On Essential Self-adjointness, Confining Potentials & the L_p-Hardy Inequality

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Abstract

Let Ω be a domain in $\mathbb{R}^m$ with non-empty boundary and let $H = -\Delta + V$ be a Schrödinger operator defined on $C_0^\infty(\Omega)$ where $V \in L_{\infty,\mathrm{loc}}(\Omega)$. We seek the minimal criteria on the potential V that ensures that H is essentially self-adjoint, i.e. that ensures the closed operator $\bar{H}$ is self-adjoint. Overcoming various technical problems, we extend the results of Nenciu & Nenciu in [1] to more general types of domain, specifically unbounded domains and domains whose boundaries are fractal. As a special case of an abstract condition we show that H is essentially self-adjoint provided that sufficiently close to the boundary

$$V(x) \;\geq\; \frac{1}{d(x)^2}\left[\,1 - \mu_2(\Omega) - \frac{1}{\ln(\,d(x)^{-1})} - \frac{1}{\ln(\,d(x)^{-1})\,\ln\ln(\,d(x)^{-1})} - \cdots\,\right], \quad (1)$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$ and the right hand side of the above inequality contains a finite number of logarithmic terms. The constant $\mu_2(\Omega)$ appearing in (1) is the variational constant associated with the L_2-Hardy inequality and is non-zero if and only if Ω admits the aforementioned inequality. Our results indicate that the existence of an L_2-Hardy inequality, and the specific value of $\mu_2(\Omega)$, depend intimately on the (Hausdorff / Aikawa) dimension of the boundary. In certain cases where Ω is geometrically simple, this constant, as well as the constant '1' appearing in front of each logarithmic term, is shown to be optimal with regards to the essential self-adjointness of H.

Foreword & Acknowledgements

I once read the foreword to someone's PhD thesis that could be paraphrased as follows:

> "From the moment I arrived at the university I knew that I was in the right place. I realized almost immediately that the area of mathematics I was researching was a fruitful one and that I could make a significant contribution to it. Although I felt challenged, I also felt confident that my abilities would allow me to succeed. Pretty soon I began to feel at home amongst the doctors and established professors at the institute..."

For the benefit of any (prospective) PhD student that may read this foreword, I would like to stress that this was **not** my experience. I remember my first meeting with my supervisors Professor Gaven Martin and Professor Boris Pavlov. They informed me that my sole task for the first year of the PhD was to read as much material as possible so that "in twelve months time we can have a meaningful conversation". At the time I remember thinking that they had underestimated me. Now I realize that they had significantly over estimated me. It took two years before I could have a meaningful conversation with them or ask them a question that they did not immediately know the answer to. At times during those first two years it was horrible. I felt like a fraud, completely out of my depth and uncertain as to whether I had the ability to succeed. Had it not been for a combination of bizarre personal circumstance, convoluted rules and regulations concerning my scholarship and sheer geographical distance, I probably would have returned home to Europe. Then after those two years had passed, slowly, things started to come together. In the end nothing of worth comes without struggle.

Looking back at my time in New Zealand, I realize that it is necessary to thank various people. First and foremost, my thanks go to my family - Mum & Dad, Margret & Chris, Vicky & Dave and Tina. Without your constant love, support and sacrifices over the years none of this would have been possible.

Next I owe a huge debt of gratitude to my supervisors Gaven & Boris. Thank you for your guidance, advice and for being unlimited wells of mathematical insight and knowledge. One of my greatest pleasures throughout the duration of the PhD has been working along side you both and observing the different approaches to mathematics that you adopt. I can only hope that some small portion of how you address the subject has rubbed off on me.

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Contents

0.1 Introduction

A densely defined, closeable, linear operator A acting upon the Hilbert space $\mathcal{H}$ is said to be essentially self-adjoint if its closure, $\bar{A}$, is self-adjoint. In this thesis we investigate the essential self-adjointness of Schrödinger operators, $H = -\Delta + V$, defined on $C_0^\infty(\Omega)$ within the Hilbert space $L_2(\Omega)$. Unless otherwise stated we will always assume that Ω is a domain (open, connected set) in $\mathbb{R}^m$ with non-empty boundary, and that the potential V is real and locally essentially bounded. Indeed, determining the criteria under which such an operator is essentially self-adjoint amounts to specifying conditions on the potential V which ensure that a particle under it's influence does not come into contact with the boundary of the domain. This is equivalent to saying that the potential V completely determines the dynamics of the system, there being no need to specify boundary conditions to 'tell the particle how to behave' should it reach the boundary.

Viewing the same problem from the perspective of classical mechanics, one would think that the condition $V(x) \to \infty$ as $x \to \partial\Omega$ would be enough to ensure the essential self-adjointness of our Schrödinger operator, since the particle under consideration would be unable to penetrate into the 'classically forbidden' region near the boundary. However, quantum mechanics allows for the possibility that the particle may 'tunnel through' this infinite potential barrier. Hence the condition that $V(x) \to \infty$ as $x \to \partial\Omega$ is not enough to ensure the essential self-adjointness of H. Instead, the potential must 'blow up' at a particular rate in order to ensure that the **probability** of finding a particle under its influence at the boundary is zero.

Yet, herein we encounter another facet of quantum mechanics which is not present within the corresponding classical setting, namely the uncertainty principle. That is to say that under the assumptions of quantum mechanics one cannot say with certainty that a particle is 'located' at the boundary. Indeed, localizing a particle to a smaller neighborhood of the boundary requires a corresponding increase in the particles total momentum, this trade off being given precise embodiment by the L_2-Hardy inequality.

As such, the main objective of this thesis is to determine the minimal rate of inflation of the potential V near to the boundary that ensures that the associated Schrödinger operator is essentially self-adjoint. We will see that this requires us to give expression to the delicate balancing act between the need of the potential to 'blow up' to prevent a particle under its influence from reaching the boundary and the uncertainty principle.

In fact, the origin of this problem pre-dates the inception of quantum mechanics. In the one dimensional case there is a well developed body of work concerning the essential self-adjointness of Schrödinger and Sturm-Liouville operators which has emerged, over the course of the last 100 years, out of Weyl's limit point - limit circle analysis. The reader is directed to [11], [14], Chapter X of [3], Chapter XIII.6 of [16] and the multitude of references supplied in [15] for an overview of results in this area. Whilst higher dimensional results have been forthcoming, these have predominantly involved domains which are geometrically simple enough for the corresponding equations to be reduced to the one dimensional case by an appropriate change of variables.

Recently, however, Nenciu & Nenciu [1] considered the essential self-adjointness of Schrödinger operators on bounded domains with C^2 boundary of co-dimension 1. Put simply, their main result can be stated as follows. If, sufficiently close to the boundary,

$$V(x) \quad \geq \quad \frac{1}{d(x)^2}\left[\, 3/4 \;-\; \frac{1}{\ln(\,d(x)^{-1})} \;-\; \frac{1}{\ln(\,d(x)^{-1})\,\ln\ln(\,d(x)^{-1})} \;-\; \cdots \,\right],$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$ and the above inequality contains a finite number of logarithmic terms, then the operator $H = -\Delta + V$ is essentially self-adjoint. Concerning the possibility of generalizing this result, Nenciu & Nenciu go on to state:

> "At the price of technicalities, one may be able to extend the results of the present note to more general situations, e.g. boundaries with components of higher co-dimension... Reducing the regularity of the boundary $\partial\Omega$ below C^2 seems to require finer analysis - in particular, of multidimensional Hardy inequalities on domains with less smooth boundary."
>
> Nenciu & Nenciu [1], page 379.

In this thesis we achieve our objective, effectively by taking up this challenge. We extend the above result of Nenciu & Nenciu to arbitrary domains, in particular to unbounded domains and domains with fractal boundaries. We arrive at the conclusion that if $H = -\Delta + V$ is defined on $C_0^\infty(\Omega)$, then H is essentially self-adjoint provided that, sufficiently close to the boundary,

$$V(x) \geq \frac{1}{d(x)^2}\left[\, 1 - \mu_2(\Omega) - \frac{1}{\ln(\,d(x)^{-1})} - \frac{1}{\ln(\,d(x)^{-1})\,\ln\ln(\,d(x)^{-1})} - \cdots \,\right], \qquad (2)$$

where $\mu_2(\Omega)$ is the variational constant associated with the L_2-Hardy inequality. The aforementioned balancing act between the 'blow up' of the potential and the uncertainty principle is clearly apparent within equation (2).

As previously suggested, to achieve this result we are forced to overcome some significant technical difficulties. For instance, if one assumes that $\partial\Omega$ is C^2, then there is a neighborhood of the boundary upon which the Euclidean distance function $d(x)$ is twice continuously differentiable (see the Appendix to Chapter 14 of [19]). In which case all the relevant analysis can take place on this neighborhood as in [1]. However, if we relax this assumption, then one is left only with the fact that $d(x)$ is once differentiable almost everywhere, courtesy of Rademacher's theorem [17] and the fact that $d(x)$ is Lipschitz.

Even more pressing is the need to evaluate the variational constant $\mu_2(\Omega)$, which is known to be non-zero if and only if the domain Ω admits an L_2-Hardy inequality. Consequently, a large part of the thesis is devoted to determining the conditions under which a domain admits an L_2-Hardy inequality and to determining the precise value of the constant $\mu_2(\Omega)$. For the moment we state only that both of these factors have an intimate dependence on the (Hausdorff / Aikawa) dimension of the boundary.

Finally, there is the question of the optimality of the potential structure described by equation (2). That is to ask whether or not it is possible to replace the constant $1-\mu_2(\Omega)$, or the constant '-1' appearing in front of each logarithmic term, with a smaller constant and still be able to guarantee the essential self-adjointness of the operator H. Although we fall short of proving a result in full generality here, we are able to show that there are domains upon which these constants are optimal, i.e. replacing them with smaller constants causes the operator H to become non-essentially self-adjoint.

0.2 Brief Overview

We have decided to structure this thesis as follows.

- Chapter 1 lays down most of the pertinent analytical foundations required by the thesis. Concepts such as the closure and the adjoint of linear operators are defined, and expanded upon, before various criteria for essential self-adjointness are given. Furthermore, by appealing to known examples, we characterize the essential self-adjointness of Schrödinger operators in terms of the inability of a particle under the influence of the associated potential to come into contact with the boundary of the domain.

- The main result of the thesis, Theorem 2.1.1, is developed within Chapter 2. However, this theorem is abstract in the sense that the conditions for essential self-adjointness expressed therein are phrased in terms of an auxiliary function $G(t)$ satisfying various conditions. Consequently, the last section of the chapter is devoted to producing two candidate functions for $G(t)$, the latter of which allows us to obtain the result described by equation (2).

- As we have said the variational constant $\mu_2(\Omega)$, appearing in equation (2), is non-zero if and only if the domain Ω admits an L_2-Hardy inequality. As such, Chapter 3 investigates the necessary and sufficient conditions required for a domain to admit an L_p-Hardy inequality in the case where $p > 1$. In particular, we characterize the existence of an L_p-Hardy inequality on a given domain in terms of the Hausdorff and Aikawa dimension of the boundary.

- Having determined the conditions required for a domain to admit an L_p-Hardy inequality, our attentions turn to evaluating the precise value of the variational constant $\mu_p(\Omega)$. Chapter 4 develops the necessary tools required to tackle this problem. Here we produce new results relating the Minkowski dimension of $\partial\Omega$ to the number of cubes appearing in sequential generations of the Whitney decomposition of the domain. As well as providing some useful corollaries, this enables us to recapture, by altogether different methods, recent results by Zubrinić [69] characterizing the Minkowski dimension of $\partial\Omega$ in terms of the integrability of the function $d(x, \partial\Omega)$.

- The ideas developed within the previous chapter come to fruition in Chapter 5, where we investigate the precise value of $\mu_p(\Omega)$ on various types of domain. The analysis here basically boils down to our ability to exploit a dimensional dichotomy concerning the integrability of the Euclidean distance function.

- Chapter 6 combines the results of Chapter 5 and Chapter 2 in order to recapture, and extend, existing results concerning the essential self-adjointness of Schrödinger operators on domains where the constant $\mu_2(\Omega)$ is known explicitly. Furthermore, by means of utilizing Weyl's limit point - limit circle analysis, we also show that the potential structure described by equation (2) is optimal on these domains.

- Finally, in Chapter 7 we produce a review of all results obtained and highlight eleven further lines of research that have emerged from the thesis.

In order to aid the flow of the disposition some material has been relegated to the appendices. Throughout the thesis the reader will be directed to the relevant information within the appendices at the appropriate time.

Chapter 1

Preliminary Material

The objective of this chapter is to provide an overview of the essential self-adjointness of Schrödinger operators defined on domains in $\mathbb{R}^m$ and to begin to elucidate the deep connection between essential self-adjointness and the L_2-Hardy constant. We will see, in Section 2.1, that this connection arises due to the uncertainty principle. Although we define the L_p-Hardy inequality in Section 1.5, this chapter is weighted more towards understanding the concept of essential self-adjointness. Indeed, after divulging the necessary analytic background we will briefly discuss some of the work that has already been done in this area.

The chapter is structured as follows. In Section 1.1 we provide a brief review of the geometric structure of Hilbert spaces and linear operators, before defining the concept of essential self-adjointness. It will be shown that if an operator is essentially self-adjoint, then it has a unique self-adjoint extension. Next, in Section 1.2, we justify our interest in essential self-adjointness by demonstrating that, from a mathematical perspective, quantum mechanics is the study of (essentially) self-adjoint operators. To do so we will have to appeal to the various axioms of the theory. Having justified our interest in essential self-adjointness, in Section 1.3 we go a step further and consider a particular type of linear operator - Schrödinger operators. Then, in Section 1.4, we review some comprehensive known results concerning the essential self-adjointness of Schrödinger operators on $\mathbb{R}^m$. This leads us to the physical interpretation that such an operator is essentially self-adjoint provided that a particle under the influence of the associated potential cannot escape to infinity in a finite amount of time. Finally, in Section 1.5 we begin the main business of the thesis - investigating the essential self-adjointness of Schrödinger operators on domains with non-empty boundary. Our intention in this section is not to go into details but to survey some of the main ideas that arise. By means of example we obtain a physical interpretation of essential self-adjointness analogous to that described in the previous section. After defining the L_p-Hardy inequality, we present a simple, well known result that exemplifies the connection between essential self-adjointness and the L_2-Hardy constant.

1.1 Hilbert Spaces & Linear Operators

The geometric structure of Hilbert spaces and linear operators acting upon these spaces is quite rich. The purpose of this short section is simply to provide a quick review of the basic features of Hilbert spaces and to define precisely what we mean by the essential self-adjointness of linear operators. Along the way it will be necessary to define concepts such as orthogonal complements, the closure and the adjoint of an operator, density, symmetry and self-adjointness. Whilst it may appear to the unaccustomed reader that Hilbert spaces are rather abstract entities, this is far from the truth. As generalizations of Euclidean space they are steeped in physical intuition and some of the later theorems we will present are deep and intricate. For reasons of brevity all theorems and lemmas in this section will be presented without proof, with only one exception. The reader interested in the proofs is directed to the following texts on functional analysis and the mathematical foundations of quantum mechanics [2], [3], [4], [5] & [6].

1.1.1 Hilbert Spaces

As is natural, we begin with the definition of inner product spaces.

Definition 1.1.1 *A complex vector space* $\mathbb{V}$ *is called an inner product space if there exists a complex valued function* $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ *so that for all* $x, y, z \in \mathbb{V}$ *and* $\alpha \in \mathbb{C}$

i) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0_{\mathbb{V}}$.

ii) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$.

iii) $\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$.

iv) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

It is not difficult to see that every inner product space $\mathbb{V}$ is a normed vector space in the induced norm $||\, x\, || = \langle x, x \rangle^{\frac{1}{2}}$. Consequently, this leads us to the following definition of Hilbert spaces.

Definition 1.1.2 *A Hilbert Space* $\mathcal{H}$ *is an inner product space that is complete in the induced norm* $||\, x\, || = \langle x, x \rangle^{\frac{1}{2}}$.

The validity of the next lemma is a simple consequence of the well known, and easily proven, Schwarz inequality

$$\left| \langle x, y \rangle \right| \leq ||\, x\, ||\, ||\, y\, || \qquad \text{for all} \qquad x, y \in \mathcal{H}. \tag{1.1}$$

Lemma 1.1.1 *Let $\mathcal{H}$ be a Hilbert space. Let $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ be sequences in $\mathcal{H}$ so that $x_n \to x$ and $y_n \to y$ in the induced norm topology. Then $\langle x_n, y_n \rangle \to \langle x, y \rangle$ as $n \to \infty$.*

We now turn to the concept of orthogonal complements. Any closed subspace of a Hilbert space $\mathcal{H}$ has another closed subspace naturally associated to it, in a sense made precise by the following definition.

Definition 1.1.3 *Let $\mathcal{H}$ be a Hilbert space and let M be a closed subspace of $\mathcal{H}$. The orthogonal complement of M, denoted by $M^{\perp}$, is the set of vectors $x \in \mathcal{H}$ so that $\langle x, y \rangle = 0$ for all $y \in M$.*

Indeed, the following theorem due to Riesz, often termed the Projection Theorem, asserts that any Hilbert space can be decomposed into the direct sum of a closed subspace and its orthogonal complement.

Theorem 1.1.2 *Let M be a closed subspace of the Hilbert space $\mathcal{H}$. Then any $x \in \mathcal{H}$ can be uniquely expressed as $x = x_1 + x_2$ where $x_1 \in M$ and $x_2 \in M^{\perp}$, so that we may write*

$$\mathcal{H} = M \oplus M^{\perp}.$$

1.1.2 Linear Operators

We now define what we mean by a linear operator acting on a Hilbert space.

Definition 1.1.4 *A linear operator A on a Hilbert space $\mathcal{H}$ is a mapping $A : D(A) \to \mathcal{H}$ where $D(A)$, the domain of A, is a subspace of $\mathcal{H}$ and where $A(x + \alpha y) = Ax + \alpha Ay$ for all $x, y \in D(A)$ and $\alpha \in \mathbb{C}$. If $D(A)$ is dense in $\mathcal{H}$, then the operator A is said to be densely defined.*

As such, a linear operator consists of both a domain of definition and a rule prescribing how the operator acts upon elements of it's domain. Two such operators A and B are said to be equivalent if $D(A) = D(B)$ and $Ax = Bx$ for all $x \in D(A)$. If $D(A) \subseteq D(B)$ and $Ax = Bx$ for all $x \in D(A)$, then B is said to be an extension of A, or equivalently A is a restriction of B. We denote the former situation by $A = B$ and the later by $A \subseteq B$.

Soon we shall see that there is a strong relationship between everywhere defined bounded operators and closed operators. In preparation for this we make the following definitions.

Definition 1.1.5 *Let A be a linear operator on the Hilbert space $\mathcal{H}$. A is said to be bounded if there exists some $M > 0$ so that $||\, Ax \,|| \leq M \,||\, x \,||$ for all $x \in D(A)$, otherwise A is said to be unbounded.*

Definition 1.1.6 *A linear operator* A *on the Hilbert space* $\mathcal{H}$ *is said to be closeable provided that for all sequences* $\{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty} \subseteq D(A)$ *so that* $f_n, g_n \to f \in \mathcal{H}$ *we have that* $\lim_{n\to\infty} Af_n = \lim_{n\to\infty} Ag_n$*. For any closeable operator* A*, we define its closure* $\bar{A}$ *as follows*

$$D(\bar{A}) = \left\{ f \in \mathcal{H} \;\middle|\; \exists \{f_n\}_{n=1}^{\infty} \subseteq D(A),\; f_n \to f,\; \{Af_n\}_{n=1}^{\infty} \text{ converges} \right\}$$

$$\bar{A}f = \lim_{n\to\infty} Af_n.$$

Clearly, $\bar{A}$ *is an extension of* A*. If* $A = \bar{A}$ *we say that* A *is closed.*

Indeed, the following lemma completely characterizes closed operators.

Lemma 1.1.3 *Let* A *be a closeable, linear operator on the Hilbert space* $\mathcal{H}$*. The following statements are equivalent:*

i) A *is closed.*

ii) if $\{f_n\}_{n=1}^{\infty} \subseteq D(A)$, $f_n \to f$ *and* $Af_n \to g$ *then* $f \in D(A)$ *and* $Af = g$.

iii) $G(A) = \left\{ \binom{x}{Ax} \;\middle|\; x \in D(A) \right\}$ *is a closed subspace of* $\mathcal{H} \times \mathcal{H}$.

We now present a simple result which we will shortly find useful.

Lemma 1.1.4 *Let* A *be a linear operator on the Hilbert space* $\mathcal{H}$*.*

i) If B *is a closed extension of* A*, then* A *is closeable.*

ii) If A *is closeable, then* $\bar{A}$ *is the smallest closed extension of* A*, i.e. if* B *is a closed extension of* A*, then* $A \subseteq \bar{A} \subseteq B$.

1.1.3 The Adjoint Operator

The notion of the adjoint of an operator will be central to all our subsequent analysis. We now take the opportunity to define it.

Definition 1.1.7 *Let* A *be a densely defined linear operator on the Hilbert space* $\mathcal{H}$*. We define the adjoint operator* A^* *as follows*

$$D(A^*) = \left\{ g \in \mathcal{H} \;\middle|\; \exists\, g^* \in \mathcal{H} \text{ so that } \langle Ax, g \rangle = \langle x, g^* \rangle \text{ for all } x \in D(A) \right\}$$

$$A^*g = g^* \qquad \text{for all} \qquad g \in D(A^*).$$

We note that in this last definition it is the density of $D(A)$ in $\mathcal{H}$ that ensures that A^* is a well defined operator. We shall find the following result concerning the nature of the adjoint useful.

Lemma 1.1.5 *Let A be a densely defined linear operator on the Hilbert space $\mathcal{H}$, then A^* is closed. Further, if A is also closeable then*

i) $A^* = \bar{A}^*$, *i.e. the adjoint of A and the adjoint of $\bar{A}$ coincide.*

ii) $D(A^*)$ *is dense in $\mathcal{H}$.*

iii) $\bar{A} = A^{**}$, *i.e. the closure of A is equal to the second adjoint of A.*

1.1.4 Symmetric & (Essentially) Self-adjoint Operators

In this section we present the notion of the symmetry, self-adjointness and essential self-adjointness of a linear operator. We begin by making the following definition.

Definition 1.1.8 *Let A be a densely defined, linear operator on the Hilbert space $\mathcal{H}$. A is said to be symmetric if $A \subseteq A^*$ so that $D(A) \subseteq D(A^*)$ and $Ax = A^*x$ for all $x \in D(A)$. Equivalently, A is symmetric if the equality $\langle Ax, y \rangle = \langle x, Ay \rangle$ holds for all $x, y \in D(A)$.*

By definition, if A is symmetric then A^* is a closed extension of A. Hence, by Lemma 1.1.4, all symmetric operators are closeable. Furthermore, if A is symmetric, then $\bar{A}$ is also symmetric. Indeed, if $x, y \in D(\bar{A})$, then there exists sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ in $D(A)$ such that $x_n \to x$ and $y_n \to y$ and so

$$\langle \bar{A}y, x \rangle = \lim_{n\to\infty} \langle Ay_n, x_n \rangle = \lim_{n\to\infty} \langle y_n, Ax_n \rangle = \langle y, \bar{A}x \rangle. \tag{1.2}$$

The following theorem, due to Hellinger-Toeplitz, states that any everywhere defined, closed operator must be bounded and vice versa.

Theorem 1.1.6 *[2, Theorem III.12]*

Let A be an everywhere defined, linear, symmetric operator on the Hilbert space $\mathcal{H}$. Then A is bounded if and only if A is closed.

We now define the self-adjointness of a linear operator.

Definition 1.1.9 *Let A be a densely defined linear operator on the Hilbert space $\mathcal{H}$. A is said to be self-adjoint if $A = A^*$, thus $D(A) = D(A^*)$ and $Ax = A^*x$ for all $x \in D(A)$.*

Some remarks are in order with regards to the definitions of symmetry and self-adjointness. First, the difference between symmetry and self-adjointness lies in the domain of the adjoint. If $D(A) \subseteq D(A^*)$, then A is symmetric, where as if $D(A) = D(A^*)$, then A is self-adjoint. Therefore, it is clear that any self-adjoint operator is symmetric. Indeed, for a bounded operator defined on the whole Hilbert space the concepts of symmetry and self-adjointness are equivalent.

However, for unbounded operators self-adjointness is a much stronger condition than symmetry. By Toeplitz's theorem, a symmetric, unbounded operator cannot be defined on the whole space. If it were, then it's closure would also be defined on the whole space and, consequently, be bounded. Thus, a symmetric unbounded operator A can only be defined on (at best) a dense subset of the Hilbert space. The domain of the adjoint of this operator may or may not turn out to be strictly greater than $D(A)$, so that it is possible that this operator is symmetric but not self-adjoint. In applications, particularly those pertaining to quantum mechanics, one usually encounters densely defined, symmetric operators that are unbounded. Consequently, one must ascertain whether the operator is self-adjoint or not. If the operator is not self-adjoint, then typically one looks for extensions of it that are. Indeed, there is an obvious natural candidate for the self-adjoint extension of any closeable operator.

Definition 1.1.10 *Let A be a densely defined, closeable, linear operator on the Hilbert space $\mathcal{H}$. A is said to be* ***essentially self-adjoint*** *if its closure is self-adjoint, i.e. if $\bar{A} = \bar{A}^* \equiv A^*$.*

The following lemma, which is an immediate consequence of the definitions, Lemma 1.1.4 and Lemma 1.1.5, asserts that if A is an essentially self-adjoint operator then its closure defines a unique self-adjoint extension of A.

Lemma 1.1.7 *Let A be a densely defined, linear, symmetric operator on the Hilbert space $\mathcal{H}$. A is essentially self-adjoint if and only if the adjoint operator A^* is symmetric, i.e. $A^* \subseteq A^{**} = \bar{A}$. Furthermore, if A is essentially self-adjoint, then $\bar{A}$ is the only self-adjoint extension of A.*

The theory of essentially self-adjoint operators on Hilbert spaces is well developed and there are many sufficient conditions for essential self-adjointness. However, throughout this thesis we shall only ever need to appeal to three such sets of conditions. Before we present the second and third criteria for essential self-adjointness (the first being given in Lemma 1.1.7 above) it is necessary to make the following definition and present an associated lemma.

Definition 1.1.11 *Let A be a linear operator on the Hilbert space $\mathcal{H}$. We define the kernel of A by $\ker(A) = \{ x \in D(A) \mid Ax = 0_{\mathcal{H}} \}$ and the range of A as follows $\mathrm{Ran}(A) = \{ y \in \mathcal{H} \mid y = Ax \text{ for some } x \in D(A) \}$.*

In particular, if A is densely defined and $\lambda \in \mathbb{C}$, then $D(A^* - \lambda) = D(A^*)$ and we have the following characterizations of the kernel of $A^* - \lambda$

$$\begin{aligned} \ker(A^* - \lambda) &= \{ \, x \in D(A^*) \; \big| \; (A^* - \lambda)x = 0_{\mathcal{H}} \, \} \\ &= \{ \, x \in \mathcal{H} \; \big| \; \big\langle (A - \bar{\lambda}) \, u, x \big\rangle = 0, \text{ for all } u \in D(A) \, \} \qquad (1.3) \\ &= \operatorname{Ran}(A - \bar{\lambda})^{\perp}. \qquad (1.4) \end{aligned}$$

The following elegant theorem, due to Von Neumann, asserts that we can always decompose the domain of the adjoint operator in terms of its kernel.

Theorem 1.1.8 *[7, Section 7.7, Lemma 1]*

Let A be a densely defined, symmetric (therefore closeable) linear operator on the Hilbert space $\mathcal{H}$. Then we may decompose the domain of the adjoint operator as

$$D(A^*) \;=\; D(\bar{A}) \oplus \ker(A^* - i) \oplus \ker(A^* + i) \qquad (1.5)$$

In particular, A is essentially self-adjoint if and only if $\ker(A^* - i) = \ker(A^* + i) = \{0_{\mathcal{H}}\}$.

We now present the final criteria we will need for essential self-adjointness. We give the proof of this criteria for two reasons. First, this result is a slight modification of the more usual 'fundamental criteria for essential self-adjointness' found in most texts on functional analysis. Secondly, the result is central to the proof of the main theorem of the thesis - Theorem 2.1.1.

Theorem 1.1.9 *Let A be a densely defined, linear symmetric operator on the Hilbert space $\mathcal{H}$. Further let there exist $C > 0$ and $\zeta \in \mathbb{R}$ so that the estimate*

$$\big\langle (A - \zeta) \, u \, , \, u \big\rangle \quad \geq \quad C \, || \, u \, ||^2$$

holds for all $u \in D(A)$. If $\ker(A^* - \zeta) = \{ 0_{\mathcal{H}} \}$, *then A is essentially self-adjoint.*

Proof. Let $\rho \in D(\bar{A})$. Then there exists a sequence $\{\rho_n\}_{n=1}^{\infty} \subseteq D(A)$ so that $\rho_n \to \rho$ and $\{(A - \zeta)\rho_n\}_{n=1}^{\infty}$ converges. Since $\rho_n \to \rho$ and $(A - \zeta)\rho_n \to (\bar{A} - \zeta)\rho$, by Lemma 1.1.1 and the assumptions of the theorem we have that

$$\big\langle (\bar{A} - \zeta)\rho \, , \, \rho \big\rangle \;=\; \lim_{n \to \infty} \big\langle (A - \zeta)\rho_n \, , \, \rho_n \big\rangle \;\geq\; \lim_{n \to \infty} C \big\langle \rho_n \, , \, \rho_n \big\rangle \;=\; C \, || \, \rho \, ||^2. \qquad (1.6)$$

Now, let $\{u_n\}_{n=1}^{\infty} \subseteq D(\bar{A})$ such that $\{f_n\}_{n=1}^{\infty} = \{(\bar{A} - \zeta)u_n\}_{n=1}^{\infty} \subseteq \operatorname{Ran}(\bar{A} - \zeta)$. Further, let us assume that $f_n \to f \in \mathcal{H}$. We aim to show that $f \in \operatorname{Ran}(\bar{A} - \zeta)$ so that $\text{Ran}(\bar{A}-\zeta)$ is a closed subspace of $\mathcal{H}$. Indeed, using equation (1.6) and the Schwarz inequality we obtain

$$
\begin{aligned}
||\, u_n - u_m \,||^2 &\leq \frac{1}{C} \left\langle \, (\bar{A}-\zeta)\,(u_n - u_m)\,,\,(u_n - u_m) \, \right\rangle \\
&\leq \frac{1}{C} \; ||\,(\bar{A}-\zeta)\,(u_n - u_m)\,|| \cdot ||\, u_n - u_m \,|| \\
&= \frac{1}{C} \; ||\, f_n - f_m \,|| \cdot ||\, u_n - u_m \,||.
\end{aligned}
$$

Consequently, $||\, u_n - u_m \,|| \leq 1/C \,||\, f_n - f_m \,|| \to 0$. Therefore, since $\mathcal{H}$ is complete, there exists $u \in \mathcal{H}$ so that $u_n \to u$. Moreover, since $\bar{A}-\zeta$ is closed, the fact that $u_n \to u$ and $(\bar{A}-\zeta)u_n \to f$ imply that $u \in D(\bar{A})$ and $(\bar{A}-\zeta)u = f$. As such, we have shown that $\text{Ran}(\bar{A}-\zeta)$ is a closed subspace of $\mathcal{H}$ and as a result of Theorem 1.1.2 we have that

$$
\begin{aligned}
\mathcal{H} &= \text{Ran}(\bar{A}-\zeta) \oplus \text{Ran}(\bar{A}-\zeta)^{\perp} \\
&= \text{Ran}(\bar{A}-\zeta) \oplus \ker(A^* - \zeta) \\
&= \text{Ran}(\bar{A}-\zeta) \qquad (1.7)
\end{aligned}
$$

where we have used the result of equation (1.4).

Since A is symmetric, in order to show that $\bar{A} = \bar{A}^* \equiv A^*$, it suffices to show that $D(A^*) \subseteq D(\bar{A})$. Indeed, let $\phi \in D(A^*)$ and $\Psi = (A^* - \zeta)\phi$. Since $\text{Ran}(\bar{A}-\zeta) = \mathcal{H}$ there exists $v \in D(\bar{A})$ so that $(\bar{A}-\zeta)v = \Psi$. Yet since A, and therefore $\bar{A}$, is symmetric we have that $A \subseteq \bar{A} \subseteq \bar{A}^* = A^*$ and hence $(A^* - \zeta)v = \Psi$. Consequently, we have that $(A^*-\zeta)v = \Psi = (A^*-\zeta)\phi$, so that $(A^*-\zeta)(v-\phi) = 0_{\mathcal{H}}$. Since $\ker(A^*-\zeta) = \{0_{\mathcal{H}}\}$ it must be the case that $\phi = v \in D(\bar{A})$. Thus, we arrive at the conclusion that $D(A^*) \subseteq D(\bar{A})$, hence $\bar{A} = A^*$ and the theorem is proven. ∎

1.2 The Axioms of Quantum Mechanics

At this point the reader may legitimately wonder **why** we are interested in the essential self-adjointness of linear operators. The simple answer to this question is that, from a mathematical perspective, quantum mechanics is the study of (essentially) self-adjoint operators. Hence, the purpose of this short section is to provide a brief overview of the mathematical formalism of quantum mechanics in order to demonstrate that self-adjoint operators lie at the heart of this theory.

First and foremost, quantum mechanics is a branch of mathematical physics dealing with physical phenomena that occur at the (sub) atomic scale, whereby the action of particles / waves is of roughly the same magnitude as Planck's constant. Pioneered by (amongst others) Planck, Heisenberg, Born, Schrödinger, Pauli, Hilbert, Dirac and von Neumann from the beginning of the twentieth century, the area has enjoyed considerable success in providing a coherent description of the dual particle like and wave like behavior of atomic matter and its interactions with energy. In order to demonstrate why self-adjoint operators are fundamental to quantum mechanics it is necessary to appeal to the various 'axioms' of the theory. The word axioms appears here in inverted commas because there is no definitive list of postulates upon which the theory is based. Typically, in any introductory textbook on quantum mechanics, a variety of abstract mathematical conditions are stated as axioms with little or no justification as to why these conditions are self-evident or provide a consistent description of reality at the (sub) atomic level. However, attempts at an a priori justification of the standard quantum mechanical model date back to the 1930's and the work of von Neumann [23]. Since then a huge amount of effort has been devoted towards attempting to 'derive' these axioms from more general elementary principles (see [24], [25], [26] and the references given in the notes to section VIII.11 of [2]). Yet...

> "despite the rather enormous literature on the 'first level' foundations of quantum mechanics there is no definitive theory of quantum axiomatics."
>
> Reed & Simon [2], page 312.

Rather than attempt to resolve this problem here we simply state, without justification, four of the more commonly stated 'axioms' of the theory that are particularly pertinent with respect to essential self-adjointness.

AXIOM I For each quantum system there is a corresponding separable Hilbert space $\mathcal{H}$. For most physical systems there is a particularly useful realization of $\mathcal{H}$ as an appropriate L_2 space.

AXIOM II The possible states of the system are equivalence classes of unit vectors in $\mathcal{H}$. Two vectors Ψ_1 and Ψ_2 are said to correspond to the same state if they differ only by a complex multiple of absolute value 1, i.e. if $\Psi_2 = e^{i\alpha}\Psi_1$. In a one particle system, the absolute square of the state vector Ψ (also known as the wavefunction) can be interpreted as a probability density function describing the probability of finding the particle in a given spacial region.

AXIOM III For every observable of the system there is a corresponding self-adjoint operator. The set of values that measurements of an observable can take are given by the spectrum of the corresponding self-adjoint operator.

AXIOM IV The dynamics of the system are governed by a strongly continuous one parameter group of unitary operators $U(t)$, which by Stone's Theorem[1] are in one to one correspondence with the self-adjoint operators on $\mathcal{H}$. If the system is in state Ψ at time t_0, then the system is in state $U(t_1)\Psi$ at time t_1.

[1]**Stone's Theorem** [2, Theorem VIII.8]: Let A be a self-adjoint operator on the Hilbert space $\mathcal{H}$. Then e^{itA} is a strongly continuous 1-parameter group of unitary operators. Conversely, if $U(t)$ is a strongly continuous one parameter group of unitary operators, then there exists a self-adjoint operator A such that $U(t) = e^{itA}$.

From these axioms alone we see that self-adjoint operators have a central role to play in almost every aspect of quantum mechanics. Indeed, given any quantum mechanical system...

"there are three general mathematical problems that arise...

1) **Self-adjointness:** In most cases, physical reasoning gives a formal expression for observables as operators on a realization of $\mathcal{H}$ as $L_2(\Omega)$. We use the word formal because domains are not specified. It is usually easy to find a domain on which a given formal expression is a well defined symmetric operator. The first problem is to prove essential self-adjointness [so that the operator defines a unique self-adjoint extension], or if the operator is not essentially self-adjoint to investigate the various self-adjoint extensions.
2) **Spectral Analysis:** The second problem is to investigate the spectra of observables and to investigate the position and multiplicity of the point spectra.
3) **Scattering:** The third problem is to describe in some way the behavior of the system for large t."

Reed & Simon [2], page 303.

Now that we have justified our interest in the essential self-adjointness of linear operators, we go one step further and consider a certain type of linear operator - Schrödinger operators.

1.3 Schrödinger Operators

For any physical system there is one observable that has a particularly distinguished status - the energy of the system. Classically, the energy of a system is thought of as the sum of its kinetic (T) and potential energy (V). For a single particle system (using suitably normalized units) this is given by

$$E \quad = \quad T \, + \, V \quad = \quad \dot{x}^{\,2} \, + \, V(x) \tag{1.8}$$

where $x(t)$ is a position vector describing the position of the particle at time t. It is worth noting that for such a classical system all the relevant information concerning the motion of the particle can be gleaned from analysis of (1.8) and Newton's famous equation

$$-\nabla V(x) \;\; = \;\; \ddot{x}. \tag{1.9}$$

Analogously, in quantum mechanics the possible energy levels of a one particle system, in the spacial region $\Omega \subseteq \mathbb{R}^m$, correspond to the spectra of the formal Schrödinger (or Hamiltonian) operator

$$H \quad = \quad -\,\Delta \; + \; V(x) \tag{1.10}$$

which we can think of as being 'derived' from equation (1.8) by replacing the classical momentum of the particle $\dot{x}$, with the momentum operator $-i\,\nabla$. Again, we use the word 'formal' as we have not yet specified a domain of definition for this operator. Hence, here and throughout the rest of the thesis, we shall denote by H the operator described by (1.10) acting on the domain $C_0^\infty(\Omega)$ within the Hilbert space $L_2(\Omega)$. Further, we will always assume that V is real and $V(x) \in L_{\infty,loc}(\Omega)$. It is obvious that H is a linear operator, and it is well known that $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$ (see [8, Theorem C.23]) so that the operator is also densely defined. Moreover, applying integration by parts twice shows that for all $u,\, \rho \in C_0^\infty(\Omega)$

$$\begin{aligned}\langle\, Hu\,,\, \rho\,\rangle \quad &= \quad -\int_\Omega \Delta u(x)\,\bar{\rho}(x)\;\;dx \quad + \quad \int_\Omega u(x)\,V(x)\,\bar{\rho}(x)\;\;dx \\ &= \quad -\int_\Omega u(x)\,\Delta\,\bar{\rho}(x)\;\;dx \quad + \quad \int_\Omega u(x)\,V(x)\,\bar{\rho}(x)\;\;dx \\ &= \quad \langle\, u\,,\, H\rho\,\rangle\end{aligned}$$

so that by definition 1.1.8, H is evidently symmetric. Given a domain $\Omega \subseteq \mathbb{R}^m$, our task will be to find the minimal requirements for the operator H to be essentially self-adjoint. As the examples in the next section show, this amounts to specifying appropriate conditions on the potential V.

1.4 Essential Self-adjointness of Schrödinger Operators on $\mathbb{R}^m$

The theory of the essential self-adjointness of Schrödinger operators on $\mathbb{R}^m$ is well developed and, in it's original incarnation, dates back to the work of Sears [10] and Titchmarsh [11] in the 1960's. Our objective in this section is simply to give an overview of known results and to use these results to develop a physical interpretation for what it means for a Schrödinger operator on $\mathbb{R}^m$ to be essentially self-adjoint. This interpretation will have a natural analogue when we consider such operators on domains with non-empty boundary. We begin by stating the following theorem due to Berezin & Shubin [9] which is effectively a higher dimensional extension of the aforementioned results by Sears and Titchmarsh.

Theorem 1.4.1 *[9, Chapter 3, Theorem 1.1]*

Let $H = -\Delta + V$ be a Schrödinger operator with domain of definition $D(H) = C_0^\infty(\mathbb{R}^m)$, where $V(x)$ is a real valued, measurable, locally bounded function on $\mathbb{R}^m$. Let $V(x) \geq -Q(|x|)$, where $Q(r)$ is an increasing, positive, continuous function on $[0,\infty)$ so that

$$\int_{r=0}^{r=\infty} \frac{1}{\sqrt{Q(2r)}}\;\;dr \quad = \quad \infty. \tag{1.11}$$

Then H is essentially self-adjoint.

In a sense Theorem 1.4.1 is not explicit. That is to say that the criteria for essential self-adjointness expressed therein is phrased in terms of an auxiliary function $Q(r)$. Clearly the stipulations of the theorem allow for the possibility that the potential V dives to minus infinity as $|x|$ becomes large. However, this begs the question - just how fast can the potential dive to minus infinity and still guarantee the essential self-adjointness of our operator? It is not hard to see that the function $Q_0(r) = ar^\alpha + b$, where $a, b > 0$, satisfies the requirements of Theorem 1.4.1 if $\alpha \le 2$ but violates equation (1.11) if $\alpha > 2$. With this as motivation, Berezin & Shubin [9] go on to prove the following corollary to Theorem 1.4.1.

Corollary 1.4.1 *[9, Section 3.1]*

Let $H = -\Delta + V$ be a Schrödinger operator with domain of definition $D(H) = C_0^\infty(\mathbb{R}^m)$ where $V(x)$ is a real valued, locally bounded function on $\mathbb{R}^m$. For any $a, b > 0$,

i) if $V(x) \ge -a\,|x|^2 - b$ then H is essentially self-adjoint, and

ii) if $V(x) \le -a\,|x|^{2+\epsilon} - b$ where $\epsilon > 0$, then H is not essentially self-adjoint.

There is a rather nice physical interpretation that one can attach to this last result. For simplicity, let us take the one dimensional case and assume that the energy level of the system under consideration is non-negative and dominates the potential, i.e. that $E \ge 0$ and $E \ge \sup\{V(x)\}$. As we have seen, classically the energy of the system is given by the sum of the potential and kinetic energies, so that $E = \big(\frac{dx(t)}{dt}\big)^2 + V(x)$. In principle we may rearrange this expression to obtain $\frac{dt}{dx} = \big(E - V(x)\big)^{-\frac{1}{2}}$, and by integrating with respect to x from zero to ∞ we arrive at the equation

$$t(\infty) - t(0) \quad = \quad \int_{x=0}^{x=\infty} \Big(E - V(x)\Big)^{-\frac{1}{2}} \, dx.$$

This last expression effectively describes the time taken for a particle under the influence of the potential V to move from the origin to infinity.

On the one hand if $V(x) \ge -a\,|x|^2 - b$, then

$$t(\infty) - t(0) \quad \ge \quad \int_{x=0}^{x=\infty} \Big(ax^2 + b + E\Big)^{-\frac{1}{2}} \, dx$$

which, by an appropriate transformation of variables, is easily seen to be divergent. We may interpret this as saying that if $V(x) \ge -a\,|x|^2 - b$, then the particle cannot escape to infinity in a finite amount of time. As such, we say that the system is **classically complete**. On the other hand if $V(x) \le -a\,|x|^{2+\epsilon} - b$, then

$$t(\infty) - t(0) \quad \le \quad \int_{x=0}^{x=\infty} \Big(a\,x^{2+\epsilon} + b + E\Big)^{-\frac{1}{2}} \, dx$$

which is convergent. Again, this may be interpreted as saying that a particle under the influence of such a potential may escape to infinity in a finite amount of time. Consequently, the system is said to be **classically incomplete**.

Although the concepts of classical completeness and essential self-adjointness are in some sense complementary, the correspondence is by no means exact. The former notion belongs to classical mechanics whilst the latter is a facet of quantum mechanics. In this respect, Rauch & Reed [12] exploit quantum effects not present in classical mechanics to produce examples of systems which are classically complete but not essentially self-adjoint and vice versa. Never the less, the inability of a particle to reach infinity (or more generally to come into contact with the boundary of a domain) is a fruitful way of thinking about the essential self-adjointness of Schrödinger operators.

Indeed, the virtue of such a line of thinking is perhaps best envisaged within the results obtained by Eastham, Evans & Mcleod in [13]. Although it is difficult to give a succinct, explicit statement of the theorems in this paper, the general gist of their main result can be summarized as follows: if there exists a sequence of 'sufficiently thick' concentric annuli that occur 'sufficiently regularly', and on these annuli the potential is 'sufficiently large', then the corresponding Schrödinger operator is essentially self-adjoint. The physical interpretation is that these hurdles provide sufficient barriers to the outward progress of the particle so that it cannot escape to infinity in a finite amount of time. Moreover if one inserts a tube of 'sufficiently low' potential that extends to infinity, then the Schrödinger operator will not be essentially self-adjoint irrespective of the nature of the potential elsewhere. The interpretation here is that the particle escapes to infinity in a finite amount of time along this tube.

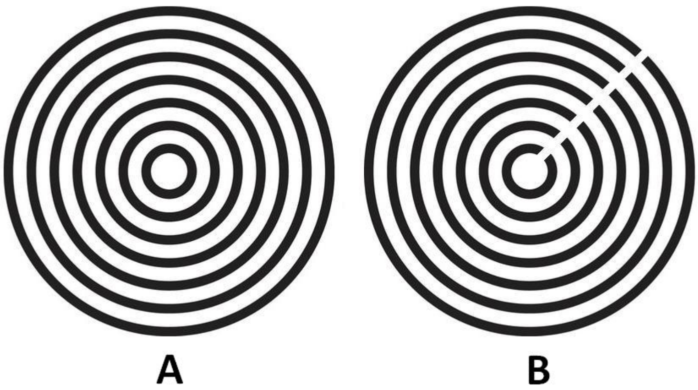

Figure 1.1: In diagram A above, the dark annuli represent regions of high potential. If these annuli are sufficiently thick and occur sufficiently regularly, then the Schrödinger operator H is essentially self-adjoint. The physical interpretation is that these annuli provide a sufficient barrier to the outward progress of the particle such that it cannot reach infinity in a finite amount of time. However, if we insert a tube of sufficiently low potential extending to infinity, as in diagram B, then H is not essentially self-adjoint. The physical interpretation here is that the particle escapes to infinity in a finite amount of time along this tube.

1.5 Essential Self-adjointness on Domains with non-Empty Boundary

We now turn to the main business of the thesis. From this point on we will only be concerned with the essential self-adjointness of Schrödinger operators on domains with non-empty boundary. Whereas the theory on $\mathbb{R}^m$ is well developed, comparatively little is known about essential self-adjointness on domains with non-empty boundary, except in the one dimensional case. In this latter setting a variety of results have been obtained by utilizing the limit point - limit circle analysis of Weyl [14]. This method relies heavily upon von Neumann's theory of deficiency indices and involves looking at the number of solutions to certain differential equations that lie in L_2. We will return to this idea in Chapter 6. For the moment we mention only that the aforementioned method has not been extended to higher dimensions.

In this section we survey some of the main ideas that arise in connection to the essential self-adjointness of Schrödinger operators on domains with non-empty boundary. We first look to obtain a physical interpretation for essential self-adjointness analogous to the interpretation described in the previous section. We arrive at this interpretation by means of explicit example. Next, we define the Euclidean distance function and the L_p-Hardy inequality before characterizing the later in terms of an associated variational problem. Finally we present a known theorem concerning the essential self-adjointness of Schrödinger operators on $\mathbb{R}^m \setminus \{0\}$ which, in some sense, encapsulates almost everything about the relationship between essential self-adjointness and the L_2-Hardy inequality.

1.5.1 A Motivating Example

Let $\Omega = (0, \infty)$ and let $H = -\frac{d^2}{dx^2}$ be the symmetric Schrödinger operator defined on $C_0^\infty(\Omega)$. The same reasoning as developed in the previous sections shows that a particle under the influence of this zero potential cannot escape to infinity in a finite amount of time. Despite this we will show that H is **not** essentially self-adjoint. This suggests that some sort of problem occurs at the boundary point at the origin.

Consider the functions $\rho_\pm(x) = e^{-\frac{1}{\sqrt{2}}(1 \pm i)\,x} \in C^\infty(\Omega) \cap L_2(\Omega)$. Two simple applications of integration by parts shows that for all $u(x) \in D(H) = C_0^\infty(\Omega)$ we have $\langle\, Hu\,,\, \rho_\pm \,\rangle = \langle\, u\,,\, -\frac{d^2}{dx^2}\rho_\pm \,\rangle$, so that $\rho_\pm \in D(H^*)$ and $H^* \rho_\pm = -\frac{d^2}{dx^2}\,\rho_\pm$. Furthermore, elementary differentiation yields $-\frac{d^2}{dx^2}\,\rho_\pm \pm i\,\rho_\pm = 0$ indicating that $\rho_\pm \in \ker(H^* \pm i)$. By Theorem 1.1.8, this is enough to show that $D(\bar{H}) \subsetneq D(H^*)$ and so H is not essentially self-adjoint.

Moreover, in Example 2 of Section X.1, Reed & Simon [3] show that the possible self-adjoint extensions of H are given by

$$H_a = -\frac{d^2}{dx^2} \qquad\qquad D(H_a) = \left\{\, \Psi \in AC^2[0,\infty] \;\middle|\; \Psi'(0) + a\Psi(0) = 0 \,\right\}$$

for all $a \in \mathbb{R}$. In other words, the self-adjoint extensions of H are obtained effectively by specifying appropriate boundary conditions at zero. The physical interpretation here is that the (zero) potential is not strong enough to prevent a particle under its influence from reaching the origin. Therefore, boundary conditions must be specified at this point to 'tell the particle how to behave' when it arrives there.

Let us now consider a Schrödinger operator $H = -\frac{d^2}{dx^2} + V(x)$ defined on $C_0^\infty(0,\infty)$, and ask the following question - what must the potential be like in the vicinity of zero in order to ensure the essential self-adjointness of H? For a moment, let us consider the corresponding classical problem of confining the particle away from the origin. This classical problem is governed by the equations (1.8) and (1.9). Let us suppose that the potential tends to infinity as it approaches the origin. Then no matter what the energy level of the system, a particle under the influence of this potential would be unable to reach zero.

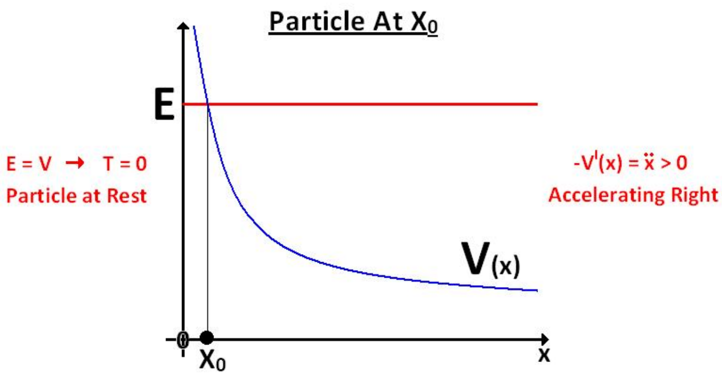

Figure 1.2: Classically, a singularity of the potential at the boundary is sufficient to ensure a particle under its influence is confined away from the boundary. In terms of the diagram above the particle is unable to penetrate into the classically forbidden region $(0, x_0)$.

At some point sufficiently close to zero the potential energy of the particle must equal its total energy. At this point its kinetic energy must be zero so that the particle is momentarily at rest. However, its acceleration, being equal to the negative of the slope of the potential must be positive. As such the particle will begin to move right and away from the boundary point at the origin.

Therefore, the classical picture suggests that if we consider a domain with non-empty boundary, then in order to ensure the essential self-adjointness of our Schrödinger operator the potential must 'blow up' as it approaches the boundary. However, in the wave-particle context of quantum mechanics, the state of the system is effectively a probability distribution describing the likelihood of finding the particle in a particular region of space. Due to the consequent quantum tunneling effect, we should expect that in order to ensure the essential self-adjointness of our operator the potential must blow up at an 'appropriate rate'. In fact, we shall soon see that the potential inflating at the rate of one over the distance to the boundary squared is sufficient to ensure essential self-adjointness. However, this only half the story. Another quantum effect - the uncertainty principle - will also come into play.

1.5.2 The Euclidean Distance to the Boundary

We have just asserted that central to the essential self-adjointness of Schrödinger operators on domains with non-empty boundary will be the blow up of the potential like one over the distance to the boundary squared. At this juncture, it would therefore seem wise to define precisely what we mean by 'the distance to the boundary' and to elucidate some of the properties of this concept.

Definition 1.5.1 *Let A and B be two sets in $\mathbb{R}^m$. We define the Euclidean distance between the two sets by the function $d : A \times B \to [0, \infty)$*

$$d(A, B) \equiv \operatorname{dist}(A, B) = \inf \left\{ r \geq 0 \;\middle|\; |x - y| \geq r,\ x \in A,\ y \in B \right\}.$$

In particular, if Ω is a domain in $\mathbb{R}^m$ with non-empty boundary, then we reserve the notation $d(x)$ for the function $d : \mathbb{R}^m \to [0, \infty)$

$$d(x) \equiv \operatorname{dist}(x, \partial\Omega) = \inf \left\{ r \geq 0 \;\middle|\; |x - y| \geq r,\ y \in \partial\Omega \right\}.$$

It is easy to see that $d(x)$ is a 1-Lipschitz function. By Rademacher's theorem (cf [17], [18, Section 3.1.2]) every Lipschitz function is once differentiable almost everywhere and its derivative is essentially bounded in magnitude by the Lipschitz constant. Thus, we immediately have that $d(x)$ is once differentiable almost everywhere and that

$$|\nabla d(x)| \leq 1. \tag{1.12}$$

In fact, wherever the derivative exists we have that $|\nabla d(x)| = 1$. In the vast majority of our subsequent analysis we will only need equation (1.12). For further information on the properties of the Euclidean distance function the reader is referred to the Appendix to Chapter 14 of [19].

1.5.3 The L_p-Hardy Inequality

As was previously eluded to, the uncertainty principle has an important role to play in the essential self-adjointness of Schrödinger operators on domains with non-empty boundary. Since the L_2-Hardy inequality is effectively an expression of the uncertainty principle it is of no huge surprise that it should factor into our analysis at some point. For completeness, and in order to simplify the exposition in subsequent chapters, we now take the opportunity to define the L_p-Hardy inequality.

Definition 1.5.2 *Let Ω be a domain in $\mathbb{R}^m$ with non-empty boundary. For $1 < p < \infty$, Ω is said to admit an L_p-Hardy inequality if there exists a finite uniform constant $C > 0$ so that the estimate*

$$\int_\Omega \frac{|\,\omega(x)\,|^p}{d(x)^p}\; dx \qquad \leq \qquad C \int_\Omega |\,\nabla\omega(x)\,|^p\; dx \tag{1.13}$$

holds for all $\omega(x) \in W^1_{p,0}(\Omega)$. The smallest constant for which equation (1.13) holds is known as the ***optimal L_p-Hardy constant*** *and is denoted by $C_{H,P}(\Omega)$.*

There is a natural variational problem associated to the existence of an L_p-Hardy inequality on a given domain. To make this idea precise we make the following definition.

Definition 1.5.3 *Let Ω be a domain in $\mathbb{R}^m$ with non-empty boundary. Define the variational constant $\mu_p(\Omega)$ by*

$$\mu_p(\Omega) \quad = \quad \inf_{\omega\,\in\, W^1_{p,0}(\Omega)} \left\{ \frac{\int_\Omega |\,\nabla\,\omega(x)\,|^p\; dx}{\int_\Omega \frac{|\,\omega(x)\,|^p}{d(x)^p}\; dx} \right\}$$

It is not hard to see that a domain Ω admits an L_p-Hardy inequality if and only if $\mu_p(\Omega) > 0$. In particular, if Ω does admit an L_p-Hardy inequality, then $C_{H,P}(\Omega) = \mu_p(\Omega)^{-1}$. As an example, consider the domain $\Omega = \mathbb{R}^m \setminus \{0\}$. We will prove, in Lemma 5.2.3, that $\mu_2(\Omega) = \left(\frac{m-2}{2}\right)^2$. As such, Ω admits an L_2-Hardy inequality for all $m \neq 2$ with optimal constant $C_{H,2}(\Omega) = \left(\frac{2}{m-2}\right)^2$.

1.5.4 Essential Self-adjointness on $\mathbb{R}^m \setminus \{0\}$

In order to make our ideas concerning the essential self-adjointness of Schrödinger operators and the L_2-Hardy inequality more precise, let us consider the domain $\mathbb{R}^m \setminus \{0\}$. The following theorem, which can be thought of as the combined result of work by Kalf & Walter [20], Schmincke [21] and Simon [22], is often referred to in the literature as the KWSS theorem.

Theorem 1.5.1 *[22, Theorem 2]*

Let $\Omega = \mathbb{R}^m \setminus \{0\}$. Let $H = -\Delta + V$ be a Schrödinger operator defined on $C_0^\infty(\Omega)$ with potential structure $V = V_1 + V_2$, $V_2 \in L_\infty(\Omega)$, $V_1 \in L_{\infty,loc}(\Omega)$ and

$$V_1(x) \quad \geq \quad -\,\frac{(m-1)\,(m-3)\; -\; 3}{4d(x)^2}.$$

Then H is essentially self-adjoint.

At first sight the relationship between essential self-adjointness and the L_2-Hardy constant within this theorem seems obscure. However, if we rewrite the condition on $V_1(x)$ as

$$V_1(x) \quad \geq \quad -\frac{(m-1)(m-3) - 3}{4\, d(x)^2} \quad = \quad \frac{1 - \left(\frac{m-2}{2}\right)^2}{d(x)^2} \tag{1.14}$$

the connection becomes apparent. Via equation (1.14) we see that Theorem 1.5.1 effectively says the following. If the potential $V(x)$ blows up at the boundary like $\frac{1}{d(x)^2}$ then the operator H is essentially self-adjoint, the physical interpretation being that this prevents a quantum particle under the influence of the potential $V(x)$ from reaching the boundary. However, we can relax this condition precisely by the value of the constant $\mu_2(\Omega)$ and still be guaranteed the essential self-adjointness of our operator. Note that if $m \geq 5$, then $\mu_2(\Omega) = \left(\frac{m-2}{2}\right)^2 > 1$ so the latter effect dominates. Hence, in this case the potential may dive to minus infinity (although not too fast) as it approaches the boundary and still be sufficiently strong to ensure essential self-adjointness.

The main theorem of this thesis, which is presented immediately in the subsequent chapter, can be thought of as generalizing this result to arbitrary domains with non-empty, possibly highly irregular, boundary.

Chapter 2

Essential Self-adjointness on Domains with non-Empty Boundary

In [1] Nenciu & Nenciu investigated the essential self-adjointness of Schrödinger operators on bounded domains in $\mathbb{R}^m$, with C^2 boundary of co-dimension 1. The objective of this chapter is to extend these result to more general domains, in particular to unbounded domains and domains with highly irregular boundary. Indeed, Theorem 2.1.1 - the main theorem of this thesis - is applicable to domains with non-empty fractal boundary, such as $\mathbb{R}^m \setminus \mathcal{C}$ where $\mathcal{C}$ is the middle thirds Cantor set. As in the example considered in the last section of the previous chapter, we will see that there are two factors governing the essential self-adjointness of Schrödinger operators; the inflation of the potential like $d(x, \partial\Omega)^{-2}$, and the value of variational constant $\mu_2(\Omega)$, as given in Definition 1.5.3.

This chapter is structured as follows. In Section 2.1, we state our main theorem and make some general remarks concerning it. In particular, we discuss the relationship between essential self-adjointness and the variational constant $\mu_2(\Omega)$ within the context of the uncertainty principle. In Section 2.2, we make various definitions and settle on some notation that will be used throughout. Then, in Section 2.3, we present some technical lemmas that will be required for the proof of the main theorem. Many of the results in this section are either slight modifications of lemmas used by Nenciu & Nenciu in [1] or are taken directly from that paper. The proof of Theorem 2.1.1 is given in Section 2.4. However, Theorem 2.1.1 is not explicit. That is to say that the criteria for essential self-adjointness expressed therein is phrased in terms of an auxiliary function $G(t)$ which satisfies various conditions. In Section 2.5, we explore two possible candidate functions for $G(t)$ which, in turn, give explicit criteria for the essential self-adjointness of the operator under consideration. Of course, there are many other possible candidate functions. However, the two chosen yield, in some sense, the simplest possible and best possible results. In Chapter 6 this will allow us to recapture, and improve, known results concerning the essential self-adjointness of Schrödinger operators.

2.1 Statement of the Main Theorem

The main theorem of this thesis is stated below.

Theorem 2.1.1 *Let* Ω *be a domain in* $\mathbb{R}^m$ *with non-empty boundary and let* $d(x) = \mathrm{dist}(x, \partial\Omega)$ *be the Euclidean distance to the boundary. Let* $0 < R < R_1 < R_2$ *, and for all* $n \in \mathbb{N}_0$ *define* $P_n = \frac{R}{\lambda^n}$ *where* $1 < \lambda < e$*. Suppose that the real valued function* $G(t) \in C^1(0,\infty)$ *satisfies the following conditions:*

1) $0 \le G'(t) \le \frac{\kappa_0}{t}$ *for all* $t > 0$, *where* κ_0 *is a positive constant.*

2) $G'(t) = 0$ *for all* $t \ge R_2$.

3) $\sum_{n=1}^{\infty} P_n^2 \, e^{-2G(P_n)} = \infty$ *for all* $R < R_1$.

4) $\sum_{n=1}^{\infty} P_n^k \, e^{-2G(P_n)} < \infty$ *for all* $R < R_1$ *and for all* $k > 2$.

(If Ω *is a bounded domain this last condition can be omitted). Define the Schrödinger operator*

$$H \;=\; -\Delta + V$$

with domain $D(H) = C_0^\infty(\Omega)$*, where* $V \in L_{\infty,loc}(\Omega)$ *is a real potential of the form*

$$V \;=\; V_1 + V_2, \qquad\qquad V_2 \in L_\infty(\Omega)$$

$$V_1(x) \quad\ge\quad G'\big(d(x)\big)^2 \;-\; \mu_2(\Omega)\, d(x)^{-2}.$$

Then the operator H is essentially self-adjoint.

Some comments about the above theorem are appropriate. First of all, let us investigate the nature of the auxiliary function $G(t)$, in terms of which our criteria for essential self-adjointness is phrased. Looking at the conditions of Theorem 2.1.1, it is clear that $G(t)$ is monotone non-increasing as $t \to 0$, and is constant for $t \ge R_2$. Furthermore, it must tend to minus infinity along the sequence $\{P_n\}_{n=0}^{\infty}$. If this were not the case then $e^{-2G(P_n)}$ would be bounded above and so the series $\sum_{n=1}^{\infty} P_n^2 \, e^{-2G(P_n)}$ would converge. Since $G(t)$ is continuous, monotone, tends to minus infinity as $t \to 0$ and is constant for all $t \ge R_2$, it must be bounded above. In particular, there exists a positive constant M_1 so that

$$e^{G(t)} \quad\le\quad M_1. \tag{2.1}$$

An obvious candidate for the function $G(t)$ will be the real valued, monotone non-decreasing function $G_0(t) \in C^1(0,\infty)$ defined by

$$G_0(t) = \begin{cases} \ln t & \text{if} \quad 0 < t \le R_1, \\ C_0 & \text{if} \quad t \ge R_2. \end{cases} \tag{2.2}$$

where $\ln R_1 < C_0 < \ln R_2$. We will show that $G_0(t)$ satisfies the conditions of Theorem 2.1.1 in Section 2.5. There, we also explore another possible candidate function for $G(t)$. However, until then the reader is encouraged to think of the function $G(t)$ as not being 'too different' from $\ln t$ for sufficiently small t.

Indeed, using this particular choice of candidate function, Theorem 2.1.1 dictates that our Schrödinger operator is essentially self-adjoint provided that

$$V_1(x) \qquad \geq \qquad \frac{1 - \mu_2(\Omega)}{d(x)^2}. \tag{2.3}$$

Within this context, let us now look at the role that the variational constant $\mu_2(\Omega)$ has to play in our criteria for essential self-adjointness. First, suppose that $\mu_2(\Omega) = 0$, so that, by the remarks following Definition 1.5.3, Ω does not admit an L_2-Hardy inequality. From equation (2.3) we clearly see that our Schrödinger operator is essentially self-adjoint if $V_1(x) \geq d(x)^{-2}$. On the other hand, if $\mu_2(\Omega) > 0$, so that Ω admits an L_2-Hardy inequality, then this criteria is reduced to $V_1(x) \geq [\,1 - \mu_2(\Omega)\,]\, d(x)^{-2}$. In other words, the existence of an L_2-Hardy inequality on a given domain relaxes the criteria for essential self-adjointness. We assert that the uncertainty principle lies behind this effect.

On a physical level, the traditional statement of the uncertainty principle is that one cannot measure the position and momentum of a particle simultaneously, because measurement of one of these variables affects the value of the other. On a mathematical level, the uncertainty principle is expressed by the fact that given any two self-adjoint operators in a Hilbert space that do not commute, it is impossible to measure the variables to which they correspond simultaneously (to an arbitrary degree of accuracy) because the product of the variances of these operators must exceed a given lower bound (see for instance [27, Section 1.3]). In a broader sense, the uncertainty principle is a term used to refer to any situation whereby it is impossible to know the value of two variables simultaneously.

Indeed, suppose that $\Omega \subseteq \mathbb{R}^m$ is a domain that admits an L_2-Hardy inequality. Furthermore, consider a particle contained within Ω described by the unit state vector $\Psi \in W^1_{2,0}(\Omega)$. An application of Hölder's inequality leads us to the following set of inequalities

$$\begin{aligned} 1 \quad &= \quad \int_\Omega d(x)\, \omega(x)\, \overline{\frac{\omega(x)}{d(x)}} \;\; dx \\ &\leq \quad \left(\int_\Omega d(x)^2\, |\, \omega(x)\,|^2 \;\; dx \right)^{\frac{1}{2}} \cdot \left(\int_\Omega \frac{|\, \omega(x)\,|^2}{d(x)^2} \;\; dx \right)^{\frac{1}{2}} \\ &\leq \quad \mu_2(\Omega)^{-\frac{1}{2}} \left(\int_\Omega d(x)^2\, |\, \omega(x)\,|^2 \;\; dx \right)^{\frac{1}{2}} \cdot \left(\int_\Omega |\, \nabla\, \omega(x)\,|^2 \;\; dx \right)^{\frac{1}{2}}. \end{aligned}$$

By squaring both sides of the last equation and re-arranging, we obtain that

$$\mu_2(\Omega) \quad \leq \quad \left(\int_\Omega d(x)^2\, |\, \omega(x)\,|^2 \;\; dx \right) \cdot \left(\int_\Omega |\, \nabla\, \omega(x)\,|^2 \;\; dx \right). \tag{2.4}$$

The second integral on the right hand side of equation (2.4) describes the total momentum of the particle, whilst the first integral gives a measure of the particle's proximity to the boundary.

Recall that the physical interpretation here is that a Schrödinger operator is essentially self-adjoint provided that a particle under the influence of the associated potential is unable to come into contact with the boundary of the domain. As such the value of $\mu_2(\Omega)$ places limits on the certainty with which we can say that a particle is 'located' at the boundary. The more highly localized a particle is near to the boundary of a domain, the greater the corresponding Dirichlet integral must be in order for equation (2.4) to hold. In other words, one cannot confine a particle to a smaller and smaller neighborhood of the boundary of a domain without producing a corresponding increase in it's total momentum.

In simplistic terms, given a domain Ω with non-empty boundary, the greater the value of $\mu_2(\Omega)$, the less certain we can be that a particle is located at the boundary. This, therefore, relaxes the criteria for the essential self-adjointness of a Schrödinger operator. Glimpses of the relationship between the uncertainty principle (in the sense described above) and essential self-adjointness can be found in [1, Section 2], [28, Section 6] and [29].

Furthermore, we note the consequence that the value of $\mu_2(\Omega)$ has for our theorem. That is to say that if $\mu_2(\Omega) < 1$, then equation (2.3) implies that the potential must tend to infinity as it approaches the boundary in order to ensure essential self-adjointness. On the other hand, if $\mu_2(\Omega) > 1$, then the same outcome is achieved even if the potential dives slowly to minus infinity as it approaches the boundary[1]. Therefore, when considering the essential self-adjointness of Schrödinger operators, two questions will dominate our thoughts. If Ω is a domain with non-empty boundary then

1. **Does Ω admit an L_2-Hardy inequality?**

2. **What is the value of $\mu_2(\Omega)$?**

In the next three chapters of this thesis we will try and answer these two questions as thoroughly as possible. Finally, we remark that we will postpone any discussion on the optimality of the hypotheses described by Theorem 2.1.1 until Chapter 6.

2.2 Definitions & Notation

In this section we fix some definitions and notation that will be used within the rest of the chapter. Throughout, Ω will denote a domain in $\mathbb{R}^m$ with non-empty boundary. Furthermore, the notation $H = -\Delta + V$ will be reserved to describe a Schrödinger

[1]For instance, consider the domain $\Omega = \mathbb{R}^m \setminus \{0\}$ so that $\mu_2(\Omega) = (\frac{m-2}{2})^2$. If $m \leq 3$, then $\mu_2(\Omega) < 1$ whereas if $m \geq 5$, then $\mu_2(\Omega) > 1$.

operator, defined on $C_0^\infty(\Omega)$, where the potential V satisfies the assumptions of Theorem 2.1.1. We begin by stating that we will establish the validity of Theorem 2.1.1 with recourse to Theorem 1.1.9. In other words, we are required to prove two things. First, we must show that there exists some $C > 0$ and some $\zeta \in \mathbb{R}$ so that the estimate $\big\langle (H - \zeta)u, u \big\rangle \;\geq\; C\,||\,u\,||^2$ holds for all $u \in D(H) = C_0^\infty(\Omega)$. Secondly, we must also show that $\ker(H^* - \zeta) = \{\,0\,\}$, i.e. that the kernel of the operator $H^* - \zeta$ contains only the function which is zero almost everywhere in Ω. The latter requirement forces us to deal with functions in the domain of the adjoint. Consequently, we will need to apply the formal operator $-\Delta + V$ to a broader class of functions than those in $C_0^\infty(\Omega)$. In preparation for this we make the following definition to ease the notation.

Definition 2.2.1 *For all functions $\gamma(x), \omega(x) \in W^1_{2,0}(\Omega)$ and for all $\zeta \in \mathbb{R}$, define the quadratic form*

$$\big\langle (-\Delta + V - \zeta)\,\omega\,,\,\gamma \big\rangle \;\equiv\; \int_\Omega \nabla\,\omega(x)\cdot\nabla\,\bar{\gamma}(x)\;dx \;+\; \int_\Omega [\,V(x) - \zeta\,]\;\omega(x)\,\bar{\gamma}(x)\;dx.$$

Note that if $\gamma, \omega \in D(H) = C_0^\infty(\Omega)$, then $\big\langle (-\Delta + V - \zeta)\,\omega\,,\,\gamma \big\rangle = \big\langle (H - \zeta)\,\omega\,,\,\gamma \big\rangle$.

Recall that the sequence of real numbers $\{P_n\}_{n=0}^\infty$ is defined by the rule $P_n = \frac{R}{\lambda^n}$ where $R > 0$ and $\lambda > 1$, so that $P_n \to 0$ as $n \to \infty$. Let $T > 0$ and define

$$\mathcal{A}_n \;=\; \{\; x \in \Omega \;\mid\; d(x) \leq P_n\,\}, \qquad \text{and} \qquad \mathcal{B}_n(T) \;=\; B(\,0, T + P_n^{-2}\,)$$

so that the region

$$\Omega_n \;=\; (\Omega \setminus \mathcal{A}_n) \cap \mathcal{B}_n$$

is an open, bounded region in Ω. In particular, since the only requirement on R and T is that they are positive, $\Omega_0 = (\Omega \setminus \mathcal{A}_0) \cap \mathcal{B}_0$ is an approximation to Ω. Evidently, for all $n \in \mathbb{N}$, we have the inclusions

$$\mathcal{A}_n \subseteq \mathcal{A}_{n-1}, \qquad B_{n-1} \subseteq B_n, \qquad \Omega_{n-1} \subseteq \Omega_n.$$

Now, for each $n \in \mathbb{N}$, let us proceed by defining the real valued, smooth functions $\phi_n(t)$ and $\theta_n(t)$ as follows

$$\phi_n(t) \;=\; \begin{cases} 0 & \text{if} \quad t \leq P_n, \\ \beta_n(t) & \text{if} \quad P_n < t \leq P_{n-1}, \\ 1 & \text{if} \quad t > P_{n-1}, \end{cases} \tag{2.5}$$

$$\theta_n(t) \;=\; \begin{cases} 1 & \text{if} \quad t \;<\; T + P_{n-1}^{-2}, \\ \sigma_n(t) & \text{if} \quad T + P_{n-1}^{-2} \;\leq\; t \;<\; T + P_n^{-2}, \\ 0 & \text{if} \quad t \;\geq\; T + P_n^{-2}, \end{cases} \tag{2.6}$$

where $\beta_n(t)$ and $\sigma_n(t)$ are monotone functions so that $|\beta_n'(t)| \le \frac{\kappa_1}{P_n}$ and $|\sigma_n'(t)| \le \kappa_2 P_n^2$. Here, κ_1 and κ_2 are absolute constants independent of n. It follows immediately from the definitions that

$$\phi_n(d(x)) = \begin{cases} 0 & \text{if } x \in \mathcal{A}_n, \\ \beta_n(d(x)) & \text{if } x \in \mathcal{A}_{n-1} \setminus \mathcal{A}_n, \\ 1 & \text{if } x \in \Omega \setminus \mathcal{A}_{n-1}, \end{cases}$$

and

$$\theta_n(|x|) = \begin{cases} 1 & \text{if } x \in \mathcal{B}_{n-1}, \\ \sigma_n(|x|) & \text{if } x \in \mathcal{B}_n \setminus \mathcal{B}_{n-1}, \\ 0 & \text{if } x \notin \mathcal{B}_n. \end{cases}$$

Definition 2.2.2 *For all $n \in \mathbb{N}$ use the functions $\phi_n(t)$ and $\theta_n(t)$ to define the cut-off functions*

$$f_n(x) = \theta_n(|x|)\, e^{G(d(x))}\, \Phi_n(d(x)). \tag{2.7}$$

Lemma 2.2.1 *If $f_n(x)$ is the function described by Definition 2.2.2, then we have that $f_n(x) \in W^1_\infty(\Omega) \cap C(\Omega)$ and $\operatorname{supp} f_n \subseteq \overline{\Omega_n}$.*

Proof. The boundedness, continuity and differentiability almost everywhere of $f_n(x)$ is inherited from the fact that each of it's component functions has these properties. As such $f_n(x) \in L_\infty(\Omega)$. If $x \notin \mathcal{B}_n$ or $x \in \mathcal{A}_n$, then $f_n(x) = 0$. Consequently it must be the case that if $f_n(x) \neq 0$, then $x \in (\Omega \setminus \mathcal{A}_n) \cap \mathcal{B}_n$, so that $\operatorname{supp} f_n \subseteq \overline{\Omega_n}$. Finally, a simple calculation yields

$$\begin{aligned} \left| \frac{\partial}{\partial x_i} f_n(x) \right|^2 &\le \sum_{j=1}^m \left| \frac{\partial}{\partial x_j} f_n(x) \right|^2 = |\nabla f_n(x)|^2 \\ &\le \left(|\theta_n'(|x|)|\, e^{G(d(x))} + G'(d(x))\, e^{G(d(x))} + \phi_n'(d(x))\, e^{G(d(x))} \right)^2 \Big|_{\overline{\Omega_n}} \\ &= \left(\kappa_2 P_n^2 M_1 + \frac{\kappa_0}{P_n} M_1 + \frac{\kappa_1}{P_n} M_1 \right)^2 < \infty, \end{aligned}$$

where in the last step we have used equation (2.1) and the fact that on $\overline{\Omega_n}$ we have $d(x) \ge P_n$ so that $0 \le G'(d(x)) \le \frac{\kappa_0}{d(x)} \le \frac{\kappa_0}{P_n}$. ∎

2.3 Technical Lemmas

In this section we give various technical lemmas that are required for the proof of Theorem 2.1.1. The first three of these lemmas are either taken directly from [1] or are minor modifications of the results therein. We begin by investigating the nature of functions in the kernel of the adjoint operator $H^* - \zeta$, where $\zeta \in \mathbb{R}$. Indeed, by a well known interior regularity result for second order elliptic differential operators (see [30, Lemma 3], [31, Page 109] or [32, Page 94]), we have the inclusion $D(H^*) \subseteq L_2(\Omega) \cap W^2_{2,loc}(\Omega)$. However for completeness we give the following lemma.

Lemma 2.3.1 *[1, Lemma 4.1]*

If H is a Schrödinger operator with potential structure defined by Theorem 2.1.1, then $\ker(H^* - \zeta) \subseteq L_2(\Omega) \cap W^2_{2,loc}(\Omega)$.

Proof. From equation (1.3) it suffices to establish that $\ker(H^* - \zeta) \subseteq W^2_{2,loc}(\Omega)$. In other words we must show that for all $\Psi \in \ker(H^* - \zeta)$ and for all compact sets $K \subseteq \Omega$ the following inequality holds

$$\sum_{|\alpha| \leq 2} \int_K |D^\alpha \Psi|^2 \, dx \quad < \quad \infty. \tag{2.8}$$

For this, let us choose an arbitrary $\Psi \in \ker(H^* - \zeta)$ and an arbitrary compact set $K \subseteq \Omega$ which, without loss of generality, we assume to be connected. Since K is compact and since $\partial\Omega$ is closed there exists some $\delta > 0$ so that $d(K, \partial\Omega) > \delta$. Therefore, we may define the sub-domain $\tilde{\Omega} = \{ x \in \mathbb{R}^m \mid d(x, K) < \delta/2 \} \subseteq \Omega$. Our intention is to show that $\Psi \in W^2_{2,loc}(\tilde{\Omega})$ so that for all compact $\tilde{K} \subseteq \tilde{\Omega}$ we have the estimate $\sum_{|\alpha|\leq 2} \int_{\tilde{K}} |D^\alpha \Psi|^2 \, dx < \infty$. As K is a compact subset of $\tilde{\Omega}$ this will establish the validity of equation (2.8) and so complete the proof.

We begin by defining the map $L : C_0^\infty(\tilde{\Omega}) \to \mathbb{C}$ by

$$L(u) \quad = \quad \langle -\Delta u \,,\, \Psi \rangle \quad = \quad -\langle (V - \zeta)\, u \,, \Psi \rangle,$$

the latter equality holding due to the fact that $\Psi \in \ker(H^* - \zeta)$ so that for all $u \in C_0^\infty(\tilde{\Omega}) \subseteq C_0^\infty(\Omega)$ we have $\langle -\Delta u + Vu - \zeta u, \Psi \rangle = 0$. We aim to show that L is a distribution (linear, bounded functional) on $C_0^\infty(\tilde{\Omega})$ - the space of test functions on $\tilde{\Omega}$. Since L is evidently linear, by Theorem 9.10 of [8], it suffices to show that for all compact $\tilde{K} \subseteq \tilde{\Omega}$ there exists a constant $C(\tilde{K}) > 0$ so that for all $u \in C_0^\infty(\tilde{K})$

$$|L(u)| \quad \leq \quad C(\tilde{K}) \sup_{\tilde{K}} |u(x)|,$$

as this implies that the restriction of L to each compact subset $\tilde{K}$ is bounded. By choosing arbitrary compact $\tilde{K} \subseteq \tilde{\Omega}$ and noting that $V \in L_\infty(\tilde{K})$ we easily obtain

$$\begin{aligned} |\, L(u)\,| &= \big|\big\langle -\Delta u\,,\, \Psi \big\rangle\big| = \big|\big\langle (\,V-\zeta\,)\,u\,,\,\Psi\big\rangle\big| \\ &\leq \left(\int_{\tilde{K}} |\,V(x) - \zeta\,|^2\, |\,u(x)\,|^2 \; dx \right)^{\frac{1}{2}} \left(\int_{\tilde{K}} |\,\Psi(x)\,|^2 \; dx \right)^{\frac{1}{2}} \\ &\leq ||\,V-\zeta\,||_{L_\infty(\tilde{K})} \;\; ||\,\Psi\,||_{L_2(\tilde{K})} \;\; |\,\tilde{K}\,|^{\frac{1}{2}} \;\; \sup_{\tilde{K}} |\,u(x)\,| \\ &= C(\tilde{K}) \; \sup_{\tilde{K}} |\,u(x)\,| \end{aligned}$$

so that L is indeed a (zero order) distribution on $C_0^\infty(\tilde{\Omega})$. Therefore, by the Riesz Representation Theorem, there exists unique $\omega \in L_2(\tilde{\Omega})$ so that for all $u \in C_0^\infty(\tilde{\Omega})$

$$L(u) \;\; = \;\; \big\langle -\Delta u\,,\,\Psi\big\rangle \;\; = \;\; \big\langle\, u\,,\,\omega\,\big\rangle. \tag{2.9}$$

Next define the regular distribution $T_{\bar{\omega}} : C_0^\infty(\tilde{\Omega}) \to \mathbb{C}$ by $T_{\bar{\omega}}(u) = \int_{\tilde{\Omega}} u(x)\,\bar{\omega}(x)\; dx$ and consider the distributional equation $-\Delta T = T_{\bar{\omega}}$. The solution to this equation is given by the regular distribution $T_{\bar{\Psi}} : C_0^\infty(\tilde{\Omega}) \to \mathbb{C}$ defined by $T_{\bar{\Psi}}(u) = \int_{\tilde{\Omega}} u(x)\,\bar{\Psi}(x)\; dx$ since from equation (2.9) we have

$$-\Delta\, T_{\bar{\Psi}}(u) \;\; = \;\; T_{\bar{\Psi}}(-\Delta u) \;\; = \;\; -\int_{\tilde{\Omega}} \Delta u(x)\,\bar{\Psi}(x)\; dx \;\; = \;\; \int_{\tilde{\Omega}} u(x)\,\bar{\omega}(x)\; dx \;\; = \;\; T_{\bar{\omega}}(u).$$

By Theorem S.2.1 of [9] this implies that $\Psi \in W^2_{2,loc}(\tilde{\Omega})$. ∎

The following lemma not only demonstrates the semi-boundedness of our Schrödinger operator but provides an inequality that will be crucial to the proof of Theorem 2.1.1.

Lemma 2.3.2 *Let H be a Schrödinger operator with potential structure described by Theorem 2.1.1. Choose any $C > 0$ and set $\zeta = -||\,V_2\,||_{L_\infty(\Omega)} - C$. Then the estimate*

$$\big\langle (-\Delta + V - \zeta)\,\omega\,,\,\omega\,\big\rangle \;-\; \int_\Omega G'\big(d(x)\big)^2\, |\,\omega(x)\,|^2 \; dx \quad \geq \quad C\; ||\,\omega\,||^2_{L_2(\Omega)}$$

holds for all $\omega(x) \in W^1_{2,0}(\Omega)$. In particular, if $\Psi \in \ker(H^* - \zeta)$ and $f_n(x)$ is the function described by Definition 2.2.2, then

$$\big\langle (-\Delta + V - \zeta)\,(\,f_n\Psi\,)\,,\,(\,f_n\Psi\,)\big\rangle - \int_\Omega G'\big(d(x)\big)^2\, |\,f_n\Psi\,|^2 \; dx \;\geq\; C\; ||\,f_n\Psi\,||^2_{L_2(\Omega)} \tag{2.10}$$

Proof. From Definition 2.2.1 and the assumptions on the structure of the potential V, the desired inequality can be obtained as follows:

$$\begin{aligned}
& \big\langle\, (-\Delta + V - \zeta)\,\omega\,,\,\omega\,\big\rangle \;-\; \int_\Omega G'(\,d(x)\,)^2\;|\,\omega(x)\,|^2\;dx \\
=\;& \int_\Omega |\,\nabla\,\omega(x)\,|^2\;dx \;+\; \int_\Omega V_1(x)\,|\,\omega(x)\,|^2\;dx \;+\; \int_\Omega V_2(x)\,|\,\omega(x)\,|^2\;dx \;-\; \zeta\;||\,\omega\,||_{L_2}^2 \\
& \qquad\qquad -\; \int_\Omega G'(\,d(x)\,)^2\,|\,\omega(x)\,|^2\;dx \\
\geq\;& \int_\Omega |\,\nabla\,\omega(x)\,|^2\;dx \;-\; \big(\zeta + ||\,V_2\,||_{L_\infty}\big)\;||\,\omega\,||_{L_2}^2 \;+\; \int_\Omega \big[\,V_1(x) \;-\; G'(\,d(x)\,)^2\,\big]\,|\,\omega(x)\,|^2\;dx \\
\geq\;& \int_\Omega |\,\nabla\,\omega(x)\,|^2\;dx \;+\; C\,||\,\omega\,||_{L_2}^2 \;-\; \mu_2(\Omega)\int_\Omega \frac{|\,\omega(x)\,|^2}{d(x)^2}\;dx \\
\geq\;& \int_\Omega |\,\nabla\,\omega(x)\,|^2\;dx \;+\; C\,||\,\omega\,||_{L_2}^2 \;-\; \int_\Omega |\,\nabla\,\omega(x)\,|^2\;dx \\
=\;& C\,||\,\omega\,||_{L_2}^2 .
\end{aligned}$$

Concerning the final assertion of the lemma, by Lemma 2.2.1 and Lemma 2.3.1 it must be that $f_n\Psi \in W^1_{2,0}(\Omega)$. Therefore, the validity of equation (2.10) is immediate. ∎

Corollary 2.3.1 *Let H be a Schrödinger operator with potential structure described by Theorem 2.1.1. Choose any $C > 0$ and set $\zeta = -||\,V_2\,||_{L_\infty(\Omega)} - C$. Then the estimate*

$$\big\langle\,(\,H - \zeta\,)\,u\,,\,u\,\big\rangle \quad\geq\quad C\;||\,u\,||_{L_2(\Omega)}^2$$

holds for all $u \in D(H) = C_0^\infty(\Omega)$.

The next lemma, which is taken directly from [1], will also lie at the heart of the proof of Theorem 2.1.1.

Lemma 2.3.3 *[1, Lemma 3.2]*

Let H be a Schrödinger operator with potential structure as defined in Theorem 2.1.1. Let $\Psi \in ker\,(\,H^* - \zeta\,)$ where $\zeta \in \mathbb{R}$ and let $f_n(x)$ be the function described in Definition 2.2.2. Then the following equality holds

$$\big\langle\,(\,-\Delta + V - \zeta\,)\,(\,f_n\,\Psi\,)\,,\,f_n\,\Psi\,\big\rangle \quad=\quad \int_\Omega |\,\nabla\,f_n\,|^2\;|\,\Psi\,|^2\;dx.$$

Proof. Definition 2.2.1 and some elementary differentiation leads us to the equations

$$\Big\langle\, (\, -\Delta \, + \, V \, - \, \zeta \,)\, (\, f_n \, \Psi \,)\, , \, f_n \Psi \,\Big\rangle$$

$$= \int_\Omega |\, \nabla \,(\, f_n \Psi \,)\,|^2 \;\; dx \quad + \quad \int_\Omega \big(\, V \, - \, \zeta \,\big)\, f_n^2 \; |\, \Psi \,|^2 \;\; dx \tag{2.11}$$

$$= \int_\Omega |\, \nabla \, f_n \,|^2 \; |\, \Psi \,|^2 \;\; dx \quad + \quad 2\, Re \bigg(\int_\Omega f_n \, \Psi \, \nabla \, f_n \, \cdot \, \nabla \overline{\Psi} \;\; dx \bigg) \tag{2.12}$$

$$+ \quad \int_\Omega |\, \nabla \, \Psi \,|^2 \, f_n^2 \;\; dx \quad + \quad \int_\Omega \big(\, V \Psi \, - \, \zeta \Psi \,\big)\, f_n^2 \; \overline{\Psi} \;\; dx.$$

Using Definition 2.2.1 once more, we see that

$$\Big\langle\, (\, -\Delta + V - \zeta \,)\, (\, f_n^2 \, \Psi \,)\, , \, \Psi \,\Big\rangle \quad = \quad \int_\Omega \nabla \,(\, f_n^2 \Psi \,) \cdot \nabla \overline{\Psi} \;\; dx \quad + \quad \int_\Omega \big(\, V \Psi - \zeta \Psi \,\big)\, f_n^2 \; \overline{\Psi} \;\; dx$$

$$= \; 2 \int_\Omega f_n \, \Psi \, \nabla \, f_n \, \cdot \, \nabla \overline{\Psi} \;\; dx \quad + \quad \int_\Omega |\, \nabla \Psi \,|^2 \, f_n^2 \;\; dx \quad + \quad \int_\Omega \big(\, V \Psi \, - \, \zeta \Psi \,\big)\, f_n^2 \; \overline{\Psi} \;\; dx$$

and so we obtain the equality

$$\int_\Omega |\, \nabla \Psi \,|^2 \, f_n^2 \;\; dx \quad + \quad \int_\Omega \big(\, V \Psi \, - \, \zeta \Psi \,\big)\, f_n^2 \; \overline{\Psi} \;\; dx$$

$$= \; \Big\langle\, (\, -\Delta \, + \, V \, - \, \zeta \,)\, (\, f_n^2 \, \Psi \,)\, , \, \Psi \,\Big\rangle \quad - \quad 2 \int_\Omega f_n \, \Psi \, \nabla \, f_n \, \cdot \, \nabla \overline{\Psi} \;\; dx.$$

Substituting this last expression into equation (2.12) yields

$$\Big\langle\, (\, -\Delta \, + \, V \, - \, \zeta \,)\, (\, f_n \, \Psi \,)\, , \, f_n \Psi \,\Big\rangle$$

$$= \int_\Omega |\, \nabla \, f_n \,|^2 \; |\, \Psi \,|^2 \;\; dx \quad + \quad 2\, Re \bigg(\int_\Omega f_n \, \Psi \, \nabla \, f_n \, \cdot \, \nabla \overline{\Psi} \;\; dx \bigg)$$

$$- \;\; 2 \int_\Omega f_n \, \Psi \, \nabla \, f_n \, \cdot \, \nabla \overline{\Psi} \;\; dx \quad + \quad \Big\langle\, (\, -\Delta \, + \, V \, - \, \zeta \,)\, (\, f_n^2 \, \Psi \,)\, , \, \Psi \,\Big\rangle$$

Now, since $\Psi \in \ker(H^* - \zeta)$ and since $f_n^2 \, \Psi \in W^1_{2,0}(\Omega)$, the density of $C_0^\infty(\Omega)$ in $W^1_{2,0}(\Omega)$ implies that

$$\Big\langle\, (\, -\Delta \, + \, V \, - \, \zeta \,)\, (\, f_n \, \Psi \,)\, , \, f_n \Psi \,\Big\rangle$$

$$= \int_\Omega |\, \nabla \, f_n \,|^2 \; |\, \Psi \,|^2 \;\; dx \quad - \quad 2\, i\, Im \bigg(\int_\Omega f_n \, \Psi \, \nabla \, f_n \, \cdot \, \nabla \overline{\Psi} \;\; dx \bigg) \tag{2.13}$$

However, by (2.11) the left hand side of equation (2.13) is evidently real. The lemma is proven. ∎

We require one further result for the proof of Theorem 2.1.1. Although this result is not hard to obtain, it does require some tedious algebra and relies heavily on the fact that $\big|\nabla |x|\big| = 1$ and $\big|\nabla d(x)\big| \leq 1$ almost everywhere.

Lemma 2.3.4 *Let $f_n(x)$ be the function described in Definition 2.2.2. Then the following estimate holds*

$$\begin{aligned} \big|\nabla f_n\big|^2 \;\leq\; & f_n^2(x)\, G'(d(x))^2 \\ & + \; e^{2G(d(x))} \left[\begin{aligned} & |\nabla\phi_n(d(x))|^2 \;+\; 2\,|\nabla\phi_n(d(x))|\; G'(d(x)) \\ +\;& |\nabla\theta_n(|x|)|^2\; \phi_n(d(x)) \\ +\;& 2\,|\nabla\theta_n(|x|)|\;\theta_n(|x|)\;\phi_n(d(x))\, G'(d(x)) \\ +\;& 2\,|\nabla\theta_n(|x|)|\cdot|\nabla\phi_n(d(x))|\;\theta_n(|x|)\,\phi_n(d(x)) \end{aligned} \right]. \end{aligned}$$

Proof.

$$\big|\nabla f_n\big|^2 \;=\; \sum_{j=1}^{m} \left(\frac{\partial}{\partial x_j} \Big[\, \theta_n(|x|)\; e^{G(d(x))}\; \phi_n(d(x)) \,\Big] \right)^2$$

and we calculate

$$\begin{aligned} \sum_{j=1}^{m} \Bigg(\left[\frac{\partial}{\partial x_j}\theta_n(|x|)\right] e^{G(d(x))}\;\phi_n(d(x)) \;+\; \theta_n(|x|)\left[\frac{\partial}{\partial x_j}\, e^{G(d(x))}\right]\phi_n(d(x)) \\ +\; \theta_n(|x|)\; e^{G(d(x))} \left[\frac{\partial}{\partial x_j}\,\phi_n(d(x))\right] \Bigg)^2 \end{aligned}$$

so that

$$\begin{aligned} \big|\nabla f_n\big|^2 \;\leq\; & \big|\nabla\theta_n(|x|)\big|^2\; e^{2G(d(x))}\;\phi_n^2(d(x)) \\ & +\; 2\,\big|\nabla\theta_n(|x|)\big|\cdot\big|\nabla e^{G(d(x))}\big|\;\theta_n(|x|)\; e^{G(d(x))}\;\phi_n^2(d(x)) \\ & +\; 2\,\big|\nabla\theta_n(|x|)\big|\cdot\big|\nabla\phi_n(d(x))\big|\;\theta_n(|x|)\; e^{2G(d(x))}\;\phi_n(d(x)) \\ & +\; \theta_n^2(|x|)\;\big|\nabla e^{G(d(x))}\big|^2\;\phi_n^2(d(x)) \\ & +\; \theta_n^2(|x|)\; e^{2G(d(x))}\;\big|\nabla\phi_n(d(x))\big|^2 \\ & +\; 2\,\theta_n^2(|x|)\; e^{G(d(x))}\;\phi_n(d(x))\,\big|\nabla e^{G(d(x))}\big|\cdot\big|\nabla\phi_n(d(x))\big|. \end{aligned}$$

Simply using the fact that ϕ_n and θ_n are both less than or equal to 1, yields

$$
\begin{aligned}
\big|\nabla f_n\big|^2 \;\le\; & \big|\nabla\theta_n(|x|)\big|^2\; e^{2G(d(x))}\;\phi_n(d(x)) \\
& + 2\,\big|\nabla\theta_n(|x|)\big|\cdot\big|\nabla e^{G(d(x))}\big|\;\theta_n(|x|)\;e^{G(d(x))}\;\phi_n(d(x)) \\
& + 2\,\big|\nabla\theta_n(|x|)\big|\cdot\big|\nabla\phi_n(d(x))\big|\;\theta_n(|x|)\;e^{2G(d(x))}\;\phi_n(d(x)) \\
& + \theta_n^2(|x|)\;\big|\nabla e^{G(d(x))}\big|^2\;\phi_n^2(d(x)) \\
& + e^{2G(d(x))}\;\big|\nabla\phi_n(d(x))\big|^2 \\
& + 2\,e^{G(d(x))}\big|\nabla e^{G(d(x))}\big|\cdot\big|\nabla\phi_n(d(x))\big|.
\end{aligned}
$$

Routine differentiation, and the fact that $|\nabla d(x)| \le 1$, allows us to calculate that $\big|\nabla e^{G(d(x))}\big| \le e^{G(d(x))}\,G'(d(x))$. Therefore, the previous estimate becomes

$$
\begin{aligned}
\big|\nabla f_n\big|^2 \;\le\; & \big|\nabla\theta_n(|x|)\big|^2\; e^{2G(d(x))}\;\phi_n(d(x)) \\
& + 2\,\big|\nabla\theta_n(|x|)\big|\;\theta_n(|x|)\;e^{2G(d(x))}\;\phi_n(d(x))\;G'(d(x)) \\
& + 2\,\big|\nabla\theta_n(|x|)\big|\cdot\big|\nabla\phi_n(d(x))\big|\;\theta_n(|x|)\;e^{2G(d(x))}\;\phi_n(d(x)) \\
& + \theta_n^2(|x|)\;e^{2G(d(x))}\;\phi_n^2(d(x))\;G'(d(x))^2 \\
& + e^{2G(d(x))}\;\big|\nabla\phi_n(d(x))\big|^2 \\
& + 2\,e^{2G(d(x))}\;\big|\nabla\phi_n(d(x))\big|\;G'(d(x))
\end{aligned}
$$

and ultimately we obtain

$$
\begin{aligned}
\big|\nabla f_n\big|^2 \;\le\; & f_n^2(x)\;G'(d(x))^2 \\
& + e^{2G(d(x))}\left[\begin{array}{l}
\quad |\nabla\phi_n(d(x))|^2 + 2\,|\nabla\phi_n(d(x))|\;G'(d(x)) \\
+ \quad |\nabla\theta_n(|x|)|^2\;\phi_n(d(x)) \\
+ 2\,|\nabla\theta_n(|x|)|\;\theta_n(|x|)\;\phi_n(d(x))\;G'(d(x)) \\
+ 2\,|\nabla\theta_n(|x|)|\cdot|\nabla\phi_n(d(x))|\;\theta_n(|x|)\;\phi_n(d(x))
\end{array}\right]
\end{aligned}
$$

as desired. ∎

2.4 Proof of Main Theorem

We are now in a position to give the proof of Theorem 2.1.1.

Proof. Choose some $C > 0$ and set $\zeta = -||V_2||_{L_\infty} - C$. We have already shown that the operator $H - \zeta$ is semi-bounded in Corollary 2.3.1. Therefore, by Theorem 1.1.9, it suffices to show that $\ker(H^* - \zeta) = \{0\}$. Noting that $P_0 < d(x)$ on Ω_0, $\Omega_0 \subseteq \Omega_{n-1}$ for all $n \geq 1$ and that $\phi_n(d(x)) = \theta_n(|x|) = 1$ on Ω_{n-1}, for all $\Psi \in \ker(H^* - \zeta)$ we have the inequalities

$$
\begin{aligned}
& e^{2G(P_0)}\, e^{-2G(P_n)} \int_{\Omega_0} |\Psi(x)|^2 \; dx \quad \leq \quad e^{-2G(P_n)} \int_{\Omega_{n-1}} \left| e^{G(d(x))}\, \Psi(x) \right|^2 \; dx \\
= \; & e^{-2G(P_n)} \int_{\Omega_{n-1}} \left| \theta_n(|x|)\; e^{G(d(x))}\; \phi_n(d(x))\; \Psi(x) \right|^2 \; dx \\
\leq \; & e^{-2G(P_n)} \int_{\Omega} |f_n(x)\, \Psi(x)|^2 \; dx.
\end{aligned}
$$

Successive applications of Lemma 2.3.2, Lemma 2.3.3 and Lemma 2.3.4 then yield

$$
\begin{aligned}
& e^{2G(P_0)}\, e^{-2G(P_n)} \int_{\Omega_0} |\Psi(x)|^2 \; dx \\
\leq \; & \frac{e^{-2G(P_n)}}{C} \left[\big\langle (-\Delta + V - \zeta)(f_n\Psi),\, f_n\Psi \big\rangle \; - \; \int_{\Omega} G'(d(x))^2 \; |f_n\Psi|^2 \; dx \right] \\
= \; & \frac{e^{-2G(P_n)}}{C} \left[\int_{\Omega} |\nabla f_n|^2\, |\Psi|^2 \; dx \; - \; \int_{\Omega} G'(d(x))^2\, |f_n\Psi|^2 \; dx \right]
\end{aligned}
$$

$$
\leq \frac{1}{C}\, e^{-2G(P_n)} \left[
\begin{array}{l}
\quad \int_\Omega G'(d(x))^2 \left| f_n\Psi \right|^2 dx \; - \; \int_\Omega G'(d(x))^2 \left| f_n\Psi \right|^2 dx \\
+ \; \int_\Omega \left|\Psi\right|^2 e^{2G(d(x))} \left[\left| \nabla \phi_n(d(x)) \right|^2 + 2 \left| \nabla \phi_n(d(x)) \right| G'(d(x)) \right] dx \\
+ \; \int_\Omega \left|\Psi\right|^2 e^{2G(d(x))} \left| \nabla \theta_n(|x|) \right|^2 \phi_n(d(x)) \; dx \\
+ \, 2 \int_\Omega \left|\Psi\right|^2 e^{2G(d(x))} \left| \nabla \theta_n(|x|) \right| \theta_n(|x|)\; \phi_n(d(x))\; G'(d(x)) \; dx \\
+ \, 2 \int_\Omega \left|\Psi\right|^2 e^{2G(d(x))} \left| \nabla\theta_n(|x|) \right| \cdot \left| \nabla \phi_n(d(x)) \right| \theta_n(|x|)\; \phi_n(d(x)) \; dx
\end{array}
\right]
$$

$$
\leq \frac{1}{C}\, e^{-2G(P_n)} \left[
\begin{array}{l}
\int_{\mathcal{A}_{n-1}\setminus\mathcal{A}_n} |\Psi|^2 e^{2G(d(x))} \left[\left| \nabla \beta_n(d(x)) \right|^2 + 2 \left| \nabla \beta_n(d(x)) \right| G'(d(x)) \right] dx \\
+ \; \int_{(\Omega\setminus\mathcal{A}_n)\cap(\mathcal{B}_n\setminus\mathcal{B}_{n-1})} |\Psi|^2 \; e^{2G(d(x))} \left| \nabla \sigma_n(|x|) \right|^2 \; dx \\
+ \, 2 \int_{(\Omega\setminus\mathcal{A}_n)\cap(\mathcal{B}_n\setminus\mathcal{B}_{n-1})} |\Psi|^2 \; e^{2G(d(x))} \left| \nabla \sigma_n(|x|) \right| G'(d(x)) \; dx \\
+ \, 2 \int_{(\mathcal{A}_{n-1}\setminus\mathcal{A}_n)\cap(\mathcal{B}_n\setminus\mathcal{B}_{n-1})} |\Psi|^2 \; e^{2G(d(x))} \left| \nabla \sigma_n(|x|) \right| \cdot \left| \nabla \beta_n(d(x)) \right| dx
\end{array}
\right]
$$

where in the last step we have used the fact that

$$
\operatorname{supp} |\nabla \phi_n(d(x))| \subseteq \mathcal{A}_{n-1}\setminus\mathcal{A}_n, \qquad \operatorname{supp} |\nabla \theta_n(|x|)| \subseteq \mathcal{B}_n\setminus\mathcal{B}_{n-1}, \qquad \operatorname{supp} \phi_n(d(x)) \subseteq \Omega\setminus\mathcal{A}_n.
$$

By the construction of the functions $\beta_n(t)$ and $\sigma_n(t)$, we see that $|\nabla \beta_n(d(x))| \leq \frac{\kappa_1}{P_n}$ and $|\nabla \sigma_n(|x|)| \leq \kappa_2 P_n^2$. Furthermore, on the regions $\mathcal{A}_{n-1} \setminus \mathcal{A}_n$ and $\Omega \setminus \mathcal{A}_n$ we have $d(x) > P_n$, so that the simple estimate $G'(d(x)) \leq \frac{\kappa_0}{d(x)} < \frac{\kappa_0}{P_n}$ must hold there. Putting all this information together, gives us the inequality

$$e^{2G(P_0)} \; e^{-2G(P_n)} \; \int_{\Omega_0} |\Psi(x)|^2 \; dx$$

$$\leq \frac{1}{C} \, e^{-2G(P_n)} \left[\begin{array}{l} + \; \int_{\mathcal{A}_{n-1}\setminus\mathcal{A}_n} |\Psi|^2 \; e^{2G(d(x))} \left[\frac{\kappa_1^2}{P_n^2} + \frac{2\kappa_0\,\kappa_1}{P_n^2} \right] dx \\ + \; \int_{(\Omega\setminus\mathcal{A}_n)\cap(\mathcal{B}_n\setminus\mathcal{B}_{n-1})} |\Psi|^2 \; e^{2G(d(x))} \; \kappa_2^2 \; P_n^4 \; dx \\ + \, 2 \int_{(\Omega\setminus\mathcal{A}_n)\cap(\mathcal{B}_n\setminus\mathcal{B}_{n-1})} |\Psi|^2 \; e^{2G(d(x))} \; \kappa_0 \; \kappa_2 \; P_n \; dx \\ + \, 2 \int_{(\mathcal{A}_{n-1}\setminus\mathcal{A}_n)\cap(\mathcal{B}_n\setminus\mathcal{B}_{n-1})} |\Psi|^2 \; e^{2G(d(x))} \; \kappa_1 \; \kappa_2 \; P_n \; dx \end{array} \right].$$

Setting $M_0 = \max\{ \kappa_1^2 \, , \, 2\kappa_0 \, \kappa_1 \}$, using the fact that $e^{2G(d(x))} \leq M_1^2$ and noting that $P_n < d(x) \leq P_{n-1}$ on the region $\mathcal{A}_{n-1} \setminus \mathcal{A}_n$, we have the estimate

$$\begin{aligned} & e^{2G(P_0)} \; e^{-2G(P_n)} \int_{\Omega_0} |\Psi|^2 \; dx \\ \leq \; & \frac{M_0}{C} \, \frac{1}{P_n^2} \int_{\mathcal{A}_{n-1}\setminus\mathcal{A}_n} |\Psi|^2 \; e^{2[\,G(P_{n-1}) - G(P_n)\,]} \; dx \\ + \; & \frac{\kappa_2^2 \, M_1^2}{C} \; P_n^4 \; e^{-2G(P_n)} \int_{(\Omega\setminus\mathcal{A}_n)\cap(\mathcal{B}_n\setminus\mathcal{B}_{n-1})} |\Psi|^2 \; dx \\ + \; & \frac{2\,\kappa_0\,\kappa_2\,M_1^2}{C} \; P_n \; e^{-2G(P_n)} \int_{(\Omega\setminus\mathcal{A}_n)\cap(\mathcal{B}_n\setminus\mathcal{B}_{n-1})} |\Psi|^2 \; dx \\ + \; & \frac{2\,\kappa_1\,\kappa_2}{C} \; P_n \int_{\mathcal{A}_{n-1}\setminus\mathcal{A}_n} |\Psi|^2 \; e^{2[\,G(P_{n-1}) - G(P_n)\,]} \; dx. \end{aligned} \tag{2.14}$$

Now, since $G(t) \in C^1(0,\infty)$ it must be the case that

$$\begin{aligned} 2\left[\, G(P_{n-1}) - G(P_n) \,\right] \; &= \; 2 \int_{P_n}^{P_{n-1}} G'(t) \; dt \; \leq \; 2\kappa_0 \int_{P_n}^{P_{n-1}} t^{-1} \; dt \\ &= \; 2\kappa_0 \; \ln\left(\frac{P_{n-1}}{P_n} \right) \; = \; \ln(\lambda^{2\kappa_0}). \end{aligned}$$

Therefore, from equation (2.14) we arrive at

$$
\begin{aligned}
& e^{2\,G(P_0)}\ e^{-2\,G(P_n)} \int_{\Omega_0} |\,\Psi\,|^2\ dx \\
\leq\ & \frac{M_0\ \lambda^{2\,\kappa_0}}{C}\ \frac{1}{P_n^2} \int_{\mathcal{A}_{n-1}\backslash\mathcal{A}_n} |\,\Psi\,|^2\ dx \\
+\ & \frac{\kappa_2^2\ M_1^2}{C}\ P_n^4\ e^{-2\,G(P_n)} \int_{(\Omega\backslash\mathcal{A}_n)\cap(\mathcal{B}_n\backslash\mathcal{B}_{n-1})} |\,\Psi\,|^2\ dx \\
+\ & \frac{2\ \kappa_0\ \kappa_2\ M_1^2}{C}\ P_n\ e^{-2\,G(P_n)} \int_{(\Omega\backslash\mathcal{A}_n)\cap(\mathcal{B}_n\backslash\mathcal{B}_{n-1})} |\,\Psi\,|^2\ dx \\
+\ & \frac{2\ \kappa_1\ \kappa_2\ \lambda^{2\,\kappa_0}}{C}\ P_n \int_{\mathcal{A}_{n-1}\backslash\mathcal{A}_n} |\,\Psi\,|^2\ dx.
\end{aligned}
\tag{2.15}
$$

Multiplying equation (2.15) through by $\frac{C\ P_n^2}{M_0\ \lambda^{2\,\kappa_0}}$ gives for all integer $n \geq 1$

$$
\begin{aligned}
& \frac{C\ e^{2\,G(P_0)}}{M_0\ \lambda^{2\,\kappa_0}}\ P_n^2\ e^{-2\,G(P_n)} \int_{\Omega_0} |\,\Psi\,|^2\ dx \quad \leq \quad \int_{\mathcal{A}_{n-1}\backslash\mathcal{A}_n} |\,\Psi\,|^2\ dx \\
+ & \left[\ \frac{\kappa_2^2\ M_1^2}{M_0\ \lambda^{2\,\kappa_0}}\ P_n^6\ e^{-2\,G(P_n)} + \frac{2\ \kappa_0\ \kappa_2\ M_1^2}{M_0\ \lambda^{2\,\kappa_0}}\ P_n^3\ e^{-2\,G(P_n)} + \frac{2\ \kappa_1\ \kappa_2}{M_0}\ P_n^3\ \right] \int_{\Omega} |\,\Psi\,|^2\ dx
\end{aligned}
$$

so that by forming the sum over all $n \geq 1$ we obtain

$$
\begin{aligned}
& \frac{C\ e^{2\,G(P_0)}}{M_0\ \lambda^{2\,\kappa_0}} \left(\sum_{n=1}^{\infty} P_n^2\ e^{-2\,G(P_n)} \right) \int_{\Omega_0} |\,\Psi\,|^2\ dx \\
\leq\ & \sum_{n=1}^{\infty} \int_{\mathcal{A}_{n-1}\backslash\mathcal{A}_n} |\,\Psi\,|^2\ dx \\
+\ & \frac{\kappa_2^2\ M_1^2}{M_0\ \lambda^{2\,\kappa_0}} \left(\sum_{n=1}^{\infty} P_n^6\ e^{-2\,G(P_n)} \right) \int_{\Omega} |\,\Psi\,|^2\ dx \\
+\ & \frac{2\ \kappa_0\ \kappa_2\ M_1^2}{M_0\ \lambda^{2\,\kappa_0}} \left(\sum_{n=1}^{\infty} P_n^3\ e^{-2G(P_n)} \right) \int_{\Omega} |\,\Psi\,|^2\ dx \\
+\ & \frac{2\ \kappa_1\ \kappa_2}{M_0} \left(\sum_{n=1}^{\infty} P_n^3 \right) \int_{\Omega} |\,\Psi\,|^2\ dx.
\end{aligned}
\tag{2.16}
$$

By hypothesis, the right hand side of equation (2.16) is finite, whilst the series $\sum_{n=1}^{\infty} P_n^2\, e^{-2G(P_n)}$ diverges. This can only be the case if $\int_{\Omega_0} |\,\Psi\,|^2\ dx = 0$. However, Ω_0 is an arbitrary approximation to Ω. By taking the limit as $R \to 0$ and $T \to \infty$ we obtain that $\int_{\Omega} |\Psi|^2\, dx = 0$. Consequently, if $\Psi \in \ker(H^* - \zeta)$ then Ψ must equal zero almost everywhere in Ω. This completes the proof. ∎

2.5 Explicit Criteria For Essential Self-adjointness

Theorem 2.1.1 is not an explicit theorem. That is to say that the conditions for essential self-adjointness expressed therein are phrased in terms of an auxiliary function $G(t)$. In this section, we construct two possible candidate functions for $G(t)$ which, in turn, allow us to obtain explicit criteria for the essential self-adjointness of our Schrödinger operator. These candidate functions are taken directly from [1].

2.5.1 The Candidate Function $G_0(t)$

We begin by showing that the candidate function described by equation (A.1) satisfies the requirements of Theorem 2.1.1.

Definition 2.5.1 *Let* $G_0(t) \in C^1(0,\infty)$ *be the real, monotone non-decreasing function defined by*

$$G_0(t) \quad = \quad \begin{cases} \ln t & \text{if} \quad 0 < t < R_1, \\ C_0 & \text{if} \quad t \geq R_2 \end{cases}$$

where $\ln R_1 \;<\; C_0 \;<\; \ln R_2$ *and where* $0 \leq G_0'(t) \leq \frac{1}{t}$.

By definition $0 \;\leq\; G_0'(t) \;\leq\; \frac{1}{t}$ for all $t > 0$ and $G_0'(t) = 0$ for all $t \geq R_2$. Furthermore, the series $\sum_{n=1}^{\infty} P_n^k \, e^{-2\,G_0(P_n)} \;=\; \sum_{n=1}^{\infty} P_n^{k-2}$ clearly diverges if $k = 2$ and converges for all $k > 2$. In a sense, $G_0(t)$ is quite a naive candidate function. However, it's virtue lies in the fact that applying Theorem 2.1.1 with $G(t)$ set equal to $G_0(t)$ yields the following simple, explicit result.

Theorem 2.5.1 *Let* Ω *be a domain in* R^m *and let* $H \;=\; -\Delta + V$ *be a Schrödinger operator defined on* $C_0^\infty(\Omega)$, *where* $V \;\in\; L_{\infty,loc}(\Omega)$ *is a real potential of the form* $V \;=\; V_1 + V_2$ *and* $V_2 \in L_\infty(\Omega)$. *If*

$$V_1(x) \quad \geq \quad \frac{1 \,-\, \mu_2(\Omega)}{d(x)^2},$$

then H *is essentially self-adjoint.*

Proof. In light of Theorem 2.1.1, all we must prove is that

$$V_1(x) \quad \geq \quad G_0'(d(x))^2 \,-\, \mu_2(\Omega)\, d(x)^{-2}.$$

However, this is immediate since the inequality

$$V_1(x) \quad \geq \quad d(x)^{-2} \,-\, \mu_2(\Omega)\, d(x)^{-2} \quad \geq \quad G_0'(d(x))^2 \,-\, \mu_2(\Omega)\, d(x)^{-2}$$

holds by definition. ∎

2.5.2 The Candidate Function $G_1(t)$

Our next candidate for the auxiliary function $G(t)$ is a lot harder to construct but allows us to obtain a stronger explicit result concerning the essential self-adjointness of Schrödinger operators. To construct this function we shall need the following barrage of definitions.

Definition 2.5.2 *Define the function* $L_1(t) = \ln(1/t)$ *for all* $t \in (\,0,\, \frac{1}{e_1}\,)$ *where* $e_1 = e$. *Then, for each* $k \in \mathbb{N}$, *recursively define the functions*

$$L_k(t) \quad = \quad \ln L_{k-1}(t)$$

for all $t \in (\,0,\, 1/e_k\,)$ *where* $e_k \;=\; e^{e_{k-1}}$.

Some remarks concerning the nature of the functions $L_k(t)$ are in order. First of all we note that $L_k(t) \geq 1$ such that $\frac{1}{L_k(t)}$ is well defined on the interval $(\,0,\, 1/e_k\,)$. Secondly, we note that the derivative of of $L_j(t)$ is given by

$$\frac{d}{dt}\, L_j(t) \qquad = \qquad -\,\frac{1}{t}\,\left(\,\prod_{k\,=\,1}^{j-1} L_k(t)\,\right)^{-1}. \tag{2.17}$$

This follows from repeated applications of the chain rule since

$$\frac{d}{dt}\, L_j(t) \;=\; \frac{d}{dt}\,\ln\big(\,L_{j-1}(t)\,\big) \;=\; \frac{1}{L_{j-1}}\,\frac{d}{dt}\,L_{j-1}(t) \;=\; \frac{1}{L_{j-1}}\,\frac{d}{dt}\,\ln\big(\,L_{j-2}(t)\,\big)$$

$$=\; \frac{1}{L_{j-1}\,L_{j-2}}\,\frac{d}{dt}\,L_{j-2}(t) \;=\cdots=\; \left(\,\prod_{k=1}^{j-1} L_k(t)\,\right)^{-1}\frac{d}{dt}\,\ln(1/t) \;=\; -\frac{1}{t}\,\left(\,\prod_{k=1}^{j-1} L_k(t)\,\right)^{-1}.$$

Definition 2.5.3 *Fix* $M \in \mathbb{N}$ *where* $2 \,\leq\, M \,<\, \infty$ *and define the function*

$$\mathcal{L}_M(t) \quad = \quad \sum_{j=2}^{M}\,\left(\,\prod_{k=1}^{j-1} L_k(t)\,\right)^{-1} \qquad \text{for all} \qquad t \in (\,0,\, 1/e_M\,).$$

We note that $\mathcal{L}_M(t) \to 0$ *as* $t \to 0^+$ *and that the interval on which* $\mathcal{L}_M(t)$ *is well defined shrinks as* $M \to \infty$.

Definition 2.5.4 *Let* $s > 0$ *and let* $f(t) \in C^1(0,s)$ *be a real valued function satisfying the conditions*

$$i)\;\; f(t) \,\geq\, 0 \qquad\qquad ii)\;\; \lim_{t\to 0^+}\, t f(t) \,=\, 0 \qquad\qquad iii)\;\; \lim_{t\to 0^+}\, \int_t^s f(u)\; du \;<\; \infty.$$

As an example, one may define $f(t) \,=\, t^{-1+\epsilon}$ *for* $t \in (0,s)$ *where* $0 < \epsilon < 1$.

Definition 2.5.5 *Let* $\mathcal{L}_M(t)$ *be the function described in Definition 2.5.3. Let* $0 < s < 1/e_M$ *and let* $f(t)$ *be the function described in Definition 2.5.4. Define the function*

$$\tilde{f}(t) \quad = \quad \begin{cases} f(t) \;+\; \frac{1}{4t}\,\mathcal{L}_M^2(t) & \text{if} \quad t < \frac{1}{e_M} \\ \quad 0 & \text{if} \quad t \geq \frac{1}{e_M} \end{cases}$$

Lemma 2.5.2 *The function* $\tilde{f}(t)$ *satisfies the conditions of Definition 2.5.4.*

Proof. That $\tilde{f}(t) \geq 0$ and $\lim_{t\to 0} t\tilde{f}(t) = 0$ are immediate from the above definitions. Clearly $\tilde{f}(t) \in C^1(0,s)$ because $f(t) \in C^1(0,s)$ and $\mathcal{L}_M(t)$ is smooth on $(0,s)$. That $\lim_{t\to 0} \int_t^s \tilde{f}(u)\,du < \infty$ results from the integrability of $f(t)$ and the integrability of $\frac{\mathcal{L}_M^2(t)}{t}$ near zero. ∎

Definition 2.5.6 *Define the monotone non-decreasing function* $h(t) \in C^1(0,\infty)$ *by*

$$h(t) \quad = \quad \begin{cases} t & \text{if} \quad 0 < t \leq R_1 \\ \alpha(t) & \text{if} \quad R_1 < t < R_2 \\ R_2 & \text{if} \quad t \geq R_2 \end{cases}$$

where $\alpha(t) \geq t$ *and* $0 \leq \alpha'(t) \leq 1+\epsilon$ *for arbitrary* $\epsilon > 0$.

Now we are finally in a position to define our second candidate function for $G(t)$.

Definition 2.5.7 *Fix an integer* $M \geq 2$ *and let* $0 < R_1 < R_2 < s < \frac{1}{e_M}$ *where* e_M *is defined by Definition 2.5.2. Let* $L_j(t)$, $\mathcal{L}_M(t)$, $\tilde{f}(t)$ *and* $h(t)$ *be the functions described by Definitions 2.5.2, 2.5.3, 2.5.5 and 2.5.6 respectively. Since* $\mathcal{L}_M(t)$ *and* $t\tilde{f}(t)$ *tend to zero as* $t \to 0^+$, *we can assume that* R_2 *has been chosen sufficiently small so that*

$$0 \quad \leq \quad \frac{1}{2} \;-\; \frac{1}{2}\mathcal{L}_M(t) \;-\; t\,\tilde{f}(t) \quad < \quad 1 \;-\; \frac{1}{2}\mathcal{L}_M(t) \;-\; t\,\tilde{f}(t) \quad \leq \quad 1 \tag{2.18}$$

for all $0 < t \leq R_2$. *Define the candidate function*

$$G_1(t) \quad = \quad \ln(\,h(t)\,) \;+\; \frac{1}{2}\sum_{j=2}^{M} L_j(\,h(t)\,) \;+\; \int_{h(t)}^{s} \tilde{f}(u)\;du.$$

The following Lemma asserts that $G_1(t)$ fulfills the conditions of Theorem 2.1.1. The analysis here is almost exactly the same as the proof of Theorem 2 in [1].

Lemma 2.5.3 *The function $G_1(t)$ satisfies the requirements of Theorem 2.1.1.*

Proof. We begin by showing that $G_1(t) \in C^1(0,\infty)$. By construction the functions $\ln t$, $L_j(t)$ and $\int_t^s \tilde{f}(u)\,du$ are all contained in $C^1(0,s)$. Therefore, since the monotone non-decreasing function $h(t)$ is contained in $C^1(0,\infty)$ and since $h(t) = R_2 < s$ for all $t \geq R_2$ it must be the case that $G_1(t) \in C^1(0,\infty)$.

Conditions 1) and 2) of Theorem 2.1.1 concern the derivative of the auxiliary function $G(t)$. Consequently, we will need an expression for $G_1'(t)$. Indeed,

$$
\begin{aligned}
G_1'(t) &= \frac{1}{h(t)}\left[\,1 + \frac{1}{2}\,h(t)\sum_{j=2}^{M} L_j'(\,h(t)\,) - h(t)\,\tilde{f}(\,h(t)\,)\,\right] h'(t) \\
&= \frac{1}{h(t)}\left[1 - \frac{h(t)}{2\,h(t)}\sum_{j=2}^{M}\left(\prod_{k=1}^{j-1} L_k(\,h(t)\,)\right)^{-1} - h(t)\,\tilde{f}(h(t))\right] h'(t) \qquad (2.19) \\
&= \frac{1}{h(t)}\left[\,1 - \frac{1}{2}\,\mathcal{L}_M(\,h(t)\,) - h(t)\,\tilde{f}(\,h(t)\,)\,\right] h'(t) \qquad (2.20)
\end{aligned}
$$

where in equation (2.19) we have used the result of (2.17). From equation (2.20) we see that if $t \geq R_2$ then $G_1'(t) = 0$ because $h'(t) = 0$ over this range. Further, for $t \in (0, R_1]$ we have $h(t) = t$ and equations (2.20) and (2.18) imply that

$$
0 \;\leq\; G_1'(t) \;=\; \frac{1}{t}\left[\,1 - \frac{\mathcal{L}_M(t)}{2} - t\,\tilde{f}(t)\,\right] \;\leq\; \frac{1}{t}.
$$

Finally, for $t \in (R_1, R_2)$, since $h(t) \geq t$ and $h'(t) \leq 1+\epsilon$ it must be the case that

$$
0 \;\leq\; G_1'(t) \;\leq\; \frac{1}{t}\left[\,1 - \frac{1}{2}\,\mathcal{L}_M(\,h(t)\,) - h(t)\,\tilde{f}(\,h(t)\,)\,\right](1+\epsilon) \;\leq\; \frac{1+\epsilon}{t}.
$$

Consequently, we have shown that $G_1(t)$ satisfies conditions 1) and 2) of Theorem 2.1.1.

It is easy to see that $G_1(t)$ also satisfies condition 4) of the theorem. Indeed, if we take any $k > 2$ then

$$
\begin{aligned}
\sum_{n=1}^{\infty} P_n^k\; e^{-2G_1(P_n)} &= \sum_{n=1}^{\infty} P_n^k\; e^{-2\left[\,\ln P_n \;+\; \frac{1}{2}\sum_{j=2}^{M} L_j(P_n) \;+\; \int_{P_n}^{s}\tilde{f}(u)\,du\,\right]} \\
&\leq \sum_{n=1}^{\infty} P_n^{k-2} \;<\; \infty.
\end{aligned}
$$

However, demonstrating that $G_1(t)$ fulfills condition 3) is a little more tricky. In this respect,

$$\sum_{n=1}^{\infty} P_n^2 \; e^{-2\,G_1(P_n)} \quad = \quad \sum_{n=1}^{\infty} P_n^2 \; e^{-2\left[\; \ln P_n \;+\; \frac{1}{2}\sum_{j=2}^{M} L_j(P_n) \;+\; \int_{P_n}^{s} \tilde{f}(u)\; du \;\right]}$$

$$= \sum_{n=1}^{\infty} \left(\prod_{j=2}^{M} e^{-L_j(P_n)} \right) e^{-2\int_{P_n}^{s} \tilde{f}(u)\; du}$$

$$\geq \left(\lim_{t \to 0^+} e^{-2\int_t^s \tilde{f}(u)\; du} \right) \sum_{n=1}^{\infty} \left(\prod_{j=2}^{M} e^{-L_j(P_n)} \right)$$

$$\geq K \sum_{n=1}^{\infty} \frac{1}{\ln(1/P_n)} \cdot \frac{1}{\ln\ln(1/P_n)} \cdots \frac{1}{\underbrace{\ln\ln\ldots\ln}_{M-1 \text{ times}}(1/P_n)}$$

$$= K \sum_{n=1}^{\infty} \frac{1}{\ln(\frac{\lambda^n}{R}) \;\cdot\; \ln\ln(\frac{\lambda^n}{R}) \;\cdots\; \underbrace{\ln\ln\ldots\ln}_{M-1 \text{ times}}(\frac{\lambda^n}{R})}$$

$$= K \sum_{n=1}^{\infty} \frac{1}{\ln_1(\frac{\lambda^n}{R}) \;\cdot\; \ln_2(\frac{\lambda^n}{R}) \;\cdot \ldots \cdot\; \ln_{M-1}(\frac{\lambda^n}{R})},$$

where we have put $\ln_0(x) = x$, $\ln_1(x) = \ln x$, $\ln_2(x) = \ln\ln x$, etc... Next, choose N sufficiently large so that for all $n \geq N$ we have $n \geq \frac{-\ln R}{1-\ln\lambda}$, or equivalently, $n \geq \ln(\lambda^n/R)$. We note that $\ln_k(\lambda^n/R) = \ln_{k-1}\ln(\lambda^n/R) \leq \ln_{k-1}(n)$, and so we arrive at the estimate

$$\sum_{n=1}^{\infty} P_n^2 \; e^{-2\,G_1(P_n)} \quad \geq \quad K \sum_{n=N}^{\infty} \frac{1}{n \,\cdot \ln_1(n) \,\cdot \ln_2(n) \,\cdot \ldots \cdot \ln_{M-2}(n)}$$

$$= \quad K \sum_{n=N}^{\infty} \frac{1}{n \,\cdot \ln(n) \,\cdot \ln\ln(n) \,\cdot \ldots \cdot \underbrace{\ln\ln\ldots\ln}_{M-2 \text{ times}}(n)}.$$

That this last expression diverges is an elementary consequence of the integral test. The lemma is now proven. ∎

Applying Theorem 2.1.1 with $G(t)$ set equal to $G_1(t)$, enables us to obtain the following result.

Theorem 2.5.4 *Let* Ω *be a domain in* $\mathbb{R}^m$ *with non-empty boundary and let* $d \equiv d(x) = \mathrm{dist}(x, \partial\Omega)$ *be the Euclidean distance to the boundary. Fix a finite number* $M \in \mathbb{N}$ *and let* $\mathcal{L}_M(t)$, $f(t)$ *and* $h(t)$ *be the functions described in Definitions 2.5.3, 2.5.4 and 2.5.6 respectively. Let* $H = -\Delta + V$ *be a Schrödinger operator defined on* $C_0^\infty(\Omega)$, *where* $V \in L_{\infty,loc}(\Omega)$ *is a real potential of the form* $V = V_1 + V_2$ *and* $V_2 \in L_\infty(\Omega)$. *If*

$$V_1(x) \quad \geq \quad h'(d)^2 \left[\frac{1}{h(d)^2} \; - \; \frac{\mathcal{L}_M\big(h(d)\big)}{h(d)^2} \; - \; \frac{f\big(h(d)\big)}{h(d)} \right] \; - \; \frac{\mu_2(\Omega)}{d^2},$$

then the operator H *is essentially self-adjoint.*

Proof. In light of Theorem 2.1.1 and Lemma 2.5.3, it suffices to show that

$$V_1(x) \quad \geq \quad G_1'(\,d(x)\,)^2 \; - \; \mu_2(\Omega)\; d(x)^{-2} \tag{2.21}$$

which, using equation (2.20), will be the case provided that

$$\begin{aligned} & h'(d)^2 \left[\frac{1}{h(d)^2} \; - \; \frac{\mathcal{L}_M(\,h(d)\,)}{h(d)^2} \; - \; \frac{f(\,h(d)\,)}{h(d)} \right] \; - \; \frac{\mu_2(\Omega)}{d^2} \\ \geq \; & h'(d)^2 \left[\frac{1}{h(d)^2} \; - \; \frac{\mathcal{L}_M(\,h(d)\,)}{h(d)^2} \; + \; \frac{\mathcal{L}_M^2(\,h(d)\,)}{4\,h(d)^2} \; + \; \frac{\mathcal{L}_M(\,h(d)\,)\,\tilde f(\,h(d)\,)}{h(d)} \right. \\ & \qquad \left. - \; \frac{2\,\tilde f(\,h(d)\,)}{h(d)} \; + \; \tilde f(\,h(d)\,)^2 \right] \; - \; \frac{\mu_2(\Omega)}{d^2}. \end{aligned}$$

In other words, we must prove the validity of the inequality

$$-\frac{f(\,h(d)\,)}{h(d)} \;\; \geq \;\; \frac{\mathcal{L}_M^2(\,h(d)\,)}{4\,h(d)^2} \; - \; \frac{2\,\tilde f(\,h(d)\,)}{h(d)} \; + \; \frac{\mathcal{L}_M(\,h(d)\,)\,\tilde f(\,h(d)\,)}{h(d)} \; + \; \tilde f(\,h(d)\,)^2. \tag{2.22}$$

From the left hand side of equation (2.18) we have that for all $0 \, < \, t \, \leq \, R_2$

$$1 \quad \geq \quad \mathcal{L}_M(t) \; + \; 2\,t\tilde f(t) \tag{2.23}$$

and by definition $\tilde f(t) \, = \, f(t) \, + \, \frac{\mathcal{L}_M^2(t)}{4t}$ for all $0 \, < \, t \, \leq \, R_2 \, < \, 1/e_M$. Therefore,

$$-\frac{f(t)}{t} \quad = \quad \frac{\mathcal{L}_M^2(t)}{4t^2} \; - \; 2\,\frac{\tilde f(t)}{t} \; + \; \frac{\tilde f(t)}{t}$$

which upon using (2.23) shows that for all $0 \, < \, t \, \leq \, R_2$,

$$\begin{aligned} -\frac{f(t)}{t} \;\; & \geq \;\; \frac{\mathcal{L}_M^2(t)}{4\,t^2} \; - \; 2\frac{\tilde f(t)}{t} \; + \; \Big(\mathcal{L}_M(t) \; + \; 2\,t\,\tilde f(t) \Big) \frac{\tilde f(t)}{t} \\ & = \;\; \frac{\mathcal{L}_M^2(t)}{4\,t^2} \; - \; 2\,\frac{\tilde f(t)}{t} \; + \; \frac{\mathcal{L}_M(t)\,\tilde f(t)}{t} \; + \; 2\,\tilde f(t)^2 \\ & \geq \;\; \frac{\mathcal{L}_M^2(t)}{4\,t^2} \; - \; 2\,\frac{\tilde f(t)}{t} \; + \; \frac{\mathcal{L}_M(t)\,\tilde f(t)}{t} \; + \; \tilde f(t)^2. \end{aligned}$$

Given that $h(t)$ is monotone non-decreasing and that $h(t) = R_2$ for all $t \geq R_2$, this establishes (2.22) and so completes the proof. ∎

Due to the number of definitions inherent in its construction Theorem 2.5.4 is cumbersome to use. However, choosing some finite $M \in \mathbb{N}$, setting $f = 0$ and recalling that $h(t) = t$ for sufficiently small t, the basic conclusion of the theorem is that the operator H is essentially self-adjoint, provided that sufficiently close to the boundary

$$
\begin{aligned}
V_1(x) &\geq \frac{1}{d(x)^2} - \frac{\mathcal{L}_M(d(x))}{d(x)^2} - \frac{\mu_2(\Omega)}{d(x)^2} \\
&= \frac{1}{d(x)^2}\Bigg[1 - \mu_2(\Omega) - \frac{1}{\ln(d^{-1})} - \frac{1}{\ln(d^{-1})\ln\ln(d^{-1})} - \dots \\
&\qquad \dots - \frac{1}{\ln(d^{-1})\ln\ln(d^{-1})\dots\underbrace{\ln\ln\dots\ln}_{M-1 \text{ times}}(d^{-1})}\Bigg]
\end{aligned}
$$

Clearly, this is a stronger result than that stated in Theorem 2.5.1.

Chapter 3

The L_p-Hardy Inequality

Let us begin this chapter by recalling the definition of the L_p-Hardy inequality made in Section 1.5.3.

Definition *Let* Ω *be a domain in* $\mathbb{R}^m$ *with non-empty boundary. For* $1 < p < \infty$, Ω *is said to admit an* L_p*-Hardy inequality if there exists a finite uniform constant* $C > 0$, *so that the estimate*

$$\int_\Omega \frac{|\,\omega(x)\,|^p}{d(x)^p}\; dx \qquad \leq \qquad C \int_\Omega |\,\nabla\omega(x)\,|^p \; dx \tag{3.1}$$

holds for all $\omega(x) \in W^1_{p,0}(\Omega)$.

In the same section, we also defined the variational constant $\mu_p(\Omega)$ by the relation

$$\mu_p(\Omega) \quad = \quad \inf_{\omega\,\in\,W^1_{p,0}(\Omega)} \left\{ \frac{\int_\Omega |\,\nabla\,\omega(x)\,|^p \; dx}{\int_\Omega \frac{|\,\omega(x)\,|^p}{d(x)^p}\; dx} \right\}.$$

Although, the L_p-Hardy inequality has been extensively studied in its own right, the reader should bear in mind that our motivation resides in the study of it's L_2 counterpart. Indeed, suppose that Ω is a domain in $\mathbb{R}^m$ and that $H = -\Delta + V$ is a Schrödinger operator defined on $C_0^\infty(\Omega)$. In the remarks following the statement of Theorem 2.1.1, we went to great lengths to demonstrate that the existence of an L_2-Hardy inequality on Ω relaxes the criteria for the essential self-adjointness of H. We also asserted that the uncertainty principle lay behind this phenomenon, in the sense that the variational constant $\mu_2(\Omega)$ (which is non-zero if and only if Ω admits an L_2-Hardy inequality) places limits on the certainty with which we can say that a particle is located at the boundary. With this in mind, in this chapter of the thesis we look to ascertain the necessary and sufficient conditions required for a domain to admit an L_p-Hardy inequality. The literature in this area is well developed and the results are widely known. Since we are not adding any new insights, we will only supply references for the stated theorems, except in the case where the proofs are short. The problem of obtaining an explicit value for $\mu_p(\Omega)$ is postponed until Chapter 5.

We have decided to structure the chapter as follows. In Section 3.1, we consider 'the original' L_p-Hardy inequality, i.e. the L_p-Hardy inequality on the domain $\mathbb{R}^m \setminus \{0\}$. This was the inequality first studied by Hardy, Littlewood and Polya in [36]. We will provide a simple, and original, proof of the existence of an L_p-Hardy inequality on this domain in the case where $p \neq m$.

Next, in Section 3.2, we give Maz'ya's characterization of the L_p-Hardy inequality on arbitrary domains in $\mathbb{R}^m$. In effect, this is a necessary and sufficient condition for a domain Ω to admit an L_p-Hardy inequality, phrased in terms of the ability to uniformly estimate the integral of the function $d(x)^{-p}$, over a compact set $K \subseteq \Omega$, by the variational p-capacity of K. The reader is referred to Appendix B for the definition of variational p-capacity and related concepts. Since the proof of Maz'ya's characterization is both elegant, and short, it is provided.

We will see in Chapter 4 that the integrability of the function $d(x)^{-p} = d(x, \partial\Omega)^{-p}$ is strongly related to 'the dimension' of the boundary $\partial\Omega$. This suggests that the existence of an L_p-Hardy inequality may have some dependence on the dimension of the boundary of a domain. Indeed, in Section 3.3, we give the result of Koskela & Zhong [51], which asserts that if Ω is a domain that admits an L_p-Hardy inequality, then the boundary must either be sufficiently fat or sufficiently thin. More specifically, if Ω is a domain which admits an L_p-Hardy inequality, then either the Hausdorff dimension of the boundary must be strictly greater than $m - p$, or the Aikawa dimension of the boundary must be strictly less than $m - p$.

The purpose of the next two sections of the chapter is to show that the necessary conditions expressed within this dimensional dichotomy are almost sufficient conditions. In Section 3.4, we show that if the complement of a domain is uniformly p-fat (a concept we will define shortly), then that domain admits an L_p-Hardy inequality. Moreover, it will also be shown that if the complement of a domain is uniformly p-fat then it must be the case that the Hausdorff dimension of the boundary exceeds $m - p$. In the borderline case where $m = p$, we also exploit a known equivalent condition for uniform p-fatness to obtain a simple sufficient condition for the existence of an L_2-Hardy inequality on planar domains. Finally, in Section 3.5, we use the analysis of Aikawa in [60] and [61] to show that if the Aikawa dimension of the boundary of a domain is less than $m - p$, then that domain admits an L_p-Hardy inequality.

However, before we begin, a word of warning is in order. In this chapter we are essentially concerned with describing the relationship between the existence of an L_p-Hardy inequality on a domain, 'the dimension' of the boundary and the 'capacity' of the complement. One should bear in mind that there are many different notions of dimension and capacity. Therefore, in order to ease the flow of the exposition, the relevant definitions of dimension and capacity have been relegated to Appendix A and Appendix B respectively. Furthermore, the use of 'maximal functions' pervades the analysis within this chapter, the definition of which is located in Appendix C. To prevent any confusion, the reader is strongly encourage to peruse these appendices before continuing.

3.1 The L_p-Hardy Inequality on $\mathbb{R}^m \setminus \{0\}$

The emergence of the classical L_p-Hardy inequality has a long and well documented history (see for instance [33]). Although many mathematicians contributed to its development over the period from 1906 to 1928, it was Hardy, Littlewood & Polya [36] who first showed that if $p > 1$, and if f is a non-negative function which is locally in $L_p(\mathbb{R})$, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\; dt \right)^p dx \qquad \leq \qquad \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p \; dx. \tag{3.2}$$

In this section we will generalize equation (3.2) to higher dimensions. Although the result is very well known, we offer an original, and simple, new proof.

Theorem 3.1.1 *Let* $\Omega = \mathbb{R}^m \setminus \{0\}$ *if* $m \geq 2$ *and* $\Omega = (0, \infty)$ *if* $m = 1$. *Then for all* $p > 1$, $p \neq m$, *the inequality*

$$\int_\Omega \frac{|\,\omega(x)\,|^p}{|\,x\,|^p} \; dx \quad \leq \quad \left| \frac{p}{m-p} \right|^p \int_\Omega |\,\nabla\omega(x)\,|^p \;\; dx \tag{3.3}$$

holds for all $\omega(x) \in W^1_{p,0}(\Omega)$, *so that* $\mu_p(\Omega) \geq \left| \frac{m-p}{p} \right|^p$.

Proof. By the density of $C_0^\infty(\Omega)$ in $W^1_{p,0}(\Omega)$, it suffices to show that equation (3.3) holds for all functions in $C_0^\infty(\Omega)$. If $p \neq 2$ let $V(x) = \frac{|\,x\,|^{-p+2}}{(m-p)\,(p-2)}$. On the other hand if $p = 2$ let $V(x) = -\frac{\ln\,|x|}{m-2}$. In either case, elementary differentiation gives that $\nabla V(x) = -\frac{1}{m-p} \frac{x}{|\,x\,|^p}$ and $\Delta V(x) = -\frac{1}{|\,x\,|^p}$. Now, let $u(x)$ be an arbitrary function in $C_0^\infty(\Omega)$. An application of integration by parts, followed by Hölder's inequality, leads us to the following set of equations.

$$\begin{aligned}
\int_\Omega \frac{|\,u(x)\,|^p}{|\,x\,|^p} \; dx &= -\int_\Omega \Delta V(x)\,|\,u(x)\,|^p \;\; dx \\
&= P \int_\Omega |\,u(x)\,|^{p-1} \;\; \nabla V(x) \;\; \nabla\,|\,u(x)\,| \;\; dx \\
&\leq P \int_\Omega |\,u(x)\,|^{p-1} \;\; |\,\nabla V(x)\,| \;\; |\,\nabla u(x)\,| \;\; dx \\
&\leq P \left(\int_\Omega |\,u(x)\,|^p \;\; |\,\nabla V(x)\,|^{\frac{p}{p-1}} \;\; dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |\,\nabla u(x)\,|^p \;\; dx \right)^{\frac{1}{p}} \\
&= \frac{p}{|\,m-p\,|} \left(\int_\Omega \frac{|\,u(x)\,|^p}{|\,x\,|^p} \;\; dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |\,\nabla u(x)\,|^p \;\; dx \right)^{\frac{1}{p}}.
\end{aligned}$$

Dividing this last expression through by $\left(\int_\Omega \frac{|u(x)|^p}{|x|^p} \, dx \right)^{\frac{p-1}{p}}$ we obtain

$$\left(\int_\Omega \frac{|u(x)|^p}{|x|^p} \, dx \right)^{\frac{1}{p}} \leq \left| \frac{p}{m-p} \right| \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}},$$

so that raising both sides to the power of p yields the result. ∎

3.2 Maz'ya's Characterization of the Hardy Inequality

Let Ω be a domain in $\mathbb{R}^m$. In [58], Maz'ya showed that the ability to uniformly estimate the integral of the function $d(x)^{-p}$ over the compact set $K \subseteq \Omega$, in terms of the variational p-capacity of K, was both a necessary and sufficient condition for Ω to admit an L_p-Hardy inequality. The reader is directed to Appendix B for the definition of variational p-capacity. More recently, Kinnunen & Korte [48] were able to give a simplified proof of this fact. We follow their analysis here to prove the following theorem.

Theorem 3.2.1 *[58, Chapter 10], [48, Theorem 2.1]*

A domain $\Omega \subseteq \mathbb{R}^m$ admits an L_p-Hardy inequality if and only if there exists a finite uniform constant $C > 0$ so that the inequality

$$\int_K \frac{1}{d(x)^p} \, dx \leq C \, C_p(K, \Omega) \tag{3.4}$$

holds for all compact sets $K \subseteq \Omega$. Here $C_p(K, \Omega)$ is the variational p-capacity of K relative to Ω.

Proof. Suppose that Ω admits an L_p-Hardy inequality. Then for any compact set $K \subseteq \Omega$, and for any $u(x) \in C_0^\infty(\infty)$ with the property that $u(x) \geq 1$ on K, there is a finite uniform constant $C > 0$ so that

$$\int_K \frac{1}{d(x)^p} \, dx \leq \int_K \frac{|u(x)|^p}{d(x)^p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx.$$

Since this holds for all such functions, it must be the case that

$$\begin{aligned} \int_K \frac{1}{d(x)^p} \, dx &\leq C \inf \left\{ \int_\Omega |\nabla u(x)|^p \, dx \;\middle|\; u \in C_0^\infty(\Omega), \; u \geq 1 \text{ on } K \right\} \\ &= C \, C_p(K, \Omega). \end{aligned}$$

This establishes the necessity.

Next we turn consider the sufficiency of the statement. Suppose that equation (3.4) holds. Let us choose arbitrary $u(x) \in C_0^\infty(\Omega)$, and for all $k \in \mathbb{Z}$ define the regions $E_k = \{\, x \in \Omega \;\big|\; |u(x)| > 2^k \,\}$. Using the monotonicity properties of variational p-capacity described in Appendix B, we derive the following set of inequalities.

$$
\begin{aligned}
\int_\Omega \frac{|u(x)|^p}{d(x)^p}\, dx &= \sum_{k=-\infty}^{\infty} \int_{E_k \setminus E_{k+1}} \frac{|u(x)|^p}{d(x)^p}\, dx \;\leq\; \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \int_{E_k \setminus E_{k+1}} \frac{1}{d(x)^p}\, dx \\
&\leq \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \int_{\bar{E}_k} \frac{1}{d(x)^p}\, dx \;\leq\; C \sum_{k=-\infty}^{\infty} 2^{(k+1)p}\, C_p(\bar{E}_k, \Omega) \\
&\leq C\, 2^p \sum_{k=-\infty}^{\infty} 2^{(k+1)p}\, C_p(\bar{E}_{k+1}, E_k).
\end{aligned}
$$

Now, let us define the functions $u_k : \Omega \to [0,1]$ by

$$
u_k(x) = \begin{cases} 1 & \text{if} \quad |\,u(x)\,| \geq 2^{k+1}, \\ 2^{-k}\,|u(x)| - 1 & \text{if} \quad 2^k < |\,u(x)\,| < 2^{k+1}, \\ 0 & \text{if} \quad |\,u(x)\,| \leq 2^k. \end{cases}
$$

Then $u_k(x) \in W^1_{p,0}(E_k) \cap C(E_k)$ and equals 1 on the compact set $\bar{E}_{k+1}$, so that by Definition B.1.2 it is an admissible test function for $C_p(\bar{E}_{k+1}, E_k)$. Continuing our previous line of argument we obtain

$$
\begin{aligned}
\int_\Omega \frac{|u(x)|^p}{d(x)^p}\, dx &\leq C\, 2^p \sum_{k=-\infty}^{\infty} 2^{(k+1)p}\, C_p(\bar{E}_{k+1}, E_k) \\
&\leq C\, 2^p \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \int_{E_k} |\nabla u_k(x)|^p\, dx \\
&= C\, 2^p \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \int_{E_k \setminus E_{k+1}} \big|\, \nabla \left(2^{-k}\,|\,u(x)\,| - 1 \right) \big|^p\, dx \\
&\leq C\, 2^{2p} \sum_{k=-\infty}^{\infty} \int_{E_k \setminus E_{k+1}} |\nabla u(x)|^p\, dx \\
&= C\, 2^{2p} \int_\Omega |\nabla u(x)|^p\, dx.
\end{aligned}
$$

Since $u(x)$ was arbitrary in $C_0^\infty(\Omega)$, which is dense in $W^1_{p,0}(\Omega)$, an approximation argument completes the proof. ∎

Maz'ya's characterization of the L_p-Hardy inequality is phrased in terms of the integrability of the distance function and variational p-capacity. We shall see in this and subsequent chapters that, within the context of the L_p-Hardy inequality, both these concepts have an intimate connection with the dimension of the boundary of a domain. This raises the question as to whether the existence of an L_p-Hardy inequality on a domain has a dependence on the dimension of the boundary. The main result in the next section confirms this fact.

3.3 A Dimensional Dichotomy

In [51], Koskela & Zhong show that if Ω is a domain in $\mathbb{R}^m$ that admits an L_p-Hardy inequality, then Ω admits an L_q-Hardy inequality for all q sufficiently close to p. More precisely, they prove the following theorem.

Theorem 3.3.1 *[51, Theorem 1.2]*

Let Ω be a domain in $\mathbb{R}^m$ that admits an L_p-Hardy inequality. Then there exists a constant $\epsilon = \epsilon(p,m)$ so that Ω admits an L_q-Hardy inequality for all $q \in (p-\epsilon,\, p+\epsilon)$.

Within the same paper, Koskela & Zhong also show that there is a dimensional dichotomy underlying the question of the existence of an L_p-Hardy inequality on a given domain. Indeed, they derive the following result (see Appendix A for the relevant definitions of dimension).

Theorem 3.3.2 *[51, Theorem 1.1], [52, Theorem 1.1]*

Let Ω be a domain in $\mathbb{R}^m$ that admits an L_p-Hardy inequality. Then there exists a constant $\epsilon = \epsilon(p,m)$ so that either

i) $\dim_{\mathcal{H}}(\partial\Omega) \geq m - p + \epsilon$, *or*

ii) $\dim_A(\partial\Omega) \leq m - p - \epsilon$.

Here $\dim_{\mathcal{H}}(\partial\Omega)$ and $\dim_A(\partial\Omega)$ denote the Hausdorff and Aikawa dimension of the boundary respectively.[1]

In effect, the above theorem states that if a domain Ω admits an L_p-Hardy inequality, then either the boundary of the domain is sufficiently fat $(\dim_{\mathcal{H}}(\partial\Omega) > m-p)$ or the boundary is sufficiently thin $(\dim_A(\partial\Omega) < m-p)$. The contrapositive of this statement asserts that if the Hausdorff or Aikawa dimension of the boundary equals $m-p$, then Ω

[1] Note that in [51], what Koskela & Zhong refer to as Minkowski dimension, we refer to as Aikawa dimension. This double notation is noted by Koskela & Zhong at the end of [51] and also by Lehrbäck in [52].

cannot admit an L_p-Hardy inequality. We note that $\mu_p(\Omega)$ must equal zero in this case. In particular, for domains with compact boundaries equation (A.4) indicates that

$$\dim_{\mathcal{H}}(\partial\Omega) \quad \leq \quad \dim_M(\partial\Omega) \quad \leq \quad \dim_A(\partial\Omega),$$

so that if $\dim_M(\partial\Omega) = m - p$, then Ω cannot admit an L_p-Hardy inequality.

Within this context, it is interesting to consider domains with boundaries for which the concepts of Hausdorff, Minkowski and Aikawa dimension diverge. For instance, if $\Omega = \mathbb{R}^m \setminus E$ where

$$E \quad = \quad \big\{\, 0 \,\big\} \cup \big\{\, (\, 1/n,\, 0,\, \ldots\, ,0\,) \;\big|\; n \in \mathbb{N} \,\big\},$$

then $\dim_{\mathcal{H}}(\partial\Omega) = 0$, $\dim_M(\partial\Omega) = 1/2$ and $\dim_A(\partial\Omega) = 1$ (see the references given at the end of Section A.5 for details). Therefore,

$$\begin{aligned} 0 \,&=\, \dim_{\mathcal{H}}(\partial\Omega) \,\leq\, m - p \qquad \text{for all} \quad p \,\leq\, m \\ 1 \,&=\, \dim_{\mathcal{A}}(\partial\Omega) \,\geq\, m - p \qquad \text{for all} \quad p \,\geq\, m - 1 \end{aligned}$$

so that Ω cannot admit an L_p-Hardy inequality for any $p \in [m-1, m]$, i.e. there is an interval on which the L_p-Hardy inequality fails to hold for this domain.

In fact, Theorem 3.3.2 is local in nature, in the sense that if $\omega \in \partial\Omega$ and $r > 0$, then either $\dim_{\mathcal{H}}\big(\, \partial\Omega \,\cap\, B(\omega, r)\,\big) \;>\; m - p$, or $\dim_A\big(\, \partial\Omega \,\cap\, B(\omega, r)\,\big) \;<\; m - p$, whenever Ω admits an L_p-Hardy inequality. As such, the boundary of a domain that admits an L_p-Hardy inequality cannot even contain an $m - p$ dimensional part. For example, if Ω is a punctured disk in $\mathbb{R}^2$, then Ω cannot admit an L_2-Hardy inequality because the boundary of the domain contains a zero dimensional part.

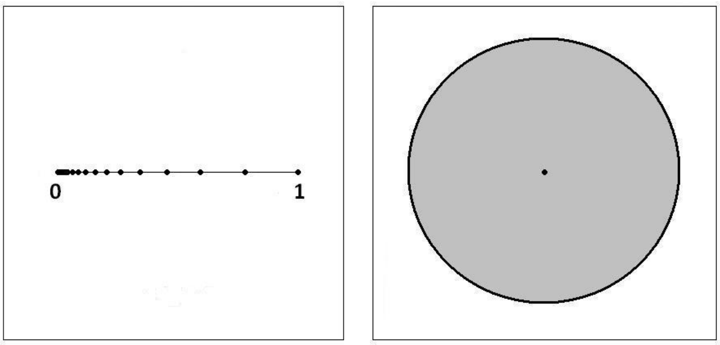

Figure 3.1: The domain $\mathbb{R}^m \setminus [\,\{\,0\,\} \cup \{\,(\,1/n,\, 0,\, \ldots\, ,0\,) \mid n \in \mathbb{N}\,\}\,]$ fails to admit an L_p-Hardy inequality for any $p \in [m-1, m]$. A punctured disk in $\mathbb{R}^2$ also fails to admit an L_2-Hardy inequality because it contains a zero dimensional part.

The objective within the next two sections of this chapter is to show that the necessary conditions for a domain to admit an L_p-Hardy inequality described by Theorem 3.3.2, are almost sufficient conditions. In Section 3.5, we will show that if Ω is a domain with thin boundary (in the sense described above) then Ω admits an L_p-Hardy inequality. Before then, in the next section we turn our attention to domains whose complements are, in a certain sense, 'thick'.

3.4 Domains with Fat Boundary

The purpose of the present section is to demonstrate that if the complement of a domain is 'sufficiently thick', then that domain admits an L_p-Hardy inequality. We begin by defining the pointwise q-Hardy inequality, a concept first introduced by Hajłasz in [38]. The reader is directed to Appendix C for the relevant definitions of maximal functions.

Definition 3.4.1 *Let Ω be a domain in $\mathbb{R}^m$. We say that Ω admits a pointwise q-Hardy inequality if there exists a finite uniform constant $C > 0$ so that the inequality*

$$|\, u(x)\,| \quad \leq \quad C\, d(x) \left(\hat{M}\, |\, \nabla u(x)\,|^q \right)^{\frac{1}{q}} \tag{3.5}$$

holds for all $u(x) \in C_0^\infty(\Omega)$ and for all $x \in \Omega$. Here $\hat{M}\,|\,\nabla u(x)\,|^q$ is the maximal function of $|\,\nabla u(x)\,|^q$.

From our perspective, the interesting thing about the pointwise q-Hardy inequality is that if a domain admits a pointwise q-Hardy inequality (for appropriate q), then that domain also admits an L_p-Hardy inequality. Indeed, this is the content of the next lemma.

Lemma 3.4.1 *[39, Proposition 3.1]*

Let Ω be a domain in $\mathbb{R}^m$ and let $1 < p < \infty$. If Ω admits a pointwise q-Hardy inequality for some $q \in (1, p)$, then Ω admits an L_p-Hardy inequality.

Proof. It suffices to prove the theorem for functions in $C_0^\infty(\Omega)$. Let us take arbitrary $u(x) \in C_0^\infty(\Omega)$ and extend the function to the whole of $\mathbb{R}^m$ by zero. Since Ω admits a pointwise q-Hardy inequality for $q \in (1, p)$, by dividing equation (3.5) through by $d(x)$, raising both sides to the power of p and integrating over the whole of $\mathbb{R}^m$ we obtain

$$\begin{aligned} \int_\Omega \frac{|u(x)|^p}{d(x)^p}\, dx \quad &\leq \quad C \int_{\mathbb{R}^m} \left(\hat{M}\,|\nabla u(x)|^q \right)^{\frac{p}{q}} dx \quad = \quad C\ ||\ \hat{M}\ |\nabla u(x)|^q\ ||_{L_{p/q}(\mathbb{R}^m)}^{p/q} \\ &\leq \quad C\ ||\ |\nabla u(x)|^q\ ||_{L_{p/q}(\mathbb{R}^m)}^{p/q} \quad = \quad C \int_\Omega |\,\nabla u(x)\,|^p\, dx \end{aligned}$$

where the last inequality holds due to Lemma C.1.1. ∎

This immediately raises the question as to whether the existence of a pointwise q-Hardy inequality, for some $q \in (1,p)$, is equivalent to the existence of an L_p-Hardy inequality. The answer to this question is negative. As we will see, there are domains (such as $\mathbb{R}^m \setminus \{0\}$ where $m > p$) which admit an L_p-Hardy inequality but which also fail to admit a pointwise q-Hardy inequality for any $q \in (1,p)$. Nevertheless, Lemma 3.4.1 dictates that the existence of a pointwise q-Hardy inequality, for $q \in (1,p)$, is sufficient to ensure the existence of an L_p-Hardy inequality. Consequently, we may inquire as to what conditions ensure the existence of a pointwise q-Hardy inequality. To answer this question, we must first define the concept of uniform p-fatness. Again, the reader is directed to Appendix B for the relevant definitions of capacity.

Definition 3.4.2 *Let E be a closed set in $\mathbb{R}^m$. We say that E is uniformly p-fat if there exists some $\gamma > 0$ so that for all $x \in E$ and for all $r > 0$*

$$\begin{aligned} C_p\big(\, E \cap \bar{B}(x,r)\,,\, B(x,2r)\,\big) \;\; &\geq \;\; \gamma\, C_p\big(\, \bar{B}(x,r)\,,\, B(x,2r)\,\big) \\ &= \;\; \gamma\, c(m,p)\, r^{m-p}. \end{aligned}$$

Here $C_p(K,G)$ is the variational p-capacity of K relative to G and $c(m,p)$ is the constant defined by expression (B.1).

We may interpret this definition as saying that a closed set $E \subseteq \mathbb{R}^m$ is uniformly p-fat provided that the variational p-capacity of E intersected with any ball centered on E is at least as great as a given fraction of the capacity of the corresponding spherical condenser. The following theorem, originally due to Lewis [41], asserts that uniformly p-fat sets exhibit a deep 'self-improving' property.[2]

Theorem 3.4.2 *[41, Theorem 1], [42, Theorem 3.7], [43, Theorem 8.2]*

If $E \subseteq \mathbb{R}^m$ is a closed, uniformly p-fat set for some $p > 1$, then there exists $q \in (1,p)$ so that E is also uniformly q-fat.

Lewis' proof of the above theorem is actually phrased in terms of Riesz p-capacity. For a proof of Theorem 3.4.2 in terms of variational p-capacity, the reader is directed to Theorem 8.2 of [43]. By means of utilizing the self-improving property of uniformly p-fat sets, both Lewis [41] and Mikkonen [43] were able to prove that if Ω is a domain in $\mathbb{R}^m$ for which $\mathbb{R}^m \setminus \Omega$ is uniformly p-fat, then Ω admits an L_p-Hardy inequality. In [42], Kinnunen and Martio were able to go one step further and proved the following theorem.

Theorem 3.4.3 *[42, Theorem 3.9 and remarks thereafter], [44, Theorem 4.4.6]*

If Ω is a domain in $\mathbb{R}^m$ so that $\mathbb{R}^m \setminus \Omega$ is uniformly p-fat for some $p \in (1,\infty)$, then Ω admits a pointwise q-Hardy inequality for some $q \in (1,p)$.

[2]This as a reflection of Theorem 3.3.1 which states that if Ω admits an L_p-Hardy inequality then it also admits an L_q-Hardy inequality for all $q \in (p-\epsilon,\, p+\epsilon)$.

Combining Theorem 3.4.3 and Lemma 3.4.1 together, we arrive at the aforementioned result of Lewis and Mikkonen.

Theorem 3.4.4 *[41, Theorem 2], [43, Theorem 8.15]*

If Ω is a domain in $\mathbb{R}^m$ so that $\mathbb{R}^m \setminus \Omega$ is uniformly p-fat for $p \in (1, \infty)$, then Ω admits an L_p-Hardy inequality.

We remark that recently Korte, Lehrbäck & Tuominen [45] have constructed a transparent proof of the above theorem. The proof is transparent in the sense that it does not rely upon the self-improving property of uniformly p-fat sets. Furthermore, this sufficient condition for the existence of an L_p-Hardy inequality turns out to hold true in a variety of general settings. In Section 3.6 of [42], Kinnunen & Martio compile the following list of domains whose complements are uniformly p-fat, and which consequently admit an L_p-Hardy inequality.

1. Every non-empty, closed set in $\mathbb{R}^m$ is uniformly p-fat for all $p > m$. Hence, if $\Omega \subsetneq \mathbb{R}^m$ is a domain, then Ω must admit an L_p-Hardy inequality for all $p > m$. In particular, if $\Omega \subsetneq \mathbb{R}$ is any open interval, then Ω must admit an L_2-Hardy inequality.
2. Every closed set satisfying the interior cone condition is uniformly p-fat for all $p \in (1, \infty)$. Consequently, if $\Omega \subsetneq \mathbb{R}^m$ is a domain for which $\mathbb{R}^m \setminus \Omega$ satisfies the interior cone condition, then Ω admits an L_p-Hardy inequality for all $p \in (1, \infty)$.
3. If Ω is a Lipschitz domain, then $\mathbb{R}^m \setminus \Omega$ is uniformly p-fat, and so must admit an L_p-Hardy inequality, for all $p \in (1, \infty)$.
4. If there is a constant $\gamma > 0$ so that $|\, B(x,r) \cap E \,| \geq \gamma |\, B(x,r) \,|$ for all $x \in E$ and all $r > 0$, then E is uniformly p-fat for all $1 < p < \infty$. Hence, if Ω is a domain in $\mathbb{R}^m$ so that $|\, B(x,r) \cap \Omega^c \,| \geq \gamma |\, B(x,r) \,|$ for all $x \in \Omega^c$, then Ω admits an L_p-Hardy inequality for all $p \in (1, \infty)$.

In fact, the uniform p-fatness of the complement of a domain turns out to be equivalent to the existence of a pointwise q-Hardy inequality for $q \in (1, p)$, and also to various other conditions, as stated in the theorem below.

Theorem 3.4.5 *[47, Theorem 1], [48, Theorem 3.7]*

If Ω is a domain in $\mathbb{R}^m$, and if $1 < p < \infty$, then the following are equivalent:

a) $\mathbb{R}^m \setminus \Omega$ is uniformly p-fat.

b) Ω admits a pointwise q-Hardy inequality for some $q \in (1, p)$.

c) There exists $\lambda \in (m-p, m]$*, and a finite uniform constant* $C > 0$*, so that the inequality*

$$\mathcal{H}^{\lambda}_{\infty}\big(\, B(\, x, 2d(x)\,) \cap \partial\Omega \big) \quad \geq \quad C\, d(x)^{\lambda}$$

holds for all $x \in \Omega$*.*

d) There exists a finite uniform constant $C > 0$ *so that the inequality*

$$\int_{B(x,r)} |u(y)|^p \; dy \quad \leq \quad C\, r^p \int_{B(x,r)} |\nabla u(y)|^p \; dy$$

holds for all $x \in \mathbb{R}^m \setminus \Omega$ *and for all* $u \in W^1_{p,0}(\Omega)$*.*

Some remarks about Theorem 3.4.5 are in order. First of all, the theorem implies that if $\mathbb{R}^m \setminus \Omega$ is uniformly p-fat, then the Hausdorff dimension of the boundary must be strictly greater than $m - p$. Indeed, if $\mathbb{R}^m \setminus \Omega$ is uniformly p-fat, then by condition c) of the theorem there exists $\lambda \in (m - p, m]$, and a constant $C > 0$, so that $\mathcal{H}^{\lambda}_{\infty}\big(B(\, x\,, 2d(x)\,) \cap \partial\Omega \big) \geq C\, d(x)^{\lambda}$ for all $x \in \Omega$. Therefore, if $\omega \in \partial\Omega$ and $r > 0$, then for all $x \in B(\omega, \frac{r}{3}) \cap \Omega$ it must be the case that $B(x, 2d(x)) \subseteq B(\omega, r)$. By the monotonicity of λ-Hausdorff content we then have that

$$\mathcal{H}^{\lambda}_{\infty}(\partial\Omega) \quad \geq \quad \mathcal{H}^{\lambda}_{\infty}\big(B(\omega, r) \cap \partial\Omega \big) \;\geq\; \mathcal{H}^{\lambda}_{\infty}\big(B(\, x\,, 2d(x)\,) \cap \partial\Omega \big) \;\geq\; C\, d(x)^{\lambda} \quad > \quad 0.$$

Hence, it must be the case that $\dim_{\mathcal{H}}(\partial\Omega) \geq \lambda > m-p$. For subsequent ease of reference, we state this result as a lemma.

Lemma 3.4.6 *[47, Page 2194]*

If Ω *is a domain in* $\mathbb{R}^m$ *and if* $\mathbb{R}^m \setminus \Omega$ *is uniformly* p*-fat for* $p \in (1, \infty)$*, then* Ω *admits an* L_p*-Hardy inequality and* $\dim_{\mathcal{H}}(\partial\Omega) > m - p$*.*

Secondly, if the complement of a domain Ω is uniformly p-fat, then since Ω must admit a pointwise q-Hardy inequality, for some $q \in (1, p)$, $\mathbb{R}^m \setminus \Omega$ must also be uniformly s-fat for all $s > p$. Consequently, if the complement of a domain is uniformly p-fat, then that domain must admit an L_s-Hardy inequality for all $s \geq p$.

Finally, we are now in a position to show that the existence of a pointwise q-Hardy inequality, for $q \in (1, p)$, is **not** equivalent to the existence of an L_p-Hardy inequality. Consider the domain $\Omega = \mathbb{R}^m \setminus \{0\}$ where $m > p$. We saw in Theorem 3.1.1 that Ω admits an L_p-Hardy inequality. However, $0 = \dim_{\mathcal{H}}(\partial\Omega) < m-p$, so that by lemma 3.4.6 $\mathbb{R}^m \setminus \Omega$ is not uniformly p-fat. Consequently, Theorem 3.4.5 implies that Ω cannot admit a pointwise q-Hardy inequality for any $q \in (1, p)$.

In the 'borderline' case where $m = p$, there is an additional condition that turns out to be equivalent to the the uniform m-fatness of the complement of a domain. However, in order to understand this condition we must first define the concept of the uniform perfectness of a set.

Definition 3.4.3 *A set* $E \subsetneq \mathbb{R}^m$ *is said to be uniformly perfect if* E *is not a singleton set and there exists a finite uniform constant* $C > 1$ *so that for all* $x \in E$ *and all* $r \in (0, \infty)$

$$E \cap \big[B(x, Cr) \setminus B(x, r) \big] \neq \emptyset$$

whenever $E \setminus B(x, Cr) \neq \emptyset$.

As such, a set $E \subseteq \mathbb{R}^m$ is uniformly perfect if for each $x \in E$, every annular neighborhood of x with fixed modulus contains at least one other point belonging to E. In [49], Korte & Shanmugalingam demonstrate that in the borderline case the uniform m-fatness of the complement of a domain is equivalent to it's uniform perfectness and unboundedness.

Theorem 3.4.7 *[49, Lemma 3.2, Theorem 3.6], [48, Theorem 4.1]*

Let Ω *be a domain in* $\mathbb{R}^m$ *where* $m \geq 2$. *Then the following are equivalent:*

a) $\mathbb{R}^m \setminus \Omega$ *is uniformly* m*-fat.*

b) Ω *admits a pointwise* q*-Hardy inequality for some* $q \in (1, m)$.

c) Ω *admits an* L_m*-Hardy inequality.*

d) $\mathbb{R}^m \setminus \Omega$ *is uniformly perfect and unbounded.*

From our perspective, there are two remarks worth making in connection with Theorem 3.4.7. First of all, the theorem implies that, in the borderline case, the existence of a pointwise q-Hardy inequality, for some $q \in (1, m)$, and the existence of an L_m-Hardy inequality are equivalent. Secondly, if Ω is a domain in $\mathbb{R}^m$, then Theorem 3.4.7 provides us with a very simple sufficient condition for the existence of an L_m-Hardy inequality. It follows immediately from Definition 3.4.3 that any connected set is uniformly perfect. Simply combining this fact with Theorem 3.4.7 leads us to the following result.

Theorem 3.4.8 *Let* Ω *be a domain in* $\mathbb{R}^m$ *where* $m \geq 2$. *If* $\mathbb{R}^m \setminus \Omega$ *is connected and unbounded, then* Ω *admits an* L_m*-Hardy inequality.*

In particular, if Ω is a domain in $\mathbb{R}^2$ and if $\mathbb{R}^2 \setminus \Omega$ is connected and unbounded, then Ω admits an L_2-Hardy inequality. For example, if we let $\Omega \subseteq \mathbb{R}^2$ be the interior of a von Koch snowflake, or an appropriately constructed 'room and corridor' type domain, then

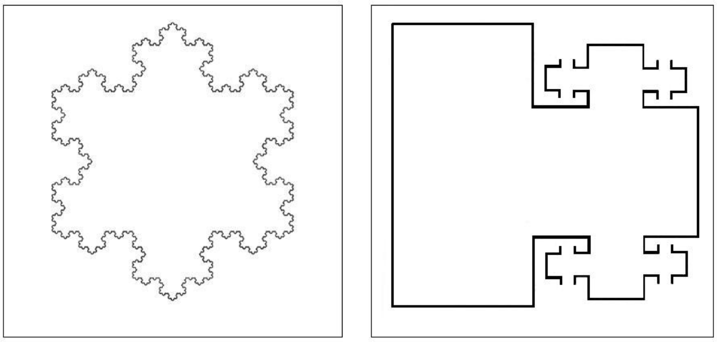

Figure 3.2: If $\Omega \subseteq \mathbb{R}^2$ is the interior of a von Koch snowflake, then Ω must admit an L_2-Hardy inequality because the complement of the domain is connected and unbounded. The same is true of an appropriately constructed 'room and corridor' type domain in $\mathbb{R}^2$.

Ω clearly admits an L_2-Hardy inequality because the complement of the domain is both connected and unbounded.

However, it is also instructive to consider examples of domains in $\mathbb{R}^m$ which **do not** admit an L_m-Hardy inequality. According to Theorem 3.4.7, such a domain can fail to admit an L_m-Hardy inequality in two ways; either if the complement of the domain is bounded or if the complement of the domain is not uniformly perfect. Indeed, let us consider the domain $\Omega = \mathbb{R}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is the usual middle thirds Cantor set. In this case Ω cannot admit an L_2-Hardy inequality because the complement of the domain, i.e. the Cantor set $\mathcal{C}$, is bounded. This example is interesting because $\dim_{\mathcal{H}}(\partial\Omega) = \dim_{\mathcal{H}}(\mathcal{C}) = \frac{\log 2}{\log 3} > 0$, illustrating the fact that the Hausdorff dimension of the boundary exceeding $m - p$ is **not** a sufficient condition to ensure the existence of an L_p-Hardy inequality, at least in the borderline case.

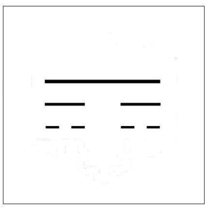

Figure 3.3: The domain $\mathbb{R}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is the usual middle thirds Cantor set, does not admit an L_2-Hardy inequality, because the complement of the domain is bounded.

3.5 Domains with Thin Boundary

As we stated in Appendix B, Riesz p-capacity is a measure of a set's ability to hold charge whilst maintaining a given potential energy. Indeed, given that Riesz p-capacity is monotonic, so that if $E_1 \subseteq E_2 \subseteq \mathbb{R}^m$, then $R_p(E_1) \leq R_p(E_2)$, one may interpret Riesz p-capacity as giving an indication of a set's size or 'thickness'. Furthermore, Riesz p-capacity is countably additive, i.e. if $E = \cup_k E_k$, then

$$R_p(E) \quad \leq \quad \sum_k R_p(E_k). \tag{3.6}$$

Under certain conditions a type of converse to equation (3.6) can be seen to hold. Suppose that Ω is an open set and that $\mathcal{W} = \{Q_j\}_{j=1}^{\infty}$ is a Whitney decomposition of Ω (see Lemma 4.1.1). We say that Riesz p-capacity is **quasiadditive** if there exists a finite uniform constant $C > 0$ so that the inequality

$$R_p(E) \quad \geq \quad C \sum_k R_p(\, E \cap Q_k \,), \tag{3.7}$$

holds for all sets $E \subseteq \mathbb{R}^m$.

In [60], and Chapter 7 of [61], Aikawa investigates the conditions required for Riesz p-capacity to admit the quasiadditivity property. In doing so, the validity of the following theorem is demonstrated.

Theorem 3.5.1 *[60, Theorem 2], [61, Theorem 7.1.2]*

Let F be a closed set having no interior points and let $1 < p < \infty$. If $\dim_A(F) < m - p$, then there exists a finite uniform constant $C > 0$ so that the inequality

$$\int_E \frac{1}{d(x,F)^p}\, dx \qquad \leq \qquad C\, R_p(E),$$

holds for all (Lebesgue) measurable sets $E \subseteq \mathbb{R}^m$.

It appears that Koskela & Zhong [51] were the first to realize the importance of this result from the perspective of the L_p-Hardy inequality. Assuming that Ω is a domain in $\mathbb{R}^m$, one may take $F = \partial\Omega$ and E to be a compact set $K \subseteq \Omega$, so that if $\dim_A(\partial\Omega) < m - p$ we immediately obtain that

$$\int_K \frac{1}{d(x)^p}\, dx \quad \leq \quad C\, R_p(K) \quad \leq \quad C\, C_p(K, \Omega). \tag{3.8}$$

Recall that the inequality on the right hand side of equation (3.8) holds due to the comparability of Riesz p-capacity and variational p-capacity (see Lemma B.2.1). By Theorem 3.2.1, the validity of equation (3.8) is sufficient to demonstrate the existence of an L_p-Hardy inequality on Ω. Consequently, we arrive at the following theorem, which was first stated by Lehrbäck in [52] following on from the remarks concluding [51].

Theorem 3.5.2 *[52, Theorem 1.2]*
If Ω is a domain in $\mathbb{R}^m$ so that $\dim_A(\partial\Omega) < m - p$, then Ω admits an L_p-Hardy inequality.

In other words, the above theorem asserts that if the boundary of a domain is sufficiently thin, then that domain admits an L_p-Hardy inequality. The proof of the theorem, as given in [60] and [61], is highly non-trivial, and draws on many different results from many different areas of analysis. We remark only that it would be extremely useful to produce a simplified proof of the theorem. Nevertheless, very recently Lehrbäck and Shanmugalingam [62] have extended the result to general metric spaces admitting a certain structure, under the additional assumption that the domain satisfies a discrete John condition. Indeed, they also show that the quasiadditivity of variational p-capacity is equivalent to the existence of an L_p-Hardy inequality via Maz'ya's characterization.

3.6 The Geometric Condition of Barbatis, Filippas & Tertikas

In the previous two sections we essentially showed that if the boundary of a domain is sufficiently fat, or sufficiently thin, then that domain admits an L_p-Hardy inequality. More specifically, we showed that for a domain $\Omega \subseteq \mathbb{R}^m$, if $\mathbb{R}^m \setminus \Omega$ is uniformly p-fat, or if $\dim_A(\partial\Omega) < m - p$, then Ω admits an L_p-Hardy inequality. Yet in a sense, these results are not explicit because they give us no information on the value of the variational constant $\mu_p(\Omega)$ (other than the fact that under these circumstances $\mu_p(\Omega) > 0$). Recall that we are primarily interested in determining the value of $\mu_p(\Omega)$, due to the central role it plays with respect to the essential self-adjointness of Schrödinger operators.

However, we do have one explicit result - Theorem 3.1.1. There it was shown that the domain $\mathbb{R}^m \setminus \{0\}$ admits an L_p-Hardy inequality and that $\mu_p(\Omega) \geq \left| \frac{m-p}{p} \right|^p$. Given the nature of the proof of Theorem 3.1.1, one may ask whether it is possible to extend this method to more general domains simply by replacing the distance to the origin with the distance to the boundary. The next theorem gives an affirmative answer to this question, provided that the domain admits a certain geometric condition. The theorem is effectively a simplified result from [34] and [35]. There Barbatis, Filippas & Tertikas determine the conditions required for a domain to admit an 'improved' L_p-Hardy inequality (which one may think of as the usual L_p-Hardy inequality with logarithmic correction terms subtracted from the right hand side). Although the result below is not new, the method of proof is original.

Theorem 3.6.1 *Let Ω be a domain in $\mathbb{R}^m$ with non-empty boundary. For $\alpha \geq 0$, define the functional $A_\alpha : \hat{C}_0^1(\Omega) \to \mathbb{C}$, where $\hat{C}_0^1(\Omega)$ denotes the set of continuous compactly supported functions that are differentiable almost everywhere on Ω, by*

$$A_\alpha[\,\phi\,] \quad = \quad \int_\Omega d(x)\, \nabla d(x) \cdot \nabla \phi(x) \quad + \quad \alpha\, \phi(x) \;\; dx.$$

If for all non-negative functions $\phi \in \hat{C}_0^1(\Omega)$ *we have that*

$$(\alpha - p)\ A_\alpha[\phi] \quad \leq \quad 0, \tag{3.9}$$

then the inequality

$$\int_\Omega \frac{|\,\omega(x)\,|^p}{d(x)^p}\ dx \quad \leq \quad \left| \frac{p}{\alpha - p} \right|^p \int_\Omega |\,\nabla\omega(x)\,|^p\ dx$$

holds for all $\omega(x) \in W^1_{p,0}(\Omega)$, *so that* $\mu_p(\Omega) \geq |\,\frac{\alpha - p}{p}\,|^p$.

Proof. First of all, if $\alpha = p$ then the result is a triviality. So let us assume that $\alpha \neq p$. If $p \neq 2$ let $V(x) = \frac{d(x)^{-p+2}}{(\alpha-p)\,(p-2)}$ and if $p = 2$ let $V(x) = -\frac{\ln(\,d(x)\,)}{\alpha-2}$. It is not difficult to show that in either case we have $\nabla\, V(x) = -\frac{d(x)^{-p+1}\,\nabla\, d(x)}{\alpha - p}$. Now for all non-negative $\phi \in \hat{C}_0^1(\Omega)$ we have that $(\alpha - p)\, A_\alpha[\phi] \leq 0$. Therefore,

$$(\alpha - p)\ \alpha \int_\Omega \phi(x)\ dx \qquad \leq \qquad -\,(\alpha - p) \int_\Omega d(x)\ \nabla d(x)\ \cdot\ \nabla\phi(x)\ dx$$

so that dividing through by $(\alpha - p)$ we obtain

$$\alpha \int_\Omega \phi(x)\ dx \qquad \lesseqgtr \qquad - \int_\Omega d(x)\ \nabla d(x)\ \cdot\ \nabla\phi(x)\ dx.$$

Here the upper inequality refers to the case where $\alpha > p$ and the lower inequality refers to the case where $\alpha < p$. Subtracting $p\ (\int_\Omega \phi(x)\ dx\,)$ from both sides of the previous equation yields

$$(\alpha - p) \int_\Omega \phi(x)\ dx \qquad \lesseqgtr \qquad - \int_\Omega d(x)\ \nabla d(x)\ \cdot\ \nabla\phi(x)\ dx \quad - \quad p \int_\Omega \phi(x)\ dx,$$

so that dividing through by $(\alpha - p)$ once more we arrive at the inequality

$$\int_\Omega \phi(x)\ dx \quad \leq \quad -\frac{1}{\alpha - p} \int_\Omega d(x)\ \nabla d(x) \cdot \nabla\phi(x)\ dx \quad - \quad \frac{p}{\alpha - p} \int_\Omega \phi(x)\ dx. \tag{3.10}$$

If $u(x) \in C_0^\infty(\Omega)$, then the non-negative function $\frac{|\,u(x)\,|^p}{d(x)^p}$ is contained in $\hat{C}_0^1(\Omega)$. As such, equation (3.10) asserts that

$$\int_\Omega \frac{|\,u(x)\,|^p}{d(x)^p}$$

$$\leq \quad -\frac{1}{\alpha - p} \int_\Omega d(x)\ \nabla d(x)\ \cdot\ \nabla\left(\frac{|\,u(x)\,|^p}{d(x)^p}\right)\ dx \quad - \quad \frac{p}{\alpha - p} \int_\Omega \frac{|\,u(x)\,|^p}{d(x)^p}\ dx$$

$$
\begin{aligned}
&= -\frac{p}{\alpha-p}\int_\Omega d(x)^{-p+1}\ \nabla d(x)\ |\,u(x)\,|^{p-1}\ \nabla\,|\,u(x)\,|\ \ dx \\
&\quad + \frac{p}{\alpha-p}\int_\Omega \frac{|\,u(x)\,|^p}{d(x)^p}\ |\,\nabla\,d(x)\,|^2\ dx \quad - \quad \frac{p}{\alpha-p}\int_\Omega \frac{|\,u(x)\,|^p}{d(x)^p}\ dx \\
&\le P\int_\Omega |\,u(x)\,|^{p-1}\ |\,\nabla\,V(x)\,|\ |\,\nabla\,u(x)\,|\ \ dx \\
&\le P\left(\int_\Omega |\,u(x)\,|^p\ |\,\nabla\,V(x)\,|^{\frac{p}{p-1}}\ dx\right)^{\frac{p-1}{p}}\ \left(\int_\Omega |\,\nabla\,u(x)\,|^p\ dx\right)^{\frac{1}{p}} \\
&= \frac{p}{|\,\alpha-p\,|}\left(\int_\Omega \frac{|\,u(x)\,|^p}{d(x)^p}\ dx\right)^{\frac{p-1}{p}}\ \left(\int_\Omega |\,\nabla\,u(x)\,|^p\ dx\right)^{\frac{1}{p}}.
\end{aligned}
$$

We may simply divide this last expression through by $\left(\int_\Omega \frac{|\,u(x)\,|^p}{d(x)^p}\ dx\right)^{\frac{p-1}{p}}$, and raise both sides of the resulting equation to the power of p, to obtain that for all $u \in C_0^\infty(\Omega)$

$$
\int_\Omega \frac{|\,u(x)\,|^p}{d(x)^p}\ dx \quad \le \quad \left|\frac{p}{\alpha-p}\right|^p \int_\Omega |\,\nabla\,u(x)\,|^p\ dx.
$$

The density of $C_0^\infty(\Omega)$ in $W_{p,0}^1(\Omega)$ now implies the result. ∎

Given a domain $\Omega \subseteq \mathbb{R}^m$, the above theorem states that if we can find a value of α so that equation (3.9) is satisfied, then this automatically provides us with an explicit lower bound for $\mu_p(\Omega)$. Therefore, a natural question arises - what sort of domains admit the geometric condition described by equation (3.9)? The next two lemmas, which are taken directly from Section 2 of [34], provide the only real known examples of domains admitting the aforementioned condition.

Lemma 3.6.2 *Let E be an affine set in $\mathbb{R}^m$ of dimension k where $0 \le k \le m-1$. Let $\Omega = \mathbb{R}^m \setminus E$ if $m \ge 2$ and $\Omega = (0,\infty)$ if $m = 1$. Then the estimate*

$$
\int_\Omega \frac{|\,\omega(x)\,|^p}{d(x)^p}\ dx \qquad \le \qquad \left|\frac{p}{m-k-p}\right|^p \int_\Omega |\,\nabla\omega(x)\,|^p\ dx
$$

holds for all $\omega(x) \in W_{p,0}^1(\Omega)$ and $p > 1$, so that $\mu_p(\Omega) \ \ge\ \left|\,\frac{m-k-p}{p}\,\right|^p$.

Proof. If $k = m-p$ then the result is trivial. As such we will assume that $k \ne m-p$. We claim that $[\ (m-k)\ -\ p\ ]\ A_{m-k}\,[\,\phi\,]\ =\ 0$ for all non-negative functions in $\hat{C}_0^1(\Omega)$. Indeed, by changing co-ordinates if necessary, we can always assume that $d(x) = (\ x_{k+1}^2\ +\ \ldots\ +\ x_m^2\ )^{\frac{1}{2}}$. Therefore,

$$
\begin{aligned}
A_{m-k}[\,\phi\,] \;&=\; \int_\Omega d(x)\,\nabla d(x)\cdot\nabla\phi(x) \;+\; (\,m-k\,)\;\phi(x)\;dx \\
&=\; \int_\Omega \frac{(\,x_{k+1}^2+\ldots+x_m^2\,)^{\frac12}}{(\,x_{k+1}^2+\ldots+x_m^2\,)^{\frac12}}\;(0,\,\ldots\,,0,\,x_{k+1}\,,\,\ldots\,,\,x_m)\cdot\nabla\phi(x)\;dx \\
&\quad + (\,m-k\,)\int_\Omega \phi(x)\;dx \\
&=\; -\,(\,m-k\,)\int_\Omega \phi(x)\;dx \;+\; (\,m-k\,)\int_\Omega \phi(x)\;dx
\end{aligned}
$$

where in the last step we have applied integration by parts. Since the claim has been proven, Theorem 3.6.1 implies that Ω must admit an L_p-Hardy inequality and that $\mu_p(\Omega)\geq\left|\,\frac{m-k-p}{p}\,\right|^p$. ∎

We note that in terms of the domain described above, the lower bound produced for $\mu_p(\Omega)$ has a strong dependence on the dimension of the boundary. Furthermore, we also note that method worked both in the case where the boundary is fat (i.e. when $k>m-p$) and the case where the boundary is thin (i.e. when $k<m-p$). The dependence of $\mu_p(\Omega)$ on the dimension of the boundary is also reflected in the following result.

Lemma 3.6.3 *Let Ω be a bounded, convex domain in $\mathbb{R}^m$ with smooth boundary of co-dimension 1. Then the estimate*

$$
\int_\Omega \frac{|\,\omega(x)\,|^p}{d(x)^p}\;dx \qquad\leq\qquad \left|\frac{p}{1-p}\right|^p \int_\Omega |\,\nabla\omega(x)\,|^p\;dx
$$

holds for all $\omega\in W^1_{p,0}(\Omega)$ and $p>1$, so that $\mu_p(\Omega)\geq\left|\,\frac{1-p}{p}\,\right|^p$.

Proof. We claim that if Ω is a bounded, convex domain in $\mathbb{R}^m$ with smooth boundary of co-dimension 1, then for all non-negative $\phi\in\hat{C}^1_0(\Omega)$ we have

$$
\big[\,m-(m-1)-p\,\big]\;A_{m-(m-1)}[\,\phi\,] \quad=\quad (\,1-p\,)\;A_1[\,\phi\,]\;\leq\;0.
$$

Given that $p>1$, in order to prove the claim it suffices to show that $A_1[\,\phi\,]\;\geq\;0$. To demonstrate this, we begin by showing that $d(x)$ is a concave function on Ω, i.e. that for all $x,y\in\Omega$ and for all $\lambda\in[0,1]$ we have that $d\big(\,\lambda x+(1-\lambda)y\,\big)\;\geq\;\lambda\,d(x)+(1-\lambda)\,d(y)$. Indeed, let us choose any $x,y\in\Omega$ and let $z=\lambda x+(1-\lambda)y$ for some $\lambda\in(0,1)$. Further, let $z_0\in\partial\Omega$ be a point that realizes the distance of z from the boundary, that is $d(z)=|z-z_0|$. Denote by T_{z_0} the hyperplane containing z_0 which is orthogonal to the vector $z - z_0$ and let x_0 , y_0 be the orthogonal projections of x and y onto T_{z_0} respectively. Since Ω is convex a simple similarity argument gives

$$d(z) \;=\; |z - z_0| \;=\; \lambda\,|x - x_0| \;+\; (1-\lambda)\,|y - y_0| \;\geq\; \lambda d(x) \;+\; (1-\lambda)\,d(y)$$

so that $d(x)$ is indeed concave. Hence, from Theorem 6.3.2 of [18] it follows that there exists a unique non-negative Radon measure μ such that for all $\Psi \in \hat{C}_0^1(\Omega)$

$$\int_\Omega \nabla d(x) \cdot \nabla\Psi(x)\;\; dx \;\;=\;\; \int_\Omega \Psi(x)\;\; d\mu.$$

In particular, letting $\Psi(x) = d(x)\,\phi(x)$ where $\phi(x) \in \hat{C}_0^1(\Omega)$ is non-negative, we obtain that

$$\begin{aligned} 0 \;\;\leq\;\; & \int_\Omega d(x)\;\phi(x)\;\; d\mu \;\;=\;\; \int_\Omega \nabla\, d(x)\;\cdot\;\nabla\left(\; d(x)\;\phi(x)\;\right)\;\; dx \\ =\;\; & \int_\Omega d(x)\;\nabla\, d(x)\;\cdot\;\nabla\,\phi(x) \;\;\;+\;\;\; \phi(x)\;\; dx \\ =\;\; & A_1[\,\phi\,] \end{aligned}$$

and the claim is proven. The result of Theorem 3.6.1 now implies that Ω admits an L_p-Hardy inequality and gives the lower bound $\mu_p(\Omega) \geq \left|\,\frac{m-(m-1)-p}{p}\,\right|^p = \left|\,\frac{1-p}{p}\,\right|^p$. ∎

In fact, in Lemma 5.2.2 and Lemma 5.2.3 we will show that the lower bounds for $\mu_p(\Omega)$ obtained in the two preceding examples actually hold with equality. This suggest that if Ω is a domain in $\mathbb{R}^m$, with Ahlfors λ-regular[3] boundary of dimension λ, then

$$\mu_p(\Omega) \quad = \quad \left|\frac{m-\lambda-p}{p}\right|^p. \tag{3.11}$$

Let us take a moment to reflect on this last assertion. First, supposing that equation (3.11) holds neatly explains why a domain $\Omega \subseteq \mathbb{R}^m$ which admits an L_p-Hardy inequality cannot have an $m-p$ dimensional boundary. Indeed, if $\lambda = m - p$, then equation (3.11) would imply that $\mu_p(\Omega) = 0$, so that, by the remarks immediately after Definition 1.5.3, Ω could not admit an L_p-Hardy inequality.

Secondly, this assertion would also explain the result of Theorem 3.3.1. That is to say that the expression on the right hand side of equation (3.11) is continuous in p, so that if $\mu_p(\Omega) > 0$ for some value of p, then $\mu_q(\Omega)$ must be greater than zero for all q sufficiently close to p. In other words, if Ω admits an L_p-Hardy inequality, then there must exist positive $\epsilon = \epsilon(p,\Omega)$ so that Ω admits an L_q-Hardy inequality for all $q \in (p-\epsilon, p+\epsilon)$.

[3] Recall that by Theorem A.5.1, the condition that the boundary is Ahlfors λ-regular ensures that the Hausdorff and Aikawa dimension of the boundary co-inside.

In a similar vein of thought, the validity of equation (3.11) would also provide a simple illustration of why a domain whose complement is uniformly p_0-fat admits an L_p-Hardy inequality for all $p \geq p_0$ (see the remarks following Theorem 3.4.6). That is to say that if $\mathbb{R}^m \setminus \Omega$ is uniformly p_0-fat, then, by Theorem 3.4.6, Ω admits an L_{p_0}-Hardy inequality and $m - \lambda - p_0 < 0$. Hence, equation (3.11) would imply that $\mu_p(\Omega) > 0$ for all $p \geq p_0$, so that Ω must also admit an L_p-Hardy inequality.

Unfortunately, the situation is not quite that simple, at least in the borderline case where $m = p$. At the end of Section 3.4, we considered the domain $\mathbb{R}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is the usual middle thirds Cantor set. Here $\lambda = \dim_{\mathcal{H}}(\partial\Omega) = \dim_{\mathcal{H}}(\mathcal{C}) = \frac{\log 2}{\log 3} \approx 0.631.$[4] On the one hand, if equation (3.11) holds, then $\mu_2(\Omega) = \frac{(\log 2/\log 3)^2}{4} > 0$, so that Ω would admit an L_2-Hardy inequality. On the other hand, the complement of this domain is bounded, so that by Theorem 3.4.7, Ω cannot admit an L_2-Hardy inequality. We can only conclude that equation (3.11) cannot be true in this case.

Nevertheless, determining the conditions under which equation (3.11) does hold would be extremely valuable. Indeed, we take up the investigation of the relationship between $\mu_p(\Omega)$ and the dimension of the boundary in Chapter 5. Before then, we must lay some ground work by investigating the relationship between the integrability of the function $d(x, \partial\Omega)^{-1}$ and the dimension of the boundary of a domain. This is essentially the content of the next chapter of the thesis.

[4]We note that by the remarks at the end of Section A.5, $\mathcal{C}$ is Ahlfors $\frac{\log 2}{\log 3}$-regular.

Chapter 4

Minkowski Dimension, Whitney Decompositions & the Integrability of the Distance Function

Let $\Omega = \mathbb{R}^m \setminus \{0\}$ if $m \geq 2$ and $\Omega = (0, \infty)$ if $m = 1$. We ask for what values of $\alpha > 0$ is the function $\frac{1}{d(x,\partial\Omega)^\alpha} = \frac{1}{|x|^\alpha}$ integrable on a neighborhood of the boundary, i.e. for which values of α is the integral

$$\int_{B(0,\delta)\setminus\{0\}} \frac{1}{|x|^\alpha}\ dx \tag{4.1}$$

finite? The obvious answer to this question is that the integral in (4.1) is convergent if and only if $\alpha < m$. This simple example illustrates a very general principle concerning the integrability of the Euclidean distance function - the thinner the boundary relative to the surrounding space (i.e. the greater the co-dimension of the boundary) the higher the degree of integrability the function $\frac{1}{d(x,\partial\Omega)}$ possesses.

In this chapter we will make this idea more precise by characterizing the integrability of the reciprocal of the distance function in terms of the (inner) Minkowski dimension of the boundary. Our motivation for studying the aforementioned relationship is that estimation of the L_p-Hardy constant boils down to a dichotomy concerning the integrability of the distance function, a fact we shall witness in the subsequent chapter. In order to develop this characterization we will prove some new results linking the (inner) Minkowski dimension of the boundary to the number of cubes appearing in the k^{th} generation of the corresponding Whitney decomposition.

The chapter is structured as follows. In Section 4.1 we discuss Whitney decompositions - partitions of open sets by non-overlapping cubes that get smaller and smaller as one approaches the boundary. We also define a new concept, that of inner γ-domains (see

Definition 4.1.2). Section 4.2 is technical in nature. There we give a Besicovitch / de Guzman type covering lemma demonstrating how it is possible to cover a bounded set by balls of equal radius that do not overlap 'too much'. We then show that if we extend the radii of these balls the overlap is still controlled. Next, in Section 4.3, we use the (inner) Minkowski dimension of the boundary to produce upper and lower bounds on the number of cubes appearing in selected generations of Whitney decompositions. These results enable us to characterize the integrability of the function $\frac{1}{d(x,\partial\Omega)}$ in terms of the dimension of the boundary in Section 4.4. That we may make such a characterization is well known (see for instance [69], [70] or [71]). However, the method of proof we develop to obtain this characterization is original. In Section 4.5, we construct a 'room and corridor' type domain to which our results can be applied. Finally, in Section 4.6, we present some lemmas, associated to the existence of an L_p-Hardy inequality on a domain, that follow easily from the results obtained in Section 4.4.

4.1 Whitney Decompositions & γ-Domains

Suppose that E is a non-empty, closed set in $\mathbb{R}^m$ and suppose that $\Omega = \mathbb{R}^m \setminus E$. In [73] Whitney showed that it is always possible to partition Ω into a countable union of non-overlapping cubes in such a way that the distance of each cube from E is comparable to its diameter. Such a partition is referred to as a Whitney decomposition of Ω. Indeed, Whitney decompositions have proven to be an extremely useful tool in many areas of analysis and geometry. In particular, the use of Whitney decompositions is especially convenient when one attempts to determine whether a function with singularities on a given closed set is integrable or not. We make these ideas precise by means of the following lemma.

Lemma 4.1.1 *[37, Chapter VI, Theorem 1]*
Let E be a non-empty, closed set in $\mathbb{R}^m$ and let $\Omega = \mathbb{R}^m \setminus E$. Then Ω can be decomposed into a collection of cubes $\mathcal{W}$, whose sides are parallel to the co-ordinate axes and which have the following properties.

i) $\Omega = \bigcup_{Q \in \mathcal{W}} Q$.

ii) *If* $Q_i, Q_j \in \mathcal{W}$ *and* $Q_i \neq Q_j$, *then* $Q_i^o \cap Q_j^o = \emptyset$, *i.e. all the cubes in $\mathcal{W}$ are non-overlapping in the sense that their interiors are disjoint.*

iii) $\operatorname{diam}(Q) \leq d(Q, \partial\Omega) \leq 4 \operatorname{diam}(Q)$ *for all* $Q \in \mathcal{W}$.

iv) *If* $Q \in \mathcal{W}$, *then* $\operatorname{diam}(Q) \in \left\{ \frac{\sqrt{m}}{2^k} \;\middle|\; k \in \mathbb{Z} \right\}$ *and* $|\,Q\,| \in \left\{ \frac{1}{2^{mk}} \;\middle|\; k \in \mathbb{Z} \right\}$.

In fact, the constants 1 and 4 appearing in part iii) of the lemma are not essential and can be replaced by arbitrary positive constants C_1 and C_2 (where $C_1 < C_2$) at the expense of changing the diameters and volumes of the cubes formed. Given the nature of the proof of Lemma 4.1.1 it is quite natural to refer to the set of cubes

$$\mathcal{W}_k \quad = \quad \left\{ Q \in \mathcal{W} \;\middle|\; \operatorname{diam}(Q) \;=\; \frac{\sqrt{m}}{2^k} \right\} \quad = \quad \left\{ Q \in \mathcal{W} \;\middle|\; |\,Q\,| \;=\; \frac{1}{2^{mk}} \right\}$$

as the k^{th} generation of cubes in the decomposition $\mathcal{W}$. We will denote an arbitrary cube in $\mathcal{W}_k$ by Q_k, and the integer $N_k = \sharp\,\mathcal{W}_k$ will represent the number of cubes in $\mathcal{W}_k$. It is plain to see that if $x \in Q_k$ then $\frac{\sqrt{m}}{2^k} \leq d(x) \leq \frac{5\sqrt{m}}{2^k}$, so that we have the inclusion

$$\mathcal{W}_k \;\subseteq\; \left\{ x \in \mathbb{R}^m \setminus E \;\middle|\; \frac{\sqrt{m}}{2^k} \;\leq\; d(x) \;\leq\; \frac{5\sqrt{m}}{2^k} \right\}. \tag{4.2}$$

The proof of Whitney's theorem is geometric in nature, and so in attempting to extend the result to general metric spaces one is left uncertain as to how to form such non-overlapping cubes. However, the theorem can be extended to metric spaces with a certain degree of regularity by replacing the idea of non-overlapping cubes with balls of uniformly bounded overlap. For further information on the properties and extensions of Whitney decompositions the reader is directed to Chapter VI of [37] and Chapter 1 of [74].

In order to prove the main result of this chapter we will need to restrict our analysis to domains which admit the following geometric property.

Definition 4.1.1 *Let Ω be a domain in $\mathbb{R}^m$ and let $\gamma > 1$. Ω is said to be a γ-domain if there exists a finite uniform constant $C > 0$ and $\hat{r} > 0$ so that the estimate*

$$|\,(\partial\Omega)_r\,| \quad \leq \quad C\;\;|\,(\partial\Omega)_{\gamma r} \setminus (\partial\Omega)_r\,|$$

holds for all $0 < r < \hat{r}$. We call the value of $\hat{r}$ the upper radial limit of the domain.

As such, $\Omega \subseteq \mathbb{R}^m$ is a γ-domain if, for sufficiently small r, it possible to estimate the volume of an r-neighborhood of the boundary from above by the volume of a γr-neighborhood excluding the volume of the r-neighborhood. We also define the related concept of an **inner** γ-domain.

Definition 4.1.2 *Let Ω be a domain in $\mathbb{R}^m$ and let $\gamma > 1$. Ω is said to be an inner γ-domain if there exists a finite uniform constant $C > 0$ and $\hat{r} > 0$ so that the estimate*

$$|\,(\partial\Omega)_r \cap \Omega\,| \quad \leq \quad C\;\;|\,[\,(\partial\Omega)_{\gamma r} \setminus (\partial\Omega)_r\,] \cap \Omega\,|$$

holds for all $0 < r < \hat{r}$. Again, we call the value of $\hat{r}$ the upper radial limit of the domain.

At this point the obvious question arises as to what sorts of domains are (inner) γ-domains. Whilst a full characterization of γ-domains is beyond the remit of this thesis, preliminary investigation suggests that domains admitting the interior cone condition and John domains are likely candidates for being γ-domains. This is a problem for further research.

4.2 Covering Lemmas

In this section we provide some technical lemmas that will form the topological basis of subsequent results. We begin with the following well known Besicovitch/de Guzman type lemma, which asserts that a bounded set in $\mathbb{R}^m$ can always be covered with a finite number of balls of equal radii in such a way that the intersection multiplicity of the balls is bounded above by a positive constant that depends only on the dimension of the underlying space. The result is effectively taken from chapter 1 of [74].

Lemma 4.2.1 *Let E be a bounded set in $\mathbb{R}^m$ and let $r > 0$. Then there exists a finite number of points $x_1, \ldots, x_N \in E$ and a positive constant $C(m)$, depending only on the dimension m, so that*

$$E \subseteq \bigcup_{j=1}^{N} B(x_j, r) \qquad \text{and} \qquad \sum_{j=1}^{N} \chi_{B(x_j,r)} \leq C(m).$$

The constant $C(m) = 4^m$ will suffice.

Proof. To each $x \in E$ associate a ball $B(x, r)$. We begin by choosing a ball $B(x_1, r)$ for some arbitrary $x_1 \in E$. If $E \setminus B(x_1, r) = \emptyset$ then the selection process is finished. However, if $E \setminus B(x_1, r) \neq \emptyset$ then one chooses a ball $B(x_2, r)$ for some $x_2 \in E \setminus B(x_1, r)$. Let us suppose that k balls have already been chosen in this manner. If $E \setminus \bigcup_{j=1}^{k} B(x_j, r) = \emptyset$ then the selection process stops. On the other hand, if $E \setminus \bigcup_{j=1}^{k} B(x_j, r) \neq \emptyset$ then one chooses a ball $B(x_{k+1}, r)$ where $x_{k+1} \in E \setminus \bigcup_{j=1}^{k} B(x_j, r)$. Since E is bounded, and since each of the balls chosen has the same radius, the process must terminate after N steps. In this way we obtain a collection of N balls with the following properties.

i) $E \subseteq \bigcup_{j=1}^{N} B(x_j, r)$ where $x_j \in E$ for all $j = 1, \ldots, N$.

ii) If $x_i \neq x_j$ then $x_i \notin B(x_j, r)$.

In other words, the center of each ball is not contained in any other ball that covers E. Therefore, for all distinct i, j from 1 to N we have that $|x_i - x_j| \geq r$, or equivalently that $B(x_i, r/2) \cap B(x_j, r/2) = \emptyset$.

Now consider any $z \in \mathbb{R}^m$. Clearly $z \in B(x_j, r)$ if and only if $x_j \in B(z, r)$. We claim that there are at most 4^m of the balls from the given cover of E that have center in $B(z, r)$. To see this, first note that if $x_j \in B(z, r)$ then $B(x_j, r) \subseteq B(z, 2r)$. For if $y \in B(x_j, r)$ then

$$|\, y - z \,| \;\leq\; |\, y - x_j \,| \;+\; |\, x_j - z \,| \;<\; r \;+\; r \;=\; 2r.$$

Therefore, the maximum number of balls from the given cover of E that can have center in $B(z, r)$ is not greater than the maximum number of balls from the given cover that are contained in $B(z,2r)$. This latter quantity is obviously equal to the maximum number of disjoint balls of radius $r/2$ that can be contained in $B(z,2r)$. Evidently, the ball $B(z,2r)$ cannot contain more than 4^m disjoint balls of radius $r/2$ because

$$4^m \; |B(y,r/2)| \quad = \quad 4^m/2^m \; |B(y,r)| \quad = \quad 2^m \; |B(z,r)| \quad = \quad |B(z,2r)|,$$

and so the claim is proven.

Consequently, z can be contained in no more than 4^m of the balls that cover E. Since z was chosen arbitrarily, this implies that $\sum_{j=1}^N \chi_{B(x_j,r)} \leq 4^m$ so that the proof is complete. ∎

The following question arises. What happens to the intersection multiplicity of the balls that cover E if we extend their radii by a factor of, say, $\beta > 1$? In particular, does the intersection multiplicity remain bounded above by a positive constant that depends only on the dimension of the underlying Euclidean space? The following lemma gives an affirmative answer to this question.

Lemma 4.2.2 *Let E be a bounded set in $\mathbb{R}^m$, $r > 0$ and $\beta > 1$. Then there exists a finite number of points $x_1, \dots, x_N \in E$ and a positive constant $C(m)$, depending only on the dimension m, so that $E \subseteq \bigcup_{j=1}^N B(x_j,r)$, $\sum_{j=1}^N \chi_{B(x_j,r)} \leq C(m)$ and*

$$\sum_{j=1}^N \chi_{B(x_j,\,\beta r)} \quad \leq \quad (1+\beta)^m \, C(m).$$

Proof. In light of the previous lemma, we know that there exists a finite collection of balls centered in E such that $E \subseteq \bigcup_{j=1}^N B(x_j,r)$ and $\sum_{j=1}^N \chi_{B(x_j,r)} \leq C(m)$. Consequently, all we must prove is that if we extend the radii of these balls by a factor of β, then their intersection multiplicity is bounded above in the specified way.

Let $\mathcal{A} = \{ B(x_j,r) \}_{j=1}^N$ be the original collection of balls that cover E, and let $\mathcal{B} = \{ B(x_j,\beta r) \}_{j=1}^N$ be the collection of balls with the same centers but whose radii are expanded by a factor of β. Consider arbitrary $z \in \mathbb{R}^m$. Then z is contained in the extended ball $B(x_j,\beta r)$ if and only if $x_j \in B(z,\beta r)$.

We claim that there are at most $(1+\beta)^m \, C(m)$ balls in $\mathcal{A}$ whose centers lie in $B(z,\beta r)$. To see this, let $\mathcal{C} = \{ B(x_j,r) \mid B(x_j,r) \in A,\; x_j \in B(z,\beta r) \}$. Since there are only N balls in $\mathcal{A}$ then there can only be a finite number of balls, say K, in $\mathcal{C}$. Now any ball in $\mathcal{C}$ is entirely contained in $B\big(z,(1+\beta)r\big)$. For if $B(x_j,r) \in \mathcal{C}$ and $y \in B(x_j,r)$ then

$$|y-z| \quad \leq \quad |y-x_j| \; + \; |x_j-z| \quad < \quad r \; + \; \beta r \quad = \quad (1+\beta)r.$$

Furthermore, since $\sum_{j=1}^N \chi_{B(x_j,r)} \leq C(m)$ we have that

$$K \; |B(x_j,r)| \; \leq \; C(m) \; |B\big(z,(1+\beta)r\big)|$$

and so it easily follows that $K \leq (1+\beta)^m C(m)$, which proves the claim.

As such, we arrive at the conclusion that z can only be contained in at most $(1+\beta)^m C(m)$ of the extended balls and, since z was chosen arbitrarily from $\mathbb{R}^m$, the proof is complete. ∎

4.3 Minkowski Dimension & Whitney Decompositions

Given a closed set E in $\mathbb{R}^m$, Lemma 4.1.1 tells us that we can always decompose $\Omega = \mathbb{R}^m \setminus E$ into a collection of non-overlapping cubes whose diameter is comparable to their distance from E. At first glance, one may be tempted to think that the partition is entirely independent of the nature of the set E. However, this is certainly not the case. The dimension of the set E will determine the number of cubes occurring in the k^{th} generation of the decomposition. Very loosely speaking, the bigger the dimension of the set E the greater the volume of the region

$$\left\{ x \in \Omega \;\middle|\; \frac{\sqrt{m}}{2^k} \leq d(x) \leq \frac{5\sqrt{m}}{2^k} \right\}$$

so that the more cubes in $\mathcal{W}_k$ are required to fill this region. This direct relationship between the dimension of the closed set and the number of cubes appearing in the k^{th} generation of the Whitney decomposition is born out by the next two results.

Indeed, the following lemma was originally proven by Martio & Vuorinen in [67] before being stated in it's present context by Edmunds & Evans [44]. We give the proof of this lemma since it is essential to our subsequent analysis.

Lemma 4.3.1 *[44, Lemma 4.3.7]*
Let Ω be a domain in $\mathbb{R}^m$ with non-empty, compact boundary. Let $\mathcal{W}$ be a Whitney decomposition of Ω and $N_k = \sharp\mathcal{W}_k$. If

$$\hat{M}_u^\lambda(\partial\Omega) = \limsup_{r\to 0^+} \frac{|(\partial\Omega)_r \cap \Omega|}{r^{m-\lambda}} < \infty,$$

then there exists $K_1 > 0$ and $k_1 \in \mathbb{N}$ so that for all integers $k \geq k_1$,

$$N_k \leq K_1 2^{\lambda k}. \tag{4.3}$$

Proof. Since $\hat{M}_u^\lambda(\partial\Omega) = \limsup_{r\to 0^+} \frac{|(\partial\Omega)_r \cap \Omega|}{r^{m-\lambda}} < \infty$ there must exists some $\kappa > 0$ and $r_0 > 0$ so that

$$\left| \left(\partial\Omega\right)_r \cap \Omega \right| \leq \kappa\, r^{m-\lambda} \qquad \text{for all} \qquad 0 < r \leq r_0.$$

Set $k_1 = \left\lceil \frac{\ln(12\sqrt{m}/r_0)}{\ln 2} \right\rceil$ and consider any integer $k \geq k_1$. We will prove the desired inequality for this value of k. To do so set $r = \frac{6\sqrt{m}}{2^k}$ such that $2r \leq r_0$. Since $\partial\Omega$ is compact, by Lemma 4.2.2, there exists a finite number of points $x_1, \ldots, x_N \in \partial\Omega$ and a positive constant $C(m)$, depending only on the dimension m, so that

i) $\partial\Omega \subseteq \bigcup_{j=1}^{N} B(x_j, r)$,

ii) $\sum_{j=1}^{N} \chi_{B(x_j,r)} \leq C(m)$,

iii) $\sum_{j=1}^{N} \chi_{B(x_j,2r)} \leq 3^m C(m)$.

We claim that every $Q_k \in \mathcal{W}_k$ is contained in at least one of the balls $B(x_j, 2r)$. To see that this is true let $x, z \in Q_k$ and let $y \in \partial\Omega$ such that $d(x) = |x-y|$. Then since $\partial\Omega \subseteq \bigcup_{j=1}^{N} B(x_j, r)$ it must be the case that $y \in B(x_j, r)$ for some $j = 1, \ldots, N$. Therefore, using Lemma 4.1.1 and the remarks thereafter, it must be the case that

$$\begin{aligned} |z - x_j| &\leq |z - x| + |x - y| + |y - x_j| \\ &< \frac{\sqrt{m}}{2^k} + \frac{5\sqrt{m}}{2^k} + \frac{6\sqrt{m}}{2^k} = \frac{12\sqrt{m}}{2^k} = 2r \end{aligned}$$

and the claim is proven.

Now let $N_{k,j}$ be the number of cubes belonging to $\mathcal{W}_k$ that are contained in $B(x_j, 2r)$. Since these cubes are non-overlapping and entirely contained in $B(x_j, 2r) \cap \Omega$, it must be the case that $N_{k,j}\,|Q_k| \leq |B(x_j, 2r) \cap \Omega|$. Therefore, recalling that $2r \leq r_0$, we have

$$\begin{aligned} N_k &\leq \sum_{j=1}^{N} N_{k,j} \leq \frac{1}{|Q_k|} \sum_{j=1}^{N} |B(x_j, 2r) \cap \Omega| \leq \frac{3^m C(m)}{|Q_k|} \left| \left(\partial\Omega\right)_{2r} \cap \Omega \right| \\ &\leq 3^m C(m)\, 2^{mk}\, \kappa \left(2r\right)^{m-\lambda} = 3^m C(m)\, 2^{mk}\, \kappa \left(\frac{12\sqrt{m}}{2^k}\right)^{m-\lambda} \\ &= K_1\, 2^{\lambda k} \end{aligned}$$

and so we have arrived at the desired inequality. ∎

By applying the same arguments it can be seen that the following result holds.

Corollary 4.3.1 *Let Ω be a domain in $\mathbb{R}^m$ with non-empty, compact boundary. Let $\mathcal{W}$ be a Whitney decomposition of $\mathbb{R}^m \setminus \partial\Omega$ and $N_k = \sharp\mathcal{W}_k$. If*

$$M_u^{\lambda}(\partial\Omega) = \limsup_{r \to 0^+} \frac{|(\partial\Omega)_r|}{r^{\,m-\lambda}} < \infty,$$

then there exists $K_1 > 0$ *and* $k_1 \in \mathbb{N}$ *so that for all integers* $k \geq k_1$*,*

$$N_k \quad \leq \quad K_1 \, 2^{\lambda k}. \tag{4.4}$$

We have obtained a result bounding the number of cubes appearing in the k^{th} generation of a Whitney decomposition from above. In order to characterize the integrability of the function $\frac{1}{d(x,\partial\Omega)}$ in terms of the Minkowski dimension of the boundary, we will need a result bounding the number of such cubes from below.

Lemma 4.3.2 *Let* Ω *be an inner* γ*-domain in* $\mathbb{R}^m$ *with non-empty, compact boundary. Let* $\mathcal{W}$ *be a Whitney decomposition of* Ω *and* $N_k = \sharp\mathcal{W}_k$*. If*

$$\hat{M}_l^{\lambda}(\partial\Omega) \quad = \quad \liminf_{r\to 0^+} \; \frac{|(\partial\Omega)_r \, \cap \, \Omega\,|}{r^{\,m-\lambda}} \quad > \quad 0,$$

then there exists $K_2 > 0$ *and* $k_2 \in \mathbb{N}$ *so that for all integers* $k \geq k_2$*,*

$$\sum_{j\,=\,k-n(\gamma)}^{k+2} N_j \quad \geq \quad K_2 \, 2^{\lambda k}. \tag{4.5}$$

Here $n(\gamma)$ *is a non-negative integer that varies proportionally with* γ*.*

Proof. Since $\hat{M}_l^{\lambda}(\partial\Omega) \;=\; \liminf_{r\to 0^+} \frac{|(\partial\Omega)_r \,\cap\,\Omega\,|}{r^{\,m-\lambda}} \;>\; 0$ there exists $\kappa > 0$ and $r_0 > 0$ so that

$$|\,\big(\partial\Omega\big)_r \,\cap\, \Omega\,| \quad \geq \quad \kappa r^{m-\lambda} \qquad \text{for all} \qquad r \,\leq\, r_0.$$

Without loss of generality we may assume that r_0 is less than the upper radial limit of the domain. Set $k_2 \;=\; \left\lceil \frac{\ln(\sqrt{m}\,/r_0)}{\ln 2} \right\rceil$ and consider any integer $k \geq k_2$. We will prove the desired inequality is true for the chosen value of k. First, set $r \,=\, \frac{\sqrt{m}}{2^k}$ so that $r \leq r_0$. Since Ω is an inner γ-domain, and since $r \leq r_0$ is less than the upper radial limit of Ω, we have that

$$|\,(\partial\Omega)_r \,\cap\, \Omega\,| \quad \leq \quad C \;\; \big|\,[\,(\partial\Omega)_{\gamma r} \setminus (\partial\Omega)_r\,] \,\cap\, \Omega\,\big|. \tag{4.6}$$

Now, from equation (4.2) it is obvious that we have the inclusion

$$[\,(\partial\Omega)_{\gamma r} \setminus (\partial\Omega)_r\,] \,\cap\, \Omega \quad \subseteq \quad \bigcup_{j=k-n(\gamma)}^{k+2} \mathcal{W}_j,$$

where $n(\gamma)$ is an appropriately chosen non-negative integer. The bigger γ is the bigger the value of $n(\gamma)$ because more generations of the decomposition are needed to obtain the desired inclusion. Continuing from equation (4.6)

$$
\begin{aligned}
|\,(\partial\Omega)_r \cap \Omega\,| \;&\leq\; C \left| \bigcup_{j=k-n(\gamma)}^{k+2} \mathcal{W}_j \right| \;\leq\; C \sum_{j=k-n(\gamma)}^{k+2} N_j\,|\,Q_j\,| \\
&\leq\; C\, \frac{1}{2^{\,m(k-n(\gamma))}} \sum_{j=k-n(\gamma)}^{k+2} N_j
\end{aligned}
$$

and since $r \leq r_0$ we have

$$
\begin{aligned}
\sum_{j=k-n(\gamma)}^{k+2} N_j \;&\geq\; C\,2^{\,mk}\,|\,(\partial\Omega)_r \cap \Omega\,| \;\geq\; C\,2^{\,mk}\,\kappa\, r^{m-\lambda} \\
&=\; C\,2^{\,mk}\,\kappa \left(\frac{\sqrt{m}}{2^k}\right)^{m-\lambda} \;\geq\; K_2\,2^{\,\lambda k}.
\end{aligned}
$$

This completes the proof. ∎

Once more, a similar argument shows the validity of the next result.

Corollary 4.3.2 *Let Ω be a γ-domain in $\mathbb{R}^m$ with non-empty, compact boundary. Let $\mathcal{W}$ be a Whitney decomposition of $\mathbb{R}^m \setminus \partial\Omega$ and $N_k = \sharp\mathcal{W}_k$. If*

$$
M_l^{\lambda}(\partial\Omega) \;=\; \liminf_{r\to 0^+} \frac{|(\partial\Omega)_r\,|}{r^{\,m-\lambda}} \;>\; 0,
$$

then there exists $K_2 > 0$ and $k_2 \in \mathbb{N}$ so that for all integers $k \geq k_2$,

$$
\sum_{j\,=\,k-n(\gamma)}^{k+2} N_j \;\geq\; K_2\,2^{\lambda k}. \tag{4.7}
$$

Here $n(\gamma)$ is a non-negative integer that varies proportionally with γ.

4.4 Integrability of the Distance Function

We now come to the main business of this chapter - investigating the relationship between the (inner) Minkowski dimension of the boundary of a domain and the integrability of the distance function. The results in this section are not entirely new (see for instance [69, Corollary 2.5 & Theorem 3.1][1]). However the method of proof is original and the results are more quantitative in the sense that they give an indication of how quickly the singularities of the function $d(x,\partial\Omega)^{-1}$ develop. Indeed, the following theorem characterizes

[1] We only became aware of the of the existence of the papers [69], [70] and [71] after having derived the results of this section for ourselves.

the the integrability of the Euclidean distance function in terms of the (inner) Minkowski dimension of the boundary. The inspiration for the proof comes from Proposition 6.1 of [68].

Theorem 4.4.1 *Let Ω be an inner γ-domain in $\mathbb{R}^m$ with non-empty, compact boundary whose inner Minkowski dimension is well defined. For $0 < \delta < \infty$ define the tubular neighborhood of the boundary by $(\partial\Omega)_\delta = \{ x \in \mathbb{R}^m \mid d(x) < \delta \}$. Then,*

$$\begin{aligned} \dim_{\hat{M}}(\partial\Omega) &= \sup \left\{ \lambda \geq 0 \;\middle|\; \int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{m-\lambda}}\, dx = \infty \right\} \\ &= \inf \left\{ \lambda \geq 0 \;\middle|\; \int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{m-\lambda}}\, dx < \infty \right\}. \end{aligned}$$

Proof. Let $\mathcal{W}$ be a Whitney decomposition of Ω and let $N_k = \sharp\mathcal{W}_k$. For notational convenience we will set $\dim_{\hat{M}}(\partial\Omega) = \lambda_M$.

First, let us suppose that $\lambda > \lambda_M = \inf\{ t \geq 0 \mid \hat{M}^t_u(\partial\Omega) < \infty \}$ and let us choose some $\lambda_1 \in (\lambda_M, \lambda)$. Consequently, we have that

$$\hat{M}^{\lambda_1}_u(\partial\Omega) = \limsup_{r \to 0^+} \frac{|(\partial\Omega)_r \cap \Omega|}{r^{m-\lambda_1}} < \infty.$$

Then, by Lemma 4.3.1, there must exist some $K_1 > 0$ and $k_1 \in \mathbb{N}$ so that for all integers $k \geq k_1$ we have

$$N_k \leq K_1 2^{\lambda_1 k}.$$

Now, let us partition the region $(\partial\Omega)_\delta \cap \Omega$ in the following way

$$(\partial\Omega)_\delta \cap \Omega \subseteq \bigcup_{k=k_1}^{\infty} \mathcal{W}_k \cup \left[[(\partial\Omega)_\delta \cap \Omega] \setminus \bigcup_{k=k_1}^{\infty} \mathcal{W}_k \right] \equiv \bigcup_{k=k_1}^{\infty} \mathcal{W}_k \cup \mathcal{B}.$$

We note that the region $\mathcal{B}$ may be empty if δ is chosen sufficiently small, and that on this region $d(x) \geq \frac{5}{8}\frac{\sqrt{m}}{2^{k_1}}$. Furthermore, since $\partial\Omega$ is compact it must also be the case that $|(\partial\Omega)_\delta \cap \Omega| < \infty$. Therefore, we have that

$$\begin{aligned} \int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{m-\lambda}}\, dx &\leq \int_{\cup_{k=k_1}^{\infty} \mathcal{W}_k} \frac{1}{d(x)^{m-\lambda}}\, dx + \int_{\mathcal{B}} \frac{1}{d(x)^{m-\lambda}}\, dx \\ &\leq \int_{\cup_{k=k_1}^{\infty} \mathcal{W}_k} \frac{1}{d(x)^{m-\lambda}}\, dx + C. \end{aligned} \tag{4.8}$$

Utilizing the standard properties of Whitney decompositions expressed in Lemma 4.1.1, we can estimate the integral term in (4.8) in the following way.

$$\int_{\cup_{k=k_1}^{\infty}\mathcal{W}_k} \frac{1}{d(x)^{m-\lambda}}\,dx \quad \leq \quad \sum_{k=k_1}^{\infty}\sum_{Q_k\in\mathcal{W}_k}\int_{Q_k}\frac{1}{d(x)^{m-\lambda}}\,dx$$

$$= \sum_{k=k_1}^{\infty} N_k\; C(m)\,\frac{2^{(m-\lambda)k}}{2^{km}} \quad \leq \quad C(m)\,K_1\sum_{k=k_1}^{\infty} 2^{\lambda_1 k}\;2^{-\lambda k}$$

$$\leq C(m)\,K_1\sum_{k=k_1}^{\infty}\left(\frac{1}{2^{\lambda-\lambda_1}}\right)^{k} \quad < \quad \infty.$$

We conclude that if $\lambda > \lambda_M$ then $\int_{(\partial\Omega)_\delta\cap\Omega}\frac{1}{d(x)^{m-\lambda}}\,dx < \infty$. This completes the first part of the proof.

Now we assume that $\lambda < \lambda_M = \inf\{\, t \geq 0 \mid \hat{M}_l^t(\partial\Omega) < \infty \,\}$ and choose $\lambda_2 \in (\lambda, \lambda_M)$. As such we have that

$$\hat{M}_l^{\lambda_2}(\partial\Omega) \quad = \quad \liminf_{r\to 0^+}\frac{|\,(\partial\Omega)_r\cap\Omega\,|}{r^{\,m-\lambda_2}} \quad > \quad 0.$$

By Lemma 4.3.2, there must exist some $K_2 > 0$ and $k_2 \in \mathbb{N}$ so that for all integers $k \geq k_2$ we have the estimate

$$\sum_{j=k-n(\gamma)}^{k+2} N_j \quad \geq \quad K_2\,2^{\lambda_2 k}. \tag{4.9}$$

Fix $k_3 > \max\left\{\,\left\lceil \frac{\ln(5\sqrt{m}/\delta)}{\ln 2}\right\rceil, k_2\,\right\}$ so that $\bigcup_{j=k_3}^{\infty}\mathcal{W}_j \subseteq \big(\partial\Omega\big)_\delta \cap \Omega$. Using equation (4.9) and the properties of Whitney decompositions, we obtain the desired result in the following manner.

$$\int_{(\partial\Omega)_\delta\cap\Omega}\frac{1}{d(x)^{\,m-\lambda}}\,dx \quad \geq \quad \sum_{j=k_3}^{\infty}\int_{\mathcal{W}_j}\frac{1}{d(x)^{\,m-\lambda}}\,dx$$

$$= \sum_{j=k_3}^{\infty}\sum_{Q_j\in\mathcal{W}_j}\int_{Q_j}\frac{1}{d(x)^{m-\lambda}}\,dx \;\geq\; C(m)\sum_{j=k_3}^{\infty} N_j\,\frac{2^{(m-\lambda)\,j}}{2^{m}j}$$

$$= C(m)\left[\;\sum_{j=k_3}^{k_3+n(\gamma)+2} N_j\,2^{-\lambda j} + \sum_{j=k_3+n(\gamma)+3}^{k_3+2n(\gamma)+5} N_j\,2^{-\lambda j} + \sum_{j=k_3+2n(\gamma)+6}^{k_3+3n(\gamma)+8} N_j\,2^{-\lambda j}\right.$$

$$\left. + \sum_{j=k_3+3n(\gamma)+9}^{k_3+4n(\gamma)+11} N_j\,2^{-\lambda j} + \cdots\;\right]$$

$$
\begin{aligned}
&= C(m) \sum_{N=0}^{\infty} \left(\sum_{j=k_3+N\,n(\gamma)+3N}^{k_3+(N+1)\,n(\gamma)+3N+2} N_j \, 2^{-\lambda j} \right) \\
&\geq C(m) \sum_{N=0}^{\infty} \left(2^{-\lambda\,[\,k_3+(N+1)\,n(\gamma)+3N+2\,]} \sum_{j=k_3+N\,n(\gamma)+3N}^{k_3+(N+1)\,n(\gamma)+3N+2} N_j \right) \\
&\geq C(m)\; K_2 \sum_{N=0}^{\infty} 2^{-\lambda\,[\,k_3+(N+1)\,n(\gamma)+3N+2\,]}\; 2^{\lambda_2\,[\,k_3+(N+1)\,n(\gamma)+3N\,]} \\
&\geq C(m)\; K_2\; 2^{-2\lambda} \sum_{N=0}^{\infty} 2^{(\,\lambda_2-\lambda\,)\;[\,k_3+(N+1)\,n(\gamma)+3N\,]} \quad = \quad \infty
\end{aligned}
$$

since $\lambda_2 > \lambda$. We conclude that if $\lambda < \lambda_M$, then the integral $\int_{(\partial\Omega)_\delta \cap \delta} \frac{1}{d(x)^{\,m-\lambda}}\, dx$ is divergent. ∎

With minor modification of the arguments, we obtain the following corollary which is a restriction of the result given by Zubrinić, in Theorem 3.1 of [69], to γ-domains.

Corollary 4.4.1 *If Ω is a γ-domain in $\mathbb{R}^m$ with non-empty, compact boundary whose Minkowski dimension is well defined, then*

$$
\begin{aligned}
\dim_M(\partial\Omega) &= \sup \left\{ \lambda \geq 0 \;\middle|\; \int_{(\partial\Omega)_\delta} \frac{1}{d(x)^{\,m-\lambda}}\, dx \;=\; \infty \right\} \\
&= \inf \left\{ \lambda \geq 0 \;\middle|\; \int_{(\partial\Omega)_\delta} \frac{1}{d(x)^{\,m-\lambda}}\, dx \;<\; \infty \right\}.
\end{aligned}
$$

At this point an natural question arises. What can be said about the integrability of the function $\frac{1}{d(x)^{m-\lambda}}$, in a tubular neighborhood of the boundary, in the critical case where λ is equal to the (inner) Minkowski dimension of the boundary? A partial answer to this question is given by the following lemma.

Lemma 4.4.2 *Let Ω be an inner γ-domain in $\mathbb{R}^m$ with non-empty, compact boundary. Furthermore, assume that $\dim_{\hat{M}}(\partial\Omega) = \lambda_M$ and that $0 < \hat{M}_l^{\lambda_M}(\partial\Omega)$. Then*

$$
\int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{\,m-\lambda}}\, dx \;<\; \infty \qquad \textit{if and only if} \qquad \lambda > \lambda_M .
$$

Proof. The validity of the sufficiency of this statement is immediate from the previous theorem. To prove the necessity of the statement we demonstrate that the contrapositive is true. In other words, we will show that if $\lambda \leq \lambda_M$, then the integral under consideration is divergent. Again, if $\lambda < \lambda_M$ this is immediate from the previous theorem. Therefore, we need only to consider the case where $\lambda = \lambda_M$. By assumption $\hat{M}_l^{\lambda_M}(\partial\Omega) > 0$ so that, from Lemma 4.3.2, there must exist $K_2 > 0$ and $k_2 \in \mathbb{N}$ so that for all integers $k \geq k_2$

$$\sum_{j=k-n(\gamma)}^{k+2} N_j \quad \geq \quad K_2\, 2^{\lambda_M k}$$

where $\mathcal{W}$ is a Whitney decomposition of Ω and $N_k = \sharp\mathcal{W}_k$. Indeed, if we set $k_3 > \max\left\{ \left\lceil \frac{\ln(5\sqrt{m}/\delta)}{\ln 2} \right\rceil, k_2 \right\}$ and follow the outline of the proof of Theorem 4.4.1, we obtain that

$$\int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{m-\lambda_M}}\, dx \qquad \geq \qquad \sum_{j=k_3}^{\infty} \int_{\mathcal{W}_j} \frac{1}{d(x)^{m-\lambda_M}}\, dx$$

$$= \quad C(m) \sum_{N=0}^{\infty} \left(\sum_{j=k_3+N\,n(\gamma)+3N}^{k_3+(N+1)\,n(\gamma)+3N+2} N_j\, 2^{-\lambda_M j} \right)$$

$$\geq \quad C(m)\ K_2 \sum_{N=0}^{\infty} 2^{-\lambda_M\,[k_3+(N+1)\,n(\gamma)+3N+2]}\ 2^{\lambda_M\,[k_3+(N+1)\,n(\gamma)+3N]}$$

$$\geq \quad C(m)\ K_2 \sum_{N=0}^{\infty} 2^{-2\lambda} \qquad = \qquad \infty.$$

The proof is now complete. ∎

Corollary 4.4.2 *Let Ω be γ-domain in $\mathbb{R}^m$ with non-empty, compact boundary. Furthermore, assume that $dim_M(\partial\Omega) = \lambda_M$ and that $0 < M_l^{\lambda_M}(\partial\Omega)$. Then*

$$\int_{(\partial\Omega)_\delta} \frac{1}{d(x)^{m-\lambda}}\, dx \quad < \quad \infty \qquad \textit{if and only if} \qquad \lambda > \lambda_M.$$

4.5 A γ-Domain with Inner Minkowski Measurable Boundary

In this section, we look to produce a non-trivial example of a domain that satisfies the hypotheses of Lemma 4.4.2. Indeed, following the procedure outlined by Evans & Harris in Section 6 of [76], we will construct a 'room and corridor' type domain that has well defined inner Minkowski dimension and which is also inner Minkowski measurable. Furthermore, we will go on to show that this domain is also an inner γ-domain with $\gamma = 2$. Our reason for investigating this particular domain will become apparent in Section 5.3. We begin by choosing some $C \in (0, 1/2)$ and $\mu > 1$. As such we clearly have that

$$C^\mu \ + \ 2C^2 \ < \ 1.$$

Next we define the monotone decreasing sequences of real numbers $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ by the rules

$$\alpha_n \;=\; C^{\mu n} \quad \text{for all} \quad n \in \mathbb{N}, \qquad \alpha_0 \;>\; \alpha_1$$
$$\beta_n \;=\; C^{n} \quad \text{for all} \quad n \in \mathbb{N}, \qquad \beta_0 \;>\; \beta_1, \qquad \beta_0 \;>\; \alpha_0.$$

We will also assume that the constants α_0 and β_0 have been chosen sufficiently large that

$$4\,\beta_0 \;+\; 2\,\alpha_0 \;-\; 15/(1\;-\;2C) \quad > \quad 0. \tag{4.10}$$

It is easily checked that these sequences satisfy the following conditions:

i) $\alpha_n \;<\; \beta_n$ for all $n = 0,\, 1,\, \ldots$

ii) There exists $\delta_1 \;>\; 0$ so that $0 \;<\; \delta_1 \;\leq\; \frac{\alpha_{n+1}}{\alpha_n} \;<\; 1$.

iii) There exists $\delta_2 \in (0,1)$ so that $0 \;<\; \frac{\beta_{n+1}}{\beta_n} \;\leq\; \delta_2 \;<\; 1$.

iv) $2\beta_{n+2} \;<\; \beta_n \;-\; \alpha_{n+1}$.

Now, let Δ_0 consist of a rectangle $Q_{0,1}$ with edge lengths $2\alpha_0 \times 2\beta_0$, and let Δ_1 consist of a single rectangle $Q_{1,1}$ with edge lengths $2\alpha_1 \times 2\beta_1$. Attach a short edge of $Q_{1,1}$ to the middle portion of a long edge of $Q_{0,1}$. Similarly, for all $n \in \mathbb{N}$, let Δ_n consist of 2^{n-1} rectangles, denoted by $Q_{n,j}$, with edge lengths $2\alpha_n \times 2\beta_n$. For each $j = 1, \ldots, 2^{n-1}$ attach a short edge of $Q_{n,j}$ to the middle portion of a long edge of a rectangle in Δ_{n-1}. Finally, let

$$\Omega \quad = \quad \left(Q_{0,1} \,\bigcup\, \left[\, \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} Q_{n,j} \,\right] \right)^{o}$$

so that Ω is the 'room and corridor' type domain depicted in Figure 4.1.

From condition iv), the fact that $2\beta_{n+2} \;<\; \beta_n \;-\; \alpha_{n+1}$ ensures that the interiors of all the rectangles are disjoint. For instance, looking at Figure 4.1, in order to prevent the rectangle $Q_{3,1}$ from intersecting the rectangle $Q_{0,1}$ we require that $2\beta_3 \;<\; l \;=\; \beta_1 - \alpha_2$. Furthermore, this condition also ensures that $|\,\Omega\,| \;<\; \infty$, since Ω can evidently be contained in the rectangle with edge lengths $2\beta_0 \;\times\; (\,2\alpha_0 \;+\; 2\beta_1\,)$.

In order to proceed, let us note that since the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are monotone decreasing, there must exist some integers $M, \tilde{M}, N, \tilde{N} \in \mathbb{N}$ so that

$$\begin{aligned} \alpha_{M+1} \;&<\; \delta \;\leq\; \alpha_M, & \qquad \beta_{N+1} \;&<\; \delta \;\leq\; \beta_N, \\ \alpha_{\tilde{M}+1} \;&<\; 2\delta \;\leq\; \alpha_{\tilde{M}}, & \qquad \beta_{\tilde{N}+1} \;&<\; 2\delta \;\leq\; \beta_{\tilde{N}}. \end{aligned} \tag{4.11}$$

Since δ can be taken to be arbitrarily small, we can always assume that we have the estimates $M + 1 \leq N$ and $\tilde{M} + 1 \leq \tilde{N}$. Furthermore, with the aid of some elementary algebra, it can be shown that the integer $M - \tilde{M} \in \{0,1\}$ so that $\tilde{M} + 1 \geq M$.

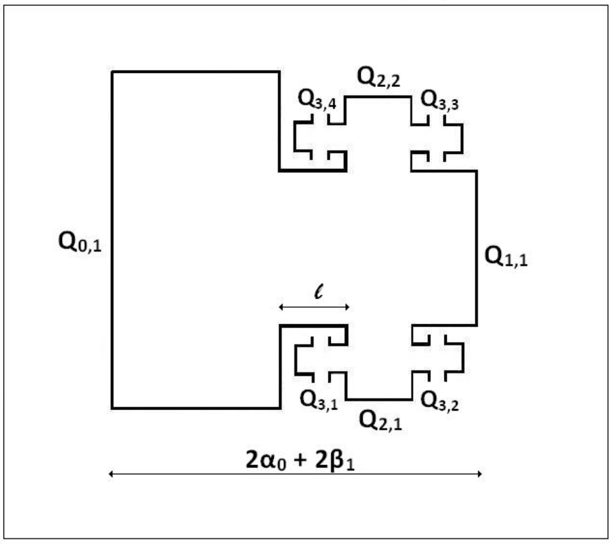

Figure 4.1: The 'room and corridor' type domain formed using the construction of Evans & Harris in [76] has inner Minkowski dimension equal to 1 and is inner Minkowski measurable. Moreover, the domain is also a γ-domain for $\gamma = 2$.

Calculating the volume of an inner δ-neighborhood of the boundary yields the equation

$$\begin{aligned} |(\partial\Omega)_\delta \cap \Omega| \;&=\; [4(\beta_0 + \alpha_0) - 2\alpha_1]\,\delta \;-\; (4 - \pi/2)\,\delta^2 \\ &+ \sum_{k=1}^{M} 2^{k-1}\Big([4(\beta_k - \alpha_{k+1}) + 2\alpha_k]\,\delta + o(\delta^2) \Big) \\ &+ 2\sum_{k=M+1}^{\infty} 2^k\,\alpha_k\,\beta_k. \end{aligned} \tag{4.12}$$

Here $o(\delta^2)$ is a positive constant with the property that $o(\delta^2) \asymp (\pi - 2)\delta^2$. Similarly, if one considers a 2δ-neighborhood of the boundary, then the following equation can easily be derived.

$$\begin{aligned} |(\partial\Omega)_{2\delta} \cap \Omega| \;&=\; 2[4(\beta_0 + \alpha_0) - 2\alpha_1]\,\delta \;-\; 4(4 - \pi/2)\,\delta^2 \\ &+ \sum_{k=1}^{\tilde{M}} 2^{k-1}\Big(2[4(\beta_k - \alpha_{k+1}) + 2\alpha_k]\,\delta + 4\,o(\delta^2) \Big) \\ &+ 2\sum_{k=\tilde{M}+1}^{\infty} 2^k\,\alpha_k\,\beta_k. \end{aligned} \tag{4.13}$$

Our first task is to bound $|(\partial\Omega)_{2\delta} \cap \Omega|$ from below. Indeed, continuing from equation (4.13) we obtain

$$
\begin{aligned}
|(\partial\Omega)_{2\delta} \cap \Omega| \;&\geq\; 2\,[\,4\,(\,\beta_0 + \alpha_0\,) - 2\alpha_1\,]\,\delta \;-\; 4\,(\,4 - \pi/2\,)\,\delta^2 \;+\; 4\,o(\delta^2) \sum_{k=1}^{\tilde{M}} 2^{k-1} \\
&\geq\; 2\,[\,4\,(\,\beta_0 + \alpha_0\,) - 2\alpha_1\,]\,\delta \;-\; 4\,(\,4 - \pi/2\,)\,\delta^2 \;+\; 2\,o(\delta^2)\,(2^M - 2) \qquad (4.14)
\end{aligned}
$$

whereupon it follows that

$$\hat{M}_l^1(\partial\Omega) \;=\; \liminf_{d\to 0^+} \frac{|(\partial\Omega)_{2\delta} \cap \Omega|}{2\delta} \;\geq\; 4\,(\,\beta_0 + \alpha_0\,) - 2\alpha_1 .$$

We can immediately conclude that $\hat{M}_l^1(\partial\Omega) > 0$ and $\underline{\dim}_{\hat{M}}(\partial\Omega) \geq 1$. Now let us estimate $|(\partial\Omega)_\delta \cap \Omega|$ from below. Following on from equation (4.12) we have

$$
\begin{aligned}
&|(\partial\Omega)_\delta \cap \Omega| \\
&\leq\; [\,4\,(\,\beta_0 + \alpha_0\,) - 2\alpha_1\,]\,\delta \;+\; 3\delta \sum_{k=0}^{\infty} 2^k \beta_k \;+\; o(\delta^2) \sum_{k=1}^{M} 2^{k-1} \;+\; 2\delta \sum_{k=0}^{\infty} 2^k \beta_k \\
&=\; [\,4\,(\,\beta_0 + \alpha_0\,) - 2\alpha_1\,]\,\delta \;+\; \frac{5\delta}{1-2C} \;+\; o(\delta^2)\,(2^M - 1) \qquad (4.15)
\end{aligned}
$$

Consequently, it follows that $\hat{M}_u^1(\partial\Omega)$ is finite and, therefore, that $\overline{\dim}_{\hat{M}}(\partial\Omega) \leq 1$ since

$$\hat{M}_u^1(\partial\Omega) \;=\; \limsup_{d\to 0^+} \frac{|(\partial\Omega)_\delta \cap \Omega|}{\delta} \;\leq\; 4\,(\,\beta_0 + \alpha_0\,) - 2\alpha_1 + \frac{5}{1-2C} \;<\; \infty .$$

As such we have succeeded in showing that $\dim_{\hat{M}}(\partial\Omega) = 1$ and that the boundary of the domain is inner Minkowski measurable given that $0 < \hat{M}_l^1(\partial\Omega) \leq \hat{M}_u^1(\partial\Omega) < \infty$. We now wish to demonstrate that Ω is a γ-domain for $\gamma = 2$. In order to do this we must show that there exists a finite, uniform constant $K > 0$ and $\delta_0 > 0$ so that the estimate

$$|(\partial\Omega)_\delta \cap \Omega| \;\leq\; K\,\big[\,|(\partial\Omega)_{2\delta} \cap \Omega| - |(\partial\Omega)_\delta \cap \Omega|\,\big]$$

holds for all $0 < \delta \leq \delta_0$. We assert that the constant $K = 2$ is sufficient for this purpose. In other words, we will show that for sufficiently small δ

$$3\,|(\partial\Omega)_\delta \cap \Omega| \;\leq\; 2\,|(\partial\Omega)_{2\delta} \cap \Omega|. \qquad (4.16)$$

From equation (4.15) we have that

$$3\,|(\partial\Omega)_\delta \cap \Omega| \;\leq\; 3\,[\,4\,(\,\beta_0 + \alpha_0\,) - 2\alpha_1\,]\,\delta \;+\; \frac{15\delta}{1-2C} \;+\; 3\,o(\delta^2)\,(2^M - 1),$$

whilst from equation (4.14) it is also true that

$$2\,|\,(\partial\Omega)_{2\delta}\cap\Omega\,| \quad \geq \quad 4\,[\,4\,(\,\beta_0+\alpha_0\,)-2\alpha_1\,]\,\delta \quad - \quad 8\,(\,4-\pi/2\,)\,\delta^2 \;+\; 4\,o(\delta^2)\,(2^M-2).$$

The validity of equation (4.16) then follows from the fact that it is always possible to find $\delta_0 > 0$ so that the expression

$$[\,4\,(\,\beta_0+\alpha_0\,)-2\alpha_1-15/(1-2C)\,]\,\delta \quad - \quad 8\,(\,4-\pi/2\,)\,\delta^2 \;+\; o(\delta^2)\,2^M \quad - \quad 5\,o(\delta^2)$$

is positive for all $0<\delta\leq\delta_0$, given the assumption of equation (4.10).

In conclusion, we have shown that the domain Ω satisfies all the requirements of Lemma 4.4.2, so that the following result is immediate.

Corollary 4.5.1 *Let Ω be a 'room and corridor' type domain as described above and let $(\partial\Omega)_\delta$ be a tubular neighborhood of the boundary. Then the integral*

$$\int_{(\partial\Omega)_\delta\cap\Omega}\frac{1}{d(x)^{2-\lambda}}\,dx \;<\; \infty \qquad \textit{if and only if} \qquad \lambda > 1.$$

4.6 Associated Results for the L_p-Hardy Inequality

In Theorem 3.2.1 the L_p-Hardy inequality was characterized in terms of the ability to uniformly estimate the integral $\int_K d(x)^{-p}\,dx$ by the variational p-capacity of the compact set K. In this respect, the following lemma may be of some use in demonstrating the existence of an L_p-Hardy inequality on a given domain. Its proof is an immediate consequence of Theorem 4.4.1.

Lemma 4.6.1 *Let Ω be an inner γ-domain in $\mathbb{R}^m$ with non-empty, compact boundary which has well defined inner Minkowski dimension. For $0<\delta<\infty$, define the tubular neighborhood of the boundary by $\big(\partial\Omega\big)_\delta = \{\,x\in\mathbb{R}^m \mid d(x)<\delta\,\}$. If $\dim_{\hat{M}}(\partial\Omega) < m-p$, then*

$$\int_{(\partial\Omega)_\delta\cap\Omega}\frac{1}{d(x)^{\,m-(m-p)}}\,dx \quad = \quad \int_{(\partial\Omega)_\delta\cap\Omega}\frac{1}{d(x)^{\,p}}\,dx \quad < \quad \infty,$$

where as if $\dim_{\hat{M}}(\partial\Omega) > m-p$, *then*

$$\int_{(\partial\Omega)_\delta\cap\Omega}\frac{1}{d(x)^{\,m-(m-p)}}\,dx \quad = \quad \int_{(\partial\Omega)_\delta\cap\Omega}\frac{1}{d(x)^{\,p}}\,dx \quad = \quad \infty.$$

The corresponding version of the above lemma phrased in terms of the usual Minkowski dimension follows from Corollary 4.4.1.

Lemma 4.6.2 *Let* Ω *be a* γ*-domain in* $\mathbb{R}^m$ *with non-empty, compact boundary which has well defined Minkowski dimension. For* $0 < \delta < \infty$*, define the tubular neighborhood of the boundary by* $\big(\partial\Omega\big)_\delta \;=\; \{\, x \in \mathbb{R}^m \mid d(x) < \delta \,\}$*. If* $\dim_M(\partial\Omega) \;<\; m - p$*, then*

$$\int_{(\partial\Omega)_\delta} \frac{1}{d(x)^{\,m-(m-p)}} \; dx \quad = \quad \int_{(\partial\Omega)_\delta} \frac{1}{d(x)^{\,p}} \; dx \quad < \quad \infty.$$

On the other hand, if $\dim_M(\partial\Omega) \;>\; m - p$*, then*

$$\int_{(\partial\Omega)_\delta} \frac{1}{d(x)^{\,m-(m-p)}} \; dx \quad = \quad \int_{(\partial\Omega)_\delta} \frac{1}{d(x)^{\,p}} \; dx \quad = \quad \infty.$$

Furthermore, Corollary 4.4.1 allows us to provide a short and simple proof of the following well known result.

Lemma 4.6.3 *[53, Theorem 1.1 & Section 3.4], [84, Theorem A.5]*

If Ω *is a* γ*-domain in* $\mathbb{R}^m$ *with non-empty, compact boundary whose Minkowski dimension is well defined, then* $\dim_M(\partial\Omega) \;\le\; \dim_A(\partial\Omega)$.

Proof. Since $dim_M(\partial\Omega) \le m,$ we may assume that $dim_A(\partial\Omega) = T < m$ otherwise there is nothing to prove. Then for any $\epsilon > 0$ it must be the case that the estimate

$$\int_{B(\omega,r)} \frac{1}{d(y)^{m-(T+\epsilon)}} \; dy \quad \le \quad A(T,\epsilon)\, r^{T+\epsilon} \tag{4.17}$$

holds for all $\omega \in \partial\Omega$ and all $r > 0$. For arbitrary $\delta > 0$, let $\{B(\omega, 2\delta)\}_{\omega\in\partial\Omega}$ be a cover of $\overline{(\partial\Omega)_\delta}$. Since $\overline{(\partial\Omega)_\delta}$ is compact we can extract a finite sub-cover so that

$$\overline{(\partial\Omega)_\delta} \;\subseteq\; \bigcup_{i=1}^{N_\delta} B(\omega_i, 2\delta) \qquad \text{where} \qquad \omega_i \in \partial\Omega \quad \text{for all} \quad i = 1, \ldots, N_\delta.$$

Then, since $dim_A(\partial\Omega) = T$ we have

$$\begin{aligned} \int_{(\partial\Omega)_\delta} \frac{1}{d(y)^{m-(T+\epsilon)}} \; dy \quad &\le \quad \sum_{i=1}^{N_\delta} \int_{B(\omega_i,2\delta)} \frac{1}{d(y)^{m-(T+\epsilon)}} \; dy \\ &\le \quad N_\delta \; A(T,\epsilon) \; (2\,\delta)^{T+\epsilon} \quad < \quad \infty. \end{aligned}$$

In light of Corollary 4.4.1, this is sufficient to show that $\dim_M(\partial\Omega) \le T + \epsilon$. Since $\epsilon > 0$ can be made arbitrarily small it must be the case that $\dim_M(\partial\Omega) \le \dim_A(\partial\Omega)$. ∎

Chapter 5

The L_p-Hardy Constant

Let Ω be a domain in $\mathbb{R}^m$ with non-empty boundary. We begin this chapter by recalling the conditions required for the Schrödinger operator $H = -\Delta + V$, defined on $C_0^\infty(\Omega)$, to be essentially self-adjoint. From the remarks following the proof of Theorem 2.5.4 we have that if M is a finite natural number and if, sufficiently close to the boundary of the domain,

$$\begin{aligned} V(x) \quad \geq \quad \frac{1}{d(x)^2} \Bigg[1 \; - \; \mu_2(\Omega) \; - \; \frac{1}{\ln(d^{-1})} \; - \; \frac{1}{\ln(d^{-1}) \, \ln\ln(d^{-1})} - \dots \\ \dots - \frac{1}{\ln(d^{-1}) \, \ln\ln(d^{-1}) \dots \underbrace{\ln\ln \dots \ln}_{M-1 \text{ times}} (d^{-1})} \Bigg], \end{aligned}$$

then H is essentially self-adjoint. The role that the variational constant $\mu_2(\Omega)$ plays in this result cannot be overstated. On the one hand if $\mu_2(\Omega) < 1$, then $V(x)$ must tend to infinity as it approaches the boundary in order to ensure that H is essential self-adjoint. On the other hand if $\mu_2(\Omega) \geq 1$, then H can be essentially self-adjoint even if the potential dives slowly to minus infinity as it approaches the boundary. Consequently, the value of the variational constant $\mu_2(\Omega)$ in some sense determines the physics underlying the problem of essential self-adjointness.

It should come as no surprise that the constant $\mu_p(\Omega)$, which is non-zero if and only if Ω admits an L_p-Hardy inequality, has an intimate dependence on the dimension of the boundary. We recall the necessary conditions for a domain to admit an L_p-Hardy inequality as expressed by Theorem 3.3.2. There it was shown that if Ω admits an L_p-Hardy inequality, then either the Hausdorff dimension of the boundary is strictly greater than $m - p$, or the Aikawa dimension of the boundary is strictly less than $m - p$. In particular, for domains with compact boundary we have

$$\dim_{\mathcal{H}}(\partial\Omega) \quad \leq \quad \dim_M(\partial\Omega) \quad \leq \quad \dim_A(\partial\Omega),$$

so that if the Minkowski dimension of the boundary equals $m - p$, then Ω cannot admit an L_p-Hardy inequality. Furthermore, these necessary conditions are almost sufficient conditions. From Theorem 3.4.5 we know that if the complement of the domain is uniformly p-fat (in which case the Hausdorff dimension of the boundary is greater than $m-p$), then the domain admits an L_p-Hardy inequality. In contrast, Theorem 3.5.2 dictates that if the Aikawa dimension of the boundary is less than $m-p$, this is again sufficient to ensure that the domain admits an L_p-Hardy inequality.

In this chapter we attempt to clarify the link between the dimension of the boundary and the value of $\mu_p(\Omega)$. Very roughly speaking, our results suggest that if 'the dimension' of the boundary of a domain is equal to λ, then $\mu_p(\Omega) = \left|\frac{m-\lambda-p}{p}\right|^p$. We begin by presenting a lemma due to Barbatis, Filippas & Tertikas [34] in the next section. This lemma will underpin all of our subsequent analysis and in some sense the inspiration for all the original results in this chapter comes from section 5 of [34]. Next, in Section 5.2, we bound the variational constant $\mu_p(\Omega)$ from above in the case where Ω is a domain in $\mathbb{R}^m$ with non-empty, smooth boundary of integer dimension k. By uniting this estimate with the results of Section 3.6, we are able to obtain explicit values for $\mu_p(\Omega)$ in certain cases where Ω is geometrically simple. The main result of this chapter comes in Section 5.3. Here we investigate the relationship between $\mu_p(\Omega)$ and the inner Minkowski dimension of the boundary. Building upon the results of Sections 4.3 and 4.4 we obtain an upper bound for $\mu_p(\Omega)$ in terms of the inner Minkowski dimension of the boundary, so extending a known result of Davies [77].

5.1 The Lemma of Barbatis, Filippas & Tertikas

In Section 5 of [34], Barbatis, Filippas & Tertikas investigate the optimality of the constants appearing in their 'improved' L_p-Hardy inequalities. One may think of the improved L_p-Hardy inequality as the usual inequality with logarithmic correction terms subtracted from the right hand side. Central to their analysis is the following lemma.

Lemma 5.1.1 *[34, Section 5]*

Let Ω be a domain in $\mathbb{R}^m$ with non-empty boundary. Let $\phi(x) \in W^1_{\infty,0}(\Omega)$ be a real valued function taking values in the interval $[0,1]$. Then the inequality

$$\begin{aligned} \mu_p(\Omega) \quad &\leq \quad \left|\frac{\alpha-p}{p}\right|^p \\ + \quad & C_p \left(\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha}\,dx\right)^{-1}\left[\left|\frac{\alpha-p}{p}\right|^{p-1}\int_\Omega \frac{|\nabla\phi(x)|}{d(x)^{\alpha-1}}\,dx + \int_\Omega \frac{|\nabla\phi(x)|^p}{d(x)^{\alpha-p}}\,dx\right] \end{aligned} \tag{5.1}$$

holds for all $\alpha \geq 0$ and for all $p > 1$. Here C_p is an absolute constant depending only on p.

Proof. By Definition 1.5.3, if $\alpha = p$, then the result is trivial. Thus we assume that $\alpha \neq p$, and set $W(x) = d(x)^{-\frac{\alpha - p}{p}}$ and $U(x) = \phi(x)\, W(x)$, so that $U(x) \in W^1_{p,0}(\Omega)$. Then

$$\begin{aligned}
\mu_p(\Omega) &= \inf_{\omega \in W^1_{p,0}(\Omega)} \left\{ \frac{\int_\Omega |\nabla \omega(x)|^p \, dx}{\int_\Omega \frac{|\omega(x)|^p}{d(x)^p} \, dx} \right\} \leq \frac{\int_\Omega |\nabla U(x)|^p \, dx}{\int_\Omega \frac{|U(x)|^p}{d(x)^p} \, dx} \\
&= \frac{\int_\Omega \left| \phi(x)\, \nabla W(x) + W(x)\, \nabla \phi(x) \right|^p \, dx}{\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha} \, dx}.
\end{aligned}$$

Using the elementary inequality

$$|a + b|^p \leq |a|^p + C_p \left(|a|^{p-1} |b| + |b|^p \right) \qquad \text{where} \qquad a, b \in \mathbb{R}^m, \;\; p > 1,$$

and the fact that $|\nabla W(x)| \leq |\frac{\alpha - p}{p}| \; d(x)^{-\alpha/p}$, we obtain the desired inequality as follows.

$$\begin{aligned}
\mu_p(\Omega) &\leq \left(\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha} \, dx \right)^{-1} \int_\Omega \phi^p(x) \left| \nabla W(x) \right|^p \, dx \\
&+ \left(\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha} \, dx \right)^{-1} C_p \int_\Omega |\nabla W(x)|^{p-1} \, W(x) \, |\nabla \phi(x)| + |\nabla \phi(x)|^p \, W^p(x) \, dx \\
&\leq \left| \frac{\alpha - p}{p} \right|^p \left(\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha} \, dx \right)^{-1} \left(\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha} \, dx \right) \\
&+ \left(\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha} \, dx \right)^{-1} C_p \left| \frac{\alpha - p}{p} \right|^{p-1} \int_\Omega |\nabla \phi(x)| \; d(x)^{-[\alpha(p-1) + \alpha - p]/p} \, dx \\
&+ \left(\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha} \, dx \right)^{-1} C_p \int_\Omega \frac{|\nabla \phi(x)|^p}{d(x)^{\alpha - p}} \, dx \\
&= \left| \frac{\alpha - p}{p} \right|^p + \left(\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha} \, dx \right)^{-1} C_p \left| \frac{\alpha - p}{p} \right|^{p-1} \int_\Omega \frac{|\nabla \phi(x)|}{d(x)^{\alpha - 1}} \, dx \\
&+ \left(\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha} \, dx \right)^{-1} C_p \int_\Omega \frac{|\nabla \phi(x)|^p}{d(x)^{\alpha - p}} \, dx.
\end{aligned}$$

The proof is now complete. ∎

As we alluded to in the introduction, our objective in this Chapter is to bound the variational constant $\mu_p(\Omega)$ from above. In effect, equation (5.1) now provides us with a very general approach for doing so. Indeed, suppose that we can find some value of α so that the integrals

$$\int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{\alpha - 1}} \, dx \qquad \text{and} \qquad \int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{\alpha - p}} \, dx,$$

are convergent, whereas the integral

$$\int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{\,\alpha}} \;\; dx$$

is divergent. The philosophy of the previous chapter suggests that setting $\alpha = m - \lambda$, where λ is 'the dimension' of the boundary, will suffice for this purpose. By constructing an appropriate sequence of functions $\{\phi_n(x)\}_{n=1}^{\infty} \subseteq W^1_{\infty,0}(\Omega)$ that exploit this dichotomy, one then aims to show that the second term on the right hand side of equation (5.1) can be made negligibly small, so that one is left with the estimate $\mu_p(\Omega) \leq \left|\frac{m-\lambda-p}{p}\right|^p$. In the subsequent two sections of this chapter, we show that this is indeed possible in the case where Ω is a domain with smooth boundary of integer dimension and where Ω is an inner γ-domain with compact boundary that is inner Minkowski measurable.

5.2 The L_p-Hardy Constant for Domains with Smooth Boundary

In [34] Barbatis, Filippas & Tertikas are concerned with ascertaining the sufficient conditions required for a domain to admit 'improved' L_p-Hardy inequalities. Moreover, they consider L_p-Hardy inequalities of the following form

$$\int_\Omega \frac{|\,\omega(x)\,|^p}{d(x,K)^p} \; dx \qquad \leq \qquad C \int_\Omega |\,\nabla\omega(x)\,|^p \; dx \tag{5.2}$$

where $\omega(x) \in W^1_{p,0}(\Omega \setminus K)$. Here the function $d(x, K)$ describes the Euclidean distance from the point $x \in \Omega$ to a closed, smooth set K of integer dimension k. In particular, K need not be the boundary of the domain. It could, for instance, be a set embedded in Ω so that $K \subseteq \Omega$. Under the assumptions that either $K \cap \Omega \neq \emptyset$, or $K = \partial\Omega$ and $k = m - 1$, it is shown in Theorem 5.1 of [34] that the variational constant $\mu_p(\Omega)$ associated to equation (5.2) can be estimated as follows

$$\mu_p(\Omega) \quad \leq \quad \left| \frac{m-k-p}{p} \right|^p. \tag{5.3}$$

The purpose of this section is to modify the analysis used in [34] to show that equation (5.3) also holds when the boundary of the domain is smooth and has integer dimension k, where $0 \leq k \leq m - 1$. Indeed, this is the content of the following lemma.

Theorem 5.2.1 *Let Ω be a domain in $\mathbb{R}^m$ with non-empty, smooth boundary of integer dimension k, where $0 \leq k \leq m - 1$. Then for all $p > 1$, we have that*

$$\mu_p(\Omega) \quad \leq \quad \left| \frac{m-k-p}{p} \right|^p.$$

In particular, if $k = m - p$, then $\mu_p(\Omega) = 0$ so that Ω does not admit an L_p-Hardy inequality.

Proof. Without loss of generality we may assume that $0 \in \partial\Omega$. Let us choose some arbitrary $T, R > 0$, and for each $n \in \mathbb{N}$ define the smooth functions $\hat{\phi}_n$, $\hat{\theta}_n : \mathbb{R} \to [0,1]$ by

$$\hat{\phi}_n(t) = \begin{cases} 0 & \text{if} \quad t \le \frac{1}{2^{n+1}} \\ \sigma_n(t) & \text{if} \quad \frac{1}{2^{n+1}} < t < \frac{1}{2^n} \\ 1 & \text{if} \quad t \ge \frac{1}{2^n} \end{cases}$$

$$\hat{\theta}_n(t) = \begin{cases} 1 & \text{if} \quad t \le T \\ \gamma(t) & \text{if} \quad T < t < T + R \\ 0 & \text{if} \quad t \ge T + R. \end{cases}$$

Here the monotone functions $\sigma_n(t)$ and $\gamma(t)$ are chosen so that $|\,\sigma_n'(t)\,| \le C_1 2^{n+1}$ and $|\,\gamma'(t)\,| \le C_2$, where C_1 and C_2 are absolute constants independent of n. Now for each $n \in \mathbb{N}$ define the functions

$$\phi_n(x) = \hat{\phi}_n(\,d(x)\,) \qquad \text{and} \qquad \theta_n(x) = \hat{\theta}_n(\,|x|\,).$$

Setting $G_n(x) = \phi_n(x)\,\theta_n(x)$ it is not hard to show that $G_n(x)$ is contained in $W^1_{\infty,0}(\Omega)$ and takes values between 0 and 1. Therefore, applying the result of Lemma 5.1.1, with $\alpha = m - k$, yields that

$$\mu_p(\Omega) \le \left| \frac{m-k-p}{p} \right|^p + \left(\int_\Omega \frac{G_n^p(x)}{d(x)^{\,m-k}}\, dx \right)^{-1} \cdot C_p \left[\left| \frac{m-k-p}{p} \right|^{p-1} \int_\Omega \frac{|\nabla G_n(x)|}{d(x)^{m-k-1}}\, dx + \int_\Omega \frac{|\nabla G_n(x)|^p}{d(x)^{m-k-p}}\, dx \right]. \tag{5.4}$$

We will now show that in the limit as $n \to \infty$ the denominator in expression (5.4) diverges whilst the numerator remains bounded. To do so we will need to define the regions

$$\mathcal{A}_n = \left\{ x \in \Omega \;\middle|\; \frac{1}{2^{n+1}} < d(x) < \frac{1}{2^n} \right\}$$

$$\Omega_n = \Omega \setminus \bigcup_{j=n}^{\infty} \mathcal{A}_j = \left\{ x \in \Omega \;\middle|\; d(x) \ge \frac{1}{2^n} \right\}.$$

Let us look at the first integral term in the numerator of (5.4). Simply by considering the level sets of the distance function, we arrive at the conclusion that this term remains bounded as $n \to \infty$ in the following manner.

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla G_n(x)|}{d(x)^{m-k-1}}\ dx \\
\leq\ & \int_{A_n \cap B(T+R)} \frac{|\nabla \sigma_n(d(x))|}{d(x)^{m-k-1}}\ dx \quad + \quad \int_{\Omega_{n+1} \cap [B(T+R) \setminus B(T)]} \frac{|\nabla \gamma(|x|)|}{d(x)^{m-k-1}}\ dx \\
\leq\ & C_1\, 2^{n+1} \int_{A_n \cap B(T+R)} \frac{1}{d(x)^{m-k-1}}\ dx \quad + \quad C_2 \int_{\Omega_{n+1} \cap [B(T+R) \setminus B(T)]} \frac{1}{d(x)^{m-k-1}}\ dx \\
=\ & C_1\, 2^{n+1} \int_{1/2^{n+1}}^{1/2^n} \int_{\{d(x)=r\} \cap B(T+R)} \frac{1}{r^{m-k-1}}\ dS\ dr \\
& + \ C_2 \int_{1/2^{n+1}}^{T+R} \int_{\{d(x)=r\} \cap [B(T+R) \setminus B(T)]} \frac{1}{r^{m-k-1}}\ dS\ dr \\
\leq\ & C\, 2^{n+1} \int_{1/2^{n+1}}^{1/2^n} \frac{r^{m-k-1}}{r^{m-k-1}}\ dr \quad + \quad C \int_{1/2^{n+1}}^{T+R} \frac{r^{m-k-1}}{r^{m-k-1}}\ dr \\
\leq\ & C\ 2^{n+1} \left[\frac{1}{2^n} \ - \ \frac{1}{2^{n+1}} \right] \quad + \quad C \quad \leq \quad M_1
\end{aligned}
$$

where M_1 is some finite constant which will depend on the geometry of $\partial\Omega$. A similar calculation shows that the second integral term in the numerator of (5.4) also remains bounded as $n \to \infty$. Indeed,

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla G_n(x)|^p}{d(x)^{m-k-p}}\ dx \\
\leq\ & C \int_{A_n \cap B(T+R)} \frac{|\nabla \sigma_n(d(x))|^p}{d(x)^{m-k-p}}\ dx \ + \ C \int_{\Omega_{n+1} \cap [B(T+R) \setminus B(T)]} \frac{|\nabla \gamma(|x|)|^p}{d(x)^{m-k-p}}\ dx \\
\leq\ & C\ 2^{(n+1)\,p} \int_{A_n \cap B(T+R)} \frac{1}{d(x)^{m-k-p}}\ dx \\
& + \ C \int_{\Omega_{n+1} \cap [B(T+R) \setminus B(T)]} \frac{1}{d(x)^{m-k-p}}\ dx \\
=\ & C\ 2^{(n+1)\,p} \int_{1/2^{n+1}}^{1/2^n} \int_{\{d(x)=r\} \cap B(T+R)} \frac{1}{r^{m-k-p}}\ dS\ dr \\
& + \ C \int_{1/2^{n+1}}^{T+R} \int_{\{d(x)=r\} \cap [B(T+R) \setminus B(T)]} \frac{1}{r^{m-k-p}}\ dS\ dr \\
\leq\ & C\, 2^{(n+1)\,p} \int_{1/2^{n+1}}^{1/2^n} \frac{r^{m-k-1}}{r^{m-k-p}}\ dr \quad + \quad C \int_{1/2^{n+1}}^{T+R} \frac{r^{m-k-1}}{r^{m-k-p}}\ dr \\
\leq\ & C\ 2^{(n+1)\,p} \Big[\ r^p \ \Big]_{2^{-n-1}}^{2^{-n}} \quad + \quad C \quad \leq \quad M_2.
\end{aligned}
$$

Again, M_2 is a finite constant which is independent of n but does depend on the geometry of $\partial\Omega$. Consequently, it remains only to show that the denominator of expression (5.4) diverges as $n \to \infty$. In this respect, since

$$\begin{aligned} \int_{\Omega} \frac{G_n^p(x)}{d(x)^{m-k}} \, dx \quad &\geq \quad \int_{B(T) \cap \Omega_n} \frac{1}{d(x)^{m-k}} \, dx \\ = \int_{\frac{1}{2^n}}^{T} \int_{\{ d(x)=r \} \cap [B(T) \cap \Omega_n]} \frac{1}{r^{m-k}} \, dS \, dr \quad &\geq \quad C \int_{\frac{1}{2^n}}^{T} \frac{r^{m-k-1}}{r^{m-k}} \, dr \\ = C \left[\ln r \right]_{\frac{1}{2^n}}^{T} \quad &\to \quad \infty \end{aligned}$$

the proof is complete. ∎

A few remarks about Theorem 5.2.1 are now in order. First, the theorem is local in nature. That is to say that all the analysis takes place inside a ball of arbitrarily small radius, centered on an arbitrary point of the boundary. As such, we only require the boundary of the domain to be smooth in the vicinity of this ball. Consequently, we may view the above theorem as a generalization of the result of Marcus, Mizel & Pinchover in [78]. There it was shown that if Ω is a domain in $\mathbb{R}^m$, and if for at least one point on the boundary there exists a tangent plane (i.e. the boundary contains an $m-1$ dimensional part), then

$$\mu_p(\Omega) \quad \leq \quad \left| \frac{m-(m-1)-p}{p} \right|^p \quad = \quad \left| \frac{1-p}{p} \right|^p .$$

Secondly, Theorem 5.2.1 provides an upper bound for $\mu_p(\Omega)$ when the boundary of the domain is smooth. In contrast, Lemma 3.6.3 and Lemma 3.6.2 provide lower bounds for the variational constant $\mu_p(\Omega)$. Combining these results allows us to obtain explicit values for $\mu_p(\Omega)$ in the following cases.

Lemma 5.2.2 *[78, Theorem 11]*

Let Ω be a bounded, convex domain in $\mathbb{R}^m$ with smooth boundary of co-dimension 1. Then the inequality

$$\int_{\Omega} \frac{|\,\omega(x)\,|^p}{d(x)^p} \, dx \quad \leq \quad \left| \frac{p}{1-p} \right|^p \int_{\Omega} |\,\nabla\omega(x)\,|^p \, dx \tag{5.5}$$

holds for all $\omega(x) \in W^1_{p,0}(\Omega)$ and $p > 1$. Moreover, $\mu_p(\Omega) = \left|\, \frac{1-p}{p} \,\right|^p$.

Proof. In Lemma 3.6.3 it was shown that equation (5.5) holds and that $\mu_p(\Omega) \geq \left|\, \frac{1-p}{p} \,\right|^p$. Therefore, it remains only to demonstrate the reverse inequality. However, the $m-1$ dimensional boundary of Ω is smooth so that, by Theorem 5.2.1, $\mu_p(\Omega) \leq \left|\, \frac{1-p}{p} \,\right|^p$. The result follows. ∎

Although the above lemma is well known, the following result for domains that are complements of affine sets is, rather surprisingly, new.

Lemma 5.2.3 *Let* $\Omega = \mathbb{R}^m \setminus E$ *where* E *is an affine set of dimension* k *and* $0 \leq k \leq m-1$. *In the case where* $m = 1$ *we assume that* $\Omega = (0, \infty)$. *Then the inequality*

$$\int_\Omega \frac{|\,\omega(x)\,|^p}{d(x)^p}\ dx \qquad \leq \qquad \left|\frac{p}{m-k-p}\right|^p \int_\Omega |\,\nabla\omega(x)\,|^p\ dx \tag{5.6}$$

holds for all $\omega(x) \in W^1_{p,0}(\Omega)$ *and* $p > 1$. *Moreover,* $\mu_p(\Omega) = |\,\frac{m-k-p}{p}\,|^p$.

Proof. We must only prove that $\mu_p(\Omega) \leq |\,\frac{m-k-p}{p}\,|^p$ since it was already shown in Lemma 3.6.2 that equation (5.6) holds and that $\mu_p(\Omega) \geq |\,\frac{m-k-p}{p}\,|^p$. However, since $\partial\Omega$ is smooth Theorem 5.2.1 implies that $\mu_p(\Omega) \leq |\,\frac{m-k-p}{p}\,|^p$. ∎

5.3 The L_p-Hardy Constant and the Inner Minkowski Dimension of the Boundary

In the previous section we saw that the ability to estimate the constant $\mu_p(\Omega)$ from above, in the case where $\partial\Omega$ is smooth, results from a dimensional dichotomy concerning the integrability of the distance function. More specifically, if $\Omega \subseteq \mathbb{R}^m$ has a smooth boundary of integer dimension k, and if $y \in \partial\Omega$, then the integrals

$$\int_{B(y,\delta)\,\cap\,\Omega} \frac{1}{d(x)^{m-k-1}}\ dx \qquad \text{and} \qquad \int_{B(y,\delta)\,\cap\,\Omega} \frac{1}{d(x)^{m-k-p}}\ dx$$

are convergent, whereas the integral

$$\int_{B(y,\delta)\,\cap\,\Omega} \frac{1}{d(x)^{m-k}}\ dx$$

is divergent. Given the result of Theorem 4.4.1, such a dichotomy is also present when we consider domains with more irregular boundary. Therefore, it is natural to ask whether we can obtain an upper bound for $\mu_p(\Omega)$, similar to that described by Theorem 5.2.1, if we relax the conditions on the smoothness and dimension of the boundary. In the case where Ω is an inner γ-domain with inner Minkowski measurable boundary, we give an affirmative answer to this question. Indeed, we are not the first to consider the relationship between the inner Minkowski dimension of the boundary and the value of the variational constant $\mu_p(\Omega)$. In [77] Davies produced the following result.

Theorem 5.3.1 *[77, Theorem 10]*

Let Ω be a domain in $\mathbb{R}^m$ with non-empty, compact boundary $\partial\Omega$. If $\partial\Omega$ is inner Minkowski measurable, and if $\dim_{\hat{M}}(\partial\Omega) = \lambda_M \geq m-2$, then

$$\mu_2(\Omega) \quad \leq \quad \left| \frac{m - \lambda_M - 2}{2} \right|^2 .$$

The following theorem can be considered an extension of this result to L_p.

Theorem 5.3.2 *Let Ω be an inner γ-domain in $\mathbb{R}^m$ with non-empty, compact boundary. Let the inner Minkowski dimension of $\partial\Omega$ be well defined and set $\lambda_M = \dim_{\hat{M}}(\partial\Omega)$. If $\partial\Omega$ is inner Minkowski measurable so that*

$$0 \quad < \quad \hat{M}_l^{\lambda_M}(\partial\Omega) \quad \leq \quad \hat{M}_u^{\lambda_M}(\partial\Omega) \quad < \quad \infty,$$

then for all $p > 1$ we have that $\mu_p(\Omega) \leq \left| \frac{m-\lambda_M-p}{p} \right|^p$. In particular, if $\lambda_M = m-p$ then Ω does not admit an L_p-Hardy inequality.

Proof. Let $\mathcal{W}$ be a Whitney decomposition of Ω and let $N_k = \sharp\,\mathcal{W}_k$. For each $k \in \mathbb{N}$ define the smooth function $\hat{\phi}_k : \mathbb{R} \to [0,1]$ by

$$\hat{\phi}_k(t) \quad = \quad \begin{cases} 0 & \text{if} \quad t \leq \frac{5/8\sqrt{m}}{2^k}, \\ \sigma_k(t) & \text{if} \quad \frac{5/8\sqrt{m}}{2^k} < t < \frac{\sqrt{m}}{2^k}, \\ 1 & \text{if} \quad \frac{\sqrt{m}}{2^k} \leq t \leq \frac{5\sqrt{m}}{2}, \\ \mu(t) & \text{if} \quad \frac{5\sqrt{m}}{2} < t < \frac{8\sqrt{m}}{2}, \\ 0 & \text{if} \quad t \geq \frac{8\sqrt{m}}{2}, \end{cases}$$

where the monotone functions $\sigma_k(t)$ and $\mu(t)$ are such that $\left|\sigma_k'(t)\right| \leq C_1 2^k$ and $\left|\mu'(t)\right| \leq C_2$. Here C_1 and C_2 are absolute constants independent of k. For each $k \in \mathbb{N}$ define the function $\phi_k : \Omega \to \mathbb{R}$ by $\phi_k(x) = \hat{\phi}_k\big(d(x)\big)$ so that the sequence $\{\phi_k(x)\}_{k=1}^{\infty}$ belongs to $W^1_{\infty,0}(\Omega)$ and each $\phi_k(x)$ takes values in the interval $[0,1]$. Therefore, if we set $\alpha = m - \lambda_M$, then it follows immediately from Lemma 5.1.1 that

$$\mu_p(\Omega) \quad \leq \quad \left| \frac{m-\lambda_M-p}{p} \right|^p \quad + \quad \left(\int_\Omega \frac{\phi_k^p(x)}{d(x)^{\,m-\lambda_M}} \; dx \right)^{-1}$$
$$\cdot \quad C_p \left[\; \left| \frac{m-\lambda_M-p}{p} \right|^{p-1} \int_\Omega \frac{|\nabla\phi_k(x)|}{d(x)^{m-\lambda_M-1}} \; dx \quad + \quad \int_\Omega \frac{|\nabla\phi_k(x)|^p}{d(x)^{m-\lambda_M-p}} \; dx \right]. \tag{5.7}$$

Since the boundary of the domain is inner Minkowski measurable, Lemma 4.4.2 indicates that the integrals

$$\int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{m-\lambda_M-1}}\, dx \qquad \text{and} \qquad \int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{m-\lambda_M-p}}\, dx$$

are finite, whereas the integral

$$\int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{m-\lambda_M}}\, dx$$

is divergent. Again, the idea behind the proof of the theorem is to exploit this dichotomy and show that in the limit as $k \to \infty$ the numerator in (5.7) stays bounded whilst the denominator tends to infinity. To demonstrate this fact it will be useful to define the following regions

$$\begin{aligned} \mathcal{A}_k &= \left\{ x \in \Omega \;\middle|\; \frac{5/8\sqrt{m}}{2^k} < d(x) < \frac{\sqrt{m}}{2^k} \right\} \subseteq \mathcal{W}_{k+1} \cup \mathcal{W}_{k+2} \\ \mathcal{B} &= \left\{ x \in \Omega \;\middle|\; \frac{5\sqrt{m}}{2} < d(x) < \frac{8\sqrt{m}}{2} \right\}. \end{aligned}$$

Since $\hat{M}_u^{\lambda_M}(\partial\Omega) < \infty$, by Lemma 4.3.1 there must exist some $K_1 > 0$ and $j_1 \in \mathbb{N}$ so that for all integers $j \geq j_1$ we have

$$N_j \leq K_1\, 2^{\lambda_M j}. \tag{5.8}$$

Similarly, given that $0 < \hat{M}_l^{\lambda_M}(\partial\Omega)$, the result of Lemma 4.3.2 implies that there exists $K_2 > 0$ and $j_2 \in \mathbb{N}$ so that for all integers $j \geq j_2$

$$\sum_{s=j-n(\gamma)}^{j+2} N_s \geq K_2\, 2^{\lambda_M j}. \tag{5.9}$$

where $n(\gamma)$ is an appropriately chosen natural number. Without loss of generality we may assume that $k \geq \max\{j_1, j_2\}$ and $j_2 \geq 1$.

We now investigate the behavior of the first integral term in the numerator of expression (5.7) as $k \to \infty$. Indeed,

$$\begin{aligned} \int_\Omega \frac{|\nabla \phi_k(x)|}{d(x)^{m-\lambda_M-1}}\, dx &= \int_{\mathcal{A}_k} \frac{|\nabla \sigma_k(d(x))|}{d(x)^{m-\lambda_M-1}}\, dx + \int_{\mathcal{B}} \frac{|\nabla \mu(d(x))|}{d(x)^{m-\lambda_M-1}}\, dx \\ &\leq \int_{\mathcal{A}_k} \frac{|\nabla \sigma_k(d(x))|}{d(x)^{m-\lambda_M-1}}\, dx + C. \end{aligned} \tag{5.10}$$

Simply by using equation (5.8), and the standard properties of Whitney decompositions expressed in Lemma 4.1.1, we can show that the remaining integral term in equation (5.10) remains bounded above as $k \to \infty$ in the following way.

$$\begin{aligned}
&\int_{\mathcal{A}_k} \frac{\left|\,\nabla\,\sigma_k\big(\,d(x)\,\big)\,\right|}{d(x)^{\,m\,-\,\lambda_M\,-\,1}}\ dx \\
\leq\ & C_1\ 2^{\,k}\left[\int_{\mathcal{W}_{k+1}} \frac{1}{d(x)^{\,m\,-\,\lambda_M\,-\,1}}\ dx \ +\ \int_{\mathcal{W}_{k+2}} \frac{1}{d(x)^{\,m\,-\,\lambda_M\,-\,1}}\ dx\right] \\
=\ & C_1\ 2^{\,k}\left[\ \sum_{Q_{k+1}\,\in\,\mathcal{W}_{k+1}} \int_{Q_{k+1}} \frac{1}{d(x)^{\,m\,-\,\lambda_M\,-\,1}}\ dx \right.\\
&\qquad \left. +\ \sum_{Q_{k+2}\,\in\,\mathcal{W}_{k+2}} \int_{Q_{k+2}} \frac{1}{d(x)^{\,m\,-\,\lambda_M\,-\,1}}\ dx\ \right] \\
\leq\ & C(m)\ 2^{\,k}\left[\ N_{k+1}\ \frac{2^{\,(m\,-\,\lambda_M\,-\,1\,)k}}{2^{\,(k+1)m}} \ +\ N_{k+2}\ \frac{2^{\,(m\,-\,\lambda_M\,-\,1\,)k}}{2^{\,(k+2)m}}\ \right] \\
\leq\ & C(m)\ K_1\ 2^{\,k}\left[\,2^{\lambda_M\,(k+1)}\ 2^{-\lambda_M k}\ 2^{-k}\ +\ 2^{\lambda_M\,(k+2)}\ 2^{-\lambda_M k}\ 2^{-k}\,\right] \\
\leq\ & C(m)\ K_1\ 2^{\,2\lambda_M}.
\end{aligned}$$

We conclude that for all $k \in \mathbb{N}$

$$\int_{\Omega} \frac{\left|\,\nabla\phi_k(x)\,\right|}{d(x)^{\,m\,-\,\lambda_M\,-\,1}}\ dx \qquad \leq \qquad M_1$$

where M_1 is some finite constant independent of k. A similar analysis, shows that the second term in the numerator of (5.7) also remains bounded above as $k \to \infty$. Indeed,

$$\int_{\Omega} \frac{\left|\,\nabla\phi_k(x)\,\right|^p}{d(x)^{\,m-\,\lambda_M\,-\,p}}\ dx \quad = \quad \int_{\mathcal{A}_k} \frac{\left|\,\nabla\,\sigma_k\big(\,d(x)\,\big)\,\right|^p}{d(x)^{\,m\,-\,\lambda_M\,-\,p}}\ dx \ +\ \int_{\mathcal{B}} \frac{\left|\,\nabla\mu\big(\,d(x)\,\big)\,\right|^p}{d(x)^{\,m\,-\,\lambda_M\,-\,p}}\ dx$$

Since the second integral term in the equation above is evidently bounded, we need only concentrate on demonstrating the boundedness of the first term. In this respect

$$\begin{aligned}
&\int_{\mathcal{A}_k} \frac{\left|\,\nabla\,\sigma_k\big(\,d(x)\,\big)\,\right|^p}{d(x)^{\,m\,-\,\lambda_M\,-\,p}}\ dx \\
\leq\ & C_1^p\ 2^{\,kp}\left[\int_{\mathcal{W}_{k+1}} \frac{1}{d(x)^{\,m\,-\,\lambda_M\,-\,p}}\ dx \ +\ \int_{\mathcal{W}_{k+2}} \frac{1}{d(x)^{\,m\,-\,\lambda_M\,-\,p}}\ dx\right]
\end{aligned}$$

$$
\begin{aligned}
&= C_1^p\ 2^{kp} \Bigg[\sum_{Q_{k+1} \in \mathcal{W}_{k+1}} \int_{Q_{k+1}} \frac{1}{d(x)^{m - \lambda_M - p}}\, dx \\
&\qquad + \sum_{Q_{k+2} \in \mathcal{W}_{k+2}} \int_{Q_{k+2}} \frac{1}{d(x)^{m - \lambda_M - p}}\, dx \Bigg] \\
&\leq C(m)\ 2^{kp} \left[N_{k+1} \frac{2^{(m - \lambda_M - p)k}}{2^{(k+1)m}} + N_{k+2} \frac{2^{(m - \lambda_M - p)k}}{2^{(k+2)m}} \right] \\
&\leq C(m)\ K_1\ 2^{kp} \left[2^{\lambda_M (k+1)}\ 2^{-\lambda_M k}\ 2^{-kp} + 2^{\lambda_M (k+2)}\ 2^{-\lambda_M k}\ 2^{-kp} \right] \\
&\leq C(m)\ K_1\ 2^{2\lambda_M}.
\end{aligned}
$$

Consequently, we have shown that for all $k \in \mathbb{N}$

$$
\int_\Omega \frac{|\nabla \phi_k(x)|^p}{d(x)^{m - \lambda_M - p}}\, dx \leq M_2
$$

where M_2 is again a finite constant independent of k.

To complete the proof it remains only to show that the denominator in expression (5.7) tends to infinity as $k \to \infty$. Using the fact that $\phi_k(x) = 1$ over the region $\cup_{j=1}^k \mathcal{W}_j$, we are led to the following set of inequalities.

$$
\begin{aligned}
\int_\Omega \frac{\phi_k^p(x)}{d(x)^{m-\lambda_M}}\, dx &\geq \sum_{j=1}^{k} \int_{\mathcal{W}_j} \frac{1}{d(x)^{m-\lambda_M}}\, dx \\
\geq \sum_{j=j_2}^{k} \int_{\mathcal{W}_j} \frac{1}{d(x)^{m-\lambda_M}}\, dx &= \sum_{j=j_2}^{k} \sum_{Q_j \in \mathcal{W}_j} \int_{Q_j} \frac{1}{d(x)^{m-\lambda_M}}\, dx \\
\geq C(m) \sum_{j=j_2}^{k} N_j \frac{2^{(m-\lambda_M)j}}{2^{mj}} &= C(m) \sum_{j=j_2}^{k} N_j\ 2^{-\lambda_M j} \\
= C(m) \Bigg[\sum_{j=j_2}^{j_2+n(\gamma)+2} N_j\ 2^{-\lambda_M j} &+ \sum_{j=j_2+n(\gamma)+3}^{j_2+2n(\gamma)+5} N_j\ 2^{-\lambda_M j} \\
+ \sum_{j=j_2+2n(\gamma)+6}^{j_2+3n(\gamma)+8} N_j\ 2^{-\lambda_M j} &+ \cdots \Bigg]. \qquad (5.11)
\end{aligned}
$$

Now let $T(k) = \lfloor \frac{k - j_2 + 1}{n(\gamma) + 3} \rfloor$, so that $T(k)$ denotes the number of 'complete' summation terms appearing in equation (5.11). We note that $T(k) \to \infty$ as $k \to \infty$. Taking into account equation (5.9) we obtain

$$
\begin{aligned}
&\int_\Omega \frac{\phi_k^p(x)}{d(x)^{\,m\,-\,\lambda_M}}\;dx \\
\geq\;\; & C(m)\,\Bigg[\,2^{-\lambda_M\,(\,j_2\,+\,n(\gamma)\,+\,2\,)}\sum_{j\,=\,j_2}^{j_2\,+\,n(\gamma)\,+\,2} N_j \;+\; 2^{-\lambda_M\,(\,j_2\,+\,2n(\gamma)\,+\,5\,)}\sum_{j\,=\,j_2\,+\,n(\gamma)\,+\,3}^{j_2\,+\,2n(\gamma)\,+\,5} N_j \\
& \quad +\; 2^{-\lambda_M\,(\,j_2\,+\,3n(\gamma)\,+\,8\,)}\sum_{j\,=\,j_2\,+\,2n(\gamma)\,+\,6}^{j_2\,+\,3n(\gamma)\,+\,8} N_j \;+\;\cdots\;\Bigg] \\
\geq\;\; & C\;K_2\,\Bigg[\,2^{-\lambda_M\,(\,j_2\,+\,n(\gamma)\,+\,2\,)}\;2^{\lambda_M\,(\,j_2\,+\,n(\gamma)\,)} \;+\; 2^{-\lambda_M\,(\,j_2\,+\,2n(\gamma)\,+\,5\,)}\;2^{\lambda_M\,(\,j_2\,+\,2n(\gamma)\,+\,3\,)} \\
& \quad +\; 2^{-\lambda_M\,(\,j_2\,+\,3n(\gamma)\,+\,8\,)}\;2^{\lambda_M\,(\,j_2\,+\,3n(\gamma)\,+\,6\,)} \;+\;\cdots\;\Bigg] \\
=\;\; & C\;K_2\,\Big[\,2^{-2\lambda_M} \;+\; 2^{-2\lambda_M} \;+\; 2^{-2\lambda_M} \;+\;\cdots\;\Big] \\
=\;\; & C\;K_2\;2^{-2\lambda_M}\;T(k)\;\to\;\infty \qquad \text{as} \qquad k\to\infty.
\end{aligned}
$$

The proof is now complete. ∎

The main drawback of Theorem 5.3.2 is that it requires the domain in question to be Minkowski measurable. Despite the fact that there has been much recent interest in the conditions required to ensure the Minkowski measurability of a set (see [85], [86] [87] and the references therein) there are still only a handful of domains known to have Minkowski measurable boundary. Indeed, although Federer [57, Section 3.2] shows that smooth m-dimensional surfaces in $\mathbb{R}^m$ are Minkowski measurable, the Minkowski measurability of a set appears to be the exception rather than the rule.

Nevertheless, in Section 4.5, we constructed a 'room and corridor' type domain in $\mathbb{R}^2$. We demonstrated that the boundary of this domain has inner Minkowski dimension equal to 1, showed that it is inner Minkowski measurable and proved that it is an inner γ-domain for $\gamma = 2$. Furthermore, the complement of this domain is both connected and unbounded, so that by Theorem 3.4.8 this domain must admit an L_2-Hardy inequality. Indeed, Ω must admit an L_p-Hardy inequality for all $p \geq 2$ since $\mathbb{R}^m \setminus \Omega$ is uniformly p-fat for all such p. Combining these facts together with Theorem 5.3.2 we arrive at the following result.

Lemma 5.3.3 *Let $\Omega \subset \mathbb{R}^2$ be the 'room and corridor' type domain described in Section 4.5. Then Ω admits an L_p-Hardy inequality for all $p \geq 2$ and the estimate $\mu_p(\Omega) \in \big(0, \,|\,\frac{1-p}{p}\,|^p\,\big]$ holds. In particular, $\mu_2(\Omega) \in (0, 1/4]$.*

Chapter 6

Explicit Results and Optimality

In this chapter we investigate the optimality of the potential structure described by Theorem 2.5.1 and Theorem 2.5.4, with regards to the essential self-adjointness of Schrödinger operators. Naturally this requires us to begin by defining precisely what we mean by 'optimality'. Crudely stated, Theorem 2.5.1 asserts that a Schrödinger operator, $H = -\Delta + V$, is essentially self-adjoint provided that $V(x) \geq [1 - \mu_2(\Omega)]\, d(x)^{-2}$. Within this context the constant $1 - \mu_2(\Omega)$ is said to be optimal if it cannot be replaced by a smaller constant. In other words, this constant is termed optimal if for every $\epsilon > 0$ and every domain $\Omega \subsetneq \mathbb{R}^m$, there exists a Schrödinger operator with potential $V(x) \leq [1 - \mu_2(\Omega) - \epsilon]\, d(x)^{-2}$ that is **not** essentially self-adjoint. Unfortunately, we are unable to prove the optimality of the aforementioned constant in this strict sense, i.e. for every conceivable domain with non-empty boundary. Nevertheless, we are able to show that the constant is optimal in a less restrictive fashion made precise by the following theorem, the validity of which follows from the upcoming Theorem 6.2.4.

Theorem 6.0.4 *There exist domains $\Omega \subsetneq \mathbb{R}^m$, whereby if $H = -\Delta + V$ is a Schrödinger operator defined on $C_0^\infty(\Omega)$ with a real, continuous potential so that $V(x) \leq [1 - \mu_2(\Omega) - \epsilon]\, d(x)^{-2}$, then H is not essentially self-adjoint.*

As we have said previously, Theorem 2.5.4 is a stronger result than Theorem 2.5.1. Indeed, the former implies that a densely defined Schrödinger operator is essentially self-adjoint provided that sufficiently close to the boundary[1]

$$
\begin{aligned}
V(x) &\geq \frac{1}{d(x)^2}\left[\, 1 \;-\; \mu_2(\Omega) \;-\; \mathcal{L}_M(\, d(x)\,)\,\right] \\
&= \frac{1}{d(x)^2}\left[\, 1 \;-\; \mu_2(\Omega) \;-\; \sum_{j=2}^{M}\left(\prod_{k=1}^{j-1} L_k(\, d(x)\,)\right)^{-1}\,\right],
\end{aligned}
$$

[1]The reader is strongly encouraged to re-familiarize themselves appropriate notation given in Section 2.5.2 before continuing.

so that

$$V(x) \;\geq\; \frac{1}{d(x)^2}\left[\,1 - \mu_2(\Omega) - \frac{1}{\ln(d^{-1})} - \frac{1}{\ln(d^{-1})\,\ln\ln(d^{-1})} - \dots \right.$$
$$\left. \dots - \frac{1}{\ln(d^{-1})\,\ln\ln(d^{-1})\dots\underbrace{\ln\ln\dots\ln}_{M-1 \text{ times}}(d^{-1})}\,\right]. \tag{6.1}$$

This raises questions as to the optimality of the constant '1' appearing in front of each logarithmic term in equation (6.1). We ask whether it is possible to replace these constants with larger constants and still guarantee the essential self-adjointness of H on **any** domain in $\mathbb{R}^m$ with non-empty boundary? Once more, we fall short of answering this question in it's entirety, but content ourselves with the following result, the legitimacy of which is immediate from Theorem 6.2.5.

Theorem 6.0.5 *There exist domains $\Omega \subsetneq \mathbb{R}^m$, whereby if $H = -\Delta + V$ is a Schrödinger operator defined on $C_0^\infty(\Omega)$, with real, continuous potential V so that sufficiently close to the boundary*

$$V(x) \;\leq\; \frac{1}{d(x)^2}\left[\,1 - \mu_2(\Omega) - \sum_{j=2}^{M-1}\left(\prod_{k=1}^{j-1} L_k(\,d(x)\,)\right)^{-1} - C\left(\prod_{k=1}^{M-1} L_k(\,d(x)\,)\right)^{-1}\,\right]$$

for some $C > 1$, then H is not essentially self-adjoint.

At this point the reader may wonder as to why we must content ourselves with demonstrating the optimality of the aforementioned constants on specific domains rather than investigating their optimality on generic domains. Indeed, the primary reason for this is that the main mathematical machinery in this chapter, Weyl's limit point - limit circle analysis, is only applicable to one dimensional Schrödinger operators. Consequently, we can only examine one dimensional Schrödinger operators, or domains geometrically simple enough for the corresponding equations to be reduced to the one dimensional case by an appropriate change of variables.

Bearing this in mind, we have decided to structure the chapter as follows. In Section 6.1 we use a result, due to von Neumann, that allows us to demonstrate that a densely defined, symmetric Schrödinger operator H is essentially self-adjoint if and only if the kernel of the the operator $H^* - i$ is trivial. In the case where H is a Schrödinger operator satisfying the conditions of Theorem 2.5.4, and therefore known to be essentially self-adjoint, this allows us to develop a result concerning the spectral properties of its closure. Our attention then turns towards one dimensional Schrödinger operators in Section 6.2. We begin by giving an overview of Weyl's limit point - limit circle analysis, before characterizing the non-essential self-adjontness of such operators in terms of the existence of square integrable solutions to a particular differential equation. Ultimately this will lead us to prove that all the constants appearing in equation (6.1) are optimal in the case where Ω is a subset of the real line. In

Section 6.3, we look at applications of this result in higher dimensions. Indeed, we prove and extend the KWSS theorem stated in Section 1.5.4, before extending related results due to Maeda [81] and Brusentsev [82]. Finally, in Section 6.4, we produce two further higher dimensional examples of domains that validate the conclusion of Theorem 6.0.5.

6.1 Decomposition of the Domain of the Adjoint

Our starting point within this chapter is to recall the result of the Theorem 1.1.8. There it was shown that if A is a densely defined, linear, symmetric operator on the Hilbert space $\mathcal{H}$, then the domain of the adjoint operator, A^*, can be decomposed as follows,

$$D(A^*) \quad = \quad D(\bar{A}) \,\oplus\, \ker(A^* - i) \,\oplus\, \ker(A^* + i). \tag{6.2}$$

By means of this decomposition von Neumann demonstrated that the self-adjoint extensions of the operator A are in one-to-one correspondence with the partial isometries acting between $\ker(A^*-i)$ and $\ker(A^*+i)$. The interested reader is directed to Section 7.7 of [7] for further details. Indeed, if $H = -\Delta + V$ is a Schrödinger operator defined on $C_0^\infty(\Omega)$, where V is a real, locally essentially bounded potential, then it is almost trivial to show that H is linear, densely defined and symmetric. Furthermore, it is also an easy exercise to prove that $\ker(H^* + i) = \overline{\ker(H^* - i)}$, where the superscript line denotes complex conjugation. Therefore, the validity of the following lemma is an immediate consequence of equation (6.2).

Lemma 6.1.1 *Let Ω be a domain in $\mathbb{R}^m$. If $H = -\Delta + V$ is a Schrödinger operator defined on $C_0^\infty(\Omega)$ where V is a real, locally essentially bounded potential, then*

$$D(H^*) \quad = \quad D(\bar{H}) \,\oplus\, \ker(H^* - i) \,\oplus\, \overline{\ker(H^* - i)},$$

so that H is essentially self-adjoint if and only if $\ker(H^* - i)$ is trivial.

From our perspective, the virtue of the above lemma is that it provides us with a criteria for determining when a Schrödinger operator is **not** essentially self-adjoint. It is easy to see that the function Ψ belongs to the kernel of the operator $H^* - i$ if and only if Ψ is a weak, square integrable solution of the equation

$$-\Delta\,\Psi(x) \;+\; V(x)\Psi(x) \;-\; i\Psi(x) \quad = \quad 0. \tag{6.3}$$

Indeed, if $\Psi \in \ker(H^* - i) \subseteq L_2(\Omega)$, then from equation (1.3) it must be the case that $\big\langle\, (H+i)\bar{u}\,,\, \Psi \,\big\rangle = 0$ for all $u \in D(H)$, or equivalently that for all $u(x) \in C_0^\infty(\Omega)$

$$\int_\Omega \big[-\Delta u(x) \;+\; V(x)\,u(x) \;-\; i\,u(x) \big]\,\Psi(x)\;dx \quad = \quad 0$$

so that Ψ is a weak solution of the aforementioned equation, as per Definition C.2.2.

Therefore, if one is able to find a square integrable solution of equation (6.3), then the kernel of the operator $H^* - i$ is non-trivial, and consequently H cannot be essentially self-adjoint. In a sense, the whole of this chapter is devoted to showing that if one imposes weaker conditions on the potential than those specified by Theorem 2.5.1 or Theorem 2.5.4, then it is possible to find such a solution. Yet, before we turn to such matters one should consider the spectral implications of Lemma 6.1.1 in the case where H is a Schrödinger operator that **does** satisfy the conditions of Theorem 2.5.4, and which is, therefore, essentially self-adjoint. The reader is directed to Appendix C for the appropriate definitions.

Theorem 6.1.2 *Let Ω be a domain in $\mathbb{R}^m$ with non-empty boundary. Let $H = -\Delta + V$ be a Schrödinger operator defined on $C_0^\infty(\Omega)$. If $V(x)$ has the potential structure described by Theorem 2.5.4, so that sufficiently close to the boundary*

$$\begin{aligned} V_1(x) \;&\geq\; \frac{1}{d(x)^2}\left[\,1 - \mu_2(\Omega) - \mathcal{L}_M(\,d(x)\,)\,\right] \\ &= \frac{1}{d(x)^2}\left[\,1 - \mu_2(\Omega) - \frac{1}{\ln(d^{-1})} - \frac{1}{\ln(d^{-1})\,\ln\ln(d^{-1})} - \cdots \right. \\ &\qquad \left. \cdots - \frac{1}{\ln(d^{-1})\,\ln\ln(d^{-1})\ldots\underbrace{\ln\ln\ldots\ln}_{M-1 \text{ times}}(d^{-1})}\,\right], \end{aligned}$$

then $\sigma(\bar{H}) \subseteq \mathcal{A} = [-||\,V_2\,||_{L_\infty},\, \infty)$. Consequently, the kernel of the operator $H^* - \lambda$ is trivial for all $\lambda \in \mathbb{C} \setminus \mathcal{A}$. In particular, there are no (strong or weak) square integrable solutions of equation (6.3).

Proof. By Theorem 2.5.4, we know that the closed operator $\bar{H}$ is self-adjoint so that $\sigma(\bar{H}) \subseteq \mathbb{R}$. Let us choose some $C > 0$ and show that the real number $\lambda = -||\,V_2\,||_{L_\infty} - C$ is contained in the resolvent set of $\bar{H}$. By Theorem C.3.1, it suffices to demonstrate that $||\,(\,\bar{H} - \lambda\,)u\,|| \geq C\,||\,u\,||$ for all $u \in D(\bar{H})$. Indeed, it is easy to see that

$$||\,(\,\bar{H} - \lambda\,)u\,||^2 \quad\geq\quad C^2\,||\,u\,||^2 \;+\; 2C\,\left[\,\langle\,\bar{H}u\,,\,u\,\rangle \;+\; ||\,V_2\,||_{L_\infty}\,\langle\,u, u\,\rangle\,\right],$$

so that it remains only to show that the second term on the right hand side of the above inequality is non-negative. In this respect, let $\{u_n\}_{n=1}^\infty$ be a sequence in $D(H) = C_0^\infty(\Omega)$ so that $u_n \to u$. Using equation (1.2) and equation (2.21), the later of which asserts that $V_1(x) \geq -\mu_2(\Omega)\,d(x)^{-2}$, we arrive at the following inequalities

$$\begin{aligned} &\langle\,\bar{H}u\,,\,u\,\rangle \;+\; ||\,V_2\,||_{L_\infty}\,\langle\,u, u\,\rangle \\ =\;& \lim_{n\to\infty}\; -\int_\Omega \Delta u_n\,\bar{u}_n\;dx \;+\; \int_\Omega V_1(x)\,|\,u_n(x)\,|^2\;dx \;+\; \int_\Omega V_2(x)\,|\,u_n(x)\,|^2\;dx \\ &\qquad + \;||\,V_2\,||_{L_\infty}\int_\Omega |\,u_n(x)\,|^2\;dx \\ \geq\;& \lim_{n\to\infty}\;\int_\Omega |\,\nabla u_n\,|^2\;dx \;-\; \mu_2(\Omega)\int_\Omega \frac{|\,\mu_n(x)\,|^2}{d(x)^2}\;dx \qquad\geq\qquad 0, \end{aligned}$$

so that λ is indeed contained in the resolvent set of $\bar{H}$. Since $C > 0$ was chosen arbitrarily, it must be the case that $\sigma(\bar{H}) \subseteq \mathcal{A} = [-||\, V_2\, ||_{L_\infty}, \infty)$. To complete the proof, let us suppose that there exists $\lambda \in \mathbb{C} \setminus \mathcal{A}$ so that $\ker(H^* - \lambda) = \ker(\bar{H} - \lambda)$ is non-trivial. Then λ would be an eigenvalue of the operator $\bar{H}$ and so belong to the spectrum of $\bar{H}$, contradicting the previous result. ∎

6.2 One Dimensional Schrödinger Operators

We now turn our attention to investigating the optimality of the potential structure described by Theorem 2.5.1 and Theorem 2.5.4 in the case where Ω is a subset of the real line. In the case of such one dimensional Schrödinger operators, a barrage of necessary and sufficient conditions for essential self-adjointness have emerged over the course of the last hundred years out of the limit point - limit circle analysis originally developed Weyl [14]. The reader is directed to Chapter XIII.6 of [16], and the multitude of references supplied in [15] for a brief overview of results. Indeed, the conclusion of Berezin & Shubin, stated in Theorem 1.4.1, can be proven by reducing to the one dimensional case and appealing to such analysis. Although, by now, the literature in this area has become somewhat cluttered, one can piece together the following result by combining the work of Titchmarsh, in chapter 2 of [11], with that of Reed & Simon, in Chapter X of [3].

Theorem 6.2.1 *Let* $\Omega = (t, s)$ *where* $-\infty \leq t < s \leq \infty$. *Let* $H = -\frac{d^2}{dx^2} + V$ *be a Schrödinger operator defined on* $C_0^\infty(\Omega)$, *where* $V = \bar{V} \in C(\Omega)$. *Consider the differential equation*

$$-\Psi''(x) \;+\; V(x)\Psi(x) \;-\; i\,\Psi(x) \;=\; 0. \tag{6.4}$$

The solution space of (6.4) is a two dimensional vector space of $C^2(\Omega)$ *functions. Let* $c \in (t, s)$ *and let* $\phi(x)$ *and* $\theta(x)$ *be two (linearly independent) solutions of equation (6.4) so that*

$$\phi(c) = 1, \qquad \phi'(c) = 0, \qquad \theta(c) = 0, \qquad \theta'(c) = 1.$$

Let $L(\cdot\,,\cdot) : \mathbb{C} \times (t, s) \to \mathbb{C}$ *be defined by* $L(z, b) = -\frac{\theta(b)z + \theta'(b)}{\phi(b)z + \phi'(b)}$, *so that for fixed* $b \neq c$ $L(z, b)$ *is a Mobius transformation mapping the real line to a circle,* C_b, *in the complex plane.*

1. *As* $b \to s$, *or as* $b \to t$, *the circle* C_b *tends to either a limit circle or to a limit point.*
2. *There exists at least one solution of equation (6.4) that is in* L_2 *at* s *(i.e which is in* L_2 *in a neighborhood of* s*).*
3. *There exists at least one solution of equation (6.4) that is in* L_2 *at* t.

4. C_b tends to a limit circle as $b \to t$ (vis s) if and only if ϕ and θ are in L_2 at t (vis s).

5. C_b tends to a limit point as $b \to t$ (vis s) if and only if neither ϕ nor θ are in L_2 at t (vis s).

6. C_b tends to a limit point as $b \to s$ and as $b \to t$ if and only if H is essentially self-adjoint.[2]

The value of the above theorem is that it reduces the problem of investigating the non-essential self-adjointness of Schrödinger operators to the problem of examining the integrability of solutions to equation (6.4) near the boundary points. In particular, if one can show that there are two linearly independent solutions of equation (6.4) that are in L_2 at t (vis s), then, since the solution space is two dimensional, all solutions of this equation must be in L_2 at t (vis s). Since Theorem 6.2.1 guarantees the existence of a continuous solution that lies in L_2 at s (vis t), this solution must be in $L_2(t, s)$ and so must belong to the kernel of $H^* - i$. In other words, if we can demonstrate that there are two linearly independent solutions of equation (6.4) that are in L_2 at t, then the corresponding Schrödinger operator H is not essentially self-adjoint. Moreover, the following lemma reduces the question of the non-essential self-adjointness of one dimensional Schrödinger operators to the problem of examining the integrability of the solutions to an even simpler differential equation.

Lemma 6.2.2 *[3, Theorem X.6]*

Let $\Omega = (t, s)$ where $-\infty \le t < s \le \infty$. Let $H = -\frac{d^2}{dx^2} + V$ be a Schrödinger operator defined on $C_0^\infty(\Omega)$, where $V = \bar{V} \in C(\Omega)$. For $\lambda \in \mathbb{C}$, consider the differential equation

$$-\Psi''(x) \quad + \quad V(x)\,\Psi(x) \quad - \quad \lambda\,\Psi(x) \quad = \quad 0. \tag{6.5}$$

If for some $\lambda_0 \in \mathbb{C}$ there are two linearly independent solutions of equation (6.5) that lie in L_2 at t (vis s), then for all $\lambda \in \mathbb{C}$ there are two linearly independent solutions that lie in L_2 at t (vis s). In particular, if there are two linearly independent solutions of the equation

$$-\Psi''(x) \quad + \quad V(x)\,\Psi(x) \quad = \quad 0 \tag{6.6}$$

*that are in L_2 at t (vis s), then H is **not** essentially self-adjoint.*

We are now almost in a position to begin our discussion on the optimality of the potential structures described by Theorem 2.5.1 and Theorem 2.5.4 in the case where Ω is a subset of the real line. However, before we can do so, we require one further result. Since the proof of this result is short, and plays an important role in our subsequent analysis, it is provided.

[2] The reader is warned that in [11] Titchmarsh considers only the domain $\Omega = (0, \infty)$ and chooses two solutions ϕ and θ so that $\phi(0) = 1$, $\phi'(0) = 0$, $\theta(0) = 0$, $\theta'(0) = 1$. This amounts to specifying self-adjoint boundary conditions at zero. To obtain the result above, one must choose $c \in (t, s)$ - see [15].

Lemma 6.2.3 *[3, Effectively Theorem X.10]*

Let $\Omega = (0, s)$ *where* $0 < s \le \infty$. *Suppose that the differential equation*

$$-\Psi''(x) \quad + \quad \tilde{V}(x)\,\Psi(x) \quad = \quad 0, \tag{6.7}$$

where $\tilde{V}(x)$ *is a real, continuous, positive potential, has two linearly independent solutions that are in* L_2 *at zero. If* $V(x) \in C(\Omega)$ *and* $V(x) \le \tilde{V}(x)$ *for all* $x \in (0, b)$, *where* $0 < b < s$, *then the differential equation*

$$-\Psi''(x) \quad + \quad V(x)\,\Psi(x) \quad = \quad 0, \tag{6.8}$$

also has two linearly independent solutions that are in L_2 *at zero.*

Proof. First of all, we note that the solution space of both equations is a two dimensional vector space of $C^2(\Omega)$ functions. Since, by assumption, there are two linearly independent solutions of equation (6.7) that are in L_2 at zero, all solutions of this equation must be in L_2 at zero. Now, let $\tilde{u}_1(x)$ be a solution of (6.7) so that $\tilde{u}_1(b/2) = 2$ and $\tilde{u}_1'(b/2) = -2$. Similarly, let $u_1(x)$ be a solution of (6.8) so that $u_1(b/2) = 1$ and $u_1'(b/2) = -1$. Then it must be the case that $\tilde{u}_1(x) \ge u_1(x)$ for all $x \in (0, b/2]$. If this were not the case then there would exist some $c \in (0, b/2)$ at which $\tilde{u}_1(c) - u_1(c) > 0$ and so that $\tilde{u}_1''(c) - u_1''(c) \le 0$. However, the later of these two inequalities implies that $\tilde{V}(c)\,\tilde{u}_1(c) \le V(c)\,u_1(c) \le \tilde{V}(c)\,u_1(c)$ so that $\tilde{u}_1(c) - u_1(c) \le 0$ providing an obvious contradiction. Since $\tilde{u}_1(x)$ is in L_2 at zero, it must be the case that $u_1(x)$ is also in L_2 at zero.

To complete the proof let $\tilde{u}_2(x)$ be a solution of (6.7) so that $\tilde{u}_2(b/2) = 2$ and $\tilde{u}_2'(b/2) = -4$. Furthermore, let $u_2(x)$ be a solution of (6.8) so that $u_2(b/2) = 1$ and $u_2'(b/2) = -2$. Applying the same reasoning as before yields that $u_2(x)$ is in L_2 at zero. Since $u_1(x)$ and $u_2(x)$ are clearly linearly independent, the proof is complete. ∎

Using the above result, we can now show that if $\Omega = (0, \infty)$, then the constant $1 - \mu_2(\Omega)$ appearing within Theorem 2.5.1 is optimal. Indeed, the following result validates the conclusion of Theorem 6.0.4 stated in the introduction to this chapter.

Theorem 6.2.4 *[3, Theorem X.10]*

Let $\Omega = (0, \infty)$ *and let* $H = -\frac{d^2}{dx^2} + V$ *be a Schrödinger operator defined on* $C_0^\infty(\Omega)$. *If* $V(x)$ *has the potential structure described by Theorem 2.5.1 so that* $V_1(x) \ge 3/4\ d(x)^{-2}$, *then* H *is essentially self-adjoint. However, if* V *is a real, continuous potential so that* $V(x) \le C\,d(x)^{-2}$ *for some* $C < 3/4$, *then* H *is not essentially self-adjoint.*

Proof. By Lemma 5.2.3, if $\Omega = (0,\infty)$, then $\mu_2(\Omega) = 1/4$. Consequently, it is immediate from Theorem 2.5.1 that if V has the potential structure described therein, so that $V_1(x) \geq 3/4\, d(x)^{-2}$, then H is essentially self-adjoint. On the other hand, suppose that $\tilde{V}(x) = C\, d(x)^{-2}$ where we may assume without loss that $0 < C < 3/4$. Then $\tilde{u}_1(x) = x^{1/2\,(1+\sqrt{1+4C}\,)}$ and $\tilde{u}_2(x) = x^{1/2\,(1-\sqrt{1+4C}\,)}$ are linearly independent solutions of the differential equation $-\Psi''(x) + \tilde{V}(x)\,\Psi(x) = 0$. Furthermore, since $C < 3/4$, both these solutions are in L_2 at zero. Using the previous lemma, if V is a real, continuous potential so that $V(x) \leq \tilde{V}(x)$, then there must exist two linearly independent solutions of the differential equation $-\Psi''(x) + V(x)\,\Psi(x) = 0$ which lie in L_2 at zero. By Lemma 6.2.2, this is enough to show that H is not essentially self-adjoint. ∎

Our next step is to show that if Ω is a small, bounded subset of $\mathbb{R}$, then the constant '1' appearing in front of each logarithmic term within Theorem 2.5.4 is optimal. We follow closely the analysis given by Nenciu & Nenciu in [1].

Theorem 6.2.5 *[1, Theorem 3]*

Fix an integer $M \geq 2$ and consider the domain $\Omega = (0, R_0)$ where $R_0 < 1/e_M$ and e_M is given by Definition 2.5.2. Let $H = -\frac{d^2}{dx^2} + V$ be a Schrödinger operator defined on $C_0^\infty(\Omega)$. If V has the potential structure described by Theorem 2.5.4, so that sufficiently close to the boundary

$$V_1(x) \quad \geq \quad \frac{1}{d(x)^2}\left[\, 3/4 \;-\; \sum_{j=2}^{M} \left(\prod_{k=1}^{j-1} L_k(\,d(x)\,) \right)^{-1} \right],$$

then H is essentially self-adjoint. However, if V is a real, continuous potential and there exists some $C > 1$ so that

$$V(x) \quad \leq \quad \frac{1}{d(x)^2}\left[\, 3/4 \;-\; \sum_{j=2}^{M-1} \left(\prod_{k=1}^{j-1} L_k(\,d(x)\,) \right)^{-1} \;-\; C\left(\prod_{k=1}^{M-1} L_k(\,d(x)\,) \right)^{-1} \right],$$

then H is not essentially self-adjoint.[3]

Proof. From Lemma 5.2.2, we have that $\mu_2(\Omega) = 1/4$. It follows immediately from Theorem 2.5.4 that if V has the potential structure described therein, so that sufficiently close to the boundary, $V_1(x) \geq \frac{1}{d(x)^2}\,[\,3/4 - \mathcal{L}_M(\,d(x)\,)\,]$, then H is essentially self-adjoint. As such, it remains only to show that if V is a real, continuous potential so that $V(x) \leq V_M(x)$, where

$$V_M(x) \quad = \quad \frac{1}{d(x)^2}\left[\, 3/4 \;-\; \sum_{j=2}^{M-1} \left(\prod_{k=1}^{j-1} L_k(\,d(x)\,) \right)^{-1} \;-\; C\left(\prod_{k=1}^{M-1} L_k(\,d(x)\,) \right)^{-1} \right],$$

[3]Throughout the chapter we use the convention that if $M = 2$, then the middle term on the right hand side of the above expression equals zero.

for some $C > 1$, then H is not essentially self-adjoint. Let us take the case where $M \geq 3$, the corresponding calculations for the case where $M = 2$ being much simpler.[4] Set

$$\Psi_M(x) \quad = \quad x^{-1/2} \left(\prod_{j=1}^{M-2} L_j(x) \right)^{-1/2} L_{M-1}(x)^{-C/2} \tag{6.9}$$

so that equation (2.17), in addition to some elementary differentiation, gives

$$\Psi_M'(x) \;=\; x^{-1} \left[-\frac{1}{2} + \frac{1}{2} \sum_{j=1}^{M-2} \left(\prod_{k=1}^{j} L_k(x) \right)^{-1} + \frac{C}{2} \left(\prod_{k=1}^{M-1} L_k(x) \right)^{-1} \right] \Psi_M(x)$$

and

$$\begin{aligned} \Psi_M''(x) \;&=\; x^{-2} \left[\frac{3}{4} - \sum_{j=2}^{M-1} \left(\prod_{k=1}^{j-1} L_k(x) \right)^{-1} - C \left(\prod_{k=1}^{M-1} L_k(x) \right)^{-1} + o(1) \right] \Psi_m(x) \\ &\equiv\; \tilde{V}_M(x)\, \Psi_M(x). \end{aligned}$$

Here $o(1)$ represents the following **positive**, higher order terms

$$\begin{aligned} &\frac{3}{4} \sum_{j=1}^{M-2} \left(\prod_{k=1}^{j} L_k(x) \right)^{-2} + \left(\frac{C^2}{4} + \frac{C}{2} \right) \cdot \left(\prod_{k=1}^{M-1} L_k(x) \right)^{-2} \\ &+ \; C \left(\prod_{k=1}^{M-1} L_k(x) \right)^{-1} \cdot \left(\sum_{j=1}^{M-2} \prod_{s=1}^{j} L_s(x)^{-1} \right) \\ &+ \; L_1^{-2}(x) \Big[\big(L_2^{-1}(x) + 1 \big) \cdot \big(L_2^{-1}(x)\, L_3^{-1}(x) + 1 \big) \cdots \\ &\qquad \cdots \big(L_2^{-1}(x)\, L_3^{-1}(x) \ldots L_{M-2}^{-1}(x) + 1 \big) - 1 \Big], \end{aligned}$$

where the last term is omitted in the case where $M = 3$. Then, by definition, $\Psi_M(x)$ is a solution of the differential equation

$$-\Psi''(x) \;+\; \tilde{V}_M(x)\, \Psi(x) \;=\; 0. \tag{6.10}$$

We note that

$$\tilde{V}_M(x) \;\geq\; \frac{1}{x^2} \left[3/4 - \sum_{j=2}^{M-1} \left(\prod_{k=1}^{j-1} L_k(x) \right)^{-1} - C \left(\prod_{k=1}^{M-1} L_k(x) \right)^{-1} \right],$$

and that the expression on the right hand side of the above inequality is equal to $V_M(x)$ for all $x \in (0, R_0/2)$. Hence, by Lemma 6.2.3, if we can show that there are two linearly

[4] If $M = 2$, then set $\Psi_M(x) = x^{-1/2}\, L_1(x)^{-C/2}$.

independent solutions of equation (6.10) which lie in L_2 at zero, then there must also be two linearly independent solutions of the equation $-\Psi''(x) + V(x)\,\Psi(x) = 0$ that lie in L_2 at zero. By Lemma 6.2.2, this is enough to show that H is not essentially self-adjoint.

We have already seen that $\Psi_M(x)$ is a solution of equation (6.10). Given that $C > 1$, it is easy to show that $\Psi_M(x)$ lies in L_2 at zero since the integral

$$\begin{aligned}\int_0^b |\,\Psi_M(x)\,|^2 \; dx &= \int_0^b x^{-1} \left(\prod_{j=1}^{M-2} L_j(x) \right)^{-1} L_{M-1}(x)^{C} \; dx \\ &= \int_0^b x^{-1} \left(\prod_{j=1}^{M-2} L_j(x) \right)^{-1} L_{M-1}(x)^{-1-\epsilon} \; dx \;=\; \frac{1}{\epsilon}\Big[L_{M-1}(x)^{-\epsilon} \Big]_0^b\end{aligned}$$

is evidently finite. A simple calculation indicates that

$$\Phi_M(x) \;=\; \Psi_M(x) \int_0^x \Psi_M(y)^{-2} \; dy \tag{6.11}$$

is also a solution of (6.10). Furthermore, since $W(\Psi_M\,,\,\Phi_M) = 1$, these two solutions are linearly independent. Hence, to complete the proof we must only determine that $\Phi_M(x)$ lies in L_2 at zero. Since $\Phi_M(x)$ is continuous, it suffices to show that this function tends to zero as $x \to 0^+$. Indeed, the validity of this latter statement is easily demonstrated. For y sufficiently small we have that

$$y \left(\prod_{j=1}^{M-2} L_j(y) \right) L_{M-1}(y)^C \;\;\le\;\; y^{-1/2},$$

given that the left hand side of the above equation converges to zero, whilst the right hand side diverges, as $y \to 0^+$. Consequently

$$\begin{aligned}\Phi_M(x) &= \Psi_M(x) \int_0^x \Psi_M(y)^{-2} \; dy \\ &= x^{-1/2} \left(\prod_{j=1}^{M-2} L_j(x) \right)^{-1/2} L_{M-1}(x)^{-C/2} \int_0^x y \left(\prod_{j=1}^{M-2} L_j(y) \right) L_{M-1}(y)^C \; dy \\ &\le x^{-1/2} \left(\prod_{j=1}^{M-2} L_j(x) \right)^{-1/2} L_{M-1}(x)^{-C/2} \int_0^x y^{-1/2} \; dy \\ &= 2 \left(\prod_{j=1}^{M-2} L_j(x) \right)^{-1/2} L_{M-1}(x)^{-C/2} \;\;\to\;\; 0\end{aligned}$$

as $x \to 0^+$. This completes the proof. ∎

6.3 Higher Dimensional Schrödinger Operators

The purpose of this section is to use the result of Theorem 6.2.4 to produce examples of higher dimensional domains upon which the constant $1 - \mu_2(\Omega)$, appearing in Theorem 2.5.1, is optimal from the perspective of the essential self-adjointness of Schrödinger operators. Although, in each case we explore, the optimality of the aforementioned constant is well known, we use the result of Theorem 2.5.4 to further relax existing criteria for essential self-adjointness on these domains. We begin by proving the KWSS theorem first described in Section 1.5.4.

Theorem 6.3.1 *[22, Theorem 2]*

Let $\Omega = \mathbb{R}^m \setminus \{0\}$ and let $H = -\Delta + V$ be a Schrödinger operator defined on $C_0^\infty(\Omega)$. If V has the potential structure described by Theorem 2.5.1, so that

$$V_1(x) \quad \geq \quad \left[1 - \left(\frac{m-2}{2} \right)^2 \right] d(x)^{-2},$$

then H is essentially self-adjoint. However, if V is a real, continuous potential so that $V(x) = C\, d(x)^{-2}$ for some $C < 1 - \left(\frac{m-2}{2} \right)^2$, then H is not essentially self-adjoint.

Proof. First we recall that Lemma 5.2.3 asserts that $\mu_2(\Omega) = \left(\frac{m-2}{2} \right)^2$. Therefore, if V has the potential structure described by Theorem 2.5.1, so that $V_1(x) \geq \left[1 - \left(\frac{m-2}{2} \right)^2 \right] d(x)^{-2}$, then H is essentially self-adjoint. On the other hand, suppose that V is a real, continuous potential and $V(x) = C\, d(x)^{-2}$ for some $C < 1 - \left(\frac{m-2}{2} \right)^2$. By Lemma 6.1.1, in order to show that H is not essentially self-adjoint, it suffices to show that there is a square integrable solution of the equation $-\Delta\Psi + \frac{C}{|x|^2}\Psi - i\,\Psi = 0$. Indeed, we will show that there is a radially symmetric, square integrable solution of this equation. In other words, we will prove that there is a solution of the equation

$$-\frac{d^2}{dr^2}\Psi(r) \quad - \quad \frac{m-1}{r}\frac{d}{dr}\Psi(r) \quad + \quad \frac{C}{r^2}\Psi(r) \quad - \quad i\,\Psi(r) \quad = \quad 0 \tag{6.12}$$

which lies in $L_2(\Omega)$. To this end, we note that, by Lemma 6.1.1 and Theorem 6.2.4, there is a solution of the one dimensional equation

$$-z''(r) \quad + \quad \frac{1}{r^2}\left[C \; + \; \frac{(m-1)(m-3)}{4} \right] z(r) \quad - \quad i\,z(r) \quad = \quad 0 \tag{6.13}$$

that lies in $L_2(0,\infty)$, if and only if $C + \frac{(m-1)(m-3)}{4} < \frac{3}{4}$, i.e. if and only if $C < 1 - \left(\frac{m-2}{2} \right)^2$. Let $z_1(r)$ be such a solution of equation (6.13). Then it is not difficult to see that the function $\Psi_1(r) = r^{-\frac{m-1}{2}} z_1(r)$ is a solution of equation (6.12), since

$$
\begin{aligned}
& - \frac{d^2}{dr^2}\, \Psi_1(r) \quad - \quad \frac{m-1}{r}\, \frac{d}{dr}\, \Psi_1(r) \quad + \quad \frac{C}{r^{\,2}}\, \Psi_1(r) \quad - \quad i\, \Psi_1(r) \\
= \quad & r^{-\frac{m-1}{2}} \left(\; - z_1''(r) \quad + \quad \frac{1}{r^{\,2}} \left[\, C \; + \; \frac{(m-1)(m-3)}{4} \,\right] z_1(r) \quad - \quad i\, z_1(r) \right) \quad = \quad 0.
\end{aligned}
$$

Furthermore, $\Psi_1(r)$ must lie in $L_2(\Omega)$ because the integral

$$
\int_\Omega \, |\, \Psi_1(r)\,|^2 \; dx \quad = \quad |\, \Sigma_{m-1}\,| \, \int_0^\infty r^{-(m-1)} \; |\, z_1(r)\,|^{\,2} \; r^{\,m-1} \; dr
$$

is evidently finite. This completes the proof. ∎

Although we have just proven the constant $1 - \mu_2(\Omega)$ is optimal in the case where $\Omega = \mathbb{R}^m \setminus \{0\}$, we may still extend the KWSS theorem simply by combining Lemma 5.2.3 and Theorem 2.5.4. Indeed, by doing so we obtain the following result.

Theorem 6.3.2 *Let* $\Omega = \mathbb{R}^m \setminus \{0\}$, *and let* $H = -\Delta + V$ *be a Schrödinger operator defined on* $C_0^\infty(\Omega)$, *where* V *has the potential structure described by Theorem 2.5.4, so that sufficiently close to the boundary*

$$
\begin{aligned}
V_1(x) \quad & \geq \quad \frac{1}{d(x)^2} \left[\, 1 \; - \; \left(\frac{m-2}{2} \right)^2 \; - \; \mathcal{L}_M(\, d(x)\,) \,\right] \\
& = \quad \frac{1}{d(x)^2} \left[\, 1 \; - \; \left(\frac{m-2}{2} \right)^2 \; - \; \frac{1}{\ln(d^{-1})} \; - \; \frac{1}{\ln(d^{-1})\, \ln\ln(d^{-1})} \; - \; \ldots \right. \\
& \qquad \left. \ldots - \; \frac{1}{\ln(d^{-1})\, \ln\ln(d^{-1}) \ldots \underbrace{\ln\ln \ldots \ln}_{M-1 \; times} (d^{-1})} \,\right].
\end{aligned}
$$

Then the operator H *is essentially self-adjoint.*

At this point a brief remark comparing the KWSS theorem to the above result is in order. In the case where $m = 4$, the KWSS theorem asserts that the Schrödinger operator H is essentially self-adjoint provided that the potential $V_1(x)$ is non-negative. In contrast, Theorem 6.3.2 implies that H is still essentially self-adjoint even if this potential dives slowly to minus infinity as it approaches the boundary at zero.

We now turn our attention to the domain $\Omega = \mathbb{R}^m \setminus E$ where E is an affine set of dimension k and $0 \leq k \leq m-1$. In [81], Maeda demonstrated that on this domain a Schrödinger operator with the potential structure described by Theorem 2.5.1 is essentially self-adjoint. Moreover, Brusentsev [82] showed that the constant $1 - \mu_2(\Omega)$ appearing in Theorem 2.5.1 is optimal on this domain. Indeed, we can recapture their results in the following manner.

Theorem 6.3.3 *[81] & [82, Section 6]*

Let $\Omega = \mathbb{R}^m \setminus E$ where E is an affine set of dimension k and $0 \leq k \leq m-1$. Let $H = -\Delta + V$ be a Schrödinger operator defined on $C_0^\infty(\Omega)$. If V has the potential structure described by Theorem 2.5.1, so that

$$V_1(x) \quad \geq \quad \left[1 - \left(\frac{m-k-2}{2} \right)^2 \right] d(x)^{-2},$$

then H is essentially self-adjoint. However, if V is a real, continuous potential so that $V(x) = C\, d(x)^{-2}$ where $C < 1 - \big(\frac{m-k-2}{2} \big)^2$, then H is not essentially self-adjoint.

Proof. We begin by noting that, according to Lemma 5.2.3, $\mu_2(\Omega) = \big(\frac{m-k-2}{2} \big)^2$. Therefore, if V has the potential structure described by Theorem 2.5.1, so that $V_1(x) \geq \big[1 - \big(\frac{m-k-2}{2} \big)^2 \big] d(x)^{-2}$, it follows immediately that H is essentially self-adjoint. Instead, let us assume that V is a real, continuous potential and that $V(x) = C\, d(x)^{-2}$ for some $C < 1 - \big(\frac{m-k-2}{2} \big)^2$. In order to show that H is not essentially self-adjoint, by Lemma 1.1.7, it suffices to show that the adjoint operator H^* is not symmetric. For notational convenience, we let

$$x' \quad = \quad (\, x_1, \, \ldots \, , x_k \,) \qquad \text{and} \qquad x'' \quad = \quad (\, x_{k+1}, \, \ldots \, , x_m \,).$$

Similarly, we define the operators $\Delta' = \sum_{j=1}^{k} \frac{\partial^2}{\partial x_j^2}$ and $\Delta'' = \sum_{j=k+1}^{m} \frac{\partial^2}{\partial x_j^2}$. By a rotation of the co-ordinate axes, if necessary, we may assume without loss of generality that $d(x) = d(x'') = \sqrt{x_{k+1}^2 + \cdots + x_m^2}$. Then, according to Theorem 6.3.1, there is a solution of the equation

$$-\,\Delta''\, \Psi(x'') \quad + \quad C\, d(x'')^{-2}\, \Psi(x'') \quad - \quad i\, \Psi(x'') \quad = \quad 0 \tag{6.14}$$

that lies in $L_2(\mathbb{R}^{m-k} \setminus \{0\})$ if and only if $C < 1 - \big(\frac{m-k-2}{2} \big)^2$. Let $\Psi_1(x'')$ be such a solution of equation (6.14), and let $\rho(x') \in C_0^\infty(\mathbb{R}^k)$. It is easily seen that the function $u(x) = \rho(x')\, \Psi_1(x'')$ is contained in $D(H^*)$. Now if H^* is symmetric, then it is obviously true that $Im \big\langle H^* u \,,\, u \big\rangle = 0$. However,

$$\begin{aligned} \big\langle H^* u \,,\, u \big\rangle \quad &= \quad \big\langle -\, \Delta' \rho\, \Psi_1 \; - \; \Delta'' \rho\, \Psi_1 \; + \; V \rho\, \Psi_1 \;,\; \rho\, \Psi_1 \big\rangle \\ &= \quad \big\langle -\, \Psi_1\, \Delta' \rho \; + \; i\, \rho\, \Psi_1 \;,\; \rho\, \Psi_1 \big\rangle \\ &= \quad \left(\int_{\mathbb{R}^k} |\, \nabla \rho(x') \,|^2 \; dx' \; + \; i \int_{\mathbb{R}^k} |\, \rho(x') \,|^2 \; dx' \right) ||\, \Psi_1(x'') \,||^2_{L_2(\mathbb{R}^{m-k} \setminus \{0\})} \end{aligned}$$

so that $Im \big\langle H^* u \,,\, u \big\rangle \neq 0$. This shows that H^* is not symmetric, and, therefore, that the operator H is not essentially self-adjoint. ∎

Once more we have proven that, on the domain $\Omega = \mathbb{R}^m \setminus E$, the constant $1 - \mu_2(\Omega)$ appearing in Theorem 2.5.1 is optimal from the perspective of the essential self-adjointness of Schrödinger operators. Nevertheless, we can extend the previous result by a simple application of Lemma 5.2.3 and Theorem 2.5.4, which taken together give the following.

Theorem 6.3.4 *Let $\Omega = \mathbb{R}^m \setminus E$ where E is an affine set of dimension k and $0 \le k \le m-1$. Let $H = -\Delta + V$ be a Schrödinger operator defined on $C_0^\infty(\Omega)$. If V has the potential structure described by Theorem 2.5.4, so that sufficiently close to the boundary*

$$\begin{aligned} V_1(x) &\ge \frac{1}{d(x)^2}\left[1 - \left(\frac{m-k-2}{2}\right)^2 - \mathcal{L}_M(\,d(x)\,)\right] \\ &= \frac{1}{d(x)^2}\left[1 - \left(\frac{m-k-2}{2}\right)^2 - \frac{1}{\ln(d^{-1})} - \frac{1}{\ln(d^{-1})\,\ln\ln(d^{-1})} - \dots \right. \\ &\qquad\qquad \left. \dots - \frac{1}{\ln(d^{-1})\,\ln\ln(d^{-1})\dots\underbrace{\ln\ln\dots\ln}_{M-1\ \textit{times}}(d^{-1})}\right], \end{aligned}$$

then H is essentially self-adjoint.

Again it is worthwhile pointing out that in the case where $m - k = 4$, the result of Maeda [81] implies that $V_1(x)$ must be non-negative in order to ensure the essential self-adjointness of the operator H, whereas the above theorem allows for the potential to dive slowly to minus infinity as it approaches the boundary.

6.4 Further Examples

In the previous section we constructed two examples of domains upon which the constant $1-\mu_2(\Omega)$ appearing in Theorem 2.5.1 is optimal. We then used the result of Theorem 2.5.4 to relax known criteria for essential self-adjointness on these domains, effectively by subtracting higher order logarithmic terms. However, the reader will have noticed that we did not show that the constant '1' appearing in front of each of these terms is optimal. The reason for this is that these logarithmic terms, e.g. $L_2(\,d(x)\,) = \ln\ln(\,1/d(x)\,)$, are only well defined for x very close to the boundary of Ω. This causes problems when one attempts to demonstrate the existence of square integrable solutions to the equation $-\Delta\,\Psi + V\,\Psi - i\,\Psi = 0$. As such, the purpose of the current section is to address this deficiency by producing examples of higher dimensional domains upon which the aforementioned constants are optimal. This is achieved within the next two theorems.

Theorem 6.4.1 *Fix an integer $M \ge 2$ and define the domain $\Omega = B(0, R_0) \subset \mathbb{R}^3$, where $0 < R_0 < 1/e_M$ and e_M is given by Definition 2.5.2. Let $H = -\Delta + V$ be a Schrödinger* *operator defined on* $C_0^\infty(\Omega)$, *where* V *is a real, continuous, radially symmetric potential and for some* $C > 1$

$$V(r) \;=\; \frac{1}{d(x)^2}\left[\,\frac{3}{4} \;-\; \sum_{j=2}^{M-1}\left(\prod_{k=1}^{j-1} L_k(\,d(x)\,)\right)^{-1} \;-\; C\left(\prod_{k=1}^{M-1} L_k(\,d(x)\,)\right)^{-1}\,\right].$$

Then H *is not essentially self-adjoint.*

Proof. First of all, let us note that Ω is a bounded, convex domain with smooth boundary of co-dimension 1 so that, by Lemma 5.2.2, $\mu_2(\Omega) = 1/4$. Furthermore, on this domain $d(x) = R_0 - |x| = R_0 - r$, so that we may re-write the potential V in the form

$$V(r) \;=\; \frac{1}{(R_0\,-\,r)^2}\left[\,\frac{3}{4} - \sum_{j=2}^{M-1}\left(\prod_{k=1}^{j-1} L_k(\,R_0 - r\,)\right)^{-1} \;-\; C\left(\prod_{k=1}^{M-1} L_k(\,R_0 - r\,)\right)^{-1}\,\right].$$

Since $C > 1$, then in exactly the same way as in the proof of Theorem 6.2.5 it is possible to show that there are two linearly independent solutions of the equation $-z''(r) + V(r)\,z(r) = 0$ that lie in L_2 at R_0.[5] Therefore, by successive applications of Lemma 6.2.2 and Theorem 6.2.1, there must be a solution of the equation

$$-\,z''(r) \quad + \quad V(r)\,z(r) \quad - \quad i\,z(r) \quad = \quad 0 \tag{6.15}$$

that lies in $L_2(0, R_0)$. Let $z_1(r)$ be such a solution of equation (6.15). Then the function $\Psi_1(r) = r^{-1}\,z_1(r)$ is a radially symmetric solution of the equation $-\Delta\Psi + V\,\Psi - i\,\Psi = 0$, since the expression

$$-\Psi_1''(r) - \frac{2}{r}\,\Psi_1'(r) + V(r)\,\Psi_1(r) - i\,\Psi_1(r) \quad = \quad r^{-1}\left[\,-\,z_1''(r) + V(r)\,z_1(r) - i\,z_1(r)\,\right]$$

must equal zero. Furthermore, Ψ_1 must be contained in $L_2(\Omega)$ because the integral

$$\int_\Omega |\,\Psi_1\,|^2\;dx \quad = \quad 4\pi\int_0^{R_0} r^{-2}\,|\,z_1(r)\,|^2\,r^{\,2}\;dr$$

is finite. By Lemma 6.1.1 this is enough to show that the operator H is not essentially self-adjoint. ∎

The following Theorem can be considered a mild extension of Theorem 6.3.3.

Theorem 6.4.2 *Fix an integer* $M \geq 2$ *and define the thin strip domain* $\Omega = \{\,x \in \mathbb{R}^m \mid 0 < x_m < R_0\,\}$, *where* $0 < R_0 < 1/e_M$ *and* e_M *is given by Definition 2.5.2. Let* $H = -\Delta + V$ *be a Schrödinger operator defined on* $C_0^\infty(\Omega)$, *where* V *is*

[5] In fact, the solutions are given by $\Psi(R_0 - r)$ and $\Phi_M(R_0 - r)$ where Ψ and Φ are defined by equation (6.9) and equation (6.11) respectively.

a real, continuous potential and for some $C > 1$

$$V(x) \;=\; \frac{1}{d(x)^2}\left[\,\frac{3}{4} \;-\; \sum_{j=2}^{M-1}\left(\prod_{k=1}^{j-1} L_k(\,d(x)\,)\right)^{-1} \;-\; C\left(\prod_{k=1}^{M-1} L_k(\,d(x)\,)\right)^{-1}\,\right].$$

Then H *is not essentially self-adjoint.*

Proof. Without any loss of generality, we may assume that on this domain $d(x) = d(x_m) = \inf\{x_m, R_0 - x_m\}$. Hence, by Theorem 6.2.5, there must be two linearly independent solutions of the one dimensional equation $-u''(x_m) + V(x_m)\,u(x_m) = 0$ that lie in L_2 at zero. As such, together Lemma 6.2.2 and Theorem 6.2.1 imply that there is a solution of the equation

$$-u''(x_m) \quad + \quad V(x_m)\,u(x_m) \quad - i\,u(x_m) \quad = \quad 0 \tag{6.16}$$

which lies in $L_2(0, R_0)$. If $u_1(x_m)$ is such a solution of equation (6.16), and if $\rho(x') \in C_0^\infty(\mathbb{R}^{m-1})$, where x' denotes the $(m-1)$-tuple $(x_1, \ldots, x_{m-1})$, then it is easy to see that the function $\Psi(x) = \rho(x')\,u_1(x_m)$ is contained in $D(H^*)$. Furthermore, using the notation $\Delta' = \sum_{j=1}^{m-1} \frac{\partial^2}{\partial x_j^2}$ and $\Delta'' = \frac{\partial^2}{\partial x_m^2}$, we arrive at the following equalities

$$\begin{aligned} &\left\langle\, H^*\Psi\,,\,\Psi\,\right\rangle \\ = \;& \left\langle\, -\,\Delta'\rho\,u_1 \,-\, \Delta''\rho\,u_1 \,+\, V\rho\,u_1\,,\,\rho\,u_1\,\right\rangle \;=\; \left\langle\, -\,u_1\,\Delta'\rho \,+\, i\,\rho\,u_1\,,\,\rho\,u_1\,\right\rangle \\ = \;& \left(\int_{\mathbb{R}^{m-1}} |\,\nabla\rho(x')\,|^2\;dx' \,+\, i\int_{\mathbb{R}^{m-1}} |\,\rho(x')\,|^2\;dx'\right)\int_0^{R_0} |\,u_1(x_m)\,|^2\;dx_m. \end{aligned}$$

Consequently, we see that $Im\left\langle\, H^*\Psi\,,\,\Psi\,\right\rangle \neq 0$ so that H^* is not symmetric. By Lemma 1.1.7, this is enough to show that H is not essentially sef-adjoint. ∎

Chapter 7

Sequitur

Throughout this chapter it will be understood that Ω denotes a domain in $\mathbb{R}^m$ with non-empty boundary. Furthermore, unless otherwise stated, H will denote the Schrödinger operator, $H = -\Delta + V$, defined on $C_0^\infty(\Omega)$, where $V = V_1 + V_2 \in L_{\infty,\mathrm{loc}}(\Omega)$, $V_2 \in L_\infty(\Omega)$ and $V_1 \in L_{\infty,\mathrm{loc}}(\Omega)$, as per the conditions of Theorem 2.1.1. We began this thesis by asking what conditions ensure that H is essentially self-adjoint, i.e. under what conditions is the closed operator $\bar{H}$ self-adjoint. Indeed, in Chapter 1 we provided a detailed description of what it means for an operator acting within a Hilbert space to be essentially self-adjoint and demonstrated that the essential self-adjointness of such a Schrödinger operator corresponds to the inability of a particle, under the influence of the associated potential V, to come into contact with the boundary of the domain.

In this respect, the most significant result of the thesis is Theorem 2.1.1. However, this theorem is abstract in nature, in the sense that the criteria for essential self-adjointness expressed therein is phrased in terms of an auxiliary function $G(t)$ satisfying various criteria. Therefore, the conclusion of Theorem 2.1.1 can be most easily understood in terms of it's more explicit realization - Theorem 2.5.1 - which asserts that H is essentially self-adjoint provided that

$$V_1(x) \quad \geq \quad \frac{1-\mu_2(\Omega)}{d(x,\partial\Omega)^2}. \tag{7.1}$$

Here $\mu_2(\Omega)$ is the variational constant associated to the L_2-Hardy inequality. We recall that this constant is non-zero if and only if the domain Ω admits an L_2-Hardy inequality.

From equation (7.1) we see that there are two factors governing the essential self-adjointness of the operator H - the inflation of the potential like $d(x,\partial\Omega)^{-2}$ and the value of the variational constant $\mu_2(\Omega)$. With regard to the former, Theorem 2.5.1 implies that given **any** domain Ω, the operator H is essentially self-adjoint provided that $V_1(x) \geq d(x,\partial\Omega)^{-2}$. One may interpret this as saying that if the potential inflates at this rate, then the probability of finding a particle under it's influence at the boundary is zero. On the other hand, if a domain admits an L_2-Hardy inequality, so that $\mu_2(\Omega) > 0$, then this relaxes the criteria for the essential self-adjointness of H. The physical reason for this is that the value of $\mu_2(\Omega)$ places limits on the certainty with which we can say that a particle is located at the boundary. Indeed, if the particle is represented by the unit state vector $\omega \in W^1_{2,0}(\Omega)$, then from equation (2.4) we have that

$$\mu_2(\Omega) \quad \leq \quad \Big(\int_\Omega d(x)^2 \,|\, \omega(x) \,|^2 \;\; dx \Big) \cdot \Big(\int_\Omega |\, \nabla\, \omega(x) \,|^2 \;\; dx \Big).$$

The first integral term on the right hand side of the above expression is a measure of the particle's proximity to the boundary, whilst the second integral represents the particles total momentum. If $\mu_2(\Omega) > 0$, the more highly localized the particle is near to the boundary the greater it's total momentum must be. In short, one cannot confine a particle to a smaller neighborhood of the boundary without producing a corresponding increase in it's total momentum.

Moreover, in Theorem 2.5.4, we were able to relax the criteria for the essential self-adjointness of the operator H even further. In simplistic terms, the aforementioned theorem asserts that H is essentially self-adjoint provided that sufficiently close to the boundary

$$V_1(x) \quad \geq \quad \frac{1}{d(x)^2} \Bigg[1 - \mu_2(\Omega) \quad - \quad \frac{1}{\ln(d^{-1})} \quad - \quad \frac{1}{\ln(d^{-1})\, \ln\ln(d^{-1})} \quad - \; \dots$$
$$\dots - \frac{1}{\ln(d^{-1})\, \ln\ln(d^{-1}) \dots \underbrace{\ln\ln\dots\ln}_{M-1 \text{ times}}(d^{-1})} \Bigg]$$

We note that if $\mu_2(\Omega) \geq 1$, then the potential may dive to minus infinity as it approaches the boundary and yet still be sufficiently strong to ensure the essential self-adjointness of H.

However, Theorem 2.5.1 and Theorem 2.5.4 are not explicit. That is to say that given an arbitrary domain Ω, we do not know the value of the variational constant $\mu_2(\Omega)$ - or even if this constant is non-zero. Consequently, from the perspective of the essential self-adjointness of the operator H, two questions remain. First, given a domain Ω is $\mu_2(\Omega) > 0$, i.e. does the domain admit an L_2-Hardy inequality? Secondly, can we find the exact value of $\mu_2(\Omega)$, or at least estimate it from below?

In Chapter 3, we answered the first of these questions conclusively, providing various necessary and sufficient conditions for the existence of an L_p-Hardy inequality. Most importantly, we characterized the L_p-Hardy inequality in terms of the 'thickness' of the boundary, or complement, of a domain. That is to say that within Theorem 3.3.2, it was shown that if Ω is a domain which admits an L_p-Hardy inequality, then either $\dim_{\mathcal{H}}(\partial\Omega) > m - p$ (the boundary is fat) or $\dim_A(\partial\Omega) < m - p$ (the boundary is thin). In particular, if the (Hausdorff, Minkowski or Aikawa) dimension of the boundary is equal to $m - p$, then Ω cannot admit an L_p-Hardy inequality. On the other hand, it was also shown that these necessary conditions are almost sufficient conditions. In Theorem 3.4.6, we stated that if the complement of a domain is uniformly p-fat, then that domain admits an L_p-Hardy inequality and $\dim_{\mathcal{H}}(\partial\Omega) > m - p$. In addition, Theorem 3.5.2 asserts that in order for a domain to admit an L_p-Hardy inequality it suffices that $\dim_A(\partial\Omega) < m - p$. Furthermore, in the borderline case where $m = p$, we found that the existence of an L_p-Hardy inequality is equivalent to the uniform perfectness and unboundedness of the complement of a domain (see Theorem 3.4.7).

Whilst we may have answered the first question comprehensively, the answer to the second question remains elusive. Nevertheless, we have made some progress in this regard. Indeed, our first real foray towards determining the value of $\mu_p(\Omega)$ came within Theorem 3.6.1. There we proved that if there exists a real number $\alpha \geq 0$ so that

$$(\alpha - p) \int_\Omega d(x)\, \nabla d(x) \cdot \nabla \phi(x) \; + \; \alpha\, \phi(x) \; dx \quad \leq \quad 0$$

for all non-negative functions $\phi(x) \in \hat{C}_0^1(\Omega)$, then $\mu_p(\Omega) \geq \left| \frac{\alpha - p}{p} \right|^p$. At present only a handful of domains are known to satisfy this geometric condition. Despite this, in Lemma 3.6.2, we showed that if $\Omega = \mathbb{R}^m \setminus E$, where E is an affine set of dimension k and $0 \leq k \leq m-1$, then $\mu_p(\Omega) \geq \left| \frac{m-k-p}{p} \right|^p$. Similarly, in Lemma 3.6.3, it was also proven that if Ω is a bounded, convex domain with smooth boundary of co-dimension 1, then $\mu_p(\Omega) \geq \left| \frac{m-(m-1)-p}{p} \right|^p = \left| \frac{1-p}{p} \right|^p$. Moreover, in Section 5.2, these lower bounds for $\mu_p(\Omega)$ were also shown to hold with equality.

The previous examples suggest that if Ω is a domain with Ahlfors λ-regular boundary of dimension λ, then

$$\mu_p(\Omega) \quad = \quad \left| \frac{m - \lambda - p}{p} \right|^p. \tag{7.2}$$

We recall that the condition of the Ahlfors λ-regularity of the boundary ensures that the Hausdorff and Aikawa dimensions of the boundary coincide (see Theorem A.5.1). Indeed, whenever equation (7.2) holds, many of the intricacies of the L_p-Hardy inequality, elucidated in Chapter 3, emerge as simple consequences of this equality.

For instance, suppose that Ω is a domain which has (Ahlfors λ-regular) boundary of dimension $\lambda = m - p$. Then, by Theorem 3.3.2, Ω cannot admit an L_p-Hardy inequality. Indeed, if equation (7.2) holds, then this fact is obvious, since $\mu_p(\Omega)$ would be equal to zero in this case. Furthermore, the right hand side of equation (7.2) is continuous in p. Therefore, if Ω admits an L_{p_0}-Hardy inequality so that $\mu_{p_0}(\Omega) > 0$, and if the aforementioned equality holds, then Ω must admit an L_p-Hardy inequality for all p sufficiently close to p_0. This is a reflection of Theorem 3.3.1, which asserts that if Ω admits an L_{p_0}-Hardy inequality, then there exists some $\epsilon = \epsilon(p_0, \Omega)$ so that Ω admits an L_p-Hardy inequality for all $p \in (p_0 - \epsilon, p_0 + \epsilon)$. Finally, if Ω is a domain with uniformly p_0-fat complement, then, by Theorem 3.4.6, Ω must admit an L_{p_0}-Hardy inequality and $m - \lambda - p_0 < 0$. If equation (7.2) also holds, then Ω must admit an L_p-Hardy inequality for all $p \geq p_0$, because $\mu_p(\Omega) > 0$ for all such p. Again, this reflects the remarks made immediately after Lemma 3.4.6, which imply that if the complement of a domain is uniformly p_0-fat then that domain admits an L_p-Hardy inequality for all $p \geq p_0$.

Unfortunately, we know that equation (7.2) does not always hold. At the end of Section 3.4, we considered the domain $\mathbb{R}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is the middle thirds Cantor set. Here, $\lambda = \dim_{\mathcal{H}}(\partial\Omega) = \dim_{\mathcal{H}}(\mathcal{C}) = \frac{\log 2}{\log 3} \approx 0.631$.[1] On the one hand, if equation (7.2) holds, then we would have that $\mu_2(\Omega) = \frac{(\log 2/\log 3)^2}{4} > 0$, so that Ω must admit an L_2-Hardy inequality. On the other hand, the complement of this domain is bounded, so that by Theorem 3.4.7, Ω cannot admit an L_2-Hardy inequality. We conclude that equation (7.2) cannot hold in this case.

In some respect, we may consider Chapter 4 and Chapter 5 of this thesis to be an attempt to establish the conditions under which equation (7.2) does hold. Indeed, in Lemma 5.1.1, we showed that if $\phi(x) \in W^1_{\infty,0}(\Omega)$ is a real valued function taking values in the interval $[0,1]$, then for all $p > 1$

$$\mu_p(\Omega) \quad \leq \quad \left| \frac{m-\lambda-p}{p} \right|^p \qquad +$$

$$\left(\int_\Omega \frac{\phi^p(x)}{d(x)^{m-\lambda}}\, dx \right)^{-1} C_p \left[\left| \frac{m-\lambda-p}{p} \right|^{p-1} \int_\Omega \frac{|\nabla\phi(x)|}{d(x)^{m-\lambda-1}}\, dx + \int_\Omega \frac{|\nabla\phi(x)|^p}{d(x)^{m-\lambda-p}}\, dx \right].$$

As such, establishing the upper bound $\mu_p(\Omega) \leq \left| \frac{m-\lambda-p}{p} \right|^p$ reduces to the problem of constructing a sequence of appropriate functions $\{\phi_k(x)\}_{k=1}^\infty$, for which the integrals

$$\int_\Omega \frac{|\nabla\phi_k(x)|}{d(x)^{m-\lambda-1}}\, dx \qquad \text{and} \qquad \int_\Omega \frac{|\nabla\phi_k(x)|^p}{d(x)^{m-\lambda-p}}\, dx$$

remain bounded, whilst the integral

$$\int_\Omega \frac{\phi_k^p(x)}{d(x)^{m-\lambda}}\, dx$$

diverges, as $k \to \infty$. In preparation for this, Chapter 4 was devoted to studying the relationship between the (inner) Minkowski dimension of the boundary of a domain and the number of cubes occurring in sequential generations of the corresponding Whitney decomposition of the domain. In particular, within Lemma 4.3.1 and Lemma 4.3.2, we were able to bound the number of such cubes above and below given the finiteness and positivity of the λ-(inner) Minkowski content of the boundary respectively. Ultimately, this enabled us to recapture, by all together new methods, the results of Zubrinić (see [69], [70] and [71]) in characterizing the (inner) Minkowski dimension of the boundary of a domain in terms of the integrability of the function $d(x, \partial\Omega)^{\lambda-m}$. More specifically, in

[1] We note that by the remarks at the end of Section A.5, $\mathcal{C}$ is Ahlfors $\frac{\log 2}{\log 3}$-regular.

Theorem 4.4.1, it was shown that if Ω is an inner γ-domain with non-empty, compact boundary of well defined inner Minkowski dimension, then

$$\begin{aligned} \dim_{\hat{M}}(\partial\Omega) &= \sup \left\{ \lambda \geq 0 \;\middle|\; \int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{\,m-\lambda}} \, dx \;=\; \infty \right\} \\ &= \inf \left\{ \lambda \geq 0 \;\middle|\; \int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{\,m-\lambda}} \, dx \;<\; \infty \right\}. \end{aligned}$$

Indeed, it was effectively this characterization which allowed us to obtain the main results of Chapter 5. In Theorem 5.2.1, it was shown that if Ω is a domain in $\mathbb{R}^m$ with non-empty smooth boundary of integer dimension k, where $0 \leq k \leq m-1$, then $\mu_p(\Omega) \leq \left| \frac{m-k-p}{p} \right|^p$. This enabled us to produce exact values for $\mu_p(\Omega)$ in the case where Ω is a bounded, convex domain with smooth boundary of co-dimension 1, as well as in the case where $\Omega = \mathbb{R}^m \setminus E$ and E is an affine set of dimension k. More importantly, in Theorem 5.3.2, we were able to relax the conditions on the smoothness of the boundary. Indeed, we demonstrated that if Ω is an inner γ-domain with non-empty, compact, inner Minkowski measurable boundary of inner Minkowski dimension λ, then $\mu_p(\Omega) \leq \left| \frac{m-\lambda-p}{p} \right|^p$. However, any further insight into the nature of the variational constant $\mu_p(\Omega)$ eluded us.

Finally, in Chapter 6, we combined the results of the previous chapter with Theorem 2.5.1 and Theorem 2.5.4 in order to recapture, and extend, existing results concerning the essential self-adjointness of Schrödinger operators on domains whereby the value of $\mu_2(\Omega)$ is known explicitly. Furthermore, by utilizing Weyl's limit point - limit circle analysis, we were able to show that the potential structure described by the aforementioned theorems is optimal on these domains.

7.1 Further Research

Having reached the end of the this thesis, one cannot help thinking that there are now more unresolved questions than when we first began. The purpose of this section is to highlight what we think are the most important of these and, where possible, indicate how they might be resolved. One may think of these questions as taking two distinct forms. Indeed, the first six questions pertain to the value of the variational constant $\mu_p(\Omega)$. In contrast, the remainder of the questions relate more to the essential self-adjointness of our Schrödinger operator H than to the L_p-Hardy inequality.

1. As indicated in the previous section, from the perspective of the essential self-adjointness of the operator H, we require lower bounds for the variational constant $\mu_p(\Omega)$. Indeed, in Theorem 3.6.1, we were able to show that if there exists a real number $\alpha \geq 0$ so that

$$(\alpha - p) \int_\Omega d(x)\, \nabla d(x) \cdot \nabla \phi(x) \;+\; \alpha\, \phi(x) \;\; dx \;\;\leq\;\; 0 \tag{7.3}$$

for all non-negative $\hat{C}_0^1(\Omega)$, then $\mu_p(\Omega) \geq \left| \frac{\alpha - p}{p} \right|^p$. In this respect, the obvious question arises as to the necessity of this condition. That is to say that if Ω is a domain which admits an L_p-Hardy inequality, then must there exist some non-negative $\alpha \neq p$ so that equation (7.3) holds?

More important is the following question. If Ω is a domain in $\mathbb{R}^m$ with Ahlfors λ-regular boundary, then (under what additional conditions) does the inequality

$$(m - \lambda - p) \int_\Omega d(x)\, \nabla d(x) \cdot \nabla \phi(x) \;+\; \big(m - \lambda\big)\, \phi(x)\; dx \;\leq\; 0$$

hold, so that $\mu_p(\Omega) \geq \left| \frac{m-\lambda-p}{p} \right|^p$? Furthermore, one may ask as to how this situation changes when we remove the condition of the Ahlfors λ-regularity of the boundary, i.e. when we consider domains with boundaries whose Hausdorff and Aikawa dimensions are not equal. The reader is directed to the end of Section 3.3 for an example of such a domain.

2. In Theorem 5.3.2, we proved that if Ω is an inner γ-domain with non-empty, compact, inner Minkowski measurable boundary of inner Minkowski dimension λ, then $\mu_p(\Omega) \leq \left| \frac{m-\lambda-p}{p} \right|^p$. However, in a sense this theorem is at odds with the philosophy of the rest of the thesis. More specifically the theorem uses the 'wrong' notion of dimension. Recall, from Theorem 3.3.2, that if Ω is a domain that admits an L_p-Hardy inequality, then either $\dim_{\mathcal{H}}(\partial\Omega) > m - p$ or $\dim_A(\partial\Omega) < m - p$. Furthermore, from Theorems 3.4.4 and 3.5.2, we know that these necessary conditions are almost sufficient conditions for the existence of an L_p-Hardy inequality. In other words, it is the Hausdorff and Aikawa dimension of the boundary that are important, not the inner Minkowski dimension.

   The only reason that inner Minkowski dimension was used in Theorem 5.3.2 was because we had previously developed a relationship between the inner Minkowski dimension of the boundary and the number of cubes appearing in sequential generations of the Whitney decomposition of Ω (see Lemmas 4.3.1 and 4.3.2). As such, it would certainly be worthwhile to obtain analogous versions of these lemmas, and consequently of Theorem 5.3.2, in terms of the Hausdorff and Aikawa dimensions of the boundary. With a bit of luck this may help us to remove the inconvenient condition of the Minkowski measurability of the boundary from the analysis.

3. Theorem 5.3.2 is also at odds with the general philosophy surrounding the L_p-Hardy inequality in another crucial way. The dimensional dichotomy harbored within Theorem 3.3.2 is local in nature. That is to say that if Ω is a domain that admits an L_p-Hardy inequality, and if $\omega \in \partial\Omega$, then either $\dim_{\mathcal{H}}\big(\partial\Omega \cap B(\omega, r)\big) > m - p$ or $\dim_A\big(\partial\Omega \cap B(\omega, r)\big) < m - p$. Consequently, we see that the boundary of a domain that admits an L_p-Hardy inequality cannot contain even an $m - p$ dimensional part.

   Strictly speaking, Theorem 5.3.2 is not local in nature. Although the analysis can easily be adapted to deal with domains whose boundaries contain well separated parts of different dimension, it is unclear how to modify the theorem when the boundary is both connected and contains parts of different dimension. One would think that this problem could be resolved by applying a method similar to that used in the proof of Theorem 5.2.1. There it was shown that if Ω is a domain in $\mathbb{R}^m$ with smooth, non-empty boundary of integer dimension $0 \leq k \leq m-1$, then $\mu_p(\Omega) \leq \left| \frac{m-k-p}{p} \right|^p$. From our current perspective, the important thing here is that the analysis all takes place on an arbitrarily small ball centered on an arbitrary point of the boundary.

4. Another failing of Theorem 5.3.2 is that it is only applicable for inner γ-domains. Although in Section 4.5 we constructed a 'room and corridor' type fractal domain that is an inner γ-domain, we have not provided a full characterization of what it means for a domain to satisfy this geometric condition. It would certainly be worthwhile determining whether or not standard types of domains, such as John domains or domains admitting the inner cone condition, are inner γ-domains.

5. Despite the aforementioned deficiencies of Theorem 5.3.2, it would still be worthwhile proving the **equality** $\mu_p(\Omega) = \left| \frac{m-\lambda-p}{p} \right|^p$, in the case where Ω is an inner γ-domain with non-empty, compact boundary of inner Minkowski dimension λ. More generally, it would be worthwhile proving the validity of this equality for any domain with fractal boundary of dimension λ, simply because of the consequences this would have for the essential self-adjointness of Schrödinger operators and for quantum mechanics as a whole. That is to say that quantum mechanics is the study of particles that are small, whose action is comparable to Planck's constant, but of a definite size. Combining the above equality with Theorem 2.5.1 would produce a quantum mechanical theorem that depends on the fractal dimension of the boundary, i.e. on a scale infinitely smaller than the diameter of an atom or electron.

6. We have stated many times that if Ω is a domain in $\mathbb{R}^m$ that admits the L_p-Hardy inequality, then either $\dim_{\mathcal{H}}(\partial\Omega) > m-p$ or $\dim_A(\partial\Omega) < m-p$. Furthermore, Theorem 3.5.2 asserts that if $\dim_A(\partial\Omega) < m-p$, then Ω admits an L_p-Hardy inequality. This immediately raises the question as to whether the condition $\dim_{\mathcal{H}}(\partial\Omega) > m-p$ is sufficient for the existence of an L_p-Hardy inequality.

   However, we already known that this condition is not sufficient, at least in the borderline case where $m = p$. If we consider the domain $\Omega = \mathbb{R}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is the middle thirds Cantor set, then, by Theorem 3.4.7, Ω cannot admit an L_2-Hardy inequality since $\mathbb{R}^2 \setminus \Omega$ is bounded, despite the fact that $\dim_{\mathcal{H}}(\partial\Omega) = \frac{\log 2}{\log 3} > 0$. Nevertheless, we make the following conjecture.

Conjecture 7.1.1 *Let Ω be a domain in $\mathbb{R}^m$. If $\dim_{\mathcal{H}}\big(\partial\Omega \cap B(\omega, r)\big) > m-p > 0$ for all $\omega \in \partial\Omega$ and $r > 0$, then Ω admits an L_p-Hardy inequality.*

7. There is also the possibility of localizing our main Theorem on the essential self-adjointness of the Schrödinger operator H. To motivate this we consider a simple example. In $\mathbb{R}^2$, let $\Omega_1 = B(0,R)$ and let $\Omega_2 = B(0,R) \setminus \{0\}$. We will call the sphere $\bar{B}(0,R) \setminus B(0,R)$ the outer boundary of Ω_2, and the origin the inner boundary of Ω_2. Since Ω_1 is a bounded, convex domain with smooth boundary of co-dimension 1, by Lemma 5.2.2, $\mu_2(\Omega_1) = \left|\frac{1-2}{2}\right|^2 = \frac{1}{4}$. Consequently, equation (7.1) asserts that the operator H is essentially self-adjoint on Ω_1 provided that $V_1(x) \geq \frac{3}{4d(x)^2}$.

   On the other hand, the boundary of Ω_2 contains a zero dimensional part so that, by Theorem 3.3.2, it must be the case that $\mu_2(\Omega_2) = 0$. As such, equation (7.1) now implies that H is essentially self-adjoint only if $V_1(x) \geq \frac{1}{d(x)^2}$. In other words, on Ω_2 the potential must inflate faster than $\frac{1}{d(x)^2}$ both on the inner **and** outer part of the boundary. It seems highly plausible that H would still be essentially self-adjoint under the weaker assumptions that $V_1(x) \geq \frac{1}{d(x)^2}$ on a neighborhood of the inner boundary but $V_1(x) \geq \frac{3}{4d(x)^2}$ on a neighborhood of the outer boundary.

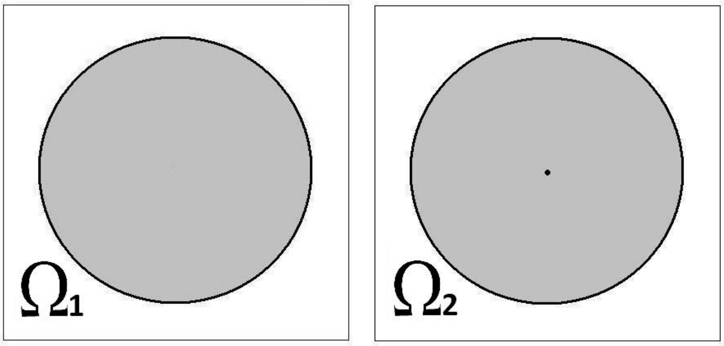

Figure 7.1: On the disk Ω_1, Theorem 2.5.1 asserts that H is essentially self-adjoint provided that $V_1(x) \geq 3/4\, d(x,\partial\Omega)^{-2}$. On the punctured disk Ω_2, Theorem 2.5.1 implies that H is essentially self-adjoint provided that $V_1(x) \geq d(x,\partial\Omega)^{-2}$, even on a neighborhood of the outer boundary. This suggests a need to localize Theorem 2.5.1 in the case where the boundary of a domain contains parts of different dimension.

   Indeed, in [81] Maeda produced an example of such a localized result. There it was shown that if $\Omega = \mathbb{R}^m \setminus F$, where F is the union of at most a countable number of well separated k_j-dimensional affine subspaces S_j, where $0 \leq k_j \leq m-1$, then the operator H is essentially self-adjoint provided that sufficiently close to each subspace

$$V_1(x) \quad \geq \quad \frac{1 - \left|\frac{m-k_j-2}{2}\right|^2}{d(x,S_j)^2}.$$

   Given an arbitrary domain Ω, this would suggest that the rate at which the potential $V_1(x)$ must inflate as $x \to \omega \in \partial\Omega$, in order to ensure the essential self-adjointness of

H, depends on the dimension of $\partial\Omega$ in the vicinity of ω. A first step towards proving this would be to develop an analogous version of Theorem 2.1.1, for a domain Ω which has a boundary with a finite number of well separated parts of different dimension. One would imagine that the more general types of Hardy inequalities studied by Barbatis, Fillipas & Tertikas in [34] and [35] would be useful within this context.

8. Let us recall the result of Corollary 1.4.1. There it was shown that if $\Omega = \mathbb{R}^m$, then the Schrödinger operator $H = -\Delta + V$, defined on $C_0^\infty(\Omega)$ is essentially self-adjoint provided that $V(x) \geq -a\,|\,x\,|^2$. The interpretation here is that a potential of this form prevents a particle under it's influence from reaching the boundary at infinity in a finite amount of time. Furthermore, in [13] Eastham, Evans & Mcleod demonstrated that the requirement that $V(x)$ is always greater than or equal to $-a\,|\,x\,|^2$ is not imperative. If there exists a sequence of annuli that occur 'sufficiently regularly', and if on these annuli the potential is 'sufficiently large', then this is enough to ensure the essential self-adjointness of the aforementioned operator. The idea here is that the annuli provide sufficient barriers to the outward progress of the particle so that it cannot escape to infinity in a finite amount of time.

   In this thesis we have considered a Schrödinger operator H defined on $C_0^\infty(\Omega)$, where Ω is a domain in $\mathbb{R}^m$ with non-empty boundary. We have found that this operator is essentially self-adjoint provided that $V(x) \geq \frac{1-\mu_2(\Omega)}{d(x)^2}$. This raises the question as to whether H would be essentially self-adjoint if the potential oscillated around the critical limit of $\frac{1-\mu_2(\Omega)}{d(x)^2}$. In other words, if, on a neighborhood of the boundary, there is only a sequence on 'layers' on which $V(x)$ is much greater than $\frac{1-\mu_2(\Omega)}{d(x)^2}$, would this be sufficient to ensure the essentially self-adjointness of H?

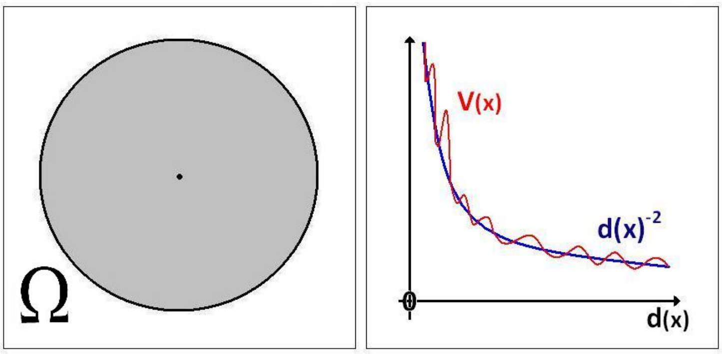

Figure 7.2: On the punctured disk $\Omega = B(0,R) \setminus \{0\}$ the operator H is essentially self-adjoint if $V(x) \geq d(x,\partial\Omega)^{-2}$. Would H still be essentially self-adjoint if the potential oscillated around the critical limit of $d(x,\partial\Omega)^{-2}$ as shown in the diagram above?

9. In Theorem 6.1.2 we showed that if $H = -\Delta + V$ is a Schrödinger operator defined on $C_0^\infty(\Omega)$ and V has the potential structure described by Theorem 2.5.4,

then the spectrum of the self-adjoint operator $\bar{H}$ is entirely contained in the interval $[-|| V_2 ||_{L_\infty}, \infty)$. Given the nature of the proof of this theorem, one would not be surprised to see some correspondence between the value of the variational constant $\mu_2(\Omega)$ and the infimum of the spectrum of $\bar{H}$. In particular, it would be interesting to investigate any relationship between $\mu_2(\Omega)$ and the spectrum of the Laplacian (defined on $C_0^\infty(\Omega)$) within this context.

10. Chapter 6 of this thesis was devoted to showing that the potential structures described by Theorem 2.5.1 and Theorem 2.5.4 are optimal from the perspective of the essential self-adjointness of Schrödinger operators. However, we were only able to demonstrate the optimality of these potential structures in the case where the domain Ω is either a subset of the real line, or geometrically simple enough to allow the corresponding equations to be reduced to the one dimensional case by an appropriate change of variables. The reason for this is that the main mathematical machinery that underpins all the results within this chapter (Weyl's limit point - limit circle analysis) is only suitable for determining the number of square integrable solutions of the one dimensional equation $-\Psi'' + V\Psi - i\Psi = 0$. It is unclear how this analysis should be extended when $\Omega \subseteq \mathbb{R}^m$ and $m \geq 2$. Whilst extending Weyl's results to higher dimensions would have important consequences both for this thesis and for quantum mechanics in general, the fact that little or no progress has been made in this regard since Weyl's original paper of 1909 (see [14]) gives an indication of how difficult this problem is.

11. All of our analysis concerning the essential self-adjointness of Schrödinger operators and the L_p-Hardy inequality has been conducted in Euclidean space. Despite this, many of the cited works regarding the L_p-Hardy inequality are set in more general metric spaces, albeit metric spaces whose structure closely mimics that of $\mathbb{R}^m$ in various key aspects (see [45], [49], [51], [53] & [62]). This raises the question as to whether our analysis, both on the nature of the variational constant $\mu_p(\Omega)$ and on the essential self-adjointness of Schrödinger operators, can be extended to such spaces, if indeed these concepts make sense at all.

Appendix A

Notions of Dimension

Roughly speaking, the dimension of a set $E \subseteq \mathbb{R}^m$,

> "indicates how much space the set occupies near to each of its points."
>
> Falconer [54], Chapter 2.

In layman's terms, 'the dimension' of a set is basically a measure of how thick the set is. For instance, it is well known that the dimension of the boundary of the von Koch snowflake is equal to $\frac{\log 4}{\log 3} \approx 1.262$. Prosaically, one may interpret this as saying that the boundary of the von Koch snowflake takes up more space than a line but less space than a plane.

The words 'the dimension' appear here in inverted commas because there are various different notions of dimension, each reflecting the structure of the set under consideration in different ways. Unfortunately, the nomenclature used in the literature to label these notions of dimension is somewhat cluttered. Notions of dimension that bear the same name sometimes turn out to assign different numerical values to the same set. In contrast, notions of dimension that bear different names often turn out to be equivalent. In order to prevent any misunderstandings, in this appendix we define all the notions of dimension used throughout the thesis and elucidate some of their properties. For further information on the concepts introduced, the reader is referred to [53], [54], [55], [56], [57] and [71].

A.1 Hausdorff Dimension

Of the many notions of dimension currently in use, the Hausdorff dimension of a set is the the oldest and probably the most important. In order to understand this concept, it is first necessary to define the idea of the λ-Hausdorff measure of a set.

Definition A.1.1 *Let E be a set in $\mathbb{R}^m$. For all $\lambda \geq 0$ we define the λ-Hausdorff measure of E by*

$$\mathcal{H}^{\lambda}(E) \;=\; \lim_{\delta\to 0^+} \inf \left\{ \sum_{j=1}^{\infty} r_j^{\lambda} \;\middle|\; E \subseteq \bigcup_{j=1}^{\infty} B(x_j, r_j),\ x_j \in \mathbb{R}^m,\ 0 < r_j \leq \delta \right\}.$$

It can easily be shown that for each set $E \subseteq \mathbb{R}^m$ there exists a unique value, $\lambda_0 \in [0, m]$, for which

$$\mathcal{H}^\lambda(E) = \begin{cases} \infty & \text{if} \quad \lambda < \lambda_0, \\ 0 & \text{if} \quad \lambda > \lambda_0. \end{cases} \tag{A.1}$$

This 'jump point' of the λ-Hausdorff measure motivates the following definition of the Hausdorff dimension of a set.

Definition A.1.2 *For a set $E \subseteq \mathbb{R}^m$ we define the Hausdorff dimension of E by*

$$\dim_{\mathcal{H}}(E) \;=\; \sup\{\, \lambda \geq 0 \;|\; \mathcal{H}^\lambda(E) \,=\, \infty \,\} \;=\; \inf\{\, \lambda \geq 0 \;|\; \mathcal{H}^\lambda(E) \,=\, 0 \,\}.$$

It is worth pointing out that a slightly different definition of Hausdorff dimension is frequently used within the literature. That is to say that one could define the λ-Hausdorff content of a set $E \subseteq \mathbb{R}^m$ as follows

$$\mathcal{H}^\lambda_\infty(E) \;=\; \inf \left\{ \sum_{j=1}^{\infty} r_j^\lambda \;\middle|\; E \subseteq \bigcup_{j=1}^{\infty} B(x_j, r_j),\ x_j \in E,\ r_j > 0 \right\}.$$

Again, it can be shown that if $\mathcal{H}^\lambda_\infty(E) = 0$, then $\mathcal{H}^t_\infty(E) = 0$ for all $t > \lambda$. Hence we can define the Hausdorff dimension of a set $E \subseteq \mathbb{R}^m$ by

$$\dim_{\mathcal{H}}(E) \;=\; \sup\{\, \lambda \geq 0 \;|\; \mathcal{H}^\lambda_\infty(E) \,>\, 0 \,\} \;=\; \inf\{\, \lambda \geq 0 \;|\; \mathcal{H}^\lambda_\infty(E) \,=\, 0 \,\}.$$

With a little bit of effort one can show that $\mathcal{H}^\lambda_\infty(E) = 0$ if and only if $\mathcal{H}^\lambda(E) = 0$, so that Hausdorff content and Hausdorff measure do indeed assign the same dimension to the set E (see [56, Proposition 4.9] or [46, Chapter 8]).

Although Hausdorff dimension is the most mathematically convenient notion of dimension to use (given that it is countably stable), calculating the Hausdorff dimension of a given set is, in general, a difficult task. Consequently, other notions of dimension have been proposed which are more computationally friendly, at the expense of having various pathological properties.

A.2 Minkowski Dimension

The idea underlying the notion of Minkowski dimension is to examine the behavior of the volume of an r-neighborhood of a set as $r \to 0$ from above. This is inherent in the following definition.

Definition A.2.1 *Let E be a bounded set in $\mathbb{R}^m$. For $0 < \delta < \infty$, let $E_\delta = \left\{ x \in \mathbb{R}^m \;\middle|\; d(x, E) < \delta \right\}$ be a δ-neighborhood of E. For $\lambda \geq 0$, the upper and lower λ-Minkowski content of E is respectively given by*

$$M^\lambda_u(E) \;=\; \limsup_{r \to 0^+} \frac{|\, E_r \,|}{r^{\,m-\lambda}} \qquad \textit{and} \qquad M^\lambda_l(E) \;=\; \liminf_{r \to 0^+} \frac{|\, E_r \,|}{r^{\,m-\lambda}}.$$

It is well known that if $M_u^\lambda(E) < \infty$, then $M_u^s(E) = 0$ for all $s > \lambda$, and that $M_u^\lambda(E) = 0$ for all $\lambda > m$. Furthermore, if $M_u^\lambda(E) > 0$, then $M_u^s(E) = \infty$ for all $s < \lambda$. Hence, there exists a unique value of $\lambda \in [0, m]$ at which the upper Minkowski content of a set 'jumps' from infinity to zero. A similar story holds true for the lower Minkowski content of a set. This naturally leads us to the following definition of upper and lower Minkowski dimension.

Definition A.2.2 *Let E be a bounded set in $\mathbb{R}^m$. We define the upper and lower Minkowski dimension of E respectively by*

$$\begin{aligned} \overline{\dim}_M(E) &= \sup\{\, \lambda \geq 0 \mid M_u^\lambda(E) = \infty \,\} &= \inf\{\, \lambda \geq 0 \mid M_u^\lambda(E) = 0 \,\} \\ \underline{\dim}_M(E) &= \sup\{\, \lambda \geq 0 \mid M_l^\lambda(E) = \infty \,\} &= \inf\{\, \lambda \geq 0 \mid M_l^\lambda(E) = 0 \,\} \end{aligned}$$

or equivalently by

$$\begin{aligned} \overline{\dim}_M(E) &= \inf\{\, \lambda \geq 0 \mid M_u^\lambda(E) < \infty \,\} \\ \underline{\dim}_M(E) &= \inf\{\, \lambda \geq 0 \mid M_l^\lambda(E) < \infty \,\}. \end{aligned}$$

When $\overline{\dim}_M(E) = \underline{\dim}_M(E)$ we say that the Minkowski dimension of E is well defined, equal to this common value and denoted by $\dim_M(E)$. If a set is known to have Minkowski dimension d, then we say that the set is Minkowski measurable if it's d-Minkowski content is positive and finite, i.e. if

$$0 < M_l^d(E) \leq M_u^d(E) < \infty.$$

Since upper (lower) Minkowski dimension is equivalent to upper (lower) box counting dimension (see [71, Proposition 2.17] and [54, Proposition 3.2]), the Minkowski dimension of a set is relatively easy to compute. However, Minkowski content is not countably additive on disjoint subsets. Consequently, Minkowski dimension has the undesirable property of assigning positive dimension to sets of isolated, but accumulating, points. As we shall see, this is a failing also shared by Aikawa and Assouad dimension.

A.3 Inner Minkowski Dimension

In this thesis we are interested in investigating the relationship between the dimension of the boundary of a domain, the L_p-Hardy inequality and the minimal criteria required to confine a particle within the interior of the domain (i.e. the problem of the essential self-adjointness of Schrödinger operators). Consequently, it is often more appropriate to consider the Minkowksi dimension of the boundary of a domain 'as viewed from the interior of the domain'. This leads us to the idea of **inner** Minkowski dimension.

Definition A.3.1 *Let Ω be a domain in $\mathbb{R}^m$ with non-empty, compact boundary. For $\lambda \geq 0$, the upper and lower inner λ-Minkowski content of $\partial\Omega$ are respectively given by*

$$\hat{M}_u^\lambda(\partial\Omega) \;=\; \limsup_{r \to 0^+} \frac{|\,(\partial\Omega)_r \,\cap\, \Omega\,|}{r^{\,m-\lambda}} \qquad\qquad \hat{M}_l^\lambda(\partial\Omega) \;=\; \liminf_{r \to 0^+} \frac{|\,(\partial\Omega)_r \,\cap\, \Omega\,|}{r^{\,m-\lambda}}.$$

Again, it is easy to see that there exists a unique value of $\lambda \in [0, m]$ at which the upper (lower) inner Minkowski content jumps from infinity to zero. This motivates the definition below.

Definition A.3.2 *Let Ω be a domain in $\mathbb{R}^m$ with non-empty, compact boundary. We define the upper and lower inner Minkowski dimension of $\partial\Omega$ respectively by*

$$\begin{aligned} \overline{\dim}_{\hat{M}}(\partial\Omega) \;&=\; \sup\{\,\lambda \geq 0 \;|\; \hat{M}_u^\lambda(\partial\Omega) = \infty\,\} \;=\; \inf\{\,\lambda \geq 0 \;|\; \hat{M}_u^\lambda(\partial\Omega) = 0\,\} \\ \underline{\dim}_{\hat{M}}(\partial\Omega) \;&=\; \sup\{\,\lambda \geq 0 \;|\; \hat{M}_l^\lambda(\partial\Omega) = \infty\,\} \;=\; \inf\{\,\lambda \geq 0 \;|\; \hat{M}_l^\lambda(\partial\Omega) = 0\,\}. \end{aligned}$$

When $\overline{\dim}_{\hat{M}}(\partial\Omega) = \underline{\dim}_{\hat{M}}(\partial\Omega)$ we say that the inner Minkowski dimension of $\partial\Omega$ is well defined, equal to this common value and denoted by $\dim_{\hat{M}}(\partial\Omega)$. If a set is known to have inner Minkowski dimension d, then we say that the set is inner Minkowski measurable if it's inner d-Minkowski content is positive and finite, i.e. if

$$0 \;<\; \hat{M}_l^d(\partial\Omega) \;\leq\; \hat{M}_u^d(\partial\Omega) \;<\; \infty.$$

It is clear that for any domain $\Omega \subset \mathbb{R}^m$ with non-empty, compact boundary, and for any $\lambda \geq 0$, we have that $\hat{M}_u^\lambda(\partial\Omega) \leq M_u^\lambda(\partial\Omega)$ and $\hat{M}_l^\lambda(\partial\Omega) \leq M_l^\lambda(\partial\Omega)$. Therefore, it follows immediately that $\dim_{\hat{M}}(\partial\Omega) \leq \dim_M(\partial\Omega)$.

A.4 Aikawa & Assouad Dimension

In Section 7 of [61], Aikawa constructed a notion of dimension which has particular relevance to the quasi-additivity properties of Riesz p-capacity. This notion of dimension, defined in terms of the integrability of the distance function, has subsequently become known as Aikawa dimension.

Definition A.4.1 *Let E be a set in $\mathbb{R}^m$. We define the Aikawa dimension of E by*

$$\dim_A(E) \;=\; \inf\left\{\, t > 0 \;\middle|\; \int_{B(x,r)} \frac{1}{d(y,E)^{m-t}}\; dy \;\leq\; A_t\, r^t, \quad \forall\, x \in E,\; r > 0 \right\}.$$

where A_t is a constant possibly depending on t.

We adopt the convention that if E has positive measure, then $\dim_A(E) = m$. This ensures that the Aikawa dimension of any set always lies in the interval $[0, m]$. In particular, if $\Omega \subsetneq \mathbb{R}^m$ is a domain with boundary $\partial\Omega$, then

$$\dim_A(\partial\Omega) \;=\; \inf\left\{ t > 0 \;\middle|\; \int_{B(x,r)} \frac{1}{d(y, \partial\Omega)^{m-t}}\, dy \;\leq\; A_t\, r^t, \quad \forall\, x \in \partial\Omega,\; r > 0 \right\}. \tag{A.2}$$

The main result in [53] is that Aikawa dimension is, in fact, equivalent to the more widely known notion of Assouad dimension. There are many equivalent definitions of Assouad dimension and the concept has appeared within the relevant literature under various different guises. We follow the definition given in [53].

Definition A.4.2 *For a set $E \subseteq \mathbb{R}^m$, let $\mathrm{Cov}(E)$ denote the set of all $\beta > 0$ for which the following covering property holds: There exists $C_\beta \geq 1$ such that, for all $0 < \epsilon < \frac{1}{2}$, each $F \subset E$ can be covered by at most $C_\beta \epsilon^{-\beta}$ balls of radius $r = \epsilon\,\mathrm{diam}(F)$. The Assouad dimension of E is then defined to be*

$$\dim_{AS}(E) \quad = \quad \inf\{\, \beta \in \mathrm{Cov}(E) \,\}.$$

Given their equivalence, we will use the terms Aikawa and Assouad dimension interchangeably throughout the thesis.

A.5 Relationships Between Dimensions

As we have already alluded to, despite the fact that Hausdorff dimension has various desirable properties of a notion of dimension, calculating the Hausdorff dimension of a given set is often extremely difficult. The notions of Minkowski and Aikawa dimension are more computationally friendly, but also admit undesirable properties. Therefore, a legitimate question is - under what conditions do these notions of dimension coincide to give us the best of both worlds?

It is well known that the Hausdorff dimension of any set is always less than or equal to it's Aikawa dimension. It is also well known that the Hausdorff dimension of a bounded set is not greater than it's lower Minkowski dimension and that it's upper Minkowski dimension is less than or equal to the Aikawa dimension of the set. Although the latter statement is effectively proven in Lemma 4.6.3, the reader is directed to [53, Section 3.4], [54, Chapter 3] and [84, Theorem A.5] for a proof of these facts. Consequently, we have that for any set $E \subseteq \mathbb{R}^m$

$$\dim_{\mathcal{H}}(E) \quad \leq \quad \dim_A(E), \tag{A.3}$$

and if E is bounded

$$\dim_{\mathcal{H}}(E) \quad \leq \underline{\dim_M}(E) \quad \leq \overline{\dim_M}(E) \quad \leq \quad \dim_A(E). \tag{A.4}$$

As such our previous question reduces to determining under what conditions equation (A.3) holds with equality. In order to answer this, we first need to make the following definition.

Definition A.5.1 *Let E be a set in $\mathbb{R}^m$. We say that E is Ahlfors λ-regular if there exists some $\lambda \in (0, m]$, and some finite uniform constant $C > 0$, so that for all $\omega \in E$ and all $0 < r < \mathrm{diam}(E)$*

$$C^{-1}\, r^{\lambda} \quad \leq \quad \mathcal{H}^{\lambda}\big(E \cap B(\omega, r)\big) \quad \leq \quad C\, r^{\lambda}.$$

The following theorem, due to Lehrbäck [52], states that if a set is Ahlfors λ-regular then its Hausdorff, Minkowski and Aikawa dimensions all coincide.

Theorem A.5.1 *[52, Lemma 2.1]*

If is E an Ahlfors λ-regular set in $\mathbb{R}^m$, then $dim_{\mathcal{H}}(E) = \dim_A(E) = \lambda$. Furthermore, if E is bounded, then $\dim_{\mathcal{H}}(E) = \dim_M(E) = \dim_A(E) = \lambda$.

One may now ask - what kinds of sets are Ahlfors λ-regular? Whilst an extensive list has been compiled by Fraser [65], the most common type of Ahlfors λ-regular sets are self similar fractals admitting the 'Open Set Condition'. Examples include the middle thirds Cantor set, the Sierpińksi triangle and the von Koch snowflake. The reader is directed to [54, Section 9.2], [65, Section 2] and [66, Example 2] for details.

However, it is also instructive to consider examples of sets for which the concepts of Hausdorff, Minkowski and Aikawa dimension diverge. For instance, it can be shown that the set

$$E \quad = \quad \big\{0\big\} \cup \big\{ \ (1/n, 0, \ldots, 0) \ \big| \ n \in \mathbb{N} \ \big\}$$

has Hausdorff dimension zero (see [54, Section 2.2]), Minkowski dimension of $\frac{1}{2}$ (see [54, Example 3.5]) and Aikawa dimension equal to 1 (see [84, Section 1]).

Appendix B

Capacity Measures

In general terms, the capacity of a set is a measure of a sets ability to hold charge whilst maintaining a given potential energy. One may again interpret the capacity of a set as being an indication of how 'thick' the set is. Just as there are many notions of dimension, there are many different notions capacity. However, in this thesis we will only need to consider two notions of capacity; the variational p-capacity and Riesz p-capacity of a set.

B.1 Variational p-Capacity

We begin with the standard definition of the variational p-capacity of a compact set.

Definition B.1.1 *Let G be an open set in $\mathbb{R}^m$, and let K be a compact subset of G. For all $p \in (1, \infty)$ we define the variational p-capacity of K, relative to G, by*

$$C_p(K, G) \quad = \quad \inf\left\{ \int_G |\nabla u(x)|^p \, dx \;\middle|\; u \in C_0^\infty(G),\ u \geq 1 \ \text{on} \ K \right\}.$$

In fact, in Chapter 2 of [40] it is shown that the same quantity is obtained if, instead of smooth functions, the infimum is taken over continuous, compactly supported functions in $W_p^1(G)$. Consequently, we have the following, equivalent, definition of variational p-capacity.

Definition B.1.2 *Let G be an open set in $\mathbb{R}^m$, and let K be a compact subset of G. For all $p \in (1, \infty)$ we may define the variational p-capacity of K, relative to G, by*

$$C_p(K, G) \quad = \quad \inf\left\{ \int_G |\nabla u(x)|^p \, dx \;\middle|\; u \in W_{p,0}^1(G) \cap C(G),\ u \geq 1 \ \text{on} \ K \right\}.$$

Although variational p-capacity has proven to be a very useful tool in many areas of analysis, calculating the variational p-capacity of a given set is often a very difficult task. Indeed, there are only a handful of sets whose variational p-capacity are known explicitly.

An example of a set whose variational p-capacity is known, is that of the so called 'spherical condenser'.

EXAMPLE: [40, Example 2.12]
If $B(x,r)$ is a ball in $\mathbb{R}^m$, then $C_p(\bar{B}(x,r)\,,\,B(x,2r)) \;=\; c(m,p)\,r^{m-p}$ where the constant $c(m,p)$ is given by

$$c(m,p) = \begin{cases} |\,\Sigma_{m-1}\,|\,\big(\ln 2\big)^{1-p} & \text{if} \quad p = m, \\ |\,\Sigma_{m-1}\,| \cdot \big|\,\frac{p-m}{p-1}\,\big|^{p-1} \cdot \big|2^{\frac{p-m}{p-1}} - 1\,\big|^{1-p} & \text{if} \quad p \neq m. \end{cases} \tag{B.1}$$

Here $|\Sigma_{m-1}|$ is the measure of the $m-1$ sphere.

In some sense, the variational p-capacity of the spherical condenser is a benchmark against which we compare the variational p-capacity of other sets. This is inherent in the definition of uniform p-fatness (see Definition 3.4.2).

The variational p-capacity of compact sets has been extensively studied, and it's properties are well known. However, in this thesis, we will only need to use the fact that variational p-capacity is increasing in it's first argument and decreasing in it's second argument. That is to say that if $K_1 \subseteq K_2$ are compact sets, and if $G_1 \subseteq G_2$ are open sets so that $K_2 \subseteq G_1$, then

$$C_p(K_1,G_2) \;\leq\; C_p(K_2,G_2), \qquad\qquad C_p(K_1,G_2) \;\leq\; C_p(K_1,G_1). \tag{B.2}$$

For a proof of this fact, and for further information on variational p-capacity, the reader is directed to [40] and [59].

B.2 Riesz p-Capacity

A second notion of the capacity of a set is provided by Riesz p-capacity. In order to understand the idea of Riesz p-capacity we first define the function $k : \mathbb{R}^m \to [0,\infty)$, by $k(x) \;=\; |\,x\,|^{1-m}$. We will often refer to this function as the **Riesz kernel**.

Definition B.2.1 *If $k(x)$ is the Riesz kernel, then for any non-negative function $f : \mathbb{R}^m \to [0,\infty)$ the convolution*

$$(k * f)(x) \qquad = \qquad \int_{\mathbb{R}^m} \frac{1}{|x-y|^{m-1}}\, f(y)\; dy$$

is well defined, in the sense that for all $x \in \mathbb{R}^m$ we have that $0 \;\leq\; (k*f)(x) \;\leq \infty$. For a set $E \in \mathbb{R}^m$ and for $1 < p < \infty$, we define the Riesz p-capacity of E by

$$R_p(E) \quad = \quad \inf \Big\{ \; ||\,f\,||^p_{L_p(\mathbb{R}^m)} \;\Big|\; \; k * f \geq 1 \quad on \quad E, \;\; f \geq 0 \;\Big\}.$$

One may ask as to the relationship between the Riesz p-capacity and the variational p-capacity of a compact set. Indeed, the following lemma asserts that the variational p-capacity of a compact set always majorizes it's Riesz p-capacity.

Lemma B.2.1 *[40, Theorem 2.38], [59, Theorems 5.5.1 & 8.2.4]*

Let G be an open set in $\mathbb{R}^m$ and let K be a compact subset of G. Then there exists a finite, uniform constant $C > 0$, so that

$$R_p(K) \quad \leq \quad C\, C_p(K, G).$$

Appendix C

Miscellaneous

In this appendix we group together various miscellaneous bits of information which, while important to the thesis, could not easily be fitted within the disposition.

C.1 Maximal Functions

Maximal functions appear everywhere in geometric analysis. Originally introduced by Hardy & Littlewood, the general role of a maximal function is to take a function f, usually contained in $L_{1,loc}(\mathbb{R}^m)$, and for each $x \in \mathbb{R}^m$ assign to f it's maximum possible average value on a neighborhood of x. Given their wide range of applications, many different forms of maximal function are in use within the literature. However, we find occasion only to use the following standard maximal function.

Definition C.1.1 *Let $f(x)$ be a given function on $\mathbb{R}^m$. We define the following maximal function associated to f.*

$$\hat{M}f(x) \;=\; \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy,$$

where $|B(x,r)|$ is the Lebesgue measure of the ball of radius r centered at x.

It is well known that if $f(x)$ is contained in $L_p(\mathbb{R}^m)$, where $1 < p < \infty$, then the maximal function of f is also in $L_p(\mathbb{R}^m)$. Indeed, the following lemma can be found in chapter 1 of [37].

Lemma C.1.1 *[37, Theorem 1]*

If $f(x) \in L_p(\mathbb{R}^m)$ for some $p \in (1, \infty)$, then $\hat{M}f(x) \in L_p(\mathbb{R}^m)$ and there exists a finite uniform constant $C = C(m,p) > 0$ so that

$$||\,\hat{M}f\,||_{L_p(\mathbb{R}^m)} \quad \leq \quad C\,||\,f\,||_{L_p(\mathbb{R}^m)}.$$

C.2 Strong & Weak Solutions

From the perspective of quantum mechanics, the space of n-times continuously differentiable functions is not quite the right space to investigate solutions of n^{th} order differential equations. This leads us to the idea of strong and weak solutions to such equations.

Definition C.2.1 *Let Ω be a domain in $\mathbb{R}^m$ and let $L = \sum_{|\alpha|\leq n} C_\alpha D^\alpha$ be a partial differential operator of order n where $C_\alpha \in C(\Omega)$. Further let $g(x)$ be a continuous function on Ω. We say that $f \in C^n(\Omega)$ is a **strong** solution of the equation $Lf = g$ if this equality holds in the sense of functions, i.e. if $Lf(x) = g(x)$ for all $x \in \Omega$.*

Definition C.2.2 *Let Ω be a domain in $\mathbb{R}^m$ and let $L = \sum_{|\alpha|\leq n} C_\alpha D^\alpha$ be a partial differential operator of order n where $C_\alpha \in L_{1,\mathrm{loc}}(\Omega)$. Further, let $g(x)$ be a locally $L_1(\Omega)$ function. We say that $f \in L_{1,\mathrm{loc}}(\Omega)$ is a **weak** solution of the equation $Lf = g$ if for all $u(x) \in C_0^\infty(\Omega)$ the following equality holds*

$$(-1)^{|\alpha|} \int_\Omega Lu(x)\, f(x)\ dx \quad = \quad \int_\Omega u(x)\, g(x)\ dx.$$

We note that any strong solution of the equation $Lf = g$ is obviously also a weak solution of the same equation.

C.3 The Spectrum of Self-adjoint Operators

In Section 1.2, we stated that one of the fundamental axioms of quantum mechanics asserts that for every observable of a system there is a corresponding self-adjoint operator. Furthermore, the values that this observable can take are given by the 'spectrum' of this operator. We now take a moment to define precisely what we mean by the spectrum of a self-adjoint operator.

Definition C.3.1 *Let A be a linear, self-adjoint operator on the Hilbert space $\mathcal{H}$. We define the resolvent set, $\rho(A)$, of the operator A by*

$$\rho(A) \quad = \quad \left\{ \lambda \in \mathbb{C} \;\middle|\; (A-\lambda)^{-1} \text{ exists and is bounded} \right\}.$$

We also define the spectrum, $\sigma(A)$, of the operator A as the complement of the resolvent set in $\mathbb{C}$, i.e. by $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

One may ask how a complex number $\lambda \in \mathbb{C}$ comes to lie in the spectrum of A. As an example, suppose that Ψ is a non-zero element of $\mathcal{H}$ and $\Psi \in \ker(A-\lambda)$ so that $(A-\lambda)\Psi = 0_{\mathcal{H}}$, i.e. Ψ is an eigenvector of A with associated eigenvalue λ. Since we also have $(A-\lambda)0_{\mathcal{H}} = 0_{\mathcal{H}}$, the operator $A-\lambda$ is not injective. Hence, if λ is an eigenvalue of A, then the inverse operator $(A-\lambda)^{-1}$ is not well defined, so that λ must belong to the spectrum of A. Indeed, we will use this fact, and those stated in the theorem below, within Section 6.1.

Theorem C.3.1 *[64, Theorems 1.2, 5.5 & 5.6]*

Let A be a linear, self-adjoint operator on the Hilbert space $\mathcal{H}$. Then

i. $\rho(A)$ *is open and* $\sigma(A)$ *is closed.*

ii. $\sigma(A) \subseteq \mathbb{R}$.

iii. The complex number λ *is contained in* $\rho(A)$ *if and only if there exists some* $C > 0$ *so that* $||\,(A-\lambda)u\,|| \geq C\,||\,u\,||$ *for all* $u \in D(A)$.

Appendix D

Notation

$\mathbb{V}$	-	An inner product space.
$0_{\mathbb{V}}$	-	The zero element of an inner product space.
$\mathcal{H}$	-	Hilbert space (an inner product space which is complete in the norm induced by the inner product).
$0_{\mathcal{H}}$	-	The zero element of a Hilbert space.
$B(x,r)$	-	A ball in $\mathbb{R}^m$ with center x and radius r.
$B(r)$	-	A ball in $\mathbb{R}^m$ centered at the origin with radius r.
Ω	-	A domain (open, connected set) in $\mathbb{R}^m$.
$\partial\Omega$	-	The closure of Ω except the interior of Ω, i.e. $\partial\Omega = \bar{\Omega} \setminus \Omega$
$C(\Omega)$	-	The set of continuous functions on Ω.
$C^n(\Omega)$	-	The set of n-times differentiable functions on Ω.
$C^\infty(\Omega)$	-	The set of smooth (infinitely differentiable) functions on Ω.
supp$f$	-	If $f : \Omega \to \mathbb{C}$, then supp$f$ is the closure of the set $\{\, x \in \Omega \mid f(x) \neq 0 \,\}$. If supp$f \subseteq \Omega$ is compact then f is said to be compactly supported.
$C_0(\Omega)$	-	The set of compactly supported continuous functions on Ω.
$C_0^n(\Omega)$	-	The set of compactly supported n-times differentiable functions on Ω.
$C_0^\infty(\Omega)$	-	The set of compactly supported smooth functions on Ω.
$\hat{C}_0^1(\Omega)$	-	The set of compactly supported functions which are differentiable almost everywhere on Ω.
p	-	A real number greater than or equal to 1 unless otherwise stated.
$L_p(\Omega)$	-	The space of functions $f : \Omega \to \mathbb{C}$ so that $\int_\Omega \mid f(x) \mid^p dx \ < \ \infty$.
$W_p^l(\Omega)$	-	The space of functions $f : \Omega \to \mathbb{C}$ so that f and all it's generalized derivatives of order less than or equal to l are in $L_p(\Omega)$.
$W_{p,0}^l(\Omega)$	-	The space of compactly supported functions $f : \Omega \to \mathbb{C}$ so that f and all it's generalized derivatives of order less than or equal to l are in $L_p(\Omega)$.

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