WHY IS THERE
AN ATHEMA
IN MATHEMATICS
??

-- A Philosophical Investigation in
Mathematics Education

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ABSTRACT

This work is a Wittgenstein-based philosophical analysis of mathematics education, primarily in the context of New Zealand secondary curriculum.

In Chapter 1.00 the aims of the Forms I-IV Syllabus are examined in detail with respect to the possible meanings of the statements contained there-in. The consequences and hidden assumptions of these meanings are elucidated.

Chapter 2.00 examines eight specific assumptions arising from Chapter 1.00 and from observation of mathematics teaching. Their consequences for mathematics education are discussed. Alternative assumptions are also considered.

Finally the history of mathematics and mathematics education in the United Kingdom, United States of America and New Zealand are summarised and then used to investigate the origins of the assumptions above.


PLEASE SIR, WHY DO WE HAVE TO LEARN THIS?

Acknowledgements

The thoughts developed in this thesis are the culmination of many years reflection and many discussions on mathematics education. However since deciding to write I have sought particular opinions and dialogue has been more specific. I therefore acknowledge in the text where particular ideas have come from, or in discussion with, other people. My thanks to them.

Three people have been of particular help throughout the writing.

David Stenhouse, my supervisor, has guided the development of the thesis - the form of Chapter 2.00 and partly Chapter 3.00 resulted from his suggestions. His comments on drafts and style have been particularly valuable. For his philosophic tutorledge I am indebted, while errors in my analysis are undoubtedly my own.

Gordon Knight and Bevan Werry have both spent some time in discussion with me over points concerning the mathematics aspect of mathematics education. My overall perspective on mathematics has developed through these talks. Specifically, Gordon has indicated the form for the introduction to Chapter 3.00 and a long discussion with him on structural ordering (section 3.11) was most useful.

I would also like to thank A.E.E. Clarke for permission to use his thesis. The beginning of section 4.2 is largely a paraphrase of this work.

I also acknowledge that the writing of this thesis has inconvenienced those with whom I live and work, in particular Ros Goldsbrough. I am grateful for their patience.

CONTENTS

1.00 INTRODUCTION

1.1 Outline of the work

It is easy to justify, in general terms, mathematics as part of a secondary curriculum in New Zealand society. But the question "Why this maths to this child now?" is not to be dismissed as a necessary evil for the whole of a successful mathematics education.

Lack of answers to the question have led through the essentially evasive tactics of individualised instruction, contract teaching, behaviour modification and other strategies. Fortunately the question persists.

This thesis is a step towards questioning the motives and practices of mathematics education at a deeper level. As such it exists primarily for the writer, however some sections (especially 3.00 and 4.00) may prove thought-provoking and instructive.

My aim in this work has been:

to examine the philosophical nature of secondary mathematics education in New Zealand by investigating possibilities within it and assumptions made by its practitioners.

More specifically the objectives are:

a) to analyse the aims of the NZ Forms I-IV Syllabus with respect to possible meanings, their consequences and hidden assumptions;
b) to discuss recurring assumptions providing alternatives where possible;
c) to investigate the origins of the assumptions in b)

Chapter 2.00 is primarily an exercise in philosophical investigation in the Wittgensteinian sense. It is a process rather than an end result. It deals with the formal aims of mathematics education as stated in the Forms I-IV Syllabus and the prescriptions to School Certificate, University Entrance and Bursary examinations. Instead of examining and arguing for or against positions adopted in these documents, I seek to investigate what possible meanings their statements contain, and to review the consequences of these meanings. This is done under the Wittgensteinian assumption that we use language in a fluid way rather than an exact, defined way. For example, in the statement of an aim the word "mathematics" may be given new connotations and denotations applicable only in that sentence or paragraph. The analysis of these meanings constitute the creative nature of language.

A philosophical investigation of the aim, therefore, should not presuppose what mathematics is, but look at what it could be so that the aim makes sense. Rather than look at what the words of the aim mean literally we should be looking at what the author could be meaning.

Consequently this chapter contains few conclusions.

Chapter 3.00 analyses several assumptions made by mathematics educators. Not all the assumptions are made by all mathematics teachers, but evidence that these assumptions are commonly made is presented. Several alternative assumptions are suggested and their consequences examined. Again dogmatic conclusions are avoided, but certain positions are preferred.

The historical development of some philosophies of mathematics education are considered in Chapter 4.00. Mathematics education has moved with the mainstream of Western mathematical thought and has adopted many philosophical positions uncritically. This chapter highlights the reasons for the incorporation of the above assumptions in our schools - reasons which are not, it is apparent, based on rational reflection, but rather on historical circumstances.

Finally the historical, social, educational and psychological theory forces on mathematics education are considered. This section links the historical development of chapter 4.00 with the assumptions in chapter 3.00.

A summary, bibliography and appendix follow, the latter being the syllabus and prescription statements from which chapter 2.00 is generated.

1.2 Context of the work

Any philosophical investigation into education exists within a particular context and rests on higher order philosophical positions and assumptions.

The context of this thesis is education in the secondary system in New Zealand. However, I consider the level or age of students to be relatively insignificant. The assumptions considered here are relevant to most formal mathematics education beyond primary school. The syllabus analysed in depth does belong to the Forms I-IV area, but a similar exercise could have been done with, say, university mathematics.

The work is specific to New Zealand in that this country is a consumer-based, high standard of living, western and literate society. The mathematics education being referred to is neither highly valued, nor necessary for survival or development - as is the case in many other countries. New Zealand seems different from other developed nations (e.g. Japan) in that a high level of formal education does not carry high status.

One of the higher order philosophical positions is that I believe education to be a process. It is neither something which happens to you, nor something you merely experience. The learner <u>creates</u> and <u>develops</u> through education. This idea is developed by Freire (1972) in the context of cultural revolution. While New Zealand society does have its 'cultures of silence' from which education can be the liberator, I am concerned here with the more general position of liberation from personal, political and cultural stagnation. Education is development, it is not a preparation for anything. Such an assumption is not limiting in the sense that all writers have assumptions at this level whether they are stated or not.


THE PATTERN
IS

2.00 A LANGUAGE-GAME ANALYSIS OF THE AIMS OF NEW ZEALAND SECONDARY MATHEMATICS.

2.0 INTRODUCTION

The bulk of this section deals with the aims of mathematics education as outlined in the Forms I to IV Mathematics Syllabus (Dept. of Education, 1972). Reference will also be made to detailed objectives for these aims which appear in a 1976 report of an in-service course (Dept of Education, 1976), and an earlier British publication along the same lines (Blakeley, 1976). The relevant portions of the first two documents appear in Appendices 1 and 2 respectively.

Sections 2.1 to 2.5 each deal with a specific aim, using the statement of the aim and the explanatory comments from the Syllabus and further explication from the In-Service Report.

Section 2.6 covers material relevant to senior mathematics classes, but not considered in 2.1 to 2.5 above. Official statements concerning these courses are found in The Education Gazette (Department of Education, 1972, 1976, 1977) -- see Appendix 3.

What is meant by a 'language-game analysis' in the title of this section?

The idea of philosophy as an investigation of language-games (hereafter abbreviated to l-g's) has developed from Wittgenstein. A paradigmatic example of the technique can be found in the collected writings of Wittgenstein on mathematics (Wittgenstein, 1964). For a fuller description than given below see Passmore's work (Passmore, 1957, Chpt XVIII).

Our use of language is such that meanings cannot be uniquely or exactly expressed. Words mean different things in different contexts, at different times, and to different people. Using language is creating language. For example, a new idea may be expressed by an old combination of words with slightly changed meanings. (Cf. the discussion of 'prime' (section 2.1) and terminology (section 3.3)).

Stenhouse investigates the notion of Wittgenstein philosophy as 'exploring possible language-games'. It is not a mere description of the use of language, but an examination of whether a particular string of words 'works' as a l-g. (Nor is it even mere words which are considered, see (Stenhouse b, p44, 45).). For example in section 2.3 it is seen that 'a liking for mathematics' and 'success in mathematics' can come to mean very similar things in many l-g's -- a situation which, I feel, most people would want to avoid.

In l-g analysis (or is it description? (see Passmore, 1957, p.426)) it is often useful to consider both a statement and its contradiction. This technique is used extensively by Wisdom (see Passmore, 1957, p.438). For example the statements:

a) "All pupils need to be able to handle numbers...in their day-to-day experience" (Dept. Education, 1972, p.5); and
b) "There is no mathematical knowledge essential for every person"

are both meaningful sentences. It is the job of the l-g analyst to discover what interpretations can be made, where such interpretations conflict, and where they are useful.

Playing l-g's could be described as exposing the rules for the use of words from the way they make up sentences; and also as exposing the meanings of sentences from the way they are made up of words.

To apply l-g analysis to the philosophy of mathematics education I shall explore possible meanings of some of the words and phrases used in the aims of the syllabus. This includes exposing the ambiguities in, and the consequences of, such meanings. I do not expect to arrive at a definitive answer as to whether these aims are (or could be) the best (or even an adequate) statement of what a Forms I to IV mathematics syllabus should achieve. Rather I hope to show what practical implications follow if we attribute various interpretations to the statements made.

Why have I chosen to use l-g analysis rather than any other form?

Stenhouse has argued in several papers (a, b, e) for a Kuhnian paradigm-shift from 'second-order' philosophical activity to active, Wittgensteinian, practical philosophy (see Stenhouse, a pp.5, 6, 8ff). Philosophy has become the domain of philosophers only, so that, for example, the l-g concept has been shorn of the practical emphasis given to it by Wittgenstein. Rather the abstract 'philosophical' aspect became important. Stenhouse sees the language-game approach as helping facilitate the paradigm-shift towards practical, creative, active attitudes in philosophy. He elaborates an example concerning Dewey (Stenhouse, a, p8), and section 3.3 argues for a creative use of mathematical symbolism seen as a l-g.

This thesis is intended to be clarificatory, but also has practical aspects. Its value lies in the extent to which both myself and the reader will have a wider range of lines of thought when considering the aims of mathematics education (see Stenhouse, e, p.5, 6). I do not wish to direct the reader towards any particular l-g (although I may, provisionally, adhere to one). Rather, by showing the value of even contradictory statements, I hope that the analysis will establish the desiribility of continuing to discuss aims, raise problems associated with any particular l-g and also indicate that such problems are resolvable and may be held over, with confidence, in the meantime (see Stenhouse, e, p7, 8). That is to say the analysis should help develop maturation in thought on mathematical/educational issues.

The analysis may also be creative by indicating new l-g's which have some utility.

On what basis can we explore l-g's?

We can call on our own experience to discover possible l-g's. This includes not only our experience with everyday language, but also our experience as teachers, and particularly as teachers of mathematics. It is only through these latter experiences that we can validate particular l-g's, (see Stenhouse, b, p49).

The only clues we have to the l-g being employed by the syllabus writers are the statements of the aims themselves and their brief explanatory paragraphs (hereafter ex.para's), (Syllabus pp4-6). These are quoted at the beginning of each section. In addition the objectives given in the In-service Report provide insight into the thinking of those involved with syllabus writing in New Zealand. They, also, are quoted where appropriate.

There is also a general statement given in the Syllabus (p4). In summary this statement describes a conception of mathematics which is eclectic (or ambiguous?): mathematics is seen as a model of the real world and as an axiomatic structure. It notes that mathematicians have been motivated to develop ideas for several reasons: their applications, their aesthetic value, their consistency, or simply their own sake. The rewards of a study of mathematics are seen to be threefold: surmounting challenges, experiencing the excitement of discovery and application, and appreciating economy and elegance.

The general aims are introduced by the words:

> "Mathematics is playing an increasingly siginificant part in many aspects of our culture and is therefore a necessary component of the general education of those who live in this culture."

The above should be taken as the context for the discussion which follows. (See also Appendix 7.2).

2.1 AIM.1.

Statement:

> "To provide mathematical experiences which enable pupils to make observations, to discover patterns and relationships, to develop concepts, to draw logical conclusions, to express thoughts accurately, and to form generalisations". (Syllabus, p.4)

Ex.para.:

> "Opportunity has already been provided...to develop an understanding of simple mathematical ideas. These ideas now form the basis of a wider range of study leading to the development of general principles which operate in mathematics. Greater emphasis is placed on the need for refining definitions and using more precise language - symbolic as well as verbal - particularly in statements from which a logical deduction may be made". (Syllabus, p.5)

This aim lists several activities/skills/thoughts which, it is claimed, lead to the development of general mathemetical principles. I shall consider each of these activities in turn, and then look at the general question: what does it mean to say 'provide mathematical experiences which enable pupils to make observations, discover patterns etc'?

The l-g for 'general principles' also needs to be investigated. This is done in section 2.2.

2.11 ...making observations,...

What does making an observation entail?

I shall assume that this is not the 'making a statement' meaning as in "He observed to his mate that it was Wednesday". Further 'to observe' does not mean literally 'to see' in this context. Seeing is something we nearly all do anyway, and it does make sense to say that a blind person 'observes' some mathematical idea.

Let us explore the l-g of 'observe' using the notion of seeing as comparison. When we see something it is not just a resting of one's eyes on the object. It also embodies attention, and a conscious registration of the object.

It makes sense to say "The book I was looking for was right before my eyes but I didn't see it". When we observe something there is also an element of conscious selection of some idea at the expense of other possible observations.

It may help to ask what is supposed to be observed.

Is it facts -- as when a number theorist observes the density of primes prior to saying that this density tends to a particular limit? (This situation is similar to the scientific cycle of hypothesise-test-observe-revise). The meaning of observe here is like that of 'record': the student recorded/observed the intercepts of the graph on the x-axis. While it is possible to teach the skill of recording, this is done neither by providing mathematical experiences, nor for the development of general mathematical principles. This aim, I feel, is saying something more.

Is it something mathematical which is to be observed? i.e. the fact has to be interpreted in a mathematical way. For example we can observe a crowd of people in several ways: as an aesthetic arrangement of colours or sound, as a sociological phenomenon, as a subject for statistical analysis, or as a mathematical collection. The movement of such a crowd can be looked at as a pendulum of given period, as a wave motion, or experienced lyrically or emotionally. The problem now is that one situation can be observed in many ways and it usually even has many mathematical interpretations. So we must admit several possible observations for a particular event. How do we teach a student to observe, and what mathematical experiences should we provide to do this?

Is there, in a particular situation, a 'right' observation to make? If so, who decides, and are all the other observations then non-mathematical? We seem to be defining what mathematical observations are -- is this what an aim should do? (See section 2.17 for further discussion on this point). The question of what is 'right' is discussed at length in section 3.8, also cf 'right' patterns below (2.12).

2.12 ...discovering patterns and relationships...

Does this phrase imply that there are 'right' patterns and relationships which are waiting to be discovered in a given situation?

Take the series: 1, 4, 9, 16, 25, 36, 49, ...
Possible discoveried might be:

D1: The series increases.

D2: Ah ha, I recognise those numbers, they are squares. This is the series of perfect squares.

D3: Look at the differences: 3, 5, 7, 9, 11, ... the series is governed by sequential odd-numbered differences.

D4: (That means that) each number of the series is the sum of a partial series of odd numbers 1 = 1; 4 = 1 + 3; 9 = 1 + 3 + 5; 16 = 1 + 3 + 5 + 7; ...

D5: The n^{th} term is $x_n = \sum_{i=1}^{n}(2i-1)$

D6: A relationship: the perfect squares are the sums of partial series of odd numbers.

D7: The mathematically expressed relationships is:

$$n^2 = \sum_{i=1}^{n}(2i-1)$$

There may well be more complex interpretations of the series. Is the aim asking for all patterns to be discovered by everyone? If not which step is enough? Alternatively will any pattern discovery fulfil the aim? If not how do we know which discovery is appropriate? Does the teacher decide (see section 3.8)? Is there a sequence of discoveries of increasing mathematical complexity, the aim implying that the next one should be discovered (see section 3.1)? Or is there a psychologically or pedagogocally appropriate sequence like Piaget's cognitive development or Bruner's special curriculum?

Perhaps the appropriate discovery depends on the previous mathematical experiences. For example, assume triangles had been the subject of several lessons, and the next subject was the interior angles of polygons. Would it be correct for a student to discover the diagonal division of any polygon into triangles and thus formula for the sum of the interior angles? And would it be wrong for another student to, say, measure the angles of a triangle, quadrilateral and pentagon and arrive at the formula that way? This is to say that the mathematical experiences define the patterns -- again see section 2.17.

Returning to the example above it helps to ask what sort of mathematical experiences might help a student to make each of the discoveries listed:

D1: Using the words 'series' and 'increases' correctly.

D2: Familiarity with number facts.

D3: Familiarity with general methods of dealing with sequences and series.

D4: ?

D5: Practice in dealing with (and possibly creating) symbolism.

D6: ?

D7: Practice in dealing with (and possibly creating) symbolism.

Our list of suitable experiences has two gaps, and they come at the most crucial points.

Certainly familiarity with mathematical terms and series, and a disposition to persevere with mathematical thoughts are both necessary. But these are hardly sufficient conditions to enable such discoveries to be made.

Several educators have attempted to describe just what is required. For example Polya, in his discussion of heuristics (Polya, 1957), and De Bono's concept of lateral thinking (De Bono, 1971) both attempt descriptions of how a person can come to new ways of observing a problem and how they can thus create new solutions.

A look at the l-g for 'discovery' in a wider context may help to clarify the discoveries in D4 and D6 above.

We say that Columbus discovered America. When did this act of discovery take place? It doesn't make sense to say "After he had seen land, then Columbus discovered America." So is the act of seeing the land the discovery? But many others saw land from Columbus' ships -- he was not even the first. What about the people living there already? Did they discover America when they were born? There is a literal sense in which we could say that these others discovered America, but when we use 'discovered' with reference to Columbus we are including his recognition of the significance of the land which he saw. And that significance was dependent on his point of view (i.e. as a European), and the discovery is attributed to him, not his sailors, because of his prior analysis of the world as round. Further, some aspect of his returning and reporting the land is also involved. Thus discovery seems to involve a point of view, a recognition of significance and possibly intention as well as the actual observation.

The aim above specifies discovery of patterns and relationships. We can think of these as already in the mathematics, as America was always there, but we must include a creative aspect: namely 'giving significance to'.

The preamble to the aims sees mathematics as (among other things) a model of the real world, i.e. aspects of a real situation are isolated and dealt with symbolically, the results being reapplicable to the original situation. Models are created, particular ones being chosen as more appropriate than others. Spherical geometry was always applicable to the earth, but it makes sense to say that Lobyachevski 'discovered' the spherical model. Did he create it? Not in the sense that the sculptor creates (although Stenhouse in this context tells about the child who asked the sculptor "How did you know the lady was inside the stone?" Perhaps creation is never an entirely original act) and yet it certainly was not all intuition. Changing the parallel axiom was the result of deductive analysis, and the beginning of a deliberate synthesis, by Lobyachevski.

For discovering patterns, then, we must include the idea of conscious recognition of significance. This implies a lack of knowledge of this significance beforehand (but what about discovery?) on the part of the student at least.

2.13 ...developing concepts...

Concept development has been much discussed, e.g. (Skemp, 1971, Chpt.2.). I shall assume that we are talking about mathematical concepts.

Let us try the l-g on a particular concept: that of a prime number.

Now I am sure that I have such a concept. Is it a definition? Definitely not:
"a number which has no integral divisors except itself and unity" is one definition,
"p is prime $\Leftrightarrow (p \in N) \wedge \neg((\exists q)((q|p) \wedge \neg(q = 1) \wedge \neg(q = p)))$ " is another.
"A prime is what you get after applying Eratosthenes seive" is a third. I could write a dozen more and none of them would be essential to <u>my</u> understanding of prime, although part of my concept is included in each one. It does make sense to say: "She knows what a prime is but cannot give a definition."

Skemp delineates between a concept and a definition in this way:

> "Definition can...be seen as a way of adding precision to the boundaries of a concept, once formed; and of stating explicitly its relation to other concepts."
> (Skemp, 1971, p.26).

But does it make sense to say: "She has a concept of a prime but cannot recognise one."? It seems not until I ask her whether 964,072,437,841 is a prime. So recognition is not the criterion.

She may reply: "I know how to find out though". If she did demonstrate that skill I would want to say that she had the concept. However there are others who could not find out whether 964,072,437,841 was prime but still had the concept e.g. a fourth former who had just begun to learn about them may have a true (but unsophisticated) concept of 'prime'.

What about being able to give an example? I think that all those with the concept of a prime could give examples -- but if I asked someone to give me a prime number and they replied "13", would I know that they had the concept?

It may have been a lucky guess. What else could I do to find out? "Give me some more". "2, 3, 5, 7, 11, 13, 17, 23, 29". I'd begin to believe, but they may have memorised them, (easily possible as a result of some third form teaching). "Why are they primes?" Now many replies are possible. Of the replies which would convince me I'm sure there are answers which would be too complex for, say, the fourth former mentioned above.

So two people may both be said to have this concept and yet one may not be comprehensible to the other. We are now getting into the 1-g of 'prime' -- so is to have a concept to be able to play a 1-g? This conclusion is argued by Stenhouse:

> "Knowing a concept is knowing the rules for the use of a word. These rules are earned mainly tacitly: we learn the language-games by hearing, seeing, and generally by participating in them". (Stenhouse, (a),p.13). (Quotation from draft material by permission of the author).

To have an experience of primes, whether it be a discovery, intuition or generalisation, changes in some way what we previously thought of as a prime. This enables us to use 'prime' in an extended way, to play a different 1-g. The cumulative exploration of such 1-g's is then our understanding of 'prime' when we hear, see or use it.

In this interpretation we must then say that the person who spouted from memory all the primes up to 97 when asked what a prime was, could be said to have a concept of a prime. For him 'prime' means just that list. The aim, then, becomes to develop (extend) such a concept, and come to some, not necessarily complete, agreement about the 1-g. (Note that if a concept is judged to be wrong it is wrong only with respect to the judger and a particular 1-g).

2.14 ...drawing logical conclusions...

It is important to look closely at what 'logical' means. On the one hand it could mean the 'common-sense', intuitive logic which we use in everyday language. On the other hand it may refer to formal, mathematical logic.

To take the second case: there is a whole branch of mathematics (Foundations -- see e.g. Hather, 1968, Cpt.2.) which deals with the contradictions arising from formal logic, and which has put in doubt the status of any conclusin based on logic alone. So a study of propositional or predicate calculus does not seem to be implied.

But if common-sense logic is what is required then why are mathematical experiences necessary?

The middle ground could be that mathematics symbolises and makes conscious what is essentially common-sense logic. This seems reasonable in the secondary school context and fits the 'model' view of mathematics mentioned in the aims. So drawing logical conclusions means using te mathematical logical symbols (e.g. $\therefore$, $\Longrightarrow$, $\equiv$, $\Longleftrightarrow$) in appropriate places. This idea is developed later (see section 3.3).

Another interpretation is that drawing logical conclusions is the way mathematics is done -- i.e. if a result is obtained logically then it is good mathematics. This is one of the assumptions dealt with in detail in section 3.4

2.15 ...expressing thoughts accurately...

There are two distinct l-g's which can be played here.

The first assumes that the thoughts are mathematical ones, i.e. the subject of the thoughts being mathematical. The aim thus means that the student can write coherent mathematics, has the skill of being able to see a mathematical relationship and write it down in an accepted form. Section 3.3 looks at the l-g for 'accepted forms'. This is the l-g many teachers play, but the second l-g is more vital in mathematics education.

The other l-g, then, involves thoughts about anything, the mathematics being the accurate expression. But if the thinker expresses what he is thinking then that, for him, is accurate. The misunderstanding comes when a reader tries to obtain the same thought from what is written. Thus 'expressing thoughts accurately' is 'expressing thoughts so that others will know what you are thinking'. But how can anyone have exactly the same thoughts as someone else? More to the point how could you ever find out? Only by expressing the thoughts and comparing them: but the expression is not the thought!

The best that can be achieved is perhaps direction indicators and the hope that the reader will follow. These indicators will presumably be the points of departure of this new line of thought from other ones. Such a l-g is exactly what the 'model' view of mathematics calls for. That this is the l-g of the writers of the aim is borne out by the appearance of objective (j) under Aim.1. in the 1976 In-Service Report (see Appendix 7.2 p153)

Here is a clear example of divergent l-g's with a phrase which looks, at first, unexceptional. The implications for mathematics education are that the first l-g makes mathematics teaching introspective and narrow, the second makes it relavant and outward-looking. This is not to say that one is better, both views are necessary for a full mathematical education. But in practice they tend to be mutually exclusive. For further discussion on world view (vs) isolation see section 3.2.

2.16 ...forming generalisations...

One l-g for this phrase equates with '...developing concepts...' which has been dealt with in section 2.13.

At least two meanings can be identified here.

On the one hand this could be a factual knowledge aim. Students must form specific generalisations which have already been decided upon. On the other hand the process of forming generalisations is required, the subject of the generalisations being irrelevant, possibly not even mathematical. These two interpretations parallel those in section 2.15.

For example, students studying graphs of linear functions may make the generalisation that the constant term gives the intercept on the dependent axis, but they should also be learning how to come to that generalisation.

Much argument about discovery learning can be analysed with respect to these l-g's, The originators of the method played the second game (or possibly both games) while the poor exponents of discovery learning, and many of its critics, played the first l-g.

2.17 ...provide mathematical experiences which will enable... etc.

Having looked at the various activities listed in the aim we now wish to know what the relevant 'mathematical experiences' are.

Questions which may help us to explore the appropriate l-g- are:

a) are mathematical experiences necessary to achieve these activities, as opposed to other sorts of experiences?
b) are the mathematical experiences referred to common to all the activities dealt with in section 2.11 - 2.16?

A rewording of the aim in the 1976 In-service Report to read:

> "To provide mathematical experiences which can enable..." (Appendix 7.2, p.)

indicates that the first question should be answered: "No". This implies that the aim does not apply in the same way to all pupils, i.e. the appropriate experiences are student-relative.

Now, to rephrase the second question, are they also activity-relative?

For making an observation the mathematical experience was seen to vary according to which observation was considered valid.

For discovering patterns a whole series of experiences were listed and for two particular patterns no experiences seemed appropriate.

For developing concepts the experience became equated with the activity. Forming a concept was the same as having experiences of the concept.

For drawing logical conclusions either experience in using logical symbolism or examples of logical sequences in mathematics were seen as appropriate. The second of these is the l-g of those who espoused Euclid as ideal mathematical subject-matter.

For each of 'expressing thoughts accurately' and 'forming generalisations' two sorts of experiences were necessary. The first were mathematical experiences in a factual sense so that, for example in teaching set theory, enough sets were joined so that the student could write a set using conventional notation and understand the (preconceived) generalisation of set union. The second sort of experiences were examples of mathematically modelling the real world.

This wide array of experiences (they certainly are activity-relative!) prompts the question: why were such diverse aspects lumped together under the same aim? If the aim is to make sense, rather than lead us to conclude that its authors were writing impressive but empty verbiage, then there must be a further purpose.

In keeping with the l-g analysis this further purpose can be seen as defining what mathematical experiences are. So the statement of the aim is not just saying that these activities are to be carried out by students, but also tells us that mathematical experiences (which presumably are what teachers should provide) are those things which lead to the specific activities mentioned.

A consequence is that teachers often will not know until afterwards whether a particular experience was mathematically helpful. But this tallies with our experiences of a classroom interaction and should not dismay us.

Nor does it make the aim circular or meaningless. In fact it emphasises (what we know but often forget) that a mathematics classroom is a feedback situation where the results of interaction define whether the experience was mathematically worthwhile at the same time as the interaction creates the result.

1+1=3

2.2 AIM.2.

Statement:

> "To develop further an understanding of the principles underlying the structure of mathematics, and the ability to apply these principles to wider fields".

Ex.para.:

> "Principles previously acquired are now extended, ..., so that pupils may see a unity and a structure within the subject as a whole. As pupils are led and encouraged to use principles in the solution of problems, they will realise that an understanding of principles adds immensely to power and economy in learning".

I shall investigate four questions:

What is a <u>principle</u> in this context?

What are the principles underlying the structure of mathematics?

What constitutes an <u>understanding</u> of the principles?

How are principles applied to wider fields?

2.21 What is a principle?

The idea of a principle carries some connotation of generalisation - a non-generalising principle does not make sense. It also includes the idea of agreement - to say that mathematicians disagreed widely on what was a mathematical principle also sounds absurd. Although there certainly have been changes in these principles (see section 2.22).

The use of the definite article: "<u>the</u> principles underlying..." indicates that the authors expect mathematics educators to agree at any one time on the principles, but allows for changes over time.

Another clue to the l-g being played is "...an understanding of principles adds immensely to power and economy in learning". Now if 'immensely' was left out then the statement is tautologous since a principle implies generalisation. But the writer wishes principles to include strongly generalising ideas only.

The statement that "principles previously acquired are now extended" suggests that either principles can become increasingly sophisticated (e.g. binary operations are extended to groups), or the same principle can be seen to apply in more areas of mathematics (e.g. the idea of binary operations applying to geometrical transformations as well as to numbers), or possibly both.

2.22 What are these particular principles?

First note that no such principles are stated in the syllabus. The 1976 In-service Report implies some in its objectives (see Appendix 7.2), but, for example, "appreciate mathematical criteria" (Objective i, ibid) does not help us to decide what these basic mathematical ideas are.

Two sorts of principles might be distinguished, dealing with either mathematical content or mathematical processes. An example of the first kind would be 'trigonometric functions are periodic'; and example of the second kind: 'mathematisation involves abstraction and symbolism'. These two kinds of principles are "applied to wider fields" in very different ways: the first as direct applications like those involved in engineering the second involves a point of view more appropriate to a practising mathematician who is considering a new problem.

The principles do have to do with the structure of mathematics.

Is there "a unity and a structure within the subject as a whole?" It is all called mathematics, but why, or what characteristics are universal, vary from Russell's:

> "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true".

to Hardy's:

> "A mathematician ... is a maker of patterns".

No universal structural characteristics are given in the syllabus, and it is presumably left to the teacher or student to develop his/her own - this is consistent with the use of 'a' not 'the' above.

However can whatever universal characteristics develop in a class be called principles? The element of agreement has disappeared. Furthermore we could never be sure that the principles could be "applied to wider fields", or indeed, that any principles were acquired in a particular case.

Another possible l-g is that principles do refer to an individual's (teacher or student) underlying understanding of what mathematics is about. The agreement component may be thought to be inevitable in the sense that mathematics is self-explanatory and anyone studying mathematics successfully is bound to recognise universal features. Again this can be seen as defining 'successful mathematics' i.e. if the principles gained are not universal, then it is not really mathematics which is being studied. This is not necessarily a static view point as new principles may be formed, argued for, and then accepted. Such was the case with Cantor's set theoretic notions which were the subject of debate and necessitated a Kuhnian paradigm shift on the part of practising mathematicians before being accepted.

What are the principles of the structure of mathematics?

The history of mathematical thought is a story of changing attitudes to this question? Ormell (1972, b) traces the various answers through ratio (Pythagoreans), space and number (Middle Ages), sets (Frege, Russell), structures (Hilbert) and language (Sematic School).

At present there is a great controversy over these principles: several positions are well-argued and consistently adhered to. And such arguments have relevance for Form I to IV level mathematics.

The relevance relates to the pupils future learning. For example although most secondary schools teach Transformation Geometry some institutes of tertiary education use Euclidean Geometry and are teaching their own courses in this from scratch, ignoring the knowledge students have already acquired. Thus changes in the accepted structure of mathematics can be detrimental if these changes are not universal. The set theory basis of 'the new maths' is seen to be only one possible basis and cannot be held to be a definitive principle of the structure of mathematics.

Ormell in his article (ibid) goes further and says that such principles are internal to mathematics and have little meaning outside of the discipline. This implies that it would be a mistake to use them as a basis for educating uninitiated mathematicians . He proposes that mathematics be seen as a science of possibility. The article (ibid) is a 1-g around this suggestion. Under this hypothesis "the ability to apply these principles to wider fields" is exactly what mathematics is about.

If principles are thought to be internal, then only mathematical process principles mentioned above, (not the mathematical content ones), are being considered.

2.23 What constitutes an understanding?

If just any level of understanding is required then the aim is vacuous since whatever understanding is achieved by a particular student will fulfill the aim. Alternatively a complete understanding by all students is unrealistic.

One apparent area of middle ground is something like: 'to develop an understanding concomitant with the capabilities of the individual' - in line with Fraser's famous statement. Yet this degenerates to the first case: an understanding within an individual's capabilities is decided by what limits to understanding he/she achieves in fact. It is conceivable that some independent indicator of understanding potential can be developed, but I doubt whether the author of the aim intended each student to be tested to determine how great an understanding is possible and then teach to that level. Even then the validator of the testing procedure can only be the level of understanding actually achieved by the students.

A look at the syllabus and curriculum rationales indicates that chronological age is taken as a broad indicator of levels of understanding - yet Piagetian studies give only the sequencing, not the achievement ages of certain levels.

The idea of sequencing combined with hierarchies of principles of increasing sophistication suggest another area of middle ground. The aim may be asking for the <u>next</u> level of sophistication to be understood. This implies not just universal principles, but a universal hierarchy of principles and the structure becomes even more rigid.

Mathematics educators may argue that the principles are near universal, but the principles are seen differently by different individuals. If we accept this, then a person's perception of a principle must be accepted as appropriate for that individual. Thus the self-validating circle is again completed.

Another relevant question is how any understanding comes about. If the principle is abstracted by the student from several mathematical examples then this suggests an individual-relative view. If the principle is stated, the terms explained and then examples given, then this supports the 'one structure' view.

The 1-g conception of the teaching of principles (cf that of concepts in section 2.13) would involve a simultaneous illustration of principles by examples and an abstraction of principles from examples - each defining the other and with no end-point ever being reached.

2.24 How are principles applied to wider fields?

The 1-g for "wider fields" is presumably any wider field rather than some particular wider fields - a reference under this aim to "new situations" (Appendix 7.2, Objective f) confirms this. The applications of the principle cannot be known in advance and this leads to time problems which are dealt with under Aim 5 (section 2.53 and section 3.7).

One consequence of this is that process principles rather than content principles are desired, since the former are more economical and time-independent. This fits with the ex.para.

How are the principles applied?

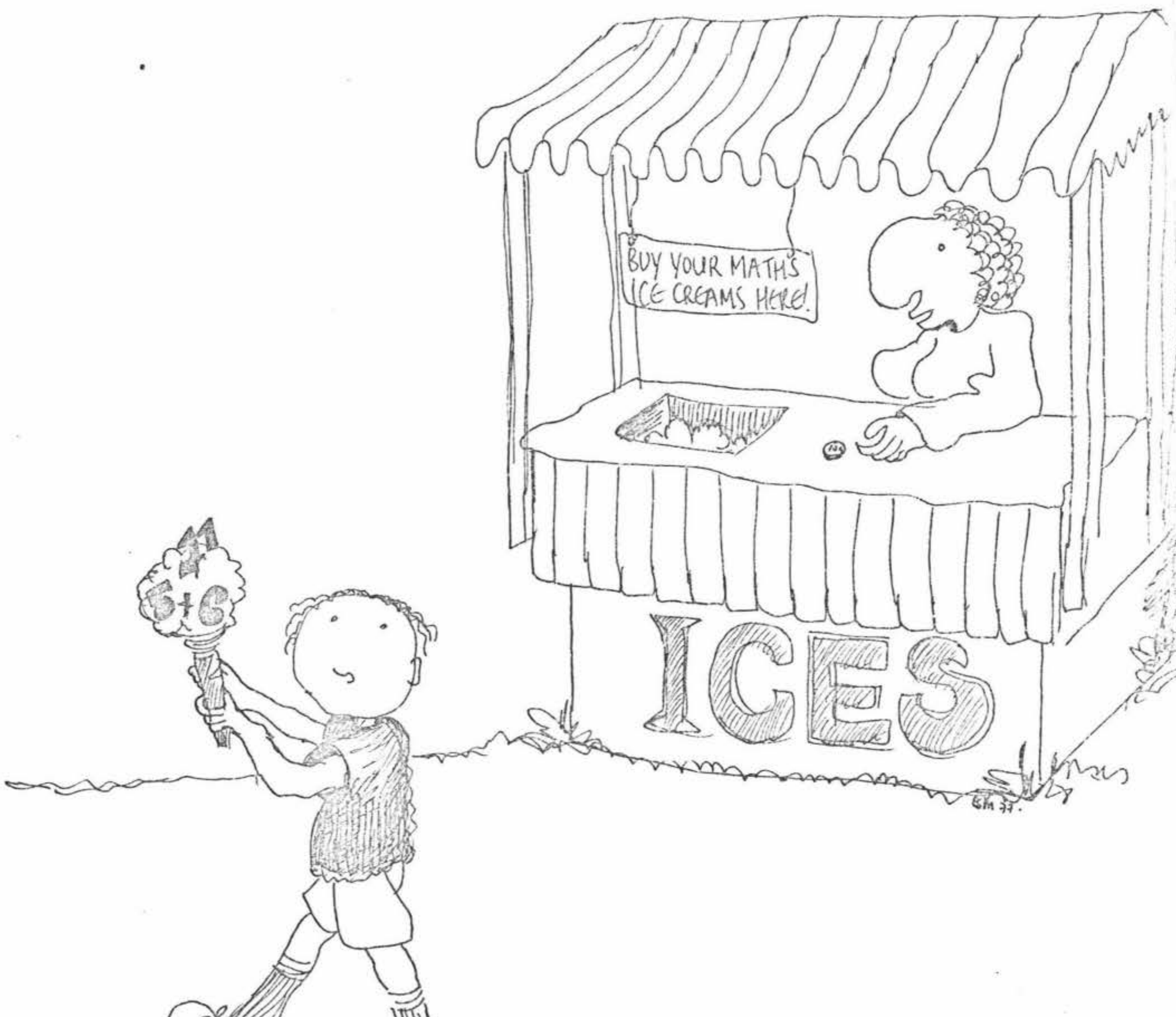
Any mathematical principle which fitted wider fields as well as mathematics would need to be very general and nearly empty of meaning, (e.g. Hardy's statement above).

The application may be mathematical, the necessary ability being the ability to see the mathematical content in an essentially non-mathematical area. This is consistent with the "economy of learning" phrase which can be taken to mean that the principles are used in the same way but in different places. An example might be the use of Venn diagrams to categorise the people involved in some situation; set inclusion and disjunction being applied to a sociological area of study.

How will the students develop this ability?

They must be familiar with the principles, but also they must be able and willing to apply them. This three-tier conception of mathematics education is considered in section 3.6. Note here that the familiarity stage is included in Aim.1. (section 2.1); and that the ability stage may possibly be achieved by using examples though not all possible situations can be studied. The willingness stage raises some ethical questions. Assuming that it is known how to make students willing to see things mathematically, is it correct to do so, do we want everyone to have this perspective? Possible reasons for a 'No' answer are that a mathematical point of view may exclude other, also valuable, points of view.

Under the l-g conception of learning principles, applying mathematical principles is not separate from developing them. When, say, we do a network analysis of a factory floor layout, this adds to the ideas we already have about network analysis as well as resolving a factory design problem. Thus learning and applying are simultaneous. The aim suggest that one first learns the principles and then applies them.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2.3 AIM.3.

Statement:

"To develop a liking for and a lasting interest in mathematics."

Ex.para.:

"The pupil's attitude towards mathematics is a critical factor in determining his success in the subject and his readiness to pursue it to higher levels. The pupil who experiences the intellectual enjoyment and personal satisfaction of discovering laws of number and space, of perceiving patterns and forms, and of finding the solution to problems will be stimulated to continue his studes. The teacher's general approach to the subject, together with provision for individual differences in needs, abilities and interests, is the key to achievement of this aim".

Discussion of this aim is divided into three overlapping areas. They centre around the answers to the questions:

What is it that students are asked to develop a liking for and a lasting interest in?

What does "success" mean in this context?

What does it mean to develop a liking for, a lasting interest in or a positive attitude towards something?

2.31 What should students like?

Several possibilities suggest themselves:

Is it the content of mathematics; the methods of mathematics; the whole field of mathematics; or the mathematical way of looking at other fields?

Is it the mathematics in the syllabus; the mathematics taught in schools (i.e. most of the syllabus plus other topics dependent on the teacher); or is it the mathematics likely to be of use in the future?

Is it all of mathematics or just that part which suits the learner?

Is the student supposed to develop a liking for mathematics as a school subject (i.e. competing against English or Social Studies); or for doing mathematics in his/her job or leisure?

The ex.para. specifies some particular experiences which suggest that students should develop a liking for the process of mathematics within the syllabus. However the key to playing this particular l-g will be the consequences of the different interpretations for the conflict between achieving this and the other aims. For example it may be that a student is fascinated by games theory and cannot be helped to a liking for any other aspect of mathematics. If the aim is thus fulfilled then trying to fulfill the other aim may turn him off mathematics altogether so that achievement of this aim is lost. Such conflict is often seen in the classroom where teachers try to buy the students' attention on, say, geometrical transformations (i.e. syllabus material) by allowing a certain time on design-making. In addition teachers try to include what is liked by students into the syllabus by a broad interpretati on of syllabus aims e.g. making tessellation patterns are 'experience in basic geometrical transformations'. These strategies are essentially ways of overcoming the conflict between this aim and the others.

2.32 What is success?

In particular what is the relationship between success in mathematics and a liking for it?

It is reasonable to say that someone who likes mathematics is more likely to be successful in any future studies, and that successful mathematicians will probably be found to like the subject. Research evidence verifies this, e.g. Naftel's investigation of various studies show correlati ns of between 0.2 and 0.4 (Naftel, 1974), between attitude and success in mathematics. Nothing in the above suggests which is the cause and which the effect.

However the ex.para. seems to say more than this: namely that attitude is a critical factor in success. This does identify attitude as the cause, and as a necessary condition.

But if school, or examination, success is meant here then the statement is empirically false: I'm sure that I could find students who had passed School Certificate University Entrance in mathematics and who hated the subject.

Successful might mean 'able to apply mathematical ideas or concepts'. Here, also, however, it would make sense to talk of someone who applied an idea or concept successfully but didn't even know that it was a mathematical idea or concept, let alone enjoyed the subject (i.e. their perception of the subject, probably gained at school). We are getting back to the question of section 2,31.

The authors of the aim do want success to involve a positive attitude. They would want to say that someone who got 80% in UE mathematics but did not enjoy the subject had not had a successful mathematical education. In other words the aim writers are defining success to be something like 'can, and is willing to, consciously apply mathematical ideas correctly'.

Similarly "readiness to pursue" further mathematical study is partly being defined by the ex.para. to include 'because the student likes the subject, not just because he wants (for example) a science degree'.

2.33 What's it like to like mathematics?

If success involves liking mathematics, what causes us to have positive attitudes towards, lasting interest in or a liking for the subject?

First let us investigate the l-g for 'personal satisfaction'.

We can try it in another context: "I got satisfaction from trying to fix my bike (even though I failed)". Could this make sense? Yes, in that if I hadn't tried I wouldn't know whether I could have fixed it or not. But would this satisfaction stimulate me to continue to try fixing my bike? Surely this depends on some measure of success. In the same way if personal satisfaction is to stimulate me further in mathematics I would need it to involve some degree of success.

Second, what does 'intellectual enjoyment' mean?

The bike fixing analogy can be used again. It seems as though a person could be stimulated to continue puddling around with their bike if they enjoyed it, whether or not they actually achieved anything. So success does not seem necessary. But is someone who is puddling around with their bike ever not achieving? They may not be fixing the specific fault which initiated the activity, but they cannot fail to become more familiar with the bike or gain experience in manipulating it. This point of view is elaborated at length in Persig's book Zen and the Art of Motorcycle Maintenance (Persig). In mathematics, too, attempting a problem can be successful although it may not be solved in that later attempts may be better.

But if either 'personal satisfaction' or 'intellectual enjoyment' involve success we have, combining with section 2.32, a circular situation! success (is part of) personal satisfaction (is part of) a liking and lasting interest (is the same as) a positive attitude (determines) success. Not very illuminating.

Continuing to investigate the l-g for 'intellectual enjoyment', we know that enjoyment is something which 'gives us a good feeling' - intellectual enjoyment could be a passive thing (i.e. in our heads) which gives us a good feeling. This good feeling is readily recognisable in ourselves even though it is difficult to describe. However can this be 'developed' by someone else? Can we learn to enjoy?

The aim writers' answer to this question can be investigated by looking at the last sentence of the ex.para.: "among other things, the pupil's individual interests are a key to developing a liking for mathematics." This seems to say that a teacher should utilise the things which interest (give enjoyment to) the pupil. In other words it is a case of wrapping the mathematics around the enjoyment, rather than the enjoyment around the mathematics. This fits with a behaviouristic model of learning - the teacher should associate mathematics with things that the student enjoys, in the hope that conditioning will occur and the enjoyment become associated with mathematics.

This can lead to conflicts described in section 2.31, where nothing about, say, the structure of mathematics (aim.2.) links with an interest of the student. If this is to be avoided what place do a student's interests play? Perhaps the use of the word 'provision' effectively means that individual needs and interests should be catered for where they do not conflict with other aims, i.e. this aim is secondary. This, however, negates the statement that 'a positive attitude is a critical factor'.

As a final comment on this aim it is explicitly recognised that the teaching process is important for this aim, and it must be asked whether a particular style of teaching which promotes a liking for and a lasting interest in mathematics will in fact be practical with respect to the other aims.

For example it is often assumed that school success is necessary for satisfaction in mathematics and thus it is tenet of teaching to arrange for some success by all pupils by finding the individual level of each student and making some tasks achievable. This could interfere with the other aims if the level of achievable tasks is too low. Further, success alone is not enough but some other factors (e.g. challenge) also need to be present. If all students are to be given sufficient (??) success, is it possible to cover the syllabus?


NOW ...... LET ME PUT IT THIS WAY. IF YOU'VE GOT THREE CALCULATORS AND I TAKE AWAY TWO OF THEM, HOW MANY ARE LEFT ?

2.4 AIM.4.

Statement:

> "To develop the basic mathematical knowledge, skills and understanding necessary for everyday living and for effective citizenship".

Ex.para.:

> "All pupils need to be able to handle numbers and spatial ideas in their day-to-day experience. In addition, as adolescents and adults, they will need mathematics to help them to understand the world they live in and to meet their obligations as citizens".

The following analysis is divided into four parts concentrating on, respectively: everyday living; effective citizenship; basic mathematical knowledge, skills and understanding; and the l-g for 'necessary'.

2.41 What is needed for everyday living?

What is everyday living?

'Everyday' has a connotation of ordinariness about it - it sounds strange to say: "His everyday activities were rather special last Wednesday", rather we would say: "He did not do his everyday activities on Wednesday". Note also that some average person is not implied: it is meaningful to say "His everyday activities were very strage to the rest of us".

Perhaps 'everyday living' encompasses all activities which could be conceived as routine for some person. However the mathematics necessary for the everyday activities of a nuclear physicist are different in scope and level from that necessary for a mutton slaughterman - obvious exceptions granted: a mutton slaughterman may do nuclear physics as a hobby.

But this aim is complementary to aim.5., which specifically mentions vocations and future studies. So 'everyday living' probably refers to non-vocational needs. Such an interpretation is backed up by the other reference to 'effective citizenship', which, in an egalitarian society such as ours is supposed to be, should not be vocationally dependent. We would then have to interpret 'day-to-day experience' and 'understand the world they live in' as phrases referring to things outside of individuals' jobs.

Does such an interpretation specify the mathematics to be learnt?

If we were to be pedantic then there is no mathematical need common to all. It could also be argued that no single mathematical fact, skill or idea is absolutely necessary for survival in the sense that its absence would entail death. However, with some vagueness, we can say that enough familiarity with numbers for bargaining, some ability to predict statistically, being able to distinguish left from right and similar numerical and spatial ideas could make everyday living considerably more comfortable. This issue is further dealt with when the l-g for 'necessary' is considered in section 2.44.

2.42 What is needed for effective citizenship?

I shall assume that the statement is made within a political context, in this case the New Zealand democracy.

Who decides what an effective citizen is like? If we ask for a consensus we will get few common attributes and thus our knowledge of mathematical prerequisites will be scanty. Are the authors of mathematics curriculae the ones to decide? It is difficult to see why - for a start they are likely to have the bias of successful mathematicians. But if these authors are working from someone else's description they do not say so. Nowhere is an effective citizen described.

Well what possible knowledge, skills and understanding might be necessary for an effective citizen?

Perhaps one merely has to be able to take part in the New Zealand democratic system, to meet one's obligations as a citizen, i.e. to vote. Neither voting, nor the knowledge that one should vote, is difficult, nor are they mathematical. However the understanding of the effect of one's vote is very complex, as is comprehending the electorial system. And neither of these appear in the syllabus.

Perhaps an effective citizen must be able to avoid being conned, be able to make decisions for the common good - a sort of utilitarian calculus. Some content of this kind appears in the introduction to statistics, but no mention of this specific aspect is made.

Perhaps effective citizens must have sufficient mathematics education to contribute to the technological and social development of the country. But is everyone supposed to work at the boundaries of development? While this sounds unrealistic I believe that it has the most potential for meaning. Educators could try to develop mathematical skills so that everyone, if put in a situation amenable to improvement by mathematical analysis, might be able to do it - from productivity or labour improvements on the shop floor, through increased efficiency in domestic or financial management, to managerial techniques. Not everyone can be expected to be at the same level, and there is the problem of teaching for future, known applications. In this respect see sections 2.24 and 3.6 on applying mathematical knowledge.

None of the above have led to specific content areas, so we must ask how this aim can have an effect on mathematics education.

2.43 What are the basic knowledge, skills and understanding referred to?

To begin with, universalising incurs severe limits in that division of labour, technology and differing activities mean that there are very few skills that literally everybody needs. For example the ability to add, subtract, multiply and divide may be suggested. But calculators do this, so must everyone be able to do long division? Perhaps everyone must then be able to use a calculator? Maybe, but has our society changed so much that all people need this skill - ten years ago very few needed it.

This aim, then, is not referring to universal needs. Nor is it referring to the lowest level of need, since some have attained this minimum level before entering Form I. If the aim is to continue to have meaning for such people, it must be understood within the 'to the best of their ability' dictum. In other words it is not achieved while an individual has potential to become a better (?) citizen.

David Stenhouse has pointed out that this contains a hidden assumption, namely 'that everyone can continue to develop'. In the present context the assumption is actually more invidious: 'that everyone can continue to develop in secondary school mathematics classes under the methods used by whatever teacher is present'.

What, in particular, is mathematical understanding?

(Section 2.23 has dealt with some ideas which will now be assumed.)

Do we understand an action if we can do it? What understanding is necessary to give change correctly or tell left from right? Can a person be said to have a mathematical understanding of Euclidean geometry if they know that the shortest distance between two points is a straight line, but do not realise that this is an axiom of Euclidean geometry? Many teachers seem to use a 1-g for 'understanding' which includes much more theoretical knowledge than is necessary for everyday living. For example, set theory is sometimes justified as being necessary to be able to categorise effectively. But this subject includes many concepts not usually explicitly used in categorising, e.g. the operation of union obeying the Commutative and Associative Laws.

Understanding can have a different meaning: namely, the ability to perceive situations in mathematical ways. It is possible that the person is not aware that this is what is happening. Further,, being able to work through to the end of the problem situation may not even be needed. Just stating or restating the problem mathematically has been said (Polya, 1957, p.6 and 209) to be indicative of mathematical understanding.

The effects of the aim are still unclear although the last suggestion for the l-g for 'understanding' leads to the field of heuristics. This topic is not, however, mentioned explicitly in the syllabus.

2.44 The l-g for 'necessary'.

I have mentioned already (section 2.41) that the extreme, survival, meaning of 'necessary' is probably not intended. However it does seem as though the authors are suggesting a compulsory minimum requirement. This can be seen as a contradiction of aim.3. - how should we teach the student who is having difficulty with this minimum, or who dislikes this aspect of mathematics?

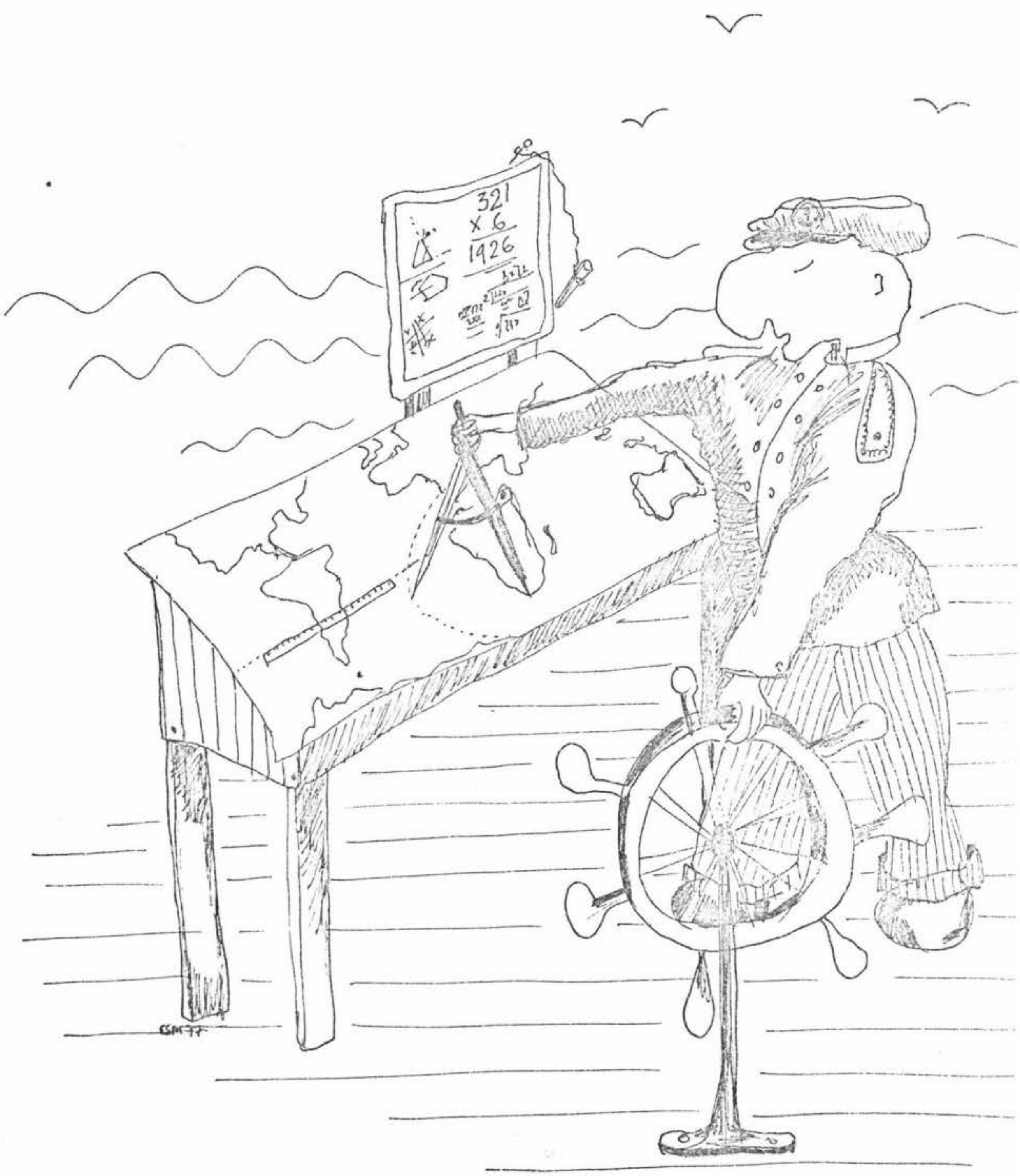
This aim is often used by teachers as a justification for, say, repeated drill of number facts, or when other justifications fail. There is little response that can be made to the statement: "You need to know it" except "What for?". The reply to this second question sometimes reveals the teacher's l-g.

Specific answers, such as "For your job" or "For your University Entrance exam" are valid if there is agreement between teacher and student that these are worth working for, but rely on the teacher's knowledge of what is necessary for, say, boilermaking -- often a questionable situation.

The reply "For everyday life", however, cannot be validated and expresses an opinion (probably based on experience, admittedly) of the teacher. Hence it is dishonest to use the expression as an authoritative statement.

Perhaps, however, the real intention of the aim is to say something about what living and being an effective citizen should involve. In other words you are not truly living, and you are not a truly effective citizen unless you have mathematical knowledge, skills and understanding. So it is not the essential mathematics which is being described, but life and citizenship.

The problem is that this is interpreted by teachers to mean that the mathematics in the syllabus makes for the good life and effective citizenship. This is a much narrower conception and again becomes an unanswerable justification for whatever the teacher wishes to teach.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2.5 AIM.5.

Statement:

> "To help pupils to appreciate the importance of mathematics in their future studies and vocations"

Ex.para.:

> "Increasing demands will be made on the mathematical competence of all pupils in their future careers. Mathematics has long been recognised as indispensible to scientists, engineers, technologists, technicians and skilled tradesmen. New applications of mathematics are being found in all fields of thought and new areas of mathematical study are being developed to meet new needs. Mathematical knowledge is important to economists and social scientists - in fact, to almost all who enter professional, administrative and business careers. These trends have influenced the development of this syllabus."

Although they overlap, the five subsections deal with, respectively:

a) the use of 'appreciate' in this aim;
b) the universal importance of mathematics in all careers;
c) the specific importance of mathematics in different careers;
d) those to whom the aim applies; and
e) other functions of the aim.

2.51 The use of 'appreciate'.

The way this aim is worded implies only that students realise that more mathematical skill and understanding is going to be necessary in their futures, not that they attain it. Thus it is a purely motivational aim i.e. it asks for an insight into how mathematics will affect their lives.

It is difficult to see what part of the syllabus or what teaching would give this. The objectives under this aim in the 1976 In-Service Report (Appendix 7.2, p.7) are little help. The first does refer to modern applications but the examples given (operations research and linear programming) are used by a tiny minority of the population. The second objective does not have anything to do with appreciating mathematics' importance and the third is as general as the statement of the aim.

Computer administration is a major contemporary mathematical influence and an awareness of its limitations and capabilities could be included here. This issue is important for the future partly because of its moral content which is often raised in non-mathematics courses (e.g. social sciences, liberal studies). Is morality a valid topic for mathematical education? Its exclusion implies a negative answer, and a pre-judgement.

2.52 The universal importance of mathematics.

The aim can be read with either the universal or specific idea of 'the importance of mathematics' in mind. The first sentence of the ex.para. seems to back up the universal sense, i.e. that (the same) mathematical competence will be required at higher levels by all students.

First, however, what evidence is there for the great (and increasing) importance of mathematics learning?

Certainly mathematics is a compulsory subject in basic education on a world-wide basis. The authority of nearly all curriculum designers is powerful evidence but it is possible that the importance of mathematics is self-perpetuating: many mathematicians mean many mathematically trained educators mean many favour compulsory mathematics education; or many people trained in mathematics means that the mathematical point of view will dominate many careers and vocations.

The counter-evidence is astounding: a large section of the population (I estimate 30%) have no grasp of Form II mathematics when they leave school. Subsequently they may, however, develop a competence in mathematics appropriate for their needs. Another 30% have some knowledge of S.C. mathematics but are unable or unwilling to use it in their lives. These people survive quite adequately, unless we define adequate survival as involving mathematics.

What is mathematical competence?

If it is what is measured in mathematics exams, the ex.para. can be interpreted as 'higher mathematics qualificatons will be required in all jobs' -- an arguable statement.

If it involves mathematical skills, then 'increasing demands' on mathematical competence means increased syllabus content -- again an arguably useful consequence.

If it involves the ability to think mathematically, then the universal interpretation implies that all pupils should see things in similar ways -- see section 2.11.

The ex.para. speaks of 'mathematics' and 'mathematical knowledge' as being indispensible for (for example) scientists and economists. Any universally useful result of mathematics education must be very general: perhaps there is a 'mathematical approach'. Is this deductive or intuitive? Both are facets of mathematics. Is it the search for order and economy? These attributes may be learnt in subjects other than mathematics. Are these important in all future studies and vocations to the extent that all students need to appreciate them?

2.53 The specific importance of mathematics.

If we interprete the aim as saying that mathematics is important for all careers but in different ways then the following questions arise.

Does the aim intend that all students should see the importance of mathematics to their own futures?

If the former, how, in Forms I to IV, are teachers or students going to know what those futures are? This looks like an impossible task. Further it necessitates one-to-one instruction.

If the latter, again there is the difficulty in knowing what the future will be like: what new studies and jobs will be available in 20 years time? This problem (raised also in section 2.24) is detailed in section 3.7. However it is modified by the realisation that the mathematics which is learnt now <u>creates</u> the future mathematical applications, needs and vocations. For example the Euclidean heritage of Western Mathematics leading to the dominance of geometry has helped to create the importance of geometry in, say, surveying. If geometry was less important then other methods would be more dominant, and if fewer geometers were available then surveying may not have developed so far. A philosophical example is the Western preoccupation of a flat earth - which is almost certainly related to the dominant influence of Euclidean geometry. Perception and analysis was done using planes and lines rather than curves, despite the fact that the circle was regarded as an ideal figure.

If the specific importance of mathematics is intended then there is the added problem of covering all possible jobs and studies - though it could be that a sample coverage would suffice. Which sample? A sample coverage will affect the job selection which then takes place, again creating the needs of the future.

2.54 To whom does this aim apply?

If it applies to <u>all</u> students then an assumption has been made that all students will take up future studies or vocatons. This implies that either the aim of education is a vocation for everyone (which is unrealistic) or that 'vocation' means whatever students subsequently occupy themselves doing - including childrearing, travelling, bumming around, labouring, and white collar and professional jobs.

The ex.para. mentions increased demands for mathematical competence in all students' future careers - i.e. there will be an increased demand for mathematical competence in whatever anyone does in the future?? This statement needs justification and amplification.

The alternative is that the aim applies only to those students who will have careers defined in some restricted sense. Specifically mentioned are scientists, engineers, technologists, technicians, skilled tradesmen, economists, social scientists or 'almost all who enter professional, administrative and business careers'.

Unless the writers think that all Form I to IV students will come under one of these headings (I estimate 50% at least do not), then the aim applies to a minority of students only. Given also that we cannot know which students will occupy such jobs and thus to whom exactly the aim applies, its usefulness is severely limited.

One possible interpretation which may escape this dilemma is that the evidence of the above groups is used as a basis for generalisation to all occupations. This is a large step to take without supporting evidence when one considers that the categories mentioned are (with 2 exceptions) all professional.

In line with the previously mentioned defining function of aims, this aim could be implying that a vocation is something which needs mathematics. That a social scientist, for example, is not a 'real' scientist unless he or she has a grasp of mathematics. While this may sound arrogant (coming as it does from mathematicians) and is arguable, I suspect there is a meaning for 'grasp of mathematics' for which the statement is true - particularly for the more technical vocations.

2.6 FORMS V, VI AND VII

This section looks briefly at changes in emphasis in the official aims for senior secondary classes. Appendix 7.3 contains the relevant statements taken from The Education Gazette 1 April 1972, 1 August 1972 and 30 September 1976.

2.61 Form V

The five aims for the School Certificate course are very similar to the aims for Forms I to IV.

Aim.2. is changed from 'to develop further an understanding of the principles underlying the structure of mathematics...' to 'to develop an understanding of the principles of mathematics...'.

It is difficult to see what the difference is between these two sorts of principles. It could be that the former is trying to express a developing attitude towards mathematics and the latter is some sort of culmination and 'final answer'. This would fit with the conception of S.C. as a terminal course for many students.

That such principles are varied has already been discussed (see section 2.22).

Aim.3. now includes 'to develop mental alertness and a spirit of inquiry...'

A spirit of inquiry is understandable, if only in the behavioural sense of asking more questions, and will not be discussed further here. However the l-g for 'mental alertness' is more complex.

We can agree that a candidate entering an examination is probably more mentally alert than someone drowsing in a chair, but is the aim asking for this state to be developed permanently? This is unlikely. Perhaps what is meant is that a person must be able to be mentally alert at appropriate times.

Is the state of mental alertness active or passive? How would a teacher judge whether a student was mentally alert - if he responded to the cues the teacher thought appropriate or the cues the student saw himself?

What is an appropriate time to be mentally alert? In examinations and a mathematics classroom, yes, but should a student, after leaving school, be constantly alert for mathematical aspects of his environment?

How can a mathematical education promote mental alertness?

Aim.4. now emphasises the computational aspects of mathematics necessary for everyday living.

Such a narrowing of the aim makes the defining aspect more explicit: an effective citizen can compute 'with understanding and efficiency'. I think that this is an arguable statement. Its effect, when one considers that 50% of the population do not pass School Certificate mathematics, is to say that most people are not effective citizens.

Aim.5. states: 'to help pupils appreciate that mathematics does underlie the modern technological society ...'

This makes the assumption that our society is a modern technological one. If the defining function of the aim is considered it also assumes that this situation should be perpetuated in that it must be prepared for.

In general then, the Form V aims are similar with a slight shift of emphasis towards a narrowing, closed conception of mathematics.

2.62 Form VI

The study of mathematics now emphasises

a) logical deduction,

b) correct use of symbols, and

c) understanding of concepts rather than computation.

The assumption has been made that motivation to learn and everyday competency have been achieved and that mathematics is now being studied for its own sake or as an academic tool for other subjects.

This confirms the defining function of Forms I-V mathematics, since Form V level is assumed to be sufficient for effective citizenship and cultural appreciation.

Note that at this level Statistics, Computing and Mechanics are taught separately, but they, too, are now academically oriented: the preamble to the statistics section of the official syllabus states:

> "It is intended that the subject (statistics) shall be treated mainly as a mathematical discipline,..."
>
> (The Education Gazette, 1 August 1972)

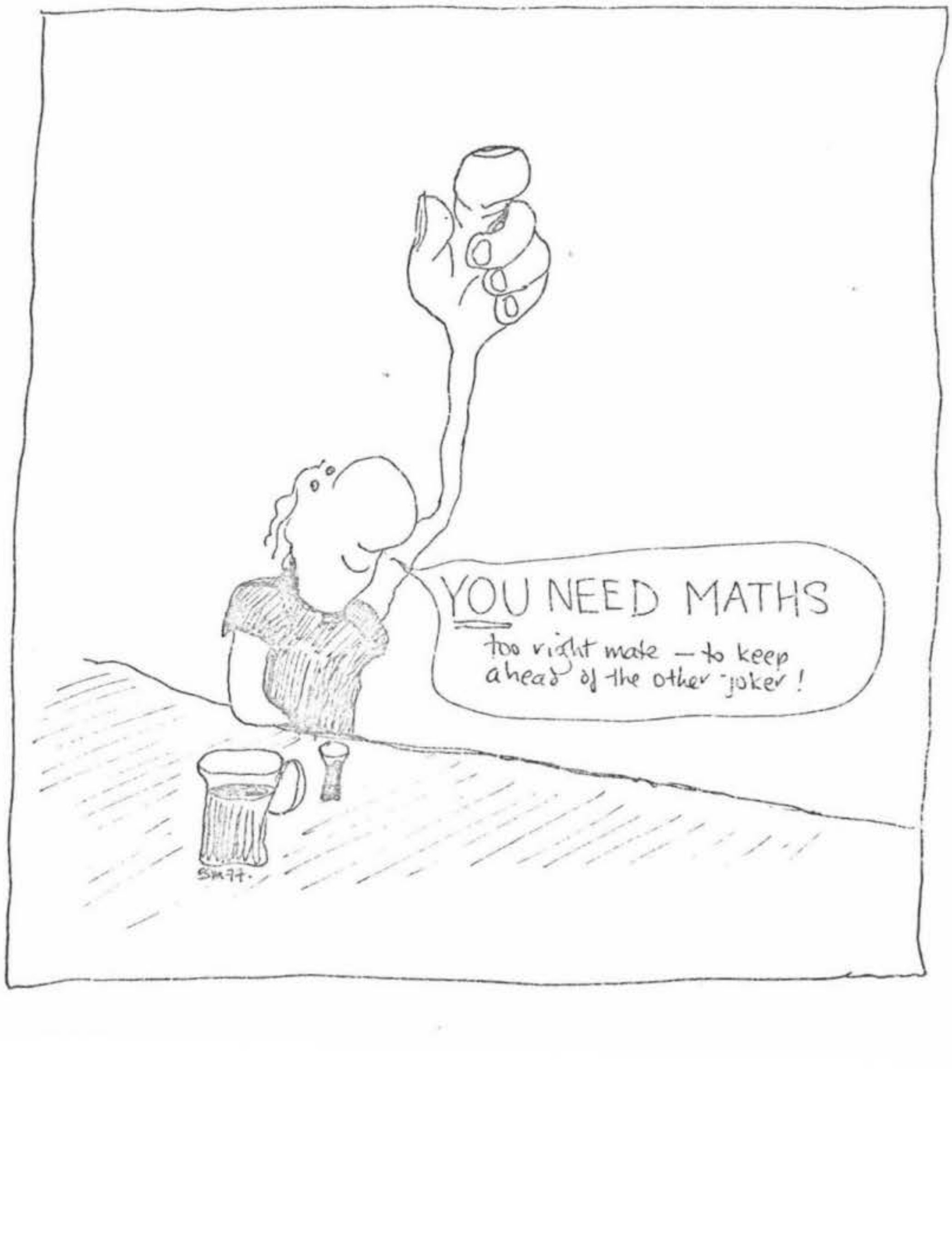
2.63 Form VII

The preamble for the University Bursaries Examination states:

> "The prescription is seen as forming part of the continuing development of the main stream of mathematics and, in particular, the ideas of function and structure....."
>
> (The Education Gazette, 1 August 1972)

Thus the academic emphasis developed in Form VI has continued and narrowed in that two organising ideas (out of many - see section 2.22) have been defined.

This will certainly result in the students' conception of mathematics being formed along these lines. The defining function of the aim is here quite strong.


YOU NEED MATHS
too right mate — to keep ahead of the other joker!
SM77.

3.00 ASSUMPTIONS IN THE PHILOSOPHY OF MATHEMATICS EDUCATION

3.0 INTRODUCTION

This chapter discusses eight specific assumptions which are commonly made by mathematics teachers. Each assumption is examined for its various interpretations in keeping with the language-game analysis in the previous chapter. The validity and consequences of adopting or rejecting the assumptions are also discussed.

The context is that of secondary education in New Zealand. However the assumptions and discussion are often relevant to primary and tertiary mathematics education.

3.01 Where have the assumptions come from?

This chapter links withthe previous one in that the aims discussed there were given various interpretations depending on the assumptions that were made. The assumptions considered here are those which occured most often (e.g. section 3.7); those which are built into the conception of mathematics given in the aims (e.g. section 3.4); and others which, depending on their acceptance, change the way in which the aims are applied (e.g. section 3.3).

The wording for each assumption is mine. In general they are overstated to make the point clearer.

Some justification for considering each particular assumption is given at the beginning of each section.

3.02 Why change assumptions?

Examination, evaluation and, if necessary, modification of assumptions needs no justification other than the possibility that existing assumptions may be wrong or now inappropriate. However there are other reasons for such effort.

One reason for examining the basic assumptions of mathematics education would be because academic standards in the subject are declining and a complete 're-think' is necessary. However this well-used justification needs some careful examination. The evidence that academic standards in mathematics are declining is meagre. The 1974 International Commission for Mathematical Instruction reported worldwide complaints by science teachers and industrialists of low standards of achievement in mathematics (see Mathematical Gazette 59:140 1975). However there are few longitudinal tests in the literature: the only New Zealand one was done in Auckland 20 years ago but did show a significant decrease in a graded mathematics test for 13 years olds and over a ten year period (Clarke, 1960, p.74); the second IEA survey is about to get under way (Education News, 3:2, March 1977) and will give a more definite picture of the situation. In America a 1969 report surveyed such tests: the chair-person said:

> "...Pooling all recent reports of mathematics testing, we found an unmistakeable trend of declining scores over the past ten years. It would be foolish to try to deny that pattern. ...the committee did not find the performance of American young people on tests...as discouraging as many critics suggest. ...declines in mathematics performance are almost uniformly accompanied by low scores in every other school subject area, suggesting much broader... explanations for the depressed (sic) performance."
>
> (The Mathematics Teacher, 1969, 62:6:441)

Also, these longitudinal tests would have to account for factors such as: shifts in maturity levels at a given age; different proportions of an age-group attending school; and shifts in curricular emphasis.

My main reason for examining assumptions stems from a more emotional standpoint. Negative student attitudes towards mathematics are common enough that a word has been coined to describe the state: mathophobic.

The extent of mathophobia has been recognised by the authors of <u>Half Our Future</u> in England (Newsom, 1963) and, in New Zealand, documented in a survey by Coxon (Coxon, 1972, p.542).

Perhaps these attitudes are in fact normal and mathematical fervour is strange? After all the call for the acceptance of a comprehensive mathematical curriculum normally comes from mathematicians: e.g. Professor A Zulauf of Waikato University said thisin his Inaugural Address:

> "...He who refuses to learn at least the rudiments of mathematics is not only denying himself the facility to acquire and communicate vitally important knowledge in virtually every sphere, of human endeavour, he is also depriving himself of much that is beautiful and enjoyable."
>
> (Zulauf, 1972)

Many such subject-centred comments can create an unreal climate of opinion.

Nevertheless I make the assumption that negative student attitudes towards mathematics can and need to be improved in secondary education. Such a view is also argued by Skemp (1971, p.114).

A second assumption I make is that teachers can affect the attitudes of students. Indeed they do, whether they intend it or not.

Such effects are partly a result of psychological theories held by the teachers, but they also result from the conception of mathematics and mathematics education held by the teachers.

I believe that, in general, the conception held by many mathematics teachers is at once too wide and too narrow. It is too wide in the sense that too much compulsory content is included in present syllabi, and it is too narrow in that the content which is included is of a kind which suits only certain kinds of learners (when other kinds of content are available). For example, closed, one-answer problems predominate when mathematics can be studied equally well with open-ended problems.

The conception of mathematics education held by a teacher is a result of assumptions, generally unstated. I have chosen to discuss the following common ones:

3.1 The different subject areas of mathematics should be taught in some logically ordered fashion.

3.2 Mathematics should be taught in isolation from other subjects since it deals only with the essentials of a problem.

3.3 A Large part of learning mathematics is learning symbol manipulation and terminology, or, at least, learning translation to and from (and games within) a mathematical language.

3.4 Deductive method is the paradigm of mathematical thought and exposition and must be dominant in mathematics education.

3.5 The maximum rigor possible is the best mathematics.

3.6 Knowledge and skills in mathematics are more important (or can be taught) than attitudes (which cannot be taught)

3.7 a) Mathematics taught in schools should aim at the general needs of all students.
b) The mathematical needs of everyone are broadly similar.

3.8 Correct answers are a large part of mathematical competence.


SEEDS OF KNOWLEDGE

3.1 SEQUENCES & SERIES: HOW SHOULD TOPICS BE ORDERED?

> 'The different subject-areas of mathematics should be taught in some logically ordered fashion.'

Note that I am not refering to the 'recurrent teaching' aspects of mathematics instruction, i.e. that one topic must be ordered within itself in abstraction and generality. Rather I wish to examine the necessary teaching of one particular skill or subject-area before another skill or subject-area because it takes some sort or precedence.

Evidence that this assumption is made is not hard to find. Despite the statement in the syllabus that "It is not intended to be a teaching sequence". (p.9), the explanatory paragraph for Aim.2. (p.5) endorses the statement in the 1976 In-Service Report that:

> "Mathematics is uniquely sequential in nature". (Appendix 7.2)

The context indicates that this is the first of the five types of orderings which I examine below - Sir Percy Nunn indicates his preference for psychological ordering (section 3.15) (Educational Studies in Mathematics 1971/72 3:322)

Five sorts of ordering will be discussed.

3.11 Structural Ordering

There is no doubt that mathematics <u>can</u> be put into a hierarchical or branching structure. Parts of this picture are very neat and uncontroversial. In the Form I-IV syllabus the progression:

sets-----relations-----transformation-----matrices

make a logical progression of ideas; i.e. within the contemporarily acknowledged structure of mathematics the material contained within the topic 'relations' necessarily precedes that.in the topic 'transformations'.

It has already been said that any structure is temporary and subjective, (section 2.22). Whether mathematics is subjective 'in reality' is more doubtful. Gordon Knight in discussion pointed out that, for example, levels of abstraction are necessarily hierarchical - e.g. the idea of 'number' precedes that of 'variable' - and that triangles consist of lines, so lines come first. The latter point I dispute, it being possible to start with the triangle as the basic unit if that is how space is perceived: the shifts from 2 to 3 dimensions can go either way, 1 dimension is not 'prior' to 2 or 3 dimensions. A cultural example of this exists in Australian aboriginal counting where the basic conceptual units are 'odd' and 'even', i.e. an aborigine would notice the difference, between 23 and 24 sheep in a flock, but not between 22 and 24.

Considering the former point, that levels of abstraction are hierarchical, David Stenhouse linked this with a paper by Ziedens (Ziedens, 1956). In this paper the argument is that the fact that we do talk in material-object language means that the terms 'real object' 'appearance', 'normal observation' have specific meanings which are not just a matter of convention. A similar argument can be applied to mathematics: the way in which we talk about mathematics defines such things as levels of abstraction and the basic units used (e.g. points, sets, logical operations): It is possible that another way of refering to mathematics would give different meanings for 'levels of abstraction' and 'basic units of mathematics'.

Two questions arise from this: first, is there another way of talking about mathematics? Second, is the assumption that mathematics is something apart from its expression, a valid one?

The first question is beyond the scope of this work. On the second David Stenhouse noted that we must appeal to language-games in ordinary language to set up mathematics. As Einstein said, there is no such thing as a line in object language. It is argued later (section 3.3) that a large part of mathematics is a language-game. Thus the structural ordering adopted depends on the language used, and can only be done by someone who knows most of the language-game. Its pedagogical use, then, in teaching a student the language-game itself is not proven. An alternative is for the student to form his/her own language-game and then put it into correspondence with the conventional one. That such a correspondence will exist is the result of the argument above based on Ziedens' ideas.

To return to the issue - is structural ordering appropriate for mathematics education? - the subjectivity of topic-ordering is borne out by looking at text-books. Kline, talking about traditional mathematics texts since 1900 says:

> "...practically all of the texts on any one of /arithmetic, geometry, algebra and trigonometry/ contain the same material and presentation; only the ordering of topics is different."
>
> (Kline, 1973, p.13)

Four further points can be mentioned concerning structural ordering.

If any structure is temporary and subjective, then students (who will be practising mathematics in the future) need not accept, learn or be bound to the structure of present day mathematicians or mathematics educators. This is especially important since, to quote Freudenthal:

> "...restructuring mathematics often means that by preference its most elementary ideas are questioned anew".
>
> (Freudenthal, 1973, p.165)

Secondly, (cf Kline, 1973, p.82.), a structural order is only worthwhile if it is recognised by students. So such ordering has no value for those who do not know what a deductive structure is, or who do not recognise the particular one being used.

Kline, while discussing set theory, has also pointed out (Kline, 1973, p.93) that structural ordering does not guarantee its appropriateness. Two mathematical examples are: a) the establishment of basic language axioms (still disputed in the foundations of mathematics) is structurally prior to propositional calculus; and, b) rigorous limit theory is prior to differential calculus - though the latter was taught and applied without it for 80 years.

Freudenthal makes the point that, as mathematicians,

> "...mathematics teachers tend to teach for the needs of future mathematicians. Mathematizing mathematics (i.e. putting it into a deductive structure) is possibly an end but it is not the start of a mathematical education"
>
> (Freudenthal, 1973, p.69)

The Bourbaki works (Bourbaki, 1966) perform the function of defining 'the' structure of mathematics, but they are written on topics which have been researched to some sort of conclusion - nowhere is new or potential work placed in the static structure.

To sum up, then, it seems that structural ordering has origins and uses of some importance, but its use in teaching and learning mathematics is doubtful.

3.12 Skills Ordering

This ordering does have a pedagogical justification. It arises when some skill (e.g. drawing a linear graph) is required before another topic (e.g. solving simultaneous equations graphically) can be taught.

However an assumption has been made that learning to do something is the same as actually doing that thing. For example, graphical solutions of simultaneous equations cannot be obtained without graphing the equations.

But we are not talking about doing graphical solutions, we are talking about teaching and learning them. The ideas involved in the graphical solution can be taught, and understood, without the pupils actually drawing the graphs: the teacher may draw the graphs so as not to be side-tracked from the point of her lesson. Furthermore it is conceivable, and often practicable, to teach, say, a topic needing graphical skills as motivation for learning how to graph equations.

There is some pedagogical use for this sort of ordering, but it is not universally applicable justification since teaching and learning mathematics is different from doing it.

3.13 Self-Generating Ordering

This ordering also has a pedagogical basis. It is described by Kline (Kline, 1973, p.74) as arising from the self-generating nature of mathematics. For example, after considering triangles (3-sided figures) one can raise similar questions about quadrilaterals (4-sided figures) or polygons (many-sided figures); or after solving simultaneous equations we could ask what happens if there are many solutions (as a result of inequality conditions) and go on to linear programming.

Like skills ordering however, this ordering is not universally applicable and has the following disadvantages.

Self-generating ordering can only be done by someone who knows the full story - it is not necessarily natural and may seem quite meaningless to someone who does not see the links which a particular teacher may make. Also there may be several possible orderings: for example solving pairs of simultaneous equations may lead on to: solving series of equations; solving simultaneous equations in 3 or more variables; or simultaneous inequations. The choice made by the teacher may not be seen as natural by the student.

Kline (Kline, 1973, p.79) argues further that this type of ordering relies on mathematical reasons for developing a new topic rather than practical applications, e.g. linear programming is taught as an extension of simultaneous equaitons rather than as a logistics or accounting problem. Thus motivational opportunities are lost.

Self-generating ordering, while often useful, neglects to link learning mathematics with using mathematics - which is also an aim of mathematics education (Aim.5., section 2.5).

3.14 Genetic Ordering

This order is that which occured in the historical development of mathematics. It was described in a document signed by 75 eminent American and Canadian mathematicians in 1962:

> "The best way to guide the mental development of the individual is to let him retrace the mental development of the race - retrace its great lines, of course, and not the thousand errors of detail. ...This genetic principle may safeguard us from a common confusion: If A is logically prior to B in a certain system, B may still justifiably precede A in teaching, especially if B has preceded A in history."
>
> (Kline, 1973, p.12.)

The justification for use of this type of ordering is made by Kline:

> "...the historical order is usually right /for learning mathematics constructively/ and that the difficulties which mathematicians experienced are just the difficulties our students will experience."
>
> (Kline, 1973, p.156.)

The genetic principle is only a principle and can only be pedagogically accepted where it proves practicable. For example computer logic can be taught before base two arithmetic, which came first historically.

There are reasons why the principle may be unsound. Our environment is different from the one in which mathematics developed, so that, for example, children in our society are constantly exposed to decimalisation and may be expected not to encounter the historical difficulty with place notation and decimal points. Similarly the study of negative numbers (experienced in freezing temperatures and debt), statistics (in advertising) and graphs (in maps) may receive help from our culture which was not present in the past. Conversely knowledge of logarithms is not as vital as it once was.

The validity of the principle in mathematics education rests on two facts. First, motivation can be aroused from the historical origins of a topic. Second, mathematics is a cumulative study so that old topics often need to be understood before historically later ones can be put into context.

3.15 Psychological Ordering

Developmental psychology provides some evidence for certain sorts of topics to be mastered before we can expect other topics to be able to be understood. For example Piaget's studies show that the concept of conservation can only be learnt at a certain stage of development, and any topics needing this concept cannot be successfully taught before that stage is reached.

The validity and usefulness of these sorts of orderings are empirical questions and are not covered here. It can be expected that the conclusions reached will be probabalistic ones rather than yes/no. The implications are that: a) different individuals will learn best under different orderings, b) other sorts of orderings may be superimposed on the psychological one, and c) ordering decisions will lie with the teacher and students at the time of learning.

3.16 Conclusions

Does the only partial validity of each of the orderings considered imply that an eclectic, pragmatic ordering of topics be developed by each teacher? Surely it is preferable to teach mathematics in a logical manner rather than an illogical one? Freudenthal states:

> "In principle it is a healthy idea not to teach isolated pieces but coherent material"
> (Freudenthal , 1973, p.74.)

However he adds: "But there is more than one kind of connection" (ibid)

There is one important consequence of accepting any sequencing. Students are often absent from class and the effect of sequencing may be lost.

Again we must beware of confusing 'mathematics' and 'learning about mathematics'. Students should come to see logical structures and should use orderings of all kinds. But there are other factors which must be taken into account when ordering topics: for example
a) a more complex problem may be used as motivation for a learning skill necessary for its solution;
b) some other motivation (consideration of a 'real' mathematical experience - see section 3.2) may call for some understanding of a topic which is not logically next on the list;
c) teacher preference - teachers may feel as though a specific topic could be next, and such subjective judgement

may have as much value as an arbitrary, if logical, sequence;

d) students' preferences - the psychology of learning provides no doubt that students learn best when they are interested or want to learn;

e) individual differences - students are 'set' for learning by prior experiences and psychological or developmental factors, none of which are constant, nor universal.

In other words the ordering of topics should be evaluated on pedagogical criteria, not dictated by a universal principle. Thus there are reasons why mathematics educators may have good ideas concerning the order of topics, but the best final result will be situation dependant.

The consequence of this is that teachers must not only be able to tap wide resources of different topics, but also be able to teach them. For example if the situation and pedagogy indicate that, say, network theory be taught to a class it is much better mathematics education (in terms of the official aims) if network theory is taught, rather than neglected because 'it is not examinable' or 'it is not in the syllabus'.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3.2 THE ASSOCIATIVE LAW: SHOULD MATHEMATICS BE TAUGHT IN ISOLATION?

> 'Mathematics should be taught in isolation from other subjects since it deals only with the essentials of a problem.'

That this assumption is made is evidenced by the structure of courses in New Zealand secondary education. Applied mathematics is a separate subject from Form VI, and statistics, numerical analysis and mechanics are examined separately. However the assumption has always been debated, references to it from the early 1900's appear in the National Council of Teachers of Mathematics 32nd Yearbook (Jones, 1970). Recently it is being questioned again. Articles on the place and philosophy of applied mathematics are becoming more numerous (e.g. Bender, 1973; Wilder, 1973; Ford & Hall, 1970; Hall & Whitcombe, 1970; Ormell, 1972a and 1972b). Recent curricular developments in New Zealand secondary schools include 'Integrated Studies' experiments.

There are three areas of analysis in the consideration of this assumption:

a) the method of mathematics teaching -- should it be problem- or mathematics-centred;

b) applied mathematics -- should this be a separate course and where does it overlap with pure mathematics;

c) integrated studies -- how much can mathematics education aims be satisfied within an integrated curriculum?

Some points to bear in mind in the following discussion are:

a) a student trained in mathematics will not necessarily go into a mathematical job;

b) the criteria on which the assumption must be judged are pedagogical ones: how are the aims of mathematics education best fulfilled; not, what mathematical training is required by existing society?;

c) the reason mathematics occupies a central position in curricula is not because it is beautiful or because it is a self-sufficient body of knowledge: it is valued because it helps man understand and master the physical and social reality (cf Kline, 1973, p.78).

3.21 Should mathematics education be problem- or mathematics-centred?

What is the difference between these approaches?

Bender (1973, p.302.), in an article advocating the former approach, distinguishes the traditional aims for each as:

	Mathematics-centred		Problem-centred
(i)	Learn some useful mathematics	(i)	Learn to use some mathematics
(ii)	Develop a feel for one or more areas of mathematics	(ii)	Develop a feel for how to apply one or more areas of mathematics
(iii)	Obtain an idea of what mathematics does and of the beauty of mathematics	(iii)	Obtain an idea of the richness of mathematics as a tool.

He elaborates:

> "The first set of goals imply that applications should be kept simple so as not to interfere with the business of learning mathematics. The latter set imply that the course must try to dig into real problems. In particular the student must learn how to set up a model and analyse it."
>
> (ibid, p.303).

The problem-centred approach is not just using problems as a source of mathematical material, which is then considered in isolation, (for a critique of this see Kline, 1973, p11 and p.76.). Rather it is a consideration of a real problem, its model, the solution of the model, and the problems inherent on a model solution (e.g. assumptions and approximations made.). For a good description of this approach in action see Hayman, 1975, p.149.

The mathematics-centred approach does not neglect real-world applications either. However such links are made primarily for motivational reasons. The aim is to abstract and thus facilitate analysis. (David Stenhouse has commented that this process is similar to that of philosophical analysis).

How do the two approaches compare?

There are several areas of argument. The first four are mentioned by Bender (1973, p.303.):

1. The mathematics-centred approach enables the students to see the beauty of mathematics. Although 'beauty of mathematics' could be the subject of an extended language-game (see section 2.3), there is no doubt that any intrinsic qualities in mathematics are played down in the problem-centred approach. It could be argued that a study of mathematics 'for its own sake' is not part of a universal curriculum and should therefore be reserved for after specialisation. There is a danger, however, that students who could get pleasure from pure mathematics will never h have the opportunity to do so.

2. The problem-centred approach may not include all the necessary material. Whether mathematics can be taugh from elementary level using this approach is an empirical question. An implication of the problem-centred approach is the abandonment of the idea that certain bits' of material must be taught, since it is not a process of finding problems to exemplify the mathematics but rather a process of learning what mathematics is present in a particular problem.. The aim 'to understand the structure of mathematics' would also have to be severly limited.

3. Setting up appropriate models of real problems involves complex mathematics and extensive background material, thus many students will be lost. 'Complex' in this criticism is defined by whether students will get lost or not and is dependent on the student, the teacher and the situation.

4. The mathematics-centred approach will enable more mathematics to be learnt since setting up models takes time and thus less content will be covered. Bender (1973, p.304) answers this criticism by noting that the important variable is 'mathematics retained in useable form' not 'mathematics covered', and under this criterion the problem-centred approach will fare better. Freudenthal (1973, p.77) makes the same point with reference to mathematics being a language

which must be attached to the 'lived through reality' of the learner. Unrelated subjects can be taught, but also forgotten just as quickly. Another argument is that the gains in attitude towards mathematics will offset the lost material (see section 3.6).

5. Many applications of mathematics were discovered in mathematics first and then applied. Wilder (1973, p.680.) quotes John von Neumann and Eugine Wigner to describe the peculiar relationship mathematics has with science -- mathematical results, initially unrelated to reality, suddenly take on a physical meaning; for example, π , the ratio of circumference to diameter, has been found to be related to population statistics, complex variable analysis and sound waves. The problem-centred approach does incorporate this aspect when results of analysis are reapplied to the initial problem. Nevertheless the historical and major instances are neglected. It can be argued that this aspect of mathematics is only for mathematicians who are on the frontiers of the subject. However mathematical discovery and subsequent application is both motivating and part of mathematics; therefore it is a legitimate part of mathematics education.

To summarise the consequences of adopting a mathematics- or problem-centred approach in terms of the aims discussed in section 2.00: the mathematics-centred approach is appropriate for aims 1, 2, 3 for some students, and possibly 4; it is inappropriate for aim 5. The problem-centred approach is appropriate for aims 1, 3, and 5; and is inappropriate for aims 2 and 4.

3.22 Applied mathematics and pure mathematics.

The image of applied mathematics which has developed in New Zealand schools has been a combination of mechanics, statistics and, recently, computer science. However there has been a developing philosophy overseas, spreading to New Zealand, of applied mathematics as a creative, model-building procedure along the lines of problem-centred teaching described above. However it is still argued whether or not this new image of applied mathematics is part of, or should be separate from, pure mathematics.

The separate point of view is expressed by Ford and Hall (1970).

Despite acceptance of an applied mathematics aim 'to understand reality mathematically', Ford and Hall differentiate between everyday applications (e.g. handling money) and more complex problems where the formulation of the mathematical entities is crucial. The subsequent manipulation of the model involves the use of mathematical problem-solving techniques -- the assumption is that these techniques have been developed in pure mathematics courses. Pure mathematics is also seen as important for applied mathematicians in order that they may be able to communicate with those who can solve their problems.

In a further paper (Hall & Whitcombe, 1970) it is noted that such model-building must be open-ended (i.e. not leading to students living and developing the subject. Such a hope is expressly one interpretation of aim 4 (see section 2.4).

The contrary viewpoint is stated by C.P. Ormell (1972a).

Using a philosophy of mathematics attributed to C.S. Pierce, Ormell makes explicit the suggestion in the new model-building philosophy of applied mathematics that it is the same as mathematics taught with a problem-centred approach. He eschews aesthetic and <u>suis generis</u> answers to the question 'Why do we teach mathematics?' (see section 2.3), and gives the Peircean answer: 'to build up a capacity to understand and handle scientific, technological and social possibilities.

In a subsequent article he says:

> "...to describe mathematics /as a study of possibilities/ is to make applied mathematics primary...and to relate pure mathematics to this. But is it appropriate to say that we are 'applying' mathematics to anything when what we are discussing is a mere possibility? If one tries to define 'applied mathematics' so that all branches of mathematics which <u>can</u> be used to describe real objects or things count as 'applied mathematics',

one finds that most of what has traditionally been known as 'pure mathematics' has to be reclassified as 'applied'. ...the pure/applied distinciton is of little use for anything but the roughest demarcation of areas in mathematics. Almost any area of mathematics can be regarded as 'applied mathematics'. But exactly the same body of results prior to interpretation can be regarded as 'pure mathematics'."

(Ormell, 1972b)

E. Blaire goes further:

"One looks for applicable mathematical models for the problem situation and sees which provides the best practical solution. What one is not claiming is that there is a discipline of applied mathematics which is governed by necessary rules so as to process all problems of reality with mathematical content. It is seeing applicable mathematics in this light that leads to rigid traditional views of applied mathematics. There is nothing there but pure mathematics, the real life problem and a way of looking at it and tackling it. What does not exist in any case is applied mathematics. One has pure mathematics which is the source of models for simulating real-life problems if interpreted in the right way."

(Blaire, 1973)

What is the point of this for secondary mathematics education?

There are two aspects: to learn some mathematics content and to learn what use this is. If we reject the latter of these aspects then we must be able to justify mathematics of itself (see section 2.2). If we accept the value of applied mathematics at all then it must be admitted from the beginning of the teaching of mathematics, not, as at present, as a scientific option after specialisation in form VI or VII. Considerations of time and organization would seem to indicate an amalgamation rather than a division.

3.23 Mathematics and other subjects in the curriculum.

Freudenthal (1973, p.72) relates the isolation of mathematics as partly responsible for the demathematicised science teaching in secondary schools: students who cannot use the mathematics they know in, say, a physics class are taught physics which requires a minimum of mathematics.

But the main question is: can the aims of mathematics education best be realised by an isolated or integrated study?

The answer to this question revolves around the cultural/scientific orientation of the mathematics teacher. In recent times the scientific/technological aspect of mathematics has predominated. Mathematicians have B.Sc. degrees rather than B.A.'s, it has been a 'boys' subject (see Clerk, 1960, p.46.), and taught as necessary for trades and apprenticeships.

Historically this has not always been the case. Egyptian mathematicians were mystics, and in the Middle Ages mathematics was linked with astrology, music, art and philosophy. This development is traced in section 4.2, and a full account is presented by Kline (1972).

Whether mathematics should be taught independently from other subjects or not seems initially to be an empirical question. We should devise an experiment and see whether or not students learn to use and understand mathematics better when it is taught in isolation. Some evidence that it is better taught separately might be inferred from recent research on the hemispheres of the brain (UNESCO, 1976). Left and right hemispheres seem to be associated with aesthetic and analytical interests respectively, and, in any individual, one hemisphere dominates.

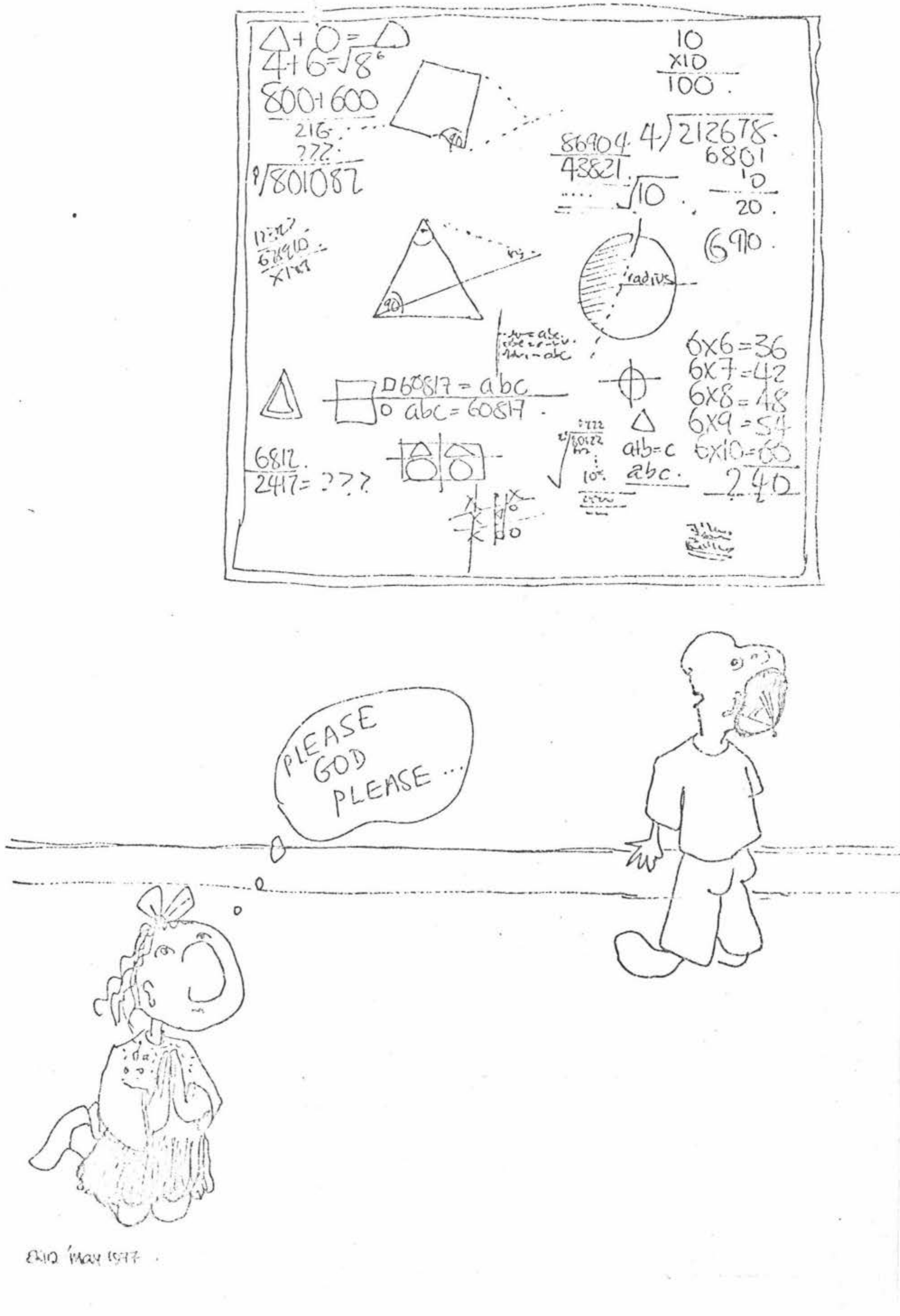
However a philosophical point arises also. The purpose of mathematics education is not just to develop abilities or skills (see section 3.6). So we must be careful that the other aspects are included in the evaluation section of the above experiment. Further we must decide whether or not we want mathematics to be something separate in the lives of most people. Certainly many do regard the study of it as separate, but that is begging the question. When we come across a problem, or game, susceptible to mathematical analysis or modelling, do we want people to 'change gear' or do we want mathematics to become a 'way of looking at things' which is part of each of us?

To clarify this point let us look at some activities. Are we conscious of doing mathematics when we:

(i) work out how many days until next Tuesday?
(ii) position ourselves to catch a ball?
(iii) read and comprehend mileages on signposts?
(iv) calculate how much change we should get?
(v) measure up the room for a carpet?
(vi) play noughts and crosses? draughts? chess?
(vii) check our bank statement?
(viii) do a tax return?
(ix) calculate the running costs of our car?
(x) do a budget?
(xi) solve a time and motion problem?

Most people would give some 'yes' and some 'no' answers. The point at which we become conscious of doing mathematics is the point at which we 'change gear'. Thus the aim of mathematics education could be seen as extending mathematics ability so that doing mathematics becomes part of how we look at the world and an automatic activity.

The consequence of this is that if mathematics is intended to be an integrated part of everybody's lives, then it should be studied within an integrated curriculum. For this the problem-centred or 'applicable' approach would be especially appropriate.


PLEASE GOD PLEASE ...

3.3 (2B)v(-(2B)): SYMBOLISM AND TERMINOLOGY

> 'A large part of learning mathematics is learning symbol manipulation and terminology, or, at least, learning translation to and from, and games within, a mathematical language.'

The roles of symbolism and terminology in mathematics education have similarities but will be discussed separately here for convenience. Thus comments in sections 3.31 and 3.32 also apply in sections 3.33 and 3.34 and vice versa.

We are concerned with the places of symbolism and terminology in learning mathematics but in each case their places in mathematics per se will be discussed first.

3.31 Symbolism in mathematics

Symbolism has many functions, mainly recording and communication ones, which have been discussed at length elsewhere (see, for example, Skemp, 1971, Chpt.5., p.68.). Skemp also argues that symbols are a necessary part of our thought processes, and that we store information using them (ibid, p.82. and p.70 respectively).

In these roles symbolism is of undoubted value (Kline, 1973, p.70). For a statistician who fully understands the signs:

$$S = \sqrt{\frac{\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}{n}}$$

is a clearer communication than the equivalent english language form. It can also aid the understanding of what is involved. However pure symbolisation and manipulation can give a new mathematical result. For example the 'Chain Rule' in calculus is attributable in part to Liebniz' notation, and appropriate symbolism was responsible for Maxwell's 'discovery' of electro-magnetic radiation (Kline, 1972, p.350).

We can go further. Many breakthroughs in mathematical thought have been a direct result of a new or different symbolisation of (in many cases) an old problem. Francois Vieta's introduction of algebraic notation was a fundamental force in the development of mathematics (Grabiner, 1974, p.357.) and Descartes' development of co-ordinate geometry can be seen as an intuitive leap to a new way of representing a point in space (Kline, 1972, p.193.). Skemp (1971, p.85.) gives a simple illustration of how appropriate symbols can lead to structural insights.

3.32 Symbolism in learning mathematics.

In secondary level mathematics symbolisation is often essential to the mathematical idea. We do not always explain the idea first and just use symbols to record or clarify it. For example in the algebraic field, induction and binomial expansion rely on symbol manipulation (see also Polya, 1957, p.134).

Thus the assumption stated above seems justified.

However translating everyday problems into mathematical ones is an important part of mathematical ability (see section 2.43). This translation must be made into symbols which will enable the problem to be solved. Existing symbols and methods of translation may not be the best ones. Indeed the skill in translation is often devising the appropriate symbols. (Note that Chinese mathematics did not develop for several centuries from the Middle Ages because its algebraic notation could not deal with more than three variables).

The danger in the above assumption, therefore, is that the following further assumption is also made:

> 'that the language, terminology and language-games of present mathematics are the best or only ones for students to learn.'

The assumption should be:

> 'a large part of learning mathematics is learning how to establish an appropriate mathematical language and to validate translations and manipulations within it.'

There is some conventional symbolism which has developed over a long period and which is extremely powerful. I do not suggest that this be abandoned, but that it must be seen as an alternative symbolisation game. For example "dy/dx" is not necessarily the first derivative of the function y with respect to the variable x. It may be, and is often used as such, but as a symbol it has the drawbacks that, for example, "f'(x)" does not have -- namely that "d" may be taken as a variable, or that it re presents a fraction and thus:

$$\frac{dy}{dx} \times \frac{dx}{dt} = \frac{dy}{dt}$$

is a statement needing no further justification. Any symbolisation has advantages and disadvantages and these must be explained. Symbols must be adopted because they are appropriate for the problems at hand, not because they are conventional -- though they are often both.

Kline (1973, p.69.), in presenting arguments against excessive symbolisation in the 'new math' says:

> "Symbolism can serve three purposes. It can communicate ideas effectively; it can conceal ideas; and it can conceal the absence of ideas."

Certainly symbolism can conceal ideas. Newton's calculus notation held up the development of calculus in Britain while in France Leibniz' symbols led directly to major breakthroughs. Intuitive ideas of space were not utilised in mathematics because the study of geometry was restricted to Euclidean axioms, methods and notation. Only when transformation geometry was introduced were students given the tools to make use of the ideas that they already had.

A final point concerning symbols and mathematics education. There is nothing wrong with highly abstract or obscure symbolism -- but only if it is practicable for those who are using it. But we are talking about students, not sophisticated mathematicians, so that the symbols introduced or developed must be suited to their needs.

3.33 Terminology in mathematics.

It does not appear at first as though terminology is as involved as symbolism with the process of mathematics.

It is often thought that the development of terminology is a concensual process and does not constitute doing mathematics. It is seen as a means to communication, not part of mathematics itself.

It is true that terminology aids precision and brevity. For example, 'average' in layman's terms can be any measure of central tendency for a data distribution. For statisticians the three common measures: mean, mode and median each have specific uses and need to be distinguished; thus the adoption of 'mean' or 'arithmetic average' to make clear what is being considered.

However terminology can also be considered as a language-game in the Wittgensteinian sense. In other words learning terminology becomes part of learning a concept. (This was explained for the term 'prime' in section 2.23. For an example using the concept of derivative see Grabiner (1974, p.363.)). This conception highlights the fact that the meanings given to apparently precise terms are constantly changing. This applies both to the meaning that we as individuals understand a term to have, and to its concensual meaning. Throughout a university mathematics degree, 'function' has had for me a basically pragmatic meaning. I could recognise a function if I saw one, could construct and use them. It is only in recent years that I have come to see a function in set-theoretic terms at all.

Furthermore the concensual understanding of 'function' has changed dramatically over the last 100 years, and is still changing, (Freudenthal, 1973, p.21.)

Notice that the earlier meaning I had of 'function' was not incorrect or useless. As a word in mathematical terminology it classified a certain group of things I would otherwise have needed much verbiage to explain.

Admittedly it was not maximally efficient -- but it still is not, and never will be. In fact analytic research partly consists of exploring the meanings of 'function' -- a new possibility may yet provide a breakthrough in mathematical thought.

3.34 Terminology in learning mathematics

The questions are: how much terminology should be part of learning mathematics; and what part should terminology play in this process?

The amount of terminology depends on the content (see section 3.6). Within each topic area the terminology depends on: a) the level; b) the language of the teacher; and c) the needs of the student (i.e. does he feel the need for a word for some idea? This, like symbolisation, might be indicated by the student developing his own terms).

Terminology is part of learning and must be used as an active means of developing a concept, not merely as a name for a concept already developed. The value of definitions in this viewpoint is that they work both ways. They illuminate the meaning of the definition sentence and the meaning of the term. The use of terminology in the classroom is not a simple communication, for it is certain that the teacher and the learner will have different language-games for any one term. Rather each use of a term is a further exaplanation in a never-ending definition of the term, or is part of a language-game. For example the statement 'Addition is a binary operation.' must be seen as helping to define 'addition' and 'binary operation'.

Freudenthal (1973, p.76.) gives a similar point of view, although in different words.

Terminology, like symbolism, should be an aid to the memory. If merely words are learnt then it is not performing any educational function, since learning mathematics consists in learning what terms can mean and where it is appropriate to use them. This is a continuous process. Consider the teacher's understanding of a term: it will change as the interaction with the student develops. For example, a difficulty experienced by a student may create a new awareness for the teacher of what a term can mean to others. Because of his experience, the teacher's language is liable to be broader and more stable than the student's. Thus the 'substance' of mathematical knowledge is reflected in the use of terminology.

```
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400READ (IO,/,A,B);
500C:=A+B;
600WRITE (RB,<"SUM OF TWO NUMBERS">);
700WRITE (IO,<I3,"+",I3,"=",I3>,A,B,C);
800END

900#
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#COMPILING 3892
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 UNEXPECTED END OF INPUT
 COMPILATION ABORTED ************************************
#SNTX
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800END.
RUN
#UPDATING
#COMPILING 3897
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#RUNNING 3901
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1
SUM OF TWO NUMBERS
  1+  1=  2
#ET=21.9 PT=0.1 IO=0.2
700WRITE(IO,<I3,"  +",I3,"  =",I3>,A,B,C);
L
100 BEGIN
200 FILE IO(KIND=REMOTE,MYUSE=IO);
300 REAL A,B,C;
400 READ (IO,/,A,B);
500 C:=A+B;
600 WRITE (IO,<"SUM OF TWO NUMBERS">);
700 WRITE(IO,<I3,"  +",I3,"  =",I3>,A,B,C);
800 END.
#
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#UPDATING
#COMPILING 3910
#ET=20.3 PT=0.8 IO=0.5
#RUNNING 3915
#?
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1
SUM OF TWO NUMBERS
  1  +  1  =  2                  <------------------ ANSWER
#ET=13.1 PT=0.2 IO=0.2

#
REMOVE
BYE
#QUEUED
#
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3.4 QUITE EFFORTLESSLY DEDUCED: DEDUCTION AND INTUITION

> 'Deductive method is the paradigm of mathematical thought and exposition and must be dominant in mathematics education.'

This assumption is such a pervasive one I shall first comment on the extent of its presence in New Zealand secondary education. Section 3.42 deals with deduction in mathematics and section 3.43 with deduction and logic in mathematics education. Finally alternatives are considered.

3.41 Basis of the Assumption

Several relevant statements are made in the Mathematics: Forms I to IV Syllabus:

> "Programmes of work based on this syllabus will stress the development of the pupils powers of ordered and systematic thinking, ..."
>
> (Dept Education, 1972a, p4)

> "Mathematics may be regarded as a logical structure built on assumptions" (ibid, p.4.)

> "Specific Objectives:
> ...Develop an understanding of the nature of proof "
>
> (ibid, p.6.)

> "Children sometimes learn intuitively. Although mathematics is essentially logical, there are many occasions when pupils see relationships and form conclusions through insight before they are able to build the necessary logical steps." (ibid, p.7)

> "The initial steps in the study of geometry are...an encouragement of intuition, discovery and inductive reasoning. A gradual growth of deductive reasoning follows, to prove apparent truths and to lead to the establishing of further properties and relationships... The emphasis is on experience leading to thought... ."
>
> (ibid, p.8)

In the last two quotes the inferior status given to intuitive over deductive reasoning is explicit.

The first part of the preamble to the University Entrance Examination Prescription states:

> "The ideas of logical deduction and the correct use of symbols should be an essential part of the teaching of this syllabus. Particular attention should be paid to the ideas of implication, converse, equivalence and the use of the counter-example."
> (Dept Education, 1972b, p.301.)

The School Certificate Prescription has similar aims to the Forms I to IV syllabus. In addition it mentions that the ability to make simple deductions will be specifically examined. No mention is made of creative, lateral or intuitive thought.

A major secondary text, the Shape of Mathematics Bk II, introduction states:

> "In our view, the important themes are functions, mathematical structure and the nature of proof. ... We ... place more emphasis on proof and disproof than one often finds in modern texts."

Such statements are notable for their lack of appreciation of the part played by intuition or creativity.

The arguments against the dominance of deduction and logic have been well espoused by, among others, Kline (1973 and 1976). Descriptions of intuitive and creative thinking in mathematics are also common (Polya, 1957 and De Bono 1971).

3.42 Deduction in Mathematics

The following is a review of criticisms from the above sources, with additional comments and examples.

In the first place mathematics did not develop deductively. In fact deductive method outside of Euclid has been common only in the last 150 years. Even Euclid's Elements was the final product of 300 years of basically intuitive thinking, and even now is recognised to rest on intuitive assumptions. It is difficult to say which mathematical 'discoveries' were the result of deduction and which intuition, but experience indicates that the major steps were intuitive at first -- deduction's place was confirmation and recording. A major counter-example pointed out by Gordon Knight is that of non-Euclidean geometry. Gauss and Lobachevsky worked from the deduced independence of axioms to the derived consequences of using alternative ones.

A second point is that, increasingly over the last 100 years, we do not agree on what is a correct proof. Until the 19th century the correctness of mathematics was linked to a belief in God and the perfection of the universe; subsequently axiomatics were developed, but recently the foundations of mathematics is a subject of considerable controversy. School level deductions are certainly not rigorous in a contemporary sense (see section 3.5) and rest on consensus/intuitive ideas of implication etc (like our everyday use of logic) rather than formal/mathematical ones. Logic is very much a language-game: in a classroom the statement:

$$3x + 4 = 10 \iff 3x = 6$$

is not based on a formalised sense of "$\iff$" but rather shows an example of how "$\iff$" can be used (or confirms its use in similar situations) as well as being a step in the solution of an equation.

This leads to a third point: the arbitrariness of deductive method. Deductive proofs are not unique -- many theorems have been proven in several different ways: there are many proofs of Pythagoras' Theorem and no agreement as to the 'best' one. Also definitions are determined by utility and then deductively justified and not vice versa. Addition of fractions was defined as:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \quad \text{and not} \quad \frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$$

because it worked that way.

Not only are proofs and results arbitrary, but so also are the axioms. The paradigmatic example is the parallel axiom in geometry. It was about 2000 years after Euclid that it was discovered that the parallel axiom (that there is only one line parallel to a given line through a given point) was independent of the other axioms, and therefore that alternative consistent geometries could be formed by making alternative assumptions. The subsequent developments opened vast new areas of mathematical thought and ultimately led to a new meaning for geometry.

Such a 'thought-shift' corresponds closely to Kuhn's concept of a scientific revolution (Kuhn, 1968) whereby advances in science are made when previous conventional wisdom is doubted and new paradigms are formed. In mathematics this could be interpreted as saying that advances result exactly when deductive methods are in some way suspended. This idea originated in a paper by David Stenhouse (d). See also in this regard Grabiner (1974).

A further point is that much mathematics either rests on, or has been inspired by, unproven results. Goldberg's conjecture and Fermat's Last Theorem are examples. More importantly the foundations of calculus were not established until 150 years after calculus was used constructively. Mathematicians using calculus knew the problems and relied on faith rather than reason. The use and formalisation of negative numbers have a similar history. There is a point of view that work using concepts which have not been formalised is not mathematics. Philosophically this is difficult to sustain since identical work done after formalisation is mathematics.

Finally, although it sounds paradoxical, deductive method is not always convincing, nor is it the basis of our belief in mathematical results.

As an example of the first case: if a proof of an intuitively strange result is given, we doubt the proof first, not our intuition. Relativity theory is a good example: many people doubted it on intuitive grounds even when they could follow the steps in its derivation.

As an example of the latter situation: recently the Four Colour Theorem was claimed to have been finally proven (after 130 years of conjecture) in a 900 page deductive exposition. Now even if this is agreed by eminent topologists to be without flaw, my belief in the theorem will be little changed. It will confirm my belief, but it will not be the basis of it. If the proof is shown to be in error my belief in the theorem will remain.

To say that deductive proof is needed to finally convince could be circular. It does make sense to say that we were convinced of something that is deductively false, so what is happening is that we are redefining 'convince' to mean something like 'believe and also be able to deductively show'. Thus deductive proofs are only part of the reason for a belief - the rest is acceptance of authority, intuitive feeling etc. A language-game perspective on this point is that what a deductive proof of an intuitively odd result achieves is not a change in belief. Rather the meanings of the words alter so that intuition does agree with the proof. 'Space' and 'speed of light' will mean different things to someone after that person has accepted relativity theory, i.e. the words will be used in different ways which will now accomodate the results of relativity.

In other words, the relationship between deduction and belief is not a one-way, necessary connection, but rather is a mutual accomodation.

The above points have attempted to show the place of deductive method in mathematics. Deduction often comes after intuitive or creative thinking as a confirmation and formalisation. Sometimes deductive proofs need to be doubted before advances are possible and sometimes they create 'truths' in combination with language and intuition.

3.43 Deduction and logic in mathematics education

The above comments are not reasons why deductive method should be relegated in the teaching and learning of mathematics. We must also examine the pedagogical characteristics.

Kline has suggested (1976, p.452.) that the human mind does not come to an understanding of ideas through deduction, but rather through intuition and thus ideas should be taught this way. I do not wish to enter the psychological learning theory debate here, but something can be said about the nature of deductive, intuitive and creative thought.

I shall assume that the deductive method is epitomized by a sequence of statements connected by allowable rules and based on axioms. We do not usually consciously think using such a method.

Creative thinking and modern heuristics (as defined in Polya, 1957, p.129.) have aspects which are directly contrary to the deductive method. Characteristics of deductive method which appear as the opposite in intuition or creative thinking are:

a) deduction is linear and usually forward-looking;
b) deduction is usually conscious, has set rules and will therefore tend to be universal rather than individual in methods and results;
c) deduction takes atomic steps;
d) deduction has no place for error as part of its method.

There are three further pedagogical reasons which make deductive method sometimes inappropriate.

First, it can lead to confusion over what are axioms and what are laws. It is not clear whether commutativity of addition is a law or an axiom. Given the language-game functions of a teaching situation it is probably both.

Second, if deduction is the sole method presented to students they can come to believe that mathematics is harder than it actually is. If the next step is not obvious the student may well ask "How did anyone ever think to do that?". In fact, of course, the deductive sequence was often found after the result.

Finally the deductive approach is often more complicated than the untuitive one. (See also the discussion of rigor - section 3.5). A topic may be learnable and useable while its justification remains too complex to attempt. For example infants can know that -5° is colder that 5° without understanding inverses. Another example is embodied in the history of calculus mentioned earlier.

What part, then, does deductive method play? Polya (1957) considers that deductive reasoning needs to be learnt since:

a) it is the method by which evidence is evaluated;
b) it gives the idea of the existence of a logical system;
c) it gives connections between bits of information, therefore making them easier to learn;
d) it provides a check on errors.

Freudenthal feels that to teach logic as a separate subject is unnecessary:

> "Rather than teaching logic, the mathematics teacher shall use logic and he shall make conscious to the learner that logic the learner is using." (1973, p.661.)

The arguments above have generally shown that making the assumption that the deductive method is dominant in mathematics education can lead to bad learning. That deduction and logic are part of mathematics is not being questioned, rather the nature of that part is being examined and its limitations exposed. Deductive method is certainly valuable, but its importance is <u>devalued</u> if its relationship with heuristics and intuition is not recognised.

Deductive method may be paradigm of mathematical exposition, it is, however, only a constituent part of mathematical thought and hence also only part of a mathematical education.

3.44 Alternatives

A case has been made, as part of the argument against deductive reasoning, for intuitive thinking. How this is to be taught is a difficult question which will not be considered here. However, if it is going to be taught, it should be taught consciously. Polya makes the same point about heuristics (1957, p.13.).

Deductive method and intuition are both part of mathematics. It seems reasonable to assume that they can be taught together. If deduction is given the specific roles mentioned above, and intuition/ heuristics/creativity are also taught or used in appropriate places, then the synthesis will be nearer to mathemtics than if deduction is make central. A consequence of this is that students would also need to be taught where and when each method is appropriate. It may be that at times the methods will need to be separated - it has already been mentioned (section 3.42) that creativity at least involves the contradiction of logically deduced results.

Kline attempts a combination in his explication of discovery approach (1975, p.154.). This is not the only way they can be combined.

3.5 RIGOUR MORTIS

'The maximum rigour possible is the best mathematics.'

3.51 What is rigour?

Rigour can be seen as the level of analysis. Generally associated with the deductive mode, it involves both the size of the steps made and the detail of the stated assumptions or axioms. Thus greater rigour can only be seen with respect to existing rigour; more detailed steps can only be made between existing ones. To the extent that rigour is associated with deduction the caveates of section 3.4 apply here.

Whether the above assumption is correct or not is really a matter for mathematicians -- I think that in the sense of mathematics as a field it is largely true. Certainly many mathematicians have spent much time investigating axiomatics (e.g. Frege, Godel, Russell, Quine); and have made interesting discoveries about hidden axioms, the interdependence of axioms and the level of rigour possible.

3.52 Rigour and mathematics education

The problem is that the assumption is made by mathematics educators, and they slip easily into the very different assumption:

> "Mathematics should be learnt with the maximum rigour possible."

This is pedagogical, not a mathematical statement and is criticised at length by Kline (1973, p.51). The following criticisms are mentioned:

1. The level of rigour acceptable among mathematicians changes. Thus Euclid, for 2000 years accepted as the paradigm of rigour, is now recognised as defective in that many 'intuitive' axioms (e.g. the 'intersection' and 'between' axioms) were not stated. Two pedagogical questions arise from this: first, if such axioms were so

intuitively obvious that they were overlooked for so long can we expect beginning mathematicians to appreciate them? Second, if rigour is time-relative then one particular level of rigour is not any better per se than any other level -- it must be justified on some other grounds.

2. Rigour developed in response to faults in what was previously thought to be a rigourous presentation. Thus in teaching a rigourous development we must ensure that the student perceives the need for it, e.g. teaching fractions as number pairs satisfying certain operations needs further explanation to a child who intuitively understands fractions on a physical level.

3. Some axioms are more complex or difficult to understand than the theorems which are proved from them. Thus it must also be explained why some propositions are axioms (starting points) and some are theorems (results). Since this disctinction is largely arbitrary, to explain rigour adequately needs some sophistication.

4. To develop rigour in any depth many trivial theorems must be proved before significant ones can be studied.

The inclusion of computer programming in mathematics does provide an argument for rigourous treatment in that computers have clearly defined limits within which they will operate effectively. However the level of rigour is variable -- between computers, between computer languages and between computer logics. This is a prime example for pointing out the relativity of rigour, and the necessity for adherence to it within suitable limits.

3.53 Teaching rigour.

What place can we give to rigour in mathematics and mathematics education?

Kline assigns rigour a place: solely the concern of professional mathematicians who wish to ensure that the deductive structures are sound.

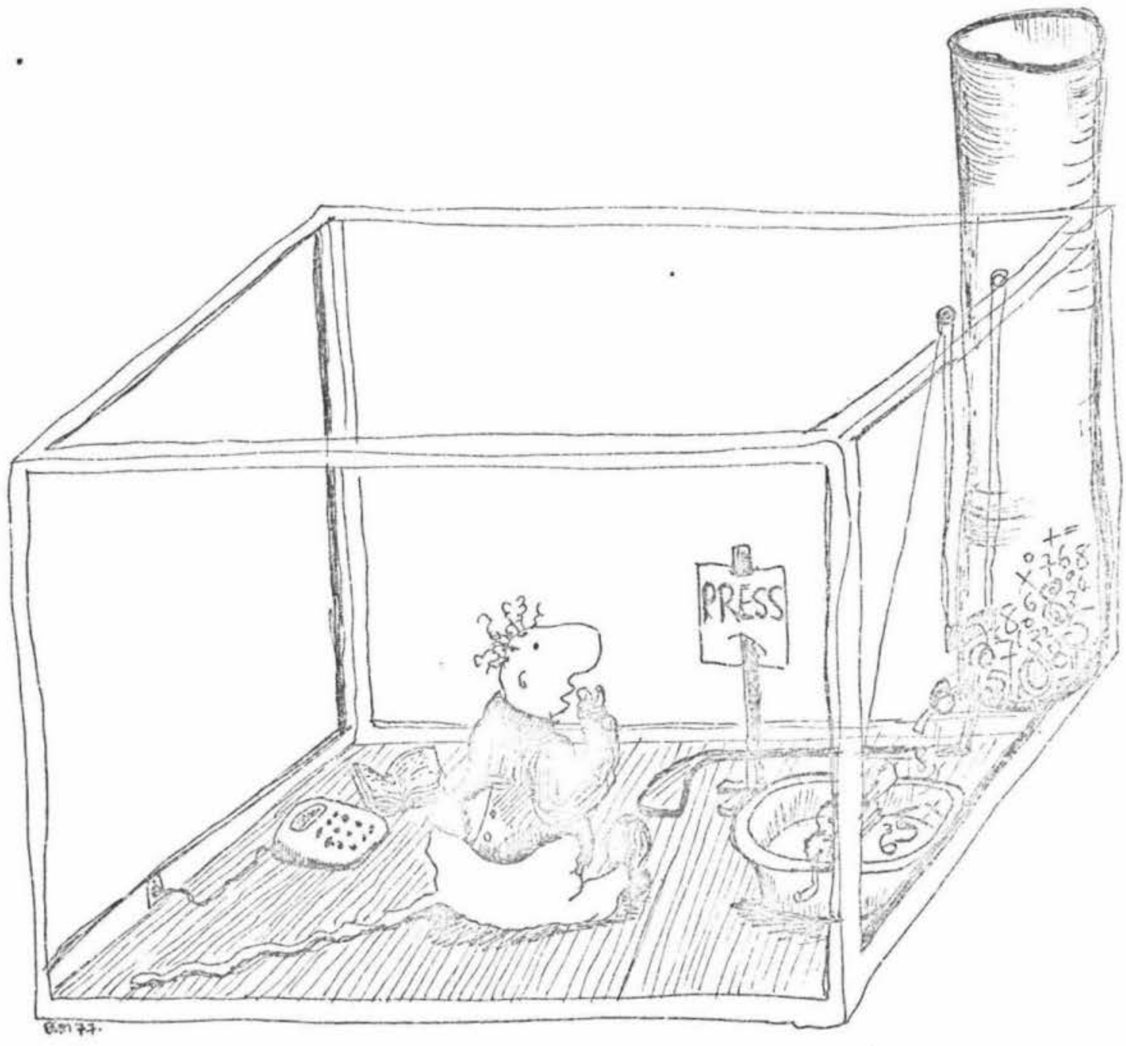
We can never know, however, an ultimate rigour; it is a response to perceived flaws. But we can be satisfied with a particular level of rigour. In mathematics education it is the student who must be thus satisfied, not the teacher. So the level of rigour should be determined by the questions asked or insecurity felt by the student about some (deductive) sequence. The teacher is in a position to generate the dissatisfaction of the student, but his/her own dissatisfaction is not sufficient justification. Rigour is part of mathematics so learning mathematics does involve learning rigour -- but learning about rigour rather than learning the most rigourous development in whatever topic is under study.

The teaching of logic is often justified as enabling students to attain greater rigour. On the above analysis this justification must be limited in value to those students who perceive the need for logic studies. Freudenthal's quote in section 3.43 is apt here.

Perhaps it would be clearer to restate the assumption as:

'Mathematics should be learnt with enough rigour to satisfy the student.'

with the additional comment that part of teaching mathematics involves generating dissatisfaction with implicit assumptions.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3.6 KNOWLEDGE, SKILLS AND ATTITUDES

> 'Knowledge and skills in mathematics are more important (or can be taught) than attitudes (which cannot be taught).'

This assumption is an unstated one. However it can be deduced from the nature of mathematics education: my feeling is that little is done in schools to promote positive attitudes towards mathematics apart from superficial motivating 'cons'. Less subjectively, the Syllabus in its aims, expresses a need for the development of positive attitudes, but in the subject matter and the examination prescriptions creativity, opportunities for individual interests and aids for the teacher in this direction are notably absent. Teaching positive attitudes is one of the most difficult aims to fulfill, partly because it conflicts with other aims and partly because it is not 'mathematical'. The lack of guidance for teachers has the same effect as the above assumption.

The language-game for 'learn' is one of the most varied in the philosophy of education. For this analysis it is convenient to identify three levels of learning corresponding to the three areas in the title of this section. Each shall be considered in turn, followed by a section looking at the consequences for mathematics education.

3.61 Learning that.

The first level is learning facts, as in "I learnt that the sine graph is periodic.".

This seems straightforward until we remember the discussion on terminology in section 3.3. Learning the 'fact' of, say, the periodicity of sine graphs constitutes learning one of the defining characteristics of sine graphs. Could we be said to know what a sine graph was if we didn't know that it was periodic? (See the discussion of primes in 2.13). It is <u>also</u> learning what is meant by 'periodic'. Even if we take the simple fact 2 + 2 = 4 we are really learning about the numbers and what they represent, rather than a unit of information.

Thus fact learning can be seen as a language-game. Consider how it is evaluated: reading-level and vocabulary are now accepted as important variables in examination papers. This is not only because students do not understand the questions, but also because vocabulary and the way in which words can be used are exactly what is being tested. Take these two test items from the Mathematics Level 6 Item Bank (Dept Education, 1973):

"6-3/39 A vertical plane is at right angles to...
A. a horizontal plane   B. an oblique plane
C. a parallel plane   D. none of these"

The meanings for several words are required, but, in particular, an objective meaning for the word 'vertical' is needed. But 'vertical' cannot be understood without its orthogonal relationship to 'horizontal': 'vertical' means 'straight up and down', but what is 'up and down'? Sooner or later an explanation involving 'sideways' or 'across' or 'horizontal' will be necessary. So the question is really about the way in which the two words relate, or the way in which we use them, not about the 'fact' of what is a vertical line.

"6-7/17 The product of two prime numbers is always...
A. another prime number   B. a composite number
C. a square number   D. an irrational number.'

Again the meanings for several words are required. But to identify the correct answer more than a knowledge of the word 'composite' is needed. The examinee must be able to identify a situation in which the word is appropriate i.e. 'composite' may not have been used in this context before, it is the relationship between 'product' and 'composite' which is crucial, nothing to do with prime numbers. The 'fact' of what is a composite number is really the use of the words 'product' and 'composite'.

3.62 Learning to.

The second level is skills learning, as in "I learnt to solve linear equations of one variable".

The behavioural understanding of 'to learn' is especially relevant here. Learning can be evaluated by performance: some behaviour appears which was not exhibited prior to learning. It is likely, however, that behavioural theorists were initially playing a much broader language-game with the word 'behaviour' than 'activity or performance'. Rather they interpreted the word as a complex of activities and dispositions.

Skills learning can be confused with fact or terminology learning. For example, part of the skill of drawing a liner graph is understanding what 'draw a graph' means. There are many fourth formers who can accurately "graph y = 3x + 2" but would not be able to "represent graphically the linear function G given by $G = \{(x,G(x)): (x \in R) \wedge (G(x) \in R) \wedge (G(x) = 3x + 2\}$", and, thanks to new maths, there may be some who can do the latter but not the former. Skill learning could be redefined to overcome this problem as: 'learning to exhibit some performance when the cue appropriate to the learner is given'.

However the problem is not solved. Consider an infant learning addition (assume that he can count to ten). Asked what 5 + 3 is the child can count things and arrive at the answer 8. At this point adding is a skill -- it involves several steps and probably some physical activity. However eventually such knowledge becomes a fact or a language-game. Does this transition take place when the addition is removed from the physical things? This is plausable until we consider repeated addition. Is 1 + 1 + 1 = 3 a fact or a skill? What about $3+5+1 = 9$? Could a child who 'knew' that $5 + 3 = 8$ and $8 + 1 = 9$ necessarily do this sum? Such an example points up one relationship between fact and skill learning: the latter can be generalisation of many 'facts', or the understanding of a language-game involving many words.

3.63 Learning to feel.

The third level is attitude learning as in "I hate mathematics".

Sentences which exhibit this type of learning often do not contain the word 'learn'. However attitudes are learnt and positive attitudes towards mathematics are part of the explicit aims of mathematics teachers.

What is meant by a 'positive attitude'? We must be careful to avoid the success ←------→ positive attitude circle described in section 2.33.

The behavioural answer is that a positive attitude is shown by a higher probability (as measured by observation of behaviour) that the student will, say, smile on entering a mathematics classroom. Alternatively psychological criteria (e.g. anxiety as measured by pulse-rate) can be used. While there is some value in this view -- the use of 'probability' especially -- it is not the whole story.

Attitude can only be gauged after its adoption whereas, if we take notice of subjective accounts of the learner, an attitude can be indicated as it is being formed. The effect of the behavioural view is to negate subjectivity. But attitudes are necessarily subjective. The behavioural evaluation can only be checked ultimately by correlating physiological or behavioural consequences with accounts of how people feel.

What contributes to positive attitudes in mathematics? I shall only consider environmental variables, inherent ones are not malleable by education, though they can, and should, be allowed for.

Success is often said to be important for a liking of the subject. Would the average fifth former increase his liking for mathematics by successfully doing 100 sums like 7 + 4 = ? It is not the actual succeeding as the other things which go with it: social prestige, worthwhileness, meeting a challenge. This is borne out by some people who seem to hate mathematics merely because it is trendy to hate it: some important people in our society seem proud of their lack of ability in mathematics.

Interest is another factor. It does not make sense to say that someone began to like something while it was boring them.

Worthwhileness incorporates some of the behavioural post-formation difficulties. If, after we have completed an activity, we look back and judge it to have been worth the effort, then a similar activity is more likely to be embarked upon than otherwise.

Relevance may seem to be a factor, but it does make sense to say, for example, "I enjoyed an article on catastrophe theory I happened to come across in the Listener, and would start to read other articles if I saw them.".

Skemp (1971) mentions two other possible factors: anxiety level and destructiveness. In general he approaches the problem from the opposite direction: he assumes a positive attitude and tries to see what turns students off. 'Insults to their intelligence' are one of his major destructive influences.

3.64 Consequences in mathematics education.

All three types of learning are necessary: it is difficult to conceive how type 2 or 3 could be achieved without some of type 1. The difference is that types 1 and 2 have many referents -- there are a myriad of possible facts and skills, but type 3 has only one referent: we want a positive attitude, not a negative one, to be learnt.

Type 1 and 2 referents can be conceptually divided into two classes: those facts and skills necessary for everyone (of age 15 say) and those not. However it is not just a question of arguing which class each fact or skill goes into. The contents of the classes change with time, place, culture and the aims of education. The dilemma is stated by Beberman (quoted in Kline, 1973, p.134) that the question as to whether a particular mathematics topic can be taught is an empirical matter, not a philosophical one; the question is: who is best able to decide what mathematics is appropriate? Are we to ignore the mathematicians' advice (he thinks that they have not been too wrong in the past), and, if so, is a teacher or student better able to know what mathematics is important?

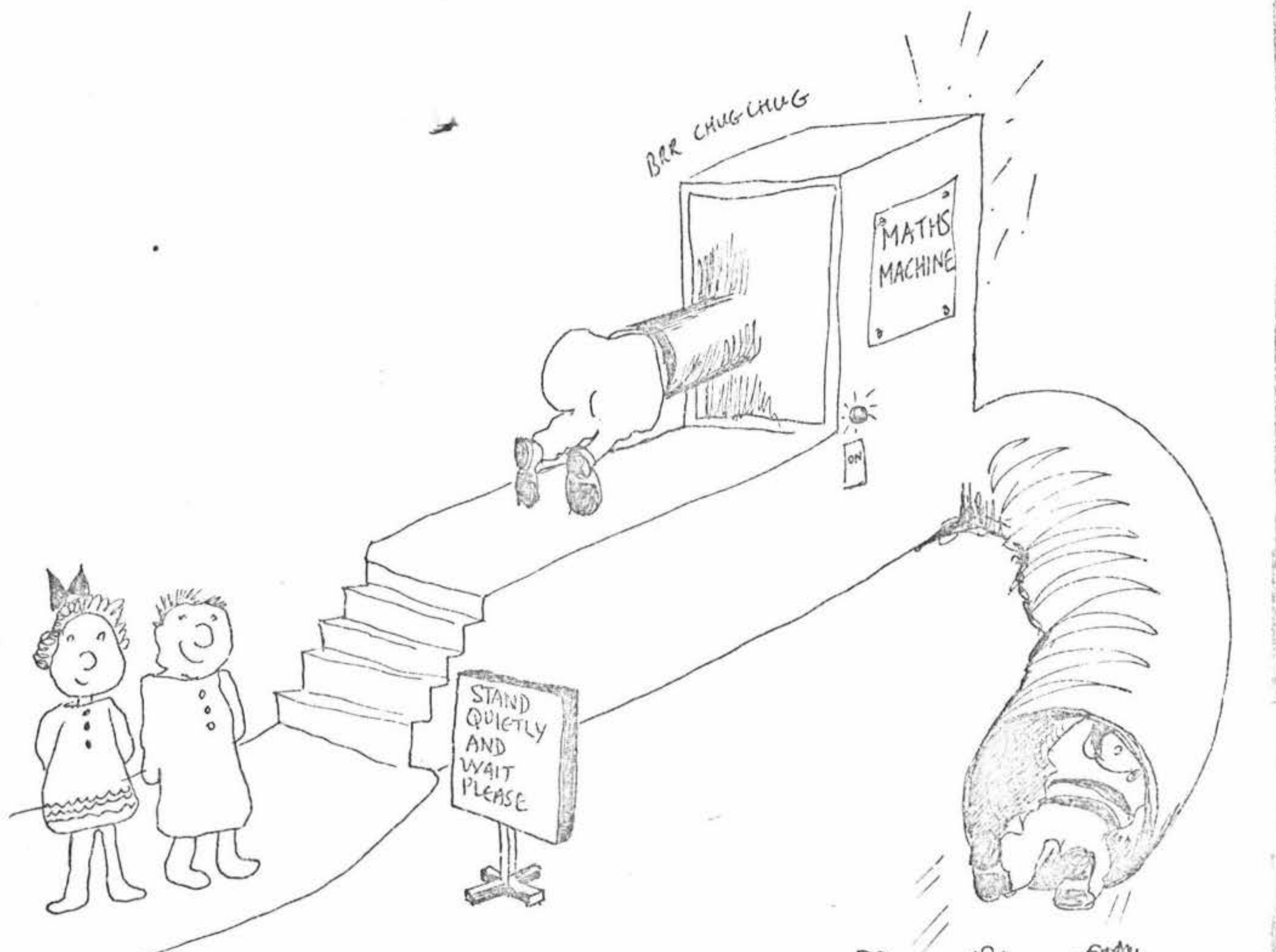
Furthermore the necessary content may change with the degree to which type 3 learning takes place. A discussion with E.L. Archer enabled me to express this idea. He suggested that 'transfer', while commonly thought of with respect to a body of content, or, at a higher level, a process or skill, should be with respect to a success feeling. Thus what is transfered is a time-saving when it comesto relearning or new learning in the same area. For example it does not matter that I have forgotten most of my form VII biology, if I do need the information my positive feelings towards the subject will mean that a) I will be prepared to make the effort, and b) that the relearning will be quicker.

The subject- and student-centred approaches to teaching can be analysed in these terms: the subject-centred approach emphasises type 1 and 2 learning in that not all factors for type 3 learning are met. Success can be arranged or interests generated for some of the students some of the time only. In other words the assumption stated at the beginning of this section is made.

The student-centred approach begins with type 3 factors; topics w hich interest the students, but suffers from the danger that the necessary class of facts and skills may not be covered. The assumption is replaced with:

a) mathematical knowledge can be transfered only if the student has a predisposition to learn and b) knowledge and skills in mathematics will automatically follow if a positive attitude has been generated.

If Skemp (1971, p.116) is right, then the student centred approach is unnecessary: facts and skills can be taught so long as they are done in such a way that the assumed positive attitudes are not destroyed.



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STAND QUIETLY AND WAIT PLEASE
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3.7 RANGE, VARIANCE & MEASURES OF CENTRAL TENDENCY

'The mathematical needs of everyone are broadly similar.'
'Mathematics taught in schools should aim at the general needs of all students'.

This section continues the discussion raised in section 2.24 and 2.53.

The assumptions have four levels of analysis:

a) that broadly similar mathematical needs exist;
b) if a), then these needs can be discovered;
c) if a) and b), then such mathematics can be taught;
d) if a), b) and c), then such mathematics should be taught.

These will be discussed in turn.

3.71 Do similar needs exist?

Similarities within mathematical content undoubtedly exist. Within the context of symbolism and terminology there is a common cultural content of mathematics. For example statistical concepts used in advertising and monetary mathematics. But also the way in which we talk about numbers and our concepts of mathematical ideas are culturally dependent, (e.g. ideas of the infinite, geometry and measurement). This is clearly seen in the historical discussion in sections 4.1 and 4.2.

Culture also affects perception and thinking. On a simple level the evidence for this is well-documented: comparative studies between African and European children in perceiving colour (Bornstein, 1973), patterns (Bentley, 1977), drawings (Deregowski, 1968) show significant differences. It is reasonable to infer subtle changes in perception and thinking within the evolution of any one culture (and hence a common base within one culture at a specific time). And the nature of mathematical thought has, and still is, changing. With the advent of computers, flow-chart logic and algorithmic procedures are becoming commonplace in our society.

Common factors in the psychology of learning are not evidence for a common mathematical education. If everyone thinks along similar lines there is no need to be taught to think like that -- though possibly patterns of human thought can be allowed for in teaching methods.

But children in our society are subject to similar environmental factors, TV being the obvious example. Peer-group organisation of schools exaggerates this.

When considering common needs we should also look at the known differences.

The level of sophistication and amount of content needed by different students will be different. Capabilities differ and post-school lives will be widely diverse. Our society contains many cultures and not all of any culture is necessary for everyone within that culture. However there is still the logical possibility of a common base of the minimum standards type.

Individual differences in ways of thinking, readiness for learning, interest, present environment, past learning and inherent capabilities are all well-documented. But none of them logically rules out the possibility of there being a common, necessary body of content and/or mathematical thinking process.

Given a common culture and mathematics' place in it, it is necessary that there be some common mathematical material: to say that two people live in the same culture but will have NO common mathematical experiences does not make sense. And therefore some common mathematical ways of dealing with these experiences are needed.

This highlights the reciprocal aspect of culture and (in this case) mathematics. Common mathematics is derived from a shared culture, but also a common culture is partly created by similaries in mathematics among a group of people. This is not circular: education creates culture as much as it transmits it.

3.72 Can common needs be discovered?

The educational time-problem presents the main difficulty. Given the context of, say, New Zealand secondary education, we are teaching for an unknown future.

There are purported ways around this.

First it can be argued that the content does not change as fast as all that. Decimal notation (and its use in monetary systems for example) has been around for several hundred years and looks as though it will last our students' lifetimes at least.

A second argument has developed in response to changing content: we should teach/learn procedures of perceiving and thinking, rather than subject material. For example the deductive method of mathematics seems constant. However it is changing also: mathematics in the 18th century was result oriented, in the 19th century it was rigour and axiom oriented (Grabiner, 1974) and now we are seeing shifts (due to computers and calls for relevance) towards algorithmic and applicable mathematics. Furthermore some of the changes in thought are quicker than the content changes: calculus has gone through all three stages mentioned above. Certainly they can change within a life-time.

It may be argued that there is still a pervading mathematical method -- but it must be rather general and a small part of the whole field of mathematics if it is to encompass even the three changes mentioned.

A third way out of the time-problem is to follow trends to discover, or make intelligent guesses about, the future common needs. As an example computer technology and its ramifications were predicted many years ago. Many mathematicians and educators have made what turn out to be extremely accurate forecasts of the impact of mathematical trends. However these may be self-fufilling and many have also been wrong. We tend to remember accurate forecasts and forget the many wrong ones.

One further attempt to solve this time-problem is to educate for a positive attitude towards mathematics making the assumption that someone who likes mathematics will be prepared to learn that which is necessary at any time. This has been discussed in section 3.64.

None of these pollings of the horns of our dilemma really solve the problem, though in combination they are enough to convince most educators that there is enough material to make compulsory education worthwhile. For a contrary view see Holt (1977). And to the extent that we educate for the present we can find out our mathematical needs.

Another difficulty in determining needs is that the perception of culture differs between groups in our society, particularly between older and younger people. The way students in secondary schools perceive mathematics is different from that of adults. Older teachers will have difficulty knowing the common needs of the younger students.

It may also be difficult to perceive the mathematical needs of a culture from <u>within</u> that culture. If we are in it then these needs may be satisfied unconsciously and recognition may be difficult. Also how will we know when we have discovered <u>all</u> the common mathematical needs?

The reciprocal nature of culture and mathematics implies that identified common material creates the needs for it. The idea that our society needs to deal with advanced technology and increased complexity leads to the education necessary to create computers when other solutions (e.g. de-industrialisation) may be possible.

3.73 Can common material be taught?

Part of this question is an empirical one, to be investigated in the psychology of mathematics learning. I shall not consider this aspect.

Another part is related to the individual differences in readiness, psychology, physiology, interests, environments and inherent capabilities of the students. Such considerations have led to diverse teaching methods: mastery learning, programmed learning, contract teaching group teaching, options, open schools and so on. Again the ability of these methods to cope with the above factors is largely an empirical matter.

However there is one philosophical problem to be considered.

Culture, including the mathematical part, is <u>created</u> by the interaction of the people within it, both among themselves, and with the present culture. Thus the common mathematical needs are created and solved by groups and individuals themselves. This may be dealt with by teaching, not mathematics directly, but rather an awareness of, and techniques for, creating and dealing with the mathematical aspects of our culture. This is very different from the mathematics taught in our schools now. For example, rather than teaching about computers or how to programme them, we should teach how to find out about (and develop) new uses for the computer. The two areas do overlap, but the latter part is neglected at present.

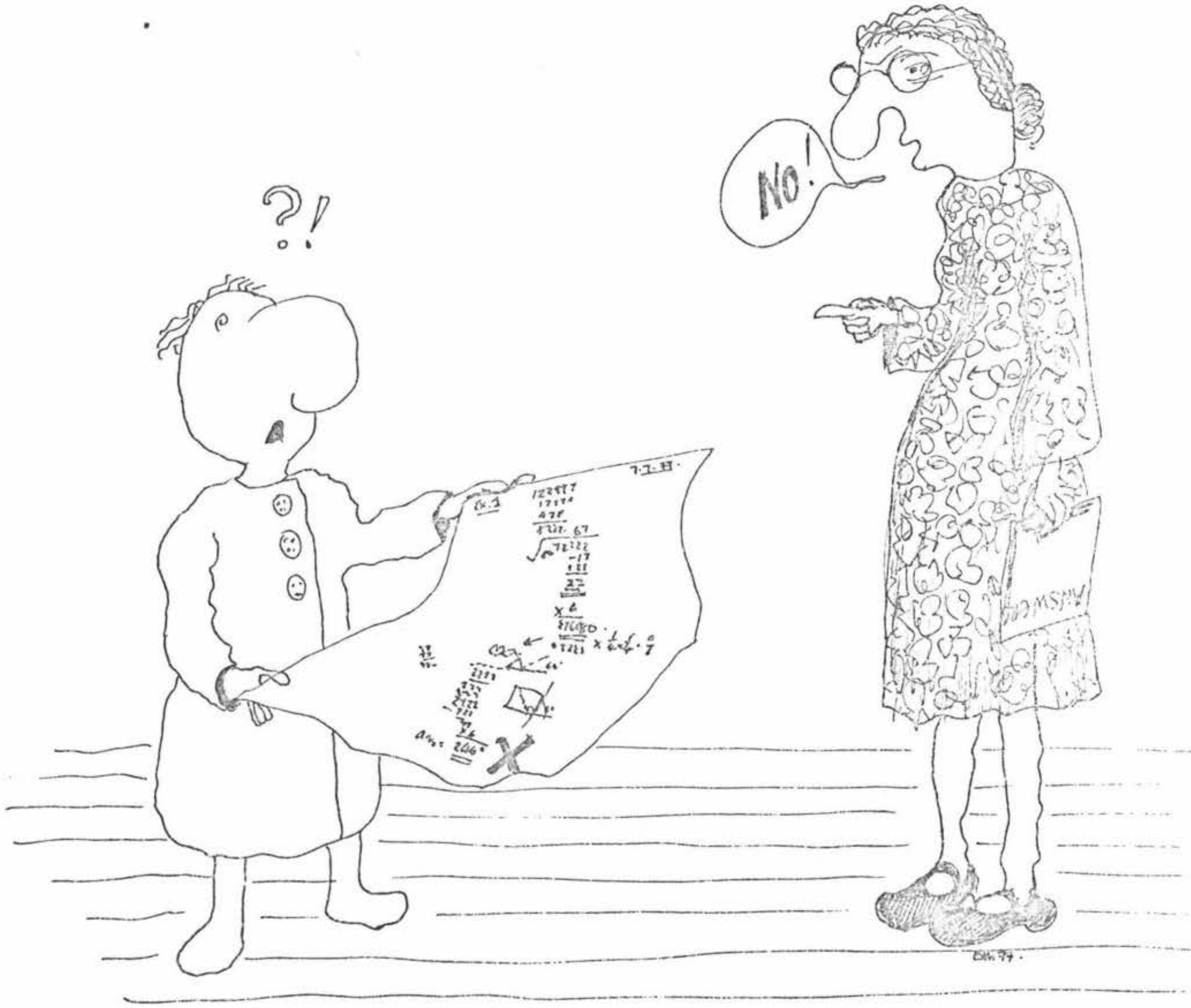
Another difficulty with teaching common material arises from the time-problem. Given that the time-problem can be partly overcome by discovering general methods and trends, it is necessary that the school system can adapt quickly to these. To the extent that it does not, the advantage is lost. For example calculus is still taught using the ways of thinking of pre-Cauchy times. Now pedagogically this may be good (i.e. the intuitive non-rigorous approach may be valuable -- see sections 3.4 and 3.5) but if modern concepts in mathematics <u>are</u> part of our culture, then they are being lost on our students.

3.74 Should common needs be taught?

Accepting the evolution of culture, may we not be doing a disservice to students by pre-setting them to the present or (given the conservative nature of change) past culture ?

Secondly, perhaps responses to mathematical needs should come from the people when they arise rather than be taught in a compulsory system before they are perceived by the learner? This view is extensively discussed by John Holt (1977).

Finally, perhaps society should change so that some needs become less needed. This is possible given the reciprocality of culture and mathematics mentioned above. It would make sense to use this power of education especially for those needs that are, in practice, not being coped with by a significant proportion of the population. The change to decimal currency can be seen in this light.


?!
No!

3.8 MY COUNT: RIGHT OR WRONG?

'Correct answers are a large part of mathematical competence.'

'Getting the right answer' has been worshipped as the prime aim of mathematics education.

On the one hand employers and others for whom accuracy is important have berated schools for turning out students who cannot perform calculations at whatever level they require. On the other hand stories of mathematicians who can never give correct change are classic.

3.81 What is a 'right' answer?

Getting the 'right' answer is largely a language-game.

This seems an exaggerated statement especially if we consider an example like 6 x 5 = ? But 6 x 5 does not necessarily equal 30. It may be equal to 26 if a duodecimal system is used, 36 in octal arithmetic, 2 in arithmetic modulo 7, 30x if 'x' stands for a variable, or something else if 'x' has been defined as some operation other than multiplication. Understanding what is required is at least as important as knowing multiplication tables.

Correct answers also depend on the question. '30' would be marked wrong by many teachers as an answer to the question: "How far would you go if you travelled at 5 km/hr for 6 hours?". Units would be required.

Method, units, statements of error factors, accompanying diagram, speed of calculation and even setting out can be factors in determining whether an answer is correct or not. Thus the teacher defines what is correct out of many alternatives. As well as performing the calculation, the student must understand what language-game is being played in a given situation. In this respect classrooms are different from 'life' where the correctness of an answer is defined by the person doing the calculation.

De Bono's discussion of lateral thinking (1971) mentions this area when he argues for specific instruction in variation of techniques and acceptable responses to some problem.

3.82 How important is it to be right?

What does a teacher mean when he/she says: "Don't worry about the answer, it's the method that counts", or "Show your working, you can get nearly all the marks for an exam question even though your answer is wrong".?

Is the teacher saying that the answer is always less important that the method, that it is mathematically speaking less important, or that on this occasion it is less important?

The first alternative, that the answer is always less important, is simply not true. Giving change in a shop is a simple example -- being correct is more important in practice than whether calculator, fingers, tables, Trachenburg method or guess was used.

The objection may be raised: "Yes, that's true in practice, but it is not how we think of someone's mathematical ability. Surely you must agree that someone who blindly guessed the correct answer is not as good at mathematics as someone who performed the correct calculations but mistakenly said 6 x 3 = 24 at the end?". I think I'd want to know a little more about how the guess was made, and whether the one who calculated did a check and estimation and so on. But it is true that we could construct an example where we would want to call the person who got the wrong answer the better mathematician. Our examinations are full of such cases.

So, sometimes at least, mathematically speaking answers are less important than method. The question now arises: if answers are not of prime importance in mathematics, what is? Several answers could be offered:
a) Method: it is generally agreed that the elegance, sophistication, abstraction, generality and speed with which an answer is produced has something to do with mathematics.
b) Thinking: was the problem laboriously or economically

solved? Were intuitive leaps followed by confirmation, or how was it thought out?

c) Verification: were rough approximations or rigourous deductive proofs made? Were errors and probabilities estimated?

All of these can be important depending on what the teacher/employer/user means when asking for mathematical competence. All can be taught, or at least demonstrated, and all should make up part of a mathematical education.

The third possible meaning for "Don't worry about the answer" is that it is not important on this occasion. This answer may be valid because, although the ultimate aim is to get correct answers, at the moment the teacher is considering only one aspect of getting the answer.

There are also pedagogical reasons why a teacher may not consider answers important. Part of teaching/learning is the development of propitious conditions for learning. Gaining a positive attitude to the material (or avoiding negative ones) may be part of this. If success or failure is a variable (see section 2.3) then de-emphasising correctness may be valuable. This is different from saying that the answer is not important in the long run, and must be recognised as such by both the teacher and the learner.


BLOODY CRAZY INVENTION........
ESM 77.

4.00 HISTORICAL DEVELOPMENT OF PHILOSOPHIES OF MATHEMATICS EDUCATION

4.0 INTRODUCTION

This chapter traces some of the major influences in the development of existing philosophies of mathematics education. This is done in order to put into perspective some of the assumptions mentioned in section 3.00, to see how they originated, and also to identify some of the problems which will have to be overcome if they are to be changed.

The development of attitudes towards mathematics education in New Zealand has been largely dependent on those in the United Kingdom and USA, and all three countries have reflected changes in educational and psychological theories as well as in mathematics itself.

Section 4.1 is a brief overview of the main philosophical notions behind Western mathematics as a subject. Our mathematics education is based almost solely on Western thought. This has been predominant in our mathematical development also. (For example Chinese mathematics virtually halted about the 13th century because of notational difficulties in dealing with more than four variables in algebraic manipulations). The main reference for this section has been Mathematics in Western Culture (Kline, 1972).

The next two sections trace the history of mathematics education in the United Kingdom and USA respectively, noting changes in social, psychological and educational factors. For section 4.3 the main reference has been NCTM 32nd Yearbook (Jones, 1970).

Next the development of New Zealand mathematics education is reviewed. Here A.E.E. Clarke's thesis (1960) has proved valuable. The effects of developments in the United Kingdom and the USA on New Zealand mathematics education are specifically noted.

The final section brings together the influences as discussed in this chapter and the assumptions mentioned in chapter 3.00. The influences are grouped under the headings: mathematical forces, social forces, educational forces, psychological forces and administrative forces. The contributions of these forces towards the formation of particular assumptions is discussed.


NEW MATHS
HARVARD UNIVERSITY

4.1 PHILOSOPHICAL CONSIDERATIONS IN THE HISTORY OF MATHEMATICS

This section traces the development of attitudes towards mathematics itself. The aim is to find the origins of some of the attitudes common in the teaching of mathematics.

4.11 B.C. - The Dark Ages

Mathematics prior to the Greek period was predominantly utilitarian, although founded on astronomy. Egyptian mathematicians (4000 - 1000 BC) were also the priests, who depended for their power and status on accurate predictions of, for example, the Nile flood. This mathematical knowledge was restricted in distribution and associated with mysticism. This is reflected in, for example, the building of temples to be sunlit on particular days, and the significance of numbers such as 3 and 7. The practical uses of mathematics in agriculture (weights and measures) and engineering (measurement and surveying) are reflected in the records which do exist (Eves, 1964, p.31). Argument or demonstration of results are absent, rather we find practical instructions and algorithms. Generalisation also is missing, multiple examples being norm. Results were all established empirically (Kline, 1976, Chpt. 2).

Babylonian mathematics was more advanced than Egyptian, particularly in arithmetic and algebra (which was linked with geometry).

Greek mathematics developed from Babylonia and can be said to have started about 600 BC with Thales. Rational, deductive argument and proofs became the <u>modus operandi</u> of a culture which had outgrown the solely utilitarian functions of mathematics in that the upper stratum of society had leisure-time in which to follow up philosophical ideas.

Coming under the influence of the philosophers (Plato, Aristotle, etc) mathematics was considered an art, and, as such, became something greater than its commercial usefulness. In eschewing empiricism the Greeks also initiated abstraction, arithmetic becoming merely an inferior skill. Geometry received most attention, probably through astronomy, but also because it lent itself to their methods. The resolution of irrational numbers was made by a geometrical interpretation, and algebra was ignored probably because of inadequate symbolism. Postulational thinking (deductions from explicit assumptions) developed prior to Euclid (300 BC). This, and the investigation of number properties are probably largely due to the Pythagoreans, a group which maintained the mystical element in mathematics, results being kept secret and membership of the society ritualised. Another development in Greek mathematics was concern for the infinitesimal, as evidenced in Zeno's paradoxes.

The Greek influence is summed up in Kline, 1976 at the end of Chpt. 3:

> "Because the Greeks converted arithmetical ideas into geometrical ones and because they devoted themselves to the study of geometry, that subject dominated mathematics until the nineteenth century,......... The Greeks ...failed to develop the number system and algebra which industry, commerce, finance, and science must have...."

By 300 BC the Greek states had fallen and Alexandria became the centre of learning. The university there was the focal point for mathematical development for 700 years. Euclid was one of the first members -- his Elements being the collected work of the Greek mathematicians put into postulational form.

This work was a major force in mathematics: it contained geometry, number theory and (geometrical) algebra, and became the model for all mathematical exposition despite the fact that Euclid probably used inductive, analytical and intuitive thinking in its formation. The identification of mathematical thinking with deduction and the importance of defining terms and stating axioms undoubtedly originates from this work.

In Alexandria scholarship flourished, and renewed an integration of mathematics with commerce, engineering and navigation. Archimedes, Eratosthenes and Hipparchus were famous for their applied mathematics before the birth of Christ. After 1 AD Heron in engineering and Ptolemy in astronomy carried on this trend.

Diophantus initiated the second stage in the development of algebra (Eves, 1964, p.158) by adopting symbols for common quantities, although this did not appear in Europe until the 15th century. Mathematics by now (300 AD) had become a means of making sense of the universe -- its potential for unifying diverse elements was recognised. For example astronomical theories assumed circular motion purely because it was the simplest. In a sense, then, mathematics had begun to determine man's thoughts.

The rise of the Roman Empire heralded a decline in European mathematics -- their practical philosophies did not allow for much mathematical development (cf their number system) although their engineering was quite sophisticated.

The Dark Ages descended on Europe and such mathematics as existed was preserved in the monastries. The focus of mathematical development shifted to India and the Islamic Empire.

4.12 The Dark Ages - Modern Mathematics

Prior to 500 AD Indian mathematics was based on religion, subsequently this base shifted to astronomy. The influence of the Far East is debated, but was probably significant. The Hindus made two important advances: they adapted the Babylonian place-value system to base 10, and they developed negative numbers. Their algebra was well advanced but geometry was weak and largely empirical. After 1100 AD Hindu mathematics remained static until modern times.

The Islamic Empire valued learning and both Greek and Hindu works were translated. Their own contributions were mainly in astronomy (trigonometry) and algebra, but they represent the link which preserved the geometry of the Greeks and the notation, arithmetic and algebra of the Hindu/Arab scholars. Through them the early European scholars received their heritage.

This retranslation was done between 1000 and 1200 by scholars who travelled to Moorish centres in Spain and North Africa. The establishment of trade between Italy and the East played an important part in the dissemination of the Hindu/Arab numeration system.

The rise of Christianity in Europe over the Medieval period accepted mathematical learning as a preparation for the study of theology. Justification of faith and the Glory of God were accepted as the reason for studying mathematics -- an influence which lasted until Newton. One prime impetus was astrology. Medicine was studied by watching the stars and medical students were taught mathematics for this purpose. Despite its conflict with the church, this practice extended to the 16th century. The European universities taught the quadrivium of arithmetic, geometry, music and astronomy. Texts began to appear in the late 15th century. There were both classical works for scholars and vernacular arithmetics intended to educate for commercial careers.

By the 15th century a renaissance in mathematical thought occured through the rediscovery of Greek mathematics combined with the rise of the merchants and artisan classes, the invention of gunpowder and the printing press, and a rejection of religious and mystical dogma. Beginning with Cardano and Tartaglia (in algebra) and Copernicus (in astronomy) there was a renewed interest in the physical world and the problems of nature. The conflict between Copernicus, Kepler, Galileo and the church epitomizes the emancipation of science from theology (Kline, 1972, Chpt. 9). Greek rationality provided the basis and Descartes, Galileo, Bacon and Kepler among others used mathematics to describe a world view.

However, rather than destroying religion, mathematics became a celebration of God and His work. Their picture of the world was of a mathematical image which had to be dicovered. Unlike the Greeks, who depended on objects and shapes, Descartes' fundamentals were extension and motion. Its aim was to build a model of the universe from these beginnings.

A change was made from explaining why things happened (and giving theological answers) to describing phenomena, isolating their fundamental properties and finally deriving the 'Laws of Nature' from basic principles in the manner of Euclid. Descartes' dynamics was the basis of Newton's Laws of Motion and together they set the stage for physical science. Newton still needed God to explain the irregularities in, for example, orbits, but by the 18th century Lagrange and Laplace had applied Newton's Laws to the solar system so well that the hypothesis of a God was unnecessary.

4.13 Modern mathematics

Modern mathematics is said to have developed from the 17th century: modern notation had become standardised, decimals and negative numbers were being used, algebra and the theory of equations were advanced and tables were being produced. Napier (logarithms), Oughtred (algebra), Galileo (dynamics), Kepler (planetary motion), Descartes (coordinate geometry), and Fermat, Pascal and Huygens (number theory and probability) were all responding to the needs of the times and mathematics became more applied. The political climate in France and England was more favourable than that in Italy and mathematical development moved north.

Mathematics owed some of its status to philosophy: Hobbes and then Locke elevated mathematical knowledge as opposed to experience. Berkeley attacked what he saw as a threat to religion and Hume further challenged the existence of material objects or mathematical truths. By 1800 Kant was using mathematical knowledge to justify a priori synthetic statements. Thus mathematical truths were central to each of the philosophers mentioned and became widely discussed.

Although its origins can be traced to Greek times, the Newton/Leibniz development of infinitesimal calculus tested the basis on which mathematical thought rested.

Neither development was rigorous enough to survive criticism of the concept of the infinitely small. However calculus was obviously useful and mathematically interesting and developed solely on faith for over a hundred years before Cauchy and Riemann rigorised analysis.

The Newton/Leibniz confrontation produced a split in English and European mathematics on two levels. English mathematicians eschewed advances in analysis made in France due to Leibniz' more flexible notation. Also, on the philosophical level, the conservative Christian ethic, in England, preserved the idea of mathematics being the tool of God, whereas the French thinkers dispensed with theological considerations entirely.

The 19th century saw several new developments.

Algebra, previously considered as symbolised arithmetic, became liberated through the new algebras of, for example, Cayley and Hamilton. Combined with the development of set theory (Cantor) and logic (Boole, De Morgan) the abstraction of mathematics began. The unifying influence of set theory created new areas (e.g. topology) and thus it came to have its high status as the basis of mathematics -- leading to the 'New Math' revolution in education.

A second development was the discovery of consistent non-Euclidean geometries by Lobachevsky, Gauss and Bolyai. This led to the increased importance of axiomatics initiated by Klein's 'Erlanger Programme' and, with set theory, contributed to the great advances mentioned above.

The arithmetisation of analysis (Cauchy, Weierstrauss) was a third development and led to much work on the foundation of real numbers.

Other 19th century advances occured in mathematical physics (Maxwell, Tait) and statistics (Graunt, Quetelet and Galton).

By the turn of the century philosophies of mathematics had developed based on questions raised by the above advances: Can mathematics be constructed on an axiomatic basis? (Peano), What structures are possible? And what can these structures achieve? (Goedel). Answers to the question of the nature of mathematics varied from logic (Russell), to structure (Hilbert), to intuition (Brouwer).

The two contemporary mathematical developments which have had time to be reflected in educational changes are computers and applied mathematics.

The astonishing technological advances in handling mathematical operations has led to the increasing importance of algorithmic procedures as well as making available new results and raising some moral questions on the uses to which computers should be put.

The explosive increase in mathematical knowledge (resulting from the unification of geometry and algebra through set theory and the arithmetisation of analysis) has been followed by increased applications. No longer is it possible to know enough mathematics to be able to bring all relevant mathematics to bear on a problem: any problem may be able to be solved using topology, analysis, numerical analysis or some other field. Thus people must have an awareness of many different areas and be able to then find the necessary information in that area, rather than become proficient in 'mathematics' and apply what they already know.

A further consequence of increased applications is that most areas of human endeavour involve some mathematics: art, science, social sciences, commerce, management and everyday universal activities (tax forms, recreation) as well as academic scientific occupations.


- GOD SAVE THE QUEEN -
Mental Arithmetic.
6+6 =
8x4 =
678
x 4
ESM 77.

4.2 MATHEMATICS EDUCATION IN THE UNITED KINGDOM

This section examines the development of attitudes towards mathematics education in the United Kingdom. These attitudes are a source of New Zealand positions (see section 4.4) and of the assumptions mentioned in chapter 3.

4.21 Up to 1700

Mathematics education in the United Kingdom began in the monastries during the Dark Ages. Then, during the period 1300-1500, the forces of commerce initiated systematic, utilitarian mathematics teaching. By the late 16th century arithmetic, geometry and trigonometry were being taught for use in navigation, gunnery, surveying, mining, fortification, astrology and astronomy. Such teaching was done on a private tutor basis, the first institutional mathematics teaching being at Gresham College in the early 17th century, followed by chairs at Cambridge and Oxford in the middle of that century. The importance of the navy resulted in several mathematical establishments being set up to teach navigation.

Mathematics teaching was not widely accepted at this time because it was still associated with sorcery: the main English mathematicians had been trained in medicine in Europe (e.g. Recorde).

Despite utilitarian origins, education was a gentleman's occupation, and a means to social not scholarly advance. Thus the applied justification for learning was not valid.

Newton's controversy with Leibniz had severe repercussions for British mathematical education starting from about 1680. The dispute led to a loyalty reaction from British mathematicians who cut themselves off from the advances being made in European mathematics. Thus while mathematics education by the beginning of the 18th century was becoming more acceptable to the public, it was not until 1800 when

Peacock introduced continental notation in the Cambridge mathematical Tripos that the standard of mathematics recovered.

Newton epitomizes another general educational influence -- that of religion. His avowed intention in studying mathematics and science was to show how wonderful God's creation was. This attitude prevailed into the 20th century.

4.22 The period 1700-1900

Private schools from 1760 onwards responded to commercial needs, but public and grammer schools did not teach mathematics until the 19th century (see Howson, 1974). The industrial revolution demonstrated the need for mathematics education so that it began to be accepted. However secondary education was still exclusively private and upper class, and mathematics was regarded as 'common'. For example, Harrow's first mathematics teacher was not considered scholarly enough to wear academic dress. Mathematics education was ancillary to the fundamentals of the classics. In keeping with this heritage, Euclid was studied exclusively although other texts were available, e.g. those by Legendre and Wolf. Little else was taught: arithmetic, decimals, proportion. By 1873 algebra and plane trigonometry had been included.

In 1871 the first mathematics teacher organisation was formed: the Association for the Improvement of Geometrical Teaching, later to become the Mathematical Association.

During the 19th century two general forces became dominant in mathematics education.

The first was examination fervour. The Tripos in Cambridge was a very rigid affair, leaving little room for imagination and resulting in a strict hierachy of candidates. It was not until 1888 that Oxford and Cambridge allowed proofs other than Euclid's, although his order was maintained until 1903. The remaining examination system then became jeopardised by a lack of uniformity in the school syllabus.

It has been suggested (Griffths, 1971) that examinations were developed originally for two reasons: to relieve the boredom of the clever/rich by competition, and to defeat patronage. Whatever its origins, the tradition was adopted by the secondary schools, partly because of the university entrance requirements. The exam system was used to improve elementary education, but by 1870 reforms were desparately needed to escape from its restrictiveness.

The second influence was the widening availability of education. The first college admitting girls opened during the 1840's and in 1870 the Forster Act provided for universal elementary education.

However, to follow an analysis of H.B. Griffths, (1971), education now began to develop into three classes: Oxbridge and the public schools for the leisured class, private and grammar schools for the petit bourgoisie, and elementary schools for the workers. Mathematics for the leisured classes consisted of Euclid, arithmetic and algebra to the neglect of other areas. This class bore the brunt of the examination system. The petit bourgoisie mathematical education was science and technology oriented, partly because of military needs. Although the grammar and secondary school organisation was conservative, competition with Europe at the turn of the century led to strong growth in science and technology. The working classes received only basic arithmetic, vocationally oriented. Their exams were originally oral tests. All three types of education were planned by the leisured class.

A further tradition in British mathematics had been firmly established by 1900: the pre-eminence of mathematical physics. Kelvin, Newton, Maxwell, Tait and Hamilton established a trend broken notably only by Cayley and Russell among British mathematicians. While no direct link to mathematics education can be sustained, it seems likely that their influence contributed to the technological/applied mathematics emphasis in British mathematics compared with the structure/ abstract emphasis in America and Europe.

4.23 The Twentieth Century

Reform, both in examinations and Euclidean rigor, began with the 20th century. The famous "Letter of the 23 Schoolmasters" (Howson, 1973) called for the arithmetisation of geometry and algebra and an experimental (drawing) emphasis in geometry. Following a period of intense communication with Europe around 1912, (unequalled until the 1960's (see Howson, 1974)), schools started teaching descriptive and experimental geometry, statistics and calculus. However the 1st World War slowed reform (rigor in the universities increased over this period) and in 1919 the Mathematical Association felt obliged to stress the utility of mathematics, the linking of mathematics and science and mathematics as a liberal art.

But school mathematics remained formal and rigorous, primarily because the examination system was still strong, but also because the teachers had generally had a conservative, rigorous education themselves.

It was about this time (the 1920's) that Godfrey, a mainstay of the reforms, began to talk of symmetry geometry -- the approach now used.

Despite the scientific stimulus of World War II, the notation of functions was the only important new idea to enter the curriculum. With the exception of an interest in practical mathematics in primary schools during the 1950's, mathematics education remained conservative. For example, even today, technical education does not feature in the A-level GCE mathematics. Furthermore it had lost some importance in 1944 when it was made an optional (not compulsory) subject for leaving certificate.

The prevailing attitude of education as a status-changer continued to hinder reform. The Butler Act in 1944 introduced secondary modern and comprehensive schools. The 11-plus system, instituted soon after WW II, channelled 20% of the pupils to grammar schools, 20% to technical schools and the rest to secondary-moderns, thus maintaining the traditional divisions. The GCE was begun in 1951 and followed the classical tradition.

Liberal views about equal opportunity for all found realisation as the same opportunity for all -- and that based on a grammar education. For example the School Mathematics Project (which began in 1961) had a very academic base although it did, finally, replace Euclidean geometry with motion geometry. Other efforts included the Nuffield Mathematics Project (primary), the Midlands Mathematical Experiment and Mathematics in Eduoation and Industry.

It was not until the late 1960's that the Certificate of Secondary Education examination attempted to fill the need for a non-academic syllabus. But uniformity of syllabi in schools and sequential teaching remain, despite the lack of centralised control of the curriculum. And because of this lack examinations have a greater effect.

Recent influences include comprehensivisation. This has resulted in teaching to a wider range of pupils, with the danger of directing teaching to the more able sections. However the public schools' monopoly of "good" education is diminishing. A further aspect is that industry and commerce are again becoming concerned. This is evidenced by the outcry over the mathematical standards of school-leavers, (Hayman, 1975). More than this, the current debate in education is public: accountability to society is a developing theme (Kerr, 1977).

Between 1945 and 1975 there has been an 80% increase in school population brought about by a higher leaving age as well as population growth. This has led to difficulty in adopting new syllabi, as well as slow improvement in educational standards. However the contemporary falling birthrate may create new opportunities: teacher supply in mathematics improved to 1972 but has declined since then (HMI, 1977).

The promoters of modern mathematics had difficulty overcoming the conservatism of the examination system, the chaos resulting from school reorganisation, teacher training problems and utility-based mathematics education in the technical and secondary-modern schools. The Newsom Report of 1963 (Newsom, 1963) approached the topic cautiously:

> "Even the 'new mathematics' may have something to offer for some of our puplis where there is a well-qualified and well-versed enthusiast on the staff".

The syllabi have been changed slowly and the texts were not widespread until the late 1960's. Modern mathematics in UK has been more practical than elsewhere. The concurrent change in teaching methods has also met with conservatism.

. The slow recognition of mathematics as important in education in the United Kingdom in the 19th century, and the recent conservatism towards overseas mathematical trends fit a hypothesis put forward by H.B. Williams (1971) that in England mathematics is a means to an end (or different ends at different times) and rarely is the end of creating mathematicians, and hence mathematics, one of them.


16 SHIPS TODAY, 3 PLANES AND THE WEATHER IN NEW YORK IS FINE.......

4.3 MATHEMATICS EDUCATION IN THE U.S.A.

4.31 Beginnings

From its beginnings in the 17th century mathematical education in America has stressed the practical applications of settlement: exploration, commerce and surveying.

However, even then, the belief in universal education was emerging -- although mathematics was not initially considered an integral part of such education. Arithmetic was first made an entrance requirement to a college in 1745 (Yale), but geometry was not necessary until after the Civil War.

In the 18th century surveying and astronomy were included amongst the mathematics courses offered in the universities. Part of the reason was practical but the scientific rise in Europe had been transmitted quickly, and mathematics became the tool of science as well as commerce.

Teaching was primarily by rule and example, although Pestalozzi's 'mental discipline' theories had some effects.

4.32 1800 to the 1930's

By the beginning of the 19th century the influence of Europe was being felt. For example Legendre's geometry text was used from 1819 onwards. French mathematics continued to be a strong influence in America. Mathematics now spread to the elementary schools.

Grammar schools declined (as did the academies in the 1850's) to be replaced by high schools. By midway through the century the principles of teacher training and free compulsory education for all had been established. A rapid growth in the school population began now and continued into the 1930's.

Initially algebra and geometry were taught at high school in response to three forces: the rise in the scientific need for mathematics felt in the universities; the continuing needs of technology and communications; and the rise of faculty psychology where both subjects were regarded as good training. However in both universities and high schools, academic and moral justifications gave way to vocational, social and practical ones -- thus special courses and electives became established.

Teacher training was first established in 1832 and the next 50 years saw a gradual change from rule/practice methods to inductive reasoning and understanding. Concern for the pedagogical problems of a large school population with varying aims led to the setting up of several investigating committees by the end of the century.

The recommendations of these committees were more oriented to college requirements than school needs. These committees did not give adequate notice to the varied needs of the enlarged school population and, although junior high schools were common by 1920, little was done for non-college bound students.

However the calls for psychologising mathematics instruction and reforming geometry were responded to. Psychologising moved through Spencer's rational theories, Dewey's experience/measurement views and David Eugine Smith's eclectisism. Later, up to World War II, Thorndike's connectionism came into vogue, mental discipline theory having been discredited. Thorndike's theory led to the fragmentation of courses together with much drilling and rote learning. Gesell's readiness theories had some effect in countering this trend. Testing of intelligence and achievement began during the early part of the 20th century, as did the idea of spiral curriculae.

In mathematics itself the German influence of Klein (functions and structure in mathematics) and Hilbert (axiomatisation of geometry) took effect. Veblen and, later, Birkoff made axiomatisation a major American interest. Such ideas quickly entered universities, teachers colleges and high schools.

In 1911 the College Entrance Examination Board was set up, providing a unifying influence on syllabi.

The establishment of junior highs led to a reorganisation of mathematics curricula towards general, non-compartmentalised courses for grades 7 and 8. Senior highs remained traditional, however, and decreasing college attendance (because of the depression) and the elective principle led to declining mathematics enrolments as well as lowered college entrance requirements. A general, compulsory, 1-year course developed which has continued to the present day and is characterised by unimaginative teaching and low interest. The Colleges, in response to a questioning of the utility of mathematics, began to develop general courses for non-mathematical specialists in the 1930's. There were three types: functional, cultural and 'modern'. This latter foreshadowed the modern reforms and dealt with such topics as set theory and non-Euclidean geometry.

4.33 The last 50 years

The depression in America inhibited mathematical reform. While student numbers continued to rise, lack of finance resulted in declining conditions and few innovations.

World War II at once highlighted the failure of classical mathematics education (i.e. draftees were seen to be below the standard expected) and raised mathematics and science to a new status (i.e. accentuated the uses and potential of advanced mathematics). Thus the post-war enrolment boom was strongest in these subjects. This put a strain on teacher resources, especially as the industrial demand for mathematicians was also boosted. For mathematics education the result was that, in the immediate post-war period, there was a backlog of reforms from the 1920's and 1930's, a manpower problem, and a heightened public interest in mathematics.

The mathematically trained man-power shortage by the post-war scientific/technological development focussed attention on outdated curricula. Industrial development and new applications stimulated problem-solving and created a demand for 'basic structure' education so that the varied fields might be accommodated. Meaning and understanding became vital in mathematics education.

Piaget's influence was felt in the 1950's, and

Gagne's heirarchical concepts, Bruner's structures of learning and Ausubel's meaningful learning theories all had exponents.

In the early 1950's several committees investigated mathematics education, one of which, the University of Illinois Committee on School Mathematics (1952) was looked upon as the initiator of 'new math'.

The Commission on Mathematics in 1955 consisting of teachers, teacher trainers and mathematicians recommended emphasising concepts, deductive reasoning, structure, inequalities, unifying ideas such as set and function, and motion geometry. These ideas were put into practice by groups such as the School Mathematics Study Group (SMSG) (Jones, 1970, p76). After the 'Sputnik' educational rethink money for research, development and application of the 'new math' was available through many sources (e.g. the National Defense Education Act 1958). Between 1950 and 1966 there were 24 national projects (8 pre-Sputnik and 5 resulting from the 1965 Elementary and Secondary Education Act) (see Crespu, 1969). The reforms emphasised understanding rather than learning content or calculating efficiency, but little content matter was in fact changed. The method of teaching also altered, reform being based on dicovery learning and precision in language (see Crespy, 1969).

The result was a complete reform of elementary mathematics, and secondary mathematics attempted to respond to calls for modern topics, for more advanced topics and for relevance. Teacher training and retraining became important, but despite massive efforts, failed to sustain the revolution that was occuring. In fact most projects badly underestimated the time needed to retrain teachers and develop materials.

Reaction began by the early 1960's. Kline and Elicher challenged the curriculum and the untested way in which reforms were adopted and spread. This reaction continues with charges of incompetence and falling interest among school-leavers. This can partly be explained by an anti-technology movement in society and seems to deny the general agreement that some aspects of the reform have been valuable. Reliable evidence on the value of the reforms is, however, lacking. Partly this is because the aims of mathematics education have changed, thus it is misleading to evaluate reforms on mathematical proficiency. However most comparison studies show no significant difference on proficiency and indicate increased understanding (Crespy, 1969). There has been a shift in public attitude back towards valueing proficiency rather than understanding (see Frand, 1976). This, together with the criticism mentioned, has led to less axiomatic programmes.

The tremendous growth in pure mathematics itself emphasised both the need for organising the structure of mathematics and for generalisation. In addition mathematicians demanded that more mathematics must be learnt in schools. One result is that professional mathematicians have become heavily involved in education.

The high level of development in mathematics has led to an excess of specialised mathematics graduates and, now, to compensate, a call has gone out for a broader-based (but still larger) mathematical education.


DONT BE A SILLY LITTLE PIGGY - OF COURSE YOU CAN COUNT UP TO TEN.
?
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4.4 MATHEMATICS EDUCATION IN NEW ZEALAND

4.41 Initial influences

Mathematics education in New Zealand has always been strongly influenced by events in the U.K., for example the influence of Percy and Nunn at the turn of the century. At that time the value of mathematics was still seen as 'training the mind' in ordered thinking and appreciating the utility of mathematics. However compartmentalisation in mathematics was being broken down and the beginnings of psychology-based developmental approach developed early this century.

The primary system worked independently -- the new syllabus introduced in 1928 had little effect in post-primary schools.

School Certificate was introduced in 1934, and the proficiency examination was abolished in 1937. However S.C. failed to provide a more realistic goal than University Entrance examination, which it was supposed to do.

At this time schools

> "had ceased to be the preserve of the intellectual elite... Post-primary education was becoming the birth-right of virtually all NZ children." (Clarke, 1960, p5)

Thus a wider range and greater numbers of children had to be catered for.

A second major influence was World War II which, with its revolutionary technological advances, increased the felt need for a good mathematics education. Technology was no longer a specialist field, but encroached on everyone's way of life.

In 1942 the Canterburry Institute for Educational Research issued a 'Report on the Teaching of Mathematics at Various Stages in the School System with Suggestions for Improvement'. Basically it asked that mathematics become unified, that it be taught by experts and that it should not be too mechanical. It also recommended practical rather than formal geometry. The schemes it presented were dry and mechanical in algebra and arithmetic, but practical and informal in geometry. There were different schemes for Trades classes (boys) and Non-examination classes (girls) in which 'easy examples' were recommended. Finally a practical mathematics course was outlined - a simple version of the full mathematics.

The practical emphases of the report, especially in geometry and the 4-way differentiation of the syllabus, were important innovations.

4.42 The Thomas Report

The Thomas Report was completed in November, 1943, but its writers were aware of two impending changes: accrediting for U.E. arrived in 1944 with the aim of making S.C., not U.E., the qualification for post-primary education. This de-emphasised academic courses, and english was made the sole compulsory subject. The other change (in 1944) was the raising of the school leaving age to 15.

In its recommendations the Thomas Report attempted to de-emphasise the narrowing effect of examinations by introducing a core curriculum which would enable the widely divergent pupil population to better prepare for life in the community. The core would not be examined externally, and was also aimed at reducing early specialisation.

The Report saw S.C. as an endpoint, not a preparation for U.E., and as within reach of most secondary pupils. The Committee envisaged 20% of pupils going on to U.E. and 40% studying S.C. mathematics, both doing the full mathematics course; the other 40% studying an elementary mathematics course. Clarke reports that by 1958 this latter elementary mathematics was taken by 31% of boys and 76% of girls.

With reference to mathematics teaching the Report emphasised utility of mathematics, concrete to abstract development, many simple examples, not so much mechanical manipulation in algebra, more aids to calculation and an informal approach to geometry.

The Thomas Report recommendations were incorporated into the Education Regulations by 1945.

4.43 1945 to the 1960's

In 1946 a new primary syllabus in mathematics was introduced. However it essentially only updated existing syllabi.

In the period to 1949 mathematics teaching declined. Although satisfactory achievement in core subjects was (and still is) a prerequisite to the award of S.C., in fact evidence suggests that this was not enforced in mathematics where substantial numbers failed to reach minimal standards and received S.C. in other subjects. Core mathematics did not lead to other examination options, no suitable text-books were available, a shortage of mathematics teachers developed and no refresher courses were provided. This grave situation was outlined in a remit presented to the NZSSA 1948 Annual Conference (Clarke, 1960, p23).

An Arithmetic Committee was set up in 1948 which criticised testing and brought down recommendations for using standardised tests. It became a national committee in 1949 and reported on the aims, texts, teacher training, organisation and refresher courses of elementary mathematics.

1950 saw the first refresher course. Core mathematics was causing serious concern and traditional methods were blamed for much of the decline. The Auckland course recommended a standard for Core mathematics, more encouragement (especially for boys!!?) to take full mathematics, modern teaching methods, more refresher courses and the production of a text-book by the Education Department.

J.H. Murdoch's book on mathematics teaching outlining 'new' methods was published later that year. It emphasised algebra as a language, classification of spatial ideas in geometry, and graphs as an important pictorial tool. This book reflected the prevailing opinion that the educational aims of mathematics were an appreciation of its social and technological utility, and awareness of the value of ordered systems.

In 1954 J.F. Sharkey produced the asked-for text-book, but the accompanying handbook was vetoed through lack of funds. Thus the book was badly misused whatever its actual value may have been.

Despite a 1956 Review Committee which endorsed the recommendations of the Thomas Report and firmly recommended the continuance of the common core, there was a growing reaction against core mathematics. Too little time had been spent on its development, it led nowhere, too few teachers (most with too low qualifications) gave it a low status and falling standards developed. This spread to the full mathematics. In 1956 an international study showed NZ to have one of the lowest mathematical educations in terms of compulsory mathematics and percentage of school time for its study. In addition Trades Certificates had increased markedly and the weaknesses of apprentices in mathematics was causing wide concern.

Public pressure finally led to the setting up of a Working Committee in 1958. Recommended were: age promotion (but not for gifted children), diagnostic tests and teachers handbooks for all texts. It also recommended a common minimal syllabus for forms I to IV to mend the break between forms II and III, and put forward the spiral curriculum idea. Weaknesses in basic skills, setting out, correct units and rigor were pointed out. It was widely felt that the time spent on mathematics was too small.

Probably the greatest single influence on mathematics education was (and still is) the teacher shortage which developed in the 1950's (see Clarke, 1960, pp93-95). Mathematics was the subject affected most. It led, not only to larger classes and poor courses offered, but also to inadequate qualifications of many mathematics teachers. Furthermore, since the introduction of core mathematics, increasing numbers of primary teachers had no mathematics qualifications (from 20% in 1947 to 64% in 1957). Thus intermediate teachers were sometimes on a par with their pupils in mathematical ability and sophistication.

NZ teacher training at that time was very narrow. Little mathematics training was given, and in-service training was poorly attended. Thus a vicious circle was set up whereby insecure primary teachers were unable to meet pupils' difficulties. The pupil then developed an aversion to the subject for his/her post-primary studies, and thus few prospective mathematics teachers result.

4.44 Modern mathematics and recent trends

In the 1960's NZ began to receive text-books from the U.S.A. (and later Britain) incorporating new mathematics topics. After a period of delay alternative S.C. and U.E syllabi were written and introduced from 1965/1966. Set theory, transformation geometry, functions and structure became accepted emphases in the syllabus. By 1976 the change-over had been completed at all levels (see diagram below).

* The content changes were produced partly to aid understanding, but also, as a result of wider applications of mathematics. Statistics, probability and computing are now covered within the mathematics syllabi.

Another result has been the formation of a continuous programme from infants to senior classes, and evolving rather than static syllabi.

Comparison is difficult, but a major primary evaluation in 1974 showed a decline in multiplication and division skills, but gains in attitude, understanding, discussion and recording. (See also Offenberger, 1976 who offers further subjective evidence for better attitudes but suggests that some useful content has been omitted). The article by Gordon and Couch also suggests that the standard of teaching has increased. The proportion of students opting for mathematics at 5th, 6th and 7th form levels has increased (McGill, 1976) but this could be due to other factors (e.g. mathematics requirements for jobs or poor mathematicians not sitting S.C.).

The teacher shortage continues and, despite refresher courses, must add to the difficulty of upgrading teaching methods.

School Certificate is still a 50% fail examination although it is accepted as a realistic goal for secondary education. This anomaly has resulted in recent years in the development of local certificates, based on syllabi which are, in general, more practically based. In 1975 experimental schemes in 5th form mathematics were implemented in two regions. They allow for a variety of (internally assessed) grades at different levels of mathematics (abstract or practical), rather than the pass/fail in one exam as in S.C.

Other changes in recent years include the development of texts written in New Zealand (notably 'The Shape of Mathematics'); the establishment of the Curriculum Development Unit; and (especially on a local level) the rise of the mathematical associations. This last has led to the local certificates mentioned above, magazines and mathematics competitions.

*(The following draws heavily from an optimistic account of the introduction of the new syllabi by the Mathematics Curriculum Officers (Gordon & Couch, 1976).)

Year	School Certificate	University Entrance	Bursary/Scholarship
1964	Mathematics A & B	Mathematics A & B	Additional Mathematics
1965	Mathematics A & B Pilot Scheme A & B		
1966	Mathematics A & B Pilot Scheme A, B & C (Transf. Geometry)	Mechanics Mathematics A & B Interim Option A & B	
		Mechanics Mathematics (1 paper) Alternative Maths	
1968		Applied Maths (Mechanics & Stats) Pure Mathematics Alternative Mathematics	
1969	Mathematics Pilot Scheme A & B (C combined with B)		
1970	Ordinary Mathematics (A & B) Alternative Mathematics (A & B)		
1971	Ord. Maths A & B Alt. Maths A & B		

Year	School Certificate	University Entrance	Bursary/Scholarship
1972	Ord. Maths A & B	Applied Mathematics Alternative Maths Pure Mathematics	
1973	Mathematics (1 paper) Ord. Maths (1 paper)	Mathematics (Applied & Alternative combined) Traditional Maths	
1974	Mathematics Ordinary Maths	Mathematics Traditional Maths	Pure Mathematics Applied Mathematics
1975	Mathematics Ordinary Maths	Mathematics Traditional Maths	Pure Mathematics Applied Mathematics
1976 & ff	Mathematics	Mathematics	Pure Mathematics Applied Mathematics

4.5 FORCES IN MATHEMATICS EDUCATION

This section attempts to link some of the influences mentioned in section 4.1 – 4.4 to the assumptions made about mathematics education in chapter 3.00. The connections are neither necessary nor sufficient, rather they are suggested as possible contributing factors and should be taken as hypotheses on which further research could be based.

Four types of forces are distinguished (although exclusive divisions are not intended): those due to mathematical developments; those due to social changes; those relating to general educational influences and those due to psychological theories.

4.51 Mathematical forces

Some aspects of mathematical education are directly attributable to episodes in the history of mathematics.

Euclid's Elements undoubtedly shaped classical education in Britain and Europe. Its effect on the content of geometry courses has gradually decreased, but the 'mathematics improves the mind / teaches you to think logically / disciplines the child' rationales are directly attributable to it. Why should Euclid have had such a great effect? Partly because it was a definitive work: a section of mathematical knowledge in completed and ordered form. Partly, also, because it was, for hundreds of years, the total knowledge available - thus, because it remained definitive over time, some extra credit was attached to it. Part of the reason that it did remain the last word in geometry was because it defined the bounds of geometry to be just this work: it was not until Bolyai and Lobachevsky investigated alternative axioms for geometry in the 19th century that new bounds were accepted.

These characteristics of the Elements are being repeated today in Bourbaki's works. It is difficult to estimate the influence of these works but there seems to be some link between Bourbaki and the 'new math' where the structure of mathematics is taught. Bourbaki writes only on mathematical topics which have been largely completed and on which it is possible to be definitive. One wonders, therefore, whether the inclusion of some piece of mathematics in Bourbaki acts towards making it definitive and closes further avenues of development, as Euclid did with geometry.

Deductive method has been valued since Greek times and its prominence in mathematics education may have been boosted by Descartes and again by modern foundationalists. However, it is not true that all famous mathematicians have made the deductive method their explicit modus operandi (see Poincare's account of his own thinking (Newman, 1956, Vol IV, p 2041)).

The development of rigor is attributed to three influences: the long-fulfilled desire to put calculus on a sound basis (and hence the increased importance of doing so); the unresolved paradoxes of the infinite and Cantor's theories; and the increasing importance of the foundation debates. These have brought rigor to the forefront of mathematicians' concern and thence into mathematics education.

The history of mathematics is inextricably linked with applications. It is surprising, therefore, that mathematics as a discipline is not more integrated. An explanation is that pure mathematics as studied in the universities has (in recent times) gained a very high academic status. The reasons are not clear but may be linked with social ones: the upper classes in Britain for example had time and money to indulge in pure research compared with the bourgeois, practical origins of applied mathematics: compare Newton (calculus, upper class) with Napier (logarithms, lower class). It is only recently (the last 50 years) that Centres of Applied Mathematics have been established and accepted. Thus the feeling has developed that 'pure' mathematics is of a higher order than 'applied' mathematics and this 'class consciousness' has been transmitted to education.

This explanation would also account for the importance of rigor, an aspect of mathematics which has only been important in pure, abstract research.

Computer development is undoubtedly affecting mathematics education. Not only has a new vocabulary developed (e.g. algorithm, flow-chart), but so also have new ways of doing mathematics (e.g. iterative solutions) and new ways of thinking (e.g. flow-chart decision making).

4.52 Social forces

The predominant social factor in education in the countries considered has been the dramatically increasing school population.

This is not simply a matter of more pupils, but also a matter of different sorts of pupils. In Britain in particular girls' rights to education were recognised long after boys', and in New Zealand the introduction of education for all meant catering for those with poorer attitudes and enviroments. Diverging extremes of standards and larger numbers affected teaching methods in general, but what effect did they have on mathematics education in particular?

In recent times the slow, but continuing, de-empasis on examinations has been one result. (The slow speed of this change may be partly attributable to a class need to retain elites and partly to the growing competitiveness of society). This in turn has opened the way for a broader curriculum towards vocational or motivating topics.

The vocational futures of students have changed so that, for example, the Victorian era of mathematics education for clerks is giving way to computer oriented jobs. (It has been estimated that, by 1980, 1 in 10 of the workforce in Britain will be associated with computers).

Large school populations have contributed to two assumptions in chapter 3.00.

The acceptance by teachers of the uniformity assumption (that students can all be taught the same way - section 3.7) has been largely a pragmatic reaction to a situation where the alternative has been impossible.

Also the increasing awareness of the need to teach positive attitudes has been the result of increasing manifestations of the dissatisfaction with mathematics.

Another social factor has been the increasing awareness of education as a national asset. Coupled with the technological boom, the industrial and scientific need for a good mathematical education is well accepted (cf the low status origins of applied mathematics education in Britain). The Sputnik boost to American mathematics has become a cliche. Such feelings coupled with increased research and funding have contributed towards the importance of pure mathematics in education through the myth of abstraction leading to generality. A myth which is plausible but false because transfer and application are not automatic.

Increased mobility of the population may be having a small effect on mathematics education, not just by requiring curricula in different areas to be able to be matched, but also by teachers and practitioners spreading their knowledge. The end result tends to uniformity and reinforces assumption 3.7.

The power of education in general (and mathematics education in particular) has increased because of the increasing complexity of society. This is evidenced by the ability to exploit and dominate people merely by withdrawing adequate education. For example in South Africa where Bantu Education has become the focus of protest, being correctly identified as one of the most powerful weapons of apartheid. As far as mathematics is concerned, the mathematisation of society has put the focus on the knowledge/skills/attitudes assumption (3.6). More mathematical skills are perceived as necessary for people to function in our technical cities -- e.g. maps, tables and statistical judgements all need to be understood. Thus skills learning becomes more important.

Another influence is the effect of unemployment for pure mathematics graduates. After the World War II boom mathematical vocations have increasingly been taken by applied mathematicians (engineers, computer scientists, etc) with the result that pure mathematicians are unemployed. In 1972 the Mathematics Magazine issued a statement warning intending pure mathematics majors of their uncertain futures (Vol 45 p.165). Most find jobs in teaching and increasingly higher qualified mathematicians are teaching at lower levels and becoming involved with curriculum change. The level of rigor, modern mathematics and symbolism have all been affected by this.

Finally society, significantly over the last century, has been undergoing an increasing rate of change. The (perhaps unconscious) need for stability may be reflected in the search for foundations of mathematics and in curriculum changes toward structure, rigor and uniformity.

4.53 Educational forces

The strongest influence arising from the practice of education in the countries considered has been examinations.

National examinations impose not only fixed curricula and content (assumptions 3.3 and 3.7) but also fixed sequences of instruction over a long period (assumption 3.1). Furthermore they determine what constitutes good mathematics (assumptions 3.4, 3.6 and 3.8).

Mathematics appears to be simple to evaluate and hence its wide use in psychological testing and educational research. However the danger is that only the easily evaluated aspects of mathematics will be tested. For example deduction rather than intuition or creativity, correct answers rather than elegant methods, facts and skills rather than attitudes. Because these are what are tested they become what is taught. Since, as yet, it is impossible to effectively evaluate "thinking" national exams are still exerting this power.

As mentioned above, the apparent ease with which mathematics ability may be analysed and each separately evaluated has led to this topic being the subject of most educational research.

This has had two effects. First the diversity of research has led to a diversity of 'new', 'experimental' or 'pilot' programmes which have been suffered by students and teachers alike. This has added to the generally poor teaching already present (see below). Secondly, the nature of the research has created the nature of the teaching: because mathematics can be analysed, it is accepted that such analysis reveals the true 'nature' of mathematics. I believe that mathematical ability is an overall 'point of view' rather than a sum of discrete skills. A clear example of analysis becoming the teaching mode is given by the Behavioural Objective school and its consequences in the classroom.

Partly because of its superficial simplicity, partly because of traditional example/exercise instructional methods, and partly because of many students' fear of mathematics, it was possible to teach mathematics badly and get away with it as a teacher. (This is less so now with large classes and different controls on students). Teacher training institutions have always been under pressure to train many teachers quickly, and thus teachers are trained to a minimum level of competence. For mathematics this minimum level is very low and thus, compared with other subjects, mathematics is taught badly. Some elements of this type of teaching are reflected in the assumptions stated: rote learning (3.6), simplistic evaluation (3.8), teaching from the book (3.1 and 3.7) and uncritical use of words and symbols (3.3).

There is another consequence of poor teacher training: the vicious circle of poor mathematics education leads to poor mathematics education leads to poor mathematicians entering teachers college leads to poor mathematics teachers leads to poor mathematics education. This has been an identified problem in mathematics in New Zealand (see section 4.43) and England (see Bellis, 1972, p180).

4.54 Psychological forces

It is not the purpose of this section to detail the implications to mathematics education of each learning and instructional theory. First some general effects of psychological theories will be mentioned and secondly some hypothetical relationships between some theories and the assumptions in chapter 3.00 will be put forward. This latter section further illustrates the divergent sources of the above assumptions.

It has already been noted that mathematics is the subject which lends itself to analysis and experimentation most easily. In addition the rise of psychological theories over the last century has led to an overconfidence that educational problems are solvable -- if only the correct theory is applied in the correct way. The hasty introduction of 'new math' (see section 4.33) can partly be attributed to these attitudes: having found better content it was assumed that it could be taught adequately if it was designed according to an appropriate theory with minimal teacher retraining. The importance of psychological theories has also generated a confidence in the possibility of all students succeeding in school: that if the correct theory is applied everyone will learn mathematics (assumption 3.7).

The way in which psychological theories have propagated is also significant. A theory is often manifest as a school (in the academic sense) and even as schools (in the physical sense). Thus education is often pursued under the influence of only one theory: examples are the influence of Dewey in America, Piaget in the New Zealand primary system, Steiner schools and Skinnerian schools (or even communities). The eclectisism of, for example, New Zealand state secondary schools is a haphazard one rather than an organised selection of the useful aspects of each theory. Thus several different (but fixed) sequences are psychologically justified (assumption 3.1); attitude learning may be neglected if an instructional method is thought to be psychologically valid (assumption 3.6); and sometimes what constitutes learning is defined by the theory, e.g. specific 'behaviour' over 'understanding' in some Skinnerian-based theories (assumption 3.8).

Some specific hypotheses about particular theories and their effects in creating the assumptions mentioned are as follows:

Thorndike's Connectionism codified the 'practice makes perfect' idea in the Law of Exercise. This strengthened the educational practice of many examples as a substitute for teaching, thus specific knowledge and skills were emphasised (assumption 3.6).

The behaviour modification practices arising from Skinner's work gave rise to programmed learning where sequences are fixed (assumption 3.1). Behaviour is discrete and using it in an instructional process emphasises the discrete aspects of the subject (assumptions 3.3 and 3.8).

Bruner, Ausubel and Gagne's theories all place emphasis on correct sequencing (assumption 3.1) and the structure of knowledge. This latter emphasis partly influenced the rise of 'new math', the content of which is based on structural assumptions. Structural ideas also relate more closely to deduction than intuition (assumption 3.4).

The current questioning of assumption 3.6 (on attitute learning) may be linked to the general rise of humanistic psychology of the Rogerian type.

Detailed influences of psychological theories on mathematics education in America are given in sections throughout the NCTM book on the history of mathematics education (Jones, 1970).

5.00 SUMMARY

There are no conclusions, or rather, every sentence is a conclusion. It is better to describe this thesis as a process. A process whereby a philosophical analysis of syllabus aims has developed into an examination of alternative meanings for the aims, an exposition of hidden assumptions and an investigation of the origins of the assumptions.

5.1 Philosophical Analysis

In the introduction to Chapter 2.00 the Wittgensteinian analysis on which the thesis is modelled is described: analysis based on the language-game conception where any statement is seen as a source of several possible meanings. The value of the analysis rests in the exploration of these meanings by examining their consequences in different contexts.

Thus in the suceeding sections the aims of the New Zealand Forms I - IV mathematics syllabus are discussed for possible meanings and the consequences examined.

The first section focuses on the meanings of 'observe', 'discover', 'concept', 'logical' 'generalisations' and 'mathematical experiences' in the context of mathematics education. The second section deals in detail with the 'principles underlying the structure of mathematics'. In particular the degree of universality is seen as important in determining the consequences of the aim.

The third section highlights the difficulties surrounding the independent definition of 'liking' mathematics and 'being successful' in mathematics. The fourth section examines the utility of mathematics for a general education. Again the universality criterion is seen to be important, as is the way in which teachers use the word 'necessary' when applied to mathematics.

Section five considers the value of mathematics and the role of applied mathematics. There is argument about how to evaluate the importance of mathematics, let alone what that importance is. The effects of the applications of mathematics depend on the student population considered.

Section six looks at differences between the syllabus for Forms I-IV and that for higher forms. Specifically examined are the narrowing of the aims (in response to examination pressures) and the increasing importance of mathematics as an academic discipline.

5.2 Assumptions

The introduction to Chapter 3.00 details the discovery of the assumptions to be examined in later sections. It then presents an argument for the need for challenging assumptions in mathematics education based on poor academic results and negative attitudes towards mathematics.

The first assumption examined is the need for sequencing in mathematics instruction. Five types of ordering are described. The types are based on: the structure of mathematics, the skills necessary for mathematics, the self-generation of mathematics, the history of mathematics and the psychology of learning. Their advantages and drawbacks are discussed, along with the issues to be examined when constructing a teaching sequence.

The second section deals with the isolation vs integration of mathematics with other subjects as epitomised by the problem-centred and mathematics-centred approaches to instruction. Comparison of these is related to how they fulfil the aims of Chapter 2.00. Applied and pure mathematics are also discussed, and an argument presented for combining them as 'applicable mathematics'.

Section 3.3 deals with the function of terminology and symbolism in mathematics and mathematics education. The case is made for the use of them as 'Wittgensteinian' tools whereby terms and symbols are recognised as having changing and developing referents. This can be used by teachers and students to enhance their learning.

Deduction and intuition are next considered, the dominance of the deductive mode examined and its usefulness questioned. Finally a case is presented for less reliance on deduction and more on intuition to achieve a more balanced and realistic mathematics education.

The fourth assumption is that maximum rigor is the best mathematics. The confusion between doing mathematics and learning mathematics is important here and rigor is seen as a method of teaching rather than a subject of it.

Section 3.6 describes the different levels of mathematics education: knowledge, skills, and attitudes. The importance of each is observed and student and subject-centred teaching evaluated using these as criteria.

The universality of mathematics education raised in sections 2.2 and 2.5 is subject to detailed examination in the seventh section. The existence, identification, teachability and value of universal aspects of mathematics are considered in turn. The reciprocal nature of mathematics and culture (parallel to that of language and it's use in Wittgensteinian philosophy) is an important consideration. The time-problem of teaching for an unknown future is also discussed.

Finally the concept of 'good mathematics' is examined. This includes a description of mathematics as a language-game.

5.3 Historical Origins

The fourth chapter examines the history of mathematics, and mathematics education in the United Kingdom, USA and New Zealand for the origins of the assumptions discussed in the previous chapters.

In the history of mathematics the influence of Euclid, the applied mathematics inspiration, the increasing desire for rigor and solid foundations, the technological advances and the explosion of knowledge are among the most important influences with contemporary effects.

In the United Kingdom the class basis of education, the influence of Newton and mathematical physicists and the overwhelming influence of examinations have all seriously affected mathematics education. The world-wide increase in school populations and the need for popular education have also been important.

American mathematics education was initially practically oriented and accepted as necessary for all. The European rather than English influence predominated and the effects of the depression, World War II and competition as a super-power have all been significant. The 'new math' movement originated here and its rise and reaction have been felt most severely in the USA.

Section 4.4 deals with New Zealand mathematics education the effects of America and Britain being most important. Most changes centred around education for all and the need for non-academic courses which would be effective. The teacher shortage is isolated as an important restriction on development. The gradual introduction of modern mathematics is detailed and recent trends of locally developed schemes noted.

The final section of the thesis reorganises the above information under the different sorts of forces which have shaped the assumptions mentioned: namely historical, social educational and psychological forces. The role of Euclid's Elements, applications of mathematics and computer development in developing the assumptions is detailed in the first part. The social forces of education for all and of education as a national asset or power source are considered as having important influences. Increasing population mobility, unemployment of mathematicians and the increasing rate of change of society are also mentioned.

The educational forces noted are examinations, the adaptability of mathematics to research, the nature of mathematics teaching as opposed to that of other subjects, and teacher training. Psychological theories have had two general effects: overconfidence in the ability to solve educational problems and the rise of theory-oriented schools rather than the eclectic use of the results of research. Specific effects of particular theories are finally hypothesised.

Thus some assumptions in mathematics education have been uncovered, discussed and their origins identified.

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6.00 BIBLIOGRAPHY

Bellis, B.T. — 'Whatever Next: Presidential Address to the Mathematical Association Annual Conference' in The Mathematical Gazette 1972 56:179

Bender, E.A. — 'Teaching Applicable Mathematics' in American Mathematical Monthly 1973 80:302

Bentley, A.M. — 'Symmetry in Pattern Reproduction by Scottish and Kenyan Children' in Journal of Cross- Cultural Psychology 1977 8:415

Blaire, E — 'Mathematics, Science of Possibility - A Critical Review' in International Journal of Mathematical Education 1973 4:413

Blakeley, B.H. et al — Why, What and How? Some Basic Questions for Mathematics Teaching, The Mathematical Association, Leicester, 1976

Boas, R.P. — 'The Cosrims Reports' in American Mathematical Monthly, 1970 77:623

Bornstein — 'Colour Vision and Colour Naming: A Psychological Hypothesis of Cultural Difference' in Psychological Bulletin 1973 80:257

Bourbaki, N — Elements of Mathematics, Addison-Wesley Publishing Co., London, 1966 ff

Clarke, A.E.E. — Mathematics in Post Primary Schools: A Survey of Mathematics in N.Z. Post Primary Schools Covering the Period 1942-1959, Unpublished thesis for MA in Education, Auckland University, 1960

Couch, L.W. — see Garden R. A.

Coxon, A.F.M. — A Comparison of the 'Pilot' and 'Traditional' Courses in Mathematics at the High School Level, Unpublished thesis for MA in Education, Otago University, 1972

Crespy, H.V. — A Study of Curriculum Development in School Mathematics By National Groups, 1950-1966, Unpublished thesis for Ed.D, Temple University, 1969. Abstract in D.A.I. 31:923-A.

De Bono, E. — The Use of Lateral Thinking, Penguin, Harmondsworth, 1971

Department of Education — 'Applied Mathematics Examination 1977' in Education Gazette, 15 June, 1977 p266

--- — Item Bank: Mathematics Level 6, Curriculum Development Unit, Wellington, 1973

--- — 'School Certificate Examination' in Education Gazette, 1 April, 1972, p118

--- — Syllabuses for Schools: Mathematics: Forms 1 to 4, Wellington, 1972

--- — Teaching of Mathematics in Forms 1-4: Hogben House Conference: November 1976, Wellington, 1976

--- — University Entrance Board: Mathematics Prescription' on Education Gazette, 1 August, 1972, p301

--- — 'University Entrance Examination' in Education Gazette, 30 September, 1976 p418

--- — 'University Entrance Examination' in Education Gazette, 15 October, 1976 p464

Deregowski, J.B. — 'Difficulties in Pictorial Depth Perception in Africa' in British Journal of Psychology, 1968, 59:175

Eves, H. — An Introduction to the History of Mathematics (Revised Edition) Holt Rinehart and Winston, New York, 1964

Ford, B. and Hall, G.G. — 'Model Building -- an Educational Philosophy for Applied Mathematics' in International Journal of Mathematical Education in Science and Technology, 1970, 1:77

Frand, J.L. — Mathematical Understanding and Mathematical Proficiency: An Analysis of Alternative Goals and Progress in Mathematics Education, Unpublished Ph.D. thesis, University of California Los Angeles, 1976. Abstract 1976. Abstract in D.A.I. 37:2695-A.

Freire, P. — Pedagogy of the Oppressed, Penguin, Harmondsworth, 1972

Freudenthal, H. — Mathematics as an Educational Task, D. Reidel Publishing Co, Dordrecht, 1973

Hall, G.G. & Whitcombe, A.R. — 'Applied Mathematics in the Secondary School' in International Journal of Mathematical

Education in Science and Technology, 1970, 1:235

Garden, R.A. & Couch, L.W. — 'School Mathematics Today' in Education, 1976, 25:3

Grabiner, J.V. — 'Is Mathematical Truth Time-Dependent?' in American Mathematical Monthly, 1974, 81:354

Griffths, H.B. — '1871: Our State of Mathematical Ignorance' in American Mathematical Monthly, 1971, 78:1067

Hall, G.G. — see Ford, B.

Hatcher, W.S. — Foundations of Mathematics, W.B. Saunders & Co, Philadephia, 1968

Hayman, M. — 'To Each According to His Needs: Presidential Address to the Mathematical Association, 1975, 59:137

Her Majesty's Inspectorate — Mathematics, Science and Modern Languages in Maintained Schools in England, DES, London, 1977 (Mathematics section reprinted in Mathematics Teaching, 1977, 79:50)

Holt, J. — Instead of Education, Penguin, Harmondsworth, 1977

Howson, A.G. — 'Mathematics -- the Fight for Recognition' in Mathematics in School, 1974, 3:6:7

Howson, A.G. — 'Charles Godfrey (1873-1924) and the Reform of Mathematical Education' in Educational Studies in Mathematics, 1973/4, 5:157

Jones P.S. et al (eds) — A History of Mathematics Education in the United States and Canada, National Council of Teachers of Mathematics 32nd Yearbook, NCTM, Washington DC, 1970

Kerr, E. — 'Some Thoughts on the Educational System and Mathematics Teaching' in The Mathematical Gazette, 1977, 61:417:157

Kline, M. — 'NACOME: Implications for Curriculum Design' in The Mathematics Teacher, 1976, 69:449

--- — Mathematics in Western Culture, Penguin, Harmondsworth, 1972

--- — Why Johnny Cant Add: The Failure of the New Math, St Martins Press, New York, 1973

Korner, S. — The Philosophy of Mathematics: An Introductory Essay, Hutchinson University Library, London, 1960

Kuhn, T.S. — The Structure of Scientific Revolutions, University of Chicago Press, Chicago, 196?

McGill, I. — 'Trends in Seventh-form Subjects 65-74' in Education, 1976, 8:11

Naftel, A.E. — Individual Instruction, Attitude and Achievement in Mathematics Learning, Unpublished thesis for MA in Education, Massey University, 1974

Newman, J.R. (ed) — The World of Mathematics (4 volumes), George, Allen & Unwin, London, 1956

Newsom, J.H. et al — Half Our Future: A Report of the Central Advisory Council for Education (England), DES, London, 1963

Offenberger, J. — 'Mathematics in Technical Institutes' in Education, 1976, 25:7:11

O'Neill, P.J. (ed) — The Shape of Mathematics, AH & AW Reed, Wellington, 1969

Ormell, C.P. (a) 'Mathematics, Applicable versus Pure-and-Applied' in International Journal of Mathematical Education in Science and Technology, 1972, 3:125

--- (b) 'Mathematics, Science of Possibility' in International Journal of Mathematical Education in Science and Technology, 1972, 3:329

Passmore, J. — A Hundred Years of Philosophy, Gerald Duckworth & Co, London, 1957

Pirsig, — Zen and the Art of Motorcycle Maintenance, Corgi

Polya, G. — How To Solve It: A New Aspect of Mathematical Method, Doubleday & Co, New York, 1957

Skemp, R.R. — The Psychology of Learning Mathematics, Penguin, Harmondsworth, 1971

Stenhouse, (a) 'Education and Knowledge: Divergent Thinking, Language Games and Practicality' in Active Philosophy in Education of Science, Allen & Unwin, London, in press

--- (b) 'Good Persons, Good Teachers and Language Games' in Educational Philosophy and Theory 1969, 1:41

--- (c) 'O' Connor's Paradox and the Teaching of Educational Philosophy' in British Journal of Educational Studies, 16:243

--- (d) 'Scientific Creativity: "Normal" or "Revolutionary"' in Australian Journal of Education, 15:171

--- (e) 'Wittgenstein as an Educational Philosopher' in Active Philosophy in Education and Science, Allen & Unwin, London, in press

Toffler, A. (ed) Learning For Tomorrow: The Role of the Future in Education Vintage Books, Random House, New York, 1974

UNESCO The UNESCO Courier, January, 1976 UNESCO, Paris, 1976

Whitcombe, A.R. see Hall, G.G.

Wilder, R.L. 'Mathematics and Its Relations to Other Disciplines' in The Mathematics Teacher, 1973 66:679

Williams, H.B. 'A History of Teachers of Mathematics' in The Mathematics Gazette, 1971, 55:272

Wittgenstein, L. Remarks on the Foundations of Mathematics (G.H. von Wright, R. Rhees and G.E.M. Anscombe -- eds) (G.E.M. Anscombe -- translator) Basil Blackwell, Oxford, 1964

Ziedins, R. 'Conditions of Observation and States of Observers' in The Philosophical Review, 1956, LXV:3

Zulauf, A. The Language of Mathematics and the Mathematics of Language: An Inaugural Lecture, University of Waikato, Hamilton, 1972

7.00 APPENDICES

7.1 Extracts from Syllabuses for Schools: Mathematics: Forms I to IV

7.2 Extracts from Teaching of Mathematics in Forms 1-4: Hogben House Course: November 1976

7.3 Extracts from The Education Gazette -- Examination Prescriptions

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7.1 EXTRACTS FROM SYLLABUSES FOR SCHOOLS: MATHEMATICS: FORMS I TO IV

p.4.: Mathematics: Forms I to IV

".... Programmes of work based on this syllabus will stress the development of the pupil's powers of ordered and systematic thinking, precision in expression of ideas by word and symbol, skill and accuracy in the manipulation of symbols; and they will seek to relate these powers and skills to situations the pupil will encounter in the world in which he lives.

The Nature of Mathematics

Mathematics may be thought of as an idealisation of the real world; a model that can be used to represent relevant objects and events, and to investigate relationships between them. It may also be regarded as a logical structure built on assumptions.

Some mathematicians have pursued ideas and developed theories and methods that have been applied successfully to a wide range of problems. Some have re-examined the basic assumptions and logical structure, and clarified them until they formed a more consistent and general system. Other mathematicians have followed challenging lines of thought for their own sake, and by so doing they have experienced the excitement of discovering patterns and relationships, of perceiving and applying basic principles, and of appreciating the economy and elegance of logic.

Mathematics is an activity. It has been constructed and developed by man and is continually evolving. This syllabus endeavours to relate the teaching of mathematics from Form I to Form 4 to this conception of the nature of the subject.

General Aims of the Syllabus

Mathematics is playing an increasingly significant part in many aspects of our culture and is therefore a necessary component of the general education of those who live in this culture.

The general aims of this syllabus are:

1. To provide mathematical experiences which enable pupils to make observations, to discover patterns and relationships, to develop concepts, to draw logical conclusions, to express thoughts accurately, and to form generalisations.

Opportunity has already been provided in all classes up to Standard 4 for children to develop an understanding of simple mathematical ideas. These ideas now form the basis of a wider range of study leading to the development of general principles which operate in mathematics. Greater emphasis is placed on the need for refining definitions and using more precise language - symbolic as well as verbal - particularly in statements from which logical deductions may be made.

2. To develop further an understanding of the principles underlying the structure of mathematics, and the ability to apply these principles to wider fields.

This aim is closely allied to the first. The Infants to Standard 4 syllabus provides for the early introduction of basic mathematical principles and for their systematic extension and refinement. The process is continued through this syllabus. Principles previously acquired are now extended, not only to the set of whole numbers, but to other sets of numbers and to other branches of mathematics, such as algebra, geometry and trigonometry, so that pupils may see a unity and structure within the subject as a whole. As pupils are led and encouraged to use principles in the solution of problems, they realise that an understanding of principles adds immensely to power and economy of learning.

3. To develop a liking for and a lasting interest in mathematics.

The pupil's attitude towards mathematics is a critical factor in determining his success in the subject and his readiness to pursue it to higher levels. The pupil who experiences the intellectual enjoyment and personal satisfaction of discovering laws of number and space, of perceiving patterns and forms, and of finding the solution to problems will be stimulated to continue his studies. The teacher's general approach to the subject, together with provision for individual differences in needs, abilities and interests, is the key to achievement of this aim.

4. To develop the basic mathematical knowledge, skills and understanding necessary for everyday living and for effective citizenship.

All pupils need to be able to handle numbers and spatial ideas in their day-today experience. In addition, as adolescents and adults, they will need mathematics to help them understand the world they live in and to meet their obligations as citizens.

5. To help pupils appreciate the importance of mathematics in their future studies and vocations.

Increasing demands will be made on the mathematical competence of all pupils in their future careers. Mathematics has long been recognised as indispensible to scientists, engineers, technologists, technicians and skilled tradesmen. New applications of mathematics are being found in all fields of thought, and new areas of mathematical study are being developed to meet new needs. Mathematical knowledge is important to economists and social scientist -- in fact, to almost all who enter professional, administrative and business careers. These trends have influenced the development of this syllabus.

Specific Objectives

The general aims of the syllabus express our mathematical aspirations in terms of the needs of pupils and society. If they are to be achieved, it is necessary to state the specific objectives towards which the performance of pupils should be directed. These objectives determine the content that is selected and the learning procedures that are used.

It is desirable that pupils should:
Know and understand mathematical facts, processes and principles.
Develop an understanding of the logical structure of mathematics and the nature of proof.
Perform computations with understanding, accuracy and efficiency.
Develop systematically their ability to solve problems.

"Develop an attitude of inquiry.
Become interested in and enjoy the study of mathematics.
Develop an appreciation of the grace, economy and breadth of application of mathematics.
Develop an awareness of the importance of mathematics in daily life.
Develop sound methods of learning and habits of independent study.

.....

Emphasis on Structure

The organisation of the content of this syllabus is based on the study of mathematical systems which make up the strucure of the subject.

1. The nature of the elements of a system and relations among them examined. These elements may be numbers, points, lines, statements, arrays.

2. Relations between different sets and elements of different sets are studied.

3. The operations on elements of the given set are studied in relation to their properties and to each other.

4. Operations on sets themselves are considered.

5. By generalisation, the principles involved in the operations are discovered, and the part they play in the structure of of the subject is realised.

...

6. The principles are applied to mathematical problems and extended to new situations.

The Study of Geometry

The initial steps in the study of geometry are practical, being based on constant employment of keen observation, handling of materials and apparatus, and encouragement of intuition, discovery and inductive reasoning. A gradual growth of deductive reasoning follows, to prove apparent truths and to lead to the establishing of further properties and relationships among these. The emphasis is on experience leading to thought and verbal expression. Various techniques are then mastered and applied to geometric problems. This culminates in a 'feeling for space', and a critical understanding and appreciation of the nature of geometry as a mathematical abstraction developed from a physical model.

Two alternative approaches to the study of geometry in Forms 3 and 4 are given in the syllabus: the traditional development derived from Euclid, and an approach through transformations of the plane. Either one may be taught, and it is appreciated that while the assumptions made and the sequence of development of the traditional approach at this elementary level are generally established, this is not so true of the alternative transformation geometry. Whichever approach is used, the treatment should be consistent and systematic.

....."

7.2 EXTRACTS FROM TEACHING OF MATHEMATICS IN FORMS 1-4: HOGBEN HOUSE COURSE: NOVEMBER 1976

p.2.: " Major Recommendations of the Course

....

3. That the syllabus should be a concise, coherent and meaningful statement of the aims, objectives and content recommended for all children in Forms 1 to 4. This implies that teachers, on reading it, know as precisely as possible what is intended with respect to aims, content, emphasis, priority, level and interdependence.

4. That the provision of alternative syllabuses in Forms 3 and 4 Geometry be discontinued and that one geometry syllabus be included.

5. That the second paragraph of "The Study of Geometry", p 8, be replaced by:
Intuitive understandings of spatial relations, similarity and congruence are readily extended and refined through a study of geometry based on transformations of the plane. Understandings in some topics such as circle geometry, on the other hand, are more readily acquired when teaching is based on the traditional Euclidean approach. It is desirable that a course in geometry based on this syllabus be a consistent one in which the elements are chosen to allow learning to proceed most easily and effectively. For this reason it is considered no longer necessary to have alternative courses in geometry in the syllabus.

...."

p.5.: " THE SYLLABUS

A Aims and Content

It was agreed that a statement was required on "The Role of Mathematics in the Curriculum" to be inserted in the syllabus after the section "The Nature of Mathematics". The purpose of such a statement was to emphasise that mathematics is not just a utility subject but one which stands in its own right and even underpins others.

A statement along the following lines was required

"The Role of Mathematics in the Curriculum".

"It is recognised that mathematics holds a unique position in the School Curriculum. The uniqueness is based on the hope that its study will result in a mind trained to think in abstract terms. This task for the teacher is an extremely difficult and demanding one and must be conducted with patience, sympathy for the growing child, and a good measure of enthusiasm.

Notwithstanding the abstract nature of much of mathematics every possible opportunity should be taken to relate it to the outside enviroment and even to allow for its integration with other school subjects where appropriate. By doing this, the utilitarian aspects of mathematics will be made more meaningful to all children, but especially to those who have had limited success in the subject. If carefully taught, mathematics should have repercussions on the spoken and written English of the children, for the training they get in orderly thinking should be extended to an orderly presentation of facts and ideas in speech and writing.

Other unique qualities include:-

- i It provides techniques to solve problems.
- ii It gives an opportunity to develop deductive and inductive reasoning.
- iii It demands precision.
- iv It gives an appreciation of the nature of a discipline.
- v It can give an immediate measure of satisfaction.
- vi It provides opportunities for convergent and divergent thought patterns.
- vii It underpins many practical aspects of other subjects.
- viii It gives pupils one component which widens their physical and social enviroment.

Mathematics is uniquely sequential in nature. It is often within the period of Forms 1-4 that the skills, abilities and attitudes of earlier years are confirmed. Teachers of the subject therefore must acknowledge this fact and ensure that the possibilities of future study in the subject are enhanced."

It was agreed that a revision of the Aims was required. It was felt that, while the nature of the subject had not changed, the requirements of society and the needs of the children had. It was also agreed that specific objectives would be more relevant if they were associated with parents' aims. We therefore have redesigned (and added 1 further aim) the section "General Aims of the Syllabus", pages 2-6, to include:

- a Aim
- b Statement of Elaboration
- c Specific goals appropriate to the Aim.

"General Aims of the Syllabus"

"Mathematics is playing an increasingly significant part in many aspects of our culture and is therefore a necessary component of the general education of those who live in this culture.

The general aims on which this syllabus is based are:-

Aim.1.
To provide mathematical experiences which can enable pupils to describe observations, discover patterns and relationships, to develop concepts, to draw logical conclusions, to communicate accurately, and to form generalisations at the appropriate level.
Statement
Para 1 Page 5
Objectives
To develop in pupils the ability to:
a detect patterns in sets of numbers, shapes and ideas;
b identify mathematical data in their observations;
c distinguish between an observation and an inference;
d weigh evidence and be prepared to change their point of view according to this evidence;
e order and process facts to lead to the solution of a problem;
f supply appropriate criteria to original data in order to organize and evaluate the material, and to draw conclusions;
g use appropriate language in recognition, description and classification: use the symbols and notation of mathematics to show it is a language with rules and vocabulary;
h distinguish facts and their interrelationships;
i apply the intellectual skills necessary to suggest hypotheses based on original data;
j understand the idea of a mathematical model.

Aim.2.
To foster an understanding of the principles underlying the structure of mathematics, and the ability to apply these principles to wöder fields.
Statement
Para 3 Page 5
Objectives
To develop in pupils the ability to:
a think flexibly and approach new situations with confidence;
b understand the fundamental number relationships;
c know and implement laws of mathematical operations;
d carry out normal mathematical procedures based on an understanding of their developments;
e appreciate the normal algorithmic processes and use them with understanding in the solution of problems;
f apply mathematical principles to new situations;
g relate mathematics to other disciplines in the curriculum;
h collect original data;
i appreciate mathematical criteria;
j analyse and evaluate new data.

Aim.3.
To encourage positive attitudes towards mathematics.
Statement
Para 5 Page 5
Objectives
To develop in pupils the ability to:
a organize their own work and learning, i.e. having learned how to learn;
b be creative - to pose their own questions, invent their own methods and symbols, and use their imagination;
c enjoy and be interested in mathematics and have a good attitude towards the subject;

d have confidence and resourcefulness when working at mathematics;
e participate willingly in mathematical activities;
f be creative and imaginative in their attitudes towards mathematics;
g gain intellectual stimulus from the study of mathematics;
h derive satisfaction from achieving conclusions to mathematical processes;
i appreciate the aesthetic appeal of mathematics.

Aim.4.
To develop the basic mathematical knowledge, skills and understanding necessary for everyday living and for effective citizenship.
Statement
Para 7 Page 5 - but to include "confidently" after the words "spatial ideas"
Objectives
To enable pupils to:
a relate mathematical techniques to everyday life, and apply certain concepts and skills in realistic problems and investigations;
b develop the confidence to analyse and evaluate new data;
c develop an attitude of critical awareness;
d develop a conviction of the general validity of mathematics;
e develop physical skills in the use of mathematical instruments and aids;
f develop familiarity and confidence with measurement and estimation.

Aim.5. (New Aim)
To rpovide a pool of mathematical skills, knowledge and understanding which will be a sound base for future study.
Statement
This aim is clearly directed at future study and emphasizing the sequential nature of the subject.
Objectives
To train pupils to:
a work with enthusiasm, determination and concentration on the subject and be well motivated to continue;
b develop sound methods of learning and habits of independent study.

Aim.6.
To help pupils to appreciate the importance of mathematics in their future studies and vocations.
Statement
Last para Page 5
Objectives
To train pupils to:
a be aware of some modern applications e.g. operations research, linear programming;
b develop the physical control to make precise drawings;
c see the subject as relevant and useful to everyday experience, and appreciate the value and contribution of mathematics to present and past societies."

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7.3 EXTRACTS FROM THE EDUCATION GAZETTE -- EXAMINATION PRESCRIPTIONS

7.31 1 April 1972

"S.C. Mathematics Prescription, 1973
(One Three-hour Paper)
Objective

To develop the pupil's capacity to live effectively in a culture that is being shaped in many of its signinicant aspects by mathematics.

This fundamental objective is expressed in the following particular aims.

(a) To provide mathematical experiences which enable pupils to make observations, to discover patterns and relationships, to develop concepts, to draw logical conclusions, to express thoughts accurately, and to form generalisations.
(b) To develop an understanding of the principles of mathematics, so that pupils may acquire an ability to apply these principles to unfamiliar or new situations.
(c) To develop mental alertness and a spirit of inquiry, a liking for and a lasting interest in mathematics.
(d) To develop the basic mathematical knowledge and skills (including the ability to perform mathematical computations with understanding and efficiency) necessary for everyday living and effective citizenship.
(e) To help pupils appreciate that mathematics does underlie the modern technological society and that mathematics will affect increasingly every vocation.

The examination will measure:
(a) a knowledge and understanding of the principles of mathematics at the level indicated by the prescription;
(b) a knowledge and understanding of techniques, and a satisfactory degree of competence in computation;
(c) a knowledge of facts and terms, and
(d) an ability to apply these principles, techniques, and facts to new situations, and to make simple deductions.

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7.32 30 September 1976

"Examinations
University Entrance Examination
Mathematics

The prescription in University Entrance mathematics below will come into force in 1977

Preamble
1. The ideas of logical deduction and correct use of symbols should be an essential part of the teaching of this syllabus. Particular attention should be paid to the ideas of implication, converse, equivalence and the use of the counter-example.
2. In teaching the syllabus, applications from the physical, social and commercial enviroments should be sought.
3. The main aim will be to establish understanding of concepts. The ability to perform intricate manipulative problems, for example on co-ordinate geometry or variation, will not be expected.
4. An approved published book of tables and standard formulae will be supplied to candidates.

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7.33 1 August 1972

"University Bursaries Examination
Pure Mathematics (One Paper)

Preamble
(1) This prescription is seen as forming part of the continuing development of the main stream of mathematics and, in particular, the ideas of function and structure.
(2) The syllabus which follows is to be read in conjunction with the explanatory notes which form, with this preamble an integral part of the prescription.
(3) While manipulation of algebraic expressions is implicit, it is not examinable as such.
(4) An approved published book of tables and standard formulae will be supplied to candidates. The use of mathematical instruments, including a slide rule, will be permitted.
(5) A choice of questions will be available to candidates in the examinaiton.

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