Minimisation of Mean Exponential Distortions and Teichmüller Theory

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ABSTRACT

This thesis studies the Cauchy boundary value problem of minimising exponential integral averages of mappings of finite distortion. Direct methods in calculus of variations provide existence theorems and we derive the Euler-Lagrange equations for minimisers of

$$\int_{\mathbb{D}} \exp(p\mathbb{K}(z,f))\, dz$$

for mappings of finite distortion $f : \mathbb{D} \to \mathbb{D}$ with prescribed boundary values. However, surprisingly, for these functionals some apriori regularity is needed before we can discuss these equations. We show by example how this can happen. We construct a mapping $f : \mathbb{D} \to \mathbb{D}$ with exponentially integrable distortion to exponent p which cannot perturbed by *any* diffeomorphism and still remain exponentially integrable with exponent p. Once enough apriori regularity is assumed for instance if a minimiser is locally quasiconformal, that is if the distortion function $\mathbb{K}(z,f)$ is locally bounded, then we use these equations to improve the regularity of the minimisers. In particular, we find that minimisers with locally bounded distortions are diffeomorphisms. Then we analyse the two extreme cases (1) $p \to 0$ and (2) $p \to \infty$. In this way we see the p-exponential problem connects the L^1 finite distortion problem, which is closely related to the classical harmonic theory in case (1), and to the Teichmüller problem, which promoted the development of quasiconformal mappings, in case (2)

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Contents

INTRODUCTION

In 1939 Teichmüller proved a famous theorem in his paper [53] (also in [54]): In the homotopy class of a diffeomorphism between closed Riemannian surfaces, there is a unique extremal mapping f which gives smallest maximal distortion $\mathbb{K}(z, f)$. Furthermore, this mapping is either conformal or has Beltrami coefficient of the form

$$\mu_f = k\frac{\overline{\Psi}}{|\Psi|},$$

where Ψ is a holomorphic function. The latter is now called a Teichmüller mapping. This work is highly valued as it was among the first times that quasiconformal mappings were connected with function theory [35]. Teichmüller's pioneering work, however, contains a lot of conjectures and incomplete proofs. Later in [1] Ahlfors studied Teichmüller's theory and gave a complete proof and a systematic introduction to this topic.

After Teichmüller and Ahlfors, the theory of quasiconformal mappings developed rapidly. Researchers extended Teichmüller's ideas and defined the problem in a more general setting. Consider the unit disk $\mathbb{D} \subset \mathbb{C}$. We assume $f_0 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a finite distortion homeomorphism. The problem we wish to study is to find the mapping that coincides f_0 along $\partial\mathbb{D}$ and minimises various distribution functionals.

In Astala, Iwaniec, and Martin's work [8], instead of the maximal dilatation (which can be regarded as the L^∞ norm of the distortion function), they studied the mean distortion- the integral of $\Psi(\mathbb{K})$ over $\mathbb{D}$. Typical choices of $\Psi(\mathbb{K})$ were $\mathbb{K}^p$, for $p = 1, 2, \ldots$. In this sense the original Teichmüller's problem can be regarded as the extremal case of the L^p problem as $p \to \infty$.

In fact Ahlfors' proof for Teichmüller's theorem was via the L^p problems. The key is that the 'minimiser' h of the inverse L^p problem yields a holomorphic combination of its first order weak derivatives,

$$\Phi(w) = \mathbb{K}(w, h)h_w(w)\overline{h_{\overline{w}}(w)},$$

called the Hopf differential, and then the classic theory of complex analysis can be applied. In particular, normal family arguments give the limit function as $p \to \infty$. Ahlfors then proved that the limit function is the extremal quasiconformal mapping of Teichmüller's problem.

However, the L^p problems themselves have a defect- the 'minimiser' might not be a true minimiser, that is not in the right space. For the inverse L^p problem, a minimising sequence h_n converges uniformly to some h; however, their inverses

f_n might not converge. So, as a 'minimiser' of the inverse L^p problem, h might be continuous but not a homeomorphism.

That is why we turn to the exponential distortion problem. The problem can be expressed as follows. Minimise the exponential mean distortion

$$\int_{\mathbb{D}} \exp[p\mathbb{K}(z, f)]dz,$$

subject to $f|_{\partial\mathbb{D}} = f_0$ and $f_0 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a homeomorphism, and

$$\int_{\mathbb{D}} \exp[p\mathbb{K}(z, f_0)]dz < \infty.$$

The exponential problems are much closer to the L^∞ case, and the most important fact is that there must exists at least one minimising homeomorphism. This was proved in [8], [29] and we will also introduce briefly in Section 1.4. In this case we can get an equation for the true minimisers and see what happens when they approach to infinity. This will be the main topic of this thesis. By the methods in calculus of variations, we derive equations with respect to the first order derivatives of the minimisers, known as the Euler-Lagrange equations, and use these to improve regularity.

However, the exponential problems also have their own problems. Unlike the L^p problem, a function with exponentially integrable distortion might not be inner variational. We will discuss and give concrete examples to explain this, and will also give some conditions to get the variation.

Overview of the Contents

Chapter 1 is devoted the backgrounds, starting with the Sobolev functions we give the definitions of the quasiconformal mappings and finite distortion functions. We focus on the Teichmüller-type problems and then record briefly the L^1 theory and the existence of the minimisers to the exponential problems. In the last part, for the sake of later applications to the regularity theory in Chapter 3, we follow and develop the theories about elliptic Beltrami equations in [6] and then conclude that the solutions to certain elliptic equations are smooth.

In Chapter 2 we will focus on the minimisation problem of the exponential distortion. The most important results are the Euler-Lagrange equations that we will get via outer and inner variations. We next give examples of functions with exponentially integrable distortions but they are not variational. Nevertheless, we will give conditions that imply a function is variational. The last part of this chapter deals with the convergence of the minimising sequence. We will get nice results of the strong convergence of the derivatives, Beltrami coefficients, distortion functions and Jacobian determinants, etc.

Chapter 3 deals with the regularity of the minimisers. We exploit the Euler-Lagrange equations we obtained in Chapter 2. Our first result is a condition that implies the extremal mapping is a C^∞-diffeomorphism. Then we go further to find the holomorphic Hopf differential and an elliptic equation for the Beltrami coefficient of the minimiser.

Chapter 4 is about the inverse problem. By variation of the inverse problem we give a weaker condition that the Hopf differential is holomorphic.

Chapter 5 proves Teichmüller's theorem. We mimic Ahlfors' proof but via the exponential problems. The result will be more general than Teichmüller and Ahlfors' original setting with Riemann surfaces. We also give a discussion about the case $p \to 0$ as it turns out to approach the L^1 problem.

In Chapter 6 we explore the minimisers between annuli and then find some nonlinear mappings that are minimisers on $\mathbb{D}$ for their own boundary values.

Chapter 7 lists possible future research. In particular, we still believe a minimiser of the exponential problem must be variational, while the uniqueness is also an interesting topic.

Here we state the main theorem that we will get in Section 3.3 (see Theorem 1.4.9 and Corollary 3.3.7):

Theorem 0.0.1 *For the exponential distortion problem, there exists a minimising homeomorphism f. Furthermore, if the distortion $\mathbb{K}(z, f)$ is locally bounded, then f is a diffeomorphism.*

1 Minimisation Problems and Beltrami Equations

1.1 Sobolev space.

The idea of Sobolev space was raised to generalise the differentiable functions. It is the main space that we will work in. A systematic introduction can be found in e.g. [13], [56].

1.1.1 Definition of Sobolev space.

Definition 1.1.1 Let $\Omega \subset \mathbb{R}^n$ be a domain, $f \in L^1_{loc}(\Omega)$, and $\alpha = (\alpha^1, \cdots, \alpha^k)$ be an index. We say $g \in L^1_{loc}(\Omega)$ is the α-th weak derivative of f, written as $g = D^\alpha f$, if

$$\int_\Omega f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega g \varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.1.1}$$

Definition 1.1.2 Let $k \geq 1$ be a positive integer, and $p \in [1, \infty]$. We say f is in the Sobolev Space $W^{k,p}(\Omega)$, if f has k-th order weak derivatives, and

$$D^\alpha f \in L^p(\Omega), \text{ for all } \alpha \text{ such that } 0 \leq |\alpha| \leq k.$$

We say $f \in W^{k,p}_{loc}(\Omega)$, if $f \in W^{k,p}(K)$ for any compact subset $K \subset \Omega$.

Theorem 1.1.3 *With the norm*

$$\|f\|_{W^{k,p}(\Omega)} := \Big(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha f(x)|^p dx \Big)^{1/p}, \quad 1 \leq p < \infty, \tag{1.1.2}$$

or, when $p = \infty$,

$$\|f\|_{W^{k,\infty}(\Omega)} := \sup_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}, \tag{1.1.3}$$

$W^{k,p}(\Omega)$ is a Banach space.

Proof. For any $p \in [1, \infty]$, we let f_j, $j = 1, 2, \ldots$ be a Cauchy sequence in $W^{1,p}(\Omega)$. Then by (1.1.2) and (1.1.3), all of f_j and $D^\alpha f_j$, $|\alpha| \leq k$ are Cauchy sequences in $L^p(\Omega)$, so they have limits f and $D^\alpha f$ in $L^p(\Omega)$, respectively. It remains to show that $D^\alpha f$ is the α-th weak derivative of f. However, this can be observed by

$$\int_\Omega f D^\alpha \varphi dx = \lim_{j \to \infty} \int_\Omega f_j D^\alpha \varphi dx = \lim_{j \to \infty} (-1)^{|\alpha|} \int_\Omega D^\alpha f_j \varphi dx = (-1)^{|\alpha|} \int_\Omega D^\alpha f \varphi dx,$$

for any $\varphi \in C_0^\infty(\Omega)$. $\square$

1.1.2 Approximation by smooth functions.

The mollification technique gives a method to approximate Sobolev functions by smooth functions. We will exploit the following $C_0^\infty(\mathbb{R}^n)$ function

$$\eta(x) := \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1; \\ 0, & |x| \geq 1, \end{cases}$$

where C is a constant adjusted so that

$$\int_{\mathbb{R}^n} \eta(x)dx = 1.$$

Next, we define

$$\eta^\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon}), \quad \varepsilon > 0.$$

Here η^ε is called the standard mollifier. For $f \in W^{k,p}_{loc}(\Omega)$, we define the convolution

$$f^\varepsilon(x) = \eta^\varepsilon * f(x) = \int_\Omega \eta^\varepsilon(x-y)f(y)dy,$$

for

$$x \in \Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}.$$

Theorem 1.1.4 *Let $f \in W^{k,p}_{loc}(\Omega)$, where $1 \leq p < \infty$. Then f^ε is a smooth sequence in $C^\infty(\Omega)$ that converges in $W^{k,p}_{loc}(\Omega)$ to $f \in W^{k,p}_{loc}(\Omega)$.*

Proof. By the theory of L^p spaces (see e.g. [50]) the sequence $f^\varepsilon \to f$ in $L^p_{loc}(\Omega)$. Meanwhile,

$$\begin{aligned} \frac{\partial}{\partial x^i}(\eta^\varepsilon * f)(x) &= \int_\Omega \frac{\partial}{\partial x^i}\eta^\varepsilon(x-y)f(y)dy \\ &= -\int_\Omega \frac{\partial}{\partial y^i}\eta^\varepsilon(x-y)f(y)dy \\ &= \int_\Omega \eta^\varepsilon(x-y)\frac{\partial}{\partial y^i}f(y)dy \\ &= \eta^\varepsilon * \frac{\partial}{\partial x^i}f(x), \end{aligned}$$

and by induction this also holds for higher order cases. Thus $D^\alpha(f^\varepsilon) = (D^\alpha f)^\varepsilon \to D^\alpha f$ in $L^p_{loc}(\Omega)$, for each $|\alpha| \leq k$. So we conclude that $f^\varepsilon \to f$ in $W^{k,p}_{loc}(\Omega)$. □

We remark that although $C^\infty(\Omega) \subset W^{k,p}(\Omega)$ is dense, the density of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ holds only when $\Omega = \mathbb{R}^n$. We use $W_0^{k,p}(\Omega)$ to denote the closure of $C_0^\infty(\Omega)$ functions in $W^{k,p}(\Omega)$.

1.1.3 Embedding theorems.

The embedding theorems give us a way to conclude that some Sobolev functions have higher integrability or continuity. The easiest case is on the real line. In fact, if $f \in W^{1,1}_{loc}(\Omega)$ is defined on a real interval $\Omega = (a, b)$, then

$$f(x) = \int_{x_0}^{x} f'(t)dt + C$$

is absolutely continuous. This property remains valid on lines when in higher dimension.

Theorem 1.1.5 *Every Sobolev function $f \in W^{k,p}_{loc}(\Omega)$ is absolutely continuous on almost every line segment in Ω parallel to the coordinate axes (ACL).*

The ACL property tells us that if f is Sobolev function, then the pointwise partial derivatives exist and are equal to the weak derivatives almost everywhere. But we should note this works only on lines. There are examples of Sobolev functions that are not continuous, see Section 1.4. Indeed, to get the continuity of f we need higher integrability of Df.

Theorem 1.1.6 (Gagliardo-Nirenberg-Sobolev Inequality) *Let $n \geq 2$, $1 \leq p < n$. Then, there is a constant $C(n, p)$ such that*

$$\|f\|_{L^{p^*}(\Omega)} \leq C\|Df\|_{L^p(\Omega)},$$

for any $f \in W^{1,p}(\Omega)$. Here

$$p^* = \frac{np}{n-p}$$

is called the Sobolev conjugate of p.

Theorem 1.1.7 (Morrey's Inequality) *Let $n < p < \infty$. Then, there is a constant $C(n, p)$ and a representative of $f \in W^{1,p}(\Omega)$, such that*

$$\|f\|_{C^{0,\gamma}(\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)},$$

where $\gamma = 1 - \frac{n}{p}$, $\|\cdot\|_{C^{0,\gamma}(\Omega)}$ is the Hölder continuity norm

$$\|f\|_{C^{0,\gamma}(\Omega)} := \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x-y|^{\gamma}}.$$

See [13] for the proofs. The embedding theorems are extremely useful not only to show the continuity of a Sobolev solution of a variational problem, but also to get the equicontiuity of a family, since here the constants C depend only on n, p but not the specific choice of the function f. We will discuss equicontiuity in later sections. At this point we can see that the following 2-dimensional functions, which are defined on a planar domain $\Omega \subset \mathbb{C}$, are continuous:

i) $W^{1,p}_{loc}(\Omega)$, where $p > 2$. This is Morrey's inequality.

ii) $W^{2,p}_{loc}(\Omega)$, where $p > 1$. By the Sobolev inequality $W^{2,p}_{loc}(\Omega) \subset W^{1,p^*}_{loc}(\Omega)$, where

$$p^* = \frac{2p}{2-p} > 2.$$

iii) $W^{3,1}_{loc}(\Omega)$. Another iteration of the Sobolev inequality gives $W^{3,1}_{loc}(\Omega) \subset W^{2,2}_{loc}(\Omega)$.

In particular, by iii), if f has arbitrary order weak derivatives, that is $f \in W^{k,1}_{loc}(\Omega)$ for any positive integer k, then f has a smooth representative.

The case $W^{1,2}_{loc}(\Omega)$ (or generally, $W^{1,n}_{loc}(\Omega)$ for $\Omega \subset \mathbb{R}^n$) is very special. In general functions in this class are not continuous. However, if a $W^{1,n}_{loc}(\Omega)$ function has finite distortion, then it is continuous [19], [29]. We will discuss this later in Section 1.4.

Harmonic functions are also an important class of smooth functions, especially in complex analysis. The next lemma, due to Weyl [55], reveals that if a function is harmonic in the sense of weak derivatives, then it is harmonic in the classical sense.

Lemma 1.1.8 (Weyl's Lemma) *Let $f : \Omega \to \mathbb{C}$ be in $L^1_{loc}(\Omega)$. If*

$$\int_\Omega f \Delta\varphi = 0, \quad \forall \varphi \in C^\infty_0(\Omega),$$

then f is harmonic.

In particular, if $\Omega \subset \mathbb{C}$, and $f \in L^1_{loc}(\Omega)$ has weak derivative $f_{\bar{z}} := \frac{1}{2}(f_x + if_y) = 0$, then f is holomorphic in Ω.

The following theorem gives an approach to the converse of the above theorems; finding the weak derivatives Df from the properties of f.

Theorem 1.1.9 *Let $1 < p \leq \infty$ and $f \in L^p_{loc}(\Omega)$. Then $f \in W^{1,p}_{loc}(\Omega)$ if and only if for any unit vector $e_i = (0, \ldots, 1, \ldots, 0)$ that is parallel to a coordinate, as $h \to 0$, the sequence*

$$F^h(x) := \frac{f(x + he_i) - f(x)}{h}$$

has a uniform $L^p(K)$ bound for any compact $K \subset \Omega$.

Proof. We only prove the 'if' part. By the Banach-Alaoglu Theorem, up to a subsequence there is a weak limit F in $L^p(\Omega)$. We prove that F is the weak derivative of f. Indeed, for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\int_\Omega F(x)\varphi(x)dx &= \lim_{h\to 0}\int_\Omega F^h(x)\varphi(x)dx \\
&= \lim_{h\to 0}\int_\Omega \frac{1}{h}\Big[f(x+he_i)\varphi(x) - f(x)\varphi(x)\Big]dx \\
&= \lim_{h\to 0}\frac{1}{h}\int_\Omega \Big[f(x)\varphi(x-he_i) - \int_\Omega f(x)\varphi(x)\Big]dx \\
&= -\int_\Omega f(x)D^i\varphi(x)dx.
\end{aligned}$$

□

Another important theorem is Green's formula, which extends to Sobolev spaces, see [2].

Theorem 1.1.10 *Let $\Omega \in \mathbb{C}$ be a Jordan domain, let $f, g \in W^{1,1}(\Omega) \cap C(\overline{\Omega})$. Then*

$$\int_\Omega f_x + g_y = \int_{\partial\Omega} f dx - g dy. \tag{1.1.4}$$

1.2 Finite distortion functions.

The theory of distortion functions is one of the core topics of geometric function theory as they are natural measures of change in a system. For a more detailed exposition, see [6],[23], and [29].

1.2.1 Quasiconformal mappings.

We now focus on the complex plane $\mathbb{C} = \mathbb{R}^2$. In classical complex analysis, the conformal mappings are those functions preserving local angles. Analytically, they satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{1.2.1}$$

where $f(x+iy) = u(x+iy) + iv(x+iy)$. In this thesis we will more frequently use the following Wirtinger derivatives:

$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\overline{z}} = \frac{1}{2}(f_x + if_y).$$

With these notations the Cauchy-Riemann equations (1.2.1) become one equation:

$$f_{\overline{z}} = 0.$$

Green's formula (1.1.4) can also be rewritten as

$$\int_\Omega f_z + g_{\bar{z}} = \frac{i}{2}\int_{\partial\Omega} f d\bar{z} - g dz, \tag{1.2.2}$$

for $f, g \in W^{1,1}(\Omega) \cap C(\overline{\Omega})$.

We next define the notion of quasiconformality. This can be approached in two ways - geometrically and analytically.

First we have the geometric definition:

Definition 1.2.1 Let $f : \Omega \to \Omega'$ be a sense-preserving homeomorphism between planar domains. At a point $z \in \Omega$, set

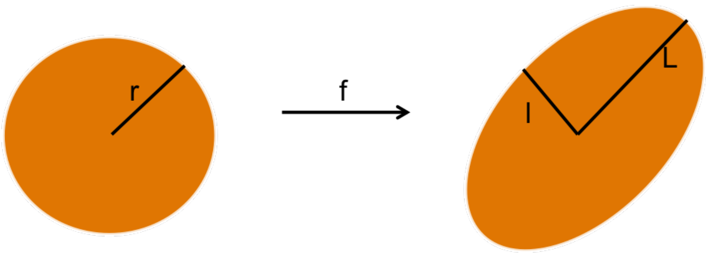
$$L(z, f, r) = \sup\{|f(z) - f(w)| : |z - w| \leq r\},$$

$$l(z, f, r) = \inf\{|f(z) - f(w)| : |z - w| \leq r\},$$

$$H(z, f) = \limsup_{r\to 0} \frac{L(z, f, r)}{l(z, f, r)}.$$

We say f is K-quasiconformal in Ω, if

$$K := \sup_{z\in\Omega} H(z, f) < \infty. \tag{1.2.3}$$



Local action of a quasiconformal mapping

The geometric definition gives us a clear picture of the local action of a quasiconformal mappings. However, in analysis we need the following definition which is more helpful in computations:

Definition 1.2.2 Suppose that $f \in W^{1,1}_{loc}(\Omega, \Omega')$ is a planar homeomorphism. We say f is quasiconformal, if

$$\max_\alpha |\partial_\alpha f(z)| \leq K \min_\alpha |\partial_\alpha f(z)|,$$

for almost every $z \in \Omega$, where

$$\partial_\alpha f(z) = \lim_{r\to 0} \frac{f(z + re^{i\alpha}) - f(z)}{r}, \quad \alpha \in [0, 2\pi).$$

We remark that the existence of $\partial_\alpha f$ is guaranteed by the Gehring-Lehto Theorem [17]:

Theorem 1.2.3 *Let $f : \Omega \to \mathbb{C}$ be continuous and open. Then f is differentiable almost everywhere in Ω if and only if f has partial derivatives almost everywhere.*

The proof for the equivalence of Definition 1.2.1 and Definition 1.2.2 is rather complicated and we refer to [17], and also see [6].

At a point of differentiability, it can be calculated that

$$\min_\alpha |\partial_\alpha f(z)| = |f_z(z)| - |f_{\overline{z}}(z)|,$$

$$\max_\alpha |\partial_\alpha f(z)| = |f_z(z)| + |f_{\overline{z}}(z)| = |Df(z)|,$$

where $|Df(z)|$ is the operator norm of Df. Then we can define the distortion function

$$K(z) = \frac{|f_z(z)| + |f_{\overline{z}}(z)|}{|f_z(z)| - |f_{\overline{z}}(z)|} = \frac{1 + |\mu(z,f)|}{1 - |\mu(z,f)|},$$

where

$$\|K(z)\|_{L^\infty(\Omega)} = K,$$

where K is that in Definition 1.2.1 and Definition 1.2.2, and

$$\mu(z,f) = \frac{f_{\overline{z}}(z)}{f_z(z)}, \text{ a.e. } z \in \Omega,$$

is called the Beltrami coefficient. We then have another description for quasiconformal mappings, which is called the Beltrami equation

$$f_z(z) = \mu(z,f) f_{\overline{z}}(z),$$

where

$$\|\mu(z,f)\|_\infty \le k < 1$$

for some $k \in [0,1)$. Note the relation of the notations k and K:

$$K = \frac{1+k}{1-k}, \quad k = \frac{K-1}{K+1}.$$

Also note the Jacobian determinant of f is

$$J(z,f) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 > 0, \text{ a.e. } z \in \Omega,$$

So we get another expression for the distortion function

$$K(z) = \frac{(|f_z(z)| + |f_{\overline{z}}(z)|)^2}{|f_z(z)|^2 - |f_{\overline{z}}(z)|^2} = \frac{|Df(z)|^2}{J(z,f)}.$$

1.2.2 Finite distortion functions.

The main objects of our study will be the finite distortion functions and the integral of the distortion function $K(z)$. However, there are some extremal cases the distortions are not uniformly bounded but still integrable. So we will release the restriction that $K(z)$ is uniformly bounded. We have the following definition of finite distortion functions:

Definition 1.2.4 Let $\Omega, \Omega' \subset \mathbb{C}$, $f \in W^{1,1}_{loc}(\Omega, \Omega')$, $J(z, f) \geq 0$ be locally integrable, and let there be a measurable function $K(z)$ such that

$$|Df(z)|^2 \leq K(z)J(z, f), \text{ a.e. } z \in \Omega.$$

Then, we say f is a finite distortion mapping, and the distortion function is defined as

$$K(z, f) = \begin{cases} \frac{|Df(z)|^2}{J(z,f)}, & J(z, f) \neq 0 \\ 1, & \text{otherwise} \end{cases} \tag{1.2.4}$$

where

$$\frac{|Df(z)|^2}{J(z, f)} = \frac{(|f_z(z)| + |f_{\overline{z}}(z)|)^2}{|f_z(z)|^2 - |f_{\overline{z}}(z)|^2} = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}.$$

1.2.3 The area formulae and Lusin's condition $\mathcal{N}$.

The area formulae deal with the problem of change of variables. The classic theory states that a sufficient condition is Lipschitz continuity. See e.g. [13]:

Theorem 1.2.5 *Let $\Omega \subset \mathbb{C}$ be a domain, let $f : \Omega \to \mathbb{C}$ be a Lipschitz homeomorphism, and let η be a nonnegative Borel measurable function on $f(\Omega)$. Then,*

$$\int_\Omega \eta(f(z))J(z, f)dz = \int_{f(\Omega)} \eta(w)dw.$$

For Sobolev functions we will need the following "Lusin's condition $\mathcal{N}$":

Definition 1.2.6 Let $f : \Omega \to \mathbb{C}$ be measurable. We say f satisfies Lusin's condition $\mathcal{N}$, if for any $E \subset \Omega$ such that $|E| = 0$, we have $|f(E)| = 0$. Suppose that f is a homeomorphism. We say f satisfies Lusin's condition $\mathcal{N}^{-1}$, if its inverse f^{-1} satisfies Lusin's condition $\mathcal{N}$.

Theorem 1.2.7 *Let $f \in W^{1,1}(\Omega, \mathbb{C})$ be a homeomorphism, and let η be a nonnegative Borel measurable function on $\mathbb{C}$. Then,*

$$\int_\Omega \eta(f(z))J(z, f)dz \leq \int_{f(\Omega)} \eta(w)dw.$$

Furthermore, equality holds if f satisfies Lusin's condition $\mathcal{N}$.

Proof. Let

$$A = \{z \in \Omega : f \text{ is differentiable at } z\}, \quad S = \Omega \setminus A.$$

By Theorem 1.2.3, $|S| = 0$. By [15, Theorem 3.1.8], A can be decomposed into countably many subsets A_i where f is Lipschitz in each of them. Then the claim follows from Theorem 1.2.5. $\square$

Theorem 1.2.8 *If $f \in W^{1,2}_{loc}(\Omega, \Omega')$ is a homeomorphism, then f satisfies Lusin's condition $\mathcal{N}$.*

Proof. Let $E \subset \Omega$ such that $|E| = 0$. We assume E is compact. Then, we can find an U such that $E \subset U \subset \Omega$, and ∂U consists of finitely many line segments that are parallel to the coordinates, and then along them f is absolutely continuous. Let $f^\varepsilon \to f$ be the approximating sequence as in Theorem 1.1.4. For the smooth functions f^ε, we have

$$\int_U J(z, f^\varepsilon) dz = |f^\varepsilon(U)|.$$

By Theorem 1.1.4 we know that First, $Df^\varepsilon \to Df$ in $L^2(U)$, so $\int_U J(z, f^\varepsilon)dz \to \int_U J(z, f)dz$; secondly, $f^\varepsilon \to f$ uniformly in $\overline{U}$, in particular, along ∂U. By the choice of U, the length of ∂U is finite, so $|f^\varepsilon(U)| - |f(U)| \to 0$ as $\varepsilon \to 0$. This proves

$$\int_U J(z, f) dz = |f(U)|.$$

On the other hand, by Theorem 1.2.7 we have

$$\int_U J(z, f)dz = \int_{U-E} J(z, f)dz \leq \int_{f(U)-f(E)} 1dx = |f(U)| - |f(E)|.$$

So we get $|f(E)| = 0$.

For noncompact $E \subset \Omega$ such that $|E| = 0$, we choose any compact $F \subset f(E)$. Then the continuity of f^{-1} gives that $f^{-1}(F)$ is compact. As we have proved, $|F| = 0$. Then the claim follows from the inner regularity of the Lebesgue measure. $\square$

Theorem 1.2.9 *Let $f : \Omega \to \Omega'$ be quasiconformal, $\eta \in L^1(\Omega')$. Then f satisfies both Lusin's conditions $\mathcal{N}$ and $\mathcal{N}^{-1}$. In particular,*

$$\int_{\Omega'} \eta(w)dw = \int_\Omega \eta(f(z))J(z, f)dz. \tag{1.2.5}$$

Proof. We note that any quasiconformal $f \in W^{1,1}_{loc}(\Omega)$ is actually in $W^{1,2}_{loc}(\Omega)$, since

$$|Df|^2 \leq KJ(z, f).$$

In fact the inverse function f^{-1} is also quasiconformal [6]. Thus it follows from Theorem 1.2.8 that both f and f^{-1} satisfy Lusin's condition $\mathcal{N}$, and then (1.2.5) follows. $\square$

1.3 Minimisation problems for distortion.

The most general setting of this cluster of problems is to find the $W^{1,1}_{loc}(\Omega, \Omega')$ homeomorphisms that minimise the energy functional

$$\int_\Omega \Phi[K(z,f)]dz,$$

where $K(z,f)$ is the distortion function as (1.2.4), $\Omega, \Omega' \subset \mathbb{C}$ are planar domains, and $\Phi : [1,\infty] \to [0,\infty]$ is a real function. Unfortunately the function $K(z,f)$ is not convex (c.f. [6, Section 21.1]) so typically we will consider $\mathbb{K}(z,f)$ as defined below.

Certainly we need some regularities for the problems. First, the shapes (in topological sense) of the domains Ω and Ω', as they are closely related to the distortion of a function. This leads us to a classification of different types of the problems. We refer to [43] for a review of this, and [3] for the historical notes. In this thesis we will focus on the Teichmüller type problems:

Let $\mathbb{D} \subset \mathbb{C}$ be the unit planar disk, $\overline{\mathbb{D}}$ be its closure. Let $f_0 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a finite distortion homeomorphism such that

$$\mathcal{E}(f_0) = \int_{\mathbb{D}} \Phi[\mathbb{K}(z, f_0)]dz < \infty, \tag{1.3.1}$$

where $\Phi : [1,\infty] \to [0,\infty]$ is a real function, and

$$\mathbb{K}(z,f) = \frac{\|Df(z)\|^2}{J(z,f)} = \frac{|f_z(z)|^2 + |f_{\overline{z}}(z)|^2}{|f_z(z)|^2 - |f_{\overline{z}}(z)|^2},$$

where

$$\|Df\|^2 = \frac{1}{2}\mathrm{tr}(Df^t Df)$$

is the mean Hilbert-Schmidt norm. Let $\mathcal{F}$ denote the class of functions $f \in W^{1,1}_{loc}(\mathbb{D})$ that are self-homeomorphisms of $\overline{\mathbb{D}}$, $f = f_0$ along $\partial\mathbb{D}$, and $\mathcal{E}(f) < \infty$.

Problem 1.3.1 *Find the minimal mappings $f \in \mathcal{F}$ such that*

$$\mathcal{E}(f) = \inf_{g \in \mathcal{F}} \mathcal{E}(g).$$

For notational ease we will sometimes write $\mu_f(z)$ for $\mu(z,f)$, $\mathbb{K}_f(z)$ for $\mathbb{K}(z,f)$, $J_f(z)$ for $J(z,f)$, etc.

1.3.1 The L^1 problem.

The L^1 distortion problem $\int_{\mathbb{D}} \mathbb{K}(z,f)dz$, in view of (1.3.1), with $\Phi(t) = t$, has been solved. It was proved in [8] that a $W^{1,2}_{loc}(\mathbb{D})$ minimiser of the L^1 problem must be the inverse of a harmonic function. Later in [24], the requirement was reduced to $W^{1,1}_{loc}(\mathbb{D})$. The key of the problem is the regularity of the inverse mapping. Many of the results will be very useful in our later study of the exponential problem. See [6], [8], [23], [24], [25], [32] for details.

Let $z \in \mathbb{D}$ be a point such that f is differentiable at it, and $Df(z)$ be invertible. Write $h = f^{-1}$, we have

$$h \circ f(z) = z.$$

Then,

$$\begin{aligned} h_w(f(z))f_z(z) + h_{\overline{w}}(f(z))\overline{f_{\overline{z}}(z)} &= 1, \\ h_w(f(z))f_{\overline{z}}(z) + h_{\overline{w}}(f(z))\overline{f_z(z)} &= 0. \end{aligned}$$

Solving this we get

$$h_w(f(z)) = \frac{\overline{f_z(z)}}{J(z,f)}, \quad h_{\overline{w}}(f(z)) = -\frac{f_{\overline{z}}(z)}{J(z,f)}. \tag{1.3.2}$$

Then

$$|\mu(f(z),h)| = \left|\frac{-f_{\overline{z}}(z)}{\overline{f_z(z)}}\right| = |\mu(z,f)|,$$

$$\mathbb{K}(f(z),h) = \frac{1+|\mu(f(z),h)|^2}{1-|\mu(f(z),h)|^2} = \frac{1+|\mu(z,f)|^2}{1-|\mu(z,f)|^2} = \mathbb{K}(z,f).$$

If we can perform the change of variables, then

$$\int_{\mathbb{D}} \mathbb{K}(z,f)dz = \int_{\mathbb{D}} \mathbb{K}(w,h)J(w,h)dw = \int_{\mathbb{D}} \|Dh(w)\|^2 dw.$$

This indicates the next theorem:

Theorem 1.3.2 *Let $\mathcal{F}$ be the class of $W^{1,1}_{loc}(\mathbb{D},\mathbb{D})$ homeomorphisms whose distortion function $\mathbb{K}(z,f) \in L^1(\mathbb{D})$. Let $f \in \mathcal{F}$. Then*

$$\int_{\mathbb{D}} \mathbb{K}(z,f)dz = \int_{\mathbb{D}} \|Df^{-1}(w)\|^2 dw. \tag{1.3.3}$$

In particular, for each given $f_0 \in \mathcal{F}$ such that $\int_{\mathbb{D}} \mathbb{K}(z,f_0)dz < \infty$, the minimisation problem

$$\min_{f\in\mathcal{F}} \int_{\mathbb{D}} \mathbb{K}(z,f)dz, \quad f|_{\partial\mathbb{D}} = f_0|_{\partial\mathbb{D}}$$

has a unique minimiser. Moreover, this extremal map is a C^∞-diffeomorphism whose inverse is harmonic in $\mathbb{D}$.

To prove this theorem we need the following lemma (See [24], [32]) for the regularities of $f \in \mathcal{F}$ and its inverse f^{-1}:

Lemma 1.3.3 *Let $f \in \mathcal{F}$ be as in Theorem 1.3.2. Then $J(z,f) > 0$ almost everywhere in $\mathbb{D}$; its inverse $h = f^{-1} \in W^{1,2}_{loc}(\mathbb{D})$ has finite distortion, that is $\mathbb{K}(w,h) < \infty$ almost everywhere in $\mathbb{D}$. In particular, h satisfies Lusin's condition $\mathcal{N}$.*

Proof of Theorem 1.3.2. We need to prove (1.3.3). On one hand, by Theorem 1.2.3 and Lemma 1.3.3, we have that f is differentiable and $J(z,f) > 0$, so (1.3.2) holds at almost every point $z \in \mathbb{D}$. By Theorem 1.2.7 we have

$$\int_{\mathbb{D}} \|Dh(w)\|^2 dw \geq \int_{\mathbb{D}} \|Dh(f(z))\|^2 J(z,f) dz = \int_{\mathbb{D}} \mathbb{K}(z,f) dz.$$

On the other hand, we set

$$\mathbb{O} = \{w \in \mathbb{D} : h \text{ is differentiable and } J(w,h) > 0\}.$$

Again by Theorem 1.2.3 and the fact that $\mathbb{K}(w,h) < \infty$, we have that $\|Dh(w)\| = 0$ almost everywhere in $\mathbb{D} \setminus \mathbb{O}$. Then,

$$\begin{aligned}
\int_{\mathbb{D}} \|Dh(w)\|^2 dw &= \int_{\mathbb{O}} \|Dh(w)\|^2 dw \\
&= \int_{\mathbb{O}} \frac{\|Df(h(w))\|^2}{J(h(w),f)} J(w,h) dw \\
&\leq \int_{h(\mathbb{O})} \mathbb{K}(z,f) dz \\
&\leq \int_{\mathbb{D}} \mathbb{K}(z,f) dz.
\end{aligned}$$

This validates (1.3.3) and then the theorem follows from the classical theories for the Dirichlet problem. We remark that it is the Radó-Kneser-Choquet Theorem (see e.g. [12]) that guarantees a harmonic function of $\mathbb{D}$ which is a homeomorphism of $\partial\mathbb{D}$ must be a C^∞-diffeomorphism. □

1.3.2 The L^p problem.

With $\Phi(t) = t^p$, $p > 1$, we obtain the L^p problem, which is to minimise

$$\int_{\mathbb{D}} \mathbb{K}^p(z,f) dz, \quad f|_{\partial\mathbb{D}} = f_0|_{\partial\mathbb{D}},$$

where $f_0 \in \mathcal{F}$ such that $\int_{\mathbb{D}} \mathbb{K}^p(z,f_0) dz < \infty$. As a function with finite L^p mean distortion must have finite L^1 mean distortion, Lemma 1.3.3 also works in this case, and then by a similar argument to the proof of Theorem 1.3.2 we can prove

Theorem 1.3.4 *Let f be a $W^{1,1}_{loc}(\mathbb{D},\mathbb{D})$ homeomorphism whose distortion function $\mathbb{K}(z,f)\in L^p(\mathbb{D})$. Then $h=f^{-1}:\mathbb{D}\to\mathbb{D}$ is a mapping of finite distortion and*

$$\int_{\mathbb{D}}\mathbb{K}^p(z,f)dz=\int_{\mathbb{D}}\mathbb{K}^p(w,h)J(w,h)dw. \tag{1.3.4}$$

We also wish to get an analogue of rest of Theorem 1.3.2- the existence, smoothness, and uniqueness of the minimiser. However, it is more complicated this time. We observe that the h side still gives higher regularity: let $\{f_j\}_{j=1}^{\infty}$ be a minimising sequence of homeomorphisms, then (1.3.4) tells us the sequence of inverses $\{h_j\}_{j=1}^{\infty}$ is also a minimising sequence for the "inverse problem". Thus, for j sufficiently large, h_j are bounded in $W^{1,2}(\mathbb{D})$:

$$\int_{\mathbb{D}}\|Dh_j(w)\|^2dw=\int_{\mathbb{D}}\mathbb{K}(w,h_j)J(w,h_j)dw\le\int_{\mathbb{D}}\mathbb{K}^p(w,h_j)J(w,h_j)dw. \tag{1.3.5}$$

We will see in the next part of this thesis that such functions are equi-continuous. Then it follows from the Arzelá-Ascoli Theorem that there is a subsequence that converges to some h locally uniformly, and h has the same modulus of continuity as h_j. However, the problem here is that the sequence f_j is not as regular as h_j. In fact, by the following computation [30] we can see f_j has a uniform $W^{1,q}(\mathbb{D})$ norm, for $q=\frac{2p}{p+1}<2$:

$$\Big[\int_{\mathbb{D}}\|Df(z)\|^{\frac{2p}{p+1}}dz\Big]^{p+1}\le\int_{\mathbb{D}}\mathbb{K}^p(z,f)dz\cdot\Big[\int_{\mathbb{D}}J(z,f)dz\Big]^p\le\pi^p\int_{\mathbb{D}}\mathbb{K}^p(z,f)dz. \tag{1.3.6}$$

Note $q\in(1,2)$ for any $p\in(1,\infty)$. However, this is not enough to get continuity- we will also discuss this in the next part. So this is the main obstacle here: for a sequence of homeomorphisms we cannot find a limit that remains to be a homeomorphism.

1.4 Exponential distortion: the existence theory.

We now introduce the main problem of the thesis: the p-exponential distortion problem. That is, in view of (1.3.1), $\Phi(t)=\exp(pt)$, with some $p>0$. Precisely, given a prescribed boundary data f_0, we wish to find the minimiser to the energy

$$\mathcal{E}_p(f)=\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]dz, \tag{1.4.1}$$

where f is a $W^{1,1}_{loc}(\mathbb{D})$ Sobolev self-homeomorphism of the unit disk $\mathbb{D}$ such that $f=f_0$ on $\partial\mathbb{D}$.

Although we only set $f\in W^{1,1}_{loc}(\mathbb{D})$, the condition $\mathcal{E}_p(f)<\infty$ implies that $\mathbb{K}(z,f)$ has a bounded $L^s(\mathbb{D})$ norm, for all $s\in(1,\infty)$. Thus (1.3.6) gives that

$f \in W^{1,q}(\mathbb{D})$ for all $q \in (1,2)$. But this is not enough for our aim to conclude that the functions have a uniform modulus of continuity. It is worth noting that by Morrey's Theorem 1.1.7, a planar $W^{1,q}_{loc}(\Omega)$ function with $q > 2$ is always continuous. Let us check the following two discontinuous examples for $q \leq 2$:

i)
$$f(z) = \log\Big(\log(1 + \frac{1}{|z|})\Big).$$
This is a $W^{1,2}(\mathbb{D})$ function. Also note it is not a finite distortion function, as $|f_z| = |f_{\bar{z}}|$ a.e.

ii)
$$f(z) = z + \frac{z}{|z|}.$$
This is a $W^{1,q}(\mathbb{D})$ finite distortion function, for any $q \in (1,2)$, but not $W^{1,2}(\mathbb{D})$.

Along with these two examples, we claim that, if a $W^{1,2}_{loc}(\Omega)$ function has finite distortion, then it must be continuous. This was First proved by Gol'dshtein and Vodop'yanov in [19], based on Reshetnyak's work [48] on the monotonic functions. But in [29] it is proved that the assumption $f \in W^{1,2}_{loc}(\Omega)$ can be even weakened, since there exists a 'gap' between $\bigcap_{q\in[1,2)} W^{1,q}_{loc}(\mathbb{D})$ and $W^{1,2}_{loc}(\mathbb{D})$. To explain this we need the following notation for Sobolev-Orlicz spaces.

1.4.1 Integrability of Df.

Definition 1.4.1 We say $P : [0,\infty] \to [0.\infty]$ is an Orlicz function, if it is convex, increasing, and $P(0) = 0$, $P(\infty) = \infty$. We say a function f is in the Orlicz space $L^P(\Omega)$, if
$$\int_\Omega P(|f|) < \infty.$$
We say a function f is in the Sobolev-Orlicz space $W^{1,P}(\Omega)$, if the weak derivative Df exists, and both f and Df are in the Orlicz space $L^P(\Omega)$.

We remark that such a Sobolev-Orlicz space $W^{1,P}(\Omega)$ is a Banach space, and analogously to the Sobolev spaces $W^{1,p}(\Omega)$ there are also approximating sequences and weak compactness. See [11].

Theorem 1.4.2 *Let $f : \mathbb{D} \to \mathbb{D}$ be a homeomorphism such that*
$$\int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]\, dz < \infty,$$
for some $p > 0$. Then, f is in the Orlicz-Sobolev space $W^{1,P}(\mathbb{D})$ with
$$P(t) = \frac{t^2}{\log(e+t)}.$$

Proof. We first claim the elementary inequality

$$ab \le a\log(a+1) + e^b - 1, \quad a, b \ge 0. \tag{1.4.2}$$

In fact, if $b < \log(a+1)$, then the inequality automatically holds. On the other hand, the function

$$b \to e^b - 1 - ab$$

is increasing in $(\log(a+1), \infty)$, so for $b \ge \log(a+1)$,

$$e^b - 1 - ab \ge e^{\log(a+1)} - 1 - a\log(a+1) \ge -a\log(a+1),$$

which verifies (1.4.2). Now since

$$\int_{\mathbb{D}} J(z,f)dz \le |\mathbb{D}| < \infty, \quad \int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]dz < \infty,$$

we only need to prove that, there is a constant C, such that

$$\frac{\|Df\|^2}{\log(e + \|Df\|)} \le C(J_f + e^{p\mathbb{K}_f}), \quad \forall z \in \mathbb{D}. \tag{1.4.3}$$

In (1.4.2) we put

$$a = \frac{J_f}{\log(e + \|Df\|)}, \quad b = p\mathbb{K}_f.$$

Then

$$\begin{aligned}
\frac{\|Df\|^2}{\log(e + \|Df\|)} &= \frac{1}{p}\Big(\frac{p\mathbb{K}_f J_f}{\log(e + \|Df\|)}\Big) \\
&\le \frac{1}{p}\Big(\frac{J_f}{\log(e + \|Df\|)}\log(1 + \frac{J_f}{\log(e + \|Df\|)}) + e^{p\mathbb{K}_f} - 1\Big) \\
&\le \frac{1}{p}\Big(\frac{2J_f}{\log[(e + \|Df\|)^2]}\log(1 + \frac{J_f}{\log(e + \|Df\|)}) + e^{p\mathbb{K}_f} - 1\Big) \\
&\le \frac{2}{p}\Big(J_f + e^{p\mathbb{K}_f}\Big).
\end{aligned}$$

This verifies (1.4.3) and then the proof is complete. □

1.4.2 Modulus of continuity.

We now consider the Sobolev-Orlicz space $W^{1,P}(\Omega)$, where $P(t) = \frac{t^2}{\log(e+t)}$. We have the following sharp result.

Theorem 1.4.3 *Let $P(t)$ be an Orlicz function satisfying the following two conditions:*

i) The divergence condition:

$$\int_1^\infty \frac{P(t)}{t^3}dt = \infty; \tag{1.4.4}$$

ii) The convexity condition:

$$t \to P(t^{\frac{5}{8}}) \text{ is convex for } t \text{ near } \infty. \tag{1.4.5}$$

Let $f \in W^{1,P}(\Omega)$ be a planar finite distortion function. Then

$$|f(z) - f(w)| \leq \rho(|z - w|),$$

where $z, w \in D(a, R) \subset D(a, 2R) \subset \Omega$ for some $a \in \Omega$, $R > 0$, and $\rho : [0, \infty) \to [0, \infty)$ is a continuous increasing function such that $\rho(0) = 0$, and depends only on $\int_\Omega P(|Df|)$.

In particular, if a sequence of finite distortion mappings $\{f_j\}$ have a uniform $W^{1,P}(\Omega)$ bound, then they are equicontinuous on any compact subset $K \subset\subset \Omega$.

Note the function $P(t) = \frac{t^2}{\log(e+t)}$ satisfies the conditions (1.4.4) and (1.4.5). A complete proof would take dozens of pages and we refer to Chapter 7 and 8 of [29]. We also record the following important lemma which is [29, Theorem 8.4.2].

Lemma 1.4.4 *Let $\{f_j\}$ be a sequence of sense-preserving mappings such that $f_j \rightharpoonup f$ in $W^{1,P}(\Omega)$, where P satisfies the conditions (1.4.4), (1.4.5). Then f is also a sense-preserving mapping, and*

$$J(z, f_j) \rightharpoonup J(z, f) \text{ in } L^1_{loc}(\Omega).$$

As stated in Problem 1.3.1. we will work in the unit disk $\mathbb{D}$ with a boundary function f_0 which is a homeomorphism. We observe a way to expand the local properties in Theorem 1.4.3 and Lemma 1.4.4 to be uniformly in $\mathbb{D}$. In fact, we can extend f_0 continuously to some $\mathbb{D}_R = D(0, R)$ $(R > 1)$ by defining

$$\tilde{f}_0(z) := \begin{cases} f_0(z) & z \in \overline{\mathbb{D}}; \\ \frac{1}{\overline{f_0(\frac{1}{\bar{z}})}}, & z \in \mathbb{D}_R - \overline{\mathbb{D}}. \end{cases}$$

We wish to keep away from the point $z_0 \in \mathbb{D}$ such that $f_0(z_0) = 0$. So choose any r such that $|z_0| < r < 1$, and set $R = 1/r$, then

$$\inf_{r<|w|\leq 1} |f_0(z)| \geq \epsilon > 0.$$

Now we can compute, for $z \in \mathbb{D}_R - \overline{\mathbb{D}}$,

$$(\tilde{f}_0)_z(z) = \frac{1}{\overline{f_0^2(\frac{1}{\bar{z}})}} \overline{(f_0)_z(\frac{1}{\bar{z}})\frac{1}{\bar{z}^2}},$$

$$(\tilde{f}_0)_{\bar{z}}(z) = \frac{1}{f_0^2(\frac{1}{\bar{z}})}\overline{(f_0)_{\bar{z}}(\frac{1}{\bar{z}})\frac{1}{z^2}}.$$

$$\mathbb{K}(z, \tilde{f}_0) = \mathbb{K}(\frac{1}{\bar{z}}, f)$$

In particular, $\tilde{f}_0$ has finite distortion in $\mathbb{D}_R - \overline{\mathbb{D}}$, and

$$\begin{aligned}\int_{\mathbb{D}_R-\overline{\mathbb{D}}} \Psi(\mathbb{K}(z, \tilde{f}_0))dz &= \int_{\mathbb{D}_R-\overline{\mathbb{D}}} \Psi(\mathbb{K}(\frac{1}{\bar{z}}, f_0)dz \\ &= \int_{\mathbb{D}-\mathbb{D}_r} \Psi(\mathbb{K}(\zeta, f_0))\frac{1}{|\zeta|^4}d\zeta \\ &\leq \frac{1}{r^4}\int_{\mathbb{D}} \Psi(\mathbb{K}(\zeta, f_0))d\zeta, \end{aligned} \tag{1.4.6}$$

for any Φ as in Problem 1.3.1. Then, for any $f \in \mathcal{F}$ as in Problem 1.3.1, as $f|_{\partial\mathbb{D}} = f_0|_{\partial\mathbb{D}}$, f can also be extended continuously with the same function $\tilde{f}_0|_{\mathbb{D}_R-\overline{\mathbb{D}}}$. Now when Theorem 1.4.3 and Lemma 1.4.4 are applied in $\Omega = \mathbb{D}_R$, as $\overline{\mathbb{D}} \subset\subset \mathbb{D}_R$, the claims are uniform in $\overline{\mathbb{D}}$.

Sometimes we will also have the form $\int_{\mathbb{D}} \Psi(\mathbb{K}(w, h_0))J(w, h_0)dw$ for the inverse functions, as we have seen in (1.3.3) and (1.3.4). In this case we can extend h_0 by the same method above. For $w \in \mathbb{D}_R - \overline{\mathbb{D}}$,

$$J(w, \tilde{h}_0) = \frac{J(\frac{1}{\bar{w}}, h_0)}{|h_0(\frac{1}{\bar{w}})|^4|w|^4} \leq \frac{1}{\varepsilon^4}J(\frac{1}{\bar{w}}, h_0),$$

where R, r, ε are similar as above but by h_0. Then, similar to (1.4.6),

$$\int_{\mathbb{D}_R-\overline{\mathbb{D}}} \Psi(\mathbb{K}(w, \tilde{h}_0))J(w, \tilde{h}_0)dw \leq \frac{1}{\varepsilon^4 r^4}\int_{\mathbb{D}} \Psi(\mathbb{K}(\zeta, f_0))d\zeta.$$

Then for any inverse function h we can also extend with $\tilde{h}_0$ thus when we want to apply Theorem 1.4.3 and Lemma 1.4.4 on them they can also be uniform on $\overline{\mathbb{D}}$.

1.4.3 Existence of minimisers.

By Lemma 1.4.4 we can generalise Theorem 1.2.8. Although we do not have $Df^\varepsilon \to Df$ in $L^2(U)$, we still have

$$\int_U J(z, f^\varepsilon)dz \to \int_U J(z, f)dz, \quad U \subset\subset \mathbb{D},$$

where f^ε is the standard mollifier. Then following the same argument as in the proof of Theorem 1.2.8 we can prove:

Theorem 1.4.5 *Let f be a $W^{1,P}_{loc}(\mathbb{D})$ self-homeomorphism of $\mathbb{D}$, where $P(t)$ is as in Theorem 1.4.3. Then f satisfies Lusin's condition $\mathcal{N}$.*

Theorem 1.4.6 *Let f be a $W^{1,1}_{loc}(\mathbb{D})$ self-homeomorphism of $\mathbb{D}$ such that $\exp[p\mathbb{K}(z,f)] \in L^1(\mathbb{D})$. Then,*

$$\int_{\mathbb{D}} \exp(p\mathbb{K}(z,f))dz = \int_{\mathbb{D}} \exp(p\mathbb{K}(w,h))J(w,h)dw.$$

Proof. This can be proved similarly to the L^1 case. However, as f satisfies Lusin's condition $\mathcal{N}$, we can prove it in an easier way. In fact, By (1.3.2), whenever $Df(z)$ exists and $J(z,f) > 0$, Dh exists at $f(z)$. We have

$$h_w(f(z)) = \frac{\overline{f_z(z)}}{J(z,f)}, \quad h_{\overline{w}}(f(z)) = -\frac{f_{\overline{z}}(z)}{J(z,f)}.$$

Thus

$$J(f(z),h) = \frac{1}{J(z,f)}, \quad \mathbb{K}(f(z),h) = \mathbb{K}(z,f).$$

By Theorem 1.2.3 and Lemma 1.3.3, this actually holds for almost every $z \in \mathbb{D}$. Also, by Theorem 1.4.5, f satisfies Lusin's condition $\mathcal{N}$. Together with Theorem 1.2.9 we get

$$\begin{aligned}\int_{\mathbb{D}} \exp(p\mathbb{K}(w,h))J(w,h)dw &= \int_{\mathbb{D}} \exp(p\mathbb{K}(f(z),h))J(f(z),h)J(z,f)dz \\ &= \int_{\mathbb{D}} \exp(p\mathbb{K}(z,f))dz.\end{aligned}$$

□

Lemma 1.4.7 *The function $x^n y^{-l}$, $n \geq l+1 \geq 1$ is convex on $(0,\infty)\times(0,\infty)$. Precisely,*

$$x^n y^{-l} - x_0^n y_0^{-l} \geq n x_0^{n-1} y_0^{-l}(x - x_0) - l x_0^n y_0^{-l-1}(y - y_0). \tag{1.4.7}$$

Proof. We claim the geometric mean inequality

$$u^\alpha v^\beta w^\gamma \leq \alpha u + \beta v + \gamma w,$$

where the numbers are all non-negative and $\alpha + \beta + \gamma = 1$. This is because of the concavity of the function $f(x) = \log x$ on $\mathbb{R}^+$. Indeed,

$$\log(u^\alpha v^\beta w^\gamma) = \alpha \log u + \beta \log v + \gamma \log w \leq \log(\alpha u + \beta v + \gamma w).$$

Now put

$$\alpha = \frac{1}{n}, \quad \beta = \frac{l}{n}, \quad \gamma = \frac{n-l-1}{n},$$
$$u = x^n y^{-l}, \quad v = x_0^n y_0^{-l-1} y, \quad w = x_0^n y_0^{-l}.$$

Then (1.4.7) follows. □

Lemma 1.4.8 *Let $\Omega \subset \mathbb{C}$ be domain, let $f : \Omega \to \mathbb{R}$ be a convex function and let $g : \mathbb{R} \to \mathbb{R}$ be convex and non-decreasing. Then $g \circ f$ is convex.*

Proof. Let $z_0 = x_0 + iy_0$, $z = x + iy$. Then,

$$\begin{aligned} g \circ f(z) - g \circ f(z_0) &\geq g'(f(z_0))(f(z) - f(z_0)) \\ &\geq g'(f(z_0))(f_x(z_0)(x - x_0) + f_y(z_0)(y - y_0)) \\ &= (g \circ f)_x(z_0)(x - x_0) + (g \circ f)_y(z_0)(y - y_0). \end{aligned}$$

□

Theorem 1.4.9 *The exponential distortion problem (1.4.1) has a minimiser.*

Proof. Let $\{f_j\}$ be a minimising sequence of self-homeomorphisms of $\mathbb{D}$ with finite exponential distortion. By Theorem 1.4.2, f_j has a uniform $W^{1,P}(\mathbb{D})$ bound for $P(t) = \frac{t^2}{\log(e+t)}$. By Theorem 1.4.3, the sequence $\{f_j\}$ form an equicontinuous family. Then, by the Arzelà-Ascoli Theorem, there is a convergent subsequence, which we still call f_j, that converges to some f which has the same modulus of continuity as f_j. By Theorem 1.4.6, the sequence of inverse functions $h_j = f_j^{-1}$ has a uniform $L^2(\mathbb{D})$ norm, since

$$p \int_{\mathbb{D}} \|Dh_j\|^2 = \int_{\mathbb{D}} p\mathbb{K}_{h_j} J_{h_j} \leq \int_{\mathbb{D}} e^{p\mathbb{K}_{h_j}} J_{h_j} = \mathcal{E}_p(f_j).$$

By Lemma 1.3.3, every h_j has finite distortion, so Theorem 1.4.3 with $P(t) = t^2$ applies on h_j, so up to a subsequence, they converge to h uniformly. By the uniform convergence we also have $h = f^{-1}$. Thus f is a homeomorphism. By the Banach-Alaoglu Theorem up to a subsequence we also have $Df_j \rightharpoonup Df$ in $L^P(\mathbb{D})$.

By Lemma 1.4.7 and Lemma 1.4.8, the function $\exp[p\frac{x^2}{y}]$ is convex. We put $x = \|Df_j(z)\|$, $y = J(z, f_j)$, $x_0 = \|Df(z)\|$, $y_0 = J(z, f)$. Then the convexity reads as

$$\begin{aligned} &\exp[p\mathbb{K}(z, f_j)] - \exp[p\mathbb{K}(z, f)] \\ \geq& 2p \exp[p\frac{\|Df(z)\|^2}{J(z, f)}] \frac{\|Df(z)\|}{J(z, f)} (\|Df_j(z)\| - \|Df(z)\|) \\ &- p \exp[p\frac{\|Df(z)\|^2}{J(z, f)}] \frac{\|Df(z)\|^2}{J^2(z, f)} (J(z, f_j) - J(z, f)). \end{aligned} \tag{1.4.8}$$

Also note that Cauchy-Schwarz inequality gives

$$\|Df_j\| - \|Df\| \geq \Big\langle \frac{Df}{\|Df\|}, Df_j \Big\rangle - \|Df\| = \Big\langle \frac{Df}{\|Df\|}, Df_j - Df \Big\rangle.$$

Set $\mathbb{D}_\varepsilon \subset\subset \Omega$ such that

$$\mathbb{D}_\varepsilon = \{z \in \mathbb{D} : \|Df(z)\| < \frac{1}{\varepsilon}, J(z,f) > \varepsilon\}, \quad \mathbb{D} = \bigcup_{\varepsilon>0} \mathbb{D}_\varepsilon.$$

Integrate both sides of (1.4.8) over $\mathbb{D}_\varepsilon$, we get

$$\begin{aligned}
&\int_{\mathbb{D}_\varepsilon} \exp[p\mathbb{K}(z,f_j)]dz - \int_{\mathbb{D}_\varepsilon} \exp[p\mathbb{K}(z,f)]dz \\
\geq & 2p \int_{\mathbb{D}_\varepsilon} \exp[p\frac{\|Df(z)\|^2}{J(z,f)}] \frac{\|Df(z)\|}{J(z,f)} \Big\langle \frac{Df(z)}{\|Df(z)\|}, Df_j(z) - Df(z) \Big\rangle dz \\
& - p \int_{\mathbb{D}_\varepsilon} \exp[p\frac{\|Df(z)\|^2}{J(z,f)}] \frac{\|Df(z)\|^2}{J^2(z,f)} (J(z,f_j) - J(z,f))dz. \qquad (1.4.9)
\end{aligned}$$

Here the right hand side of (1.4.9) converges to 0, as the first term follows from the weak convergence of Df_j, and the second term follows from Lemma 1.4.4. This proves

$$\liminf_{j\to\infty} \int_{\mathbb{D}_\varepsilon} \exp[p\mathbb{K}(z,f_j)]dz \geq \int_{\mathbb{D}_\varepsilon} \exp[p\mathbb{K}(z,f)]dz.$$

Now let $\varepsilon \to 0$. By Theorem 1.2.3 and Lemma 1.3.3, we have

$$|\bigcup_{\varepsilon>0} \mathbb{D}_\varepsilon| = |\mathbb{D}|.$$

Thus

$$\liminf_{j\to\infty} \int_{\mathbb{D}} \exp[p\mathbb{K}(z,f_j)]dz \geq \int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]dz.$$

This proves that f is a minimiser of the problem. $\square$

1.5 Beltrami equations and elliptic systems.

In this section we consider the following problem: for given μ, find the solutions to the Beltrami equation

$$f_{\overline{z}}(z) = \mu(z) f_z(z). \qquad (1.5.1)$$

First we need the notion of complex potentials.

1.5.1 Complex potentials.

For $f \in L^p(\mathbb{C})$, $p \geq 1$, the Cauchy transform is defined by the rule

$$\mathcal{C}(f)(z) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\tau)}{z-\tau} d\tau; \qquad (1.5.2)$$

while the Beurling transform is

$$\mathcal{S}(f)(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\tau)}{(z-\tau)^2}d\tau, \tag{1.5.3}$$

where the integral is the Cauchy principal value. By the Cauchy formula (1.2.2) it is not hard to see

$$\frac{\partial \mathcal{C}}{\partial \bar{z}} = Id, \quad \frac{\partial \mathcal{C}}{\partial z} = \mathcal{S}. \tag{1.5.4}$$

The Beurling transform $\mathcal{S}$ can be given as a Fourier multiplier

$$\widehat{\mathcal{S}(f)}(\xi) = m(\xi)\hat{f}(\xi), \tag{1.5.5}$$

where

$$m(\xi) = \frac{\bar{\xi}}{\xi}.$$

Thus $\mathcal{S}$ maps $L^2(\mathbb{C})$ to $L^2(\mathbb{C})$, with the norm $\|\mathcal{S}\|_2 = 1$, see [9]. By (1.5.5) we also see that $\mathcal{S}$ maps $L^p(\mathbb{C})$ to $L^p(\mathbb{C})$, for every $1 < p < \infty$. In particular,

$$\mathcal{S}(f_{\bar{z}}) = f_z, \quad f \in W^{1,p}(\mathbb{C}),$$

see [6]. The key problem here is determining the operator norm

$$\mathbf{S}_p := \sup_{\|f\|_p=1} \|\mathcal{S}(f)\|_p.$$

In fact, this problem has not been completely solved, but we have the following conjecture by Iwaniec [27]:

$$\mathbf{S}_p = \begin{cases} 1/(p-1), & 1 < p < 2; \\ p-1, & p \geq 2. \end{cases} \tag{1.5.6}$$

Nevertheless, we know

$$\lim_{p\to 2} \mathbf{S}_p = 1. \tag{1.5.7}$$

Thus for any $k < 1$, there is a pair $(Q(k), P(k))$, where

$$1 < Q(k) < 2 < P(k) < \infty, \tag{1.5.8}$$

such that

$$k\mathbf{S}_p < 1, \text{ for every } p \in (Q(k), P(k)). \tag{1.5.9}$$

We remark that although the values of $\mathbf{S}_p$ remains to be a conjecture, it is proved in [6, Theorem 14.0.4] that if $|\mu| \leq k < 1$, the operator $\mathbf{I} - \mu\mathcal{S}$ is invertible in $L^q(\mathbb{C})$ for any $q \in (1+k, 1+\frac{1}{k})$. This accords with (1.5.6).

1.5.2 The basic Beltrami equation.

We call a solution f for (1.5.1) a principal solution, if it is in $W^{1,2}_{loc}(\mathbb{C})$ and

$$f(z) = z + \mathcal{O}(1/z), \text{ as } z \to \infty. \tag{1.5.10}$$

Consider the inhomogeneous equation

$$\sigma_{\overline{z}}(z) = \mu(z)\sigma_z(z) + \varphi(z), \tag{1.5.11}$$

where $|\mu| \leq k\chi_{\mathbb{D}}$, $\varphi \in L^p(\mathbb{C})$, for some $0 \leq k < 1$ and $p \in (1+k, 1+\frac{1}{k})$. Define

$$\sigma(z) = \mathcal{C}((\mathbf{I} - \mu\mathcal{S})^{-1}\varphi). \tag{1.5.12}$$

Thus

$$\sigma_z = \mathcal{S}((\mathbf{I} - \mu\mathcal{S})^{-1}\varphi) = \mathcal{S}\varphi + \mathcal{S}\mu\mathcal{S}\varphi + \mathcal{S}\mu\mathcal{S}\mu\mathcal{S}\varphi + \cdots, \tag{1.5.13}$$

$$\sigma_{\overline{z}} = (\mathbf{I} - \mu\mathcal{S})^{-1}\varphi = \varphi + \mu\mathcal{S}\varphi + \mu\mathcal{S}\mu\mathcal{S}\varphi + \cdots = \mu\sigma_z + \varphi. \tag{1.5.14}$$

Then (1.5.11) is satisfied. Here the convergence of the series are guaranteed by (1.5.9). Furthermore, let f and g both satisfy the conditions, then

$$\big(f - g\big)_{\overline{z}}(z) = \mu(z)\big(f - g\big)_z(z).$$

With the conditions that f and g decay as $\mathcal{O}(1/z)$ and are holomorphic near ∞, $f - g$ could only be 0. We then conclude

Theorem 1.5.1 *Let $0 \leq k < 1$, $|\mu| \leq k\chi_{\mathbb{D}}$, and $\varphi \in L^p(\mathbb{C})$, where $p \in (1+k, 1+\frac{1}{k})$, and μ, φ are both compactly supported. Then, the equation*

$$\sigma_{\overline{z}}(z) = \mu(z)\sigma_z(z) + \varphi(z) \tag{1.5.15}$$

admits a unique solution σ with $D\sigma \in L^p(\mathbb{C})$, and σ decays at ∞ as $\mathcal{O}(1/z)$. In particular, if $p > 2$, then $\sigma \in W^{1,p}(\mathbb{C})$.

We come back to equation (1.5.1). In fact we only need to set σ to be the solution as in Theorem 1.5.1 that satisfies

$$\sigma_{\overline{z}}(z) = \mu(z)\sigma_z(z) + \mu(z).$$

Then

$$f(z) = \sigma(z) + z \tag{1.5.16}$$

is the unique principal solution to (1.5.1).

Theorem 1.5.2 *Let $0 \leq k < 1$, $|\mu| \leq k\chi_{\mathbb{D}}$. Then, there is a unique principal solution f to the Beltrami equation*

$$f_{\overline{z}}(z) = \mu(z) f_z(z), \quad a.e. \text{ on } \mathbb{C}.$$

and the solution $f \in W^{1,2}_{loc}(\mathbb{C})$ is a K-quasiconformal homeomorphism of $\mathbb{C}$, where

$$K = \frac{1+k}{1-k}.$$

Furthermore, if $\mu \in C_0^\infty(\mathbb{C})$, the principal solution f is a C^∞-diffeomorphism. In particular, we have $|f_z| > 0$ in $\mathbb{C}$.

Proof. The existence and uniqueness have already been proved. To see that f is a homeomorphism, we note that $|\mu(z)| \leq k$ gives

$$J(z, f) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 \geq (1-k)|f_z(z)|^2 > 0,$$

where $|f_z(z)| > 0$ follows from the expressions (1.5.13), (1.5.16). So f is a local homeomorphism. By (1.5.10), f is analytic near ∞, then it follows from the monodromy theorem (see e.g. [34]) that f is a global homeomorphism. If $\mu \in C_0^\infty(\mathbb{C})$, from the expressions (1.5.12)-(1.5.16), we see that

$$(D\sigma)_{\overline{z}} = D(\sigma_{\overline{z}}) = D(\mu\sigma_z + \varphi) = \mu(D\sigma)_z + (D\mu)\sigma_z + D\varphi.$$

So this is again an equation with the form of (1.5.11) that satisfies the conditions to prove $D\sigma \in W^{2,p}(\mathbb{C})$. Then inductively we get $\sigma \in C^\infty(\mathbb{C})$. This proves that $f = \sigma + z$ is a C^∞-diffeomorphism. $\square$

We now consider the other solutions to (1.5.1), and wish to conclude that some of them are also C^∞-diffeomorphisms.

Theorem 1.5.3 (Stoilow Factorization) *Let $0 \leq k < 1$, $|\mu| \leq k\chi_{\mathbb{D}}$, and $f \in W^{1,1}_{loc}(\mathbb{D})$ be a homeomorphic solution to the Beltrami equation*

$$f_{\overline{z}}(z) = \mu(z) f_z(z), \quad z \in \mathbb{D}.$$

Let g be any other $W^{1,2}_{loc}(\mathbb{D})$ solution. Then, there is a holomorphic function $\Phi : \mathbb{D} \to \mathbb{C}$ such that

$$g(z) = \Phi(f(z)), \quad z \in \mathbb{D}.$$

Proof. Let $h = f^{-1}$, $\Phi = g \circ h$. At almost every point $z \in \Omega$,

$$h_w(f(z)) = \frac{\overline{f_z}(z)}{J(z,f)}, \quad h_{\overline{w}}(f(z)) = -\frac{f_{\overline{z}}(z)}{J(z,f)}.$$

As a quasiconformal mapping f satisfies Lusin's condition $\mathcal{N}$, so for almost every $w \in \Omega'$,

$$\begin{aligned}\Phi_{\overline{w}}(w) =& g_z(h(w))h_{\overline{w}}(w) + g_{\overline{z}}(h(w))\overline{h_w}(w)\\ =& g_z(z)h_{\overline{w}}(f(z)) + g_{\overline{z}}(z)\overline{h_w}(f(z)) = 0.\end{aligned}$$

Finally, Theorem 1.4.3 gives that g is continuous, and then so is Φ. Now we can apply Weyl's lemma 1.1.8 to get that Φ is holomorphic. $\square$

Theorem 1.5.4 *Let $0 \le k < 1$, μ be C^∞-smooth, $|\mu| \le k\chi_{\mathbb{D}}$, and f is a $W^{1,1}_{loc}(\mathbb{D})$ homeomorphism which is a solution to the Beltrami equation*

$$f_{\overline{z}}(z) = \mu(z) f_z(z), \quad z \in \mathbb{D}.$$

Then f is a C^∞-diffeomorphism in $\mathbb{D}$.

Proof. First note that the condition $|\mu| \le k\chi_{\mathbb{D}}$ implies that f is quasiconformal, so we actually have $f \in W^{1,2}_{loc}(\mathbb{D})$. Set $\Omega \subset\subset \mathbb{D}$ be compactly contained. Choose an $\eta \in C_0^\infty(\mathbb{D})$ such that $\eta = 1$ in Ω. Then, by Theorem 1.5.2 $\tilde{\mu} = \eta\mu \in C_0^\infty(\mathbb{C})$ admits a principal solution g that is a C^∞-diffeomorphism. In particular, $g|_\Omega$ is a solution to

$$g_{\overline{z}}(z) = \mu(z) g_z(z), \quad z \in \Omega.$$

Since $f|_\Omega$ is another $W^{1,2}_{loc}(\Omega)$ solution, we then have, on $g(\Omega)$,

$$\Phi = f \circ g^{-1}$$

is a holomorphic homeomorphism. Thus Φ is biholomorphic from $g(\Omega)$ to $f(\Omega)$ and then $f|_\Omega$ is also a C^∞-diffeomorphism. As this holds for any compactly contained subset, f is then a C^∞-diffeomorphism in $\mathbb{D}$. $\square$

1.5.3 Elliptic equations.

A natural generalisation of (1.5.1) is the following problem

$$f_{\overline{z}}(z) = \mu(z) f_z(z) + \nu(z)\overline{f_z}(z), \quad |\mu| + |\nu| \le k < 1. \tag{1.5.17}$$

This is basically same as (1.5.1)- we only need to use $\mu\mathcal{S} + \nu\overline{\mathcal{S}}$ to replace the $\mu\mathcal{S}$ in that problem. (1.5.17) is still linear- ultimately we want to study the fully non-linear situation

$$f_{\overline{z}}(z) = \mathcal{H}(z, f(z), f_z(z)). \tag{1.5.18}$$

To make the problem solvable we need the following regularity:

Definition 1.5.5 Problem (1.5.18) is called an elliptic system, if the function $\mathcal{H}(z, w, \zeta)$ satisfies

1) The homogeneity condition. That is,

$$\mathcal{H}(z, w, 0) = 0, \quad a.e.(z, w) \in \mathbb{C} \times \mathbb{C};$$

2) The uniform ellipticity condition. That is, there is a $k \in [0, 1)$ such that for almost every $(z, w) \in \mathbb{C} \times \mathbb{C}$ and every $\zeta, \xi \in \mathbb{C}$,

$$\big|\mathcal{H}(z, w, \zeta) - \mathcal{H}(z, w, \xi)\big| \leq k|\zeta - \xi|;$$

3) The function

$$z \to \mathcal{H}(z, f(z), f_z(z))$$

is measurable.

Various cases have been discussed in [6]. Here we only introduce the specific case of autonomous equations

$$f_{\overline{z}}(z) = \mathcal{H}(f_z(z)), \quad z \in \mathbb{D}. \tag{1.5.19}$$

Theorem 1.5.6 *Let the function*

$$\zeta \to \mathcal{H}(\zeta): \quad \mathbb{D} \to \mathbb{C}$$

be C^∞-smooth and elliptic, as in Definition 1.5.5. Let f be a $W^{1,2}_{loc}(\mathbb{D})$ solution for equation (1.5.19). Then $f \in C^\infty(\mathbb{D})$.

We separate the proof into several parts.

Lemma 1.5.7 (Caccioppoli-type Estimate) *Let f be a $W^{1,s}_{loc}(\mathbb{D})$ solution for the problem*

$$f_{\overline{z}}(z) = \mu(z) f_z(z), \quad \|\mu\|_{L^\infty(\mathbb{D})} = k < 1, \tag{1.5.20}$$

where $s > 1 + k$. Then $f \in W^{1,p}_{loc}(\mathbb{D})$ for any $p \in (1 + k, 1 + \frac{1}{k})$ and it satisfies

$$\|\eta Df\|_{L^p(\mathbb{C})} \leq C(p) \|f D\eta\|_{L^p(\mathbb{C})} \tag{1.5.21}$$

for any $\eta \in C_0^\infty(\mathbb{D})$.

Proof. Let $\eta \in C_0^\infty(\mathbb{D})$ be any test function. Then,

$$(\eta f)_{\overline{z}} - \mu(\eta f)_z = (\eta_{\overline{z}} - \mu\eta_z) f := \varphi \in L^{s^*}(\mathbb{C}), \tag{1.5.22}$$

where $s^* = \frac{2s}{2-s} > 2$, where we assumed that $1 + k < s < 2$, without loss of generality. By Theorem 1.5.1, ηf is the unique $W^{1,2}_{loc}(\mathbb{C})$ solution for the Beltrami equation (1.5.22) that vanishes at ∞ as $\mathcal{O}(1/z)$. Thus

$$(\eta f)_{\overline{z}} = (\mathbf{I} - \mu\mathcal{S})^{-1}\varphi \in L^q(\mathbb{C}), \tag{1.5.23}$$

$$(\eta f)_z = \mathcal{S}((\mathbf{I} - \mu\mathcal{S})^{-1}\varphi) \in L^q(\mathbb{C}), \tag{1.5.24}$$

for any $q < \min\{1 + \frac{1}{k}, s^*\}$. By Theorem 1.1.7, f is then continuous, thus in $L^p_{loc}(\mathbb{D})$, for any $p \in (1+k, 1+\frac{1}{k})$, and then (1.5.21) follows from (1.5.22)-(1.5.24). □

Lemma 1.5.8 *Let f be a $W^{1,s}_{loc}(\mathbb{D})$ solution to equation (1.5.19), where $\mathcal{H}$ is elliptic, $s > 1 + k$. Then $f \in W^{2,p}_{loc}(\mathbb{D})$, for all $p \in (1 + k, 1 + \frac{1}{k})$.*

Proof. Fix any small $h > 0$ and a unit coordinate e_i. We define

$$F^h(z) = \frac{f(z + he_i) - f(z)}{h}, \quad d(z, \partial\mathbb{D}) < h$$

Then

$$F^h_z(z) = \frac{f_z(z + he_i) - f_z(z)}{h},$$
$$F^h_{\overline{z}}(z) = \frac{f_{\overline{z}}(z + he_i) - f_{\overline{z}}(z)}{h}.$$

The condition of ellipticity gives that,

$$|f_{\overline{z}}(z + he_i) - f_{\overline{z}}(z)| \le k|f_z(z + he_i) - f_z(z)|.$$

Thus

$$F^h_{\overline{z}} \le kF^h_z. \tag{1.5.25}$$

By Lemma 1.5.7, we have the Caccioppoli-type Estimate

$$\|\eta DF^h\|_{L^p(\mathbb{C})} \le C(p)\|F^h D\eta\|_{L^p(\mathbb{C})},$$

for any $\eta \in C^\infty_0(\mathbb{D})$ and $p \in (1 + k, 1 + \frac{1}{k})$. Also by Lemma 1.5.7, we know $f \in W^{1,p}_{loc}(\mathbb{D})$ for any $p \in (1 + k, 1 + \frac{1}{k})$. Then, by the 'only if' part of Theorem 1.1.9, the right hand side is uniformly bounded, then so is the left hand side. Then, by the 'if' part of Theorem 1.1.9,

$$f_z, f_{\overline{z}} \in W^{1,p}_{loc}(\mathbb{D}).$$

So it follows that $f \in W^{2,p}_{loc}(\mathbb{D})$. □

Lemma 1.5.9 *If $\mathcal{H} \in C^1(\mathbb{D})$ satisfies the ellipticity condition:*

$$|\mathcal{H}(\zeta) - \mathcal{H}(\xi)| < k|\zeta - \xi|, \quad \forall \zeta, \xi \in \mathbb{D}. \tag{1.5.26}$$

Then

$$|\mathcal{H}_\zeta(\zeta)| + |\mathcal{H}_{\overline{\zeta}}(\zeta)| \le k, \quad \forall \zeta \in \mathbb{D}. \tag{1.5.27}$$

Proof. As we discussed in Section 1.2.1, $|\mathcal{H}_\zeta(\zeta)| + |\mathcal{H}_{\overline{\zeta}}(\zeta)|$ is the maximal directional derivative $\partial_\alpha \mathcal{H}$ of $\mathcal{H}$ at ζ, for some $\alpha = [0, 2\pi)$. That is,

$$|\mathcal{H}_\zeta(\zeta)| + |\mathcal{H}_{\overline{\zeta}}(\zeta)| = |\partial_\alpha \mathcal{H}(\zeta)| = \lim_{h\to 0} \Big| \frac{\mathcal{H}(\zeta + he^{i\alpha}) - \mathcal{H}(\zeta)}{h} \Big| \leq k.$$

□

Lemma 1.5.10 *Let f be a $W^{2,s}_{loc}(\mathbb{D})$ solution to equation (1.5.19), where $\mathcal{H}$ is elliptic and $\xi \to \mathcal{H}(\xi)$ is C^∞-smooth. Then $f \in C^\infty(\mathbb{D})$.*

Proof. Since the second weak derivatives exist, we may compute

$$(f_x)_{\overline{z}}(z) = \mathcal{H}_w(f_z(z))(f_x)_z(z) + \mathcal{H}_{\overline{w}}(f_z(z))\overline{(f_x)_z(z)}. \tag{1.5.28}$$

Note it is same for f_y. This becomes an equation for $g = f_x$:

$$g_{\overline{z}}(z) = \mu(z) g_z + \nu(z)\overline{g_z(z)}, \tag{1.5.29}$$

where

$$|\mu(z)| + |\nu(z)| = |\mathcal{H}_w(f_z(z))| + |\mathcal{H}_{\overline{w}}(f_z(z))| \leq k.$$

Then (1.5.29) is again an elliptic equation and then $g = f_x \in W^{2,s}_{loc}(\mathbb{D})$- note this is a little bit more complicated than Lemma 1.5.8 as there are also z-terms involved, but we refer to [6, Theorem 8.7.1]. As the same argument works on f_y, we then have $f \in W^{3,s}(\mathbb{D}) \subset C^2(\mathbb{D})$, so both $\mu = \mathcal{H}_w(f_z)$ and $\nu = \mathcal{H}_{\overline{w}}(f_z)$ are in C^1. Thus we can differentiate both sides of (1.5.28) again, which gives

$$(f_{xx})_{\overline{z}}(z) = \mathcal{H}_w(f_z(z))(f_{xx})_z(z) + \mathcal{H}_{\overline{w}}(f_z(z))\overline{(f_{xx})_z(z)} + \varphi(z), \tag{1.5.30}$$

where $\varphi(z)$ is composed by lower-order terms thus is continuous, and the equation is again elliptic. And then inductively, if we have shown that $f \in W^{p+1,s}(\mathbb{D})$, then we have an elliptic equation as (1.5.30) but for p-derivatives of f, thus by Lemma 1.5.8 we get $f \in W^{p+2,s}(\mathbb{D})$. So we finally conclude that f is C^∞-smooth. □

This completes the proof for Theorem 1.5.6.

1.6 The exponential Beltrami equation.

We introduce the way to find a solution for the Beltrami equation if μ is not uniformly bounded by $k < 1$, but $\int_{\mathbb{D}} \exp[p\frac{1+\|\mu\|^2}{1-\|\mu\|^2}] < \infty$, for some $p > 0$. In this case we define

$$\mu_m(z) = \begin{cases} \mu(z), & \text{if } |\mu(z)| \leq 1 - \frac{1}{m}; \\ (1-\frac{1}{m})\frac{\mu(z)}{|\mu(z)|}, & \text{otherwise.} \end{cases}$$

Then, by Theorem 1.5.2 each μ_m admits a unique principal solution $f_m : \mathbb{C} \to \mathbb{C}$, and the distortions $\mathbb{K}(z, f_m)$ are monotonically increasing to $\frac{1+\|\mu(z)\|^2}{1-\|\mu(z)\|^2}$ pointwise. By Theorem 1.4.2 and Theorem 1.4.3, up to a subsequence f_m converge uniformly to some f, and Df_m converge weakly to Df in $L^P_{loc}(\mathbb{C})$, for $P(t) = \frac{t^2}{\log(e+t)}$. So the limit function f gives the following theorem:

Theorem 1.6.1 *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function such that $|\mu| \leq \chi_{\mathbb{D}}$, and the distortion function*

$$\mathbb{K}(z) = \frac{1 - |\mu(z)|^2}{1 + |\mu(z)|^2}$$

is p-exponentially integrable in $\mathbb{D}$, that is

$$\int_{\mathbb{D}} \exp[p\mathbb{K}(z)]dz < \infty, \quad p > 0.$$

Then, there is a sense-preserving homeomorphism $f \in W^{1,P}(\mathbb{C})$, where $P(t) = \frac{t^2}{\log(e+t)}$, such that

$$f_{\overline{z}}(z) = \mu(z) f_z(z), \quad z \in \mathbb{C}.$$

When f is restricted to $\mathbb{D}$, we can map $f(\mathbb{D})$ back to $\mathbb{D}$ by a Riemann mapping Φ. As $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, $\partial f(\mathbb{D}) = f(\partial\mathbb{D})$ is a Jordan curve, so Φ can be extended to $\Phi : \overline{f(\mathbb{D})} \to \overline{\mathbb{D}}$ (Carathéodory's theorem, see e.g. [34]). Thus $\Phi \circ f|_{\overline{\mathbb{D}}}$ is a homeomorphism that maps $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$. We record this as follows.

Theorem 1.6.2 *Let μ be as in Theorem 1.6.1. Then, there is a sense-preserving homeomorphism $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that*

$$f_{\overline{z}}(z) = \mu(z) f_z(z), \quad z \in \mathbb{D}.$$

2 Minimisation of Exponential Distortion

In this chapter we will focus on the minimisation of exponential distortion. Let us restate the problem:

Let $p > 0$. The $\exp(p)$ mean distortion of a finite distortion self-homeomorphism of $\overline{\mathbb{D}}$ is defined as

$$\mathcal{E}_p(f) := \int_{\mathbb{D}} \exp[p\mathbb{K}(z, f)]dz. \tag{2.0.1}$$

Let $f_0 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a finite distortion homeomorphism such that $\mathcal{E}_p(f_0) < \infty$. We set

$$\begin{aligned} \mathcal{F}_p := \Big\{ & f \in W^{1,1}_{loc}(\mathbb{D}) : f \text{ is a homeomorphism from } \overline{\mathbb{D}} \text{ to } \overline{\mathbb{D}}, \\ & \mathcal{E}_p(f) < \infty, \text{ and } f|_{\partial\mathbb{D}} = f_0|_{\partial\mathbb{D}} \Big\}. \end{aligned} \tag{2.0.2}$$

Problem 2.0.1 *Find the minimal mappings $f \in \mathcal{F}_p$ such that*

$$\mathcal{E}_p(f) = \min_{g\in\mathcal{F}_p} \mathcal{E}_p(g).$$

As has been shown in Theorem 1.4.9 such minimal mappings must exist. In the following several chapters we will discuss the equations and establish regularity results.

2.1 A linear minimiser.

Theorem 2.1.1 *Let $L : \mathbb{D} \to \mathbb{C}$ be an injective linear map. Then L is the unique minimiser for the exponential problem among all the finite distortion homeomorphisms with the same boundary values as L.*

Proof. By Lemma 1.4.7 and Lemma 1.4.8, the function $\exp(pX^2J^{-1})$ is convex, for $X, J > 0$. That is,

$$\exp(p\frac{X^2}{J}) - \exp(p\frac{X_0^2}{J_0}) \geq p\exp(p\frac{X_0^2}{J_0})[\frac{2X_0}{J_0}(X - X_0) - \frac{X_0^2}{J_0^2}(J - J_0)]. \tag{2.1.1}$$

Let h be any finite distortion homeomorphism with the same boundary values as L. Let $\varepsilon > 0$ be arbitrarily given. We put $X = \|Dh\|$, $J = J_h + \varepsilon$, $X_0 = \|DL\|$, $J_0 = J_L + \varepsilon$ into (2.1.1), and integrate both sides. By the linearity of L we know DL is a constant. Thus

$$\begin{aligned} &\int_{\mathbb{D}} \exp(p\frac{\|Dh\|^2}{J_h + \varepsilon}) - \int_{\mathbb{D}} \exp(p\frac{\|DL\|^2}{J_L + \varepsilon}) \\ \geq & p\exp(p\frac{\|DL\|^2}{J_L + \varepsilon})[\frac{2\|DL\|}{J_L + \varepsilon}\int_{\mathbb{D}}(\|Dh\| - \|DL\|) - \frac{\|DL\|^2}{(J_L + \varepsilon)^2}\int_{\mathbb{D}}(J_h - J_L)]. \end{aligned} \tag{2.1.2}$$

We claim

$$\int_{\mathbb{D}} (\|Dh\| - \|DL\|) \geq 0, \tag{2.1.3}$$

and

$$\int_{\mathbb{D}} (J_h - J_L) \leq 0. \tag{2.1.4}$$

For (2.1.3), by the Cauchy-Schwarz inequality we have

$$\begin{aligned}\int_{\mathbb{D}} \|Dh\| - \|DL\| &\geq \int_{\mathbb{D}} \Big\langle \frac{DL}{\|DL\|}, Dh \Big\rangle - \|DL\| \\ &= \Big\langle \frac{DL}{\|DL\|}, \int_{\mathbb{D}} Dh - DL \Big\rangle = 0;\end{aligned}$$

For (2.1.4), by Theorem 1.2.9 we have

$$\int_{\mathbb{D}} J_L = |L(\mathbb{D})| \geq \int_{\mathbb{D}} J_h.$$

So we conclude

$$\int_{\mathbb{D}} \exp(p\frac{\|Dh\|^2}{J_h + \varepsilon}) - \int_{\mathbb{D}} \exp(p\frac{\|DL\|^2}{J_L + \varepsilon}) \geq 0, \quad \forall \varepsilon > 0. \tag{2.1.5}$$

We let $\varepsilon \to 0$ in (2.1.5). Note the integrand on the left hand side of (2.1.5) has an integrable dominator, namely

$$\Big| \exp(p\frac{\|Dh\|^2}{J_h + \varepsilon}) - \exp(p\frac{\|DL\|^2}{J_L + \varepsilon}) \Big| \leq \exp[p\mathbb{K}(z,h)] + \exp[p\mathbb{K}(z,L)] \in L^1(\Omega).$$

Thus by the dominated convergence theorem we get

$$\int_{\mathbb{D}} \exp(p\frac{\|Dh\|^2}{J_h}) - \int_{\mathbb{D}} \exp(p\frac{\|DL\|^2}{J_L}) = \lim_{\varepsilon \to 0} \int_{\mathbb{D}} \exp(p\frac{\|Dh\|^2}{J_h + \varepsilon}) - \int_{\mathbb{D}} \exp(p\frac{\|DL\|^2}{J_L + \varepsilon}) \geq 0.$$

Also note the equality holds only when

$$\|Dh\| = \Big\langle \frac{DL}{\|DL\|}, Dh \Big\rangle.$$

That is,

$$Dh = kDL,$$

for some $k \in \mathbb{C}$. However, as the boundary values are fixed, this happens only when $h = L$. □

Similar to Theorem 1.6.2, with a conformal Φ we can also map $L(\mathbb{D})$ back to $\mathbb{D}$, and $\Phi \circ L$ extends to $\overline{\mathbb{D}}$ as a homeomorphism.

Corollary 2.1.2 *Let $L : \mathbb{D} \to \mathbb{C}$ be a linear homeomorphism, and $\Phi : L(\mathbb{D}) \to \mathbb{D}$ be a conformal mapping. Then $f_0 = \Phi \circ L : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is the unique minimiser for Problem 2.0.1 with the boundary values $f = f_0$ along $\partial\mathbb{D}$.*

2.2 Equations of variational functions.

We come back to the general case. The aim of this section is to get the distributional equations for the minimisers of Problem 2.0.1. We will do this in two ways- outer and inner variations. Although inner variations have better regularities than outer variations (which we will see below), we will adopt both of them, since then we will have more equations for sufficiently regular minimisers.

2.2.1 Outer variation.

Let $\mathcal{E}_p$, $\mathcal{F}_p$ be as in (2.0.1), (2.0.2), and $f \in \mathcal{F}_p$. Define

$$f^t(z) := f(z) + t\varphi(z), \quad \varphi \in C_0^\infty(\mathbb{D}). \tag{2.2.1}$$

We then get a family of functions f^t, $t \in (-t_0, t_0)$ for some $t_0 > 0$, and each f^t gives an energy

$$F(t) := \mathcal{E}_p(f^t) = \int_{\mathbb{D}} \exp[p\mathbb{K}(z, f^t)]dz. \tag{2.2.2}$$

Definition 2.2.1 We say an $f \in \mathcal{F}_p$ is outer variational, if for every φ and f^t defined as in (2.2.1), there is a $t_0 > 0$ such that the family $\{f^t : t \in (-t_0, t_0)\}$ satisfies the following two conditions:

i) Each f^t is a candidate solution for the problem, namely, $f^t \in \mathcal{F}_p$. In our problem, this means that f^t must be a finite distortion homeomorphism such that $\exp[p\mathbb{K}(z, f^t)] \in L^1(\mathbb{D})$, and it has the same boundary values as f.

ii) The function $F(t)$ is differentiable at 0.

If f is an outer variational minimiser, we then have

$$F'(0) = 0. \tag{2.2.3}$$

This follows from Fermat's theorem on stationary points: otherwise there is a $t \in (-t_0, t_0)$ such that

$$\mathcal{E}_p(f^t) = F(t) < F(0) = \mathcal{E}_p(f),$$

which contradicts the assumption that f is a minimiser of the problem.

Unfortunately, not all of these requirements are automatically satisfied in our problem. To see this, we compute

$$f_z^t = f_z + t\varphi_z, \quad f_{\overline{z}}^t = f_{\overline{z}} + t\varphi_{\overline{z}}. \tag{2.2.4}$$

$$
\begin{aligned}
J_{f^t} &= |f_z^t|^2 - |f_{\overline{z}}^t|^2 \\
&= |f_z + t\varphi_z|^2 - |f_{\overline{z}} + t\varphi_{\overline{z}}|^2 \\
&= J(z,f) + t^2 J(z,\varphi) + 2t\Re e[f_z\overline{\varphi_z} - f_{\overline{z}}\overline{\varphi_{\overline{z}}}].
\end{aligned}
\tag{2.2.5}
$$

For an $f \in \mathcal{F}_p$ we only know it is a homeomorphism (but not necessarily a diffeomorphism), so it is possible that $J(z_0, f) = 0$ at some point $z_0 \in \mathbb{D}$. Then for any $\varphi \in C_0^\infty(\mathbb{D})$ such that $J(z_0,\varphi) < 0$, we have $J(z_0, f^t) < 0$, for some $t > 0$ or some $t < 0$. In this case f^t is not even a homeomorphism.

Later in Section 2.5 we will prove that if f is a C^1-diffeomorphism, then it is outer variational. But at this step we wish to get some equations for variational minimisers. In fact, as we have seen in (2.2.5), if f is outer variational, $J(z,f)$ must be uniformly bounded from below in any compact subset of $\mathbb{D}$, and then so is $|f_z|$, as $J(z,f) = |f_z|^2 - |f_{\overline{z}}|^2$. This validates the following computations at almost every $z \in \mathbb{D}$:

$$
\begin{aligned}
\mu_{f^t} &= \frac{f_{\overline{z}}^t}{f_z^t} = \frac{f_{\overline{z}} + t\varphi_{\overline{z}}}{f_z + t\varphi_z}, \\
\frac{\partial}{\partial t}\Big|_{t=0}\mu_{f^t} &= \frac{\varphi_{\overline{z}} f_z - \varphi_z f_{\overline{z}}}{f_z^2} = \frac{1}{f_z}(\varphi_{\overline{z}} - \varphi_z\mu_f), \\
\frac{\partial}{\partial t}\Big|_{t=0}|\mu_{f^t}|^2 &= 2\Re e[(\frac{\partial}{\partial t}\Big|_{t=0}\mu_{f^t})\cdot\overline{\mu_f}] = 2\Re e(\frac{\varphi_{\overline{z}}}{f_{\overline{z}}} - \frac{\varphi_z}{f_z})|\mu_f|^2, \\
\frac{\partial}{\partial t}\Big|_{t=0}\mathbb{K}_{f^t} &= \frac{2\frac{\partial}{\partial t}\Big|_{t=0}|\mu_{f^t}|^2}{(1-|\mu_f|^2)^2} = \frac{4\Re e(\frac{\varphi_{\overline{z}}}{f_{\overline{z}}} - \frac{\varphi_z}{f_z})|\mu_f|^2}{(1-|\mu_f|^2)^2}, \\
\frac{\partial}{\partial t}\Big|_{t=0}\exp[p\mathbb{K}_{f^t}] &= p\exp[p\mathbb{K}_f]\frac{4\Re e(\frac{\varphi_{\overline{z}}}{f_{\overline{z}}} - \frac{\varphi_z}{f_z})|\mu_f|^2}{(1-|\mu_f|^2)^2}.
\end{aligned}
$$

Put them into (2.2.3), and since at almost every $z \in \mathbb{D}$, $\exp[p\mathbb{K}(z,f^t)$ is real analytic near $t = 0$ as a function of t, we get

$$
\begin{aligned}
0 = F'(0) =& \frac{\partial}{\partial t}\Big|_{t=0}\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f^t)]dz \\
=& \int_{\mathbb{D}} pe^{p\mathbb{K}(z,f)}\cdot\frac{\partial}{\partial t}\Big|_{t=0}\mathbb{K}(z,f^t)dz \\
=& 4p\int_{\mathbb{D}}\frac{e^{p\mathbb{K}(z,f)}}{(1-|\mu_f(z)|^2)^2}\Re e(\frac{\varphi_{\overline{z}}(z)}{f_{\overline{z}}(z)} - \frac{\varphi_z(z)}{f_z(z)})|\mu_f(z)|^2 dz.
\end{aligned}
$$

In fact this holds for every $\varphi \in C_0^\infty(\mathbb{D})$. So we put $i\varphi$ into the formula, and then get the same equation for the imaginary parts. Hence

$$
\int_{\mathbb{D}}\frac{e^{p\mathbb{K}(z,f)}}{(1-|\mu_f(z)|^2)^2}\Big[\frac{\varphi_{\overline{z}}(z)}{f_{\overline{z}}(z)} - \frac{\varphi_z(z)}{f_z(z)}\Big]|\mu_f(z)|^2 dz = 0.
$$

After rearrangement we can write this as

$$\int_{\mathbb{D}} \frac{|\mu_f(z)|^2 e^{p\mathbb{K}(z,f)}}{f_z(z)[1-|\mu_f(z)|^2]^2} \varphi_z(z) dz = \int_{\mathbb{D}} \frac{|\mu_f(z)|^2 e^{p\mathbb{K}(z,f)}}{f_{\overline{z}}(z)[1-|\mu_f(z)|^2]^2} \varphi_{\overline{z}}(z) dz. \tag{2.2.6}$$

This equation is usually called an Euler-Lagrange equation.

Theorem 2.2.2 *Let f be a minimiser of Problem 2.0.1. If f is outer variational, then it satisfies the Euler-Lagrange equation (2.2.6).*

2.2.2 Inner variation.

For the inner variation we define a family of functions

$$g^t(z) = z + t\varphi(z), \quad \varphi \in C_0^\infty(\mathbb{D}), \tag{2.2.7}$$

We calculate

$$\mu_{g^t} = \frac{g^t_{\overline{z}}}{g^t_z} = \frac{t\varphi_{\overline{z}}}{1+t\varphi_z},$$
$$J_{g^t} = |1+t\varphi_z|^2 - t^2|\varphi_{\overline{z}}|^2.$$

So we can assume that $|t|$ is so small that the Jacobian $J(z, g^t) > 0$ in $\mathbb{D}$ then we can apply the inverse mapping theorem to get every g^t is a local diffeomorphism. However, in $\mathbb{D} - supp(\varphi)$, g^t is the identity map. Thus topology guarantees that g^t is a global diffeomorphism. This is the monodromy theorem.

Lemma 2.2.3 *For each $\varphi \in C_0^\infty(\mathbb{D})$ there is a $t_0 > 0$ such that for any $t \in (-t_0, t_0)$, $g^t(z) = z + t\varphi$ is a C^∞-diffeomorphism of $\mathbb{D}$ which extends by the identity on the boundary $\mathbb{S}$.*

By Lemma 2.2.3, the inverse $(g^t)^{-1}$ exists and is also a C^∞-diffeomorphism. Thus we can define, for $f \in \mathcal{F}_p$,

$$f^t(w) = f \circ (g^t)^{-1}(w), \tag{2.2.8}$$

$$F(t) = \mathcal{E}_p(f^t) = \int_{\mathbb{D}} \exp[p\mathbb{K}(w, f^t)] dw.$$

We will see that the family f^t has better regularity than that in the outer variation.

Lemma 2.2.4 *Let g^t, f^t be as in (2.2.7), (2.2.8), $t \in (-t_0, t_0)$ be as in Lemma 2.2.3. Then f^t is a finite distortion homeomorphism.*

Proof. First, f^t is a homeomorphism as both f and g^t are. As g^t is a diffeomorphism we may compute directly that, for almost every $w \in \mathbb{D}$ and $z = (g^t)^{-1}(w)$,

$$Df^t(w) = Df(z) \cdot D(g^t)^{-1}(w) = Df(z) \cdot (Dg^t(z))^{-1}, \tag{2.2.9}$$

$$J(w, f^t) = J(z, f)J(w, (g^t)^{-1}) = \frac{J(z, f)}{J(z, g^t)}. \tag{2.2.10}$$

As $g^t(z) = z$ in $\mathbb{D} - supp(\varphi)$ we have a uniform upper bound for $|(Dg^t(z))^{-1}|$ and a uniform lower bound for $J(z, g^t)$. Precisely,

$$\sup_{z\in\mathbb{D}} |(Dg^t(z))^{-1}| = \sup_{z\in supp(\varphi)} |(Dg^t(z))^{-1}| < \infty,$$

$$\inf_{z\in\mathbb{D}} J(z, g^t) = \inf_{z\in supp(\varphi)} J(z, g^t) > 0.$$

Then, by (2.2.9) we get $f^t \in W^{1,1}_{loc}(\mathbb{D})$. By (2.2.10), $J(w, f^t) = 0$ if only if $J(z, f) = 0$, which implies that $Df(z) = 0$, as f has finite distortion. Thus $Df^t(w) = 0$ by (2.2.9). This proves that f^t has finite distortion. □

We recall the conditions in Definition 2.2.1: $f^t \in \mathcal{F}_p$ and $F(t)$ is differentiable at 0. Analogously we can define the inner variational functions.

Definition 2.2.5 We say an $f \in \mathcal{F}_p$ is inner variational, if for every φ and f^t defined as in (2.2.8), there is a $t_0 > 0$ such that the family $\{f^t : t \in (-t_0, t_0)\}$ satisfies the following conditions:

i) Each $f^t \in \mathcal{F}_p$;

ii) The function $F(t) = \mathcal{E}_p(f^t)$ is differentiable at 0.

Although by Lemma 2.2.4 every f^t is a finite distortion homeomorphism, it is not automatically in the space $\mathcal{F}_p$, as the distortion function $\mathbb{K}(z, f^t)$ might not be p-exponentially integrable. We will explain this with examples in Section 2.3. For now we assume that f is an inner variational minimiser and calculate the Euler-Lagrange equation.

First we compute the distortion of $f \circ (g^t)^{-1}$:

$$\begin{aligned}(f^t)_w(g^t(z)) &= f_z(z)((g^t)^{-1})_w(g^t(z)) + f_{\overline{z}}(z)\overline{((g^t)^{-1})_{\overline{w}}}(g^t(z)) \\ &= f_z(z)\frac{\overline{g^t_z}(z)}{J(z, g^t)} - f_{\overline{z}}(z)\frac{\overline{g^t_{\overline{z}}}(z)}{J(z, g^t)},\end{aligned}$$

$$
\begin{aligned}
(f^t)_{\overline{w}}(g^t(z)) &= f_z(z)((g^t)^{-1})_{\overline{w}}(g^t(z)) + f_{\overline{z}}(z)\overline{((g^t)^{-1})_w}(g^t(z)) \\
&= -f_z(z)\frac{g^t_{\overline{z}}(z)}{J(z,g^t)} + f_{\overline{z}}(z)\frac{g^t_z(z)}{J(z,g^t)},
\end{aligned}
$$

$$
\mu_{f^t}(g^t(z)) = \frac{(f^t)_{\overline{w}}(g^t(z))}{(f^t)_w(g^t(z))} = \frac{\mu_f(z) - \mu_{g^t}(z)}{1 - \mu_f(z)\overline{\mu_{g^t}}(z)}\frac{g^t_z(z)}{\overline{g^t_z}(z)},
$$

$$
\begin{aligned}
|\mu_{f^t}(g^t(z))|^2 &= \frac{\mu_f(z) - \mu_{g^t}(z)}{1 - \mu_f(z)\overline{\mu_{g^t}}(z)}\frac{\overline{\mu_f(z)} - \overline{\mu_{g^t}(z)}}{1 - \mu_f(z)\overline{\mu_{g^t}}(z)} \\
&= \frac{|\mu_f|^2 + |\mu_{g^t}|^2 - 2\Re e(\mu_f\overline{\mu_{g^t}})}{1 + |\mu_f|^2|\mu_{g^t}|^2 - 2\Re e(\mu_f\overline{\mu_{g^t}})} \\
&= 1 + \frac{|\mu_f|^2 + |\mu_{g^t}|^2 - 1 - |\mu_f|^2|\mu_{g^t}|^2}{1 + |\mu_f|^2|\mu_{g^t}|^2 - 2\Re e(\mu_f\overline{\mu_{g^t}})},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{K}(g^t(z), f\circ(g^t)^{-1}) &= \frac{1 + |\mu_{f\circ(g^t)^{-1}}(g^t(z))|^2}{1 - |\mu_{f\circ(g^t)^{-1}}(g^t(z))|^2} \\
&= \frac{1 + |\mu_f|^2|\mu_{g^t}|^2 + |\mu_f|^2 + |\mu_{g^t}|^2 - 4\Re e(\mu_f\overline{\mu_{g^t}})}{1 + |\mu_f|^2|\mu_{g^t}|^2 - |\mu_f|^2 - |\mu_{g^t}|^2} \\
&= \frac{(1 + |\mu_f|^2)(1 + |\mu_{g^t}|^2) - 4\Re e(\mu_f\overline{\mu_{g^t}})}{(1 - |\mu_f|^2)(1 - |\mu_{g^t}|^2)} \\
&= \mathbb{K}(z,f)\mathbb{K}(z,g^t)[1 - \frac{4\Re e(\mu_f\overline{\mu_{g^t}})}{(1 + |\mu_f|^2)(1 + |\mu_{g^t}|^2)}]. \qquad (2.2.11)
\end{aligned}
$$

Differentiate this at $t = 0$ we get

$$
\frac{\partial}{\partial t}\Big|_{t=0} J(z, g^t) = 2\Re e(\varphi_z), \qquad (2.2.12)
$$

$$
\frac{\partial}{\partial t}\Big|_{t=0}\mu_{g^t} = \varphi_{\overline{z}},
$$

$$
\frac{\partial}{\partial t}\Big|_{t=0}|\mu_{g^t}|^2 = 0,
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0}|\mu_{f^t}(g^t(z))|^2 &= \frac{\partial}{\partial t}\Big|_{t=0}\Big[1 + \frac{|\mu_f|^2 + |\mu_{g^t}|^2 - 1 - |\mu_f|^2|\mu_{g^t}|^2}{1 + |\mu_f|^2|\mu_{g^t}|^2 - 2\Re e(\mu_f\overline{\mu_{g^t}})}\Big] \\
&= 2\Re e\Big[\overline{\mu_f}\frac{\partial}{\partial t}\Big|_{t=0}\mu_{g^t}\Big](|\mu_f|^2 - 1) \\
&= 2\Re e(\overline{\mu_f}\varphi_{\overline{z}})(|\mu_f|^2 - 1),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0}\mathbb{K}(g^t(z),f^t) &= \frac{\partial}{\partial t}\Big|_{t=0}\Big[\frac{1+|\mu_{f\circ(g^t)^{-1}}(g^t(z))|^2}{1-|\mu_{f\circ(g^t)^{-1}}(g^t(z))|^2}\Big] \\
&= \frac{2\frac{\partial}{\partial t}\Big|_{t=0}|\mu_{f^t}(g^t(z))|^2}{(1-|\mu_f(z)|^2)^2} \\
&= \frac{4\Re e(\overline{\mu_f}\varphi_{\overline{z}})}{|\mu_f|^2-1}.
\end{aligned} \tag{2.2.13}
$$

We put (2.2.12), (2.2.13) into the equation $F'(0)=0$. Then,

$$
\begin{aligned}
0 =&\frac{\partial}{\partial t}\Big|_{t=0}\int_{\mathbb{D}}\exp[p\mathbb{K}(w,f^t)]dw \\
=&\frac{\partial}{\partial t}\Big|_{t=0}\int_{\mathbb{D}}\exp[p\mathbb{K}(g^t(z),f^t)]J(z,g^t)dz \\
=&\int_{\mathbb{D}}p\exp[p\mathbb{K}(g^t(z),f^t)]J(z,g^t)\frac{\partial}{\partial t}\mathbb{K}(g^t(z),f^t)dz\Big|_{t=0} \\
&+\int_{\mathbb{D}}\exp[p\mathbb{K}(g^t(z),f^t)]\frac{\partial}{\partial t}J(z,g^t)dz\Big|_{t=0} \\
=&4p\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]\frac{\Re e(\overline{\mu_f(z)}\varphi_{\overline{z}}(z))}{|\mu_f(z)|^2-1}dz \\
&+2\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]\Re e(\varphi_z(z))dz.
\end{aligned}
$$

We then get the Euler-Lagrange equation

$$
\int_{\mathbb{D}}\exp[p\frac{1+|\mu_f(z)|^2}{1-|\mu_f(z)|^2}]\Re e(\varphi_z(z))dz = 2p\int_{\mathbb{D}}\frac{\Re e(\overline{\mu_f(z)}\varphi_{\overline{z}}(z))}{1-|\mu_f(z)|^2}\exp[p\frac{1+|\mu_f(z)|^2}{1-|\mu_f(z)|^2}])dz.
$$

As $\varphi\in C_0^\infty(\mathbb{D})$ is arbitrarily chosen, the above equation also holds for $i\varphi$. It follows that for every $\varphi\in C_0^\infty(\mathbb{D})$,

$$
\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]\varphi_z(z)dz = 2p\int_{\mathbb{D}}\frac{\overline{\mu_f(z)}}{1-|\mu_f(z)|^2}\exp[p\mathbb{K}(z,f)])\varphi_{\overline{z}}(z)dz. \tag{2.2.14}
$$

So we conclude

Theorem 2.2.6 *Let f be a minimiser for Problem 2.0.1. If f is inner variational, then it satisfies the Euler-Lagrange equation (2.2.14).*

2.3 Examples of non-inner variational functions.

As we indicated in the last section there are functions f with p-exponentially integrable distortion, but for arbitrarily small t, the distortion of $f^t = f\circ(g^t)^{-1}$ is not p-exponentially integrable. In this section we will construct concrete examples to explain this.

2.3.1 A basic example.

We start with the function

$$F(r) = \frac{1}{r^2 \log^2(\frac{2}{r})}, \quad 0 < r < 1. \tag{2.3.1}$$

This function satisfies

$$\int_0^1 F(r) r dr < \infty,$$

but for any $q > 1$,

$$\int_0^1 F^q(r) r dr = \infty.$$

We wish to find an f such that

$$e^{\mathbb{K}(z,f)} = \frac{e}{r^2 \log^2(\frac{2}{r})}, \quad r = |z|. \tag{2.3.2}$$

Note then

$$\mathbb{K}(z, f) = 1 - 2\log(r \log \frac{2}{r}).$$

For continuous $\rho : [0, 1] \to [0, 1]$ we can define

$$f(z) = \frac{z}{|z|}\rho(|z|).$$

Such an f is called a radial stretching ([6, Section 2.6]). We can compute its Beltrami coefficient as

$$\mu_f(z) = \frac{z}{\bar{z}} \frac{|z|\dot{\rho}(|z|) - \rho(|z|)}{|z|\dot{\rho}(|z|) + \rho(|z|)}. \tag{2.3.3}$$

Then,

$$\mathbb{K}(z, f) = \frac{1 + |\mu(z)|^2}{1 - |\mu(z)|^2} = \frac{(r\dot{\rho} + \rho)^2 + (r\dot{\rho} - \rho)^2}{(r\dot{\rho} + \rho)^2 - (r\dot{\rho} - \rho)^2} = \frac{1}{2}(\frac{r\dot{\rho}}{\rho} + \frac{\rho}{r\dot{\rho}}).$$

Solving the formula, we get

$$\frac{r\dot{\rho}}{\rho} = \mathbb{K}(z, f) \pm \sqrt{\mathbb{K}^2(z, f) - 1}.$$

We choose the larger answer, which is no smaller than 1. Then,

$$\frac{r\dot{\rho}}{\rho} = 1 - 2\log(r \log \frac{2}{r}) + \sqrt{[1 - 2\log(r \log \frac{2}{r})]^2 - 1},$$

$$\frac{1}{\rho}d\rho = \frac{1 - 2\log(r\log\frac{2}{r}) + \sqrt{[1 - 2\log(r\log\frac{2}{r})]^2 - 1}}{r}dr,$$

$$\log\rho(r) = \int_1^r \frac{1 - 2\log(s\log\frac{2}{s}) + \sqrt{[1 - 2\log(s\log\frac{2}{s})]^2 - 1}}{s}ds,$$

$$\rho(r) = \exp[\int_1^r \frac{1 - 2\log(s\log\frac{2}{s}) + \sqrt{[1 - 2\log(s\log\frac{2}{s})]^2 - 1}}{s}ds]. \tag{2.3.4}$$

Note this satisfies $\rho(1) = e^0 = 1$, and $\log\rho(0) \leq \int_1^0 \frac{1}{s}ds = -\infty$, so $\rho(0) = 0$ as required. So we have found a self-homeomorphism of $\mathbb{D}$:

$$f(z) = \frac{z}{|z|}\rho(|z|), \tag{2.3.5}$$

where ρ is as (2.3.4), and then the distortion $\mathbb{K}(z, f)$ satisfies (2.3.2).

Now recall the composition formula (2.2.11):

$$\mathbb{K}(g^t(z), f \circ (g^t)^{-1}) = \mathbb{K}(z, f)\mathbb{K}(z, g^t)[1 - \frac{4\Re e(\mu_f\overline{\mu_{g^t}})}{(1 + |\mu_f|^2)(1 + |\mu_{g^t}|^2)}].$$

We are interested in the sign of the term $\Re e(\mu_f\overline{\mu}_{g^t})$. If it is non-positive, we then have

$$\mathbb{K}(g^t(z), f \circ (g^t)^{-1}) \geq \mathbb{K}(z, f)\mathbb{K}(z, g^t).$$

Observe that for each pair of complex numbers $z = a + bi$ and $w = c + di$, we have

$$\Re e(z\bar{w}) = ac + bd.$$

That is to say, at least one of the following is non-positive:

$$\Re e(z\bar{w}), \quad \Re e(zw), \quad \Re e(-z\bar{w}), \quad \Re e(-zw).$$

On the other hand, by (2.3.3) we can see that for a radial stretching $f(z) = \frac{z}{|z|}\rho(|z|)$,

$$\mu_f(z) = \frac{z}{\bar{z}}\frac{|z|\dot{\rho}(|z|) - \rho(|z|)}{|z|\dot{\rho}(|z|) + \rho(|z|)},$$

$$\mu_f(iz) = \frac{iz}{\overline{iz}}\frac{|z|\dot{\rho}(|z|) - \rho(|z|)}{|z|\dot{\rho}(|z|) + \rho(|z|)} = -\mu_f(z),$$

$$\mu_f(\bar{z}) = \frac{\bar{z}}{z}\frac{|z|\dot{\rho}(|z|) - \rho(|z|)}{|z|\dot{\rho}(|z|) + \rho(|z|)} = \overline{\mu_f(z)},$$

$$\mu_f(i\bar{z}) = \frac{i\bar{z}}{\overline{i\bar{z}}}\frac{|z|\dot{\rho}(|z|) - \rho(|z|)}{|z|\dot{\rho}(|z|) + \rho(|z|)} = -\overline{\mu_f(z)}.$$

Also, when t is sufficiently small, $\mu_{g^t} = \frac{t\varphi_{\overline{z}}}{1+t\varphi_z}$ is continuous on $\mathbb{D}$. Assume that $\Re e(\mu_{g^t}(0)) \neq 0$ and $\Im m(\mu_{g^t}(0)) \neq 0$, and by choosing r' small, there is a neighbourhood $A = \mathbb{D}(0, r')$ in which $\Re e(\mu_{g^t})$ and $\Im m(\mu_{g^t})$ do not change their signs, and

$$\inf_{z\in A} |\mu_{g^t}(z)| \geq \varepsilon_1 > 0.$$

Then,

$$q := \inf_{z\in A} \mathbb{K}(z, g^t) > 1.$$

Also, $J(z, g^t) = |1 + t\phi_z|^2 - t^2|\phi_{\overline{z}}|^2$ gives that

$$J(z, g^t) > \frac{1}{2},$$

if t is sufficiently small. Combining these facts we finally conclude

$$\begin{aligned}
\int_{\mathbb{D}} \exp[\mathbb{K}(w, f\circ(g^t)^{-1})]dw &= \int_{\mathbb{D}} \exp[\mathbb{K}(z, f)\mathbb{K}(z, g^t) - \frac{4\Re e(\mu_f(z)\overline{\mu}_{g^t}(z))}{(1-|\mu_f(z)|^2)(1-|\mu_{g^t}(z)|^2)}]J(z, g^t)dz \\
&\geq \frac{1}{8}\int_A \exp[\mathbb{K}(z, f)\cdot \inf_{z\in A}\mathbb{K}(z, g^t)]dz \\
&= \frac{1}{8}\int_A [\frac{e}{|z|^2\log^2(\frac{2}{|z|})}]^q dz \\
&= \frac{1}{8}\int_0^{2\pi} d\theta \int_0^{r'} [\frac{e}{r^2\log^2(\frac{2}{r})}]^q \cdot rdr \\
&\geq \frac{\pi}{4}\int_0^{r'} [\frac{e}{r\log^2(\frac{2}{r})}]^q dr \\
&= \infty.
\end{aligned}$$

We come back to consider the condition $\Re e(\mu_{g^t}(0))\Im m(\mu_{g^t}(0)) \neq 0$. From the expression $\mu_{g^t}(z) = \frac{t\varphi_{\overline{z}}(z)}{1+t\varphi_z(z)}$ we can compute

$$\Re e(\mu_{g^t}) = \frac{1}{2}(\frac{t\varphi_{\overline{z}}}{1+t\varphi_z} + \overline{\frac{t\varphi_{\overline{z}}}{1+t\varphi_z}}) = \frac{t\Re e(\varphi_{\overline{z}}) + t^2\Re e(\varphi_{\overline{z}}\overline{\varphi_z})}{|1+t\varphi_z|^2},$$

$$\Im m(\mu_{g^t}) = \frac{1}{2i}(\frac{t\varphi_{\overline{z}}}{1+t\varphi_z} - \overline{\frac{t\varphi_{\overline{z}}}{1+t\varphi_z}}) = \frac{t\Im m(\varphi_{\overline{z}}) + t^2\Im m(\varphi_{\overline{z}}\overline{\varphi_z})}{|1+t\varphi_z|^2}.$$

So for t small, the condition $\Re e(\mu_{g^t}(0))\Im m(\mu_{g^t}(0)) \neq 0$ is satisfied if only

$$\Re e(\varphi_{\overline{z}}(0))\Im m(\varphi_{\overline{z}}(0)) \neq 0. \tag{2.3.6}$$

We record this as follows.

Theorem 2.3.1 *The Sobolev homeomorphism f defined by (2.3.4)-(2.3.5) has exponentially integrable distortion $\mathbb{K}(z,f)$. However, for any $\varphi \in C_0^\infty(\mathbb{D})$ that satisfies (2.3.6) and g^t defined as (2.2.7), there is a $t_0 > 0$ such that for any nonzero $t \in (-t_0, t_0)$,*

$$\int_{\mathbb{D}} \exp[\mathbb{K}(w, f\circ (g^t)^{-1})]dw = \infty.$$

2.3.2 A totally non-variational function.

We wish to remove condition (2.3.6) and find a Sobolev homeomorphism f that is non-variational with respect to any non-zero $\varphi \in C_0^\infty(\mathbb{D})$. To this end we need the following lemma:

Lemma 2.3.2 *In the unit disk $\mathbb{D}$ there are a countable dense subset $\{z_k\}$, disjoint Borel sets S_1, S_2, S_3, S_4, and a positive number $\delta > 0$ with the property that for every point z_k, there is an $R_k > 0$ such that for any $r \in (0, R_k)$ and $i = 1, 2, 3, 4$,*

$$\frac{|S_i \cap D(z_k, r)|}{|D(z_k, r)|} > \delta.$$

We postpone the proof to the Section 2.3.3. Now let $F(r)$ be as defined in (2.3.1). This time we set

$$\mathbb{K}(z,f) = 1 + \frac{1}{p}\log\big(\sum_k \frac{1}{2^k} F(|z-z_k|)\chi_{D(z_k, dist(z_k,\partial\mathbb{D}))}\big),$$

where $\{z_k\} \subset \mathbb{D}$ is a dense subset as in Lemma 2.3.2. Then $\exp[p\mathbb{K}(z,f)] \in L^1(\mathbb{D})$. Indeed,

$$\int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]dz \le \sum_k \frac{2\pi e^p}{2^k}\int_0^1 F(r) r dr < \infty.$$

The absolute value of the Beltrami coefficient that corresponds to $\mathbb{K}(z,f)$ is

$$|\mu_f(z)| = \sqrt{\frac{\mathbb{K}(z,f)-1}{\mathbb{K}(z,f)+1}}. \tag{2.3.7}$$

By Theorem 1.6.2 we may set $\mu_f(z) = |\mu_f(z)|e^{i\theta(z)}$ for any measurable function $\theta : \mathbb{D} \to [0, 2\pi)$ and then find a homeomorphism $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ that has Beltrami coefficient μ_f on $\mathbb{D}$. Recall the composition formula (2.2.11):

$$\mathbb{K}(g^t(z), f\circ(g^t)^{-1}) = \mathbb{K}(z,f)\mathbb{K}(z,g^t)[1 - \frac{4\Re e(\mu_f\overline{\mu}_{g^t})}{(1+|\mu_f|^2)(1+|\mu_{g^t}|^2)}]. \tag{2.3.8}$$

Put $g^t = z + t\varphi$ into consideration we can compute

$$\mu_{g^t}(z) = \frac{t\varphi_{\overline{z}}(z)}{1+t\varphi_z(z)}, \tag{2.3.9}$$

$$\Re e(\mu_f \overline{\mu}_{g^t}) = \frac{1}{|1+t\varphi_z|^2}\Big(t\Re e(\varphi_{\overline{z}}\overline{\mu_f}) + t^2\Re e(\varphi_{\overline{z}}\overline{\mu_f \varphi_z})\Big). \tag{2.3.10}$$

We now determine the argument of μ_f. Let S_1, S_2, S_3, S_4 as in Lemma 2.3.2, and set

$$\mu_f(z) = \begin{cases} |\mu_f(z)|, & z \in S_1; \\ -|\mu_f(z)|, & z \in S_2; \\ i|\mu_f(z)|, & z \in S_3; \\ -i|\mu_f(z)|, & z \in \mathbb{D} - \bigcup_{i=1}^3 S_i \supset S_4. \end{cases}$$

So all of them have the density of δ near every z_k. We put $\pm|\mu_f|$ and $\pm i|\mu_f|$ into (2.3.10), respectively, and get

$$\Re e(\mu_f \overline{\mu}_{g^t}) = \frac{|\mu_f|}{|1+t\varphi_z|^2}\Big(t\Re e(\varphi_{\overline{z}}) + t^2\Re e(\varphi_{\overline{z}}\overline{\varphi_z})\Big), \quad z \in S_1; \tag{2.3.11}$$

$$\Re e(\mu_f \overline{\mu}_{g^t}) = -\frac{|\mu_f|}{|1+t\varphi_z|^2}\Big(t\Re e(\varphi_{\overline{z}}) + t^2\Re e(\varphi_{\overline{z}}\overline{\varphi_z})\Big), \quad z \in S_2; \tag{2.3.12}$$

$$\Re e(\mu_f \overline{\mu}_{g^t}) = \frac{|\mu_f|}{|1+t\varphi_z|^2}\Big(t\Im m(\varphi_{\overline{z}}) + t^2\Im m(\varphi_{\overline{z}}\overline{\varphi_z})\Big), \quad z \in S_3; \tag{2.3.13}$$

$$\Re e(\mu_f \overline{\mu}_{g^t}) = -\frac{|\mu_f|}{|1+t\varphi_z|^2}\Big(t\Im m(\varphi_{\overline{z}}) + t^2\Im m(\varphi_{\overline{z}}\overline{\varphi_z})\Big), \quad z \in S_4. \tag{2.3.14}$$

There are several cases depending on φ:

i) $\varphi_{\overline{z}} \equiv 0$ in $\mathbb{D}$. Since $\varphi \in C_0^\infty(\mathbb{D})$, this happens only when $\varphi \equiv 0$ in $\mathbb{D}$, and then $g^t(z) = z$, $f \circ (g^t)^{-1} = f$.

ii) If $\Re e(\varphi_{\overline{z}})$ is not the constant 0 in $\mathbb{D}$, say $\Re e(\varphi_{\overline{z}})(z_0) > 0$ at some point $z_0 \in \mathbb{D}$. Then by the smoothness of φ there is an open neighbourhood A where $\Re e(\varphi_{\overline{z}}) \geq \varepsilon_1 > 0$. In view of (2.3.9) and (2.3.11), there is a $t_0 < 0$ such that for any $t \in (t_0, 0)$, in $A \cap S_1$ we have

$$|\mu_{g^t}| \geq \varepsilon_2 t > 0, \tag{2.3.15}$$

$$\Re e(\mu_f \overline{\mu}_{g^t}) < 0, \tag{2.3.16}$$

$$J(z, g^t) = |1+t\phi_z|^2 - t^2|\phi_{\overline{z}}|^2 > \frac{1}{2}. \tag{2.3.17}$$

Then by (2.3.15), (2.3.16) and (2.3.8),

$$\mathbb{K}(g^t(z), f \circ (g^t)^{-1}) \geq q\mathbb{K}(z, f).$$

By the density of $\{z_k\}$ there is a $z_k \in A$. We choose a small disk $D(z_k, r_0) \subset A$ where $r_0 < R_k$ as in Lemma 2.3.2. Combine all the arguments above we now can compute

$$\begin{aligned}
\int_{\mathbb{D}} \exp[p\mathbb{K}(w, f \circ (g^t)^{-1})]dw &= \int_{\mathbb{D}} \exp[p\mathbb{K}(g^t(z), f \circ (g^t)^{-1})J(z, g^t)]dz \\
&\geq \frac{1}{2}\int_{D(z_k,r_0)\cap S_1} \exp[pq\mathbb{K}(z, f)]dz \\
&\geq \frac{\delta}{2}\int_{D(z_k,r_0)} \exp[pq\mathbb{K}(z, f)]dz \\
&\geq \pi\delta \int_0^{r_0} (\frac{e^p}{2^k}F(r))^q r dr \\
&\geq C\int_0^{r_0} F^q(r) r dr = \infty.
\end{aligned}$$

If $\Re e(\varphi_{\overline{z}})(z_0) < 0$, then by (2.3.12) the same result follows.

iii) If $\Im m(\varphi_{\overline{z}})$ not the constant 0 in $\mathbb{D}$. Then we exploit (2.3.13), (2.3.14), and the same result follows.

By above arguments we can conclude:

Theorem 2.3.3 *Given $p > 0$ there is a homeomorphism $f : \mathbb{D} \to \mathbb{D}$ such that $\exp[p\mathbb{K}(z, f)] \in L^1(\mathbb{D})$ with the property that for any non-constant $\varphi \in C_0^\infty(\mathbb{D})$, there is a $t_0 > 0$ such that for any nonzero $t \in (-t_0, t_0)$, $\exp[p\mathbb{K}(g^t(z), f \circ (g^t)^{-1})] \notin L^1(\mathbb{D})$, where $g^t = z + t\varphi$.*

In fact we can even generalise Theorem 2.3.3 to the complex coefficient case. That is, set

$$g^\eta = z + \eta\varphi, \quad \eta \in \mathbb{C}, \quad \varphi \in C_0^\infty(\mathbb{D}).$$

In this case

$$\Re e(\mu_f \overline{\mu}_{g^\eta}) = \frac{1}{|1 + \eta\varphi_z|^2}\Big(\Re e(\eta\varphi_{\overline{z}}\overline{\mu_f}) + \Re e(\eta^2\varphi_{\overline{z}}\overline{\mu_f\varphi_z})\Big). \tag{2.3.18}$$

Again we choose μ_f and S_1, S_2, S_3, S_4 same as before. Write $\eta = |\eta|e^{i\alpha}$, then (2.3.18) reads as

$$\Re e(\mu_f \overline{\mu}_{g^\eta}) = \frac{|\mu_f|}{|1 + \eta\varphi_z|^2}\Big(|\eta|\Re e(e^{i\alpha}\varphi_{\overline{z}}) + |\eta|^2\Re e(e^{2i\alpha}\varphi_{\overline{z}}\overline{\varphi_z})\Big), \quad z \in S_1,$$

and analogously for S_2, S_3, S_4 as (2.3.12)-(2.3.14).

Then, for any non-constant φ, we may find a neighbourhood in $\mathbb{D}$ where either $\Re e(e^{i\alpha}\varphi_{\bar{z}})$ or $\Im m(e^{i\alpha}\varphi_{\bar{z}})$ is nonzero. So by the same argument as before, there is an $\varepsilon > 0$ that for any η with $|\eta| < \varepsilon$,

$$\int_{\mathbb{D}} \exp[p\mathbb{K}(g^\eta(z), f \circ (g^\eta)^{-1})]dz = \infty.$$

For every fixed φ, the number ε depends only on α. If we let α vary in $[0, 2\pi]$, then $e^{i\alpha}\varphi_{\bar{z}}$ and $e^{2i\alpha}\varphi_{\bar{z}}\overline{\varphi_z}$ move continuously w.r.t. α, thus $\varepsilon = \varepsilon(\alpha)$ can be chosen as a continuous function of α. Now since $[0, 2\pi]$ is compact, $\varepsilon(\alpha)$ admits a positive minimum value. We then have proved:

Theorem 2.3.4 *GIven $p > 0$, there is a homeomorphism $f : \mathbb{D} \to \mathbb{D}$ such that $\exp[p\mathbb{K}(z, f)] \in L^1(\mathbb{D})$ with the property that for any non-constant $\varphi \in C_0^\infty(\mathbb{D})$, there is an $\varepsilon > 0$ such that for any nonzero $\eta \in \mathbb{C}$ with $|\eta| < \varepsilon$, $\exp[p\mathbb{K}(g^\eta(z), f \circ (g^\eta)^{-1})] \notin L^1(\mathbb{D})$, where $g^\eta = z + \eta\varphi$.*

2.3.3 The construction of density.

We prove Lemma 2.3.2 in this part. As Step 1, we start with the first point $p^1 = (0, 0)$ and choose the disk sectors

$$S_1^1 = \{z \in D(p^1, \frac{1}{2^5}) : 0 < \arg(z - p^1) < \frac{\pi}{2}\},$$

$$S_2^1 = \{z \in D(p^1, \frac{1}{2^5}) : \frac{\pi}{2} < \arg(z - p^1) < \pi\},$$

$$S_3^1 = \{z \in D(p^1, \frac{1}{2^5}) : \pi < \arg(z - p^1) < \frac{3\pi}{2}\},$$

$$S_4^1 = \{z \in D(p^1, \frac{1}{2^5}) : \frac{3\pi}{2} < \arg(z - p^1) < 2\pi\}.$$

We construct inductively. At Step $n \geq 2$, we choose the points $p_{j,l}^n = (\frac{j}{2^{n-1}}, \frac{l}{2^{n-1}})$, for any integers $j, l \in [1 - 2^{n-1}, 2^{n-1} - 1]$ such that $D(p_{j,l}^n, \frac{1}{2^{5n}}) \subset \mathbb{D}$, and $p_{j,l}^n$ has not been chosen in the previous steps. Define the sector unions

$$S_1^n = \bigcup_{j,l}\{z \in D(p_{j,l}^n, \frac{1}{2^{5n}}) : 0 < \arg(z - p_{j,l}^n) < \frac{\pi}{2}\},$$

$$S_2^n = \bigcup_{j,l}\{z \in D(p_{j,l}^n, \frac{1}{2^{5n}}) : \frac{\pi}{2} < \arg(z - p_{j,l}^n) < \pi\},$$

$$S_3^n = \bigcup_{j,l}\{z \in D(p_{j,l}^n, \frac{1}{2^{5n}}) : \pi < \arg(z - p_{j,l}^n) < \frac{3\pi}{2}\},$$

$$S_4^n = \bigcup_{j,l} \{z \in D(p_{j,l}^n, \frac{1}{2^{5n}}) : \frac{3\pi}{2} < \arg(z - p_{j,l}^n) < 2\pi\}.$$

Write

$$S^n = S_1^n \cup S_2^n \cup S_3^n \cup S_4^n.$$

We now define S_i as the set such that $z \in S_i^n$ for some n but not in S^m for any $m \geq n+1$. Precisely,

$$S_i = \bigcup_{n=1}^{\infty} \Big(S_i^n \cap \bigcap_{m=n+1}^{\infty} (S^m)^c \Big), \quad i = 1, 2, 3, 4.$$

We claim that the points p^1, $p_{j,l}^n$ and the sets S_i satisfy the requirements.

We estimate the total area of $\bigcup_{n \geq 1} S^n$. At each Step n, we have no more than 2^{4n} points, and each disk has area $\frac{\pi}{2^{10n}}$. Thus

$$\Big| \bigcup_{n=1}^{\infty} S^n \Big| \leq \sum_{n=1}^{\infty} \frac{2^{4n}}{2^{10n}} \pi < \frac{\pi}{2^5}. \tag{2.3.19}$$

Fix any point $p = p_{j,l}^n$. Then

$$\{z \in D(p, \frac{1}{2^{5n}}) : 0 < \arg(z - p) < \frac{\pi}{2}\} \subset S_1^n.$$

Let $r < \frac{1}{2^{5n}}$ be an arbitrary number. Let N be the largest integer such that $\frac{1}{2^N} > r$. Consider the sector

$$F := \{z \in D(p, r) : 0 < \arg(z - p) < \frac{\pi}{2}\} \subset S_1^n.$$

Note that by the choice of N, $F \cap S^m = \emptyset$ for any integer m such that $n+1 \leq m \leq N-1$. So we consider the disk $D(p, \frac{1}{2^N})$. Analogously to (2.3.19) we have

$$|D(p, \frac{1}{2^N}) \cap \bigcup_{n \geq N} S^n| \leq \frac{1}{2^{2N}} \cdot \frac{\pi}{2^5}.$$

On the other hand, by the choice of N we have $\frac{1}{2^{N+1}} \leq r < \frac{1}{2^N}$. So

$$|F| \geq \frac{1}{4} \cdot \frac{\pi}{2^{2N+2}}.$$

Thus

$$\frac{|D(p,r) \cap S_1|}{|D(p,r)|} \geq \frac{|F \cap S_1|}{|D(p, \frac{1}{2^N})|} \geq \frac{\frac{\pi}{2^{2N+4}} - \frac{\pi}{2^{2N+5}}}{\frac{\pi}{2^{2N}}} = \frac{1}{32}.$$

It is symmetric for S_2, S_3, S_4. This proves Lemma 2.3.2.

2.4 A condition for an outer variational function.

In this section we prove:

Theorem 2.4.1 *Let $f \in \mathcal{F}_p$ be a C^1-diffeomorphism. Then f is outer variational.*

Lemma 2.4.2 *Let $f \in \mathcal{F}_p$ be a C^1-diffeomorphism and $\varphi \in C_0^\infty(\mathbb{D})$. Then, there is a $t_0 > 0$ such that $f^t = f + t\varphi \in \mathcal{F}_p$ for all $t \in (-t_0, t_0)$.*

Proof. For a C^1-diffeomorphism f we have that $J(z,f)$ is continuous and positive everywhere in $\mathbb{D}$. Let $\varphi \in C_0^\infty(\mathbb{D})$. Then, in the compact subset $supp(\varphi)$, there is an $\varepsilon_1 > 0$ such that $J(z,f) \geq \varepsilon_1$. Now by (2.2.5), there is a $t_0 > 0$ such that

$$J(z, f^t) \geq \varepsilon_2 > 0, \quad z \in \mathbb{D}, \quad t \in (-t_0, t_0).$$

Thus each f^t is a local C^1-diffeomorphism, and similar to Lemma 2.2.3 we know f^t is in fact a global C^1-diffeomorphism.

To see $\exp[p\mathbb{K}(z, f + t\varphi)] \in L^1(\mathbb{D})$, we compute

$$\begin{aligned}
\exp[p\mathbb{K}(z,f^t)] &= \exp[p\frac{|f_z^t|^2 + |f_{\overline{z}}^t|^2}{|f_z^t|^2 - |f_{\overline{z}}^t|^2}] \\
&= \exp[p\mathbb{K}(z,f)] \exp\Big[p\Big(\frac{|f_z + t\varphi_z|^2 + |f_{\overline{z}} + t\varphi_{\overline{z}}|^2}{|f_z + t\varphi_z|^2 - |f_{\overline{z}} + t\varphi_{\overline{z}}|^2} - \frac{|f_z|^2 + |f_{\overline{z}}|^2}{|f_z|^2 - |f_{\overline{z}}|^2}\Big)\Big] \\
&= \exp[p\mathbb{K}(z,f)] \exp\Big[\frac{4pt\Re e[\overline{f_{\overline{z}} f_z}(f_z\varphi_{\overline{z}} - f_{\overline{z}}\varphi_z)] + 2pt^2(|f_z\varphi_{\overline{z}}|^2 - |f_{\overline{z}}\varphi_z|^2)}{J(z,f^t)J(z,f)}\Big] \\
&= \exp[p\mathbb{K}(z,f)] \exp[tM(z,t)], \qquad (2.4.1)
\end{aligned}$$

where

$$|M(z,t)| \leq M < \infty, \quad \forall z \in \mathbb{D}, \quad t \in (-t_0, t_0).$$

So the integrability of $\exp[p\mathbb{K}(z,f^t)]$ follows from that of $\exp[p\mathbb{K}(z,f)]$. □

Lemma 2.4.3 *Let $f \in \mathcal{F}_p$ be a C^1-diffeomorphism. Then*

$$F(t) = \int_{\mathbb{D}} \exp[p\mathbb{K}(z, f + t\varphi)]dz$$

is differentiable at 0.

Proof. Set

$$G(z,t) = \frac{1}{t}\Big(\exp[p\mathbb{K}(z, f + t\varphi)] - \exp[p\mathbb{K}(z,f)]\Big).$$

For almost every $z \in \mathbb{D}$, the function $\exp[p\mathbb{K}(z, f + t\varphi)]$ is differentiable at $t = 0$, thus $G(z,t)$ is continuous at $t = 0$, with

$$G(z,0) := \frac{\partial}{\partial t}\Big|_{t=0} \exp[p\mathbb{K}(z, f + t\varphi)].$$

Our aim is to show that there exists a dominating function

$$|G(z,t)| \leq G_0(z) \in L^1(\mathbb{D}), \quad \forall t \in (-t_0, t_0). \tag{2.4.2}$$

Then, by the dominated convergence theorem,

$$F'(0) = \lim_{t\to 0} \int_{\mathbb{D}} G(z,t)dz = \int_{\mathbb{D}} G(z,0)dz \leq \int_{\mathbb{D}} G_0(z)dz.$$

So $F'(0)$ exists and is finite.

By (2.4.1),

$$\begin{aligned} |G(z,t)| &= \left|\frac{1}{t}\Big(\exp[p\mathbb{K}(z, f + t\varphi)] - \exp[p\mathbb{K}(z, f)]\Big)\right| \\ &\leq \exp[p\mathbb{K}(z,f)]\frac{\exp(|t|M) - 1}{|t|}. \end{aligned}$$

We consider the function

$$a(s) = \exp(sM), \quad s > 0.$$

This is a strictly increasing convex function. Thus

$$\frac{\exp(|t|M) - 1}{|t|} = \frac{a(|t|) - a(0)}{|t|} \leq a'(|t|) = M\exp(|t|M) \leq M\exp(t_0 M).$$

So we can choose

$$G_0(z) = \exp[p\mathbb{K}(z,f)]M\exp(t_0 M),$$

and then (2.4.2) is proved, which completes the proof. □

By Theorem 2.2.2 and Theorem 2.4.1 we get

Theorem 2.4.4 *Let f be a minimiser for Problem 2.0.1 which is a C^1-diffeomorphism. Then, f satisfies the Euler-Lagrange equation (2.2.6).*

2.5 A condition for an inner variational function.

Similarly to the outer variational case, if we assume that f is a C^1-diffeomorphism, the result that f is inner variational follows. However, in this section we will give a weaker condition:

Theorem 2.5.1 *Suppose $f \in \mathcal{F}_p$ is as in (2.0.2), and suppose there is $q > p$ such that*

$$\int_A \exp[q\mathbb{K}(z, f_0)]dz < \infty, \tag{2.5.1}$$

for any compact $A \subset \mathbb{D}$. Then f is inner variational.

We fix a $\varphi \in C_0^\infty(\mathbb{D})$ and choose a compact $A(\varphi)$ such that

$$supp(\varphi) \subset A(\varphi) \subset \mathbb{D}.$$

Recall the composition formula (2.2.11):

$$\mathbb{K}(g^t(z), f \circ (g^t)^{-1}) = \mathbb{K}(z, f)\mathbb{K}(z, g^t)[1 - \frac{4\Re e(\mu_f \overline{\mu}_{g^t})}{(1 + |\mu_f|^2)(1 + |\mu_{g^t}|^2)}]. \tag{2.5.2}$$

We consider the absolute value of the last term in the brackets, with small t,

$$\begin{aligned}\Big|\frac{4\Re e(\mu_f \overline{\mu}_{g^t})}{(1 + |\mu_f|^2)(1 + |\mu_{g^t}|^2)}\Big| &\le 4|\mu_f \overline{\mu}_{g^t}| \le 4|\mu_{g^t}| \\ &= \frac{4|t\varphi_{\overline{z}}|}{|1 + t\varphi_z|} \le 8\|\varphi_{\overline{z}}\|_\infty |t| \chi_{A(\varphi)}. \end{aligned}\tag{2.5.3}$$

Also,

$$\mathbb{K}(z, g^t) = \frac{|1 + t\varphi_z|^2 + t^2|\varphi_{\overline{z}}|^2}{|1 + t\varphi_z|^2 - t^2|\varphi_{\overline{z}}|^2} = 1 + \frac{2t^2|\varphi_{\overline{z}}|^2}{|1 + t\varphi_z|^2 - t^2|\varphi_{\overline{z}}|^2}. \tag{2.5.4}$$

Put (2.5.3) and (2.5.4) back to (2.5.2), we find that, for any $q > p$ and given $\varphi \in C_0^\infty(\mathbb{D})$, there is a sufficiently small $t_0 > 0$ such that for any $t \in (-t_0, t_0)$,

$$\mathbb{K}(g^t(z), f \circ (g^t)^{-1}) < \mathbb{K}(z, f) + (\frac{q}{p} - 1)\chi_{A(\varphi)}\mathbb{K}(z, f). \tag{2.5.5}$$

Also note the Jacobian

$$J(z, g^t) = |1 + t\varphi_z|^2 - t^2|\varphi_{\overline{z}}|^2 = 1 + 2t\Re e(\varphi_z) + t^2 J(z, \varphi). \tag{2.5.6}$$

So we have $J(z, g^t) = 1$ in $\mathbb{D} - A(\varphi)$ and $J(z, g^t) \to 1$ uniformly in $A(\varphi)$ as $t \to 0$. Then we may assume that $J(z, g^t) < 2$ in $A(\varphi)$, for any $t \in (-t_0, t_0)$. Combine these facts we find that the assumption (2.5.1) implies

$$\begin{aligned}\int_{g^t(A(\varphi))} \exp[p\mathbb{K}(w, f \circ (g^t)^{-1})]dw &= \int_{A(\varphi)} \exp[p\mathbb{K}(g^t(z), f \circ (g^t)^{-1})]J(z, g^t)dz \\ &\le 2\int_{A(\varphi)} \exp[q\mathbb{K}(z, f])dz < \infty, \end{aligned}\tag{2.5.7}$$

and

$$\int_{\mathbb{D}-g^t(A(\varphi))} \exp[p\mathbb{K}(w, f\circ(g^t)^{-1})]dw = \int_{\mathbb{D}-A(\varphi)} \exp[p\mathbb{K}(z,f])dz. \tag{2.5.8}$$

So we get the following lemma:

Lemma 2.5.2 *Let $f \in \mathcal{F}_p$ satisfy (2.5.1). Then, for any $\varphi \in C_0^\infty(\mathbb{D})$ and f^t defined as in (2.2.8), there is a $t_0 > 0$ such that for any $t \in (-t_0, t_0)$,*

$$F(t) = \int_{\mathbb{D}} \exp[p\mathbb{K}(w, f^t)]dw < \infty.$$

This together with Lemma 2.2.4 proves that $f^t \in \mathcal{F}_p$, which is the first condition for the variational functions. We next prove that the second condition is also satisfied:

Lemma 2.5.3 *Let $f \in \mathcal{F}_p$ satisfy (2.5.1). Then,*

$$F(t) = \int_{\mathbb{D}} \exp[p\mathbb{K}(w, f\circ(g^t)^{-1})]dw$$

is differentiable at 0.

Proof. We consider the following function

$$G(z,t) := \frac{1}{t}\Big(\exp[p\mathbb{K}(g^t(z), f^t)]J(z,g^t) - \exp[p\mathbb{K}(z,f)]\Big). \tag{2.5.9}$$

Similar to the proof of Lemma 2.4.3, we only need to find a dominating function

$$|G(z,t)| \le G_0(z) \in L^1(\mathbb{D}), \quad \forall t \in (-t_0, t_0). \tag{2.5.10}$$

To this end we rewrite (2.5.2)-(2.5.8) as

$$\mathbb{K}(g^t(z), f\circ(g^t)^{-1}) = \mathbb{K}(z,f)(1+tM(z,t)), \tag{2.5.11}$$

$$J(z,g^t) = 1 + tN(z,t), \tag{2.5.12}$$

where

$$|M(z,t)| \le M\chi_{A(\varphi)}, \quad |N(z,t)| \le N\chi_{A(\varphi)}.$$

are uniformly bounded. Then

$$\begin{aligned}
|G(z,t)| &= \Big|\frac{1}{t}\big(\exp[p\mathbb{K}(z,f)(1+tM(z,t))](1+tN(z,t)) - \exp[p\mathbb{K}(z,f)]\big)\Big| \\
&= \Big|\frac{1}{t}\big(tN(z,t)\exp[p\mathbb{K}(z,f)(1+tM(z,t))] + \exp[p\mathbb{K}(z,f)](\exp[p\mathbb{K}(z,f)tM(z,t)] - 1)\big)\Big| \\
&\le N\chi_{A(\varphi)}\exp[q\mathbb{K}(z,f)] + \exp[p\mathbb{K}(z,f)]\big(\frac{\exp[p\mathbb{K}(z,f)|t|M\chi_{A(\varphi)}] - 1}{|t|}\big).
\end{aligned} \tag{2.5.13}$$

We consider the function, for almost every fixed $z \in A(\varphi)$,

$$a(s) = \exp[sp\mathbb{K}(z,f)M], \quad s > 0.$$

This is a strictly increasing convex function. Thus

$$\begin{aligned}
\frac{\exp(|t|p\mathbb{K}(z,f)M) - 1}{|t|} &= \frac{a(|t|) - a(0)}{|t|} \\
&\leq a'(|t|) \\
&= p\mathbb{K}(z,f)M \exp[|t|p\mathbb{K}(z,f)M] \\
&\leq p\mathbb{K}(z,f)M \exp(t_0 Mp\mathbb{K}(z,f)) \\
&\leq pM \exp(\frac{q}{p}\mathbb{K}(z,f)), \qquad (2.5.14)
\end{aligned}$$

where t_0 is chosen sufficiently small. So by (2.5.13) and (2.5.14), we can choose

$$G_0(z) := (pM + N)\exp[q\mathbb{K}(z,f)]\chi_{A(\varphi)}.$$

Then (2.5.10) follows and the lemma is proved. $\Box$

Now Theorem 2.5.1 follows from lemma 2.5.2 and Lemma 2.5.3 immediately. By Theorem 2.2.6 we also get the following result:

Theorem 2.5.4 *Let f be a minimiser of Problem 2.0.1, and* $\exp[q\mathbb{K}(z,f)] \in L^1_{loc}(\mathbb{D})$ *for some $q > p$. Then, f satisfies the Euler-Lagrange equation (2.2.14).*

2.6 Strong convergence of minimising sequence.

As we introduced in Section 1.4, if we let f_j be a minimising sequence, that is $\int_{\mathbb{D}} \exp[p\mathbb{K}(z,f_j)]dz$ is decreasing to

$$\min_{g\in\mathcal{F}_p} \int_{\mathbb{D}} \exp[p\mathbb{K}(z,g)]dz,$$

for $\mathcal{F}_p$ as (2.0.2), then up to a subsequence there is a limit function f of f_j in $\mathcal{F}_p$ such that

$$\int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]dz = \lim_{j\to\infty} \int_{\mathbb{D}} \exp[p\mathbb{K}(z,f_j)]dz = \min_{g\in\mathcal{F}_p} \int_{\mathbb{D}} \exp[p\mathbb{K}(z,g)]dz.$$

Recall we have the convergence as

$$f_j \to f \text{ uniformly in } \mathbb{D}, \text{ weakly in } W^{1,P}(\mathbb{D}) \text{ for } P(t) = \frac{t^2}{\log(e+t)},$$

$$J(z,f_j) \rightharpoonup J(z,f) \text{ weakly in } L^1(\mathbb{D}).$$

Meanwhile, for the inverse sequence $h_j = f_j^{-1}$, $h = f^{-1}$, we have

$$h_j \to h \text{ uniformly in } \mathbb{D}, \text{ weakly in } W^{1,2}(\mathbb{D}),$$

$$J(w, h_j) \rightharpoonup J(w, h) \text{ weakly in } L^1(\mathbb{D}).$$

In this section we introduce a way to improve this convergence.

Before we start with the convergence we remark that as stated in Lemma 1.3.3, the integrability of $\mathbb{K}(z, f)$ is enough to ensure that a homeomorphism f has $J(z, f) > 0$ a.e.. Note it is not enough to say $J(w, h) > 0$ in the L^1 case, but in the exponential case this will be true .

Theorem 2.6.1 *Let $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a homeomorphism such that*

$$\int_{\mathbb{D}} \exp[p\mathbb{K}(z, f)]dz < \infty,$$

and set $h = f^{-1}$. Then, for almost every $w \in \mathbb{D}$,

$$J(w, h) > 0. \tag{2.6.1}$$

Proof. By Theorem 1.2.3, at almost every point $z \in \mathbb{D}$,

$$J(z, f) < \infty, \quad J(z, f)J(f(z), h) = 1,$$

thus

$$J(f(z), h) > 0.$$

Then the claim follows from Theorem 1.4.5 that f satisfies Lusin's condition $\mathcal{N}$. $\square$

There are some other facts that we wish to state here. First, if a sense-preserving homeomorphism f has finite mean p-exponential distortions, then its Jacobian has an $L \log L$ bound, and $|Df|$ has higher integrability than the $P(t) = \frac{t^2}{\log(e+t)}$ we had before. Precisely, for all $\alpha \in (0, \frac{p}{2})$ we have

$$\int_{\mathbb{D}} J(z, f) \log^{\alpha}(e + J(z, f))dz \leq M_1 < \infty, \tag{2.6.2}$$

$$\int_{\mathbb{D}} |Df(z)|^2 \log^{\alpha-1}(e + |Df(z)|)dz \leq M_2 < \infty. \tag{2.6.3}$$

Note here our previous $P(t) = \frac{t^2}{\log(e+t)}$ serves as a lower bound for $p > 0$. Meanwhile, for the inverse h we have

$$\int_{\mathbb{D}} J(w, h) \log(e + J(w, h))dw \leq M_3 < \infty. \tag{2.6.4}$$

Secondly, for any $s \in (0, \frac{p}{2})$, the inverse h has a $L^s(\mathbb{D})$ bound for $\mathbb{K}(w,h)$, that is

$$\int_{\mathbb{D}} \mathbb{K}^s(w,h)dw \leq M_4 < \infty. \tag{2.6.5}$$

Thirdly, the reciprocal of J_f is also controlled in the following sense: for all $C > 0$ and $0 < \gamma < \frac{1}{2}$,

$$\int_{\mathbb{D}} \exp\Big(C \log^{\gamma}\big(e + \frac{1}{J(z,f)}\big)\Big) \leq M_5 < \infty.$$

Here all of M_1, M_2, M_3, M_4, M_5 depend only on $\int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]dz$.

See [5], [18], [23], [24], [29], [47] for details.

2.6.1 Convergence theorems.

For our purpose to improve the convergence we need the following theorems from functional analysis. See e.g. [16], [44] for the proofs.

Theorem 2.6.2 (Radon-Riesz Theorem) *Every uniformly convex Banach space $\mathcal{B}$ is a Radon-Riesz space. That is, for any sequence x_j such that $x_j \rightharpoonup x$ weakly in $\mathcal{B}$, if the norms $\|x_j\|_{\mathcal{B}} \to \|x\|_{\mathcal{B}}$, then $x_j \to x$ strongly in $\mathcal{B}$. In particular, every L^p space with $1 < p < \infty$ is a Radon-Riesz space.*

Theorem 2.6.3 (Vitali Convergence Theorem) *Let $\Omega \subset \mathbb{R}^n$ be finite. Let a function sequence $f_j \to f$ pointwise (a.e.) in Ω, and f_j are equi-integrable, then $f_j \to f$ strongly in $L^1(\Omega)$.*

The term 'equi-integrable' means, for any $\varepsilon > 0$, there is a $\delta > 0$ such that for any j and any measurable $\Omega_\delta \subset \Omega$ such that $|\Omega_\delta| < \delta$,

$$\int_{\Omega_\delta} |f_j| d\mu < \varepsilon.$$

A quick test to gain the equi-integrability is as follows. If there is a convex increasing function $\Psi : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to\infty} \frac{\Psi(t)}{t} = \infty$, and $\int_\Omega \Psi(|f_j|)d\mu$ are uniformly bounded, then f_j are equi-integrable. As an example, if a sequence $f_j \to f$ pointwise, and $\|f_j\|_{L^p(\Omega)}$ are uniformly bounded for some $p > 0$, then $f_j \to f$ strongly in $L^q(\Omega)$ for any $q \in (0,p)$.

2.6.2 Strong convergence of $\exp[p\mathbb{K}(z,f_j)]$.

Let f_j be a minimising sequence and f be the limit as we stated at the beginning of this section. Recall we have the property

$$\int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]dz = \lim_{j\to\infty} \int_{\mathbb{D}} \exp[p\mathbb{K}(z,f_j)]dz. \tag{2.6.6}$$

Theorem 2.6.4 *For the minimising sequence f_j and the limit f,*

$$\exp[p\mathbb{K}(z,f_j)] \to \exp[p\mathbb{K}(z,f)] \text{ strongly in } L^1(\mathbb{D}).$$

Proof. We consider the sequence

$$\exp[\frac{p}{2}\mathbb{K}(z,f_j)+\frac{p}{2}\mathbb{K}(z,f)].$$

On one hand, it follows from polyconvexity that

$$\liminf_{j\to\infty}\int_{\mathbb{D}}\exp[\frac{p}{2}\mathbb{K}(z,f_j)+\frac{p}{2}\mathbb{K}(z,f)]dz \geq \int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]dz.$$

On the other hand, by Hölder's inequality we have

$$\begin{aligned}
&\limsup_{j\to\infty}\int_{\mathbb{D}}\exp[\frac{p}{2}\mathbb{K}(z,f_j)]\exp[\frac{p}{2}\mathbb{K}(z,f)]dz\\
\leq&\limsup_{j\to\infty}\Big(\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f_j)]dz\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]dz\Big)^{\frac{1}{2}}\\
=&\Big(\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)dz]\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]dz\Big)^{\frac{1}{2}}\\
=&\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]dz.
\end{aligned}$$

So we have

$$\lim_{j\to\infty}\int_{\mathbb{D}}\exp[\frac{p}{2}\mathbb{K}(z,f_j)+\frac{p}{2}\mathbb{K}(z,f)]dz = \int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]dz.$$

Together with (2.6.6),

$$\begin{aligned}
&\lim_{j\to\infty}\int_{\mathbb{D}}\Big(\exp[\frac{p}{2}\mathbb{K}(z,f_j)]-\exp[\frac{p}{2}\mathbb{K}(z,f)]\Big)^2dz\\
=&\lim_{j\to\infty}\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f_j)]dz+\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]dz-2\int_{\mathbb{D}}\exp[\frac{p}{2}\mathbb{K}(z,f_j)+\frac{p}{2}\mathbb{K}(z,f)]dz\\
=&\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]dz+\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]dz-2\int_{\mathbb{D}}\exp[p\mathbb{K}(z,f)]dz\\
=&0.
\end{aligned}$$

This proves

$$\exp[\frac{p}{2}\mathbb{K}(z,f_j)] \to \exp[\frac{p}{2}\mathbb{K}(z,f)] \text{ strongly in } L^2(\mathbb{D}),$$

which is equivalent to the claim. □

We remark that the last theorem can also be proved by the observation

$$\exp[\frac{p}{2}\mathbb{K}(z,f)] \leq M_0(z), \quad a.e.,$$

where $M_0(z)$ is the weak limit of $\exp[\frac{p}{2}\mathbb{K}(z,f_j)]$ in $L^2(\mathbb{D})$. Then,

$$\begin{aligned}\int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]dz \leq \int_{\mathbb{D}} M_0^2(z)dz &\leq \lim_{j\to\infty}\int_{\mathbb{D}} \exp[p\mathbb{K}(z,f_j)]dz \\ &= \int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]dz.\end{aligned}$$

This forces $\exp[\frac{p}{2}\mathbb{K}(z,f)] = M_0(z)$, so $\exp[\frac{p}{2}\mathbb{K}(z,f_j)]$ converges to $\exp[\frac{p}{2}\mathbb{K}(z,f)]$ weakly in $L^2(\mathbb{D})$, and then the strong convergence follows from the Radon-Riesz theorem 2.6.2.

2.6.3 The inverse sequence.

It follows from Theorem 2.6.4 that, up to a subsequence, we can say

$$\exp[p\mathbb{K}(z,f_j)] \to \exp[p\mathbb{K}(z,f)] \text{ pointwise in } \mathbb{D},$$

and then

$$|\mu(z,f_j)| \to |\mu(z,f)| \text{ pointwise in } \mathbb{D}.$$

We now turn to the inverse functions $h_j = f_j^{-1}$, $h = f^{-1}$. Observe that for almost every $w \in \mathbb{D}$,

$$|\mu_{f_j}(h_j(w))| = |\mu_{h_j}(w)|, \quad |\mu_f(h(w))| = |\mu_h(w)|,$$

since f_j, f and h_j, h are differentiable almost everywhere, $J(z,f_j) > 0$, $J(z,f) > 0$ almost everywhere, and f_j, f have Lusin's property $\mathcal{N}$.

Lemma 2.6.5 *With the notation above, there is a subsequence μ_{h_j} such that $|\mu_{h_j}(w)| \to |\mu_h(w)|$, for almost every $w \in \mathbb{D}$.*

Proof. Let $q \in [1,\infty)$, φ be any uniformly continuous function in $\mathbb{D}$. Then,

$$\begin{aligned}\lim_{j\to\infty}\int_{\mathbb{D}} |\mu_{f_j}(h_j(w))|^q\varphi(w)dw &= \lim_{j\to\infty}\int_{\mathbb{D}} |\mu_{f_j}(z)|^q J(z,f_j)\varphi(f_j)dz \\ &= \int_{\mathbb{D}} |\mu_f(z)|^q J(z,f)\varphi(f)dz \\ &= \int_{\mathbb{D}} |\mu_f(h(w))|^q\varphi(w)dw. \qquad (2.6.7)\end{aligned}$$

We explain the convergence in detail. In fact,

$$
\begin{aligned}
&\int_{\mathbb{D}} |\mu_{f_j}(z)|^q J(z,f_j)\varphi(f_j(z)) - |\mu_f(z)|^q J(z,f)\varphi(f(z))dz \\
=&\int_{\mathbb{D}} |\mu_{f_j}(z)|^q J(z,f_j)\varphi(f_j(z)) - |\mu_{f_j}(z)|^q J(z,f_j)\varphi(f(z))dz \qquad (2.6.8)\\
+&\int_{\mathbb{D}} |\mu_{f_j}(z)|^q J(z,f_j)\varphi(f(z)) - |\mu_f(z)|^q J(z,f_j)\varphi(f(z))dz \qquad (2.6.9)\\
+&\int_{\mathbb{D}} |\mu_f(z)|^q J(z,f_j)\varphi(f(z)) - |\mu_f(z)|^q J(z,f)\varphi(f(z))dz. \qquad (2.6.10)
\end{aligned}
$$

Here (2.6.8) $\to 0$ is obvious, as $\varphi(f_j) \to \varphi(f)$ uniformly, and $|\mu_{f_j}(z)|^q J(z,f_j) \leq J(z,f_j)$ have a uniform $L^1(\mathbb{D})$ bound π. (2.6.10) $\to 0$ is also easy, as $J(z,f_j) \rightharpoonup J(z,f)$ in $L^1(\mathbb{D})$. To prove (2.6.9) $\to 0$, we recall (2.6.2) that $J(z,f_j)$ are uniformly bounded in a Zygmund space:

$$
\int_{\mathbb{D}} J(z,f_j)\log^{\alpha}(e+J(z,f_j))dz \leq M, \qquad (2.6.11)
$$

for some $M < \infty$. On the other hand, $|\mu_{f_j}|^q \to |\mu_f|^q$ strongly in the *Exp*-space, then it follows from Hölder's inequality for Orlicz conjugates (see [29], P78) that (2.6.9) $\to 0$.

To be precise, we have the following elementary inequality:

$$
a^{\frac{\alpha}{2}} b \leq b\log^{\alpha}(e+b) + C(\alpha)\exp(a), \quad a,b,\alpha > 0.
$$

Let $\varepsilon > 0$ be arbitrarily small. Then,

$$
\begin{aligned}
&\Big|\int_{\mathbb{D}} |\mu_{f_j}(z)|^q J(z,f_j)\varphi(f(z)) - |\mu_f(z)|^q J(z,f_j)\varphi(f(z))dz\Big| \\
\leq&\|\varphi\|_\infty \int_{\mathbb{D}} \big||\mu_{f_j}(z)|^q - |\mu_f(z)|^q\big| J(z,f_j)dz \\
\leq&\|\varphi\|_\infty \Big(\varepsilon\int_{\mathbb{D}} J(z,f_j)\log^{\alpha}(e+J(z,f_j))dz + C(\alpha)\int_{\mathbb{D}} \exp\Big[\big|\frac{1}{\varepsilon}(|\mu_{f_j}(z)|^q - |\mu_f(z)|^q)\big|^{\frac{2}{\alpha}}\Big]dz\Big).
\end{aligned}
$$

Note here $\exp\Big[\big|\frac{1}{\varepsilon}(|\mu_{f_j}(z)|^q - |\mu_f(z)|^q)\big|^{\frac{2}{\alpha}}\Big] \to 0$ pointwise, and they are dominated by $\exp[(\frac{2}{\varepsilon})^{\frac{2}{\alpha}}]\chi_{\mathbb{D}}$. So let $j \to \infty$ we get

$$
\limsup_{j\to\infty}\Big|\int_{\mathbb{D}} |\mu_{f_j}(z)|^q J(z,f_j)\varphi(f(z)) - |\mu_f(z)|^q J(z,f_j)\varphi(f(z))dz\Big| \leq \varepsilon\|\varphi\|_\infty M.
$$

But this holds for every $\varepsilon > 0$, so the claim (2.6.9) $\to 0$ follows.

So we have validated (2.6.7) for any uniformly continuous φ in $\mathbb{D}$. In particular, this holds for every $\varphi \in C_0^\infty(\mathbb{D})$ and $q = 1$. So

$$|\mu_{f_j}(h_j)| \to |\mu_f(h)| \text{ in } \mathcal{D}'(\mathbb{D}). \tag{2.6.12}$$

Now choose any $q > 1$. Since $|\mu_{f_j}(h_j)|^q$ are dominated by $\chi_{\mathbb{D}}$, up to a subsequence we have $|\mu_{f_j}(h_j)|$ converges weakly in $L^q(\mathbb{D})$. By (2.6.12), the limit could only be $|\mu_f(h)|$. So we get

$$|\mu_{f_j}(h_j)| \rightharpoonup |\mu_f(h)| \text{ weakly in } L^q(\mathbb{D}).$$

In (2.6.7) we can also choose $\varphi = \chi_{\mathbb{D}}$, then

$$\lim_{j\to\infty} \int_{\mathbb{D}} |\mu_{f_j}(h_j(w))|^q dw = \int_{\mathbb{D}} |\mu_f(h(w))|^q dw.$$

So by the Radon-Riesz theorem, we get

$$|\mu_{f_j}(h_j)| \to |\mu_f(h)| \text{ strongly in } L^q(\mathbb{D}),$$

and then there is a pointwise convergent subsequence. $\square$

We next observe that the same process in the last part about the sequence f_j can also be applied on the inverse sequence h_j. Analogously to Lemma 2.6.4 we can prove:

Lemma 2.6.6 *Let h_j be any minimising sequence of the inverse p-exponential problem and let h be the limit function. Then,*

$$\exp[p\mathbb{K}(w,h_j)]J(w,h_j) \to \exp[p\mathbb{K}(w,h)]J(w,h) \textit{ strongly in } L^1(\mathbb{D}).$$

Passing to a subsequence, Lemma 2.6.5 and Lemma 2.6.6 give us that

$$|\mu_{h_j}(w)| \to |\mu_h(w)| \text{ pointwise},$$

$$\exp[p\mathbb{K}(w,h_j)]J(w,h_j) \to \exp[p\mathbb{K}(w,h)]J(w,h) \text{ pointwise}.$$

Then together they give

$$J(w,h_j) \to J(w,h) \text{ pointwise},$$

$$|(h_j)_w| \to |h_w|, \quad |(h_j)_{\overline{w}}| \to |h_{\overline{w}}| \text{ pointwise}.$$

$$\mathbb{K}(w,h_j) \to \mathbb{K}(w,h) \text{ pointwise}.$$

In fact these can be improved. By (2.6.4), (2.6.5), and the fact that $\|Dh_j\|_{L^2(\mathbb{D})}$ are uniformly bounded, we can apply the Vitali convergence theorem 2.6.3 and get

$$\mathbb{K}(w,h_j) \to \mathbb{K}(w,h) \text{ strongly in } L^q(\mathbb{D}), \quad 0 < q < p;$$

$J(w,h_j)\log^\beta(e+J(w,h_j)) \to J(w,h)\log^\beta(e+J(w,h))$ strongly in $L^1(\mathbb{D})$, $\quad 0 \leq \beta < 1$.

In particular,

$$J(w,h_j) \to J(w,h) \text{ strongly in } L^1(\mathbb{D}).$$

And

$$|(h_j)_w| \to |h_w|, \quad |(h_j)_{\overline{w}}| \to |h_{\overline{w}}| \text{ strongly in } L^q(\mathbb{D}), \quad 0 < q < 2.$$

In particular,

$$\int_{\mathbb{D}} |(h_j)_w|^q \to \int_{\mathbb{D}} |h_w|^q, \quad \int_{\mathbb{D}} |(h_j)_{\overline{w}}|^q \to \int_{\mathbb{D}} |h_{\overline{w}}|^q, \quad 0 < q < 2.$$

As is in the settings, we also have

$$(h_j)_w \rightharpoonup h_w, \quad (h_j)_{\overline{w}} \rightharpoonup h_{\overline{w}} \text{ weakly in } L^q(\mathbb{D}), \quad 0 < q < 2.$$

So by the Radon-Riesz theorem 2.6.2 we get

$$(h_j)_w \to h_w, \quad (h_j)_{\overline{w}} \to h_{\overline{w}} \text{ strongly in } L^q(\mathbb{D}), \quad 0 < q < 2.$$

Again up to a subsequence we have

$$(h_j)_w \to h_w, \quad (h_j)_{\overline{w}} \to h_{\overline{w}} \text{ pointwise in } \mathbb{D}.$$

This implies

$$\mu(w,h_j) \to \mu(w,h) \text{ pointwise in } \mathbb{D}.$$

and this also implies

$$\mu(w,h_j) \to \mu(w,h) \text{ strongly in } L^q(\mathbb{D}), \quad 0 < q < \infty.$$

2.6.4 Convergence of f_j terms.

We now come back to f_j and consider the sequence $J(z,f_j)$ and $J(z,f)$. As $0 < J(w,h_j), J(w,h) < \infty$,

$$J(z,f_j) = \frac{1}{J(f_j(z),h_j)}, \quad J(z,f) = \frac{1}{J(f(z),h)}$$

for almost every $z \in \mathbb{D}$.

Lemma 2.6.7 . *With the notation above we have that $J(z,f_j) \to J(z,f)$ strongly in $L^1(\mathbb{D})$.*

Proof. Consider the sequence $J^{\frac{1}{2}}(z, f_j)$. They are uniformly bounded in $L^2(\mathbb{D})$, so there is a weakly convergent subsequence in $L^2(\mathbb{D})$. But for any $\varphi \in C_0^\infty(\mathbb{D})$, as $J(w, h_j) \to J(w, h)$ strongly in $L^1(\mathbb{D})$,

$$\begin{aligned}\lim_{j\to\infty} \int_{\mathbb{D}} J^{\frac{1}{2}}(z, f_j)\varphi(z)dz &= \lim_{j\to\infty} \int_{\mathbb{D}} J^{\frac{1}{2}}(w, h_j)\varphi(h_j(w))dw \\ &= \int_{\mathbb{D}} J^{\frac{1}{2}}(w, h)\varphi(h(w))dw \\ &= \int_{\mathbb{D}} J^{\frac{1}{2}}(z, f)\varphi(z)dz.\end{aligned}$$

This proves

$$J^{\frac{1}{2}}(z, f_j) \rightharpoonup J^{\frac{1}{2}}(z, f) \text{ in } L^2(\mathbb{D}).$$

So again the Radon-Riesz theorem gives

$$J^{\frac{1}{2}}(z, f_j) \to J^{\frac{1}{2}}(z, f) \text{ in } L^2(\mathbb{D}),$$

which is equivalent to the claim. □

Now again up to a subsequence,

$$J(z, f_j) \to J(z, f) \text{ pointwise,}$$

$$|(f_j)_z| \to |f_z|, \quad |(f_j)_{\overline{z}}| \to |f_{\overline{z}}| \text{ pointwise.}$$

By the same arguments as for the inverse sequence h_j, we have the strong convergence of $(f_j)_z$, $(f_j)_{\overline{z}}$, μ_{f_j}, $J(z, f_j)$ in some $L^q(\mathbb{D})$ or Orlicz spaces, respectively.

We close this part by collecting all the facts and recording as follows.

Theorem 2.6.8 *Let $\{f_j\}j = 1^\infty$ be any minimising sequence of the p-exponential Problem 2.0.1, and f be a weak limit of f_j in $W^{1,P}(\mathbb{D})$, $P(t) = \frac{t^2}{\log(e+t)}$. Then, there is a subsequence (which we still call f_j) such that the following convergence holds both pointwise and strongly:*

$$\exp[p\mathbb{K}(z, f_j)] \to \exp[p\mathbb{K}(z, f)] \text{ in } L^1(\mathbb{D}).$$

$$J(z, f_j)\log^\alpha(e + J(z, f_j)) \to J(z, f)\log^\alpha(e + J(z, f)) \text{ in } L^1(\mathbb{D}), \quad 0 \le \alpha < \frac{p}{2}.$$

$$|Df_j|^2 \log^{\beta-1}(e + |Df_j|) \to |Df|^2 \log^{\beta-1}(e + |Df|) \text{ in } L^1(\mathbb{D}), \quad 0 \le \beta < \frac{p}{2}.$$

$$\mu_{f_j} \to \mu_f \text{ in } L^q(\mathbb{D}), \quad 0 < q < \infty.$$

$$\exp\Big(C\log^\gamma\big(e+\frac{1}{J(z, f_j)}\big)\Big) \to \exp\Big(C\log^\gamma\big(e+\frac{1}{J(z, f)}\big)\Big) \text{ in } L^1(\mathbb{D}), \quad C > 0, \quad 0 < \gamma < \frac{1}{2}.$$

For the inverse sequence $h_j = f_j^{-1}$, $h = f^{-1}$,

$$\exp[p\mathbb{K}(w,h_j)]J(w,h_j) \to \exp[p\mathbb{K}(w,h)]J(w,h) \ \textit{in} \ L^1(\mathbb{D}).$$

$$J(w,h_j)\log^\alpha(e+J(w,h_j)) \to J(w,h)\log^\alpha(e+J(w,h)) \ \textit{in} \ L^1(\mathbb{D}), \quad 0 \le \alpha < 1.$$

$$Dh_j \to Dh \ \textit{in} \ L^q(\mathbb{D}), \quad 0 < q < 2.$$

$$\mu_{h_j} \to \mu_h \ \textit{in} \ L^q(\mathbb{D}), \quad 0 < q < \infty.$$

$$\mathbb{K}(w,h_j) \to \mathbb{K}(w,h) \ \textit{in} \ L^q(\mathbb{D}), \quad 0 < q < p.$$

3 The Regularity Theory

In the last chapter we studied Problem 2.0.1, to find the minimisers of the p-conformal energy functionals. The inner variation leads us to the following Euler-Lagrange equation (2.2.14) that for any $\varphi \in C_0^\infty(\mathbb{D})$,

$$\int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]\varphi_z(z)dz = \int_{\mathbb{D}} \exp[p\mathbb{K}(z,f)]\frac{2p\overline{\mu_f(z)}}{1-|\mu_f(z)|^2}\varphi_{\overline{z}}(z)dz.$$

In this chapter we set $p \geq 1$ and will show that under certain circumstances, the distributional solutions to this equation are C^∞-smooth.

3.1 Gaining ellipticity.

We start with the Euler-Lagrange equation (2.2.14) and rewrite it as

$$\int_{\mathbb{D}} (e^{p\mathbb{K}_f(z)} - e^p)\varphi_z(z)dz = \int_{\mathbb{D}} e^{p\mathbb{K}_f(z)}\frac{2p\overline{\mu_f(z)}}{1-|\mu_f(z)|^2}\varphi_{\overline{z}}(z)dz. \tag{3.1.1}$$

We set

$$a(z) = e^{p\mathbb{K}_f(z)} - e^p, \tag{3.1.2}$$

$$\nu^*(z) := \frac{2p\overline{\mu_f(z)}e^{p\mathbb{K}_f(z)}}{(1-|\mu_f(z)|^2)(e^{p\mathbb{K}_f(z)} - e^p)}. \tag{3.1.3}$$

Then (3.1.1) reads as

$$\int_{\mathbb{D}} a(z)\varphi_z(z)dz = \int_{\mathbb{D}} a(z)\nu^*(z)\varphi_{\overline{z}}(z)dz. \tag{3.1.4}$$

We wish to show that there is a number K such that

$$|\nu^*| \geq K > 1. \tag{3.1.5}$$

Consider the function

$$|\nu^*|(t) := \frac{2pte^{p\frac{1+t^2}{1-t^2}}}{(1-t^2)(e^{p\frac{1+t^2}{1-t^2}} - e^p)}, \quad t \in [0,1]. \tag{3.1.6}$$

We need to determine the minimum value of it. Put

$$s = p\frac{t^2}{1-t^2}, \quad s \in [0,\infty).$$

Then

$$t = \sqrt{\frac{s}{p+s}}.$$

Put this back to (3.1.6), we get

$$|\nu^*|(s) = 2\sqrt{s(p+s)}\frac{e^{2s}}{e^{2s}-1} = \sqrt{s(p+s)}\Big(1+\coth[s]\Big), \quad s\in[0,\infty). \tag{3.1.7}$$

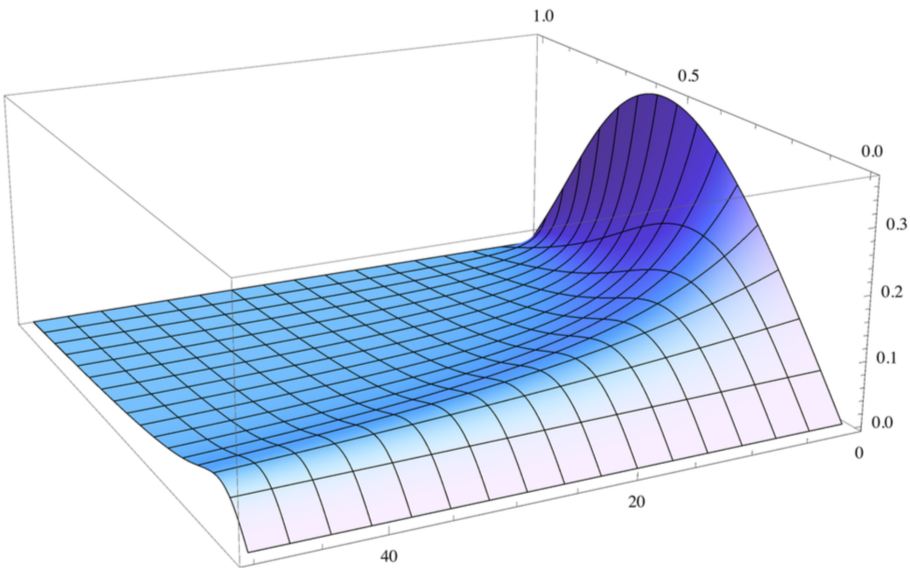
Note that near 0 we have

$$1+\coth[s] = \frac{1}{s}+1+\frac{s}{3}-\frac{s^3}{45}+O(s^5).$$

In fact we find

$$\lim_{s\to 0^+}|\nu^*|(s) = +\infty, \qquad \lim_{s\to+\infty}|\nu^*|(s) = +\infty,$$

and it gains a minimum at somewhere inside $(0,+\infty)$. Set $\nu = 1/\nu^*$, the graph of $|\nu|(t)$ is as follows.



The graph of $|\nu| = 1/|\nu^*|$.

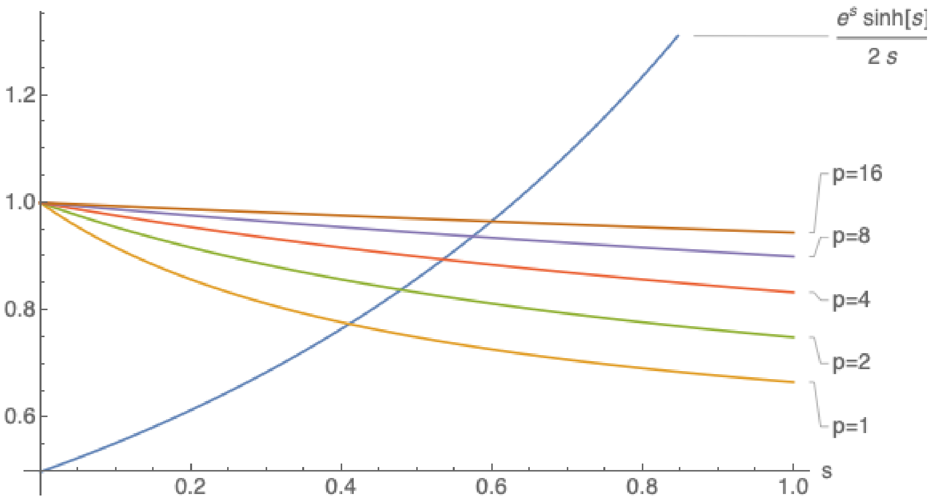
So we consider the derivative

$$\frac{d}{ds}\sqrt{s(p+s)}\Big(1+\coth[s]\Big) = \frac{(p+2s)(1+\coth[s]-2s(p+s)\mathrm{csch}^2[s])}{2\sqrt{s(p+s)}}.$$

Rearrange it we get the equation for $|\nu^*|'(s)=0$:

$$\frac{e^s\sinh[s]}{2s} = 1-\frac{s}{p+2s}. \tag{3.1.8}$$

For $s \in [0, \infty)$, the left hand side is strictly increasing from $\frac{1}{2}$ to ∞, while the right hand side is strictly decreasing from 1 to $\frac{1}{2}$. Therefore there is a unique root s_p for equation (3.1.8). Furthermore, if $s > 0$ is fixed, the right hand side is strictly increasing as p increasing. This implies that when $p \geq 1$ is increasing, the root s_p of equation (3.1.8) is increasing. This can be observed in the following graph.



The graphs of $\frac{e^s \sinh[s]}{2s}$ and $1 - \frac{s}{p+2s}$ for $p = 1, 2, 4, 16, \infty$.

When $p = 1$, we find the root of the equation

$$\frac{e^s \sinh[s]}{2s} = 1 - \frac{s}{1+2s}$$

is approximately $s_1 \approx 0.410026\ldots$; when $p = \infty$, the root of the equation

$$\frac{e^s \sinh[s]}{2s} = 1$$

is approximately $s_\infty \approx 0.628216\ldots$.

We rewrite (3.1.8) to show p by s_p:

$$p = \frac{2s_p^2}{2s_p - e^{s_p} \sinh[s_p]} - 2s_p. \tag{3.1.9}$$

p	k_p		p	k_p
1	0.367978 ...		6	0.175753 ...
2	0.282803 ...		7	0.163723 ...
3	0.238952 ...		8	0.153876 ...
4	0.210894 ...		9	0.145621 ...
5	0.190920 ...		∞	0

Table 1: The values k_p with $\|\nu_p\|_\infty \le k_p$
.

We put (3.1.9) into (3.1.7), then get the minimum value

$$\begin{aligned} k_p^* &= \sqrt{s_p(p+s_p)}\Big(1+\coth[s_p]\Big) \\ &= s_p\sqrt{\frac{e^{s_p}\sinh[s_p]}{2s_p - e^{s_p}\sinh[s_p]}}\cdot\frac{e^{s_p}}{\sinh[s_p]} \\ &= \frac{s_p e^{2s_p}}{\sqrt{2s_p e^{s_p}\sinh[s_p] - (e^{s_p}\sinh[s_p])^2}}. \end{aligned}$$

We consider its reciprocal

$$k_p = 1/k_p^* = \sqrt{\frac{e^{2s_p}-1}{4s_p^2 e^{4s_p}}(4s_p + 1 - e^{2s_p})}.$$

Now it is easy to see that both of the functions $\frac{e^{2s_p}-1}{4s_p^2 e^{4s_p}}$ and $(4s_p + 1 - e^{2s_p})$ are decreasing while $s_p > 0.4$ is increasing. So k_p is a decreasing function of s_p. We list some of the values of k_p for $p = 1, 2, \ldots,$ see Table 1.

So now we have validated (3.1.5). We come back to (3.1.3) and consider its reciprocal

$$\nu(z) = 1/\nu^*(z) = \frac{(1-|\mu_f(z)|^2)(e^{p\mathbb{K}_f(z)} - e^p)}{2p\overline{\mu_f(z)}e^{p\mathbb{K}_f(z)}} \tag{3.1.10}$$

By (3.1.5) we have that for each fixed $p \ge 1$,

$$\|\nu\|_\infty \le k_p < 1. \tag{3.1.11}$$

We now can get a quasiconformal mapping as follows.

Theorem 3.1.1 *There is a quasiconformal $h : \mathbb{D} \to \mathbb{D}$ such that*

$$h_{\bar{z}}(z) = \nu(z)h_z(z). \tag{3.1.12}$$

Proof. By Theorem 1.5.2 there is a principal solution $\tilde{h} : \mathbb{C} \to \mathbb{C}$ that satisfies (3.1.12) in $\mathbb{D}$. We restrict it to $\mathbb{D}$, then there is a Riemann mapping $\Psi : \tilde{h}(\mathbb{D}) \to \mathbb{D}$. Set
$$h = \Psi \circ \tilde{h} : \mathbb{D} \to \mathbb{D},$$
then h satisfies the conditions. $\square$

3.2 Improved regularity.

Set $g = h^{-1}$. By (1.3.2) we have that at almost every point $w \in \mathbb{D}$,
$$g_w(w) = \frac{\overline{h_z(g(w))}}{J(g(w), h)}, \quad g_{\overline{w}}(w) = -\frac{h_{\overline{z}}(g(w))}{J(g(w), h)}. \tag{3.2.1}$$
Put (3.2.1) into (3.1.12), we find that g satisfies the equation
$$g_{\overline{w}}(w) = -\nu(g(w))\overline{g_w(w)}. \tag{3.2.2}$$
By [6, Theorem 13.2.3], since h and g are $\frac{1+k_p}{1-k_p}$- quasiconformal, they are in the space $W^{1,r}_{loc}(\mathbb{D})$, for all $r \in [1, 1 + \frac{1}{k_p})$.

Now recall (3.1.4):
$$\int_{\mathbb{D}} a(z)\varphi_z(z)dz = \int_{\mathbb{D}} a(z)\nu^*(z)\varphi_{\overline{z}}(z)dz,$$
where
$$a(z) = e^{p\mathbb{K}_f(z)} - e^p,$$
$$\nu^*(z) := \frac{2p\overline{\mu_f(z)}e^{p\mathbb{K}_f(z)}}{(1 - |\mu_f(z)|^2)(e^{p\mathbb{K}_f(z)} - e^p)}.$$
By Theorem 1.2.9, we can change the variables with $z = g(w)$, then the left hand side becomes
$$\begin{aligned}
&\int_{\mathbb{D}} a(g(w))J(w, g)\varphi_z(g(w))dw \\
=&\int_{\mathbb{D}} a(g(w))(1 - |\nu(g(w))|^2)|g_w(w)|^2\varphi_z(g(w))dw \\
=&-\int_{\mathbb{D}} a(g(w))(1 - |\nu(g(w))|^2)\nu^*(g(w))g_w(w)[\varphi_z(g(w))g_{\overline{w}}(w)]dw,
\end{aligned} \tag{3.2.3}$$
while the right hand side is
$$\begin{aligned}
&\int_{\mathbb{D}} a(g(w))\nu^*(g(w))J(w, g)\varphi_{\overline{z}}(g(w))dw \\
=&\int_{\mathbb{D}} a(g(w))(1 - |\nu(g(w))|^2)\nu^*(g(w))|g_w(w)|^2\varphi_{\overline{z}}(g(w))dw \\
=&\int_{\mathbb{D}} a(g(w))(1 - |\nu(g(w))|^2)\nu^*(g)g_w(w)[\varphi_{\overline{z}}(g(w))\overline{g_w(w)}]dw.
\end{aligned} \tag{3.2.4}$$

Put them together we get

$$\int_{\mathbb{D}}[a(g(w))(1-|\nu(g(w))|^2)\nu^*(g(w))g_w(w)](\varphi\circ g)_{\overline{w}}(w)dw=0,\quad \forall\varphi\in C_0^\infty(\mathbb{D}). \tag{3.2.5}$$

We now wish to conclude that, under certain conditions,

$$\Phi(w):=a(g(w))(1-|\nu(g(w))|^2)\nu^*(g(w))g_w(w) \tag{3.2.6}$$

is holomorphic. In view of Weyl's lemma 1.1.8 we need the following two conditions:

$$\int_{\mathbb{D}}\Phi(w)\psi_{\overline{w}}(w)dw=0,\quad \forall\psi\in C_0^\infty(\mathbb{D}), \tag{3.2.7}$$

and

$$\Phi(w)\in L^1_{loc}(\mathbb{D}). \tag{3.2.8}$$

We know that $h\in W^{1,r}_{loc}(\mathbb{D})$, for all $r\in[1,1+\frac{1}{k_p})$. Note this implies that $\psi\circ h\in W^{1,r}_0(\mathbb{D})$, for any $\psi\in C_0^\infty(\mathbb{D})$. In fact, we may compute

$$\begin{aligned}\Big(\int_{\mathbb{D}}|(\psi\circ h)_z(z)|^r dz\Big)^{\frac{1}{r}} &= \Big(\int_{\mathbb{D}}|\psi_w(h(z))h_z(z)+\psi_{\overline{w}}(h(z))\overline{h_{\overline{z}}}(z)|^r dz\Big)^{\frac{1}{r}}\\ &= \Big(\int_V|\psi_w(h(z))h_z(z)+\psi_{\overline{w}}(h(z))\overline{h_{\overline{z}}(z)}|^r dz\Big)^{\frac{1}{r}}\\ &\le \|D\psi\|_\infty^{\frac{1}{r}}\|Dh\|_{L^r(V)}<\infty,\end{aligned}$$

where V is a compact set such that $h^{-1}(supp(\psi))\subset\subset V\subset\subset\mathbb{D}$. Also note $\psi\circ h$ is compactly supported in $\mathbb{D}$, so it is in $W^{1,r}_0(\mathbb{D})$. It is same for $(\psi\circ h)_{\overline{z}}$. We thus conclude:

Lemma 3.2.1 *For any $\psi\in C_0^\infty(\mathbb{D})$ and $1\le r<1+\frac{1}{k_p}$ we have $\psi\circ h\in W^{1,r}_0(\mathbb{D})$.*

We now deal with condition (3.2.7). Fix any $\psi\in C_0^\infty(\mathbb{D})$. By (3.2.5), for any $\varphi\in C_0^\infty(\mathbb{D})$ we have

$$\begin{aligned}&\int_{\mathbb{D}}\Phi(w)\psi_{\overline{w}}(w)dw\\ &=\int_{\mathbb{D}}\Phi(w)\Big(\psi_{\overline{w}}(w)-(\varphi\circ g)_{\overline{w}}(w)\Big)dw\\ &=\int_{\mathbb{D}}\Phi(h(z))\Big(\psi_{\overline{w}}(h(z))-\varphi_z(z)g_{\overline{w}}(h(z))-\varphi_{\overline{z}}(z)\overline{g_w(h(z))}\Big)J(z,h)dz\\ &=\int_{\mathbb{D}}\Phi(h(z))\Big(\psi_{\overline{w}}(h(z))J(z,h)+\varphi_z(z)h_{\overline{z}}(z)-\varphi_{\overline{z}}(z)h_z(z)\Big)dz.\end{aligned} \tag{3.2.9}$$

Consider the function $\psi\circ h$:

$$(\psi\circ h)_z(z)=\psi_w(h(z))h_z(z)+\psi_{\overline{w}}(h(z))\overline{h_{\overline{z}}(z)},$$

$$(\psi \circ h)_{\overline{z}}(z) = \psi_w(h(z))h_{\overline{z}}(z) + \psi_{\overline{w}}(h(z))\overline{h_z(z)}.$$

Thus

$$\psi_{\overline{w}}(h(z))J(z,h) = (\psi \circ h)_{\overline{z}}(z)h_z(z) - (\psi \circ h)_z(z)h_{\overline{z}}(z).$$

Put this back to (3.2.9) we get

$$\begin{aligned}
&\int_{\mathbb{D}} \Phi(w)\psi_{\overline{w}}(w)dw \\
=&\int_{\mathbb{D}} \Phi(h(z))\Big((\psi \circ h)_{\overline{z}}(z)h_z(z) - (\psi \circ h)_z(z)h_{\overline{z}}(z) + \varphi_z(z)h_{\overline{z}}(z) - \varphi_{\overline{z}}(z)h_z(z)\Big)dz \\
=&\int_{\mathbb{D}} \Phi(h(z))\Big(h_z(z)[(\psi \circ h)_{\overline{z}}(z) - \varphi_{\overline{z}}(z)] + h_{\overline{z}}(z)[\varphi_z(z) - (\psi \circ h)_z(z)]\Big)dz.
\end{aligned} \tag{3.2.10}$$

This holds for all $\varphi \in C_0^\infty(\mathbb{D})$. In $W^{1,r}(\mathbb{D})$, for any $r \in (1, 1 + \frac{1}{k_p})$, we may choose a sequence $\varphi^j \in C_0^\infty(\mathbb{D})$ such that $\varphi^j \to \psi \circ h$. Then by (3.2.10) it follows that, for some V such that $h^{-1}(supp(\psi)) \subset\subset V \subset\subset \mathbb{D}$,

$$\Big|\int_{\mathbb{D}} \Phi(w)\psi_{\overline{w}}(w)dw\Big| < C\|\Phi(h)h_z\|_{L^{r^*}(V)}\|\varphi^j - \psi \circ h\|_{W^{1,r}(V)} \to 0,$$

if only we have

$$\Phi(h(z))h_z(z) \in L^{r^*}_{loc}(\mathbb{D}), \quad r^* > 1 + k_p. \tag{3.2.11}$$

We now seek for the condition for (3.2.11). In fact, by the expressions (3.1.2), (3.1.3) and (3.2.6),

$$\begin{aligned}
\Phi(h(z))h_z(z) &= a(z)(1 - |\nu(z)|^2)\nu^*(z)g_w(h(z))h_z(z) \\
&= a(z)\nu^*(z)(1 - |\nu(z)|^2)\frac{\overline{h_z(z)}}{J(z,h)}h_z(z) \\
&= a(z)\nu^*(z) = \frac{2p\overline{\mu_f(z)}}{(1 + |\mu_f(z)|^2)}\mathbb{K}_f(z)e^{p\mathbb{K}_f(z)}.
\end{aligned}$$

So the $L^{r^*}_{loc}(\mathbb{D})$ integrability of $\Phi(h)h_z$ is guaranteed by that of $\exp(p\mathbb{K}_f)$. We conclude:

Lemma 3.2.2 *Suppose there is an $s > 1 + k_p$ such that*

$$\exp(p\mathbb{K}_f) \in L^s_{loc}(\mathbb{D}). \tag{3.2.12}$$

Then,

$$\int_{\mathbb{D}} \Phi(w)\psi_{\overline{w}}(w)dw = 0, \quad \forall \psi \in C_0^\infty(\mathbb{D}).$$

We now turn to condition (3.2.8). For any compact $A \subset \mathbb{D}$,

$$\begin{aligned}
\int_A |\Phi(w)|dw &= \int_A \Big|a(g(w))(1-|\nu(g(w))|^2)\nu^*(g(w))g_w(w)\Big|dw \\
&= \int_{g(A)} \Big|a(z)(1-|\nu(z)|^2)\nu^*(z)g_w(h(z))J(z,h)\Big|dz \\
&\le C\int_{g(A)} \mathbb{K}_f(z)\exp(p\mathbb{K}_f(z))|h_z(z)|dz \\
&\le C\|\mathbb{K}_f \exp(p\mathbb{K}_f)\|_{L^{r^*}(g(K))}\|h\|_{W^{1,r}(g(K))}.
\end{aligned}$$

So we find the finiteness of the last equation is again guaranteed by the same condition (3.2.12).

Now we can close with the final result of this section:

Theorem 3.2.3 *In the Euler-Lagrange equation (2.2.14), if*

$$\exp[p\mathbb{K}(z,f)] \in L^s_{loc}(\mathbb{D}),$$

for some

$$s > 1 + k_p.$$

Then

$$\Phi(w) = a(g(w))(1-|\nu(g(w))|^2)\nu^*(g(w))g_w(w)$$

is holomorphic.

3.3 Smoothness of minimisers.

In this section we will always assume that the condition in Theorem 3.2.3 is satisfied, so that

$$\Phi(w) = a(g(w))(1-|\nu(g(w))|^2)\nu^*(g(w))g_w(w)$$

is holomorphic.

By the simply connectedness of $\mathbb{D}$, we can choose a holomorphic function Ψ on $\mathbb{D}$ which is an anti-derivative of Φ. We set

$$F(z) = \Psi(h(z)). \tag{3.3.1}$$

Then

$$F_z(z) = \Phi(h(z))h_z(z) = a(z)\nu^*(z) = e^{p\mathbb{K}(z,f)}\frac{2p\overline{\mu_f(z)}}{1-|\mu_f(z)|^2}, \tag{3.3.2}$$

$$F_{\overline{z}}(z) = \Phi(h(z))h_{\overline{z}}(z) = a(z) = e^{p\mathbb{K}(z,f)} - e^p. \tag{3.3.3}$$

Then the distributional equation (3.1.1) reads as

$$\int_{\mathbb{D}} F_{\overline{z}}(z)\varphi_z(z)dz = \int_{\mathbb{D}} F_z(z)\varphi_{\overline{z}}(z)dz. \tag{3.3.4}$$

Note here $\Phi(h) \in L^\infty_{loc}(\mathbb{D})$ and both $h_z, h_{\overline{z}} \in L^r_{loc}(\mathbb{D})$, for all $r \in [1, 1+\frac{1}{k_p})$. In particular, $F \in W^{1,2}_{loc}(\mathbb{D})$. Write $|\mu_f| = t$, then

$$|F_z| = e^{p\frac{1+t^2}{1-t^2}} \frac{2pt}{1-t^2}, \tag{3.3.5}$$

$$F_{\overline{z}} = e^{p\frac{1+t^2}{1-t^2}} - e^p. \tag{3.3.6}$$

By (3.3.6),

$$t = \sqrt{\frac{\log[F_{\overline{z}} + e^p] - p}{\log[F_{\overline{z}} + e^p] + p}}. \tag{3.3.7}$$

Put (3.3.7) into (3.3.5) we get

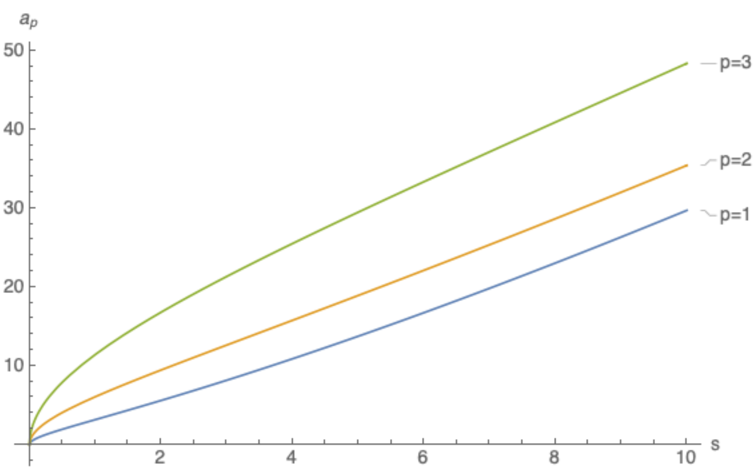
$$|F_z| = (F_{\overline{z}} + e^p)\sqrt{\log^2[F_{\overline{z}} + e^p] - p^2}. \tag{3.3.8}$$

We write it as

$$|F_z| = a_p(F_{\overline{z}}), \tag{3.3.9}$$

where

$$a_p(s) = (s+e^p)\sqrt{\log^2(s+e^p) - p^2}, \quad s \geq 0. \tag{3.3.10}$$



The graphs of $a_p(s)$ for $p = 1, 2, 3$.

For each fixed $p \geq 1$, $a_p(s)$ is a strictly increasing function with $a_p(0) = 0$, and

$$a_p'(s) = \sqrt{\log^2(s + e^p) - p^2} + \frac{\log(s + e^p)}{\sqrt{\log^2(s + e^p) - p^2}}.$$

Note that $a_p'(0) = a_p'(+\infty) = +\infty$. In fact $a_p'(s)$ has a minimum at some point $s_p > 0$ and is decreasing in $(0, s_p)$, then increasing in $(s_p, +\infty)$. To see this, we substitute the variables by

$$x = \log^2(s + e^p), \quad x \geq p.$$

Rewrite $b_p(x) = a_p'(s)$ as

$$b_p(x) = \sqrt{x^2 - p^2} + \frac{x}{\sqrt{x^2 - p^2}}. \tag{3.3.11}$$

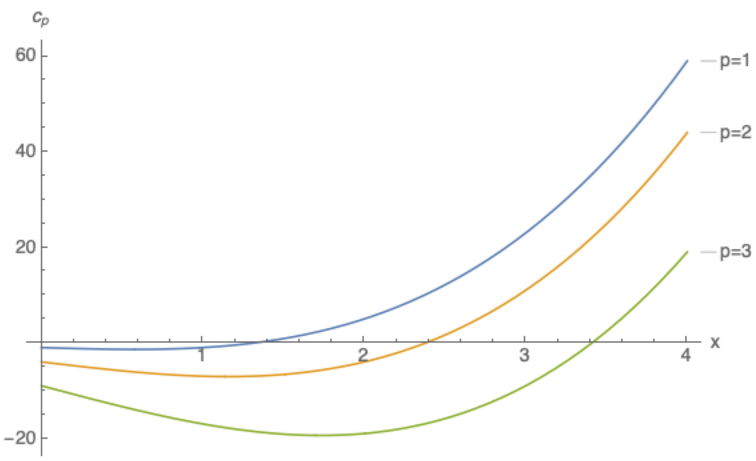
Then

$$b_p'(x) = \frac{x^3 - p^2 x - p^2}{(x^2 - p^2)^{3/2}}.$$

In the domain $x \geq p$ the cubic function

$$c_p(x) := x^3 - p^2 x - p^2 \tag{3.3.12}$$

is monotonically increasing and meets a unique zero point x_p.



The graphs of $c_p(x)$ for $p = 1, 2, 3$.

Let x_p be the minimiser of $b_p(x)$ and we wish to estimate the minimal value $m_p := b_p(x_p) = a_p'(s_p)$. Observe that in the cubic equation (3.3.12), if p is increasing, then the zero point x_p is increasing. At x_p we have

$$x_p^3 - p^2 x_p - p^2 = 0.$$

Thus

$$p^2 = \frac{x_p^3}{1 + x_p}.$$

Put this into (3.3.11) we get

$$m_p = b_p(x_p) = \sqrt{1 + x_p} + \frac{x_p}{\sqrt{1 + x_p}}.$$

This is also an increasing function. We then conclude that, when p increases, x_p increases, and then m_p increases.

So we consider the case that $p = 1$. In this case we have,

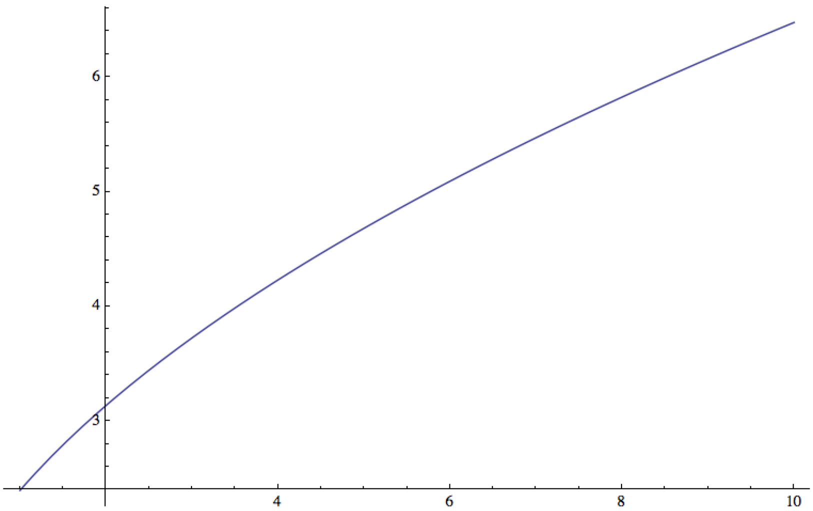
$$c_1(x) = x^3 - x - 1.$$

The only root in $(1, +\infty)$ is

$$x_1 = \left(\frac{3\sqrt{3} + \sqrt{23}}{6\sqrt{3}}\right)^{\frac{1}{3}} + \left(\frac{3\sqrt{3} - \sqrt{23}}{6\sqrt{3}}\right)^{\frac{1}{3}} \approx 1.3247179572447,$$

which gives

$$m_1 = b_1(x_1) = \sqrt{1 + x_1} + \frac{x_1}{\sqrt{1 + x_1}} \approx 2.3935395417626.$$



The graphs of m_p for $p \geq 1$.

We have therefore proved

Lemma 3.3.1 *For any $p \geq 1$ and $s \geq 0$,*

$$a_p'(s) = \sqrt{\log^2(s + e^p) - p^2} + \frac{\log(s + e^p)}{\sqrt{\log^2(s + e^p) - p^2}} > 2.$$

We now can set $s = \mathcal{A}_p(t)$ as the inverse mapping of $a_p(s)$. Then,

$$F_{\overline{z}} = \mathcal{A}_p(|F_z|), \tag{3.3.13}$$

$$\mathcal{A}_p'(t) < \frac{1}{2}. \tag{3.3.14}$$

In the sense of Definition 1.5.5, (3.3.13) is an elliptic equation, as there is a $0 \leq k < 1$ such that

$$|\mathcal{A}_p(|\zeta|) - \mathcal{A}_p(|\xi|)| \leq \mathcal{A}_p'(t_0)||\zeta| - |\xi|| \leq k|\zeta - \xi|.$$

So now it follows from Lemma 1.5.8 that

$$F \in W^{2,2}_{loc}(\mathbb{D}). \tag{3.3.15}$$

We remark that our case is a little bit different with Theorem 1.5.6, where the equation was set with $f_{\overline{z}}(z) = \mathcal{H}(f_z(z))$, but here in our case it is the absolute value $|F_z|$. To solve this problem we note that, by squaring both sides of (3.3.8), we get

$$|F_z|^2 = (F_{\overline{z}} + e^p)^2[\log^2(F_{\overline{z}} + e^p) - p^2]. \tag{3.3.16}$$

So we can rewrite it as

$$|F_z|^2 = \tilde{a}_p(F_{\overline{z}}), \tag{3.3.17}$$

where

$$\tilde{a}_p(s) = (s + e^p)^2[\log^2(s + e^p) - p^2], \quad s \geq 0. \tag{3.3.18}$$

For each fixed $p \geq 1$, $\tilde{a}_p(s)$ is also a strictly increasing function with $\tilde{a}_p(0) = 0$, and

$$\tilde{a}_p'(s) = 2(s + e^p)[\log^2(s + e^p) - p^2 + \log(s + e^p)].$$

We note that $\tilde{a}_p'(s)$ is an increasing function, and $\tilde{a}_p'(0) = 2pe^p > 2$, for any $p \geq 1$. Thus $\tilde{a}_p$ is invertible so we can define $\mathcal{B}_p = \tilde{a}_p^{-1}$, where $\mathcal{B}_p$ is also a C^∞ smooth function. Then

$$F_{\overline{z}} = \mathcal{B}_p(|F_z|^2), \tag{3.3.19}$$

where $\mathcal{B}_p$ is a C^∞-smooth function. Note then $\mathcal{B}_p(t^2) = \mathcal{A}_p(t)$, thus

$$\mathcal{A}_p'(t) = 2t\mathcal{B}_p'(t^2). \tag{3.3.20}$$

By (3.3.15) we can differentiate both sides of (3.3.19) by x, and get

$$(F_x)_{\overline{z}} = \mathcal{B}_p'(|F_z|^2)\overline{F_z}(F_x)_z + \mathcal{B}_p'(|F_z|^2)F_z\overline{(F_x)_z}, \tag{3.3.21}$$

where

$$\mathcal{B}_p'(|F_z|^2)|\overline{F_z}| + |\mathcal{B}_p'(|F_z|^2)|F_z| \leq \mathcal{A}_p'(|F_z|) \leq k.$$

Thus (3.3.21) is again an elliptic equation for the function F_x, and then $F \in W^{3,2}_{loc}(\mathbb{D})$. So we can differentiate it again and then

$$(F_{xx})_{\overline{z}} = \mathcal{B}_p'(|F_z|^2)\overline{F_z}(F_{xx})_z + \mathcal{B}_p'(|F_z|^2)F_z\overline{(F_{xx})_z} + \varphi(z),$$

where $\varphi(z)$ is composed by lower-order terms, and the equation is again elliptic. Now the argument is inductive and same as in the proof of Theorem 1.5.6, and then we conclude that F is C^∞-smooth.

So we have proved:

Lemma 3.3.2 *Let F be a $W^{1,2}_{loc}(\mathbb{D})$ solution to*

$$F_{\overline{z}} = \mathcal{A}_p(|F_z|),$$

where $\mathcal{A}_p = a_p^{-1}$, a_p is as (3.3.10). Then F is C^∞-smooth. In particular, both

$$F_z(z) = e^{p\mathbb{K}(z,f)}\frac{2p\overline{\mu_f(z)}}{1-|\mu_f(z)|^2}, \quad F_{\overline{z}}(z) = e^{p\mathbb{K}(z,f)} - e^p$$

are C^∞-smooth.

Theorem 3.3.3 *Assume that f satisfies equation (2.2.14) and condition (3.2.12). Then the Beltrami coefficient μ_f is a $C^\infty(\mathbb{D})$ function.*

Proof. By (3.3.2)-(3.3.8),

$$\begin{aligned}\mu_f &= \frac{\overline{F_z}}{|F_z|}\sqrt{\frac{\log[F_{\overline{z}}+e^p]-p}{\log[F_{\overline{z}}+e^p]+p}} \\ &= \frac{\overline{F_z}}{(F_{\overline{z}}+e^p)[\log(F_{\overline{z}}+e^p)+p]}.\end{aligned}$$

So it is C^∞-smooth since both F_z and $F_{\overline{z}}$ are. $\Box$

We now turn to f.

Lemma 3.3.4 *If a finite distortion homeomorphism $f : \mathbb{D} \to \mathbb{D}$ has C^1-smooth Beltrami coefficient μ_f and p-integrable distortion $\mathbb{K}(z,f)$, that is*

$$\int_{\mathbb{D}} \mathbb{K}^p(z,f)dz < \infty,$$

for some $p > 0$, then $|\mu_f| < 1$ in $\mathbb{D}$.

Proof. Let $|\mu_f(z_0)| = 1$ for some $z_0 \in \mathbb{D}$. For notational ease we set $z_0 = 0$. We consider the function $|\mu_f|$, which is then also C^1-smooth in a disk $D(0,\delta)$. As $|\mu_f| \leq 1$, we have $|\mu_f|_x(0) = |\mu_f|_y(0) = 0$. Then by Taylor's expansion,

$$|\mu_f(z)| \geq 1 - M|z|^2, \quad z \in D(0,\delta),$$

where

$$M = \sup_{z \in D(0,\delta)} |D^2|\mu_f|| < \infty.$$

Then,

$$\begin{aligned} \int_{\mathbb{D}} \Big(\frac{1+|\mu_f|^2}{1-|\mu_f|^2}\Big)^p &\geq \frac{1}{2^p} \int_{\mathbb{D}} \frac{1}{(1-|\mu_f|)^p} \\ &\geq \frac{1}{(2M)^p} \int_{D(0,\delta)} \frac{1}{|z|^{2p}} = \infty, \end{aligned}$$

which gives the contradiction. □

Theorem 3.3.5 *Assume that f satisfies equation (2.2.14) and condition (3.2.12). Then f is a C^∞-diffeomorphism.*

Proof. Let $\Omega \subset\subset \mathbb{D}$ be compactly contained. By Lemma 3.3.4, there is a k such that

$$|\mu_f(z)| \leq k < 1, \quad \forall z \in \Omega.$$

Then, by Theorem 3.3.3 and Theorem 1.5.4, f is a C^∞-diffeomorphism in Ω. Since this works for any $\Omega \subset\subset \mathbb{D}$, f is a C^∞-diffeomorphism in $\mathbb{D}$. □

Together with Theorem 2.5.1 this can also be stated as:

Theorem 3.3.6 *Let f be a minimiser of Problem 2.0.1 such that condition (3.2.12) is satisfied, then f is a C^∞-diffeomorphism.*

Now the following particular case follows.

Corollary 3.3.7 *Let f be a minimiser of Problem 2.0.1 such that $\mathbb{K}_f \in L^\infty_{loc}(\mathbb{D})$, then f is a C^∞-diffeomorphism.*

3.4 Further regularity.

We now assume that $f : \mathbb{D} \to \mathbb{D}$ is a C^1-diffeomorphism. By Theorem 2.4.4, in this case we have equation (2.2.6), which is

$$\int_{\mathbb{D}} \frac{|\mu_f(z)|^2 e^{p\mathbb{K}(z,f)}}{f_z(z)[1-|\mu_f(z)|^2]^2} \varphi_z(z)dz = \int_{\mathbb{D}} \frac{|\mu_f(z)|^2 e^{p\mathbb{K}(z,f)}}{f_{\overline{z}}(z)[1-|\mu_f(z)|^2]^2} \varphi_{\overline{z}}(z)dz, \quad \forall \varphi \in C_0^\infty(\mathbb{D}). \tag{3.4.1}$$

We write

$$A(z)=\frac{|\mu_f(z)|^2e^{p\mathbb{K}(z,f)}}{(1-|\mu_f(z)|^2)^2}.$$

Then (3.4.1) reads as

$$\int_{\mathbb{D}}\frac{A}{f_z}\varphi_z=\int_{\mathbb{D}}\frac{A}{f_{\overline{z}}}\varphi_{\overline{z}},\quad \forall\varphi\in C_0^\infty(\mathbb{D}).$$

Define

$$G=\mathcal{C}\Big(\frac{A}{f_z}\Big),\quad H=\mathcal{C}^*\Big(\frac{A}{f_{\overline{z}}}\Big),$$

where $\mathcal{C}$ is the Cauchy transform as (1.5.2), and $\mathcal{C}^*$ is the conjugate defined by $\mathcal{C}^*\eta=\overline{\mathcal{C}\overline{\eta}}$. Then, by the definition of weak derivatives,

$$\int_{\mathbb{D}}G\varphi_{z\overline{z}}=-\int_{\mathbb{D}}G_{\overline{z}}\varphi_z=-\int_{\mathbb{D}}H_z\varphi_{\overline{z}}=\int_{\mathbb{D}}H\varphi_{z\overline{z}},\quad \forall\varphi\in C_0^\infty(\mathbb{D}).$$

So it follows from Weyl's lemma 1.1.8 that $\phi:=G-H$ is harmonic in $\mathbb{D}$. Thus $G_z=H_z+\phi_z$, $H_{\overline{z}}=G_{\overline{z}}-\phi_{\overline{z}}$ are both in $C(\mathbb{D})$, so both $G,H\in C^1(\mathbb{D})$. In particular, $G_z-H_z=\phi_z$ is holomorphic in $\mathbb{D}$. Let ψ be an anti-derivative of ϕ_z, and define

$$g:=G-\psi\in C^1(\mathbb{D}).$$

Then,

$$g_z=G_z-\phi_z=H_z=\frac{A}{f_{\overline{z}}},\quad g_{\overline{z}}=G_{\overline{z}}=\frac{A}{f_z},$$

$$\mu_g=g_{\overline{z}}/g_z=\frac{A}{f_z}\Big/\frac{A}{f_{\overline{z}}}=\mu_f.$$

Set $h=f^{-1}$, $\Phi=g\circ h$. Then,

$$\begin{aligned}\Phi_{\overline{w}}(w)&=g_z(h(w))h_{\overline{w}}(w)+g_{\overline{z}}(h(w))\overline{h_w}(w)\\&=-g_z(h(w))\frac{f_{\overline{z}}(h(w))}{J(h(w),f)}+g_{\overline{z}}(h(w))\frac{f_z(h(w))}{J(h(w),f)}\\&=0.\end{aligned}$$

So Φ is holomorphic, and then its derivative $\Phi'(w)$ is also holomorphic. We compute

$$\begin{aligned}\Phi'(w)&=g_z(h(w))h_w(w)+g_{\overline{z}}(h(w))\overline{h_{\overline{w}}}(w)\\&=\frac{A(h(w))}{f_{\overline{z}}(h(w))}h_w(w)+\frac{A(h(w))}{f_z(h(w))}\overline{h_{\overline{w}}}(w)\\&=A(h(w))J(w,h)\Big[-\frac{h_w(w)}{h_{\overline{w}}(w)}+\frac{\overline{h_{\overline{w}}}(w)}{\overline{h_w}(w)}\Big]\\&=\frac{|\mu_f(h(w))|^2e^{p\mathbb{K}(h(w),f)}}{(1-|\mu_f(h(w))|^2)^2}J(w,h)\Big[-\frac{1}{\mu_h(h(w))}+\overline{\mu_h(h(w))}\Big]\\&=-e^{p\mathbb{K}(w,h)}h_w(w)\overline{h_{\overline{w}}}(w)\end{aligned}$$

So we get a holomorphic function

$$\Psi(w) := e^{p\mathbb{K}(w,h)} h_w(w)\overline{h_{\overline{w}}}(w). \tag{3.4.2}$$

This is called the Hopf differential. We also note that, as a holomorphic function in the simply connected domain $\mathbb{D}$, Ψ is either the constant zero or non-zero almost everywhere. Then, in the first case, h is conformal, and then so is f; in the latter case, we have the following at almost everywhere:

$$\frac{\overline{\Psi}}{|\Psi|} = \frac{\overline{h_w} h_{\overline{w}}}{|h_w h_{\overline{w}}|} = \frac{\mu_h}{|\mu_h|}.$$

This gives the following theorem:

Theorem 3.4.1 *Let f be a minimiser of Problem 2.0.1 which is a C^1-diffeomorphism. Then, either f is conformal or its inverse h satisfies*

$$\mu_h = |\mu_h| \frac{\overline{\Psi}}{|\Psi|}, \tag{3.4.3}$$

where Ψ is a holomorphic function.

We come back to (3.4.2) and assume that f is a C^2-diffeomorphism. So we can differentiate (3.4.2) again, and get

$$0 = \Psi_{\overline{w}} = pe^{p\mathbb{K}_h}(\mathbb{K}_h)_{\overline{w}} h_w \overline{h_{\overline{w}}} + e^{p\mathbb{K}_h} h_{w\bar{w}} \overline{h_{\overline{w}}} + e^{p\mathbb{K}_h} h_w \overline{h_{w\bar{w}}}.$$

Equivalently,

$$0 = 4pe^{p\mathbb{K}_h}(\mathbb{K}_h)_{\overline{w}} h_w \overline{h_{\overline{w}}} + e^{p\mathbb{K}_h} \Delta h \overline{h_{\overline{w}}} + e^{p\mathbb{K}_h} h_w \overline{h_{w\overline{w}}},$$

$$0 = 4pe^{p\mathbb{K}_h}(\mathbb{K}_h)_{w} \overline{h_w} h_{\overline{w}} + e^{p\mathbb{K}_h} \overline{h_{w\overline{w}}} h_{\overline{w}} + e^{p\mathbb{K}_h} \overline{h_w} h_{w\overline{w}}.$$

Hence

$$0 = 4pe^{p\mathbb{K}_h} h_w h_{\overline{w}} [(\mathbb{K}_h)_w \overline{h_w} - (\mathbb{K}_h)_{\overline{w}} \overline{h_{\overline{w}}}] + e^{p\mathbb{K}_h} h_{w\overline{w}} (|h_w|^2 - |h_{\overline{w}}|^2),$$

$$0 = 4ph_w h_{\overline{w}} [(\mathbb{K}_h)_w f_z(h) - (\mathbb{K}_h)_{\overline{w}} \overline{f_{\overline{z}}}(h)] + h_{w\overline{w}},$$

$$0 = 4ph_w h_{\overline{w}} [\mathbb{K}(f,h)]_z + h_{w\overline{w}}. \tag{3.4.4}$$

We set

$$\lambda(z) = e^{p\mathbb{K}(z,f)}. \tag{3.4.5}$$

Then (3.4.4) reads as

$$h_{w\overline{w}} + (\log\lambda)_z(h) h_w h_{\overline{w}} = 0. \tag{3.4.6}$$

This is the tension equation [40] for the metric $\lambda(z)|dz|^2$. So we conclude:

Theorem 3.4.2 *Let f be a minimiser of Problem 2.0.1 which is a C^2-diffeomorphism. Then the inverse mapping $h = f^{-1} : (\mathbb{D}, E) \to (\mathbb{D}, \lambda)$ is harmonic, where E is the Euclidean metric, and λ is as in (3.4.5).*

Having the tension equation (3.4.6) in hand, we will next obtain a first order nonlinear equation for μ_f. For the ease of notation, in the rest of this section we will write μ for μ_f and $\mathbb{K}$ for $\mathbb{K}_f$.

Theorem 3.4.3 *If a C^2-diffeomorphism h satisfies equation (3.4.6), then the Beltrami coefficient μ of its inverse f satisfies*

$$\mu_z - \overline{\mu}\mu_{\overline{z}} = -\mu(\rho + \overline{\mu\rho}), \tag{3.4.7}$$

where

$$\rho = \frac{1}{4}(\log \lambda)_z. \tag{3.4.8}$$

Proof. By (1.3.2), we have

$$\mu(h) = \frac{f_{\overline{z}}(h)}{f_z(h)} = -\frac{h_{\overline{w}}}{\overline{h_w}}.$$

Thus

$$\begin{aligned} h_{w\bar{w}} &= -[\mu(h)\overline{h_w}]_w \\ &= -\mu_z(h)|h_w|^2 - \mu_{\overline{z}}(h)\overline{h_w h_{\overline{w}}} - \mu(h)\overline{h_{w\bar{w}}}. \end{aligned} \tag{3.4.9}$$

We put (3.4.6) into (3.4.9), and get

$$-\rho(h)h_w h_{\overline{w}} = -\mu_z(h)|h_w|^2 - \mu_{\overline{z}}(h)\overline{h_w h_{\overline{w}}} + \mu(h)\overline{\rho(h)h_w h_{\overline{w}}},$$

$$\begin{aligned} \rho(h)\mu(h) &= -\mu_z(h) + \mu_{\overline{z}}(h)\overline{\mu(h)} - \mu(h)\overline{\rho(h)\mu(h)}, \\ \mu_z(z) - \overline{\mu(z)}\mu_{\overline{z}}(z) &= -\mu(z)\big(\rho(z) + \overline{\mu(z)\rho(z)}\big). \end{aligned}$$

□

We now put (3.4.5) and (3.4.8) into (3.4.7). Then

$$\rho = \frac{1}{4}(\log \lambda)_z = p\mathbb{K}_z,$$

$$\mu_z - \overline{\mu}\mu_{\overline{z}} = -p\mu(\mathbb{K}_z + \overline{\mu}\mathbb{K}_{\overline{z}}), \tag{3.4.10}$$

where

$$\mathbb{K}_z = \frac{2(\mu_z\overline{\mu} + \mu\overline{\mu_{\overline{z}}})}{(1 - |\mu|^2)^2}, \quad \mathbb{K}_{\overline{z}} = \frac{2(\mu_{\overline{z}}\overline{\mu} + \mu\overline{\mu_z})}{(1 - |\mu|^2)^2}.$$

Recall we have $|\mu| < 1$. Thus (3.4.10) becomes

$$(1-|\mu|^2)^2(\mu_z - \overline{\mu}\mu_{\overline{z}}) = -2p\mu[(\mu_z\overline{\mu} + \mu\overline{\mu_{\overline{z}}}) + \overline{\mu}(\mu_{\overline{z}}\overline{\mu} + \mu\overline{\mu_z})].$$

Rearrange the terms we get

$$[1+2(p-1)|\mu|^2+|\mu|^4]\mu_z + 2p|\mu|^2\mu\overline{\mu_z} = [1-2(p+1)|\mu|^2+|\mu|^4]\overline{\mu}\mu_{\overline{z}} - 2p\mu^2\overline{\mu_{\overline{z}}}. \quad (3.4.11)$$

We write

$$A(t) = 1 + 2(p-1)t^2 + t^4,$$
$$B(t) = 2pt^2,$$
$$C(t) = 1 - 2(p+1)t^2 + t^4.$$

Then (3.4.11) can be shown as

$$A(|\mu|)\mu_z + B(|\mu|)\mu\overline{\mu_z} = C(|\mu|)\overline{\mu}\mu_{\overline{z}} - 2p\mu^2\overline{\mu_{\overline{z}}}.$$

We also have the conjugates for both sides

$$A(|\mu|)\overline{\mu_z} + B(|\mu|)\overline{\mu}\mu_z = C(|\mu|)\mu\overline{\mu_{\overline{z}}} - 2p\overline{\mu}^2\mu_{\overline{z}}.$$

Eliminate $\overline{\mu_z}$ terms we get

$$(A^2 - B^2|\mu|^2)\mu_z = (AC + 2p|\mu|^2B)\overline{\mu}\mu_{\overline{z}} - (BC + 2pA)\mu^2\overline{\mu_{\overline{z}}}.$$

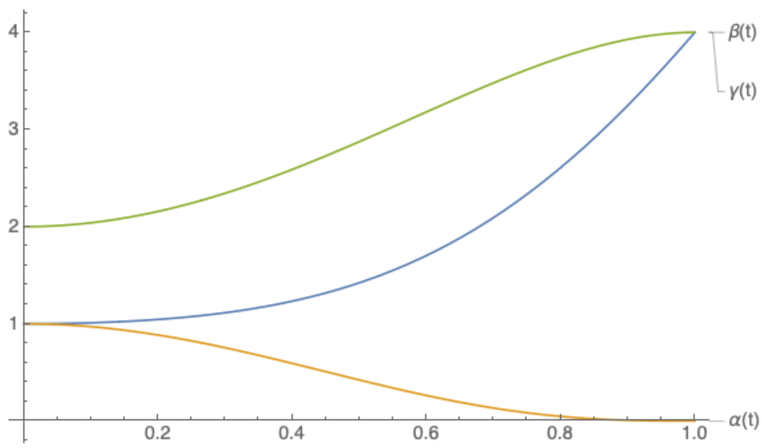
We rewrite it as

$$\gamma(|\mu|)\mu_z = \alpha(|\mu|)\overline{\mu}\mu_{\overline{z}} - \beta(|\mu|)\mu^2\overline{\mu_{\overline{z}}}, \quad (3.4.12)$$

where

$$\gamma(t) = 1 + (4p-3)t^2 + (4p^2 - 4p + 3)t^4 - t^6,$$
$$\alpha(t) = (1-t^2)^3,$$
$$\beta(t) = 2p(1 + 2pt^2 - t^4).$$

We check the behaviours of γ, α, β for $t \in [0, 1)$ and find they are all positive, and for each fixed $p \geq 1$, β and γ are increasing, while α is decreasing.



The graphs of $\alpha(t)$, $\gamma(t)$, $\beta(t)$, for $p = 1$.

We next consider

$$
\begin{aligned}
&\gamma(t) - ta(t) - t^2\beta(t) \\
=&1 + (4p-3)t^2 + (4p^2 - 4p + 3)t^4 - t^6 - t(1-t^2)^3 - 2pt^2(1 + 2pt^2 - t^4) \\
=&(1-t^2)^2(1 - t + t^2(2p-1) + t^3).
\end{aligned}
$$

Note the last term is positive, for each $p \geq 1$ and $t \in [0, 1)$. So we conclude that

$$ta(t) + t^2\beta(t) < \gamma(t), \quad t \in [0, 1). \tag{3.4.13}$$

Then in the sense of Definition 1.5.5, we have

Theorem 3.4.4 *For all $p \geq 1$, equation (3.4.12) is elliptic in any compactly contained domain $\Omega \subset\subset \mathbb{D}$. In particular, the conjugate of the Beltrami coefficient $\overline{\mu_f}$ is locally quasiregular.*

4 The Inverse Exponential Problem

As we introduced in Section 1.3.1, the L^1 case was solved by converting the distortion problem of f into the Dirichlet problem of its inverse f^{-1}, and then everything follows from the classic harmonic analysis. As an analogue, we have already seen in Theorem 1.4.6 that for any $f \in W^{1,1}_{loc}(\mathbb{D})$ that has p-exponentially integrable distortion, we can also turn the problem to the inverse one:

$$\int_{\mathbb{D}} \exp(p\mathbb{K}(z,f))dz = \int_{\mathbb{D}} \exp(p\mathbb{K}(w,f^{-1}))J(w,f^{-1})dw.$$

In this chapter we study the inverse problem and get some more properties of the minimisers. The problem can be stated as follows.

Let $p > 0$, the inverse $\exp(p)$ mean distortion of a finite distortion self-homeomorphism of $\overline{\mathbb{D}}$ is defined as

$$\mathcal{E}_p^{-1}(h) := \int_{\mathbb{D}} \exp[p\mathbb{K}(w,h)]J(w,h)dw. \tag{4.0.1}$$

Let $h_0 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a finite distortion homeomorphism such that $\mathcal{E}_p^{-1}(h_0) < \infty$. We set

$$\begin{aligned} \mathcal{H}_p := \Big\{ & h \in W^{1,1}_{loc}(\mathbb{D}) : h \text{ is a homeomorphism from } \overline{\mathbb{D}} \text{ to } \overline{\mathbb{D}}, \\ & \mathcal{E}_p^{-1}(h) < \infty, \text{ and } h|_{\partial\mathbb{D}} = h_0|_{\partial\mathbb{D}} \Big\}. \end{aligned} \tag{4.0.2}$$

Problem 4.0.1 *Find the minimal mappings $h \in \mathcal{H}_p$ such that*

$$\mathcal{E}_p^{-1}(h) = \min_{g \in \mathcal{H}_p} \mathcal{E}_p^{-1}(g).$$

4.1 Variation equations for inverse exponential problem.

We assume that h is variational and calculate the variation formulae.

4.1.1 Outer variation.

We set

$$h^t(w) = h(w) + t\varphi(w), \quad \varphi \in C_0^\infty(\mathbb{D}).$$

Following Section 2.2, we calculate

$$h^t_w = h_w + t\varphi_w, \quad h^t_{\overline{w}} = h_{\overline{w}} + t\varphi_{\overline{w}},$$

$$J_{h^t} = J_h + t^2 J_\varphi + 2t\Re e(\overline{h_w}\varphi_w - \overline{h_{\overline{w}}}\varphi_{\overline{w}}),$$

$$\frac{\partial}{\partial t}\Big|_{t=0} J_{h^t} = 2\Re e(\overline{h_w}\varphi_w - \overline{h_{\overline{w}}}\varphi_{\overline{w}}),$$

$$\frac{\partial}{\partial t}\Big|_{t=0}\mathbb{K}_{h^t} = \frac{2\frac{\partial}{\partial t}\Big|_{t=0}|\mu_{h^t}|^2}{(1-|\mu_h|^2)^2} = \frac{4\Re e(\frac{\varphi_{\overline{w}}}{h_{\overline{w}}} - \frac{\varphi_w}{h_w})|\mu_h|^2}{(1-|\mu_h|^2)^2}.$$

Then

$$\begin{aligned}
0 =& \frac{\partial}{\partial t}\Big|_{t=0}\int_{\mathbb{D}} e^{p\mathbb{K}(w,h^t)} J(w,h^t)dw \\
=& \int_{\mathbb{D}} pe^{p\mathbb{K}(w,h)} J(w,h^t)\cdot \frac{\partial}{\partial t}\Big|_{t=0}\mathbb{K}(w,h)dw + \int_{\mathbb{D}} e^{p\mathbb{K}(w,h)}\cdot\frac{\partial}{\partial t}\Big|_{t=0} J(w,h^t)dw \\
=& 4p\int_{\mathbb{D}} \frac{e^{p\mathbb{K}(w,h)}}{1-|\mu_h(w)|^2}\Re e(\frac{\varphi_{\overline{w}}(w)}{h_{\overline{w}}(w)} - \frac{\varphi_w(w)}{h_w(w)})|h_{\overline{w}}(w)|^2 dw \\
&+ 2\int_{\mathbb{D}} e^{p\mathbb{K}(w,h)}\Re e(\overline{h_w(w)}\varphi_w(w) - \overline{h_{\overline{w}}(w)}\varphi_{\overline{w}}(w))dw.
\end{aligned}$$

This also applies on $i\varphi$, so we get

$$0 = 2p\int_{\mathbb{D}} \frac{e^{p\mathbb{K}(w,h)}}{1-|\mu_h(w)|^2}(\frac{\varphi_{\overline{w}}(w)}{h_{\overline{w}}(w)} - \frac{\varphi_w(w)}{h_w(w)})|h_{\overline{w}}(w)|^2 dw + \int_{\mathbb{D}} e^{p\mathbb{K}(w,h)}(\overline{h_w(w)}\varphi_w(w) - \overline{h_{\overline{w}}(w)}\varphi_{\overline{w}}(w))dw,$$

$$\int_{\mathbb{D}} e^{p\mathbb{K}(w,h)}\overline{h_{\overline{w}}(w)}(\frac{2p}{1-|\mu_h(w)|^2} - 1)\varphi_{\overline{w}}(w)dw = \int_{\mathbb{D}} e^{p\mathbb{K}(w,h)}\overline{h_w(w)}(\frac{2p|\mu_h(w)|^2}{1-|\mu_h(w)|^2} - 1)\varphi_w(w)dw,$$

$$\int_{\mathbb{D}} e^{p\mathbb{K}(w,h)}[(\mathbb{K}(w,h)+1)p-1)]h_{\overline{w}}(w)\varphi_w(w)dw = \int_{\mathbb{D}} e^{p\mathbb{K}(w,h)}[(\mathbb{K}(w,h)-1)p-1)]h_w(w)\varphi_{\overline{w}}(w)dw.$$

This is the Euler-Lagrange equation for outer variational minimisers of the inverse exponential Problem 4.0.1.

4.1.2 Inner variation.

Again we set

$$g^t(w) = w + t\varphi(w), \quad h^t(\zeta) = h\circ(g^t)^{-1}(\zeta).$$

Then

$$(g^t)_w(w) = 1 + t\varphi_w(w), \quad (g^t)_{\overline{w}}(w) = t\varphi_{\overline{w}}(w),$$

$$\mu_{h\circ(g^t)^{-1}}(\zeta) = \frac{(h\circ(g^t)^{-1})_{\overline{\zeta}}(\zeta)}{(h\circ(g^t)^{-1})_{\zeta}(\zeta)} = \frac{\mu_h(w) - \mu_{g^t}(2)}{1-\mu_h(w)\overline{\mu_{g^t}}(w)}\frac{(g^t)_w(w)}{\overline{(g^t)_w(w)}},$$

$$J(\zeta,h^t) = \frac{J(w,h)}{J(w,g^t)}.$$

Thus

$$\frac{\partial}{\partial t}\Big|_{t=0}|\mu_{h\circ(g^t)^{-1}}(g^t(w))|^2 = 2\Re e(\varphi_{\overline{w}}(w)\overline{\mu_h(w)})\Big(|\mu_h(w)|^2 - 1\Big),$$

$$\frac{\partial}{\partial t}\Big|_{t=0}\mathbb{K}(g^t(w), h\circ (g^t)^{-1}) = \frac{4\Re e(\varphi_{\overline{w}}(w)\overline{\mu_h(w)})}{|\mu_h(w)|^2-1}.$$

$$\begin{aligned}
0 =&\frac{\partial}{\partial t}\Big|_{t=0}\int_{\mathbb{D}} \exp[p\mathbb{K}(\zeta, h^t)]J(\zeta, h^t)d\zeta\\
=&\frac{\partial}{\partial t}\Big|_{t=0}\int_{\mathbb{D}} \exp[p\mathbb{K}(g^t(w), h\circ (g^t)^{-1})]J(g^t(w), h^t)J(w, g^t)dw\\
=&\int_{\mathbb{D}} \frac{4\Re e[\varphi_{\overline{w}}(w)\overline{\mu_h(w)}]}{|\mu_h(w)|^2-1} p\exp[p\mathbb{K}(w,h)]J(w,h)dw\\
=&-4p\int_{\mathbb{D}} \Re e[h_w(w)\overline{h_{\overline{w}}(w)}\varphi_{\overline{w}}(w)]\exp[p\mathbb{K}(w,h)]dw.
\end{aligned}$$

Then

$$\int_{\mathbb{D}} \exp[p\mathbb{K}(w,h)]h_w(w)\overline{h_{\overline{w}}(w)}\varphi_{\overline{w}}(w)dw = 0, \quad \forall \varphi\in C_0^\infty(\mathbb{D}). \tag{4.1.1}$$

4.1.3 Hopf differential.

We observe that the kernel of (4.1.1) is the Hopf differential in (3.4.2):

$$\Psi(w) = \exp[p\mathbb{K}(w,h)]h_w(w)\overline{h_{\overline{w}}(w)}.$$

So (4.1.1) reads as

$$\int_{\mathbb{D}} \Psi(w)\varphi_{\overline{w}}(w)dw = 0, \quad \forall \varphi\in C_0^\infty(\mathbb{D}).$$

Then, by Weyl's lemma, Ψ is holomorphic if only it is locally integrable. We note

$$\Psi(w) = \exp(p\mathbb{K}(w,h))h_w(w)\overline{h_{\overline{w}}(w)} = \mathbb{K}(w,h)\exp[p\mathbb{K}(w,h)]J(w,h)\frac{\overline{\mu_h(w)}}{1+|\mu_h(w)|^2},$$

where

$$\frac{|\mu_h(w)|}{1+|\mu_h(w)|^2}\le \frac{1}{2}.$$

So the local integrability only requires

$$\mathbb{K}(w,h)\exp(p\mathbb{K}(w,h))J(w,h)\in L^1_{loc}(\mathbb{D}). \tag{4.1.2}$$

On the other hand, by a similar argument to in Section 2.5, we can prove the following analogue of Theorem 2.5.4:

Theorem 4.1.1 *If h is a minimiser of Problem 4.0.1, and there is a $q>p$ such that*

$$\exp(q\mathbb{K}(w,h))J(w,h)\in L^1_{loc}(\mathbb{D}), \tag{4.1.3}$$

then h satisfies the Euler-Lagrange equation (4.1.1).

Note the condition (4.1.2) is covered by (4.1.3). Furthermore, by Theorem 1.4.5, both f and h satisfy Lusin's condition $\mathcal{N}$, thus in exactly the same way as in Theorem 1.4.6, we can prove that

$$\int_{h(A)} \exp(q\mathbb{K}(z,f))dz = \int_A \exp(q\mathbb{K}(w,h))J(w,h)dw,$$

for any $q > 0$ and compact subset $A \subset \mathbb{D}$. Thus condition (4.1.3) is equivalent to

$$\exp[q\mathbb{K}(z,f)] \in L^1_{loc}(\mathbb{D}).$$

Recall that in (3.4.2) we got the Hopf differential $\Psi(w)$ under the assumption that f is a C^∞-diffeomorphism, which is guaranteed by our earlier assumption (3.2.12): $\exp[p\mathbb{K}(z,f)] \in L^s_{loc}(\mathbb{D})$ for some

$$s > 1 + k_p.$$

So here we get a weaker condition for this:

Theorem 4.1.2 *Let f be a minimiser of Problem 2.0.1, and $h = f^{-1}$. If $\exp[q\mathbb{K}(z,f)] \in L^1(\mathbb{D})$ for some $q > p$, then the Hopf differential*

$$\Psi(w) = \exp(p\mathbb{K}(w,h))h_w(w)\overline{h_{\overline{w}}(w)}$$

is holomorphic.

5 The Extremal Teichmüller Problem

We recall the Teichmüller problem: Let $f_0 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a quasiconformal mapping. Let $\mathcal{F}_\infty$ be the class of all quasiconformal mappings from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$ such that $f = f_0$ along $\partial\mathbb{D}$, and $\mathcal{E}_\infty(f)$ be the maximal distortion of f in $\mathbb{D}$, that is

$$\mathcal{E}_\infty(f) = \|\mathbb{K}(z,f)\|_{L^\infty(\mathbb{D})}.$$

Problem 5.0.1 *Find the minimal mappings $f \in \mathcal{F}_\infty$ such that*

$$\mathcal{E}_\infty(f) = \min_{g\in\mathcal{F}_\infty} \mathcal{E}_\infty(g). \tag{5.0.1}$$

Theorem 5.0.2 (Teichmüller's Existence Theorem) *For every quasiconformal $f_0 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, there is an extremal mapping $f \in \mathcal{F}_\infty$ that satisfies (5.0.1) and either f is conformal or the Beltrami coefficient of its inverse $h = f^{-1}$ satisfies*

$$\mu_h = k\frac{\overline{\Psi}}{|\Psi|},$$

where $0 < k < 1$ is a constant and Ψ is a holomorphic mapping.

5.1 A minimising sequence.

We will solve this problem by pushing $p \to \infty$ in the exponential problems $\mathcal{E}_p$. Let f_p be a minimiser of the the p-exponential problem, $h_p = f_p^{-1}$, and assume that the condition in Theorem 4.1.1 is satisfied. That is, for each p we have the holomorphic Hopf differential

$$\Psi_p = \exp(p\mathbb{K}_{h_p})(h_p)_w\overline{(h_p)_{\overline{w}}} = \exp(p\mathbb{K}_{h_p})\mathbb{K}_{h_p}J_{h_p}\frac{\overline{\mu_{h_p}}}{1+|\mu_{h_p}|^2}. \tag{5.1.1}$$

Lemma 5.1.1 *Let $0 < p \le q < \infty$. For every f such that $\int_{\mathbb{D}} \exp[q\mathbb{K}_f] < \infty$,*

$$\frac{1}{p}\log[\frac{1}{\pi}\int_{\mathbb{D}}\exp(p\mathbb{K}_f)] \le \frac{1}{q}\log[\frac{1}{\pi}\int_{\mathbb{D}}\exp(q\mathbb{K}_f)].$$

Proof. This is equivalent to

$$[\frac{1}{\pi}\int_{\mathbb{D}}\exp(p\mathbb{K}_f)]^{\frac{1}{p}} \le [\frac{1}{\pi}\int_{\mathbb{D}}\exp(q\mathbb{K}_f)]^{\frac{1}{q}}.$$

Meanwhile, it follows from Hölder's inequality that

$$\int_{\mathbb{D}}\exp(p\mathbb{K}_f) \le \pi^{\frac{q-p}{q}}[\int_{\mathbb{D}}\exp(q\mathbb{K}_f)]^{\frac{p}{q}}.$$

□

Lemma 5.1.2

$$\mathcal{E}_\infty(f) = \lim_{p\to\infty} \frac{1}{p} \log[\frac{1}{\pi}\mathcal{E}_p(f)],$$

if either side is finite.

Proof. On one hand,

$$\frac{1}{p}\log[\frac{1}{\pi}\int_{\mathbb{D}} \exp(p\mathbb{K}_f)] \le \frac{1}{p}\log[\exp(p\mathcal{E}_\infty(f))] = \mathcal{E}_\infty(f).$$

On the other hand, for any M such that $0 < M < \mathcal{E}_\infty(f)$, we set

$$E = \{z \in \mathbb{D} : \mathbb{K}_f(z) \ge M\}.$$

From the definition of $\mathcal{E}_\infty(f)$ we have $m := |E| > 0$. Then,

$$\begin{aligned}\lim_{p\to\infty}\frac{1}{p}\log[\frac{1}{\pi}\int_{\mathbb{D}} \exp(p\mathbb{K}_f)] &\ge \lim_{p\to\infty}\frac{1}{p}\log[\frac{1}{\pi}\int_{E} \exp(p\mathbb{K}_f)] \\ &\ge \lim_{p\to\infty}\frac{1}{p}\log[\frac{m}{\pi}(\exp(pM))] \\ &= M.\end{aligned}$$

□

Lemma 5.1.3 *There is a subsequence of $\{f_p\}$ (which we still call f_p) that converges uniformly and weakly in $W^{1,P}(\mathbb{D})$ to some f_∞, where $P(t) = \frac{t^2}{\log(e+t)}$; while h_p converges uniformly and weakly in $W^{1,2}(\mathbb{D})$ to $h_\infty = f_\infty^{-1}$.*

Proof. Let f_0 be as in the setting of Problem 5.0.1. Then, for each $p \ge 1$, as f_p is a minimiser of the p-exponential problem,

$$\log[\frac{1}{\pi}\mathcal{E}_1(f_p)] \le \frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_p)] \le \frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_0)] \le \mathcal{E}_\infty(f_0).$$

Then, it follows from Theorem 1.4.2 that f_p has a uniform $W^{1,P}(\mathbb{D})$ norm. Similarly, for h_p we have

$$\log[\frac{1}{\pi}\|Dh_p\|_{L^2(\mathbb{D})}] = \frac{1}{\pi}\mathcal{E}_1^{-1}(h_p) = \frac{1}{\pi}\mathcal{E}_1(f_p) \le \mathcal{E}_\infty(f_0). \tag{5.1.2}$$

Then the claims follow the same reasoning as in the proof of Theorem 1.4.9. □

Theorem 5.1.4 *Let f_p and f_∞ be as in Lemma 5.1.3. Then,*

$$\mathcal{E}_\infty(f_\infty) = \lim_{p\to\infty}\frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_p)] < \infty. \tag{5.1.3}$$

In particular, f_∞ is a minimiser of Problem 5.0.1.

Proof. Let $1 \le p \le q < \infty$. Then, by Lemma 5.1.1 and Lemma 5.1.2,

$$\begin{aligned}
\frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_p)] &\le \frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_q)] \\
&\le \frac{1}{q}\log[\frac{1}{\pi}\mathcal{E}_q(f_q)] \\
&\le \frac{1}{q}\log[\frac{1}{\pi}\mathcal{E}_q(f_0)] \\
&\le \mathcal{E}_\infty(f_0). \qquad (5.1.4)
\end{aligned}$$

So $\frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_p)]$, $p \ge 1$ is a bounded increasing sequence. This proves the existence of the limit in (5.1.3). As we proved in Theorem 1.4.9, for each fixed $k > 0$ we have

$$\mathcal{E}_k(f_\infty) \le \liminf_{p\to\infty} \mathcal{E}_k(f_p).$$

Together with Lemma 5.1.1 we get

$$\frac{1}{k}\log[\frac{1}{\pi}\mathcal{E}_k(f_\infty)] \le \liminf_{p\to\infty} \frac{1}{k}\log[\frac{1}{\pi}\mathcal{E}_k(f_p)] \le \lim_{p\to\infty}\frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_p)].$$

Then, by Lemma 5.1.2,

$$\mathcal{E}_\infty(f_\infty) = \lim_{k\to\infty}\frac{1}{k}\log[\frac{1}{\pi}\mathcal{E}_k(f_\infty)] \le \lim_{p\to\infty}\frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_p)]. \qquad (5.1.5)$$

Conversely, for each p,

$$\frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_p)] \le \frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_\infty)] \le \mathcal{E}_\infty(f_\infty). \qquad (5.1.6)$$

Now (5.1.3) follows from (5.1.5) and (5.1.6). Finally, in (5.1.4) we can replace f_0 by any $g \in \mathcal{F}_\infty$ and the same inequality holds, so

$$\mathcal{E}_\infty(f_\infty) = \lim_{p\to\infty}\frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_p)] \le \mathcal{E}_\infty(g), \quad \forall g \in \mathcal{F}_\infty.$$

This proves the second claim. □

5.2 The equation for inverse function.

Let f_∞ be as in Lemma 5.1.3. We have already shown it is a minimiser of Problem 5.0.1. In this section we prove it satisfies the conditions in Theorem 5.0.2. First of all, if $\mathcal{E}_\infty(f_\infty) = 1$, then f_∞ is a conformal mapping, which turns back the first case in Theorem 5.0.2. So from now on we will always assume that $\mathcal{E}_\infty(f_\infty) > 1$. Set

$$k := \sqrt{\frac{\mathcal{E}_\infty(f_\infty) - 1}{\mathcal{E}_\infty(f_\infty) + 1}} > 0. \qquad (5.2.1)$$

Recall we have the holomorphic sequence

$$\Psi_p = \exp(p\mathbb{K}_{h_p})(h_p)_w\overline{(h_p)_{\overline{w}}} = \exp(p\mathbb{K}_{h_p})\mathbb{K}_{h_p}J_{h_p}\frac{\overline{\mu_{h_p}}}{1+|\mu_{h_p}|^2}.$$

Define

$$C_p(s) := \frac{1}{p}\log[\frac{1}{\pi}\int_{\mathbb{D}}\exp(ps\mathbb{K}_{h_p})|(h_p)_w(h_p)_{\overline{w}}|], \quad 0<s<1,$$

and

$$C_p(1) := \frac{1}{p}\log[\frac{1}{\pi}\int_{\mathbb{D}}|\Psi_p|]$$

Lemma 5.2.1 *With the notation above we have the following two inequalities.*
i) For any $s\in(0,1)$,

$$\limsup_{p\to\infty} C_p(s) \leq \mathcal{E}_\infty(f_\infty). \tag{5.2.2}$$

ii)

$$\liminf_{p\to\infty} C_p(1) \geq \mathcal{E}_\infty(f_\infty). \tag{5.2.3}$$

Proof. i) For every fixed s we can choose p so large that

$$\begin{aligned}\int_{\mathbb{D}}\exp(ps\mathbb{K}_{h_p})|(h_p)_w(h_p)_{\overline{w}}| &= \int_{\mathbb{D}}\exp(ps\mathbb{K}_{h_p})\mathbb{K}_{h_p}J_{h_p}\frac{|\mu_{h_p}|}{1+|\mu_{h_p}|^2}\\ &\leq \frac{1}{2}\int_{\mathbb{D}}\exp(p\mathbb{K}_{h_p})J_{h_p}\\ &= \frac{1}{2}\mathcal{E}_p^{-1}(h_p) = \frac{1}{2}\mathcal{E}_p(f_p).\end{aligned}$$

Then the claim follows from (5.1.3).

ii) We observe that by elementary computation,

$$\exp(p\mathbb{K}_{h_p}) \leq \mathbb{K}_{h_p}\exp(p\mathbb{K}_{h_p})\frac{|\mu_{h_p}|}{1+|\mu_{h_p}|^2}\frac{1+\delta_p^2}{\delta_p} + \exp(p\frac{1+\delta_p^2}{1-\delta_p^2}), \quad \forall\delta_p\in(0,1).$$

Multiply by J_{h_p} and integrate both sides over $\mathbb{D}$, we get

$$\mathcal{E}_p(f_p) = \mathcal{E}_p^{-1}(h_p) \leq \frac{1+\delta_p^2}{\delta_p}\int_{\mathbb{D}}|\Psi_p(w)|dw + \pi\exp(p\frac{1+\delta_p^2}{1-\delta_p^2}).$$

For sufficiently large p we can find a $\delta_p\in(0,1)$ such that

$$\pi\exp(p\frac{1+\delta_p^2}{1-\delta_p^2}) = \frac{1}{2}\mathcal{E}_p(f_p).$$

Then
$$\frac{1+\delta_p^2}{1-\delta_p^2} = \frac{1}{p}\log[\frac{1}{\pi}\mathcal{E}_p(f_p)] - \frac{1}{p}\log 2 \to \mathcal{E}_\infty(f_\infty).$$
That is,
$$\lim_{p\to\infty}\delta_p = \sqrt{\frac{\mathcal{E}_\infty(f_\infty)-1}{\mathcal{E}_\infty(f_\infty)+1}} = k.$$
So for every p,
$$\int_{\mathbb{D}}|\Psi_p(w)| \geq \frac{\delta_p}{1+\delta_p^2}\cdot\frac{1}{2}\mathcal{E}_p(f_p) \geq \frac{\delta_p}{4}\mathcal{E}_p(f_p),$$
and then
$$\frac{1}{p}\log[\frac{1}{\pi}\int_{\mathbb{D}}|\Psi_p(w)|] \geq \frac{1}{p}\log[\frac{\delta_p}{4\pi}\mathcal{E}_p(f_p)].$$
Let $p\to\infty$ then (5.2.3) follows. $\Box$

Lemma 5.2.2
$$\lim_{p\to\infty}\int_{\mathbb{D}}\Big||(h_p)_{\overline{w}}| - k|(h_p)_w|\Big| \to 0. \tag{5.2.4}$$

Proof. Choose any $\varepsilon > 0$ which is so small that both $k(1+\varepsilon)$ and $k(1-\varepsilon)$ are in $(0,1)$, and set
$$E_p := \{w\in\mathbb{D} : |\mu_{h_p}(w)| > k(1+\varepsilon)\},$$
$$F_p := \{w\in\mathbb{D} : |\mu_{h_p}(w)| < k(1-\varepsilon)\}.$$
First,
$$\begin{aligned}\pi\exp[pC_p(s)] &= \int_{\mathbb{D}}\exp(ps\mathbb{K}_{h_p})|(h_p)_w(h_p)_{\overline{w}}| \\ &\geq \int_{E_p}\exp(ps\mathbb{K}_{h_p})|(h_p)_w(h_p)_{\overline{w}}| \\ &\geq \exp[ps\frac{1+k^2(1+\varepsilon)^2}{1-k^2(1+\varepsilon)^2}]k(1+\varepsilon)\int_{E_p}|(h_p)_w|^2.\end{aligned}$$

Rearrange this to obtain
$$\frac{1}{\pi}\int_{E_p}|(h_p)_w|^2 \leq \frac{1}{k(1+\varepsilon)}\Big(\exp[C_p(s) - s\cdot\frac{1+k^2(1+\varepsilon)^2}{1-k^2(1+\varepsilon)^2}]\Big)^p. \tag{5.2.5}$$
Here we can choose $s<1$ so near to 1 that
$$s\cdot\frac{1+k^2(1+\varepsilon)^2}{1-k^2(1+\varepsilon)^2} > \mathcal{E}_\infty(f_\infty).$$

By (5.2.2), we know that for p sufficiently large,

$$C_p(s) - s\frac{1+k^2(1+\varepsilon)^2}{1-k^2(1+\varepsilon)^2} < 0.$$

Then,

$$\Big(\exp[C_p(s) - s\frac{1+k^2(1+\varepsilon)^2}{1-k^2(1+\varepsilon)^{2s}}]\Big)^p \to 0.$$

So we conclude that

$$\lim_{p\to\infty}\int_{E_p} |(h_p)_w|^2 = 0. \tag{5.2.6}$$

In E_p the condition $|\mu_{h_p}(w)| \geq k(1+\varepsilon)$ gives

$$|(h_p)_{\overline{w}}| - k|(h_p)_w| \geq k(1+\varepsilon)|(h_p)_w| - k|(h_p)_w| = k\varepsilon|(h_p)_w| \geq 0.$$

On the other hand, since $|(h_p)_{\overline{w}}| \leq |(h_p)_w|$,

$$0 \leq |(h_p)_{\overline{w}}| - k|(h_p)_w| \leq (1-k)|(h_p)_w|.$$

Then

$$(|(h_p)_{\overline{w}}| - k|(h_p)_w|)^2 \leq (1-k)^2|(h_p)_w|^2.$$

So by (5.2.6),

$$\lim_{p\to\infty}\int_{E_p}(|(h_p)_{\overline{w}}| - k|(h_p)_w|)^2 = 0.$$

Then the Cauchy-Schwarz inequality gives

$$\lim_{p\to\infty}\int_{E_p}\Big||(h_p)_{\overline{w}}| - k|(h_p)_w|\Big| = 0. \tag{5.2.7}$$

We now turn to F_p. Set

$$\Xi_p = \frac{\Psi_p}{|\Psi_p|_{L^1(\mathbb{D})}} = \frac{\Psi_p}{\pi\exp[pC_p(1)]}. \tag{5.2.8}$$

Then $\{\Xi_p, p \geq 1\}$ form a normal family and then up to a subsequence there is a holomorphic limit

$$\Psi = \lim_{p\to\infty}\Xi_p. \tag{5.2.9}$$

We compute

$$\begin{aligned}\int_{F_p}|\Xi_p| &= \frac{1}{\pi\exp[pC_p(1)]}\int_{F_p}|\exp(p\mathbb{K}_{h_p})(h_p)_w(h_p)_{\overline{w}}| \\ &\leq \frac{1}{\pi\exp[pC_p(1)]}\exp[p\frac{1+k^2(1-\varepsilon)^2}{1-k^2(1-\varepsilon)^2}]k(1-\varepsilon)\int_{F_p}|(h_p)_w|^2.\end{aligned}$$

Using a similar argument to the E_p case we have

$$\frac{1}{\exp[pC_p(1)]}\exp[p\frac{1+k^2(1-\varepsilon)^2}{1-k^2(1-\varepsilon)^2}] \to 0.$$

Also recall $\|Dh_p\|_{L^2(\mathbb{D})}$ are uniformly bounded as in (5.1.2). Thus

$$\lim_{p\to\infty}\int_{F_p}|\Xi_p| = 0. \tag{5.2.10}$$

Equation (5.2.10) leads us to two cases: either $|F_p| \to 0$ or the limit function $\Psi = 0$ throughout $\mathbb{D}$, as it is holomorphic. But the latter happens only when

$$h_{\overline{w}} = \lim_{p\to\infty}(h_p)_{\overline{w}} = 0,$$

which returns to the case $\mathcal{E}_\infty(f_\infty) = 1$. So for our case we have

$$\lim_{p\to\infty}\int_{F_p}\Big||(h_p)_{\overline{w}}| - k|(h_p)_w|\Big| = 0. \tag{5.2.11}$$

Finally, in $\mathbb{D} - E_p - F_p$,

$$k(1-\varepsilon) \le |\mu_{h_p}| \le k(1+\varepsilon).$$

Then

$$\begin{aligned}
&\lim_{p\to\infty}\int_{\mathbb{D}-E_p-F_p}\Big||(h_p)_{\overline{w}}| - k|(h_p)_w|\Big| \\
=&\lim_{p\to\infty}\int_{\mathbb{D}-E_p-F_p}|(h_p)_w|\Big||\mu_{h_p}| - k\Big| \\
\le& k\varepsilon\lim_{p\to\infty}\int_{\mathbb{D}-E_p-F_p}|(h_p)_w| \\
\le& \pi^{\frac{1}{2}}k\varepsilon\|Dh_p\|^{\frac{1}{2}}_{L^2(\mathbb{D})}.
\end{aligned}$$

Again we have that $\|Dh_p\|_{L^2(\mathbb{D})}$ are uniformly bounded. Then (5.2.4) follows as ε can be arbitrarily small. $\square$

Lemma 5.2.3 *Let Ψ be the holomorphic limit of Ξ_p as defined in (5.2.8), (5.2.9). Then,*

$$\mu_h = k\frac{\overline{\Psi}}{|\Psi|}. \tag{5.2.12}$$

Proof. By (5.2.4),

$$\begin{aligned}\int_{\mathbb{D}} \Big|(h_p)_{\overline{w}}\frac{\Xi_p}{|\Xi_p|} - k(h_p)_w\Big| &= \int_{\mathbb{D}} \Big|(h_p)_{\overline{w}}\frac{\Psi_p}{|\Psi_p|} - k(h_p)_w\Big| \\ &= \int_{\mathbb{D}} \Big|\frac{(h_p)_w|(h_p)_{\overline{w}}|}{|(h_p)_w|} - k(h_p)_w\Big| \\ &= \int_{\mathbb{D}} \Big||(h_p)_{\overline{w}}| - k|(h_p)_w|\Big| \to 0. \qquad (5.2.13)\end{aligned}$$

We set

$$\mathbb{D}_\varepsilon := \{w \in \mathbb{D} : |\Psi(w)| > \varepsilon\}.$$

Then in each $\mathbb{D}_\varepsilon$ we have

$$\frac{\Xi_p}{|\Xi_p|} \to \frac{\Psi}{|\Psi|}$$

locally uniformly. Since Ψ is holomorphic, we have $|\mathbb{D} - \mathbb{D}_\varepsilon| \to 0$ as $\varepsilon \to 0$. Then, as $\|Dh_p\|_{L^2(\mathbb{D})}$ are uniformly bounded, it follows that for any $\varepsilon > 0$ and compact $A \subset \mathbb{D}$,

$$\begin{aligned}&\lim_{p\to\infty} \int_{\mathbb{D}_\varepsilon \cap A} \Big|(h_p)_{\overline{w}}\frac{\Psi}{|\Psi|} - k(h_p)_w\Big| \\ &\leq \lim_{p\to\infty} \int_{\mathbb{D}_\varepsilon \cap A} \Big|(h_p)_{\overline{w}}\big(\frac{\Psi}{|\Psi|} - \frac{\Xi_p}{|\Xi_p|}\big)\Big| + \int_{\mathbb{D}_\varepsilon \cap A} \Big|(h_p)_{\overline{w}}\frac{\Xi_p}{|\Xi_p|} - k(h_p)_w\Big| \to 0.\end{aligned}$$

By Lemma 5.1.3 we know

$$(h_p)_{\overline{w}}\frac{\Psi}{|\Psi|} - k(h_p)_w \rightharpoonup h_{\overline{w}}\frac{\Psi}{|\Psi|} - kh_w$$

in $L^2(\mathbb{D})$. So we conclude

$$h_{\overline{w}}\frac{\Psi}{|\Psi|} - kh_w = 0.$$

As h is quasiconformal we have $|h_w| > 0$ a.e. Thus

$$\mu_h = k\frac{|\Psi|}{\Psi} = k\frac{\overline{\Psi}}{|\Psi|}$$

almost everywhere in $\mathbb{D}_\varepsilon \cap A$. By the arbitrariness of ε and A this holds almost everywhere in $\mathbb{D}$. □

5.3 When $p \to 0$.

We consider the case $p \to 0$ for the $\mathcal{E}_p$ problems, and it will turn out that the limit is the L^1 minimising problem connecting harmonic mappings and quasiconformal mappings.

Theorem 5.3.1 *Let f be a finite distortion function such that for some $p_0 > 0$,*

$$\int_{\mathbb{D}} \exp(p_0 \mathbb{K}_f) < \infty.$$

Then

$$\lim_{p\to 0} \frac{1}{p} \log[\frac{1}{\pi} \int_{\mathbb{D}} \exp(p\mathbb{K}_f)] = \frac{1}{\pi} \int_{\mathbb{D}} \mathbb{K}_f.$$

Proof. Assume $p \leq p_0$. By Jensen's inequality we have

$$\frac{1}{\pi} \int_{\mathbb{D}} \exp(p\mathbb{K}_f) \geq \exp(\frac{p}{\pi} \int_{\mathbb{D}} \mathbb{K}_f), \quad \forall p > 0.$$

This proves one direction. For the other direction, we note

$$\exp(p\mathbb{K}_f) = \sum_{j=0}^{\infty} \frac{p^j \mathbb{K}_f^j}{j!}.$$

By the monotone convergence theorem,

$$\begin{aligned}
\frac{1}{\pi} \int_{\mathbb{D}} \exp(p\mathbb{K}_f) &= \frac{1}{\pi} \sum_{j=0}^{\infty} \frac{p^j}{j!} \int_{\mathbb{D}} \mathbb{K}_f^j \\
&= 1 + \frac{p}{\pi} \int_{\mathbb{D}} \mathbb{K}_f + \frac{p^2}{\pi} \Big(\sum_{j=2}^{\infty} \frac{p^{j-2}}{j!} \int_{\mathbb{D}} \mathbb{K}_f^j \Big) \\
&\leq 1 + \frac{p}{\pi} \int_{\mathbb{D}} \mathbb{K}_f + \frac{p^2}{\pi} \Big(\sum_{j=2}^{\infty} \frac{p_0^{j-2}}{j!} \int_{\mathbb{D}} \mathbb{K}_f^j \Big) \\
&\leq 1 + \frac{p}{\pi} \int_{\mathbb{D}} \mathbb{K}_f + \frac{p^2}{\pi p_0^2} \Big(\int_{\mathbb{D}} \exp(p_0 \mathbb{K}_f) \Big).
\end{aligned}$$

We write

$$M = \frac{1}{\pi p_0^2} \Big(\int_{\mathbb{D}} \exp(p_0 \mathbb{K}_f) \Big).$$

For every $\varepsilon > 0$, we can choose p so small that $pM < \varepsilon$. That is,

$$\frac{1}{\pi} \int_{\mathbb{D}} \exp(p\mathbb{K}_f) \leq 1 + p\Big(\frac{1}{\pi} \int_{\mathbb{D}} \mathbb{K}_f + \varepsilon \Big).$$

It follows that

$$\begin{aligned}
\log[\frac{1}{\pi} \int_{\mathbb{D}} \exp(p\mathbb{K}_f)] &\leq \log \Big[1 + p\Big(\frac{1}{\pi} \int_{\mathbb{D}} \mathbb{K}_f + \varepsilon \Big) \Big] \\
&\leq p\Big(\frac{1}{\pi} \int_{\mathbb{D}} \mathbb{K}_f + \varepsilon \Big),
\end{aligned}$$

and then

$$\frac{1}{p} \log[\frac{1}{\pi} \int_{\mathbb{D}} \exp(p\mathbb{K}_f)] \leq \frac{1}{\pi} \int_{\mathbb{D}} \mathbb{K}_f + \varepsilon.$$

This proves the other direction. $\square$

6 Exponential Minimisers between Annuli and Examples between Disks

In Theorem 2.1.1 we have seen that a linear mapping is a unique minimiser for its own boundary values. The main target of this chapter is to find more extremal mappings from $\mathbb{D}$ to $\mathbb{D}$. However, we will start with the annuli.

6.1 An equation for exponential minimisers between annuli.

It is proved in [41] that the minimisers among homeomorphisms of finite distortion between two annuli (with no restriction on the boundary values) are radial stretchings. In particular, for the exponential problem, the minimiser $f(\rho e^{i\theta}) = F(\rho)e^{i\theta}$ satisfies

$$\rho^2(1 - \frac{F^2(\rho)}{\rho^2 F_\rho^2(\rho)}) \exp[\frac{p}{2}(\frac{\rho F_\rho(\rho)}{F(\rho)} + \frac{F(\rho)}{\rho F_\rho(\rho)})] = \alpha, \tag{6.1.1}$$

where α is a constant. This is [43, Theorem 3.4] but we put our distortion function $\exp(p\mathbb{K}_f)$ into it.

We will find a parametric formula for $F(\rho)$. Write

$$a = \frac{\rho F_\rho(\rho)}{F(\rho)}. \tag{6.1.2}$$

Since f is sense-preserving we have $a \geq 0$ a.e. in its domain. Then (6.1.1) reads as

$$\rho = C_1 \frac{a}{\sqrt{|a^2-1|}} \exp[-\frac{p}{4}(a + \frac{1}{a})], \tag{6.1.3}$$

where C_1 is a positive constant. By (6.1.2) we can also compute F. Note

$$(\log F)_\rho = \frac{F_\rho}{F} = \frac{a}{\rho}.$$

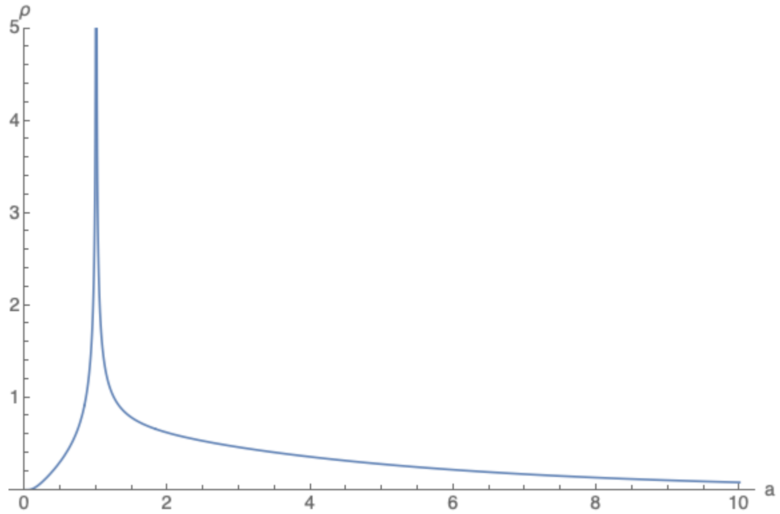
Using this and (6.1.3), we obtain

$$\begin{aligned}
\log F &= \int \frac{a(\rho)}{\rho} d\rho = \int \frac{a}{\rho(a)} \rho_a(a) da \\
&= \int \frac{(1 + a^4 - 2a^2)p + 4a}{4a - 4a^3} da \\
&= \log\Big(C_2 a^{\frac{p}{4}} \frac{\sqrt{a+1}}{\sqrt{|a-1|}} \exp(-\frac{a^2 p}{8})\Big).
\end{aligned}$$

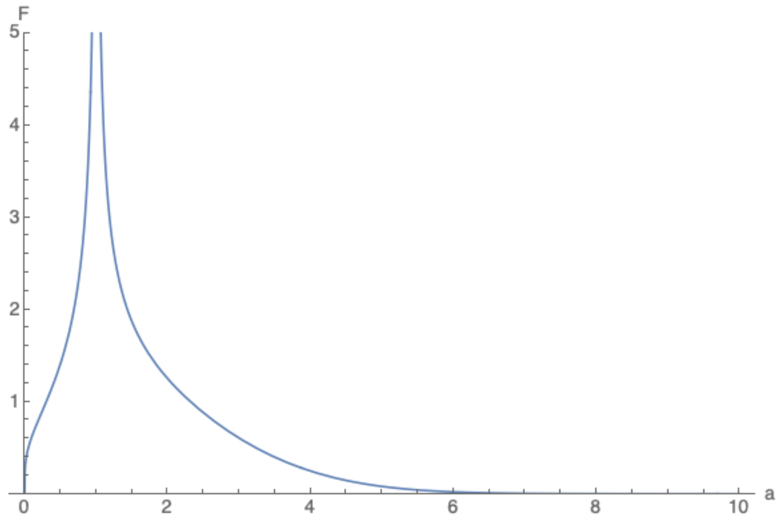
So we obtain the relation of F and a:

$$F = C_2 a^{\frac{p}{4}} \frac{\sqrt{a+1}}{\sqrt{|a-1|}} \exp(-\frac{a^2 p}{8}), \tag{6.1.4}$$

where C_2 is also a positive constant. Now (6.1.3) and (6.1.4) together give the function of $F(\rho)$. This is well-defined. Examples are shown in the graphs.



The graph of $\rho(a)$, with $p = 1$, $C_1 = 1$.



The graph of $F(a)$, with $p = 1$, $C_2 = 1$.

So either with $0 < a < 1$ or $1 < a < \infty$, by adjusting C_1, C_2 we can get a family of functions $F(\rho)$ that satisfy (6.1.1). We claim these functions, together with the conformal mappings, are all of the minimisers between annuli.

Theorem 6.1.1 *Consider the p-exponential Nitsche-type problem for mappings between annuli $A(r,R)$ and $A(r',R')$, where $p>0$, and*

$$0<r<R<\infty, \quad 0<r'<R'<\infty.$$

i) If $\frac{R}{r}=\frac{R'}{r'}$, then there is a unique minimiser

$$z \to Cz, \quad C=\frac{R'}{r'}\Big/\frac{R}{r}.$$

ii) If $\frac{R}{r}>\frac{R'}{r'}$, then there is a unique minimiser $\rho e^{i\theta} \to F(\rho)e^{i\theta}$, where

$$\rho = C_{1,1}\frac{a}{\sqrt{1-a^2}}\exp[-\frac{p}{4}(a+\frac{1}{a})],$$

$$F=C_{1,2}\frac{\sqrt{1+a}}{\sqrt{1-a}}a^{\frac{p}{4}}\exp(-\frac{a^2p}{8}),$$

where $0<a<1$, and $C_{1,1}$, $C_{1,2}$ are adjusted so that F maps $(r,R)\to(r',R')$.

iii) If $\frac{R}{r}<\frac{R'}{r'}$, then there is a unique minimiser $\rho e^{i\theta} \to F(\rho)e^{i\theta}$, where

$$\rho = C_{2,1}\frac{a}{\sqrt{a^2-1}}\exp[-\frac{p}{4}(a+\frac{1}{a})],$$

$$F=C_{2,2}\frac{\sqrt{a+1}}{\sqrt{a-1}}a^{\frac{p}{4}}\exp(-\frac{a^2p}{8}),$$

where $1<a<\infty$, and $C_{2,1}$, $C_{2,2}$ are adjusted so that F maps $(r,R)\to(r',R')$.

Proof. It is proved in [41] that for each pair of annuli there is a unique minimiser that satisfies (6.1.1), so it must be with the form in either one of the cases described in the theorem. So we only need to prove the claims with respect to $\frac{R}{r}$ and $\frac{R'}{r'}$. The conformal case is obvious. For the case $a=\frac{\rho F_\rho}{F}<1$, we have

$$(\log F)_\rho = \frac{F_\rho}{F} < \frac{1}{\rho}.$$

Then

$$\log\frac{F(R)}{F(r)} = \int_r^R (\log F)_\rho(\rho)d\rho < \int_r^R \frac{1}{\rho}d\rho = \log\frac{R}{r}.$$

Thus

$$\frac{R'}{r'} = \frac{F(R)}{F(r)} < \frac{R}{r}.$$

And similarly we can prove the case $a>1$. □

6.2 Extend to the origin.

The minimisers obtained in the last section can also be extended to the origin as a homeomorphism, which we will see below. Then, they become homeomorphisms between disks. In particular, as minimisers between annuli, they satisfy the distributional equations, but away from the origin. Precisely, every function in Theorem 6.1.1 satisfies the following:

$$\int_{\mathbb{D}} \exp(\mathbb{K}_f)\varphi_z = 2p \int_{\mathbb{D}} \frac{\overline{\mu_f}}{1-|\mu_f|^2} \exp(\mathbb{K}_f)\varphi_{\overline{z}}, \quad \forall \varphi \in C_0^\infty(\mathbb{D}^*). \tag{6.2.1}$$

where $\mathbb{D}^*$ is the punctured disk $\mathbb{D} \setminus \{0\}$. Unfortunately, there is no way to extend this to those $\varphi \in C_0^\infty(\mathbb{D})$ such that $0 \in \mathrm{Supp}(\varphi)$, as we will see. Nevertheless, we still have the pointwise property

$$\Big[\exp(p\mathbb{K}_f)\Big]_z = \Big[\frac{2p\overline{\mu_f}}{1-|\mu_f|^2} \exp(p\mathbb{K}_f)\Big]_{\overline{z}}, \quad a.e. \quad z \in \mathbb{D}. \tag{6.2.2}$$

This can also be checked by putting the functions into the equation.

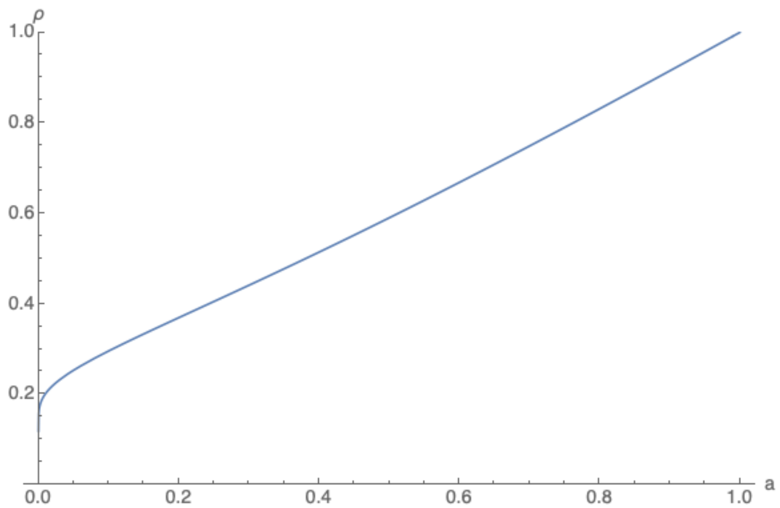
6.2.1 $0 < a < 1$ case.

We first consider the $0 < a < 1$ case.

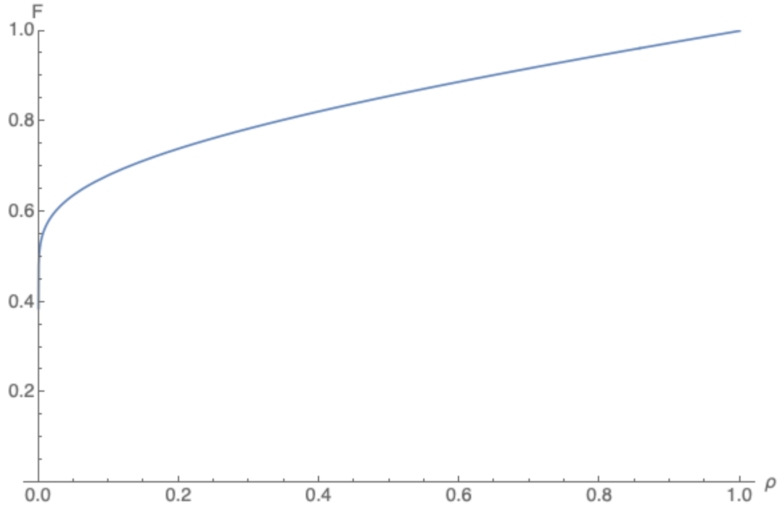
$$\rho = C_1 \frac{a}{\sqrt{1-a^2}} \exp[-\frac{p}{4}(a + \frac{1}{a})], \tag{6.2.3}$$

$$F = C_2 \frac{\sqrt{1+a}}{\sqrt{1-a}} a^{\frac{p}{4}} \exp(-\frac{a^2 p}{8}). \tag{6.2.4}$$

Note that $\rho = 0$ if and only if $a = 0$, if and only if $F = 0$. See the graphs below.



The graph of the function $F(\rho)$, with $p = 1$, $C_1 = 1$ and $C_2 = 0.316876....$



The graph of the function $F(\rho)$, with $p = 1$, $C_1 = 10$ and $C_2 = 1.07214....$

From the graphs we can also see that if C_1 is small, the distortion is small, and the function is close to the identity map $f(z) = z$. In fact we can let C_1 move in $(0, \infty)$, and adjust C_2 to get $F(1) = 1$. Then we get a family of functions $f(\rho e^{i\theta}) = F(\rho)e^{i\theta} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, where f is a self-homeomorphism of $\overline{\mathbb{D}}$, and $f(z) = z$ along $\partial\mathbb{D}$. For a radial stretching we have the distortion

$$\mathbb{K}_f = \frac{1}{2}(\frac{\rho F_\rho}{F} + \frac{F}{\rho F_\rho}) = \frac{1}{2}(a + \frac{1}{a}).$$

This implies that $\mathbb{K}_f(0) = \infty$ and is finite anywhere else. So away from the origin f is quasiconformal, but $\mathbb{K}_f(z)$ blows up when z approaches 0. Note $\|Df(z)\|$ also blows up at 0:

$$\lim_{\rho\to 0} F_\rho(\rho) = \lim_{a\to 0} \frac{F_a(a)}{\rho_a(a)} = \lim_{a\to 0} \frac{C_2}{C_1} a^{\frac{p}{4}}(1 + a)e^{\frac{(-a^3+2a^2+2)p}{8a}} = +\infty.$$

So f is not a diffeomorphism over $\mathbb{D}$.

We next compute the $\exp(p)$ distortion energy of these functions. Write

$$\rho_1(a) := \frac{a}{\sqrt{1 - a^2}} \exp[-\frac{p}{4}(a + \frac{1}{a})].$$

Then for

$$\rho(a) = C_1 \frac{a}{\sqrt{1 - a^2}} \exp[-\frac{p}{4}(a + \frac{1}{a})] = C_1\rho_1(a),$$

the mean $\exp(p)$-distortion over $\mathbb{D}$ is

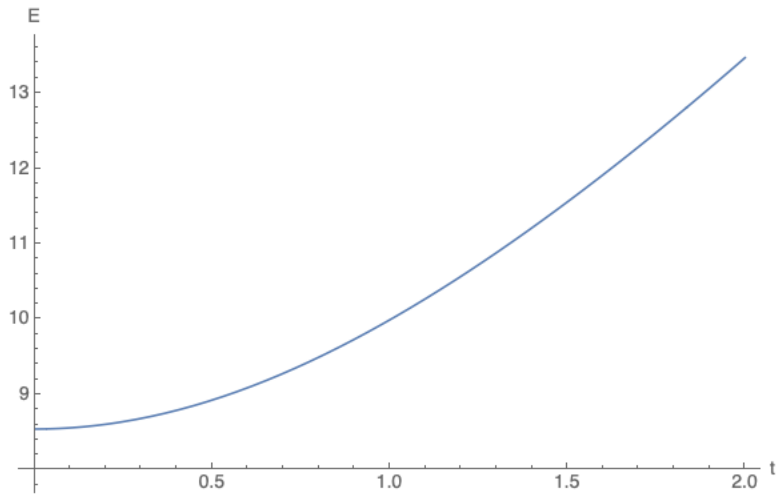
$$\begin{aligned}\int_{\mathbb{D}} \exp[p\mathbb{K}_f(z)]dz &= 2\pi \int_0^1 \exp[p\mathbb{K}_f(\rho)]\rho d\rho \\ &= 2\pi \int_0^{\rho_1^{-1}(1/C_1)} \exp[p\mathbb{K}_f(a)]C_1\rho_1(a)C_1(\rho_1)_a(a)da \\ &= 2\pi C_1^2 \int_0^{\rho_1^{-1}(1/C_1)} \frac{(a^4-2a^2+1)p+4a}{4(a^2-1)^2}da < \infty.\end{aligned}$$

However, $\mathbb{K}_f \exp(p\mathbb{K}_f)$ is not integrable over $\mathbb{D}$:

$$\begin{aligned}\int_{\mathbb{D}} \mathbb{K}_f(z)\exp[p\mathbb{K}_f(z)]dz &= 2\pi \int_0^1 \mathbb{K}_f(\rho)\exp[p\mathbb{K}_f(\rho)]\rho d\rho \\ &= 2\pi \int_0^{\rho_1^{-1}(1/C_1)} \mathbb{K}_f(\rho)\exp[p\mathbb{K}_f(a)]C_1\rho_1(a)C_1(\rho_1)_a(a)da \\ &= \pi C_1^2 \int_0^{\rho_1^{-1}(1/C_1)} (a+\frac{1}{a})\frac{(a^4-2a^2+1)p+4a}{4(a^2-1)^2}da \\ &\geq \pi C_1^2 \int_0^{\rho_1^{-1}(1/C_1)} \frac{p}{4a}da = \infty.\end{aligned}$$

This also explains why (6.2.1) with $0 \in \mathrm{Supp}(\varphi)$ cannot be satisfied.

We now consider what happens if C_1 moves in $(0,\infty)$. Call $E_p(t)$ the energy of the function with $C_1 = t$. The graph is as follows:



The graph of $E_p(t)$ with $p = 1$.

We observe that the limit as $t \to 0$ is πe- the minimal energy which is given by the identity $f(z) = z$. In fact this can be proved by a limit computation:

Lemma 6.2.1 *With the notation above we have*

$$\lim_{t\to 0} E_p(t) = \pi e^p, \quad \lim_{t\to\infty} E_p(t) = \infty.$$

Proof. Recall

$$E_p(t) = 2\pi t^2 \int_0^{\rho_1^{-1}(1/t)} \exp[p\mathbb{K}_f(a)]\rho_1(a)(\rho_1)_a(a)da.$$

We consider

$$\begin{aligned}
&\frac{d}{dt}\int_0^{\rho_1^{-1}(1/t)} \exp[p\mathbb{K}_f(a)]\rho_1(a)(\rho_1)_a(a)da\\
=&\exp[p\mathbb{K}_f(\rho_1^{-1}(1/t))]\frac{1}{t}(\rho_1)_a(\rho_1^{-1}(1/t))\Big[\frac{d}{dt}\Big(\rho_1^{-1}(1/t)\Big)\Big]\\
=&\frac{1}{t}\exp[p\mathbb{K}_f(\rho_1^{-1}(1/t))](\rho_1)_a(\rho_1^{-1}(1/t))\frac{1}{(\rho_1)_a(\rho_1^{-1}(1/t))}\Big[\frac{d}{dt}\Big(\frac{1}{t}\Big)\Big]\\
=&-\frac{1}{t^3}\exp[p\mathbb{K}_f(\rho_1^{-1}(1/t))].
\end{aligned}$$

So

$$\begin{aligned}
\lim_{t\to 0} E_p(t) &= 2\pi \lim_{t\to 0}\frac{\int_0^{\rho_1^{-1}(1/t)} \exp[p\mathbb{K}_f(a)]\rho_1(a)(\rho_1)_a(a)da}{\frac{1}{t^2}}\\
&= 2\pi \lim_{t\to 0}\frac{\frac{d}{dt}\int_0^{\rho_1^{-1}(1/t)} \exp[p\mathbb{K}_f(a)]\rho_1(a)(\rho_1)_a(a)da}{-2\frac{1}{t^3}}\\
&= \pi \lim_{t\to 0} \exp[p\mathbb{K}_f(\rho_1^{-1}(1/t))]\\
&= \pi \lim_{a\to 1} \exp[\frac{p}{2}(a+\frac{1}{a})]\\
&= \pi e^p.
\end{aligned}$$

A similar computation works for the case $t\to\infty$ and we get

$$\begin{aligned}
\lim_{t\to\infty} E_p(t) &= \pi \lim_{t\to\infty} \exp[p\mathbb{K}_f(\rho_1^{-1}(1/t))]\\
&= \pi \lim_{a\to 0} \exp[\frac{p}{2}(a+\frac{1}{a})] = \infty.
\end{aligned}$$

□

The fact $\lim_{t\to 0} E_p(t) = \pi e^p$ implies that those functions with $C_1 \to 0$ form a minimising sequence of the $p-$exponential distortion problem. Then, by Theorem 1.4.9, it contains a subsequence that converges uniformly to the minimiser-which is the identity map.

Collecting all the above facts we have proved the following theorem.

Theorem 6.2.2 *There is a sequence of homeomorphisms f_j that are radial stretchings defined by (6.2.3) and (6.2.4), with $C_1 = \frac{1}{j}$, C_2 adjusted to $F(1) = 1$, and it satisfies the following properties:*

i) It is a minimising sequence of the $p-$exponential distortion problem $\mathbb{D} \to \mathbb{D}$ with boundary values $f(z) = z$ along $\partial\mathbb{D}$. In particular, f_j converges to the identity map uniformly in $\overline{\mathbb{D}}$.

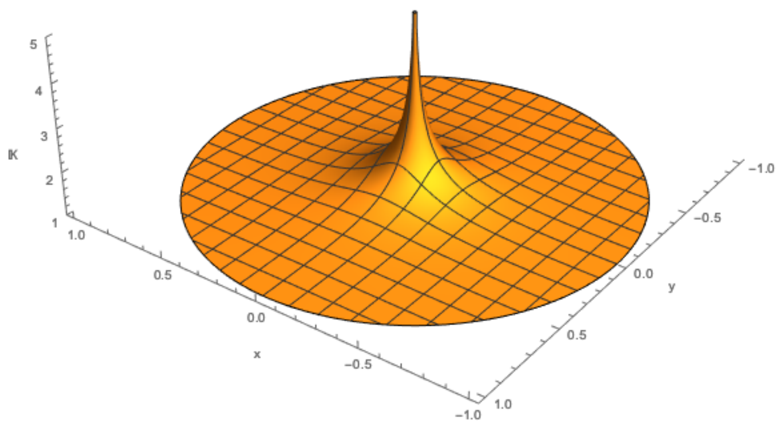
ii) Each f_j has

$$\int_{\mathbb{D}} \mathbb{K}_{f_j} \exp(p\mathbb{K}_{f_j}) = \infty.$$

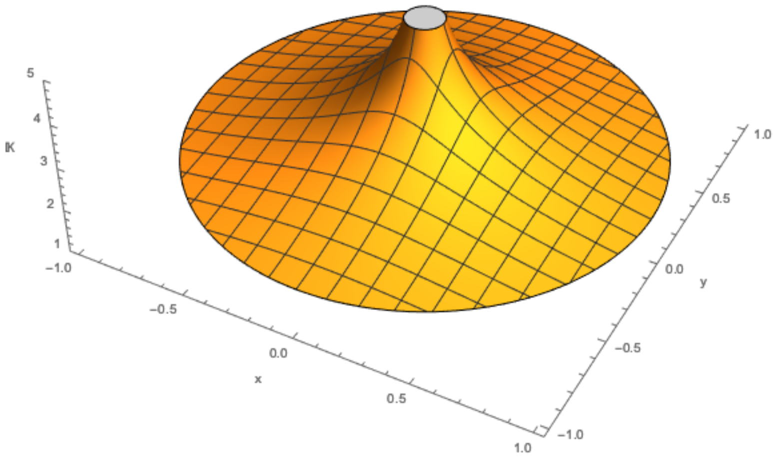
In particular, they are not inner variational.

iii) Each f_j satisfies the pointwise equation

$$\Big[\exp(p\mathbb{K}_{f_j})\Big]_z = \Big[\frac{2p\overline{\mu_{f_j}}}{1-|\mu_{f_j}|^2}\exp(p\mathbb{K}_{f_j})\Big]_{\overline{z}}, \quad a.e. \quad z \in \mathbb{D}.$$



The graph of $\mathbb{K}(z, f)$ in $\mathbb{D}$, with $p = 1$, $C_1 = 1$.



The graph of $\mathbb{K}(z, f)$ in $\mathbb{D}$, with $p = 1$, $C_1 = 10$.

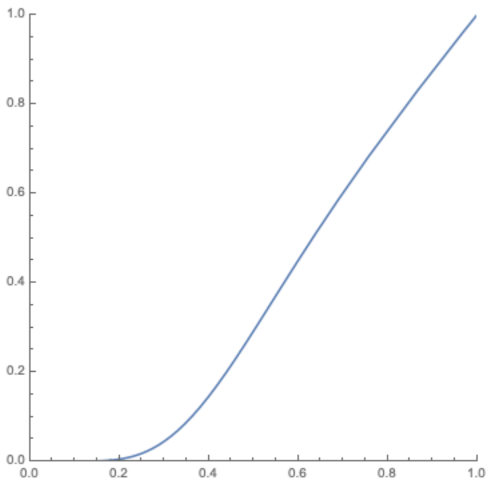
6.2.2 $1 < a < \infty$ case.

In this case we have

$$\rho = C_1 \frac{a}{\sqrt{a^2 - 1}} \exp[-\frac{p}{4}(a + \frac{1}{a})],$$

$$F = C_2 \frac{\sqrt{1 + a}}{\sqrt{a - 1}} a^{\frac{p}{4}} \exp(-\frac{a^2 p}{8}).$$

Then $\rho = 0$ if and only if $a = \infty$, if and only if $F = 0$. So this can also be extended to 0 as a homeomorphism.



The graph of the function $F(\rho)$, with $p = 1$, $C_1 = 1$ and $C_2 = 0.382442...$.

This time $F_\rho(0) = 0$, as the graph shows. In fact,

$$\lim_{\rho\to 0} F_\rho(\rho) = \lim_{a\to\infty} \frac{F_a(a)}{\rho_a(a)} = \lim_{a\to\infty} \frac{C_2}{C_1} a^{\frac{p}{4}}(1+a)e^{\frac{(-a^3+2a^2+2)p}{8a}} = 0.$$

But the energy is not finite:

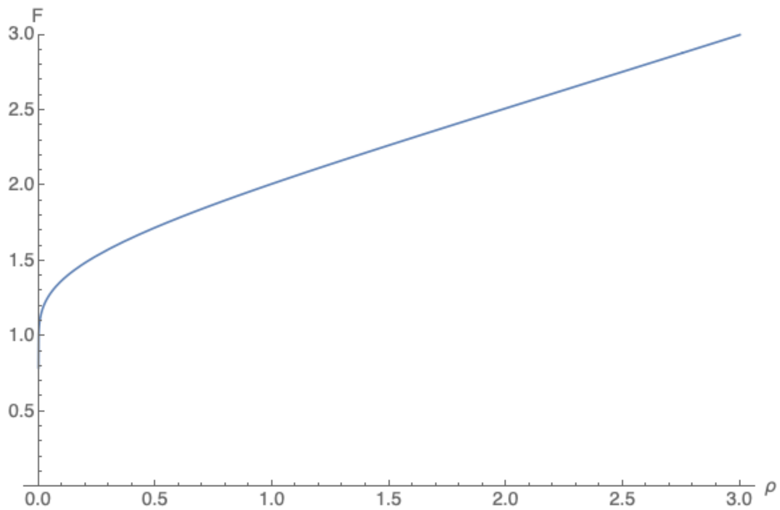
$$\begin{aligned}\int_0^1 \exp[p\mathbb{K}_f(\rho)]\rho d\rho &= \int_{+\infty}^{\rho^{-1}(1)} \exp[p\mathbb{K}_f(a)]\rho(a)\rho_a(a)da \\ &= \int_{\rho^{-1}(1)}^{\infty} \frac{(a^4-2a^2+1)p+4a}{4(a^2-1)^2} da = \infty.\end{aligned}$$

6.3 Examples between disks.

We now exploit the functions obtained above to find some extremal mappings between disks. Consider the function with $p = 1$, $0 < a < 1$, $C_1 = 10$, and C_2 adjusted to $F(3) = 3$. In fact we need $C_2 = 2.156632....$ So the function is

$$\rho(a) = 10\frac{a}{\sqrt{1-a^2}} \exp[-\frac{1}{4}(a+\frac{1}{a})],$$

$$F(a) = (2.156632...)a^{\frac{1}{4}} \frac{\sqrt{1+a}}{\sqrt{1-a}} \exp(-\frac{a^2}{8}).$$



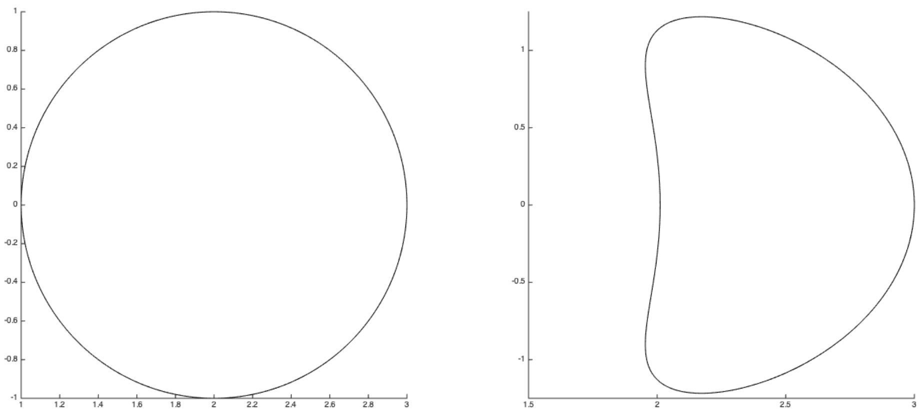
The graph of $F(\rho)$.

6.3.1 Away from the origin.

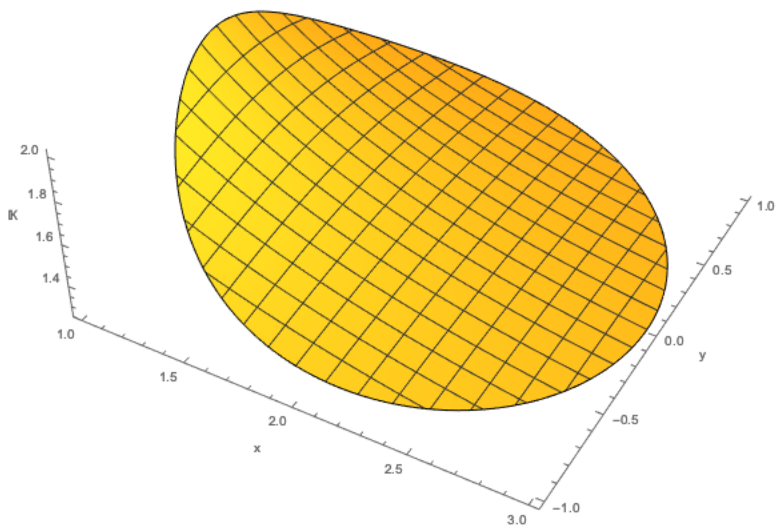
Away from the origin we get quasiconformal minimisers for their own boundary values. We consider the disk

$$D(2,1) := \{z : |z-2| < 1\}.$$

Then f maps $D(2,1)$ to some simply connected subdomain of $\mathbb{D}^*$. Next, by a conformal mapping Φ, the image $f(D(2,1))$ can be mapped back to $\mathbb{D}$, and then with a translation, $\tilde{f} = \Phi \circ f|_{D(2,1)}$ works as a quasiconformal minimiser from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$, for its own boundary values.



The graph of $D(2,1)$ under f.

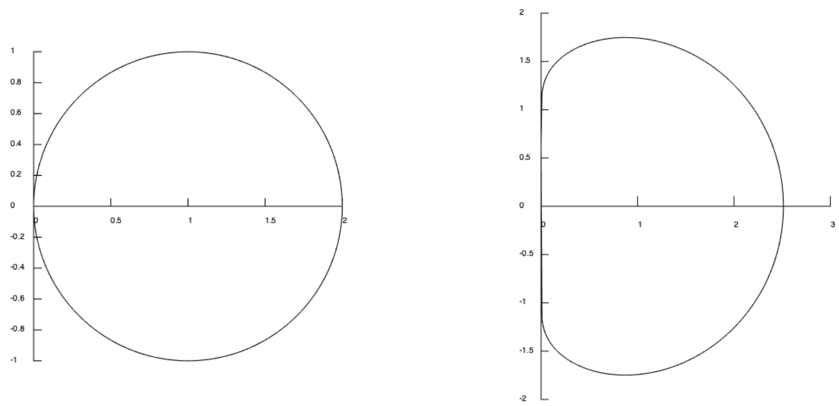


The graph of $\mathbb{K}(z, f)$ in $D(2,1)$.

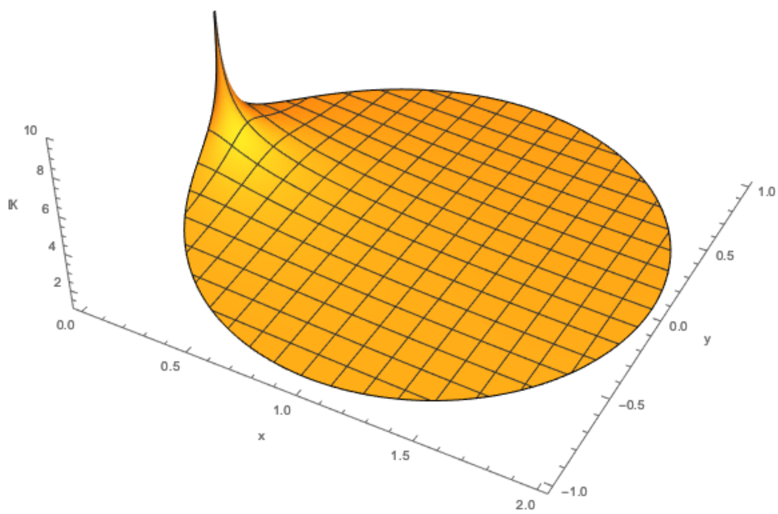
6.3.2 Disks passing the origin.

Let the function f be as above. We consider a disk that passes through 0. Consider

$$D(1,1) := \{z : |z-1| < 1\}.$$



The graph of $D(1,1)$ under f.



The graph of $\mathbb{K}(z, f)$ in $D(1,1)$.

We can see that the distortion is small when away from 0 but increases dramatically when approaching it. Again by a conformal mapping Φ we can map $f(D(1,1))$ back to $\mathbb{D}$. Note here although $\|Df\|$ blows up at a point on the boundary, f is still a homeomorphism.

As $\tilde{f} = \Phi \circ f$ satisfies the inner variational equation for every $\varphi \in C_0^\infty(D(1,1))$, it is a critical point. Furthermore, for any subdomain that is away from the origin $\tilde{f}$ is a unique minimiser, so we believe it is a minimiser in $D(1,1)$, for its own boundary values. However, at the moment we can not prove this since we do not know how to handle the origin.

Conjecture 6.3.1 $\tilde{f} = \Phi \circ f|_{D(1,1)}$ is a minimiser for its boundary values.

We finally conclude that with a translation, $\tilde{f}$ works as an example of homeomorphism from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$ such that

i) $\tilde{f}$ is a diffeomorphism and locally quasiconformal in $\mathbb{D}$.

ii) At a point z_0 of the boundary $\partial\mathbb{D}$, $\mathbb{K}_{\tilde{f}}(z_0) = \infty$, and $\|D\tilde{f}(z_0)\| = \infty$.

iii) $\tilde{f}$ satisfies the inner variational equation

$$\int_{\mathbb{D}} \exp(\mathbb{K})\varphi_z = \int_{\mathbb{D}} \frac{2\overline{\mu}}{1 - |\mu|^2} \exp(\mathbb{K})\varphi_{\overline{z}}, \quad \forall \varphi \in C_0^\infty(\mathbb{D}).$$

Furthermore, if Conjecture 6.3.1 holds, then $\tilde{f}$ is a minimiser for its boundary values.

7 Further Research

7.1 The uniqueness of minimisers.

In Section 3.4 we have seen that the inverse minimiser h satisfies the tension equation (3.4.6)

$$h_{w\overline{w}} + (\log \lambda)_z(h) h_w h_{\overline{w}} = 0,$$

with $\lambda(z) = e^{p\mathbb{K}(z,f)}$. In view of [40], this is the equation of a harmonic function between certain Riemannian surfaces. It was studied by Schoen-Yau [51] and then followed by others in more general settings, for example, Li-Tam [36], [37], that under certain circumstances in a negative curvature space the minimiser is unique. However, in our case things are more complicated. First, our space itself depends on the minimiser h. Secondly, we cannot allocate the sign of the Gaussian curvature of our metric. In fact we can compute that the curvature is

$$\mathcal{K} = \frac{2}{\lambda^3}(\lambda_z \lambda_{\overline{z}} - \lambda \lambda_{z\bar{z}}).$$

Put $\lambda(z) = e^{p\mathbb{K}(z,f)}$ into consideration we get

$$\mathcal{K} = \frac{-2p\mathbb{K}_{z\bar{z}}}{\lambda}.$$

So the sign of the Gaussian curvature $\mathcal{K}$ depends on that of $\mathbb{K}_{z\bar{z}}$. In particular, the space $(\mathbb{D}, \lambda)$ has non-positive curvature if $\mathbb{K}$ is Euclidean subharmonic. We can also compute that

$$\begin{aligned}
\mathbb{K}_{z\bar{z}} &= \frac{2}{(1-|\mu|^2)^2}(|\mu|^2)_{z\bar{z}} + \frac{4}{(1-|\mu|^2)^3}(|\mu|^2)_z(|\mu|^2)_{\overline{z}} \\
&= \frac{2}{(1-|\mu|^2)^3}\Big[(1-|\mu|^2)(|\mu|^2)_{z\bar{z}} + 2(|\mu|^2)_z(|\mu|^2)_{\overline{z}}\Big],
\end{aligned}$$

but we cannot get anything simply about its sign in this expression.

We recall that in Ahlfors' paper [1], he gave the uniqueness theorem for the original Teichmüller's problem with respect to quadratic differentials on Riemann surfaces. However, in the more general setting, there are examples of boundary values for the L^∞ extremal quasiconformal problems where the minimisers are not unique [35], [52]. So we are not sure whether the uniqueness holds in the exponential case or not (though we tend to believe it does). We remark this problem is also open in the L^p $(p \geq 2)$ cases [8].

7.2 Variational minimisers.

In Chapter 2 we gave the equations for variational minimisers of the exponential problems. However, it becomes a critical problem that the homeomorphisms with exponentially integrable distortion might not be variational. To solve the problem we gave different conditions to guarantee that the function is variational. However, we still believe that the minimisers must be inner variational. This is hopefully to be solved if we can discover more properties of the truncated exponential problems (as we will introduce below), or with the help of other auxiliary functionals.

7.2.1 Truncated exponential problem.

To get the holomorphic Hopf differential $\Psi = e^{p\mathbb{K}_h} h_w \overline{h_{\overline{w}}}$, one may consider the truncated inverse problems

$$E_m^{-1}(h) := \int_{\mathbb{D}} \sum_{j=1}^{m} \frac{p^j \mathbb{K}^j(w,h)}{j!} J(w,h)dw, \quad h|_{\partial\mathbb{D}} = h_0|_{\partial\mathbb{D}}.$$

This functional converges to $\int_{\mathbb{D}} \exp(p\mathbb{K}_h)J_h$ as $m \to \infty$, and each one is a linear combination of some inverse L^p distortions, so each E_m^{-1} admits a continuous minimiser h_m that has the holomorphic Hopf differential

$$\Psi_m = \sum_{j=1}^{m} \frac{p^j \mathbb{K}_{h_m}^{j-1}}{(j-1)!} (h_m)_w \overline{(h_m)_{\bar{w}}}.$$

This sequence has a uniform $W^{1,2}(\mathbb{D})$ norm, since

$$\int_{\mathbb{D}} \|Dh_m(w)\|^2 dw = \int_{\mathbb{D}} \mathbb{K}(w,h_m)J(w,h_m)dw \le E_m^{-1}(h_m) \le E_m^{-1}(h_0) \le \mathcal{E}_p^{-1}(h_0).$$

So there is a limit function h such that $h_m \rightharpoonup h$ in $W^{1,2}(\mathbb{D})$. In fact the above computation holds not only for h_0 but for all $g \in \mathcal{H}_p$, which gives

$$\limsup_{m\to\infty} E_m^{-1}(h_m) \le \min_{g\in\mathcal{H}_p} \mathcal{E}_p^{-1}(g).$$

On the other hand, for the limit function h we have that for each fixed k,

$$E_k^{-1}(h) \le \liminf_{m\to\infty} E_k^{-1}(h_m) \le \liminf_{m\to\infty} E_m^{-1}(h_m),$$

where the first inequality follows from polyconvexity. Then, by Fatou's Lemma,

$$\begin{aligned}
\mathcal{E}_p^{-1}(h) &= \int_{\mathbb{D}} \exp[p\mathbb{K}(w,h)]J(w,h)dw \\
&= \int_{\mathbb{D}} \lim_{k\to\infty} \sum_{j=0}^{k} \frac{p^j \mathbb{K}^j(w,h)}{j!} J(w,h)dw \\
&\le \liminf_{k\to\infty} \int_{\mathbb{D}} \sum_{j=0}^{k} \frac{p^j \mathbb{K}^j(w,h)}{j!} J(w,h)dw \\
&\le \liminf_{m\to\infty} E_m^{-1}(h_m) \le \min_{g\in\mathcal{H}_p} \mathcal{E}_p^{-1}(g).
\end{aligned}$$

So h is a continuous minimiser of the exponential problem. By the same method we can prove that

$$\|\Psi\|_{L^1(A)} \le \liminf_{m\to\infty} \|\Psi_m\|_{L^1(A)}, \text{ for all } A \subset\subset \mathbb{D}.$$

So $\Psi \in L^1_{loc}(\mathbb{D})$ if only we can show that the holomorphic functions Ψ_m form a normal family. But unfortunately we cannot show this at the moment. Another problem of this method is that although we have the limit function h is a minimiser, the sequence h_m might not be a minimising sequence: as $m \to \infty$, we have $E_m^{-1}(h_m) \to \mathcal{E}_p^{-1}(h)$, but $\mathcal{E}_p^{-1}(h_m)$ might blow up to $+\infty$.

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