Steady Size Distributions in Cell Populations

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Abstract

In any population of cells, individual cells grow for some period of time and then divide into two or more parts, called daughters. To describe this process mathematically, we need to specify functions describing the growth rate, size at division, and proportions into which each cell divides. In this thesis, it is assumed that the growth rate of a cell can be determined precisely from its size, but that both its size at division and the proportions into which it divides may be described stochastically, by probability density functions whose parameters are dependent on cell size and age (or birth-size). Special cases are also considered where all cells with the same birth-size divide at the same size, or where all cells divide exactly in half.

We consider a population of cells growing and dividing steadily, such that the total cell population is increasing, but the proportion of cells in any size class remains constant. In Chapter 1, equations are derived which need to be solved in order to deduce the shape of the steady size distribution (or steady size/age or size/birth-size distributions) from any given growth rate and probability distributions describing the division rate and division proportions. In the general case, a Fredholm-type integral equation is obtained, but if the probability of cell division depends on cell size only (i.e. not age or birth-size), and all cells divide into equal-sized daughters, then we obtain a functional differential equation.

In two special cases, the resulting equations simplify considerably, and it is these cases which are explored further in this thesis. The first is where the probability of a cell dividing in any instant of time is a constant, independent of cell age or size. In Chapter 2, the functional differential equation resulting when cells divide into equal-sized daughters is solved for the special case where the growth rate is constant, and in an appendix the case where the growth rate is described by a power law is dealt with. The second case which simplifies is where the time-independent part of the growth rate of a cell is proportional to cell size. This case is particularly important, as it is a good first-order approximation to the real cell growth rate in some structured tissues, and in some bacteria. The special case in which this leads to a functional differential equation is discussed in Chapter 3, and the integral equation arising in the general case is dealt with in Chapter 4. Finally, the conditions under which the integral operator in Chapter 4 will be both square-integrable and non-factorable are discussed in Chapter 5. It is shown that if these conditions are satisfied then a unique, stable, steady size distribution will exist.

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  - Try something even if the chances of success look poor, or
  - Solve a simpler problem then look for ways to generalise!

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