CONTRIBUTIONS TO APPLIED PROBABILITY

A thesis presented for the degree of

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Jeffrey Joseph Hunter

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ABSTRACT

This thesis covers a selection of published research papers, manuscripts and book chapters of the contributions that the author has made in the field of applied probability. The overall themes focus on the development of the theory and applications of Markov renewal processes, Markov chains, generalized matrix inverses, queueing models, and two dimensional renewal processes. The presentation highlights some strong interconnections between many of the topics including Markov renewal theory and Markov chains to queueing models; generalized matrix inverses to Markov chains and Markov renewal processes; and correlated bivariate processes as two-dimensional renewal processes and arrival processes to queueing models. Many of the research papers have appeared in the "Advances in Applied Probability" or in "Linear Algebra and its Applications" highlighting the strong interdisciplinary links and contributions that the author has made to the both the fields of Applied Probability and Linear Algebra.

ACKNOWLEDGEMENTS

There are many who have had a considerable influence on my activities in the field of applied probability field. I wish to place a debt of gratitude to my PhD supervisor Professor Walter L Smith, at the University of North Carolina at Chapel Hill. His challenging and inspired supervision exposed me to the fields of queueing theory (with a direction towards the area of parallel models) and semi-Markov and Markov renewal processes. Professor Ross Leadbetter, a member of my PhD supervision committee and a fellow New Zealander, provided me with the grounding in the finer points of measure theoretic probability as well as courses that provided a firm theoretical base on which a variety of stochastic processes are formulated. I value his contribution. Professor Ralph Disney, whom I met at a Queueing Theory Conference at Kalamazoo in 1973, has had a considerable influence on my research career. I spent sabbatical leaves with him at Virginia Polytechnic Institute and State University in 1980 and 1987 and visited him at Texas A & M at College Station in 1994. While we never wrote a joint paper we had numerous discussions regarding the use of Markov renewal processes in queueing models. I wish to express my sincere thanks to these aforementioned colleagues for their mentoring, support and advice over many years. Their friendship is something that I value.

There are others that I wish to acknowledge. Professor George Styan at McGill (1988), Professor Uma Prabhu at the Mathematical Sciences Institute, Cornell University (1988); Professor Shayle Searle, Cornell University (1988); Professor Marcel Neuts at Purdue University (1973) and later at the University of Arizona (1988); Professor Frank Kelly at University of Cambridge (1992); Professor Frank Ball at Nottingham University (2002) have also hosted me for parts of overseas leaves. I thank them for their advice at those times. They played significant roles in allowing me to share and discuss many of my ideas and gain much from them. I wish to acknowledge my colleagues at the University of Auckland in the 1970's and 1980's, especially Professors John Butcher, George Seber and Alastair Scott and thank them for their support, encouragement and friendship.

Finally, the most important contributor has been my wife, Hazel, She has been with me over my entire career, accompanied me on most of my overseas trips but above all she has been there as a rock and soul-mate. I wrote my books and completed the bulk of my research at the time our children, Mark and Michelle, were growing up. The whole family supported me but at the same time we had a lot of fun and I don't think I neglected taking an interest and support in their numerous activities and education.

My research has in many ways focused on fundamental problems within applied probability. I am reminded of the words of Wally Smith – "the mathematics that lasts is the mathematics that is beautiful". It has been this underlying beauty that I have always striven for even if I have not always achieved the simple and elegant ways of expressing results.

Jeffrey J Hunter

NOTE 1:
The first paper in Section 1 of the compilation, "On the renewal density matrix of a semi-Markov process", was based upon research for the candidate's PhD degree, and published after the completion of that degree. It is not submitted for assessment but is included to set the scene for the candidates continuing research on semi-Markov and Markov renewal processes.

NOTE 2:
The compilation submitted does not cover all the published work of the candidate. Those works that do not fit the general theme of the compilation have been omitted.

In particular, only one chapter from Volume 2 of the candidates two-volume work on "*Mathematical Techniques of Applied Probability*" published by Academic Press, in 1983, has been included. This chapter, on discrete-time queueing models, which is widely sited, highlights an area that links with the general thrust of the candidates research.

NOTE 3:
All the material submitted is the sole authorship of the candidate, except for one paper in Section 7 (Hunter, J.J. and Titchener, M.R.: "The synchronization process for variable length T – codes". Published in *I.E.E. Proceedings, E, Comput. & Digital Tech.*, in 1986).) The contribution of the candidate in this paper is estimated at 50%.

NOTE 4:
The following abbreviations are used in referencing abstracts and reviews of the published material:

MR:	Mathematical Reviews
Zbl:	Zentralblatt fur Mathematik und ihre Grenzgebiete
IAOR:	International Abstracts in Operations Research
STMA:	Statistical Theory and Methods Abstracts
CiteSeer:	Scientific Literature Digital Library (http://citeseer.ist.psu.edu/)

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INTRODUCTION

The general thrust of the author's research has been in the theory and application of Markov renewal processes, Markov chains, queueing theory and two dimensional renewal processes.

There are strong interconnections between these areas. A Markov renewal process always has an embedded Markov chain present as an underlying component. Markov renewal processes and Markov chains are often present in queueing models. An important component of the research has been in the application of the mathematical techniques of generalized matrix inverses in the context of Markov chains and Markov renewal processes. An underlying theme has also been in the modeling of bivariate and two-dimensional processes often in a queueing or renewal theoretic setting.

The contributions have been separated into eight inter-related general applied probability themes:

- The development of the theory of Markov renewal processes. (Section 1)
- The study of queueing models with correlated arrivals. (Section 2)
- The theory and application of two dimensional renewal processes (Section 3)
- The application of Markov renewal theory to queueing models. (Section 4)
- The development of the theory and application of generalized matrix inverses to stochastic models where an embedded Markov chain is present. (Section 5)
- The study of "discrete time" queueing models. (Section 6)
- The application of Markov chain theory. (Section 7)
- The development of some techniques for estimating casualties due to multiple weapons. (Section 8)

There are strong interconnections between many of the sections. Sections 1 and 4 are linked by the theory of "Markov renewal processes". Sections 2, 4 and 6 have a common focus on "queueing models". Sections 5 and 7 have a common "Markov chain" link. Sections 2 and 3 have a linking through the modeling of "bivariate" processes – either as inputs to queueing service facilities or as a two dimensional renewal process. Section 8 is a small stand-alone section with a probabilistic background.

Each section of this thesis is prefaced by the list of the papers included in each section together with a preamble giving a brief synopsis and the main contributions and relevance of the papers.

Many of the research papers have appeared in the "Advances in Applied Probability", the leading journal in the researcher's field of applied probability, or in "Linear Algebra and its Applications", a major international journal in Linear Algebra. The connections between these two fields has come about, in the main, through the identification of the role that the mathematical tool of generalized matrix inverses can play in the solution of problems in applied probability where a Markov chain is present. The publications in these two journals highlight the strong interdisciplinary links and contributions that the author has made to both of the fields of applied probability and linear algebra. (See Sections 1 and 5)

The author has also made contributions to development of the theory and applications of two dimensional renewal processes (Section 3) and the application of Markov renewal theory to queueing models (Section 4).

Included in this compilation is one chapter form the author's two-volume work on "Mathematical Techniques of Applied Probability". This chapter, on discrete time queueing models, has been widely cited. It contained new results developed by the author and was one of the first surveys of this now quite significant field of application, especially in the field of computer communications (Section 6).

Section 1: Markov Renewal Theory

1. Hunter, J.J. "On the renewal density matrix of a semi-Markov process", *Sankhya: The Indian Journal of Statistics, Series A*, **31**, 281 - 308, (1969). (Web of Science - 2 citations; MR **46**,10088; Zbl **186**, 511; STMA **11**,1019)

2. Hunter, J.J. "On the moments of Markov renewal processes", *Advances in Applied Probability*, **1** (2), 188 - 210, (1969). (Web of Science - 40 citations; CiteSeer – 5 citations; MR **40**, 8143 - 1 review citation; Zbl **184**, 213; STMA **11**, 1017)

3. Hunter, J.J. "The equivalence of moments of Markov renewal processes", *unpublished manuscript*, (1970).

Paper 1 provides necessary and sufficient conditions for the convergence of the renewal density matrix of a semi-Markov process. These are based upon the earlier work of Walter L Smith who derived such conditions for the simpler renewal density function associated with a renewal process.

This paper led to more in-depth studies of Markov renewal processes and their associated semi-Markov processes – see in particular Section 4.

Paper 2 details the derivation of the asymptotic values of the first two moments of Markov renewal processes using known renewal theoretic results. This method of approach utilized Kemeny and Snell's "fundamental matrix" of a finite irreducible Markov chain", the Z matrix, and identified for the first time that it was in fact a generalized matrix inverse of I – P, where P is the transition matrix of the embedded Markov chain. As a by-product of the derivation, explicit expressions for the moments of the first passage time distributions in the associated semi-Markov process were derived.

This paper served as a motivator for a more detailed and thorough study of generalized inverses and their use in various applied probability problems where an embedded Markov chain is present – see in particular Section 5.

This paper is often quoted as containing the "classical result on the mean recurrence time in an irreducible positive recurrent Markov renewal process" (cf. p174, Neuts M.F., "Structural Stochastic Matrices of M/G/1 type and Their Applications", Marcel Dekker, N.Y., 1989). The paper has a significant number of citations due to the wide applicability of many of the results contained in the paper.

Paper 3 shows that three different techniques appearing in the literature for finding asymptotic expressions for the matrix renewal function of a Markov renewal process (derived by Kshirsagar and Gupta, Keilson, and Hunter) all lead to the same results. While these results are not generally available in the public domain, F.Ball and R.K. Milne have referenced this paper in their paper on "Simple derivations of properties of counting processes associated with Markov Renewal Processes", Research Report 03-02, University of Nottingham, Statistics Division, 2003.

Section 2: Queueing Models with Correlated Arrivals

1. Hunter, J.J.: “Two queues in parallel”, *The Journal of the Royal Statistical Society, Series B*, **31** (3), 432 - 445, (1969). (Web of Science - 5 citations, Zbl **186**, 247; STMA **11**, 1018)

2. Hunter, J.J. “Further studies on two queues in parallel”, *Australian Journal of Statistics*, **13** (2), 83 - 93, (1971). (Web of Science -1 citation; Zbl **249**, 60051; IAOR **13**, 11517, STMA**13**, 1100)

3. Hunter, J.J.: “Two queues in parallel with exponential type semi-Markovian inputs”, *Opsearch, The Journal of the Operational Research Society of India*, **14** (1), 29 - 37, (1977). (**MR 56**, 9728; IAOR **17**, 16854)

4. Hunter, J.J.: “Queueing and storage systems in parallel with correlated inputs", *New Zealand Operational Research*, **9** (2), 119 - 136, (1981) (MR **83a**, 60166)

5. Hunter, J.J.: “Markovian queueing systems with correlated arrivals”, *Research Letters in the Information & Mathematical Sciences*, **7**, 1-18, (2004).

The papers in this section all have a common theme of attempting to model queueing systems with correlated arrival processes either to two service facilities in parallel or to a single service facility.

Papers 1 and 2 focus on a queueing system consisting of two queues, each with single independent Markovian servers fed by a correlated bivariate Poisson input. The intention was to derive an expression for the stationary distribution of the joint queue lengths. The first paper obtains a functional equation for the joint probability generating function of the joint queue lengths when unlimited queue lengths are permitted. Structural and computational procedures are also found for the joint stationary probabilities in the case of finite waiting rooms. The follow-up paper 2 explored more general explicit expressions for the joint queue length distributions under the imposition of finite waiting rooms. It also considered the effects that the correlation on the arrival streams has on the correlation between the queue size distributions leading to some general conjectures concerning these relationships.

Paper 3 also considered on a queueing system consisting of two queues, each with single independent Markovian servers. In this system however the arrivals are of two types, which are generated by a two state semi-Markov process with arrivals of one type being assigned to one queue with a dedicated server and arrivals of type 2 assigned to a second queue with its own server. Expressions for both the marginal and the joint equilibrium queue length distributions were developed.

Paper 4 was motivated by the observation that queues or dams in close proximity are generally fed by correlated arrivals or inflows that typically have a general joint distribution. In this paper a discrete time environment is imposed and the joint stationary queue lengths for the Geometric/D/1 scenario and the related dam storage level distributions under a unit release at the end of each time interval are explored. Infinite and finite waiting room models are considered. Interconnections between the results for the queueing and dam environments are explored.

Paper 5 was motivated by a consideration how variable can the queue length distributions of a single server queueing system behave under the imposition of a first order correlation structure for the arrival process, under the assumption of exponential marginal distributions. The extreme cases follow from results of Fréchet involving bounds on the joint distributions of bivariate random variables and used earlier by the author in his papers on two dimensional renewal processes (See Section 3.)

Section 3: Two-dimensional Renewal Theory

1. Hunter, J.J.: "Renewal theory in two dimensions: Basic results", *Advances in Applied Probability*, **6** (2), 376 - 391, (1974). (Web of Science - 14 citations; MR **49**, 11649a; Zbl **284**, 60080; STMA **16**, 1160)

2. Hunter, J.J.: "Renewal theory in two dimensions: Asymptotic results", *Advances in Applied Probability*, 6 (3), 546 - 562, (1974). (Web of Science - 14 citations; MR **49**, 11649b - 1 review citation; Zbl **316**, 60058; STMA **16**, 1595)

3. Hunter, J.J.: "Renewal theory in two dimensions: Bounds on the renewal function", *Advances in Applied Probability*, **9** (3), 527 - 541, (1977). (Web of Science - 2 citations; MR **58**, 24585; Zbl **375**, 60096; STMA **20**, 475)

4. Hunter, J.J.: "Mathematical Techniques for Warranty Analysis", Chapter 7, (pp157-190), *Product Warranty Handbook, An Integrated Approach to the Analysis of Warranty Policies*, W.R. Blischke and D.N.P. Murthy, Editors, Marcel Dekker, ISBN 0-8247-8955-5, (1996) (Web of Science -3 citations)

The first three papers provide a comprehensive development of the theory of two dimensional renewal processes.

Paper 1 provides a unified theory, based primarily on bivariate generating functions and bivariate Laplace transforms. Explicit expressions for the two-dimensional renewal density, the two dimensional renewal functions, the correlation between the marginal univariate renewal counting processes and other related quantities were derived.

Paper 2 gives some useful asymptotic results for the bivariate renewal counting process, the distribution of the two-dimensional renewal counting process and the two dimensional renewal function. The paper also contains a comprehensive bibliography on multi-dimensional renewal theory.

Paper 3 provides a set of bounds on the bivariate renewal function, utilizing the Fréchet bounds for joint distributions.

There has been recent interest in two dimensional renewal theory and Papers 1 and 2 are widely cited. They now serve as a standard reference for determining the effectiveness of two dimensional warranty policies where decisions are based on two characteristics, typically time and distance (e.g. the number of kilometres travelled by a motor vehicle and the age of the vehicle, typically say 50,000 km or 3 years). These types of warrant policies are highlighted in Paper 4, which is a major survey of appropriate techniques employed in Warranty Analysis.

Section 4: Markov Renewal Theory applied to Queueing Models

1. Hunter, J.J.: “Queue length processes, from different viewpoints“, *New Zealand Operational Research Proceedings of the 17th Annual Conference ORSNZ*, 33 - 40, (1981).

2. Hunter, J.J.: “Filtering of Markov renewal queues, I: Feedback queues”, *Advances in Applied Probability*, **15** (2), 349 - 375, (1983). (Web of Science - 8 citations; MR **85**f, 60137a; Zbl **508**, 60076; STMA **25**, 1260)

3. Hunter, J.J.: “Filtering of Markov renewal queues, II: Birth-death queues”, *Advances in Applied Probability*, **15** (2), 376 - 391, (1983). (Web of Science - 6 citations; MR **85**f, 60137b - 1 review citation; Zbl **508**, 60077; STMA **25**, 1261)

4. Hunter, J.J.: “Filtering of Markov renewal queues, III: Semi-Markov processes embedded in feedback queues”, *Advances in Applied Probability*, **16** (2), 422 - 436, (1984). (Web of Science - 4 citations; MR **85**k, 60129 - 2 review citations; Zbl **535**, 60084; STMA **26**,1638)

5. Hunter, J.J.: “Filtering of Markov renewal queues, IV: Flow processes in feedback queues”, *Advances in Applied Probability*, **17** (2), 386 - 407, (1985). (Web of Science - 4 citations; MR **86**j, 60207; Zbl **561**, 60097; STMA **27**, 1477)

6. Hunter, J.J.: “Sojourn time problems in feedback queues”, *QUESTA: Queueing Systems, Theory and Applications*, **5**, 55 - 76, (1989). (Web of Science - 3 citations; MR **91**e, 60273; Zbl **685**, 60094)

7. Hunter, J.J.: “Birth-death queues with feedback”, *New Zealand Operational Research*, **13** (1), 39 - 49, (1985). (MR **86**a, 60119)

8. Hunter, J.J.: “Flow processes in birth-death queues”, *Proceedings of the Pacific Statistical Congress, 1985* I.S. Francis, B.F.J. Manly and F.C. Lam (Editors), Elsevier Science (Amsterdam), 195 - 197, (1986).

9. Hunter, J.J.: "The non - renewal nature of the quasi - input process in the M/G/1 queue", *Journal of Applied Probability*, **23** (3), 803 - 811, (1986). (Web of Science - 1 citation; MR **87**m, 60211; Zbl **624**, 60109)

In the study of queueing processes one learns early that the behaviour of a queueing system depends very much on how you look at the queueing system. Different results are typically obtained, for instance, for the stationary queue length distribution immediately following a “departure” in an M/GI/1 queue, the stationary queue length distribution immediately before an “arrival” in a GI/M/1 queue, and the continuous time versions of these aforementioned discrete time processes. The behaviour of queue length processes, from different viewpoints, in the same queueing model is explored in Paper 1. It is shown that the viewpoints of different observers, identified as the “arrival doorman”, the “server” and the “departure doorman typically lead to different queue length distributions.

These observations provided the motivation to consider examining a general class of queueing models where the queue length process can be modelled as a Markov Renewal Process. A wide class of queueing models, both in discrete and continuous time, satisfies this restriction. By utilizing Cinlar's "filtering" technique, whereby starting with the underlying Markov renewal process for the queueing model, the behaviour of processes embedded at different epoch types can be shown to also have a Markov renewal structure. This observation is explored in depth in Papers 2 to 8. The general feedback model is formulated in Paper 2 where we exhibit the structure of the Markov renewal kernels for each of the different embeddings with epoch types of "arrivals", "departures", "feedbacks", "inputs", "outputs", and "externally observed epochs". From this Markov renewal theory is utilized to deduce the key properties from the underlying Markov renewal kernel.

In Paper 2 the stationary and limiting distributions for each of a variety of embedded discrete time processes, the embedded Markov chains, is derived for feedback queues. These results are applied to birth-death queues with instantaneous state dependent feedback including the special cases of M/M/1/N and M/M/1 queues with instantaneous Bernoulli feedback.

In Paper 3, the results of paper 2 are extended to consider the associated embedded continuous time semi-Markov processes. The limiting distributions of the queue length processes in both continuous and discrete time are derived and interrelationships between them are examined in the case of continuous-time birth-death queues including the M/M/1/N and M/M/1 variants. Results for discrete-time birth-death queues are also derived.

In Paper 4 the equivalent continuous time queue length results for feedback queues are derived, complementing the results of Paper 3 in the non-feedback situation. The derivation utilized expressions for the moments of the appropriate mean holding times for each embedding.

In Paper 5 the nature of the flow processes in such feedback queues are explored via a derivation of explicit elemental expressions of the semi-Markov kernels for each embedded process. In particular it is shown that the inter-event distributions for the arrival process and the departure process are the same, with an equivalent result for the inputs and outputs. Conditions under which any of the flow processes are renewal processes, or more particularly Poisson processes, are also investigated.

Sojourn time problems in queueing networks are notoriously difficult to handle by analytic methods. Paper 6 illustrates the powerful use of Markov renewal theory to determine an expression for the distribution and moments of the sojourn time, the time a typical customer spends in a general Markov renewal queue with state dependent feedback. This paper makes extensive use of the general theory developed in the quartet of papers mentioned above related to the filtering of Markov renewal processes, extending this to the derivation of the appropriate first passage time distribution in a more general Markov renewal process. The paper also includes a survey of the literature on sojourn time problems in single node feedback queueing systems.

Paper 7 provides a general non-technical descriptive overview of the key results in Papers 2 and 5. The theoretical details were eliminated and the presentation focused on a general birth-death queue with instantaneous state dependent feedback.

Paper 8, gives a more general technical overview of the results from Papers 2 to 5 focusing on the flow process results for Birth-death queues.

Paper 9 rounds out this section by showing that the M/GI/1 queue can be analyzed using Markov renewal theory to show that the quasi - input process (i.e. the times that customers enter the service facility) is never a renewal process, unlike the arrival process (which is a Poisson process), except in the trivial case of instantaneous service.

Section 5: Generalized Inverses applied to Markovian Processes

1. Hunter, J.J.: "Generalized inverses and their applications to applied probability problems", *Linear Algebra and its Applications*, **45**, 157 - 198, (1982). (Web of Science -16 citations; CiteSeer – 3 citations; MR **83**j, 60064; Zbl **493**, 15003)

2. Hunter, J.J.: "Characterizations of generalized inverses associated with Markovian kernels", *Linear Algebra and its Applications*, **102**, 121 - 142, (1988). (Web of Science - 4 citations; MR **89**e, 60131; Zbl **667**, 15005)

3. Hunter, J.J.: "Parametric forms for generalized inverses of Markovian kernels and their applications", *Linear Algebra and its Applications*, **127**, 71 - 84, (1990). (Web of Science - 1 citation; MR **91**f, 60116; Zbl **695**, 15004)

4. Hunter, J.J.: "Stationary distributions of perturbed Markov chains", *Linear Algebra and its Applications*, **82**, 201 - 214, (1986). (Web of Science - 4 citations; CiteSeer – 1 citation; MR **87**m, 60148; Zbl **608**, 60062)

5. Hunter, J.J.: "The computation of stationary distributions of Markov chains through perturbations", *Journal of Applied Mathematics and Stochastic Analysis*, **4** (1), 29 - 46, (1991). (CiteSeer – 1 citation; MR **91**m, 60128; Zbl **719**, 60069)

6. Hunter, J. J.: "Stationary distributions and mean first passage times in Markov chains using generalized inverses", *Asia - Pacific Journal Operational Research*, **9**, 145 - 153, (1992). (MR **93**i, 60127; Zbl **772**, 60050)

7. Hunter, J.J.: "A Survey of Generalized Inverses and their use in Stochastic Modelling", *79-90, Advances in Probability and Stochastic Processes, A Volume in Honor of Professors R.P. Pakshirajan, G. Sankaranarayanan & S.K. Srinivasan,* A. Krishnamoorthy, N. Raju and V. Ramaswami (Editors), Notable Publications Inc., New Jersey, USA. ISBN: 0-9665847-2-4, (2001).

8. Hunter, J.J.: "Stationary distributions and mean first passage times in perturbed Markov chains", *Research Letters in the Information & Mathematical Sciences*, **3**, 85-98 (2002).

9. Hunter, J.J.: "Generalized Inverses, stationary distributions and mean first passage times with applications to perturbed Markov chains", *Research Letters in the Information & Mathematical Sciences*, **3**, 99-116 (2002)

10. Hunter, J.J.: "Mixing times with applications to perturbed Markov chains", *Research Letters in the Information & Mathematical Sciences*, **4**, 35-49 (2003)

Generalized inverses of singular matrices play an important role are providing general solutions to singular sets of linear equations. Generalized inverses of singular matrices are not unique. The underlying defining characterization is that A^- is a generalized inverse of the matrix A if A^- satisfies $A A^- A = A$. Other more restrictive conditions may be imposed leading, for example, to the unique Moore-Penrose generalized inverse.

Paper 1 gave the first general survey of the use of generalized inverses of the Markovian kernel I – P where P is the transition matrix of a finite irreducible discrete time Markov chain. The paper explores their use in a variety of applied probability problems – obtaining general procedures for finding stationary distributions, moments of first passage times and asymptotic forms for the moments of occupation times in a wide class of processes such as Markov chains in discrete and continuous time, semi - Markov processes and Markov renewal processes. The motivation for using generalized inverses to tackle these problems followed from the derivation of Kemeny and Snell's "Fundamental matrix" Z as a generalized inverse of I – P (as shown in paper 2 of Section 1) and the observation that many results in Markovian theory depend upon the solution of systems of linear equations. Generalized inverses can thus be used to obtain general structural results for the solution of such systems of equations. This formed the basis of many of the results in Paper 1.

It is well known that different multi-condition generalized inverses have characterisations that involve choices of various parameters. In Paper 2 we provide a systematic investigation of the various multi-condition generalized inverses of I – P. Partitioned forms for the g-inverses are also presented based on a full-rank factorisation of I – P. Special well known cases such as the group inverse and the Moore-Penrose inverse are also given.

In Paper 3 it is shown that it is possible to reduce the number of arbitrary parameters to a minimum for each class of g-inverses associated with the Markovian kernel I - P and, in fact, provide unique parametric forms for each such class. This provides a very useful tool for classifying an arbitrary given g-inverse of I - P. These results are utilized in deriving techniques for obtaining the moments of first passage times in the underlying Markov chain.

In Paper 4, techniques for updating the stationary distribution of a finite irreducible Markov chain following a rank one perturbation of its transition matrix are explored. These are based upon the generalized inverse techniques of Paper 1. A variety of situations where such perturbations may arise are presented together with suitable procedures for the derivation of the related stationary distributions.

In Paper 5, an algorithmic procedure for the determination of the stationary distribution of a Markov chain is developed. The technique is based upon a succession of rank one perturbations of the trivial doubly stochastic matrix whose known steady state distribution is updated at each stage to finally yield the required stationary probability vector.

In Paper 6 different forms of generalized inverses of I - P are used to simultaneously find the stationary distributions and mean first passage times of irreducible discrete time Markov chains.

Paper 7 provides a general survey of generalized inverses and their use in stochastic modelling. These summarise the evolution of the results, derived mainly by the author, from those initially presented in Paper 1.

In Paper 8 it is shown that the stationary distributions of perturbed finite irreducible discrete time Markov chains are intimately connected with the behaviour of associated mean first passage times. This interconnection is explored through the use of generalized matrix inverses. Some interesting qualitative results regarding the nature of the relative and absolute changes to the stationary probabilities are obtained together with some improved bounds.

Paper 9 continues the theme of Paper 8. It is shown that when two perturbations of the transition probabilities in a single row are carried out the differences between the stationary probabilities in the unperturbed and perturbed situations are easily expressed in terms of a reduced number of mean first passage times. Using this procedure we provide an updating procedure for mean first passage times to determine changes in the stationary distributions under successive perturbations. Simple procedures for determining both stationary distributions and mean first passage times in a finite irreducible Markov chain are also given. Techniques used in the paper are based upon the application of generalized matrix inverses.

In Paper 10 a measure of the "mixing time" or "time to stationarity" in a finite irreducible discrete time Markov chain is considered. The statistic, based upon the stationary distribution and the family of mean first passage times of the Markov chain, is shown to be independent of the initial state that the chain starts in, is minimal in the case of a periodic chain, yet can be arbitrarily large in a variety of situations. An application concerning the affect perturbations of transition probabilities have on the stationary distributions of Markov chains leads to a new bound, involving the "mixing time", for the 1-norm of the difference between the stationary probability vectors of the original and the perturbed chain. When the "mixing time" is large the stationary distribution of the initial chain is shown to be very sensitive to perturbations of the transition probabilities.

Section 6: Discrete Time Queueing Models

Hunter, J.J.: "*Mathematical Techniques of Applied Probability, Volume 1, Discrete Time Models: Basic Theory*". Academic Press, New York, N.Y. (Operations Research and Industrial Engineering Series) pp. xiii + 239, (ISBN 0-12-361801-0 (v. 1), (1983). (Web of Science - 36 citations; CiteSeer – 4 citations; MR **86**d, 60001a; Zbl **539**, 60064)

Hunter, J.J.: "*Mathematical Techniques of Applied Probability, Volume 2, Discrete Time Models: Techniques and Applications*". Academic Press, New York, N.Y. (Operations Research and Industrial Engineering Series) pp. xiii + 286, (ISBN 0-12-361802-9 (v. 2), (1983). (Web of Science - 86 citations; CiteSeer – 27 citations; MR **86**d, 60001b; Zbl **539**, 60065)

Both Volumes are listed in The Mathematical Association of America - Basic Library List http://www.maa.org/bll/probablility.htm

Rather than present both volumes of these works for consideration, only Chapter 9 of Volume 2 on "Discrete Time Queueing Models" is included in the submitted material.

This chapter has been widely cited as a key reference for much recent activity in this field. Such discrete time queueing models have applications to computer communication. Novel features include the use of the geometric pattern, either as "inter-arrivals" or "service times. This chapter includes a general survey of the Geometric/Geometric/1, Geometric/G/1, and GI/Geometric/1 queueing models with extensions to compound or bulk arrivals and/or batch servicing.

It also pioneers the introduction of different arrival systems, viz. the "early arrival system", the "late arrival system with immediate access" and the "late arrival system with delayed access". These all describe how arrivals occur at the beginning or the end of each discrete time slot prior to assignment (at the beginning or end of a time slot) to the service facility. The queue length distributions for models with different arrival systems are typically different as illustrated by specific examples in this Chapter. Interconnections between the different distributions are explored for both within and between different queueing models.

As well as queue length distributions, features such as classifications of states in the embedded Markov chains, properties of departure processes, first passage time distributions, limiting waiting time and queueing time distributions, busy period distributions, and the number of customers served in a busy period are all explored.

Section 7: Markov chain applications

1. Hunter, J.J.: "On the occurrence of the sequence SF in Markov dependent Bernoulli trials", *Mathematical Chronicle*, 2 (3), 131 - 136, (1973). (MR 50, 3278; Zbl **269**, 60051)

2. Hunter, J.J. and Titchener, M.R.: "The synchronization process for variable length T – codes". *I.E.E. Proceedings, E, Comput. & Digital Tech.*, **133** (1), 54 - 64, (1986). (Web of Science - 12 citations, CiteSeer – 2 citations)

Paper 1 is a minor work illustrating how recurrent event processes can be used to explore the occurrence of patterns in Markov dependent Bernoulli trials. Expressions for the probability distribution and expected value of the number of occurrences of the special pattern a success followed by a failure are obtained.

Paper 2 describes the application of discrete time Markov chains to the analysis of the time to sychronisation of a T-code decoding system. The details of the coding theory were developed by Mark Titchener (50%). The stochastic process analysis, showing firstly that the T-codes are statistically synchronisable (with probability one) and secondly deriving expressions for the expected sychronisation delay, was the work of the first named author (50%).

Section 8: Casualty Estimation Techniques

1. Hunter, J.J.: "An analytical technique for estimating casualties due to the direct effects of a multiple nuclear weapon attack on a city", *Technical Report RM - 230 - 2*, Operations Research and Economics Division, Research Triangle Institute, N.C., U.S.A. (1965).

2. Hunter, J.J.: "An analytical technique for urban casualty estimation from multiple nuclear weapons", *Operations Research*, **16** (6), 1096 - 1108, (1967). (Web of Science - 3 citations, IAOR **8**, 6708)

3. Hunter J.J.: "Analytical and numerical integration casualty estimation technique", *Technical Report RM - 873 - 1*, Engineering Division, Research Triangle Institute, N.C., U.S.A. (1974).

This section is a short stand-alone section and contains some problems that the author was exposed to, as a post-graduate student, during summer employment when he was employed by the Operations Research Division of the Research Triangle Institute Research Triangle as part of a contract with Office of Civil Defense of the Unites States Department of the Army. The work initiated at that time was eventually presented at an annual meeting of the Operations Research Society of America, published in the journal of the society and worked on again later in the 1970's when I spent a leave in North Carolina.

Paper 1 describes a model for estimating the total number of direct effects casualties due to multiple weapon attack on a single population centre. The specifications for implementation of the model are the coordinates of the designated ground zeros in relation to the centre of the population, the casualty type, the weapon parameters, the aiming error for each weapon, and the city population parameters. The objective of the research was to derive an analytical model that could be used in sensitivity analyses to determine the relative importance of the individual parameters.

Paper 2 summarises the key features of the model following release by the Office of Civil Defense for publication in open literature. This paper also includes some further generalizations involving extensions of the model to consider elliptical normal population distributions, uniform population distributions, elliptical normal aiming error distributions, and modified casualty functions.

Paper 3 presents some improved casualty estimation techniques to reduce the computing time for a large number of weapons and consideration of different casualty functions. Two main theoretical ideas were explored – one introducing bounds using Bonferroni type inequalities and one using numerical integration procedures as opposed to analytical procedures.