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THE PARAMETER SPACE BOUNDARY FOR ESCAPE
AND CHAOS IN THE DUFFING TWIN-WELL
OSCILLATOR

A THESIS PRESENTED IN PARTIAL
FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE
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Abstract

The Duffing ‘twin-well’ oscillator is investigated both experimentally and theoretically. The construction of a physical, nonlinear air-track oscillator with an ultrasound position detection system permits observation of a wide range of oscillatory behaviours, including chaotic motion, on a human scale (amplitudes of \sim metre).

Phase space and Poincaré sections are constructed in real time and, in the case of chaos, Lyapunov exponents determined. The range of control space conditions which give rise to chaos is investigated. In particular, the boundaries between chaotic and periodic motion are measured experimentally.

An analytic description of the primary boundaries of interest is constructed via a harmonic-balance generated solution to the governing differential equation and a perturbation style stability analysis. Successful theoretical prediction of the chaos boundary is achieved without recourse to numerical methods.

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