r, "\mathrm{d}"] \arrow[rrrr, bend right, "\star"'] & \Omega^1(M) \arrow[r, "\mathrm{d}"] \arrow[u, shift left, "\sharp"] \arrow[rr, bend right, "\star"'] & \cdots \arrow[r, "\mathrm{d}"] & \Omega^{n-1}(M) \arrow[r, "\mathrm{d}"] & \Omega^n(M) \arrow[r] & 0
\end{tikzcd}
$$

Figure 1: The de Rham complex of a Riemannian manifold M. The upper sequence corresponds to the lower sequence with identifications given by the vertical arrows (which are isomorphisms). Likewise, the upper curved arrows corresponds to the lower curved arrows.

Next, notice that $\mathcal{A}\xi = \mathcal{L}\,\mathrm{d}\star \mathrm{i}_\xi\mathrm{vol}$, so $\mathcal{J}\xi = \mathrm{d}\star\mathrm{i}_\xi\mathrm{vol} = \mathrm{d}\star\star\xi^\flat = \mathrm{d}\xi^\flat$. Now, the Hamiltonian form of the Euler-Arnold equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mu} = -\pounds_\xi\bar{\mu}, \quad \bar{\mu} = \mathcal{J}\xi = \mathrm{d}\xi^\flat. \tag{9a}$$

and the Lagrangian form is

$$\mathrm{d}\dot{\xi}^\flat = -\pounds_\xi\,\mathrm{d}\xi^\flat. \tag{9b}$$

The requirement for solutions of equation (9b) to also fulfill equation (8b) is that $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ is totally geodesic inside $\mathrm{Diff}_{\mathrm{vol}}(M)$. The exact condition for this is given in [8], in the case when M has no boundary. It is extended to the case when M has a boundary, and for a possibly altered H^1_α metric (corresponding to the *averaged Euler fluid equation*), in Theorem 15 below.

(e) There is another vorticity formulation, which is perhaps the most elegant form of the ideal hydrodynamic fluid equation. Again, the requirements is that M is a 3–manifold, and we consider the subgroup $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M)$. As pairing space we chose $\mathfrak{g}^\bullet = \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$, and the pairing operator is defined by

$$\langle\mathcal{L}\bar{\mu},\xi\rangle = \langle\bar{\mu},\mathrm{curl}^{-1}\xi\rangle_{L^2}.$$

It follows from Lemma 19 that curl^{-1} is well defined on $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$. Further, using Theorem 17, it follows that $\mathrm{ad}^\bullet_\xi(\bar{\mu}) = -\mathrm{ad}_\xi(\bar{\mu}) = [\xi,\bar{\mu}]_{\mathfrak{X}}$, since the inner product $\langle\cdot,\mathrm{curl}^{-1}\cdot\rangle_{L^2}$ is bi-invariant. Moreover, $\mathcal{J}\xi = \mathrm{curl}\,\xi$, so the Hamiltonian form of the Euler-Arnold equation takes the Lax pairing form

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mu} = -[\xi,\bar{\mu}]_{\mathfrak{X}}, \quad \bar{\mu} = \mathcal{J}\xi = \mathrm{curl}\,\xi. \tag{10a}$$

The corresponding Lagrangian form of the equation is

$$\mathrm{curl}\,\dot{\xi} = -[\xi,\mathrm{curl}\,\xi]_{\mathfrak{X}}. \tag{10b}$$

From the de Rham complex of a 3–manifold (see Figure 2) it follows that the previous form **(d)** is obtained from these equations by applying the flat operator followed by the Hodge star. Again, we remark that these equations give solutions corresponding to solutions of the full Euler fluid equation only in the case when $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ is totally geodesic in $\mathrm{Diff}_{\mathrm{vol}}(M)$.

3 Totally Geodesic Subgroups

First, recall the definition of totally geodesic:

Definition 3.1. Let M be a Riemannian manifold, with metric $\langle\!\langle\cdot,\cdot\rangle\!\rangle$, and let $N \subset M$ be a submanifold with the induced metric. Then N is called *totally geodesic in M with respect to* $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ if each geodesic of N, embedded in M, is also a geodesic of M.

Now, let G be a Lie group equipped with a left (or right) invariant metric. Let H be a Lie subgroup of G, i.e., a topologically closed submanifold which is closed under the group multiplication inherited from G. The main ambition of our paper is to investigate conditions on the left invariant metric $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathcal{A}}$ under which a given subgroup H is totally geodesic in G. Due to left invariance, it is enough to consider the Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$:

Lemma 1. *H is totally geodesic in G if and only if all solutions $\xi(t) \in \mathfrak{h}$ to the Euler-Arnold equation on $\mathfrak{h}$ are also solutions to the Euler-Arnold equation on $\mathfrak{g}$.*

Proof. Let $i : H \to G$ be the inclusion map, let L_h^H and L_g^G for $h \in H$ and $g \in G$ be the left translation maps on H and G respectively. Then $i \circ \mathsf{L}_h^H = \mathsf{L}_{i(h)}^G \circ i$, so $Ti \circ T\mathsf{L}_h^H = T\mathsf{L}_{i(h)}^G \circ Ti$.

Now, let $\xi(t) \in \mathfrak{g}$ be the solution to the Euler-Arnold equation on $\mathfrak{g}$ with initial data $\xi(0) = T_{e^G} i\psi_0$ for $\psi_0 \in \mathfrak{h}$ (e^G is the identity element in G). Further, let $\psi(t) \in \mathfrak{h}$ be the solution to the Euler-Arnold equation on $\mathfrak{h}$ with initial data $\psi(0) = \psi_0$. Let $g(t) \in G$ and $h(t) \in H$ be the corresponding geodesic curves. We need to prove that $i(h(t)) = g(t)$ if and only if $\xi(t) = T_{e^H} i\psi(t)$. The curve $h(t)$ fulfils $\dot{h}(t) = T_{e^H}\mathsf{L}_{h(t)}^H \psi(t)$ with $h(0) = e^H$. Applying Ti from the right and using the identity derived above we get

$$\frac{\mathrm{d}}{\mathrm{d}t} i(h(t)) = T_{h(t)} i \circ T_{e^H}\mathsf{L}_{h(t)}^H \psi(t) = T_{e^G}\mathsf{L}_{i(h(t))}^G \circ T_{e^H} i\psi(t), \qquad i(h(0)) = e^G.$$

Thus, $i(h(t))$ fulfils the same reconstruction equation as $g(t)$ if and only if $\xi(t) = T_{e^H} i\psi(t)$, so the result follows by uniqueness of solutions. □

Thus, we say that a subalgebra $\mathfrak{h}$ is totally geodesic in $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ if solutions to the Euler-Arnold equation for $\mathfrak{h}$ are also solution to the Euler-Arnold equation for $\mathfrak{g}$. Whether this holds or not depends upon an interplay between the choice of subalgebra $\mathfrak{h}$ and the choice of metric $\langle\cdot,\cdot\rangle_{\mathcal{A}}$. As a basic tool we have the following result, of which $1 \leftrightarrow 4$ is stated in [8]:

Theorem 2. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and let $\mathfrak{h}^{\perp_{\mathcal{A}}}$ the orthogonal complement of $\mathfrak{h}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$. The following statements are equivalent:*

1. *$\mathfrak{h}$ is totally geodesic in $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{A}}$.*
2. *$\langle\xi,[\xi,\eta]\rangle_{\mathcal{A}} = 0$ for all $\xi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}^{\perp_{\mathcal{A}}}$.*
3. *$\langle\psi,[\xi,\eta]\rangle_{\mathcal{A}} + \langle\xi,[\psi,\eta]\rangle_{\mathcal{A}} = 0$ for all $\xi,\psi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}^{\perp_{\mathcal{A}}}$.*
4. *$\mathrm{ad}_\xi^{\top_{\mathcal{A}}}(\xi) \in \mathfrak{h}$ for all $\xi \in \mathfrak{h}$.*
5. *$\mathrm{ad}_\xi^{\top_{\mathcal{A}}}(\psi) + \mathrm{ad}_\psi^{\top_{\mathcal{A}}}(\xi) \in \mathfrak{h}$ for all $\xi,\psi \in \mathfrak{h}$.*

Proof. We first prove $1 \leftrightarrow 2$. Let $\xi(t)$ be a solution to the Euler-Arnold equation (2) on $\mathfrak{h}$. Consider the weak formulation (2a). Every test function can be uniquely written $\eta = \eta_1 + \eta_2$ with $\eta_1 \in \mathfrak{h}$ and $\eta_2 \in \mathfrak{h}^{\perp_{\mathcal{A}}}$. Since $\xi(t) \in \mathfrak{h}$ for all t it holds that $\langle\dot{\xi}(t),\eta_2\rangle_{\mathcal{A}} = 0$. Thus, in order for $\xi(t)$ to be totally geodesic, i.e., fulfil the Euler-Arnold equation (2a) on $\mathfrak{g}$, a sufficient condition is that $\langle\xi(t),[\xi(t),\eta_2]\rangle_{\mathcal{A}} = 0$ for all $\eta_2 \in \mathfrak{h}^\perp$. Since the initial condition $\xi(0) \in \mathfrak{h}$ is arbitrary the condition is also necessary. Next, $2 \leftrightarrow 3$ follows since the bi-linear form $Q_\eta(\xi,\psi) := \langle\xi,[\psi,\eta]\rangle_{\mathcal{A}}$ fulfils $Q_\eta(\xi,\xi) = 0$ if and only if Q_η is skew-symmetric. Lastly, $2 \leftrightarrow 4$ and $3 \leftrightarrow 5$ follows from the definition of $\mathrm{ad}_\xi^{\top_{\mathcal{A}}}$ and the fact that $\mathfrak{g}$ is spanned by $\mathfrak{h} \oplus \mathfrak{h}^{\perp_{\mathcal{A}}}$. □

A geometric interpretation of the result in Theorem 2 is that $\mathfrak{h}$ is totally geodesic if and only if $[\xi,\eta]\perp_{\mathcal{A}}\xi$ for all $\xi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}^{\perp_{\mathcal{A}}}$. That is, $[\xi,\eta]$ must belong to the hyperplane which is $\mathcal{A}$–orthogonal to ξ.

Remark 3.1. Notice that Theorem 2 is valid also in the case when $\mathfrak{g}$ is an infinite dimensional Fréchet Lie algebra. Indeed, a Fréchet Lie subalgebra $\mathfrak{h}$ is (by definition) a topologically closed linear subspace of $\mathfrak{g}$ which is closed under the bracket. Thus, every $\eta \in \mathfrak{g}$ admits a unique decomposition $\eta = \eta_1 + \eta_2$ with $\eta_1 \in \mathfrak{h}$ and $\eta_2 \in \mathfrak{h}^{\perp_{\mathcal{A}}}$.

Example 3.1 (Rigid body, cont.)**.** Consider again Example 2.1. A one dimensional subalgebra of $\mathfrak{so}(3)$ is given by $\mathfrak{h} = \{\xi \in \mathfrak{so}(3); \xi = ae_1, a \in \mathbb{R}\}$. Since it is one dimensional the bracket is trivial, so the Euler-Arnold equation on $\mathfrak{h}$ reduce to $\dot{\xi} = 0$, i.e., all solutions are stationary.

From Theorem 2 we obtain that $\mathfrak{h}$ is totally geodesic if and only if $\mathrm{ad}_{\xi}^{\top_{\mathcal{A}}}(\xi) \in \mathfrak{h}$ for all $\xi \in \mathfrak{h}$. Expressed in $\mathrm{ad}^{\bullet}$ this means $\mathrm{ad}_{e_1}^{\bullet}(J\hat{e}_1) = b\hat{e}_1$ for some $b \in \mathbb{R}$. (Since solutions are stationary we know that $b = 0$.) Explicitly, this reads $\hat{e}_1 \times J\hat{e}_1 = b\hat{e}_1$, which happens if and only if $\hat{e}_1$ and $J\hat{e}_1$ are parallel, i.e., $\hat{e}_1$ is an eigenvector of J. Indeed, it is well known that the only stationary solutions to the Euler-Arnold equation on $\mathfrak{so}(3)$ are given by the set of eigenvectors of the inertia matrix J.

3.1 Derivation Using the Second Fundamental Form

In this section we give a different derivation of Theorem 2, based on computing the second fundamental form. This derivation gives more geometrical insight to the process.

To begin with, recall the following well known result (see e.g. [13]):

Theorem 3. *Let N be a submanifold of a Riemannian manifold M with metric g. Then N is totally geodesic in M if and only if the second fundamental form of N vanishes identically.*

Thus, an alternative approach for deriving Theorem 2 is to compute the second fundamental form of the subgroup $H \subset G$. Again, $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathcal{A}}$ denotes a left invariant metric on G, and we use the same notation for its restriction to H. Let e denote the identity element of G. Recall that the second fundamental form is a symmetric tensor on TH, given by

$$\Pi(X, Y) := (\nabla_X Y)^{\perp_{\mathcal{A}}}.$$

Because the metric is invariant, this tensor is determined by its values on the tangent vectors to H at the identity. It therefore follows that we can determine the tensor by computing a formula for

$$\langle\!\langle \nabla_{X_\xi} Y_\eta, Z_\psi \rangle\!\rangle_{\mathcal{A},e},$$

where X_ξ denotes the left invariant vector field on G whose value at e is ξ, and similarly for Y_η and Z_ψ. Indeed, we have the following result:

Proposition 4. *For left invariant vector fields X_ξ, Y_η on G the following formula holds,*

$$\nabla_{X_\xi} Y_\eta(e) = \frac{1}{2}\big([\xi, \eta] - \mathrm{ad}_{\xi}^{\top_{\mathcal{A}}}(\eta) - \mathrm{ad}_{\eta}^{\top_{\mathcal{A}}}(\xi)\big). \tag{11}$$

Proof. Starting with the defining identity for the connection,

$$\begin{aligned} 2\langle\!\langle \nabla_X Y, Z \rangle\!\rangle_{\mathcal{A}} &= \pounds_X \langle\!\langle Y, Z \rangle\!\rangle_{\mathcal{A}} + \pounds_Y \langle\!\langle Z, X \rangle\!\rangle_{\mathcal{A}} - \pounds_Z \langle\!\langle X, Y \rangle\!\rangle_{\mathcal{A}} \\ &\quad - \langle\!\langle Y, [X, Z] \rangle\!\rangle_{\mathcal{A}} - \langle\!\langle Z, [Y, X] \rangle\!\rangle_{\mathcal{A}} + \langle\!\langle X, [Z, Y] \rangle\!\rangle_{\mathcal{A}} \end{aligned}$$

and replacing X by X_ξ, Y by Y_η and Z by Z_λ and noting that the first three terms then vanish due to left invariance of the vector fields and metric, we then have, evaluating at the identity,

$$\begin{aligned} 2\langle\!\langle \nabla_{X_\xi} Y_\eta, Z_\lambda \rangle\!\rangle_{\mathcal{A},e} &= -\langle\!\langle Y_\eta, [X_\xi, Z_\lambda] \rangle\!\rangle_{\mathcal{A},e} - \langle\!\langle Z_\lambda, [Y_\eta, X_\xi] \rangle\!\rangle_{\mathcal{A},e} + \langle\!\langle X_\xi, [Z_\lambda, Y_\eta] \rangle\!\rangle_{\mathcal{A},e} \\ &= -\langle \eta, [\xi, \lambda] \rangle_{\mathcal{A}} - \langle \lambda, [\eta, \xi] \rangle_{\mathcal{A}} + \langle \xi, [\lambda, \eta] \rangle_{\mathcal{A}} \\ &= \langle -\mathrm{ad}_{\xi}^{\top_{\mathcal{A}}} \eta, \lambda \rangle_{\mathcal{A}} - \langle [\eta, \xi], \lambda \rangle_{\mathcal{A}} - \langle \mathrm{ad}_{\eta}^{\top_{\mathcal{A}}} \xi, \lambda \rangle_{\mathcal{A}} \\ &= \langle [\xi, \eta] - \mathrm{ad}_{\xi}^{\top_{\mathcal{A}}} \eta - \mathrm{ad}_{\eta}^{\top_{\mathcal{A}}} \xi, \lambda \rangle_{\mathcal{A}} \end{aligned}$$

which proves the result. $\square$

Using this result we obtain a different proof of Theorem 2 above:

Another proof of Theorem 2. From equation (11) we have that the fundamental form, Π of H is given by, for $\xi, \psi \in \mathfrak{h}$

$$\Pi(X,Y)(e) = (\nabla_X Y)^{\perp_{\mathcal{A}}} = \frac{1}{2}([\xi,\psi] - \mathrm{ad}_\xi^{\top_{\mathcal{A}}}\psi - \mathrm{ad}_\psi^{\top_{\mathcal{A}}}\xi)^{\perp_{\mathcal{A}}} \tag{12}$$

where X and Y are arbitrary vector fields extending ξ and ψ. From this equation it follows that the fundamental form vanishes if and only if the pairing of the right hand side of (12) with every element $\eta \in \mathfrak{h}^{\perp_{\mathcal{A}}}$ vanishes. Thus, the second fundamental form is zero if and only if for all $\xi, \psi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}^{\perp_{\mathcal{A}}}$ it holds that $\langle([\xi,\psi] - \mathrm{ad}_\xi^{\top_{\mathcal{A}}}\psi - \mathrm{ad}_\psi^{\top_{\mathcal{A}}}\xi), \eta\rangle_{\mathcal{A}} = 0$. Since $[\xi,\psi] \in \mathfrak{h}$, this holds if and only if for all $\xi, \psi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}^{\perp_{\mathcal{A}}}$, it holds that $\langle \mathrm{ad}_\xi^{\top_{\mathcal{A}}}\psi + \mathrm{ad}_\psi^{\top_{\mathcal{A}}}\xi, \eta\rangle_{\mathcal{A}} = 0$. This is equivalent to $\langle\psi, [\xi,\eta]\rangle_{\mathcal{A}} = -\langle\xi, [\psi,\eta]\rangle_{\mathcal{A}}$ for all $\xi, \psi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}^{\perp_{\mathcal{A}}}$. In particular, letting $\psi = \xi$, this last equation says that $\langle\xi, [\xi,\eta]\rangle_{\mathcal{A}} = -\langle\xi, [\xi,\eta]\rangle_{\mathcal{A}}$ for all $\eta \in \mathfrak{h}^{\perp_{\mathcal{A}}}$ and therefore $\langle\xi, [\xi,\eta]\rangle_{\mathcal{A}} = 0$. This yields condition (2) in Theorem 2. □

3.2 Coordinate Form of Theorem 2

We now work out the consequence of Theorem 2 in terms of the structure constants of the Lie algebra and the symmetric matrix of the inner product. That is, we investigate the condition for a subalgebra to be totally geodesic from a coordinate point of view.

Let $\mathfrak{g}$ be of finite dimension n, and let $\mathfrak{h}$ be a subalgebra of dimension $m < n$. Further, let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$ such that $\mathfrak{h}$ is spanned by $e_1, \ldots, e_m$. The corresponding Cartesian coordinate vectors in $\mathbb{R}^n$ are denoted with bold symbols $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. We denote by C_i the matrix representation of ad_{e_i} in the given basis. In terms of the structure constants c_{ij}^k for the bracket we have $\boldsymbol{e}_k^\top C_i \boldsymbol{e}_j = c_{ij}^k$.

For every inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ on $\mathfrak{g}$ there corresponds a symmetric matrix $A = (a_{ij})$ defined by $a_{ij} = \langle e_i, e_j\rangle_{\mathcal{A}}$. The following result is a statement of Theorem 2 in terms of the matrices C_j and A:

Proposition 5. *The subalgebra* $\mathfrak{h}$ *is totally geodesic with respect to* $(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathcal{A}})$ *if and only if*

$$\boldsymbol{e}_i^\top A C_j A^{-1}\boldsymbol{e}_k + \boldsymbol{e}_j^\top A C_i A^{-1}\boldsymbol{e}_k = 0 \qquad \textit{for all} \qquad \begin{cases} i,j & \in \{1,\ldots,m\} \\ k & \in \{m+1,\ldots,n\} \end{cases}.$$

Proof. From Theorem 2 it follows that $\mathfrak{h}$ is totally geodesic if and only if

$$\langle\xi, [\xi,\eta]\rangle_{\mathcal{A}} = 0 \qquad \forall\, \xi \in \mathfrak{h} \quad \text{and} \quad \eta \in \mathfrak{h}^\perp.$$

This is true if and only if the kernel of each of the one-forms $\phi_\xi : \eta \mapsto \langle\xi, [\xi,\eta]\rangle_{\mathcal{A}}$, with $\xi \in \mathfrak{h}$, contains $\mathfrak{h}^\perp$. Since $\xi \mapsto \phi_\xi$ is quadratic, this holds if and only of ϕ_ξ is of the form

$$\phi_\xi = \sum_{i,j,k=1}^{m} \xi^i \xi^j b_{ijk} \langle e_k, \cdot\,\rangle_{\mathcal{A}} \tag{13}$$

for some tensor b_{ijk}. On the other hand, direct expansion of ϕ_ξ in the basis gives

$$\phi_\xi = \sum_{i,j=1}^{m} \xi^i\xi^j \langle e_i, [e_j,\cdot]\rangle_{\mathcal{A}} = \sum_{i,j=1}^{m} \xi^i\xi^j \langle \mathrm{ad}_{e_j}^\top e_i, \cdot\rangle_{\mathcal{A}} = \sum_{k=1}^{n}\sum_{i,j=1}^{m} \xi^i\xi^j b'_{ijk}\langle e_k,\cdot\rangle_{\mathcal{A}} \tag{14}$$

where $b'_{ijk} = \boldsymbol{e}_k^\top A^{-1} C_j^\top A \boldsymbol{e}_i$. (Notice that $A^{-1}C_j^\top A$ is the matrix representation of $\mathrm{ad}_{e_j}^{\top_{\mathcal{A}}}$.) Comparing equation (13) and equation (14) we get the condition

$$\sum_{i,j=1}^{m} \xi^i\xi^j \boldsymbol{e}_k^\top A^{-1} C_j^\top A \boldsymbol{e}_i = 0, \qquad \forall\, k \in \{m+1,\ldots,n\}.$$

This is true for all $\boldsymbol{\xi} \in \mathbb{R}^m$ if and only if the condition in the theorem is fulfilled. □

Notice that if the basis $e_1, \ldots, e_n$ diagonalises A (i.e. $A = \mathrm{Id}$ in the basis), then the condition reads $\boldsymbol{e}_i^\top C_j \boldsymbol{e}_k + \boldsymbol{e}_j^\top C_i \boldsymbol{e}_k = 0$. From skew-symmetry of the bracket we get $\boldsymbol{e}_i^\top C_k \boldsymbol{e}_j + \boldsymbol{e}_j^\top C_k \boldsymbol{e}_i = 0$. In turn, this implies that the leading $m \times m$–block of the C_k matrices are skew-symmetric for $k \in m+1, \ldots, n$. The equivalent non-coordinate statement is that $\langle \mathrm{ad}_\eta \xi, \psi \rangle_{\mathcal{A}} + \langle \mathrm{ad}_\eta^{\top_{\mathcal{A}}} \xi, \psi \rangle_{\mathcal{A}} = 0$ for all $\eta \in \mathfrak{h}^{\perp_{\mathcal{A}}}$ and $\xi, \psi \in \mathfrak{h}$, which again recovers Theorem 2.

4 Totally Geodesic Metrics

In this section the setting is the following: given a Lie group G and a subgroup H, find a right invariant Riemannian metric such that H is totally geodesic. Thus, throughout this section the condition in Theorem 2 is interpreted as a condition on the inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ in order for the subalgebra $\mathfrak{h}$ of H to be totally geodesic in the algebra $\mathfrak{g}$ of G.

4.1 Construction with Invariant Form

As a start, consider first the case when the full algebra $\mathfrak{g}$ is finite dimensional and semisimple. In particular, this implies that the Killing form, denoted $\langle \cdot, \cdot \rangle_{\mathcal{K}}$, is non-degenerate (but not necessarily negative definite). Recall that the Killing form is *bi-invariant*, i.e., it fulfils the relation $\langle [\xi, \psi], \eta \rangle_{\mathcal{K}} = -\langle \psi, [\xi, \eta] \rangle_{\mathcal{K}}$ for all $\xi, \psi, \eta \in \mathfrak{g}$. If the corresponding self-adjoint isomorphism $\mathcal{K} : \mathfrak{g} \to \mathfrak{g}^*$ is used as pairing operator, the Lagrangian form of the Euler-Arnold equation 4b takes the "rigid body form"

$$\mathcal{J}\dot{\xi} = -[\xi, \mathcal{J}\xi], \qquad \text{where} \qquad \mathcal{J} = \mathcal{K}^{-1}\mathcal{A}, \tag{15}$$

which is a direct consequence of the bi-invariant property. Indeed, it holds that $\langle \psi, \mathrm{ad}_\xi(\eta) \rangle_{\mathcal{K}} = -\langle \mathrm{ad}_\xi(\psi), \eta \rangle_{\mathcal{K}}$ so $\mathrm{ad}_\xi^\bullet = -\mathrm{ad}_\xi$. Notice that both $\mathcal{K}\mathcal{J}$ and $\mathcal{A}\mathcal{J}$ are self-adjoint operators, so $\mathcal{J}$ is self-adjoint with respect to both $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{A}}$.

Remark 4.1. If, in addition to being semisimple, $\mathfrak{g}$ is also compact, then its Killing form is negative definite. Thus, we may use $\langle \cdot, \cdot \rangle_{\mathcal{A}} = \langle - \cdot, \cdot \rangle_{\mathcal{K}}$ as choice of inner product. Since it is bi-invariant, it follows that *all* subalgebras in $\mathfrak{g}$ fulfil the condition in Theorem 2, i.e., all subalgebras are totally geodesic with respect to a bi-invariant inner product. The dynamics in this case is trivial. Indeed, we have $\mathcal{J} = -\mathrm{Id}$, so the Euler-Arnold equation (15) reduce to $-\dot{\xi} = [\xi, -\xi] = 0$. For the rigid body this happens when all moments of inertia are equal. From a geometric point of view, bi-invariance of the metric implies that geodesics are given by the group exponential.

A direct consequence of Theorem 2 is that $\mathfrak{h}$ being totally geodesic in a semisimple Lie algebra $\mathfrak{g}$ is equivalent to

$$\mathcal{J}^{-1}[\xi, \mathcal{J}\xi] \in \mathfrak{h} \quad \forall \xi \in \mathfrak{h}. \tag{16}$$

In particular this is always true if $\mathfrak{h}$ is an invariant subspace of $\mathcal{J}$, i.e., $\mathcal{J}\mathfrak{h} = \mathfrak{h}$.

We now continue with a generalisation of these ideas, which will lead to a recipe for the construction of totally geodesic metrics.

Definition 4.1. Let $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ be an inner product on $\mathfrak{g}$. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called *easy totally geodesic in* $\mathfrak{g}$ *respect to* $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ if $\mathrm{ad}_{\mathfrak{h}}(\mathfrak{h}^{\perp_{\mathcal{A}}}) \subset \mathfrak{h}^{\perp_{\mathcal{A}}}$, i.e., $\mathfrak{h}^{\perp_{\mathcal{A}}}$ is invariant under $\mathrm{ad}_{\mathfrak{h}}$.

As the name implies, easy totally geodesic is a special case of totally geodesic:

Proposition 6. *Let* $\mathfrak{h}$ *be a subalgebra of* $\mathfrak{g}$ *which is easy totally geodesic in* $\mathfrak{g}$ *with respect to* $\langle \cdot, \cdot \rangle_{\mathcal{A}}$. *Then* $\mathfrak{h}$ *is totally geodesic in* $\mathfrak{g}$ *with respect to* $\langle \cdot, \cdot \rangle_{\mathcal{A}}$.

Proof. Let $\xi, \psi \in \mathfrak{h}$ and $\eta \in \mathfrak{h}^{\perp_{\mathcal{A}}}$. Easy totally geodesic implies $\langle [\xi, \eta], \psi \rangle_{\mathcal{A}} = \langle \mathrm{ad}_\xi(\eta), \psi \rangle_{\mathcal{A}} = 0$, since $\mathrm{ad}_\xi(\eta) \in \mathfrak{h}^{\perp_{\mathcal{A}}}$ and $\psi \in \mathfrak{h}$. The result now follows from Theorem 2 by taking $\psi = \xi$. $\square$

We now develop a method for constructing inner products for which a given subalgebra is easy totally geodesic. The construction generalises the approach described above, where the Killing form was used as a pairing.

Definition 4.2. Let $\mathfrak{h}$ be a subalgebra and V a subspace of $\mathfrak{g}$. A symmetric bi-linear form $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ on $\mathfrak{g}$ is called $\mathrm{ad}_{\mathfrak{h}}$*–invariant on* V if

$$\langle \mathrm{ad}_\xi(\eta),\psi\rangle_{\mathcal{K}} + \langle \eta, \mathrm{ad}_\xi(\psi)\rangle_{\mathcal{K}} = 0 \qquad \forall\, \xi\in\mathfrak{h} \quad \text{and} \quad \forall\, \eta,\psi\in V.$$

If $V=\mathfrak{g}$ we simply call $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ $\mathrm{ad}_{\mathfrak{h}}$*–invariant.*

Notice that bi-invariance is equivalent to $\mathrm{ad}_{\mathfrak{g}}$–invariance. Also notice that $\mathrm{ad}_{\mathfrak{h}}$–invariance is equivalent to $\langle \mathrm{Ad}_h(\xi),\mathrm{Ad}_h(\eta)\rangle_{\mathcal{K}} = \langle\xi,\eta\rangle_{\mathcal{K}}$ for all $h\in H$.

Given an $\mathrm{ad}_{\mathfrak{h}}$–invariant form on $\mathfrak{g}$, which is non-degenerate on $\mathfrak{h}$, we can construct a large class of inner products on $\mathfrak{g}$ for which $\mathfrak{h}$ is totally geodesic. Indeed, we have the following result:

Theorem 7. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h}\subset\mathfrak{g}$ a subalgebra. Further, let $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ be an $\mathrm{ad}_{\mathfrak{h}}$–invariant form on $\mathfrak{g}$, such that its restriction to $\mathfrak{h}$ is non-degenerate, and let $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ be an inner product on $\mathfrak{g}$. Then:*

1. *If $\mathfrak{h}^{\perp_{\mathcal{K}}} = \mathfrak{h}^{\perp_{\mathcal{A}}}$, then $\mathfrak{h}$ is easy totally geodesic in $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{A}}$.*[1]
2. *If $[\mathfrak{h},\mathfrak{h}]=\mathfrak{h}$ and $\mathfrak{h}$ is easy totally geodesic in $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{A}}$, then $\mathfrak{h}^{\perp_{\mathcal{K}}} = \mathfrak{h}^{\perp_{\mathcal{A}}}$.*

Proof. In general, $\mathfrak{g}$ have the two decompositions $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}^{\perp_{\mathcal{A}}}$ and $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}^{\perp_{\mathcal{K}}}$. The inertia operator $\mathcal{A}:\mathfrak{g}\to\mathfrak{g}^*$ can be decomposed as $\mathcal{A}\xi = \mathcal{A}_1\xi_1 + \mathcal{A}_2\xi_2$, where $\xi=\xi_1+\xi_2$ are the unique components in the decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}^{\perp_{\mathcal{A}}}$, and $\mathcal{A}_1:\mathfrak{h}\to\mathfrak{h}^*$, $\mathcal{A}_2:\mathfrak{h}^{\perp_{\mathcal{A}}}\to(\mathfrak{h}^{\perp_{\mathcal{A}}})^*$ are invertible operators. Further, the operator $\mathcal{K}:\mathfrak{g}\to\mathfrak{g}^*$ can be decomposed as $\mathcal{K}\xi = \mathcal{K}_a\xi_a + \mathcal{K}_b\xi_b$, where $\xi=\xi_a+\xi_b$ are the unique components in the decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}^{\perp_{\mathcal{K}}}$, and $\mathcal{K}_a:\mathfrak{h}\to\mathfrak{h}^*$ is invertible.

We first prove assertion *1.* Let $\xi,\psi\in\mathfrak{h}$ and $\eta\in\mathfrak{h}^{\perp_{\mathcal{A}}}$. Then

$$\begin{aligned}
\langle[\xi,\eta],\psi\rangle_{\mathcal{A}} &= \langle\mathcal{A}\psi,[\xi,\eta]\rangle = \langle\mathcal{A}_1\psi,[\xi,\eta]\rangle \\
&= \langle\mathcal{K}_a\underbrace{\mathcal{K}_a^{-1}\mathcal{A}_1}_{\mathcal{J}_1}\psi,[\xi,\eta]\rangle = \langle\mathcal{K}\mathcal{J}_1\psi,[\xi,\eta]\rangle \\
&= \langle\mathcal{J}_1\psi,[\xi,\eta]\rangle_{\mathcal{K}} = -\langle[\xi,\mathcal{J}_1\psi],\eta\rangle_{\mathcal{K}} = 0
\end{aligned}$$

where the last equality follows since $[\xi,\mathcal{J}_1\psi]\in\mathfrak{h}$ and $\eta\in\mathfrak{h}^{\perp_{\mathcal{A}}}=\mathfrak{h}^{\perp_{\mathcal{K}}}$. Thus, $\langle[\xi,\eta],\psi\rangle_{\mathcal{A}}=0$ for all $\xi,\psi\in\mathfrak{h},\eta\in\mathfrak{h}^{\perp_{\mathcal{A}}}$, which means that $[\xi,\eta]\in\mathfrak{h}^{\perp_{\mathcal{A}}}$ for all $\xi\in\mathfrak{h},\eta\in\mathfrak{h}^{\perp_{\mathcal{A}}}$, i.e., $\mathfrak{h}$ is easy totally geodesic.

Next we prove assertion *2.* Again, let $\xi,\psi\in\mathfrak{h}$ and $\eta\in\mathfrak{h}^{\perp_{\mathcal{A}}}$. Then, since $\mathfrak{h}$ is easy totally geodesic, it holds that

$$\begin{aligned}
0 &= \langle[\xi,\eta],\psi\rangle_{\mathcal{A}} = \langle\mathcal{A}\psi,[\xi,\eta]\rangle = \langle\mathcal{A}_1\psi,[\xi,\eta]\rangle \\
&= \langle\mathcal{K}_a\mathcal{K}_a^{-1}\mathcal{A}_1\psi,[\xi,\eta]\rangle = \langle\mathcal{J}_1\psi,[\xi,\eta]\rangle_{\mathcal{K}} = -\langle[\xi,\mathcal{J}_1\psi],\eta\rangle_{\mathcal{K}}.
\end{aligned}$$

Thus, since $\mathcal{J}_1:\mathfrak{h}\to\mathfrak{h}$ is non-degenerate and ξ,ψ is arbitrary, it must hold that $\langle[\mathfrak{h},\mathfrak{h}],\eta\rangle_{\mathcal{K}}=0$ for all $\eta\in\mathfrak{h}^{\perp_{\mathcal{A}}}$. Using now that $[\mathfrak{h},\mathfrak{h}]=\mathfrak{h}$, we get $\langle\mathfrak{h},\eta\rangle_{\mathcal{K}}=0$ for all $\eta\in\mathfrak{h}^{\perp_{\mathcal{A}}}$. Since every element in $\mathfrak{h}^{\perp_{\mathcal{K}}}$ also fulfils this, and since $\mathfrak{h}^{\perp_{\mathcal{A}}}$ and $\mathfrak{h}^{\perp_{\mathcal{K}}}$ are isomorphic, it holds that $\mathfrak{h}^{\perp_{\mathcal{A}}}=\mathfrak{h}^{\perp_{\mathcal{K}}}$, which proves the result. □

[1]Here, $\mathfrak{h}^{\perp_{\mathcal{K}}} = \{\eta\in\mathfrak{g};\langle\eta,\mathfrak{h}\rangle_{\mathcal{K}}=0\}$ denotes the *generalised* orthogonal complement with respect to $\langle\cdot,\cdot\rangle_{\mathcal{K}}$.

From the first part of Theorem 7 we obtain a recipe for constructing easy totally geodesic inner products. Indeed, take any inner product of the form

$$\langle \xi, \psi \rangle_{\mathcal{A}} = \langle \xi_1, \psi_1 \rangle_{\mathcal{A}_1} + \langle \xi_2, \psi_2 \rangle_{\mathcal{A}_2},$$

where $\xi = \xi_1 + \xi_2$ and $\psi = \psi_1 + \psi_2$ are the unique components in the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp_{\mathcal{K}}}$.

From the second part of Theorem 7 we see that if $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$ we can totally characterise the inner products making $\mathfrak{h}$ easy totally geodesic. For example, in the finite dimensional case we have the following:

Corollary 8. *Let $\mathfrak{g}$ be an n–dimensional Lie algebra, and let $\mathfrak{h} \subset \mathfrak{g}$ be an m–dimensional semisimple subalgebra. Denote by $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ the Killing form on $\mathfrak{g}$. Then $\mathfrak{h}$ is easy totally geodesic in $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ if and only if $\mathfrak{h}^{\perp_{\mathcal{A}}} = \mathfrak{h}^{\perp_{\mathcal{K}}}$. Further, the set of inner products on $\mathfrak{g}$ making $\mathfrak{h}$ easy totally geodesic defines a manifold of dimension $(n+1)n/2 - (n-m)m$.*

Proof. Since $\mathfrak{h}$ is semisimple, the Killing form restricted to $\mathfrak{h}$ is non-degenerate. Further, $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$ since $\mathfrak{h}$ is semisimple. Thus, it follows from Theorem 7 that $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ makes $\mathfrak{h}$ easy totally geodesic in $\mathfrak{g}$ if and only if $\mathfrak{h}^{\perp_{\mathcal{A}}} = \mathfrak{h}^{\perp_{\mathcal{K}}}$. Further, every inertia operator $\mathcal{A}$ with $\mathfrak{h}^{\perp_{\mathcal{A}}} = \mathfrak{h}^{\perp_{\mathcal{K}}}$ takes the form

$$\mathcal{A}\xi = \mathcal{A}_1 \xi_1 + \mathcal{A}_2 \xi_2,$$

where $\mathcal{A}_1 : \mathfrak{h} \to \mathfrak{h}^*$ and $\mathcal{A}_2 : \mathfrak{h}^{\perp_{\mathcal{K}}} \to (\mathfrak{h}^{\perp_{\mathcal{K}}})^*$ are self-adjoint linear operators. The set of such pairs $(\mathcal{A}_1, \mathcal{A}_2)$ forms a linear space of dimension $(n+1)n/2 - (n-m)m$. The subset of such pairs having positive definite eigenvalues is thus a manifold of dimension $(n+1)n/2 - (n-m)m$. $\square$

In contrast to the non-easy totally geodesic case, the following universality result holds for easy totally geodesic subalgebras:

Proposition 9. *Let $\mathfrak{g}$ be a Lie algebra, and let $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ be an inner product. Further, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and $\mathfrak{k}$ a subalgebra of $\mathfrak{h}$. If $\mathfrak{k}$ is easy totally geodesic in $\mathfrak{h}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{A}}$, then $\mathfrak{k}$ is easy totally geodesic in $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{A}}$.*

Proof. Let $\mathfrak{k}^{\perp_{\mathcal{A}}}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{A}}$. Then $\mathrm{ad}_{\mathfrak{k}}(\mathfrak{k}^{\perp_{\mathcal{A}}}) \subset \mathfrak{k}^{\perp_{\mathcal{A}}}$ since $\mathfrak{k}$ is easy totally geodesic in $\mathfrak{g}$. Now, since $\mathfrak{h}$ is a subalgebra it holds that

$$\mathrm{ad}_{\mathfrak{k}}(\mathfrak{k}^{\perp_{\mathcal{A}}} \cap \mathfrak{h}) \subset \mathfrak{k}^{\perp_{\mathcal{A}}} \cap \mathfrak{h}$$

which proves the theorem. $\square$

Example 4.1. Let $\mathfrak{g} = \mathfrak{gl}(3)$ and $\mathfrak{h} = \mathfrak{so}(3)$. A basis $e_1, \ldots, e_9$ for $\mathfrak{gl}(3)$ is given by

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The first three elements gives the standard basis of $\mathfrak{so}(3)$. It is straightforward to check that the symmetric matrix representing the Killing form with respect to this basis is diagonal with entries $(-1,-1,-1,1,1,1,1,1,0)$. Thus, the orthogonal complement $\mathfrak{so}(3)^{\perp_{\mathcal{K}}}$ is the subspace generated by $e_4, \ldots, e_9$. Now, from Theorem 7 it follows that $\mathfrak{so}(3)$ is easy totally geodesic in $\mathfrak{gl}(3)$ for any inertia operator $\mathcal{A} : \mathfrak{so}(3) \to \mathfrak{so}(3)^*$ which is represented by a 3×3 and 6×6 block diagonal matrix with respect to the basis $e_1, \ldots, e_9$.

Given such an inner product, the weak form of the Euler-Arnold equations in the decomposition $\xi = \xi_1 + \xi_2$ relative to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp_{\mathcal{K}}}$ is

$$\begin{aligned} \langle \dot{\xi}_1, \eta_1 \rangle_{\mathcal{A}_1} &= \langle \xi_1, [\xi_1, \eta_1] \rangle_{\mathcal{A}_1} + \langle \xi_2, [\xi_2, \eta_1] \rangle_{\mathcal{A}_2}, \qquad && \forall\, \eta_1 \in \mathfrak{h} \\ \langle \dot{\xi}_2, \eta_2 \rangle_{\mathcal{A}_2} &= \langle \xi, [\xi_2, \eta_2] \rangle_{\mathcal{A}} + \langle \xi_2, [\xi_1, \eta_2] \rangle_{\mathcal{A}_2}, \qquad && \forall\, \eta_2 \in \mathfrak{h}^{\perp_{\mathcal{K}}}. \end{aligned}$$

Notice that the coupling between the $\mathfrak{so}(3)$ and $\mathfrak{so}(3)^{\perp_{\mathcal{A}}}$ variables is non-trivial. That is, the full Euler-Arnold equation of $(\mathfrak{gl}(3), \langle\cdot,\cdot\rangle_{\mathcal{A}})$ for a block diagonal inertia operator $\mathcal{A}$ as above does not simply decouple into one "rotation" part and one "non-rotation" part. Also notice that $\xi_2 = 0$ implies $\dot{\xi}_2 = 0$, and the equation reduces to the Euler-Arnold equation for ξ_1, as expected from the totally geodesic property. In contrast, $\xi_1 = 0$ does not imply $\dot{\xi}_1 = 0$, nor $\dot{\xi}_2 = 0$.

Notice that the algebra of trace free matrices $\mathfrak{sl}(3)$ is spanned by the basis elements $e_1, \ldots, e_8$. Thus, using Proposition 9, we get that $\mathfrak{so}(3)$ is easy totally geodesic also as a subalgebra of $\mathfrak{sl}(3)$ for any of the constructed inner products restricted to $\mathfrak{sl}(3)$.

4.2 Semidirect Products

Consider the semidirect product $G \circledS V$ of the group G with the vector space V, with group multiplication given by

$$(g, v) \cdot (h, u) := (gh, g \cdot v + u),$$

where $g \cdot v$ denotes the linear action (representation) of G on V. The Lie algebra of $G \circledS V$ is denoted $\mathfrak{g} \circledS V$, and the corresponding Lie bracket on $\mathfrak{g} \circledS V$ is given, in terms of the Lie bracket on $\mathfrak{g}$, by

$$[(\xi, v), (\eta, u)] = ([\xi, \eta], \xi \cdot u - \eta \cdot v)$$

where $\xi \cdot v$ indicates the infinitesimal action of $\mathfrak{g}$ on V from the action of G on V.

The unit element in $G \circledS V$ is $(e, 0)$. There are two natural subgroups, $G \circledS \{0\}$ and the normal subgroup $\{e\} \circledS V$. Their Lie algebras are given correspondingly by $\mathfrak{g} \circledS \{0\}$ and $\{0\} \circledS V$. An inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ on $\mathfrak{g} \circledS V$ is called a *split metric* if $(\mathfrak{g} \circledS \{0\})^{\perp_{\mathcal{A}}} = \{0\} \circledS V$.

Theorem 10. *It holds that:*

1. *The subalgebra $\mathfrak{g} \circledS \{0\}$ is easy totally geodesic in $\mathfrak{g} \circledS V$ with respect to any split metric $\langle\cdot,\cdot\rangle_{\mathcal{A}}$.*
2. *The subalgebra $\{0\} \circledS V$ is totally geodesic in $\mathfrak{g} \circledS V$ with respect to an inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ on $\mathfrak{g} \circledS V$ if and only if G acts on V by isometries with respect to $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ restricted to V.*

Proof. For $(\xi, 0) \in \mathfrak{g} \circledS \{0\}$ and $(0, v) \in (\mathfrak{g} \circledS \{0\})^{\perp_{\mathcal{A}}}$, we have

$$[(\xi, 0), (0, v)] = (0, \xi \cdot v) \in \{0\} \circledS V = (\mathfrak{g} \circledS \{0\})^{\perp_{\mathcal{A}}}.$$

Thus, we conclude that the subalgebra $\mathfrak{g} \circledS \{0\}$ is easy totally geodesic in $\mathfrak{g} \circledS V$ for any split metric.

Next consider the subalgebra $\{0\} \circledS V$. For $v \in V$, $(\eta, u) \in (\{0\} \circledS V)^{\perp_{\mathcal{A}}}$ we again compute the obstruction to the vanishing of the second fundamental form

$$\langle (0, v), [(0, v), (\eta, u)] \rangle_{\mathcal{A}} = \langle (0, v), (0, -\eta \cdot v) \rangle_{\mathcal{A}} = -\langle v, \eta \cdot v \rangle_{\mathcal{A}|_V},$$

where $\langle\cdot,\cdot\rangle_{\mathcal{A}|_V}$ is the restriction of $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ to V. Now, let $g(t)$ be a curve in G such that $g(0) = e$ and $\dot{g}(0) = \eta$. Then

$$\langle v, \eta \cdot v \rangle_{\mathcal{A}|_V} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \langle g(t) \cdot v, g(t) \cdot v \rangle_{\mathcal{A}|_V}.$$

The right hand side vanishes for all $\eta \in \mathfrak{g}$ and $v \in V$ if and only if G acts on V by isometries with respect to $\langle\cdot,\cdot\rangle_{\mathcal{A}|_V}$. □

Example 4.2 (Magnetic Groups)**.** For any group G we can consider the action of G on $\mathfrak{g}^*$ by coadjoint action, which is a linear representation of G. We can then form the semidirect product, $G \circledS \mathfrak{g}^*$, so that $\mathfrak{g}^*$ plays the role of V in the previous development.

This example arises physically in magnetohydrodynamics, where the group is given by $G = \mathrm{Diff}_{\mathrm{vol}}(M)$, thus forming the semidirect product $\mathrm{Diff}_{\mathrm{vol}}(M) \circledS \mathfrak{X}_{\mathrm{vol},t}(M)^*$. The inner product is given by $((\xi, a), (\eta, b)) \mapsto \langle \xi, \eta \rangle_{L^2} + \langle a, b \rangle_{L^2}$. Since this is a split metric, Theorem 7 asserts that

$\mathfrak{X}_{\mathrm{vol},t}(M) \circledS \{0\}$ is easy totally geodesic. From a physical point of view, it means that if the magnetic field is initially zero, then it remains zero and the flow reduces to the Euler fluid.

We now wish to construct the Euler equations generally for a split metric on the magnetic group $G \circledS \mathfrak{g}^*$. This is the content of the following theorem, given e.g. in [14, 4]:

Theorem 11. *Let $\mathcal{A} : \mathfrak{g} \to \mathfrak{g}^*$ be an inertia tensor on $\mathfrak{g}$. The right Euler-Arnold equation on the Lie algebra of the magnetic group $G \circledS \mathfrak{g}^*$ associated to the split metric $\langle(\xi,\mu),(\eta,\sigma)\rangle_{\mathcal{A}} := \langle \xi,\eta\rangle_{\mathcal{A}} + \langle \mu,\sigma\rangle_{\mathcal{A}^{-1}}$ is given by*

$$\begin{aligned} \dot{\xi} &= -\mathrm{ad}_{\xi}^{\top_{\mathcal{A}}}(\xi) + \mathrm{ad}_{\mathcal{A}^{-1}(\mu)}^{\top_{\mathcal{A}}} \circ \mathcal{A}^{-1}\mu \\ \dot{\mu} &= \mathcal{A} \circ \mathrm{ad}_{\xi} \circ \mathcal{A}^{-1}\mu \end{aligned} \tag{17}$$

Proof. The weak form of the right Euler equations on $\mathfrak{g} \circledS \mathfrak{g}^*$ for the split metric is given by

$$\begin{aligned} \langle \frac{\mathrm{d}}{\mathrm{d}t}(\xi,\mu),(\eta,\sigma)\rangle_{\mathcal{A}} &= -\langle(\xi,\mu),[(\xi,\mu),(\eta,\sigma)]\rangle_{\mathcal{A}} \\ &= -\langle(\xi,\mu),([\xi,\eta],\xi\cdot\sigma-\eta\cdot\mu)\rangle_{\mathcal{A}} \\ &= -\langle \xi,[\xi,\eta]\rangle_{\mathcal{A}} - \langle \mu,\xi\cdot\sigma-\eta\cdot\mu\rangle_{\mathcal{A}^{-1}} \\ &= -\langle \mathrm{ad}_{\xi}^{\top_{\mathcal{A}}}(\xi),\eta\rangle_{\mathcal{A}} - \langle \mu,\xi\cdot\sigma\rangle_{\mathcal{A}^{-1}} + \langle \mu,\eta\cdot\mu\rangle_{\mathcal{A}^{-1}} \end{aligned}$$

We now isolate σ and η, respectively, in the final two terms as follows. For the second term we have

$$\begin{aligned} \langle \mu,\xi\cdot\sigma\rangle_{\mathcal{A}^{-1}} &= \langle \mu,-\mathrm{ad}_{\xi}^{*}(\sigma)\rangle_{\mathcal{A}^{-1}} \\ &= \langle -\mathrm{ad}_{\xi}^{*}(\sigma),\mathcal{A}^{-1}\mu\rangle \\ &= \langle \sigma,-\mathrm{ad}_{\xi}\circ\mathcal{A}^{-1}\mu\rangle \\ &= \langle \sigma,-\mathcal{A}\circ\mathrm{ad}_{\xi}\circ\mathcal{A}^{-1}\mu\rangle_{\mathcal{A}^{-1}}, \end{aligned}$$

and for the last term we have

$$\begin{aligned} \langle \mu,\eta\cdot\mu\rangle_{\mathcal{A}^{-1}} &= \langle \mu,-\mathrm{ad}_{\eta}^{*}(\mu)\rangle_{\mathcal{A}^{-1}} \\ &= \langle -\mathrm{ad}_{\eta}^{*}(\mu),\mathcal{A}^{-1}\mu\rangle_{\mathcal{A}} \\ &= \langle \mu,-\mathrm{ad}_{\eta}\circ\mathcal{A}^{-1}\mu\rangle \\ &= \langle \mu,\mathrm{ad}_{\mathcal{A}^{-1}\mu}(\eta)\rangle \\ &= \langle \mathcal{A}^{-1}(\mu),\mathrm{ad}_{\mathcal{A}^{-1}\mu}(\eta)\rangle_{\mathcal{A}} \\ &= \langle \mathrm{ad}_{\mathcal{A}^{-1}(\mu)}^{\top_{\mathcal{A}}}\circ\mathcal{A}^{-1}\mu,\eta\rangle_{\mathcal{A}}. \end{aligned}$$

We then conclude that

$$\langle \frac{\mathrm{d}}{\mathrm{d}t}(\xi,\mu),(\eta,\sigma)\rangle_{\mathcal{A}} = \langle -\mathrm{ad}_{\xi}^{\top_{\mathcal{A}}}(\xi) + \mathrm{ad}_{\mathcal{A}^{-1}\mu}^{\top_{\mathcal{A}}}\circ\mathcal{A}^{-1}\mu,\eta\rangle_{\mathcal{A}} + \langle \mathcal{A}\circ\mathrm{ad}_{\xi}\circ\mathcal{A}^{-1}(\mu),\sigma\rangle_{\mathcal{A}^{-1}}.$$

Since this must hold for all (η,σ), the equations of the statement of the Theorem now follow. □

Let us now investigate directly from the Euler equations of the preceding Theorem 11, the conditions for the subgroups $G \circledS \{0\}$ and $\{0\} \circledS \mathfrak{g}^*$ to be totally geodesic. It is clear from the equations that $\mathfrak{g} \circledS \{0\}$ is invariant under the flow of equations (17). For the Euler flow to be tangent to $\{0\} \circledS \mathfrak{g}^*$ however, we require that for all $\xi \in \mathfrak{g}$, $\langle \mathrm{ad}_{\mathcal{A}^{-1}(\mu)}^{\top_{\mathcal{A}}},\xi\rangle_{\mathcal{A}} = 0$. But we have

$$\begin{aligned} \langle \mathrm{ad}_{\mathcal{A}^{-1}\mu}^{\top_{\mathcal{A}}}\circ\mathcal{A}^{-1}\mu,\xi\rangle_{\mathcal{A}} &= \langle \mathcal{A}^{-1}\mu,-\mathrm{ad}_{\xi}\circ\mathcal{A}^{-1}\mu\rangle_{\mathcal{A}} \\ &= -\langle \xi\cdot\mu,\mathcal{A}^{-1}\mu\rangle = -\langle \mu,\xi\cdot\mu\rangle_{\mathcal{A}^{-1}} \end{aligned}$$

and this last term vanishes (for all ξ and all μ) if and only if $\mathfrak{g}$ acts by isometries on $\mathfrak{g}^*$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{A}^{-1}}$, in conjunction with Theorem 7 above.

Example 4.3 (Rigid body in fluid)**.** Another example in physics in given by Kirchhoff's equations for a rigid body in a fluid. Here, $G = \mathrm{SO}(3)$ and $V = \mathbb{R}^3$, thus forming the special Euclidean group $\mathrm{SO}(3) \circledS \mathbb{R}^3$. The SO(3) variable describes the orientation of the body, and the $\mathbb{R}^3$ variable the translational position of its centre of mass. The Lie algebra is given by $\mathfrak{so}(3) \circledS \mathbb{R}^3$, and the inner product (describing the total kinetic energy) is of the form

$$((\xi, u), (\eta, v)) \mapsto \langle \xi, \eta \rangle_{\mathfrak{I}} + m\langle u, v \rangle_{L^2} + Q(\xi, v) + Q(\eta, u),$$

where $\mathfrak{I} : \mathfrak{so}(3) \to \mathfrak{so}(3)^*$ is the rotational moments of inertia operator, $m > 0$ is the effective mass, and Q is positive and bilinear (depending on the geometry of the body). Thus, the inner product is generally *not* a split metric, so $\mathfrak{so}(3) \circledS \{0\}$ is typically *not* totally geodesic. Physically this implies that if the initial velocity of the centre of mass of a rotating rigid body in a fluid is zero, it will generally *not* remain zero (due to interaction with the fluid). However, since $m\langle g \cdot u, g \cdot v \rangle_{L^2} = m\langle u, v \rangle_{L^2}$ for all $g \in \mathrm{SO}(3)$ it holds that SO(3) acts on $\mathbb{R}^3$ by isometries. Thus, it follows from Theorem 7 that $\{0\} \circledS V$ is totally geodesic, meaning that an initially non-rotating rigid body moving in a fluid will remain non-rotating.

5 Diffeomorphism Group Examples

Theorem 7 is valid also for infinite dimensional Fréchet Lie algebras. In the finite dimensional case, as we have seen, one can always use the Killing form as $\mathrm{ad}_{\mathfrak{h}}$–invariant form, so the only requirement is that the Killing form is non-degenerate on $\mathfrak{h}$, which is equivalent to the subalgebra $\mathfrak{h}$ being semisimple. In the infinite dimensional case the situation is more difficult: one has to explicitly find an infinite dimensional $\mathrm{ad}_{\mathfrak{h}}$–invariant form on $\mathfrak{g}$. In this section we give some examples.

5.1 Isometries and H^1_α metric

Let (M, g) be a Riemannian manifold. Further, let $\langle \cdot, \cdot \rangle_{L^2}$ be the inner product on $\Omega^k(M)$ given by (7). Recall the flat operator $^\flat : \mathfrak{X}(M) \to \Omega^1(M)$, the differential $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$, and the co-differential $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$.

Definition 5.1. The left (right) H^1_α metric on $\mathrm{Diff}(M)$ is the left (right) invariant metric given by left (right) translation of the corresponding H^1_α inner product on $\mathfrak{X}(M)$ given by

$$\langle \xi, \eta \rangle_{H^1_\alpha} := \langle \xi^\flat, \eta^\flat \rangle_{L^2} + \alpha \langle \mathrm{d}\xi^\flat, \mathrm{d}\eta^\flat \rangle_{L^2} + \alpha \langle \delta\xi^\flat, \delta\eta^\flat \rangle_{L^2}.$$

Remark 5.1. The H^1_α metric as defined here contains $1 + n(n-1)/2$ partial derivative terms. In some literature, an H^1_α metric is defined such that it contains all the n^2 partial derivative terms.

Let $\mathfrak{X}_{\mathrm{iso}}(M) = \{\xi \in \mathfrak{X}(M); \pounds_\xi \mathsf{g} = 0\}$ denote the Killing vector fields on M. Let $\mathrm{Diff}_{\mathrm{iso}}(M) \subset \mathrm{Diff}(M)$ denote the subgroup of isometries. The corresponding subalgebra is given by the tangential Killing vector fields $\mathfrak{X}_{\mathrm{iso},t}(M) = \mathfrak{X}_{\mathrm{iso}}(M) \cap \mathfrak{X}_t(M)$.

Proposition 12. *Let (M, g) be an n dimensional Riemannian manifold. Then the H^1_α inner product on $\mathfrak{X}_t(M)$ is $\mathfrak{X}_{\mathrm{iso}}(M)$–invariant.*

For the proof of this proposition we need the following result:

Lemma 13. *Let $\beta \in \Omega^k(M)$ and $\xi \in \mathfrak{X}_{\mathrm{iso}}(M)$. Then $\pounds_\xi \star \beta = \star \pounds_\xi \beta$.*

Proof. Let $\alpha \in \Omega^k(M)$. The Hodge star fulfils (by definition) $\alpha \wedge \star\beta = \mathsf{g}^\flat(\alpha, \beta)\mathrm{vol}$, where $\mathsf{g}^\flat$ is the inner product on $\Omega^k(M)$, induced by g (see e.g. [19, Chap. 2]). It is straightforward to check that

$\pounds_\xi \mathsf{g} = 0$ implies $\pounds_\xi \mathsf{g}^\flat = 0$. Thus,

$$\pounds_\xi(\alpha \wedge \star\beta) = \pounds_\xi(\mathsf{g}^\flat(\alpha,\beta)\mathrm{vol})$$
$$\Updownarrow$$
$$\pounds_\xi\alpha \wedge \star\beta + \alpha \wedge \pounds_\xi \star \beta = \underbrace{(\pounds_\xi \mathsf{g}^\flat)}_{0}(\alpha,\beta)\mathrm{vol} + \mathsf{g}^\flat(\pounds_\xi\alpha,\beta)\mathrm{vol}$$
$$+ \mathsf{g}^\flat(\alpha,\pounds_\xi\beta)\mathrm{vol} + \mathsf{g}^\flat(\alpha,\beta)\underbrace{\pounds_\xi\mathrm{vol}}_{0}$$
$$\Updownarrow$$
$$\alpha \wedge \pounds_\xi \star \beta = \mathsf{g}^\flat(\alpha,\pounds_\xi\beta)\mathrm{vol} = \alpha \wedge \star\pounds_\xi\beta,$$

where we use that $\pounds_\xi \mathsf{g}^\flat = 0$ since ξ is a Killing vector field, and $\pounds_\xi\mathrm{vol} = 0$ since every Killing vector field is divergence free. Now, since α is arbitrary, we get that the Lie derivative $\pounds_\xi$ commutes with the Hodge star operator on $\Omega^k(M)$. □

Proof of Proposition 12. For vector fields $\xi,\eta \in \mathfrak{X}(M)$, it holds that $\pounds_\xi\eta^\flat = (\pounds_\xi\eta)^\flat + (\pounds_\xi\mathsf{g})(\eta,\cdot)$. Thus, if $\xi \in \mathfrak{X}_{\mathrm{iso}}(M)$ and $\eta,\psi \in \mathfrak{X}_t(M)$ it holds that

$$\langle \mathrm{ad}_\xi(\eta),\psi\rangle_{H^1_\alpha} = -\langle[\xi,\eta]_{\mathfrak{X}},\psi\rangle_{H^1_\alpha} = -\langle\pounds_\xi\eta^\flat,\psi^\flat\rangle_{L^2} - \alpha\langle \,\mathrm{d}\pounds_\xi\eta^\flat,\,\mathrm{d}\psi^\flat\rangle_{L^2} - \alpha\langle\delta\pounds_\xi\eta^\flat,\delta\psi^\flat\rangle_{L^2}. \quad (18)$$

We show that each of these terms are invariant. For the first term in (18) we get

$$\begin{aligned}\langle\pounds_\xi\eta^\flat,\psi^\flat\rangle_{L^2} &= \int_M \pounds_\xi\eta^\flat \wedge \star\psi^\flat = \int_M \pounds_\xi(\eta^\flat \wedge \star\psi^\flat) - \int_M \eta^\flat \wedge \pounds_\xi \star \psi^\flat \\ &= \int_M \mathrm{d}\mathrm{i}_\xi(\eta^\flat \wedge \star\psi^\flat) - \int_M \eta^\flat \wedge \pounds_\xi \star \psi^\flat \\ &= \int_{\partial M} i^*(\mathrm{i}_\xi(\eta^\flat \wedge \star\psi^\flat)) - \int_M \eta^\flat \wedge \pounds_\xi \star \psi^\flat \\ &= \int_{\partial M} \mathrm{i}_{i^*\xi}(i^*(\eta^\flat) \wedge i^*(\star\psi^\flat)) - \int_M \eta^\flat \wedge \pounds_\xi \star \psi^\flat\end{aligned}$$

where we have used Cartan's magic formula and Stokes theorem to get the boundary terms, and $i : \partial M \to M$ is the natural inclusion. Since ψ is tangential to the boundary ∂M, it holds that $i^*(\star\psi^\flat) = 0$ (see [1, Sect. 7.5]). Thus, the boundary terms vanishes. Next, using Lemma 13 we get $\langle\pounds_\xi\eta^\flat,\psi^\flat\rangle_{L^2} = -\langle\eta^\flat,\pounds_\xi\psi^\flat\rangle_{L^2}$.

For the second term in (18) we recall the identity $\langle \,\mathrm{d}\alpha,\beta\rangle_{L^2} = \langle\alpha,\delta\beta\rangle_{L^2} + \int_{\partial M}\alpha \wedge \star\beta$, which holds for any $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$. Using this formula, Lemma 13, that $\pounds_\xi$ commutes with d, and the same calculation as for the first term, we get

$$\begin{aligned}\langle \,\mathrm{d}\pounds_\xi\eta^\flat,\,\mathrm{d}\psi^\flat\rangle_{L^2} &= \langle\delta\,\mathrm{d}\pounds_\xi\eta^\flat,\psi^\flat\rangle_{L^2} + \int_{\partial M} i^*(\,\mathrm{d}\pounds_\xi\eta^\flat \wedge \star\psi^\flat) \\ &= \langle\star\,\mathrm{d}\star\,\mathrm{d}\pounds_\xi\eta^\flat,\psi^\flat\rangle_{L^2} + \int_{\partial M} i^*(\,\mathrm{d}\pounds_\xi\eta^\flat) \wedge \underbrace{i^*(\star\psi^\flat)}_{0} \\ &= \langle\pounds_\xi \star\,\mathrm{d}\star\,\mathrm{d}\eta^\flat,\psi^\flat\rangle_{L^2} = \langle\pounds_\xi\delta\,\mathrm{d}\eta^\flat,\psi^\flat\rangle_{L^2} \\ &= -\langle\delta\,\mathrm{d}\eta^\flat,\pounds_\xi\psi^\flat\rangle_{L^2} \\ &= -\langle \,\mathrm{d}\eta^\flat,\,\mathrm{d}\pounds_\xi\psi^\flat\rangle_{L^2} + \int_{\partial M} i^*(\,\mathrm{d}\eta^\flat \wedge \star\pounds_\xi\psi^\flat) \\ &= -\langle \,\mathrm{d}\eta^\flat,\,\mathrm{d}\pounds_\xi\psi^\flat\rangle_{L^2} + \int_{\partial M} i^*(\,\mathrm{d}\eta^\flat) \wedge i^*(\star\pounds_\xi\psi^\flat) \\ &= -\langle \,\mathrm{d}\eta^\flat,\,\mathrm{d}\pounds_\xi\psi^\flat\rangle_{L^2}.\end{aligned}$$

The last boundary term vanish since

$$i^*(\star \pounds_\xi \psi^\flat) = i^*(\pounds_\xi \star \psi^\flat) = i^*(\,\mathrm{d}\mathrm{i}_\xi \star \psi^\flat) + i^*(\mathrm{i}_\xi\,\mathrm{d} \star \psi^\flat) = \,\mathrm{d}\mathrm{i}_{i^*\xi}\,\underbrace{i^*(\star\psi^\flat)}_{0} + \mathrm{i}_{i^*\xi}\,\mathrm{d}\,\underbrace{i^*(\star\psi^\flat)}_{0}.$$

Likewise, for the third term in (18) we get

$$\begin{aligned}\langle \delta \pounds_\xi \eta^\flat, \delta\psi^\flat\rangle_{L^2} &= \langle\,\mathrm{d}\delta\pounds_\xi\eta^\flat, \psi^\flat\rangle_{L^2} = \langle \pounds_\xi\,\mathrm{d}\delta\eta^\flat, \psi^\flat\rangle_{L^2} \\ &= -\langle\,\mathrm{d}\delta\eta^\flat, \pounds_\xi\psi^\flat\rangle_{L^2} = -\langle \delta\eta^\flat, \delta\pounds_\xi\psi^\flat\rangle_{L^2}.\end{aligned}$$

Altogether, we now have

$$\begin{aligned}\langle \mathrm{ad}_\xi(\eta), \psi\rangle_{H^1_\alpha} &= -\langle [\xi,\eta]_{\mathfrak{X}}, \psi\rangle_{H^1_\alpha} \\ &= \langle \eta^\flat, \pounds_\xi\psi^\flat\rangle_{L^2} + \alpha\langle\,\mathrm{d}\eta^\flat,\,\mathrm{d}\pounds_\xi\psi^\flat\rangle_{L^2} + \alpha\langle\delta\eta^\flat, \delta\pounds_\xi\psi^\flat\rangle_{L^2} \\ &= \langle \eta, [\xi,\psi]_{\mathfrak{X}}\rangle_{H^1_\alpha} = -\langle \eta, \mathrm{ad}_\xi(\psi)\rangle_{H^1_\alpha},\end{aligned}$$

which proves the theorem. □

As a consequence, we now have:

Corollary 14. $\mathrm{Diff}_{\mathrm{iso}}(M)$ *is easy totally geodesic in* $\mathrm{Diff}(M)$ *with respect to the* H^1_α *metric. In fact, if* $\xi \in \mathfrak{X}_{\mathrm{iso}}(M)$*, then* ξ *is a stationary solution to the* $(\mathrm{Diff}(M), \langle\!\langle\cdot,\cdot\rangle\!\rangle_{H^1_\alpha})$ *Euler-Arnold equation.*

Proof. The first assertion follows from Proposition 12 and Proposition 6. Further, if $\xi \in \mathfrak{X}_{\mathrm{iso}}(M)$ the weak Euler-Arnold equation is

$$\langle \dot\xi, \eta\rangle_{H^1_\alpha} = \langle \mathrm{ad}_\xi(\eta), \xi\rangle_{H^1_\alpha} = \langle \eta, \mathrm{ad}_\xi(\xi)\rangle_{H^1_\alpha} = 0,$$

for any $\eta \in \mathfrak{X}(M)$. Thus, ξ is a stationary solution. □

5.2 Exact Volume Preserving Diffeomorphisms and H^1_α metric

In this section we extend a result in [8], which, on a compact Riemannian manifold without boundary, gives a condition for the subgroup of exact volume preserving diffeomorphisms, corresponding to the Lie subalgebra of exact divergence free vector fields, to be totally geodesic with respect to the L_2 metric. We extend the result to compact Riemannian manifolds with boundary and H^1_α metric.

Let (M, g) be a Riemannian n–manifold with boundary. Recall that the exact divergence free vector fields on M are given by

$$\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M) = \{\xi \in \mathfrak{X}_{\mathrm{vol}}(M); \exists\alpha \in \Omega^{n-2}(M) \text{ s.t. } \mathrm{i}_\xi \mathrm{vol} = \,\mathrm{d}\alpha\}.$$

It is straightforward to check that it is a subalgebra. Indeed, if $\xi,\eta \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ then

$$\mathrm{i}_{[\xi,\eta]}\mathrm{vol} = \pounds_\xi \mathrm{i}_\eta \mathrm{vol} + \mathrm{i}_\eta\,\underbrace{\pounds_\xi \mathrm{vol}}_{0} = \pounds_\xi\,\mathrm{d}\alpha = \,\mathrm{d}\pounds_\xi\alpha,$$

so $\mathrm{i}_{[\xi,\eta]}\mathrm{vol}$ is exact. The subgroup of $\mathrm{Diff}_{\mathrm{vol}}(M)$ corresponding to $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M) = \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M) \cap \mathfrak{X}_t(M)$ is denoted $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M)$.

Theorem 15. $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ *is totally geodesic in* $\mathrm{Diff}_{\mathrm{vol}}(M)$ *with respect to the* H^1_α *metric if and only if*

$$\langle \mathrm{i}_\xi\,\mathrm{d}\xi^\flat, \gamma\rangle_{L^2} = 0$$

for all $\xi \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ *and* $\gamma \in \mathcal{H}^1(M)$.

Proof. By the flat operator, the space $\mathfrak{X}_{\mathrm{vol},t}(M)$ corresponds to the tangential co-closed 1–forms $\mathsf{D}^1_t(M)$, and $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ corresponds to the co-exact tangential 1–forms $(\delta\Omega^2(M))_t = \delta\Omega^2_t(M)$. From the Hodge decomposition for manifolds with boundary (see [1, Sect. 7.5]) it follows that the L^2 orthogonal complement of $\delta\Omega^2_t(M)$ in $\Omega^1_t(M)$ is given by the tangential closed 1–forms $\mathsf{C}^1_t(M)$. Thus, the L^2 orthogonal complement of $\delta\Omega^2_t(M)$ in $\mathsf{D}^1_t(M)$ is given by $\mathsf{C}^1_t(M)\cap\mathsf{D}^1_t(M)$, which are the tangential harmonic fields $\mathcal{H}^1_t(M)$. Since $\delta\gamma = 0$ and $\mathrm{d}\gamma = 0$ for any harmonic field, it follows that $\mathcal{H}^1_t(M)$ is the orthogonal complement of $\delta\Omega^2_t(M)$ also with respect to H^1_α.

As computed in Example 2.2 **(c)**, it holds that ad^*_ξ represented on $\mathsf{D}^1_t(M)$ takes the form $\mathrm{ad}^\bullet_\xi(\psi) = P(\pounds_\xi\psi^\flat)$, where P is the L^2 orthogonal projection $\Omega^1_t(M) \to \mathsf{D}^1_t(M)$. Now, from Theorem 2 we get that $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ is totally geodesic in $\mathfrak{X}_{\mathrm{vol},t}(M)$ if and only if

$$\begin{aligned}
0 &= \langle \mathrm{ad}^{\top_\mathcal{A}}_\xi(\xi), \gamma^\sharp\rangle_{H^1_\alpha} = \langle \mathcal{A}^{-1}\mathrm{ad}^*_\xi(\mathcal{A}\xi), \gamma^\sharp\rangle_{H^1_\alpha} \\
&= \langle P(\pounds_\xi\xi^\flat), \gamma\rangle_{H^1_\alpha} = \langle P(\pounds_\xi\xi^\flat), \gamma\rangle_{L^2} \\
&= \langle \pounds_\xi\xi^\flat + \mathrm{d}p, \gamma\rangle_{L^2} = \langle \pounds_\xi\xi^\flat, \gamma\rangle_{L^2}
\end{aligned}$$

for all $\xi \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$, and all $\gamma \in \mathcal{H}^1_t(M)$. The last equality follows since $\mathrm{d}\Omega^0(M)$ is orthogonal to $\mathcal{H}^1_t(M)$. Next we have

$$\begin{aligned}
\langle \pounds_\xi\xi^\flat, \gamma\rangle_{L^2} &= \langle \mathrm{i}_\xi\,\mathrm{d}\xi^\flat, \gamma\rangle_{L^2} + \langle\, \mathrm{d}\mathrm{i}_\xi\xi^\flat, \gamma\rangle_{L^2} \\
&= \langle \mathrm{i}_\xi\,\mathrm{d}\xi^\flat, \gamma\rangle_{L^2} + \langle \mathrm{i}_\xi\xi^\flat, \underbrace{\delta\gamma}_{0}\rangle_{L^2} + \int_M i^*(\mathrm{i}_\xi\xi^\flat)\wedge\underbrace{i^*(\star\gamma)}_{0} \\
&= \langle \mathrm{i}_\xi\,\mathrm{d}\xi^\flat, \gamma\rangle_{L^2}
\end{aligned}$$

which proves the theorem. □

Remark 5.2. In the case when $\alpha = 0$ and M has no boundary, Theorem 15 amounts to statement $1 \leftrightarrow 5$ of Theorem 1 in [8].

5.3 Maximal Torus of Volume Preserving Diffeomorphisms

Consider the finite cylinder $M = S^1 \times [0,1]$, coordinatised by (θ, z) and equipped with the natural Riemannian structure. In [5, 6, 7] the group of volume preserving diffeomorphisms on M is studied. In particular, it is shown in [7] that the maximum Abelian subgroup of $\mathrm{Diff}_{\mathrm{vol}}(M)$ is given by

$$\mathcal{T} = \{\phi \in \mathrm{Diff}_{\mathrm{vol}}(M); \phi(\theta, z) = (\theta + f(z), z), f \in C^\infty([0,1], S^1)\}.$$

The corresponding algebra is given by

$$\mathfrak{t} = \{\xi \in \mathfrak{X}_{\mathrm{vol},t}(M); \xi(\theta, z) = T'(z)\partial_\theta, T \in C^\infty([0,1], \mathbb{R})\}.$$

It is also shown in [7] that $\mathcal{T}$ is totally geodesic in $\mathrm{Diff}_{\mathrm{vol}}(M)$ with respect to the L^2 inner product. Using our framework, we now show the slightly stronger result that it actually is easy totally geodesic.

Since M is a 2–manifold, the metric together with the induced volume form equips M with the structure of a Kähler manifold. Thus, since the volume form is the symplectic form, the algebra of tangential divergence free vector fields on M is equal to the space of tangential symplectic vector fields on M. Furthermore, by the flat map, the space of tangential divergence free vector fields on M is isomorphic to the tangential co-closed 1–forms on M, i.e., $\mathfrak{X}_{\mathrm{vol},t}(M)^\flat = \mathsf{D}^1_t(M)$. It is a result in [7] that $\mathsf{D}^1_t(M) = \delta\Omega^2_t(M)$, i.e., that every tangential co-closed 1–form on the finite cylinder is co-exact. In turn, this implies that $\mathfrak{X}_{\mathrm{vol},t}(M)$ consists of tangential Hamiltonian vector fields on M. Notice that a Hamiltonian vector field is tangential if and only if the corresponding Hamiltonian function is constant when restricted to each connected component $(\partial M)_i$ of the boundary.

Next, it is straightforward to compute the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{X}_{\mathrm{vol},t}(M)$. Indeed, let $\xi_T \in \mathfrak{t}$, and consider an element $\xi_H = \frac{\partial H}{\partial z}\partial_\theta - \frac{\partial H}{\partial \theta}\partial_z \in \mathfrak{X}_{\mathrm{vol},t}(M)$. Now

$$\langle \xi_T, \xi_H \rangle_{L^2} := \int_M \mathsf{g}(\xi_T, \xi_H)\mathrm{vol} = \int_M \star\, \mathrm{d}T \wedge \star\star\, \mathrm{d}H = \int_M \mathrm{d}T \wedge \star \mathrm{d}H.$$

Using a Fourier expansion we see that $\langle \xi_T, \xi_H \rangle_{L^2} = 0$ for all $\xi_T \in \mathfrak{t}$ if and only if H is of the form

$$H(\theta, z) = \mathrm{const} + \sum_{k=1}^{\infty} a_k(z)\cos(k\theta) + b_k(z)\sin(k\theta). \tag{19}$$

Thus, the L^2 orthogonal complement of $\mathfrak{t}$ in $\mathfrak{X}_{\mathrm{vol},t}(M)$ is given by

$$\mathfrak{t}^\perp = \mathfrak{r} = \big\{\xi_H \in \mathfrak{X}_{\mathrm{vol},t}(M); H(\theta,z) = \sum_{k=1}^{\infty} a_k(z)\sin(k\theta) + b_k(z)\cos(k\theta),$$
$$a_k, b_k \in C^\infty([0,1],\mathbb{R})\big\}.$$

Now, let $\xi_T \in \mathfrak{t}$ and $\xi_H \in \mathfrak{r}$. Then $\mathrm{ad}_{\xi_T}(\xi_H) = [\xi_T, \xi_H] = \xi_{\{T,H\}}$, where $\{T, H\} = -\frac{\partial H}{\partial \theta}\frac{\partial T}{\partial z}$. It is straightforward to check that $\frac{\partial H}{\partial \theta}\frac{\partial T}{\partial z}$ is of the form (19). Thus, $\mathrm{ad}_{\mathfrak{t}}(\mathfrak{r}) \subset \mathfrak{r}$, so $\mathfrak{t}$ is easy totally geodesic in $\mathfrak{X}_{\mathrm{vol},t}(M)$.

5.4 Bi-invariant Form on $\mathfrak{X}_{\mathrm{Ham}}(M)$

Let (M, ω) be a symplectic manifold with boundary. The Hamiltonian vector fields $\mathfrak{X}_{\mathrm{Ham}}(M)$ are the tangential symplectic vector fields which have a globally defined Hamiltonian. Consider the following symmetric bi-linear form on $\mathfrak{X}_{\mathrm{Ham}}(M)$:

$$(\xi_H, \xi_G) \mapsto \int_M HG\,\omega^n =: \langle \xi_H, \xi_G \rangle_{\mathrm{Ham}} \tag{20}$$

where H, G are normalised such that $\int_M H\,\omega^n = \int_M G\,\omega^n = 0$. If $\Phi \in \mathrm{Diff}_{\mathrm{Sp}}(M)$ then $\mathrm{Ad}_\Phi(\xi_H) = \xi_{\Phi^* H} \in \mathfrak{X}_{\mathrm{Ham}}(M)$, since Φ preserves the symplectic structure. Now,

$$\begin{aligned}
\langle \mathrm{Ad}_\Phi(\xi_H), \mathrm{Ad}_\Phi(\xi_G)\rangle_{\mathrm{Ham}} &= \int_M (\Phi^*H)(\Phi^*G)\,\omega^n = \int_M (\Phi^*H)(\Phi^*G)(\Phi^*\omega)^n \\
&= \int_M (\Phi^*H)(\Phi^*G)\Phi^*\omega^n = \int_M \Phi^*(HG\,\omega^n) \\
&= \int_M HG\,\omega^n = \langle \xi_H, \xi_G\rangle_{\mathrm{Ham}}
\end{aligned}$$

where we have used that $\Phi^*\omega = \omega$. Thus, since the Lie algebra of $\mathrm{Diff}_{\mathrm{Sp}}(M)$ is $\mathfrak{X}_{\mathrm{Sp},t}(M)$ we have the following result, which is given for boundary-free manifolds in [17] and [10]:

Proposition 16. *The bi-linear form* (20) *defines an* $\mathrm{ad}_{\mathfrak{X}_{\mathrm{Sp},t}(M)}$*–invariant inner product on* $\mathfrak{X}_{\mathrm{Ham}}(M)$.

5.5 Bi-invariant Form on $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$

Let (M, g) is a Riemannian 3–manifold. The exact divergence free vector fields $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ are the vector fields on M that have globally defined vector potentials. That is, $\xi \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ implies that $\xi = \mathrm{curl}\,\psi$ for some $\psi \in \mathfrak{X}(M)$. Equivalently, in the language of differential forms, $\xi \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)$

$$
\begin{tikzcd}
 & \mathcal{F}(M) \arrow[r, "\mathrm{grad}"] \arrow[rrr, bend left, "\mathrm{Id}"] \arrow[d, leftrightarrow, "\mathrm{Id}"'] & \mathfrak{X}(M) \arrow[r, "\mathrm{curl}"] \arrow[r, bend left, "\mathrm{Id}"] \arrow[d, shift left, "\flat"] & \mathfrak{X}(M) \arrow[r, "\mathrm{div}"] \arrow[d, "\xi\mapsto \mathrm{i}_\xi \mu"] & \mathcal{F}(M) \arrow[d, "f\mapsto f\mathrm{vol}"] & \\
0 \arrow[r] & \Omega^0(M) \arrow[r, "\mathrm{d}"] \arrow[rrr, bend right, "\star"'] & \Omega^1(M) \arrow[r, "\mathrm{d}"] \arrow[r, bend right, "\star"'] \arrow[u, shift left, "\sharp"] & \Omega^2(M) \arrow[r, "\mathrm{d}"] & \Omega^3(M) \arrow[r] & 0
\end{tikzcd}
$$

Figure 2: The de Rham complex for a Riemannian 3–manifold.

implies that $\mathrm{i}_\xi \mathrm{vol} = \mathrm{d}\alpha$, for some $\alpha \in \Omega^1(M)$ which is unique up to closed 1-forms. Now, let $\mathrm{i}_\xi \mathrm{vol} = \mathrm{d}\alpha$ and $\mathrm{i}_\eta \mathrm{vol} = \mathrm{d}\beta$ and consider the following bi-linear form

$$(\xi, \eta) \mapsto \int_M \alpha \wedge \mathrm{d}\beta =: \langle \xi, \eta \rangle_{\mathrm{hel}} \tag{21}$$

sometimes called *cross helicity.* This form is symmetric and independent of the choice of α (see [4, Sect. III.1D]). Equivalently, in terms of the curl operator we have

$$\langle \xi, \eta \rangle_{\mathrm{hel}} = \int_M \mathsf{g}(\xi, \mathrm{curl}^{-1} \eta) \,\mathrm{vol}.$$

The following result is given in [3] (see also [16] and [4, Sect. III.1D]). We give here a different proof, based on the Hodge decomposition theorem.

Theorem 17. *Let* (M, g) *be a Riemannian 3–manifold with boundary. Then* (21) *defines an* $\mathrm{ad}_{\mathfrak{X}_{\mathrm{vol},t}(M)}$*–invariant non-degenerate symmetric bi-linear form on* $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$*.*

We define $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},n}(M) = \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M) \cap \mathfrak{X}_n(M)$. For the proof we need the following results:

Lemma 18. $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ *is an ideal in* $\mathfrak{X}_{\mathrm{vol},t}(M)$*.*

Proof. Let $\xi \in \mathfrak{X}_{\mathrm{vol},t} M$ and $\eta \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$. It holds that $[\xi, \eta] \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ if and only if $\mathrm{i}_{[\xi,\eta]}\mathrm{vol} \in \mathrm{d}\Omega^2_n(M)$. Since $\eta \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ it holds that $\mathrm{i}_\eta \mathrm{vol} = \mathrm{d}\alpha$, for some (non-unique) $\alpha \in \Omega^1_n(M)$. Now,

$$\mathrm{i}_{[\xi,\eta]}\mathrm{vol} = \pounds_\xi \mathrm{i}_\eta \mathrm{vol} - \mathrm{i}_\eta \underbrace{\pounds_\xi \mathrm{vol}}_{0} = \pounds_\xi \,\mathrm{d}\alpha = \mathrm{d}\pounds_\xi \alpha \in \mathrm{d}\Omega^1_n(M),$$

which proves the result. □

Lemma 19. *Let* $(\mathrm{d}\Omega^1(M))_t$ *denote the tangential exact* 2*–forms. The operator* $\mathrm{d}\star$ *is an* L^2 *self-adjoint isomorphism* $(\mathrm{d}\Omega^1(M))_t \to \mathrm{d}\Omega^1_n(M)$*. Equivalently,* curl *is an* L^2 *self-adjoint isomorphism* $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},n}(M) \to \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$*.*

Proof. By the generalised Hodge decomposition, we get $\Omega^2(M) = \mathrm{d}\Omega^1(M) \oplus \mathsf{D}^2_t(M)$, where $\mathsf{D}^2_t(M)$ are the co-closed tangential 2–forms; see [1, Sect. 7.5]. Further, $(\mathrm{d}\Omega^1(M))_n = \mathrm{d}\Omega^1_n(M)$ since d commutes with the pull-back of the inclusion $i : \partial M \to M$. Since $\mathfrak{X}_t(M) \simeq \Omega^2_n(M)$ and $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M) \simeq \mathrm{d}\Omega^1(M)$, by the isomorphism given by contraction with the volume form, it holds that $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M) = \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M) \cap \mathfrak{X}_t(M) \simeq \mathrm{d}\Omega^1(M) \cap \Omega^2_n(M) = \mathrm{d}\Omega^1_n(M)$ with the same isomorphism.

By the de Rham complex (see Figure 1) we see that the operator curl corresponds to the operator $\mathrm{d}\star$ on $\Omega^2(M)$. The kernel of $\mathrm{d}\star$ contains $\mathsf{D}^2_t(M)$. Indeed, if $\beta \in \mathsf{D}^2_t(M)$, then $\mathrm{d} \star \beta =$

$\star\star\,\mathrm{d}\star\beta = \star\delta\beta = 0$, since β is co-closed. Further, it is clear that the image of $\mathrm{d}\star$ is equal to $\mathrm{d}\Omega^1(M)$ (surjectivity follows since $\star$ is an isomorphism).

Next, $\mathrm{d}\star$ maps $(\mathrm{d}\Omega^1(M))_t$ isomorphically to $\mathrm{d}\Omega^1_n(M)$. We first show that the image of $\mathrm{d}\star$ on $(\mathrm{d}\Omega^1(M))_t$ is contained in $\mathrm{d}\Omega^1_n(M)$. If $\mathrm{d}\alpha \in (\mathrm{d}\Omega^1(M))_t$, then

$$i^*(\mathrm{d}\star\,\mathrm{d}\alpha) = i^*(\star\delta\,\mathrm{d}\alpha) = 0$$

where the last equality follows since δ maps tangential to tangential. Thus, $\mathrm{d}\star\,\mathrm{d}\alpha$ is normal. Next, we show surjectivity. Let $\mathrm{d}\beta \in \mathrm{d}\Omega^1_n(M)$. It follows from a variant of the Hodge decomposition theorem for manifolds with boundary (see [1, Sect. 7.5]), that $\beta = \delta\gamma + p$, with $\delta\gamma \in \delta\Omega^2(M)$ and p a normal closed 1–form. Since β and p are normal, it must hold that $\delta\gamma \in (\delta\Omega^2(M))_n$. We now have

$$\mathrm{d}\beta = \mathrm{d}(\delta\gamma + p) = \mathrm{d}\underbrace{\delta\gamma}_{\text{normal}} = \mathrm{d}\star\underbrace{\star\delta\gamma}_{\text{tangential}}\ , \tag{22}$$

where we have used that the Hodge star maps normal forms to tangential forms, and co-exact forms to exact forms. Thus, any $\mathrm{d}\beta \in \mathrm{d}\Omega^1_n(M)$ is the image under $\mathrm{d}\star$ of an element in $(\mathrm{d}\Omega(M))_t$, so the map $\mathrm{d}\star : (\mathrm{d}\Omega^1(M))_t \to \mathrm{d}\Omega^1_n(M)$ is surjective. Furthermore, $\star\delta\gamma$ in equation (22) is unique since: $\star$ is an isomorphism $(\mathrm{d}\Omega^1(M))_t \to (\delta\Omega^2(M))_n$, $\delta\gamma$ is unique by the Hodge decomposition, and d is non-degenerate on $(\delta\Omega^2(M))_n$. Indeed,

$$\langle\,\mathrm{d}\delta\gamma, \alpha\rangle_{L^2} = \langle\delta\gamma, \delta\alpha\rangle_{L^2} + \int_{\partial M}\underbrace{i^*(\delta\gamma)}_{0}\wedge i^*(\star\alpha) = \langle\delta\gamma, \delta\alpha\rangle_{L^2}$$

which is zero for all $\delta\alpha \in (\delta\Omega^2(M))_n$ if and only if $\delta\gamma = 0$. Altogether, we now have that $\mathrm{d}\star : (\,\mathrm{d}\Omega^1(M))_t \to \mathrm{d}\Omega^1_n(M)$ is bijective, i.e., it is an isomorphism. Notice also that $\mathrm{d}\star$ is self-adjoint with respect to the L^2 inner product. Indeed, if $\mathrm{d}\alpha, \mathrm{d}\beta \in (\mathrm{d}\Omega^1(M))_t$, then

$$\begin{aligned}\langle\,\mathrm{d}\star\,\mathrm{d}\alpha,\,\mathrm{d}\beta\rangle_{L^2} &= \langle\star\,\mathrm{d}\alpha, \delta\,\mathrm{d}\beta\rangle_{L^2} + \int_{\partial M} i^*(\star\,\mathrm{d}\alpha)\wedge\underbrace{i^*(\star\,\mathrm{d}\beta)}_{0}\\ &= \langle\star\,\mathrm{d}\alpha, \star\,\mathrm{d}\star\,\mathrm{d}\beta\rangle_{L^2} = \langle\,\mathrm{d}\alpha,\,\mathrm{d}\star\,\mathrm{d}\beta\rangle_{L^2}.\end{aligned}$$

For vector fields the result implies that curl is an isomorphism from the exact divergence free vector fields normal to the boundary, to $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$. □

Proof of Theorem 17. Due to Lemma 19, the form (21), which we denote $\langle\cdot,\cdot\rangle_{\mathcal{K}}$, is a well defined symmetric non-degenerate bi-linear form on $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$. Indeed, $\mathrm{d}\star$ is a L^2 self-adjoint isomorphism, so $(\mathrm{d}\star)^{-1}$ is also L^2 self-adjoint. Thus, if $\xi, \eta \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$, then

$$\begin{aligned}\langle\xi,\eta\rangle_{\mathcal{K}} &= \langle\mathrm{curl}^{-1}\,\xi,\eta\rangle_{L^2} = \langle(\mathrm{d}\star)^{-1}\mathrm{i}_\xi\mathrm{vol}, \mathrm{i}_\eta\mathrm{vol}\rangle_{L^2}\\ &= \langle\mathrm{i}_\xi\mathrm{vol}, (\mathrm{d}\star)^{-1}\mathrm{i}_\eta\mathrm{vol}\rangle_{L^2} = \langle\xi, \mathrm{curl}^{-1}\,\eta\rangle_{L^2} = \langle\eta,\xi\rangle_{\mathcal{K}}.\end{aligned}$$

We are now ready to show $\mathrm{ad}_{\mathfrak{X}_{\mathrm{vol},t}(M)}$–invariance. Let $\xi \in \mathfrak{X}_{\mathrm{vol},t}(M)$ and let $\eta, \psi \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$. First, it follows from Lemma 18, that $\mathrm{ad}_\xi(\eta) \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$, so even though $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ is only defined on $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$, it makes sense for it to be $\mathrm{ad}_{\mathfrak{X}_{\mathrm{vol},t}(M)}$–invariant. Denote by $\mathrm{d}\alpha = \mathrm{i}_\eta\mathrm{vol}$ and $\mathrm{d}\beta = \mathrm{i}_\psi\mathrm{vol}$

the corresponding elements in $\mathrm{d}\Omega^2_n(M)$. Now,

$$\begin{aligned}
\langle \mathrm{ad}_\xi(\eta), \psi\rangle_{\mathcal{K}} &= \langle -[\xi,\eta], \psi\rangle_{\mathcal{K}} = -\langle \mathrm{i}_{[\xi,\eta]}\mathrm{vol}, (\mathrm{d}\star)^{-1}\mathrm{i}_\psi \mathrm{vol}\rangle_{L^2} \\
&= -\langle \mathrm{i}_{[\xi,\eta]}\mathrm{vol}, (\mathrm{d}\star)^{-1}\,\mathrm{d}\beta\rangle_{L^2} = -\langle \mathrm{i}_{[\xi,\eta]}\mathrm{vol}, (\mathrm{d}\star)^{-1}\,\mathrm{d}\star\star\beta\rangle_{L^2} \\
&= -\langle \mathrm{i}_{[\xi,\eta]}\mathrm{vol}, \star\beta\rangle_{L^2} = -\int_M \mathrm{i}_{[\xi,\eta]}\mathrm{vol}\wedge\beta \\
&= -\int_M \pounds_\xi\,\mathrm{d}\alpha\wedge\beta + \int_M \mathrm{i}_\eta \underbrace{\pounds_\xi \mathrm{vol}}_{0}\wedge\beta \\
&= \int_M \mathrm{d}\alpha\wedge\pounds_\xi\beta + \int_M \pounds_\xi(\,\mathrm{d}\alpha\wedge\beta) \\
&= \int_M \mathrm{d}\alpha\wedge\pounds_\xi\beta + \int_{\partial M} i^*(\mathrm{i}_\xi(\,\mathrm{d}\alpha\wedge\beta)) \\
&= \int_M \mathrm{d}\alpha\wedge\pounds_\xi\beta + \int_{\partial M} \mathrm{i}_{i^*\xi}(\underbrace{i^*(\,\mathrm{d}\alpha)}_{0}\wedge i^*(\beta)) \\
&= \int_M \alpha\wedge\,\mathrm{d}\pounds_\xi\beta + \int_{\partial M} \underbrace{i^*(\alpha)}_{0}\wedge i^*(\pounds_\xi\beta) \\
&= \int_M \alpha\wedge\pounds_\xi\,\mathrm{d}\beta = \langle\star\alpha, \pounds_\xi\,\mathrm{d}\beta\rangle_{L^2} \\
&= \langle(\mathrm{d}\star)^{-1}\,\mathrm{d}\star\star\alpha, \pounds_\xi\,\mathrm{d}\beta\rangle_{L^2} = \langle\,\mathrm{d}\alpha, \pounds_\xi\,\mathrm{d}\beta\rangle_{\mathcal{K}} \\
&= \langle\eta, [\xi,\psi]\rangle_{\mathcal{K}} = -\langle\eta, \mathrm{ad}_\xi(\psi)\rangle_{\mathcal{K}},
\end{aligned}$$

which proves the result. □

Remark 5.3. Notice that normality of $\mathrm{d}\alpha$, but never of $\mathrm{d}\beta$, is used in the proof above. Further, we never use that ξ is tangential. Thus, $\langle\eta,\psi\rangle_{\mathcal{K}}$ is well defined for $\eta\in\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ and $\psi\in\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)$, and we have $\langle \mathrm{ad}_\xi(\eta),\psi\rangle_{\mathcal{K}} = -\langle\eta,\mathrm{ad}_\xi(\psi)\rangle_{\mathcal{K}}$ for any $\xi\in\mathfrak{X}_{\mathrm{vol}}(M)$. In particular, this allows us to write the L^2 inner product on $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ using $\langle\cdot,\cdot\rangle_{\mathcal{K}}$. Indeed, we have

$$\begin{aligned}
\langle\eta,\psi\rangle_{L^2} &= \langle \mathrm{i}_\eta\mathrm{vol}, \mathrm{i}_\psi\mathrm{vol}\rangle_{L^2} = \langle\,\mathrm{d}\star(\mathrm{d}\star)^{-1}\mathrm{i}_\eta\mathrm{vol}, \mathrm{i}_\psi\mathrm{vol}\rangle_{L^2} \\
&= \langle\star(\mathrm{d}\star)^{-1}\mathrm{i}_\eta\mathrm{vol}, \delta\mathrm{i}_\psi\mathrm{vol}\rangle_{L^2} + \int_{\partial M}\underbrace{i^*(\star(\mathrm{d}\star)^{-1}\mathrm{i}_\eta\mathrm{vol})}_{0}\wedge i^*(\star\mathrm{i}_\psi\mathrm{vol}) \\
&= \langle(\mathrm{d}\star)^{-1}\mathrm{i}_\eta\mathrm{vol},\,\mathrm{d}\star\mathrm{i}_\psi\mathrm{vol}\rangle_{L^2} = \langle\eta,\mathrm{curl}\,\psi\rangle_{\mathcal{K}}
\end{aligned}$$

where the boundary term vanishes since $\star(\mathrm{d}\star)^{-1}$ is a map $\mathrm{d}\Omega^1_n(M)\to\Omega^1_n(M)$.

With this result we get a characterisation of subalgebras of $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ which are easy totally geodesic with respect to the L^2 inner product.

Theorem 20. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$. Then $\mathfrak{h}$ is easy totally geodesic in $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$ with respect to the L^2 inner product if and only if* $\mathrm{ad}_{\mathfrak{h}}(\mathrm{curl}\,\mathfrak{h})\subseteq\mathfrak{h}$.

Proof. Let V be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M)$. Then $\mathfrak{h}$ is easy totally geodesic if and only if $\langle\mathrm{ad}_{\mathfrak{h}}(V),\mathfrak{h}\rangle_{L^2}=\{0\}$. Now,

$$\langle\mathrm{ad}_{\mathfrak{h}}(V),\mathfrak{h}\rangle_{L^2} = \langle\mathrm{ad}_{\mathfrak{h}}(V),\mathrm{curl}\,\mathfrak{h}\rangle_{\mathcal{K}} = -\langle V,\mathrm{ad}_{\mathfrak{h}}(\mathrm{curl}\,\mathfrak{h})\rangle_{\mathcal{K}}.$$

This proves sufficiency. To get necessity, we need to show that $\mathrm{ad}_{\mathfrak{h}}(\mathrm{curl}\,\mathfrak{h})\subset\mathfrak{X}_{\mathrm{vol},t}(M)$, because then $\langle V,\mathrm{ad}_{\mathfrak{h}}(\mathrm{curl}\,\mathfrak{h})\rangle_{\mathcal{K}}=\{0\}$ implies $\mathrm{ad}_{\mathfrak{h}}(\mathrm{curl}\,\mathfrak{h})\subseteq\mathfrak{h}$. But this follows since $\mathfrak{X}_{\mathrm{vol},t}(M)$ is an ideal in $\mathfrak{X}_{\mathrm{vol}}(M)$. Indeed, if $\xi \in \mathfrak{X}_{\mathrm{vol}}(M)$ and $\eta \in \mathfrak{X}_{\mathrm{vol},t}(M)$ then

$$\begin{aligned} i^*(\mathrm{i}_{[\xi,\eta]}\mathrm{vol}) &= i^*(\pounds_\xi \mathrm{i}_\eta \mathrm{vol}) \\ &= i^*(\,\mathrm{d}\mathrm{i}_\xi \mathrm{i}_\eta \mathrm{vol} + \mathrm{i}_\xi\, \mathrm{d}\mathrm{i}_\eta \mathrm{vol}) \\ &= \mathrm{d}\mathrm{i}_{i^*\xi}\, \underbrace{i^*(\mathrm{i}_\eta \mathrm{vol})}_{0} + \mathrm{i}_{i^*\xi}\,\mathrm{d}\, \underbrace{i^*(\mathrm{i}_\eta \mathrm{vol})}_{0} = 0, \end{aligned}$$

where, as usual, $i : \partial M \to M$ is the natural inclusion. Thus, $\mathrm{i}_{[\xi,\eta]}\mathrm{vol}$ is normal, which is equivalent to $[\xi,\eta] \in \mathfrak{X}_{\mathrm{vol},t}(M)$. □

As an example, let M now be a three dimensional contact manifold, with contact form $\theta \in \Omega^1(M)$. For details on contact manifolds, see [18, Sect. 11]. In our context, it is enough to recall the following properties:

- M carries a natural contact Riemannian structure;
- the volume form is given by $\theta \wedge \mathrm{d}\theta$;
- the *Reeb vector field* is given by $\xi_R = \theta^\sharp$. We assume K–contact structure (cf. [18, Sect. 11]), i.e., that the Reeb vector field is Killing. This is the common case, although it is not always true.

Consider the subgroup of exact contact diffeomorphisms $\mathrm{Diff}^{\mathrm{ex}}_\theta(M) = \{\phi \in \mathrm{Diff}(M); \phi^*\theta = \theta\}$. It is shown by Smolentsev [18, Sect. 11.2] that $\mathrm{Diff}^{\mathrm{ex}}_\theta(M)$ is a subgroup of $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M)$. Now, using Theorem 20, we give the following new example of an easy totally geodesic subgroup of $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M)$:

Corollary 21. $\mathrm{Diff}^{\mathrm{ex}}_\theta(M)$ *is easy totally geodesic in* $\mathrm{Diff}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ *with respect to the* L^2 *metric.*

Proof. The algebra of $\mathrm{Diff}^{\mathrm{ex}}_\theta(M)$ is given by $\mathfrak{X}^{\mathrm{ex}}_{\theta,t}(M) = \{\xi \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol},t}(M); \pounds_\xi\theta = 0\}$. Let $\xi \in \mathfrak{X}^{\mathrm{ex}}_{\theta,t}(M)$. We first show that $\pounds_{\mathrm{curl}\,\xi}\theta = 0$, and then use Theorem 20. Since $\theta = \xi_R^\flat$, and since ξ_R is a Killing vector field, it holds that $\pounds_{\mathrm{curl}\,\xi}\theta = [\mathrm{curl}\,\xi, \xi_R]^\flat$. From [18, Sect. 11.2] we have that $\mathrm{curl}\,\xi = (f - \Delta f)\xi_R + \xi_R \times \mathrm{grad}\, f$, where $f = \mathrm{i}_\xi\theta$ is the *contact Hamiltonian.* We recall that the Reeb vector field conserves all contact Hamiltonians. Also, we have $[\xi_R, \xi_R \times \mathrm{grad}\, f] = 0$. Thus, it remains to show $[(f - \Delta f)\xi_R, \xi_R] = 0$. But this follows since both f and Δf are contact Hamiltonians, so

$$[\xi_R, (f-\Delta f)\xi_R] = \pounds_{\xi_R}(f-\Delta f)\xi_R = (\underbrace{\pounds_{\xi_R} f}_{0} - \underbrace{\pounds_{\xi_R}\Delta f}_{0})\xi_R + (f-\Delta f)\underbrace{\pounds_{\xi_R}\xi_R}_{0}.$$

Thus, $[\mathrm{curl}\,\xi, \eta] \in \mathfrak{X}^{\mathrm{ex}}_{\theta,t}(M)$, for any $\eta \in \mathfrak{X}^{\mathrm{ex}}_{\theta,t}(M)$, and the result follows from Theorem 20. □

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