Word problems in teaching and learning algebra

A thesis presented in partial fulfilment of the requirements for the degree of

Master of Educational Studies

in

Mathematics

at Massey University, Palmerston North, New Zealand.

Philip R. Bennett

2002

Abstract

This research seeks to examine student understanding of algebra and how teachers facilitate algebraic learning for the purpose of improving learning outcomes. Based on the earlier work of Nathan and Koedinger (2000a; 2000c; 2001), the role of word problems in particular is investigated in relation to student development of algebraic understanding and technique.

The year 10 students surveyed displayed particularly low levels of algebraic thinking and poor algebraic skill. The results show that as the structural complexity of problems increased, student understanding diminished and there was a clear shift in student choice of strategy. The use of calculators showed a significant increase in algebraic proficiency, supporting the view that beginning algebra students find it difficult to focus simultaneously on the algebraic and arithmetical aspects of problems. Story problems with result unknown and start unknown complexity solicited a greater proportion of informal strategies than equation problem counterparts. When students chose to use algebra, it was predominantly for problems in an equation format.

The results indicate a disparity between what is being taught and what is being learned. This may be explained in part by the apparent philosophical conflict in teacher beliefs, where importance is placed both on achieving success in algebraic technique, and also on encouraging student driven solution methods. In order to capture student interest, teachers endorse the use of informal strategies by students through advocating word problems as applications of the real world and promoting a goal oriented approach to problem solving. Findings from this study suggest that in order to promote algebraic thinking teachers should present problems for which algebraic means of finding a solution is both preferred and optimal. Students should be made explicitly aware of the purpose for a particular set task, such as word problems, and monitored carefully in their choice of strategy.

Acknowledgements

This research was supervised by Dr. Glenda Anthony, Head of Department for Technology, Science and Mathematics Education. I am grateful for her constant encouragement and expert guidance. Additional help, for which I am grateful, was provided by Dr Mitchell Nathan of the University of Colorado.

Approval for the surveys conducted as part of this research was granted by the Massey University Human Ethics Committee. My sincere appreciation goes to the students and teachers who generously gave up time to participate in this study.

Support for this research was provided through a Mathematics, Science and Technology Teachers Study Award funded by the Ministry of Education.

Contents

3 Methodology

4 RESULTS

List of tables

List of figures

1 INTRODUCTION

1.1 Background

Concern over student development of algebraic knowledge and skill has been increasing in New Zealand over a number of years. In 1997 the chief examiner of the national School Certificate examination, provided a review of year 11 performance that was scathing towards students' algebraic performance: "Many candidates (including those who otherwise scored well) were extremely weak in this area [algebra]. Skills commonly taught at 4th form [year 10] were often poorly done." (NZQA, 1997, p. 1). The chief examiner was particularly critical of students' use of the "guess and check" technique, noting that many students were unsuccessful in its implementation, and warned that it was not an appropriate technique for those students wishing to advance in mathematics.

The 1998 School Certificate examination included a separate algebra section for the first time and student results highlighted further the lack of competence in basic algebraic techniques (NZQA, 1998). Results for 1999 showed equally disturbing results with few students gaining full marks and many not even attempting an answer (NZQA, 1999). "Algebra continues to be the 'Achilles heel' of mathematics for many candidates" (NZQA, 1999, p. 5). Examination results in 2000 finally showed some improved student performance with the majority of candidates solving linear equations successfully (NZQA, 2000).

The problems experienced in New Zealand with teaching and learning algebra are mirrored world-wide. In the U.S.A approximately 30 percent of high school graduates that matriculate to college are required to take remedial algebra before they can begin college work (Kirst, 2000, cited in NRC, 2002a). In evaluating Advanced Placement AB Calculus 1998 examination results, the College Entrance Examination Board speculated that the poor performance observed on basic calculus was due to poor algebra skills rather than lack of understanding of the calculus (NRC, 2002b).

While algebra is not traditionally introduced until the ninth grade in the USA, by 1999, 18 percent of seventh and eighth grade students were enrolled in algebra courses (NRC, 2002b). The move to develop foundational algebraic skills at an earlier age is complicated by the fact that some grade eight teachers taking algebra courses are not certified to teach mathematics[1] (NRC, 2002b).

With the aim of addressing these concerns, algebra is highlighted in the '*Principles and Standards for School Mathematics*' developed by the National Council of Teachers of Mathematics (NCTM, 2000). The standards endorse the explicit inclusion of algebraic instruction prior to middle school and high school, in recognition both of the role of algebra in unifying the school mathematics curriculum through links with number, geometry and data analysis, as well as the importance of algebraic competence for life skills and further education (NCTM, 2000). Particular concerns highlighted by NCTM are concepts of a variable and equality.

A particularly thorny issue when teaching algebra relates to the use of word problems, which are a common tool for learning to apply mathematics to real world situations. Instruction in arithmetic and algebra typically makes use of word problems. The debate revolves around the timing of introduction of word problems within the algebra topic. Word problems are traditionally used after the teaching of algebraic rules in a symbolic setting, which is near the end of the algebra topic in the New Zealand mathematics curriculum. Recently, however, it has been found that word problems written explicitly for practising elementary algebra have merely elicited arithmetic techniques from students (Nathan & Koedinger, 2000a). One response to this is the suggestion that algebra be introduced through elementary word problems, allowing students to use informal strategies to find solutions.

The tremendous usefulness of algebraic methods and ideas in supporting mathematical work in many fields underscores the importance for students in achieving algebraic competency. Educators need to understand what algebra is, and how students construct algebraic knowledge in order to ensure that learning is directed appropriately. Nathan, Koedinger and colleagues (Nathan & Koedinger 2000b; 2000c; 2000a; Koedinger, Alibali, Nathan, 2001) investigated teachers' beliefs of students' algebra development and reasoning with the view to "[helping] teachers recognize the rationale behind their reasoning and, where it may be needed, develop professional development activities to

[1] approximately 18 percent of middle school teachers were not certified to teach mathematics in 1999

reform their beliefs so they are in closer accord with student performance" (Nathan, 2001).

This research is based on the work of Nathan and Koedinger (2000b; 2000a; 2001) and seeks to examine both student understanding of algebra and how teachers facilitate algebraic learning for the purpose of improving learning outcomes. After first reviewing the nature of algebraic thought, the cognitive obstacles faced by students are examined along with teacher understandings of algebra, in order to evaluate the effectiveness of algebraic instruction. The role of word problems in particular is investigated in relation to student development of algebraic understanding and technique.

1.2 Objectives

The objective of this study is to investigate the way in which algebra word problems are used in New Zealand schools. The qualitative methodology employed here is characterized by a "hypothesis-free orientation" that raises questions about the teaching and learning of algebra without presupposing answers (Burns, 2000). This approach is based on the assumption that student behaviour should be studied from within the subjective experience of the individual. Algebra word problems are therefore viewed as the object or phenomenon under study for which the key research questions intend to provide insight into their use by teachers and influence on student learning.

The key research questions to be explored are:

1) What strategies do New Zealand students use for solving word problems for which algebra could be used?
   a) Do students have greater success in solving word problems or solving symbolic problems?
   b) Do students prefer to use informal arithmetical strategies or algebraic strategies?

2) How do teachers view the role and difficulty of word problems in relation to symbolic representations?
   a) What do teachers believe about the difficulty of algebra word problems compared to similar symbolic algebra problems?

b) Do teachers support the use of strategies invented by students to solve word problems?
c) To what extent are teachers influenced by text books for teaching algebra?

This research adopts an interpretive paradigm and uses a phenomenological approach to assess student strategies and teacher beliefs.

2 LITERATURE REVIEW

2.1 Introduction

There has been much written about algebra within learning environments. The plethoras of writings for educational purposes are diverse and deal with concerns of what algebra is, in addition to approaches towards the teaching and learning of algebra. Perspectives of what algebra is range from a mathematical language and tool, to algebra as a way of thinking. Algebra is also perceived to be an inherent property of particular kinds of problems. The overlaps between algebraic thinking, algebraic techniques and algebraic problems create complications in establishing effective means for the teaching of algebra. In schools, algebra is often taught as a stand-alone topic, yet in senior high school algebra often permeates most other topic areas. To ensure that effective teaching of algebra takes place, it is important to understand how students develop an understanding of algebra and algebraic thinking to the point where students choose to apply algebraic techniques when solving problems.

2.2 The historical development of algebra

Understanding the historical development of algebra provides insight into both the nature of what algebra is, as well as the processes involved in the learning of algebra. Algebra was originally developed through an extension of the operations and rules of arithmetic. Whereas arithmetic stops at the processing of individual numbers, algebra allows for a problem to be generalised for a variety of possibilities. In learning algebra, students have been considered to extend their knowledge of arithmetic in much the same way that algebra was developed historically.

The historical development of algebra has been summarised as consisting of three defining stages (Boyer, 1968; Cajori, 1917; Harper, 1987; Kieran, 1992b; Rojano 1996; Sfard & Linchevski, 1994; Wright, 1998) :

1. The rhetorical stage where no symbols were used. Only natural language provided both the description and path to a solution for problems.
2. The syncopated stage where letters symbolised specific unknown quantities. Letters were used to abbreviate particular problems, rather than provide a generalisation for alternative quantities. Diophantus (C250 AD) is credited as

the initiator of using symbols. Letters were initially used to represent the unknown values in problems.

3. The symbolic stage of algebra where letters are used to express both givens as well as unknowns, leading to problems being generalised. This development occurred in the 1500's after the works of Diophantus were translated for the European mathematicians. Vieta (1540 – 1603) is credited as the first to produce works with solely symbolic representations of problems.

These stages highlight a progression according to the methodologies used in solving problems. Parallel to the development of techniques is a transformation of the perceptions of the problems – a development of algebraic thinking. The way problems were viewed progressed from a primarily procedural view of individual mathematical operations, towards a structural view of a whole system with interactions between known and unknown values.

In ancient cultures, rhetorical methods were used for problems that would readily be classed as algebra today. The Egyptians dealt with problems that can today be written symbolically in the form ax + bx = c. The Babylonians comfortably solved quadratic equations algebraically and also solved some cubics using tables of values (Boyer, 1968). For example, a Babylonian problem and solution found in Boyer (1968):

Problem: Find the side of the square if the area less the side is 14,30 (the number system is sexagesimal)

rewritten with symbolic structure $x^2 - px = q$ (where $p = 1$ and q=14,30)

Solution: Take half of one, which is 0;30 , and multiply 0;30 by 0;30 , which is 0;15; add this to 14,30 to get 14,30;15. This is the square of 29;30. Now add 0;30 to 29;30 and the result is 30, the square of the side.

rewritten with symbolic structure $x = \sqrt{(p/2)^2 + q} + p/2$

The rhetorical approach was *procedural*, with algebraists focusing on the interaction between quantities. Problems consisted of real everyday-life quantities. The arithmetical operations on specific quantities were unwound to find the unknown quantities. The objectives in solving problems involved the search for unique numerical solutions to unique problems.

The introduction of symbols did not immediately change the way problems were approached. Diophantus approached problems in a procedural manner, and is recognised as using a different technique for each of his 189 problems (Harper, 1987). Even when equations began to use fully symbolic notation after the 16^{th} century, many mathematicians explained their solution paths rhetorically. Mathematicians such as Descartes, Pascal, Fermat, Leibniz and Newton, who depended on the development of symbolic algebra, still kept the use of symbols to a minimum (Cajori, 1928).

A *structural* view of algebra, where a whole system can be considered a single object, started when the use of symbols were extended to the use of known quantities as well as unknown quantities. Vieta, in demonstrating his use of fully symbolic algebra on one of the problems of Diophantus, produced a generalised solution that highlighted the problem's underlying structure. The problem (cited in Harper, 1987) is stated as follows:

> "If you are given the sum and difference of any two numbers, show that you can always find what the two numbers are"

Diophantus solved this problem by choosing an arbitrary pair of numbers for the sum and difference and then letting one of the unknown numbers be represented by a symbol:

> Let the sum be 100 and the difference 40
>
> Let the smaller number be represented by x
>
> Then the larger number is $x + 40$
>
> So $2x + 40 = 100$
>
> So $x = 30$
>
> And the two numbers are 30 and 70

Vieta in introducing symbols for the initial given values, as well as the goal unknown values, demanded a variable perspective to be taken of the letters:

Let the sum be a and the difference b

Let the smaller number be represented by x

Then the larger number is $x + \mathrm{b}$

So $2x + \mathrm{b} = \mathrm{a}$

So $x = (\mathrm{a} - \mathrm{b})/2$

And the two numbers are $(\mathrm{a} - \mathrm{b})/2$ and $(\mathrm{a} + \mathrm{b})/2$

The use of symbols for all aspects of a problem enables a specific problem to be generalised and used for other similar situations. "A generalisation and deepening of concept became possible only after the form of presentation had been altered" (Cajori, 1928, p. 228). Diophantus was able to replicate his procedure for any pair of numbers presented, but not until two numbers were trialled could he be absolutely sure that a solution would evidence itself. With the fully symbolic notation, Vieta displayed greater mathematical rigour, allowing all possible numbers to be explored thoroughly and simultaneously. It was through the use of symbols for both known and unknown values that a variable nature became evident in the letters, later resulting in the development of functionality (Sfard & Linchevski, 1994).

A structural conception of algebraic symbols allows an idea to be recognised at a glance and manipulated as a single object. Vieta's solution above can be looked at as the representation of a single number (structural) or as a set of operations (procedural) required to obtain the desired results. Gray and Tall (1994, cited in Thomas & Tall, 2001) introduced the term *procept* to acknowledge that a symbolic algebraic expression possesses both procedural and conceptual (structural) qualities simultaneously. The notion of procept is an "amalgam of three things—process, symbol and concept" (Kota & Thomas, 1997, p. 36).

Sfard and Linchevski (1994) observe that modern algebra relies heavily on a structural perspective and consider the transition from a purely procedural approach towards a structural one to be a reification of algebra. Sfard and Linchevski's theory of reification states that a structural perspective increases one's efficiency in understanding and manipulating problems. A structural view is seen as advancement in algebraic thought and can be considered a level of understanding requisite for students of algebra. Thomas and Tall (2001) assert that advanced algebraic thinking necessary for manipulation algebra requires flexibility in moving at will between a procedural and structural view of algebraic expressions. "We contend that manipulation algebra can only be given flexible meaning if the algebraic expressions can be seen both as evaluation processes and as manipulable concepts. They then become procepts" (Thomas & Tall, 2001, p. 594). This versatile thinking which integrates procedural and structural views is a cognitive activity that imposes ideas onto the symbol set to allow the meaningful reordering of algebraic symbols (Crowley, Thomas, & Tall, 1994; Thomas, 1994).

Although the development of a fully symbolic algebra induced advancement towards a structural view, the use of symbols is not a prerequisite for algebraic activity (McNamara, 1996; Sfard & Linchevski, 1994). The term "algebra' was first used in the 12th century when the Arabic writings of Alchwarizmî were translated into Latin for the European community. Alchwarizmî titled his treatise "Alderscebr Walmulcâbala" meaning restoration and opposition (Cajori, 1917). On translation, the first word was transformed to algebra whilst the second word was discarded. The treatise contained explanations of elementary operations for linear and quadratic equations. The method of restoration transposes negative terms to the other side of an equation, whereas the method of opposition discards like terms that occur on both sides of an equation. For example

$5x^2 = 6 + 2x + 3x^2$ becomes $2x^2 = 6 + 2x$.

Although Alchwarizmî used rhetorical representations when solving problems, the naming of his treatise according to his methodology indicates an intention to generalise the mechanics in manipulating problems. Algebra in this sense seeks to understand the interaction between the quantities in a problem and combine both known and unknown values in a variety of ways, such that each retains its integrity.

While the introduction of symbols enhanced the efficiency in manipulating quantities, algebra is, however, essentially a cognitive activity, "...something which goes on inside our heads" (Hewitt, 1985, p. 15). According to Hewitt, symbols are one way of representing algebraic activity—they are not algebra in their own right. Just as with words in a language, the meaning of a symbol is derived from its relation to other symbols with which it is placed (McNamara, 1996). The setting surrounding the reason for a symbol's placement is what governs the cognitive activity induced within an observer.

Symbolisation condensed the mechanical processes of algebra, allowing manipulation to be carried out with a minimum of thinking (Bell, 1940). During the development of symbolic algebra, the focus on the symbols progressed from one of recognising that each symbol represented a particular quantity with dimensions, towards a dimension free view of the symbols as being separated from reality (Rojano, 1996).

Most early mathematics prior to the nineteenth century depended primarily on geometry and geometric intuition (Wilder, 1968). Algebra was seen as a procedural tool rather than a representational form of problems. Geometry, however, failed to explain areas such as the artificial numbers that were increasingly becoming commonplace in mathematics. By the eighteenth century the interest in geometry declined as methods became more and more algebraic (Gray, 1987).

The abstraction of algebra through symbolisation allowed for the advancement of algebra as a discipline in its own right. The development of set theory and a desire to formalise those numbers that failed to conform to geometric intuition—the artificial numbers—enabled algebra to grab the attention of mathematicians away from geometry. Much work was accomplished in arithmetising the real number system. Analysts such as Weierstrass, Dedekind, and Cantor set about basing the real numbers on the natural numbers and their arithmetic (Wilder, 1968). Arithmetical operations became the focus of attention and the numbers started to lose their need to measure something. It was through the arithmetisation of the real numbers that further advancements in the generalising of arithmetic by using algebra were made possible (Wilder, 1968). For example, before the formalisation of negative numbers the equations $ax^2 + bx = c$, and $ax^2 = bx + c$ were presented as two distinct problems (Bell, 1940). Only with the use of negative numbers can the above equations be reduced to the single equation $ax^2 + bx + c = 0$.

Algebra has developed through history from a methodology for solving real world problems, towards abstract generalisations of problems that can often be void of any geometric representation. The introduction of symbolism has helped to create a uniform language that reduces cognitive load on the working memory of algebraists to allow for greater efficiency in the manipulation of expressions. Through symbolism, expressions possess a process-product duality allowing them to be considered as dynamic or static, depending on the objectives of the observer. Algebra has become a symbolic system, a calculus, and a representational system (Wheeler, 1996).

2.3 Cognitive obstacles in the learning of algebra

When students begin to learn algebra in school, several cognitive obstacles are encountered including; the symbolic language of algebra, the ability to visualise the structure of problems, the ability to abstract real world contexts algebraically, and the conventions of algebraic manipulation. The many factors presented within a problem (such as the kind of numbers used, the symbols used, contextual information, and semantic representation) can influence the techniques that students use, and ultimately their success in finding favourable solutions. When presented with problematic situations, students will interpret the problem and select solution strategies according to the proficiency of their algebraic knowledge in conjunction with their knowledge of the elements represented within the problem.

2.3.1 The Letters

Küchemann (1981) found that when presented with elementary symbolic problems of increasing difficulty, students' interpretation of what the letters represented affected their success in answering the questions. Six hierarchical interpretations were distinguished:

1. Letter evaluated
2. Letter not used
3. Letter used as an object
4. Letter used as a specific unknown
5. Letter used as a generalised number
6. Letter used as a variable

It was found that those students who interpreted the letters in a manner according to the first three categories had little success beyond the most elementary of problems presented (an example of a level one problem was: $a + 5 = 8$).

Küchemann (1981, p. 105) argued that for "...any real understanding of even the beginnings of algebra [students] need to be able to cope with items that require the use of a letter as a specific unknown".

Further research has shown that students who spontaneously and intuitively manipulate the numerical parts of an equation while completely ignoring the letters have cognitive difficulties that stem from deficiencies within their understandings of arithmetic operations (Linchevski & Herscovics, 1996; Linchevski & Livneh, 1999). In a study attempting to establish the reasons why students developed poor understandings of the letters used in algebra, MacGregor and Stacey (1996b) found that many of the misconceptions held by students were due to learning experiences.

2.3.2 Expressions, Equations and the Equals Sign

When applying techniques of algebraic manipulation, students often find difficulty distinguishing between the equivalence held by a single expression after it has changed form, and the required equivalence held by variables when solving an equation. For example, if $x + x = 6$ implies that $2x = 6$, the expression $x + x$ is equivalent to $2x$ and the left hand side of the first statement retains equality with the left hand side of the second statement. In comparison, if $2x = 6$ implies that $x = \frac{6}{2}$, only the value of the variable retains equality, and the expressions between each statement are not equal. Kieran (1992a) refers to the above distinction as a difference in the systemic structure between expressions and equations. Students must cope with two kinds of equality-equivalence: a) the equality of the left and right hands sides of an equation, and b) the equivalence of successive equations (Kieran, 1997).

The systemic structure that underpins equations permits the performance of the same operation to both sides of an equation and allows for alternative ways of expressing subtractions and divisions as additions and multiplications (Kieran, 1992a). Students often struggle in their ability to communicate the changes they have made to an equation or an expression, regularly confusing the systemic structure of each. Subsequently students often misuse algebraic language, particularly the equals sign.

At the procedural level of understanding, many students are unable to treat expressions as a static entity and will often turn expressions into equations to emphasise that the expression must equal something—a number (Kieran, 1990). Also, when simplifying algebraic expressions, students can have difficulty facing a lack of closure and will often try to simplify expressions in such a way as to combine all parts into a single entity (Booth, 1988). This error is called conjoining, for example, students will write $2x + 3$ as $5x$ or 5. Thomas (1994) considers conjoining to be due to differences between arithmetic and algebraic symbolism, for example 35 in arithmetic means 3 tens and 5 ones, whereas in algebra 3n means 3 multiplied by n. A preference to use an arithmetic schema is what trips students into conjoining 3 + n to 3n.

During the learning of arithmetic, students are often given a series of practice problems such as 12 + 5 =____, where it is expected that the answer is placed in the space to the right of the equals sign. With the goal of finding a single numerical answer, arithmetic is portrayed predominantly as a procedural discipline. The equals sign triggers anticipation that the answer comes next in a sequence of operations. Students usually carry over this procedural view, learnt doing arithmetic, into their learning of algebra (Stacey & MacGregor, 1997). If students expect to see a single entity on the right hand side of an equation, the equals sign begins to confuse them when teachers expect them to view it more as a symbol of equivalence. Falkner and Carpenter (1999) in a study of first and second grade children found that many children displayed a good understanding of equality but have difficulty relating this to symbolic representations involving the equals sign. The cognitive obstacle that students face is a unidirectional view of the equals sign that is often read as 'gives' (Kieran, 1992b).

Students that are susceptible to a unidirectional view of the equals sign have difficulty understanding equations such as $15 = 2x + 7$ where the right hand side is more complex than the left hand side. A left to right reading of an equation is a natural and automatic occurrence by students where the equation is perceived as a temporal event rather than as a static entity (Pirie & Martin, 1997). A temporal state versus a static state can be directly related to the structural/procedural duality in algebraic thinking.

Tall (2001) asserts that to overcome the unidirectional view of the equals sign, students must visualise equations as concepts to manipulate rather than processes that need doing

—they must visualise *procepts*. Understanding equivalence is essential to overcoming procedural preferences where equivalence includes both rearranging (e.g., $2(x + b) = 2x + 2b$), and substitution of new symbols (e.g., $2(p + 1) = 5$ is equivalent to $2n = 5$) (Thomas, 1994). Thomas and Tall (1991, cited in Thomas & Tall, 2001) developed a programme for students involving 'evaluation algebra', where computers were used by students to assign numbers for variables and create expressions that the computer would evaluate. It was found that students performing 'evaluation algebra' were able to focus on the equivalence of algebraic expressions, which lead towards the development of proceptual thinking before moving onto manipulation algebra.

Word problems create added difficulties for students in the development and understanding of expressions and equations. Two categories of difficulties arise: the recognition of relationships between quantities represented and the ability to write them down correctly, and the writing of equations in a form that can be solved to find unknown values (MacGregor & Stacey, 1996a).

Research exploring the characteristics of expert algebra problem solvers compared to novice algebra learners has found few differences in the procedural techniques employed in manipulating equations (Chaiklin, 1992; Lewis, 1981). However, differences do occur in the way experts use equations to represent and solve problems, and the meaningful interpretations they make. Novices have difficulty discriminating what is important during their initial encounter with a problem (Kieran, 1992a). In contrast, experts are able to recognise important relationships between variables and form useful equations that highlight underlying mathematical structures.

The formation of equations from words has received a lot of attention. The reversal error is particularly well documented. Clement and colleagues investigated the reversal error where the following question was presented to first year university students (Clement, 1982; Clement, Lochhead & Monk, 1981)

> Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at the university." Use S for the number of students and P for the number of professors.

Of the students who failed to create the correct equation, 68% of errors were reversal errors, where the response was 6S = P rather than the correct response of S = 6P.

In examining this issue, Herscovics (1989) explored two lines of possible student thinking as causes for reversal errors; syntactic and semantic. Syntactic translation of the problem text translates the sequence of words directly into a corresponding sequence of symbols. For example, 6S = P is read as six times as many students as professors. When using semantic translation students interpret the meaning behind the text as the relative size between groupings. Students often illustrate their semantic interpretation through drawings (see Figure 2.1) (Herscovics, 1989).

Figure 2.1 Semantic translation of the student professor problem, (Herscovics, 1989).

The equation produced in both syntactic and semantic translations treats the letters used as labels or objects rather than as representations of numbers. Davis (1984) likened the use of letters as labels similar to the comparison of equivalent unit quantities (1 m = 100 cm). The equals sign is misused by students with the equation representing a correspondence or association rather than equality (MacGregor & Stacey, 1993). The equation 6S = P is read as six students *to* one professor. Student thinking in this case involves proportionality reasoning rather than algebraic (Herscovics, 1989).

The understanding that students have of the problematic situation is not necessarily in error, rather it is their use of the algebraic language that is incorrect. Fisher (1988) attempted to discourage students from using the letters as labels by requiring that N_s and N_p be used to represent the number of students and professors. However it was found that student performance was even lower. It was concluded that the more complex notation did reduce literal mapping errors by students, but that more complex difficulties with symbolic language surfaced in their place.

Not only do students write incorrect mathematical statements, they are often unable to interpret correctly the mathematical statements they have written. Interpretation is important in order to be able to check the sensibility of equations. In a study by Lopez-Real (1996), the following problem was presented to secondary students in Hong Kong:

> Mrs Chang has N oranges and M apples. Write a sentence to explain the meaning of the following statement: N = 3M

In addition to the construction of equations, the study sought to explore the occurrence of reversal error in the interpretation of equations. It was found that translating equations into every day language proved more difficult for students than expected. Although not all errors were attributed to reversal error, 76% of responses were found incorrect, as opposed to 50% incorrect for a corresponding question requiring equation construction. Lopez-Real (1996) concluded that a greater amount of time needs to be spent instructing students to become mathematically literate, where they develop the ability to understand mathematics as well as the ability to do mathematics.

2.4 How students solve problems

The cognitive processes involved in approaching word problems consist of: reading the problem, problem comprehension, problem representation, solution search and implementation (Ellerton & Clements, 1992; Reed, 1999). Mayer (1982; 1983) separated these cognitive processes into two distinct stages; translation and solution. The components of these two processes are illustrated in Figure 2.2.

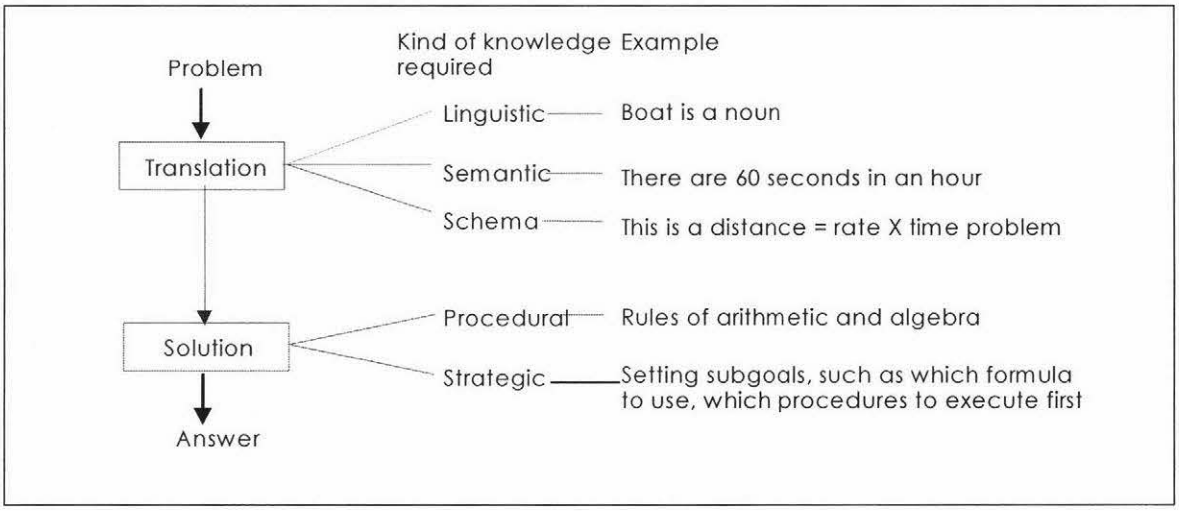


Figure 2.2: Analysis of mathematical problem solving (Mayer, 1983, p. 355)

2.4.1 Linguistic and Semantic issues

Algebra story problems contain linguistic, contextual and mathematical aspects, all of which must be addressed by the student (Zevenbergen, 2000). Language and contextual information are areas where student knowledge is primarily developed outside of the mathematics classroom. As students come to the mathematics classroom with a variety of language and contextual capability, problems requiring knowledge in these areas often have a discriminating effect. Language proficiency is difficult to disentangle from social and cultural factors, including socio-economic status, gender and ethnicity (MacGregor & Price, 1999). Even so, achievement in mathematics has been linked to language proficiency (Secada, 1992, cited in MacGregor & Price, 1999). Cummins, Kintsch, Ruesser, & Weimer (1988) found that not only did poor linguistic knowledge produce poor solution performance but also that the nature of problem text directly influenced the strategies used and errors produced by students.

In addition to language difficulties, the context of word problems can affect student achievement in mathematics. Low socio-economic groups have been found to be disadvantaged by problems embedded in real world contexts (Cooper & Dunne, 1998). Gender disadvantages also occur due to context, with many traditional word problems having a gender bias against girls (Boaler, 1994). In a study by Forbes (2000) on the fairness of assessment within New Zealand schools, it was found that contextual information produces discrimination between ethnic groupings, with Maori students being particularly disadvantaged.

In a study by Sebrechts and colleagues, however, it was found that linguistic and contextual information only played a minor role in student success during problem solving (Sebrechts, Enright, Bennett, & Martin, 1996). The study sought to investigate the strategies used by students to solve algebra word problems, and to establish the success of those strategies used. The results showed that the surface context framing a problem and the linguistic structure appeared to influence the strategy used by students, but only 11% of errors could be attributed to linguistic or contextual determinants. In a number of cases students solved for something other than what was requested, and in other instances given values were misused or additional assumptions were made that were not part of the problem statement. The above statistic, however, does not include those responses that were incomplete or provided only a guessed solution, which together accounted for 22% of errors. Further research would be needed to establish whether low literacy levels and general knowledge were contributing factors for the indeterminate responses.

It is clear that deficient knowledge in either language or context increases cognitive demands during problem solving which may lead to a greater likelihood for the occurrence of errors. To comprehend an algebra word problem, a correspondence must be made between the underlying formal equations and the student's informal understanding of the situation described in the problem (Nathan, Kintsch, & Young, 1992). Poor language and comprehension skills create demands on cognition that can interfere with the mathematical aspects of problem solving, causing errors and hindering the acquisition of effective problem solving schemata.

Cognitive load theory (Sweller, 1990; 1994; 1999) explores the interference of language and context aspects on schema development. Sweller contends that problems containing features that are unessential to the mathematical goals of learning are detrimental to learning. Unnecessary features consist of elements that frame the way a problem is presented, including language, situational contextual information, or the nature of the task goal as perceived by students. Cognitive load is increased through the need to integrate seemingly helpful information (such as text in conjunction with diagrams), before sense can be made of the problem (split attention). Alternatively, cognitive load can be reduced by trimming a problem of excess information that is superfluous (redundancy effect) (Chandler & Sweller, 1992; Sweller, 1999; Sweller & Chandler, 1994). Sweller's studies found that low achieving students in particular are adversely affected by unnecessary features presented in classroom problems.

In order to focus primarily on the learning of mathematical concepts Sweller and associates (Carroll, 1994; Cooper, & Sweller, 1987; Leung, Low & Sweller, 1997; Sweller, 1990; 1994; 1999; Sweller & Chandler, 1994) advocate that algebraic procedures be taught within context-free worked examples so as to prevent the possibility of split attention caused by unfamiliar contexts. For learning to occur effectively and efficiently, it is recommended that low ability students, in particular, be explicitly stepped through unfamiliar procedures. Sweller recognises that goal-free problems provide greater learning opportunity for students by encouraging students to focus on problem states rather than problem goals, promoting forward working strategies as used by expert problem solvers rather than backward means-ends analysis commonly used on traditional problems. The research conducted by Sweller and associates indicate that the use of worked examples, explicitly stepping students through a variety of problems, achieves the same benefits as goal-free problems without the same consumption of time (Sweller, 1999).

The use of context-free worked examples, however, is a view contrary to those held by recent curriculum reforms which promote problem solving within authentic real world situations. In reviewing recent mathematics curricula projects funded by the National Science Foundation, Phillips and Lappan (1998, p. 11) state that "problem-based curricula put quantitative reasoning in the forefront and thereby provide the basis from which to investigate patterns of regularity among rates of change between variables".

As previously discussed, when students encounter word problems they can translate the text directly (syntactically) or attempt to interpret the essence of the text (semantically). Artificial intelligence research in the nineteen sixties saw the development of a computer programme called STUDENT (Bobrow, 1968). Originally seeking to develop a computer programme that would recognise and interpret natural language, STUDENT used algebra word problems to investigate the ways in which text might be interpreted for decision-making purposes. It was found that computer algorithms could be developed to solve standard algebra word problems. Direct translation was used to recognise key connective words and separators such as commas in sentences. A database of problem forms that typified the type and position of connective phrases was then used to choose the appropriate mathematical operations and assign numerical values to variables.

STUDENT, however, did not interpret the contextual nature of problems at all, and was even able to solve nonsense situations as long as the mathematics followed standard problems. Gibberish situations could be used, or situations that contradict reality could be used. For example, STUDENT did not recognise that quarters have a higher value to dimes and was able to solve the following problem without any conflict:

> The number of quarters a man has is seven times the number of dimes that he has. The value of the dimes exceeds the value of the quarters by two dollars and fifty cents. How many has he of each coin?

Verschaffell, DeCorte and Greer and others (Mekhmandarov & Meron, 1996; Greer, 1997; Verschaffel & De Corte, 1997; Verschaffel, De Corte, & Lasure, 1994; Verschaffel, Greer, & De Corte, 2000; Wyndhamn & Säljö, 1997; Yoshida, Verschaffel, & De Corte, 1997) have made extensive investigations into how students make sense of word problems and found that school students, like the computer programme, are prone to ignore every day knowledge when problems are presented within the mathematics classroom. In research primarily concerned with arithmetic story problems, students were found to be mechanical and thoughtless in their problem solving approach. When problems presented to students were either absurd, or required every day knowledge from outside the problem statement. It was found that many ignored the contextual information and dealt solely with the numbers present in the problem text. For example, research findings noted that many students generated answers to unanswerable questions such as, 'there are 26 sheep and 10 goats on a ship, how old is the captain?' Incorrect sense making was also noticed for many division with remainder problems such as, 'an army bus holds 36 soldiers, if 1128 soldiers are being bussed to their training site, how many buses are needed?' In division with remainder problems like the one presented, many students offer solutions which do not consider the discrete nature of the objects in question. The answer often given in this question is 31 $\frac{1}{3}$ buses.

Verschaffell, Greer and De Corte (2000) consider student beliefs about mathematics to be the main reason why students solve problems in a non-realistic way. Students' expectations have undergone an enculturation in the mathematics classroom where it is perceived that knowledge from everyday life is often not needed for the tasks undertaken. Traditional mathematics classrooms contribute to the development of learning the school game where students learn things peculiar to the classroom but not useful in the real world (Boaler, 1999). Likewise, Lave (1993) asserts that traditional word problems form a genre within the mathematics classroom completely separate from other parts of life, requiring students to learn the language of 'word-problemese' (p. 77).

Boaler proposes that students become attuned to their learning environment in terms of the classroom community, and to the extent that this is artificial relative to a real-world context, students develop perceptions of boundaries and barriers in the formation and use of mathematics knowledge. Mayer (1992) highlights three typical detrimental beliefs that students have about problem solving:

1. Formal mathematics has little or nothing to do with real life.
2. Mathematics problems can always be solved in less than ten minutes.
3. Only geniuses are capable of discovering or creating mathematics.

From Boaler's proposition, these student beliefs are likely to reflect a classroom community where the pattern of mathematical behaviour is not representative of the world outside of the classroom. In such a learning environment it is plausible that the learning of mathematical concepts can be somewhat divorced from sense making and depend to a greater degree on the regularities of the classroom community (Boaler, 1999). Boaler's research on situated theory supports the views of Verschaffel et al. (2000) that many students build up a schema of what a word problem is, devoid of the need to search for deficiencies or contradictions or even connections within everyday life.

In addition to the linguistic proficiencies necessary for the translation of text in word problems, is the translation of algebraic notations. Algebraic notation is a mathematical language with its own grammatical rules and conventions (Stacey & MacGregor, 1997). Students with a high proficiency in algebraic language are also highly competent in solving algebraic problems (MacGregor & Price, 1999). MacGregor and Price found that students translated symbolic notations much the same way that they translate text. Effective translation involves metalinguistic awareness where students are able to reflect on and analyse language. MacGregor and Price highlight three metalinguistic competencies used by students in translating symbolic notations: *Symbol awareness,* where numerals, letters, and other signs are symbols and can be manipulated regardless of any original real-world referents, and groups of symbols can be regarded as a single symbol; *Syntax awareness* includes recognition of correct mathematical statements (for example, the correct use of the equals sign); *Ambiguity awareness* includes the ability to recognise multiple interpretations within mathematical expressions depending on structural interpretations.

2.4.2 Schema theory

The content of a story problem plays an important roll in student performance when translating and solving story problems. Hinsley, Hayes and Simon (1977) showed that similar to the computer programme STUDENT, high school students are able to categorise typical problems effectively before attempting to solve them. Considerable agreement was found between students on the categories used for algebra word problems, with up to 18 distinct categories or schemes highlighted. Problems were categorised as: triangle, distance-rate-time, averages, scale conversion, ratio, interest, area, max-min, mixture, river current, probability, number, work, navigation, progressions (two variations), physics, and exponentials. The ability of students to categorise problems is evidence that students use schemata to solve problems.

Mayer (1981) further sought to develop the categorisation of algebra word problems and distinguished between the form of the underlying structure of the *solution equation* and the form of the *story line*. The two forms are not mutually exclusive when a distinction is made between the simple surface story of a problem and the deeper relatedness of many stories used in school problems to an underlying mathematical formula. Blessing and Ross (1996) highlighted the interdependence of context and mathematical structure by presenting students with problems whose context was atypical for the problems mathematical structure. They found that students had difficulty in both solving and categorising atypical problems. Even experienced problems solvers were less accurate and slower in finding solutions.

By surveying exercise problems from twelve major text books in California, Mayer (1981) highlighted eight primary families of source formulae (for example the 'time rate' family depends on the formula "amount = rate X time"). Each family contains similar contexts, such as distance-rate-time, work, or motion problems within the time rate family. The categories within each family contain a variety of versions of the source formula, ranging from a simple arrangement towards a more complex use of the source formula. Motion problems, for example, had thirteen different templates, including vehicles approaching from opposite directions, overtaking one another, round trips, speed changes and so on. The research on problem categorisation suggests that experienced problem solvers use schemata to translate problem information for the express reason of applying familiar solution strategies (Mayer, Larkin, & Kadane, 1984).

Expert problem solvers may possess efficient organisations of problem and solution schemata. Novices, however, have been found to develop defective categorisation structures. Silver (1979; 1981) asked students to both solve and categorise story problems and found that poor problem solvers tended to group problems based primarily on surface content such as trains, buses or money, rather than any underlying mathematical formulae. Boote (1998) comments that novice physics students behave similarly, by categorising physics word problems according to surface content rather than according to the principals of physics, and argues that expert categorisation of a problem indicates an ability to solve it.

During the planning stages of problem solving, familiar aspects may be recognised that can aid the solving process in activating appropriate schemata. If an appropriate scheme is held by the solver, the search for a solution is often bypassed as solution procedures are activated from memory. Gick (1989, cited in Reed, 1999, p. 37) illustrated the interaction of prior knowledge in problem solving as expressed in figure 2.3.

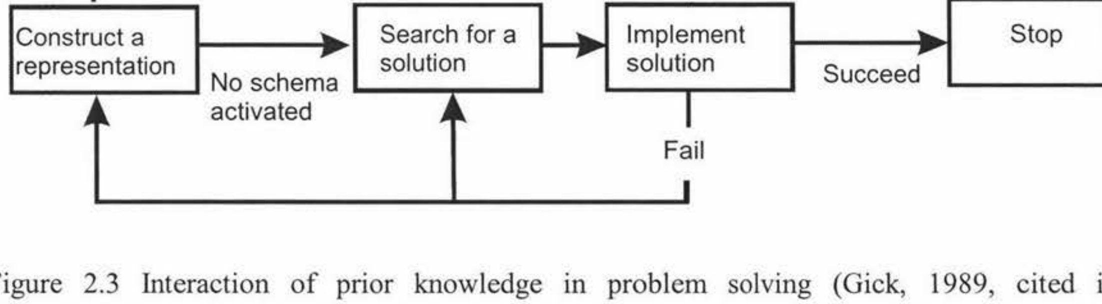


Figure 2.3 Interaction of prior knowledge in problem solving (Gick, 1989, cited in Reed, 1999, p. 37).

Searching for a solution when no schemata exist often requires a repertoire of heuristics that the student can draw from, in order to muddle through in search of a solution. To determine the strategic and tactical approaches used, Hall, Kibler, Wenger and Truxaw (1989) observed the workings of 85 undergraduate computer science students when solving algebra story problems. It was found that many approaches did not utilise prior knowledge in the form of problem category schemata as suggested by Hinsley et al. (1977) or Mayer et al. (1984). Many students were found to proceed with difficulty and construct solutions using reasoning activities that were only partially connected to any formalism. Hall et al (1989) did not refute the work of Hinsley et al. (1977) or Mayer et al. (1984), rather it was suggested that the students studied were not currently expert in solving algebra story problems and therefore did not draw from any story problem schemata.

2.4.3 Processes for developing solutions

In a tertiary course on problem solving at Berkeley, Schoenfeld (1992; 1998) promotes the solving of non-routine problems for which it is unlikely that students possess any relevant schemata. Schoenfeld advocates the use of Pólya type heuristics such as analogy, auxiliary elements, decomposing and recombining, induction, specialisation, variation, and working backwards (Pólya, 1957, cited in Schoenfeld, 1992). Rule of thumb techniques assist in developing a plan of action in unfamiliar situations and are best used in conjunction with metacognitive skills such as monitoring and control of decision making. The construction of some form of situational model, making inferences within that model in order to comprehend and solve a problem in a controlled trial and error fashion, has been referred to as model-based reasoning (Hall, Kibler, Wenger, & Truxaw, 1989).

At the implementation phase of problem solving, strategic knowledge is required to manipulate the information present into a favourable solution. The skills that high school students use are often not those taught to them by their teachers. Students also vary the methods they use depending on the nature of the question being asked. The types of algebraic situations formally experienced in schools include the development of expressions and equations, and the simplification and manipulation of expressions and equations. Word problems are often used as a vehicle for students to experience the developing of equations for the purposes of generating a solution through manipulation.

2.4.3.1 Techniques used in simplifying expressions

Demby (1997), in a study on students aged between 13 and 15 in Poland, found that the procedures used by students in manipulating variables can be varied. The analysis of both written responses and oral explanations identified seven types of procedures used by students: Automisation (where students went straight to a correct result without being able to explain how); Formulae (such as $(a+b)(a-b) = a^2-b^2$); Guessing-Substituting; Preparatory Modification of the expression (for example, replacing differences with sums and negative coefficients); Concretisation (treating symbols as concrete objects); Rules (either rules presented by the teacher or those made up by the student); and Quasi-rules (unstable rules not used consistently and often made up on the spot by the student). These procedures describe the methods used by students and not necessarily those taught by their teachers: "It was found that many students used procedures that had never been shown during any lesson and were absent in their textbooks." (Demby, 1997, p. 61)

2.4.3.2 Techniques used in solving equations

In the solving of equations, Kieran (1992b) summarises seven techniques as: using number facts, using counting techniques, cover-up, undoing (unwinding or working backwards), trial-and-error substitution (guess and check or modelling), transposing (change side—change sign), performing the same operation to both sides (balancing). Only transposing and balancing techniques are considered to be formal, and are commonly taught in schools.

Within the formal methods of transposing and balancing, Mayer (1992) highlights three basic strategies required for the manipulation of unknowns and variables:

- Attraction—where multiple instances are moved into a position where they can be combined (eg. moving from $16 + 2R = 3R - 24$ to $16 = 3R - 2R - 24$).
- Collection—where multiple instances are combined (eg. moving from $16 = 3R - 2R - 24$ to $16 = R - 24$).
- Isolation—where numbers and unknowns are separated to opposite sides of the equals sign (eg. moving from $16 = R - 24$ to $16 + 24 = R$).

2.4.3.3 Techniques used in solving algebra word problems

In the observations of Hall, Kibler, Wenger, and Truxaw (1989) the techniques used by tertiary students to solve algebra story problems were categorised as: Annotation (including diagrams, retrieval of formulae, lists of givens); algebra (the use of equations that need to be manipulated); model-based reasoning (simulations, guess and check); ratio (use of non-algebraic proportioning or scaling); unit (reasoning based on the units of quantities), and procedure (the use of a memorised sequence of actions without any algebraic reference or understanding). Algebraic techniques were found to be most common amongst the tertiary students, however even at this level 73% of students used the informal method of model-based reasoning at least once. The use of annotations were found to be governed by the surface content of the story, with for example, diagrams being prevalent in motion problems and lists of givens being prevalent in work problems.

Sebrechts and colleagues analysed the written responses of 51 undergraduate students to 20 algebra word problems (Sebrechts, Enright, Bennett, & Martin, 1996). Four major solution strategies were identified: equation formulation, ratio set-up, simulation, and unsystematic (unidentifiable methods). The use of diagrams, descriptions and formulae were termed collateral strategies, and were utilised primarily by high achieving students. Unsystematic approaches were more common amongst low achieving students. Equation formulation was found to be the most prominent strategy used by this group of students and resulted in the greatest success.

Some types of questions in the study by Sebrechts, Enright, Bennett, and Martin (1996) were found to solicit particular strategies more than others. The question: "A person invests $10,000, some at 8 percent per year and some at 10 percent per year. The annual income from this investment is $870. How much was invested at 8 percent?", was predominantly solved using simulation. However the question: "When walking, a certain person takes 16 complete steps in 10 seconds. At this rate, how many complete steps does the person take in 72 seconds?" was predominantly solved with a ratio set-up. Sebrechts et al. (1996) concluded that the choice of strategy was influenced by either the story content and linguistic structure, or the quantitative structure within a problem. It was also noticed that simulation strategies tended to be used in problems with multiple instances of a variable, where as using equations would have been the most useful technique.

One of the major differences between arithmetic and algebra problems is the difference in problem representation. When setting up an equation from words, the problem situation must represent rather than the solution operations (Kieran, 1990). For example: "When 4 is added to 3 times a certain number, the sum is 40. Find the number." To find the answer here requires the subtraction of 4 and then a division by 3. However representing the problem structure algebraically requires the operations of multiplication and addition ($3x + 4 = 40$). To set up the equation successfully, students must think in the opposite way that they would solve it. Unfortunately, however, high school students almost never choose to write and use equations that represent the mathematical structure of word problems (Kieran, 1997).

2.4.3.4 Comparison of techniques for word problems and symbolic equations

Nathan and Koedinger (2000a) presented a series of problems to students that consisted of both word problems and symbolic equations. Observations of written protocols sought to establish whether the same techniques were used on equivalent problems presented in both words and equations. A sample of the problems used is presented in Table 2.1 (Nathan & Koedinger, 2000a).

The problems presented to students by Nathan and Koedinger (2000a) varied in difficulty according to two major factors: the position of the unknown quantity in the problem and the linguistic presentation. Three classes of presentation (symbolic-equations, word-equations and story-problems) were combined with two positions of the unknown (start-unknown and result unknown).

The solution strategies observed to these questions were classified into four categories; arithmetic, algebraic, guess-and-test (including simulations), and unwinding (backtracking). Arithmetic methods are the direct application of arithmetic operations. All of the result unknown problems can be solved with arithmetic and so result unknown problems were considered arithmetic level problems and not algebraic. The start unknown problems were considered to require algebraic techniques of creating and manipulating equations and were therefore considered algebraic level problems.

Table 2.1 Problem combinations used by Nathan and Koedinger, 2000a.

Problem type	**Problem statement**
Start unknown	
Story-problem	When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour did Ted make?
Word-equation	Starting with some number, if I multiply it be 6 and then add 66, I get 81.90. What number did I start with?
Symbolic-equation	Solve for x: 6x + 66 = 81.90
Result unknown	
Story-problem	When Ted got home from his waiter job, he took the $81.90 he had earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?
Word-equation	Starting with 81.90. if I subtract 66 and then divide by 6, I get a number. What is it?
Symbolic-equation	Solve for x: (81.90 – 66) ÷ 6 = x

Nathan and Koedinger (2000a) found that students did not use the same techniques for both word problems and their symbolic equivalents. When presented with an algebraic problem in symbolic form, students predominantly attempted to use algebraic techniques. Equivalent problems, however, in both word-equation and story formats, tended to solicit the more informal strategies of guess-and-test or backtracking. It was also found that students were more successful with word-equations and story problems using informal strategies than they were with symbolic equations using algebra.

2.4.4 Influences on techniques used

The choice of strategy used by students is influenced by both the difficulty of mathematical structure and the numbers present in a problem and the way it is presented. Küchemann (1983) attributed students' use of intuitive methods over formal methods to the kind of numbers present. In a survey of 16 year old students in Sweden by Ekenstam and Nilsson (cited in Küchemann, 1983), the question $\frac{30}{x}=6$ met a success rate of 82%, whereas the question $\frac{4}{x}=3$ only had 48% successful responses. Küchemann (1983) concluded that greater success occurred in the first equation due to the use of positive whole numbers. Familiar numbers allow problems to be solved easily by intuition and negatively influence the emerging perceptions students have towards the usefulness of formal algebraic methods.

The context used for framing a word problem has been found to invoke the use of informal strategies when the context is familiar to students. For example problems that involve the use of money are often more familiar to students and are therefore more likely to invoke informal strategies. In replicating a study done on the informal strategies used by Brazilian children (Carraher, Carraher, & Schliemann, 1987) Baranes, Perry and Stigler (1989) inferred that context alone did not invoke informal strategy use, but rather a more complex interaction occurred between a combination of context and the numbers commonly used within that context. For example, particular kinds of numbers and number formats are used with money, therefore when problems use numbers that are common to money situations, students are able to use their knowledge of money to assist in solving the problem.

Another influence on the kind of techniques that students use is the mathematical structure of a problem. The mathematical structure of a linear equation consisting of one occurrence of a single unknown quantity is recognised to predominantly invoke intuitive solving strategies. With the single occurrence of an unknown, only the known quantities need to be operated on (Stacey & MacGregor, 1999). Problems such as $4x + 3 = 19$ can be solved using formal algebra, but students often choose not to use algebraic methods. Sfard and Linchevski (1994) observe that a defining factor in categorising equations of the form $ax + b = c$ as arithmetical is situated in the common unidirectional view for the equals sign. It would seem that "many students, described as *process-oriented*, are locked into an arithmetic, procedural way of thinking even when doing algebra" (Thomas & Kota, 1996, p. 564, emphasis in original). For these reasons, linear equations with a single occurrence of the unknown have often been termed arithmetic equations, with the term algebraic equations reserved for problems possessing a mathematical structure where there are multiple occurrences of the unknown (Filloy & Rojano, 1989, cited in Stacey & MacGregor, 1999).

In addition to any influences inherent in the problems presented to students, Thomas and Kota (1996) investigated the affective factors of anxiety and self-concept in the preferences students have to using arithmetical techniques over algebraic ones. In particular, girls were found to increase the use of algebraic techniques as their self-concept increased and anxiety decreased.

In order to ensure that students use formal algebraic techniques, the problems presented to them should promote algebraic thinking rather than arithmetic thinking. Stacey and MacGregor (1999) distinguish the differences in algebraic thinking in Table 2.2.

Table 2.2 Characteristics of Arithmetic and Algebraic Thinking (Stacey & MacGregor, 1999, p. 29)

Arithmetic thinking	Algebraic thinking
Operating from knowns to unknowns	Operating with and on unknowns
Unknowns transient, representing intermediate stages	Unknown(s) identified and stable throughout the problem
Equation (if any) as formula for calculating answer	Equation as description of relationships
Chains of successive calculations	Chains of logically linked equalities or inequalities
Intermediate quantities have a ready interpretation	Intermediate quantities may not have a ready interpretation

As word problems in particular tend to solicit intuitive methods, Stacey and MacGregor (1999) emphasise the need to use word problems with a mathematical structure containing multiple unknowns to encourage algebraic thinking. The example in Table 2.3 illustrates the difference between a single unknown word problem traditionally used and a multiple unknown word problem.

Table 2.3 Characteristics between a single unknown and multiple unknown word problems (Stacey & MacGregor, 1999, p. 27)

Problem	Equation
To rent a car from Tiger costs $100 per day and 20 cents per km. How far can I drive, if the most I can afford to pay is $240?	$0.20x + 100 = 240$
To rent a car from Tiger costs $100 per day and 20 cents per km. To rent a car from Kangaroo costs $120 per day and 15 cents per km. For what distance are the costs of each company the same?	$0.20x + 100 = 0.15x + 120$

In a recent study by Koedinger, Alibali and Nathan (2001), earlier research (Nathan & Koedinger, 2000a) was extended to include the use of problems that contain multiple occurrences of an unknown. The tertiary students to which problems were presented replicated the use of informal strategies for elementary problems. However, the study found that students using arithmetical techniques had little success with the problems of greater algebraic complexity (multiple unknown problems). Those students who were successful with multiple unknown problems used formal algebraic techniques. This finding agrees with the assertion of Sfard and Linchevski (1994) that the technique of backtracking which is commonly used successfully on equations of the form $ax + b = c$, no longer works for multiple unknown problems such as $ax + b = cx + d$. It is therefore advantageous for students to be encouraged to learn to think in algebraic ways in order for them to advance to more complex problems.

2.5 Teaching Algebra

Algebra is a difficult subject to learn. Within past New Zealand School Certificate examinations, algebra has consistently been the area of poorest performance by students (NZQA, 1997; 1998; 1999; 2000). In recognising that algebra is an integral part of the mathematics curriculum, there is much debate as to the best approach to take in teaching beginning algebra to ensure that students develop secure algebraic thinking and overcome the cognitive obstacles confronted.

MacGregor and Stacey (1999) recognise that many of the difficulties students experience while learning algebra are developed much sooner than their first algebra lesson. Some practical strategies are offered to teachers in the elementary school to help set the ground work necessary for students to develop algebraic thinking as follows:

- Understanding the equality – set tasks that promote a bi-directional view of the equals sign.
- Recognising the operations – Promote the understanding of arithmetical operations (such as multiplication being comprised of repeated addition).
- Use a wide range of numbers – Use large or fractional numbers in problems. Allow students to use calculators so that attention is focused on the operations used.

- Describing patterns and functions – Encourage students to look for a functional relationship between two variables instead of just looking at changes in one variable.

Likewise, Carpenter and Franke (2001) argue that algebraic reasoning should be developed at an earlier age through arithmetic activities. In a study of students in grades one through six, they found that using number sentences in flexible ways (such as exploring that $78 - 49 + 49 = 78$ is the same as $78 - 0 = 78$), enabled students to gain a better understanding of the nature and constraints of arithmetic operations and the equals sign. Students with a more flexible understanding of number sentences were found to readily transition into algebra and using variables to make generalisations.

Traditionally, algebra has been taught to students through the presentation of a series of lessons, first learning the rules of symbolic manipulation before being introduced to word problems. Students are then expected to translate word problems into equations to which they can apply the manipulation skills they have learned.

In "Approaches to Algebra", edited by Bednarz, Kieran and Lee (1996), a variety of authors explore a range of teaching options in an attempt to give meaning to algebra. Four approaches emerge as: generalising, problem-solving, modelling, and functions.

Generalising promotes the use of patterns (numeric or geometric) and the laws of numerical relations to produce expressions in a general symbolic form. Generalising demands the immediate use of letters as variables (Lee, 1996).

Problem-solving involves the solving of specific problems or classes of problems. Problem-solving allows students to experience the full activity of beginning with a problem, forming an equation, solving it, and interpreting the result (Bell, 1996). Problem-solving is based on the historical development of algebra emphasised so far in this review, building on students' prior arithmetic knowledge as they transition into algebraic thinking.

Modelling emphasises the processes used to examine a relationship between the key variables in a situation. Symbols, Tables, and/or graphs are used to represent observations or measurements. After a model has been developed to represent a situation, it is then tested for validity (Janvier, 1996).

A *functional* approach uses variables to describe real world quantities whose values change. The key concept emphasised in a functional approach, in addition to the use of variables rather than unknowns, is to observe how a value incurs change on its corresponding functional value (Kieran, Boileau, & Garançon, 1996). Problems presented to students teach them to work and reason with a variety of functional representations, including graphs, numerical, and symbolic representations.

In summarising the four approaches, Wheeler (1996) acknowledges that they are not independent of each other, and notes that the objectives are for students to learn how to solve problems, to model situations, to handle functions, and to make generalisations. When choosing a teaching approach, Thornton (2001) reminds teachers that the power of algebra is in its ability to represent situations in a variety of ways, and that each alternative representation can lead to new insights for the same situation. The choice before teachers, when introducing algebraic situations, is to decide which approach to begin with in order to better promote algebraic thinking. It is hoped that students will eventually experience and learn all of the skills utilised across all approaches.

Whichever approach is used, the algebraic concepts being taught include teaching students to read, write and operate with symbolic notations, and encouraging them to choose these notations in representing relationships and solving problems (Bell, 1996). The differences between each approach are subtle and require a high degree of teacher understanding and careful lesson preparation in order for students to gain maximum benefit.

2.5.1 The role of teacher knowledge

The effectiveness of a teacher rests largely on the knowledge that they possess. A seminal study by Shulman (1987, cited in McGee & Penlington, 2001) defined seven components of teacher knowledge:

- content knowledge,
- general pedagogical knowledge,

- curriculum knowledge,
- pedagogical content knowledge,
- knowledge of students and their characteristics,
- knowledge of educational contexts, and
- knowledge of educational ends and purposes and their philosophical and historical backgrounds.

Ball and Bass (2000) argue that in terms of mathematical knowledge, the exact content required to be an effective teacher is not well understood. Fennema and Franke (1992) found that more classroom discussion occurred during problem solving when the teacher was more familiar with the mathematics content involved. Similarly, Carlson (1991, cited in Fennema and Franke, 1992) noticed that in science classes, when teachers had low content knowledge they gave more time to desk work than to experimental work.

Teacher mastery of a mathematical concept, however, is a necessary but not sufficient condition to develop student thinking. Teachers also require pedagogical content knowledge, that is, a combination of content knowledge with subject specific pedagogy knowledge that enables teachers to anticipate and respond to learning challenges in practice (Ball & Bass, 2000). Pedagogical content knowledge includes subject specific content knowledge, understanding typical interpretations or learning difficulties that students face in relation to specific subject matter, and possessing knowledge of alternative explanations to assist students to jump over learning hurdles. For example, Tirosh, Even, and Robinson's (1998) exploration of teacher awareness of students conjoining algebraic expressions found that not all teachers were aware of this tendency. It was also observed that teachers within their study approached the teaching of algebra in a traditional manner where rules without justification were taught, and it was suggested that teachers should be sensitive to the ways in which students make sense of subject matter during instruction.

While pedagogical content knowledge helps teachers to deal with regularities in patterns of student learning, it may not always be sufficient in practice due to endemic uncertainty from the limitless situational possibilities that a teacher may be faced with in the classroom (Ball & Bass 2000). Ball and Bass (2000, p. 89, 97) define a need for "*pedagogically useful mathematical* understanding" that goes beyond *what* is known to define "*how* subject matter must be understood in order to be usable in teaching". Responding effectively to students in real time requires a teacher to be both flexible and adaptive. Ma (1999, cited in Ball & Bass, 2000) proposes that flexibility is achieved when a teacher is able to reorganise both content and pedagogy knowledge in order to respond to a particular learning or situational context. In this vein, 'decompression' is defined as the ability to "work backwards from mature and compressed understanding of the content to unpack its constituent elements" (Cohen, cited in Ball & Bass, 2000, p. 98).

A teacher's knowledge influences the decisions they make in the classroom, decisions regarding their control on both student learning and social activity. Knowledge, however, is not the only determinant of teacher behaviour. As Schoenfeld (1998, p. 84) puts it: "The 'action' – what the teacher decides to do, and why – is a function of the teacher's beliefs, knowledge, and goals." From this viewpoint is follows that a teacher's philosophy of mathematics is a determinant of their chosen pedagogy (Thompson, 1992). It has been noted, for example, that teachers who hold to the constructivist perspective tend to be guided by student interests and abilities rather than following a strict curriculum and textbook (NCTM, 1994).

2.5.2 The role of teacher beliefs

There has been much written on the distinction between knowledge and belief and the transition from one to the other (Pajares, 1992; Thompson, 1992; Woods, 1996). Beliefs have been described as attitudes, values, judgements, axioms, opinions, perceptions, perspectives, dispositions and rules of practice. Fennema and Franke (1992) consider beliefs to be a facet of teacher knowledge. Moreover, it has been observed that often teachers treat their beliefs as knowledge (Grossman, Wilson & Shulman, 1989, cited in Thompson, 1992). In an attempt to distinguish between knowledge and beliefs, dualities have been used such as conceptions/ preconceptions, theoretical knowledge/ practical knowledge, context free knowledge/ situated knowledge. The use of such numerous descriptive terms has blurred any straightforward distinction between knowledge and beliefs in educational research.

Perhaps the simplest way to define a belief, as distinct from something known, is in relation to individual subjectivity. Lea et al. (1987) describe beliefs as an individual's thoughts about the relationship between an object or behaviour and its characteristics. In contrast to knowledge which requires consensual agreement, beliefs can be held with varying degrees of conviction, so that while one belief may be fundamental, others may be peripheral (Fennema & Franke, 1992; Fishbein & Ajzen, 1975; Green, 1971, cited in Thompson, 1992). This emphasises that beliefs are a form of knowledge that is unique to an individual.

As a form of knowledge, beliefs are developed over time through the adoption of values, life experiences and the finding of new evidence. This means that with the application of further information, beliefs can become knowledge and knowledge can be relegated to belief. Classroom interactions will play a dynamic role in teacher beliefs as a teacher constantly evaluates student knowledge and behaviour in order to assess student understanding and the effectiveness of teaching practice. Much of this fluid process is based on the teacher's subjective interpretations of teacher practice and student response, interpretations which are in turn substantially dependent on the set of knowledge and beliefs held (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997).

Understanding what a teacher believes can be a good indicator of the practices they use in the classroom. Askew, Brown, Rhodes, Johnson, and Wiliam (1997, p. 20), for example, suggest that teacher beliefs about "what it is to be a numerate pupil", "pupils and how they learn to become numerate", and "how best to teach pupils to become numerate", all influence the teaching of numeracy. These types of beliefs are philosophically driven, and relate to conceptions of the nature and meaning of mathematics. Ernest (1988, cited in Thompson, 1992) defined three philosophical views of mathematics as being: the problem-solving view where mathematics is a process of constant enquiry and distillation of findings, and adding to a body of knowledge; the Platonist view of mathematics as an immutable body of knowledge that is discovered rather than created; and the instrumentalist view of mathematics as a utilitarian tool bag of disparate rules and facts that can be applied to achieve particular outcomes. The philosophy of mathematics held by a teacher will play a significant role in the approach to teaching that is ultimately adopted.

Askew, Brown, Rhodes, Johnson, and Wiliam (1997) discuss three different sets of teacher beliefs in terms of orientations toward teaching mathematics: connectionist, transmission and discovery. The connectionist perspective emphasises linkages between differing aspects of mathematics in order to gain meaning of mathematics. This is believed to be achieved through dialogue between teacher and pupils in order to understand student thinking and provide access for students to the teacher's knowledge. Transmission emphasises teaching more than learning, with correct procedures and routines promoted. Discovery encourages pupils to work things out for themselves. Discovery is goal driven with less importance placed on strategy choice.

Kuhs and Ball (1986, cited in Thompson, 1992) present four pedagogical dispositions, which although similar to the connection/transmission/discovery framework, focus more on the nature of teacher instructional practice:

1. *Student-centred*

   The student-centred perspective of constructivism involves methods of inquiry where problem solving is held in high esteem and self-generated ideas are encouraged.

2. *Content-centred, with a focus on understanding*

   The content-centred perspective, considered a Platonist view, gives greater emphasis to the objects, concepts, and ideas of mathematics, in conjunction with the logical relationships amongst them. Tasks presented by the teacher may be similar to those given by a teacher with a student-centred view, but would be organised to a greater extend around mathematical structures rather than student concerns.

3. *Content-centred with a focus on performance*

   A content-centred approach that emphasises performance reflects an instrumentalist philosophy of mathematics. Instruction here emphasises the correct way to do things. Lessons focus on mathematical rules and procedures, necessary for getting the right answers to problems. Drill is used until a student is able to demonstrate a proficient automated performance of their mathematics.

4. *Classroom-centred*

   The classroom-centred paradigm approach is isolated from a particular mathematical philosophy to the extent that it assumes that mathematical content is pre-determined. Effective teaching and learning is determined by a teacher who can skilfully explain mathematical ideas and manage the classroom environment. Well-structured lessons with predetermined tasks and feedback are required of the teacher, while the student's role is to listen attentively and follow instructions.

These separate sets of beliefs about teaching practice are unlikely to completely define a given teacher, but it is likely that a teacher is disposed predominantly toward one of these approaches.

Although beliefs are dynamic, they are resilient and resistant to imposed change without the application of sufficient new information. Pajares (1992) states that beliefs about teaching are usually well established by the time a person leaves school, and suggests that the beliefs of adults rarely change significantly. Moreover, Sullivan (1987, cited in Mayers, 1994) found that personal experiences of learning mathematics are the most significant determinant of *student-teacher* beliefs about mathematics teaching. It follows that such beliefs are usually based on and consistent with the pedagogy experienced by student-teachers, as the sum of collective influences of classroom and teacher experiences over many schooling years.

An example of the resistance of beliefs to change is shown in Buzeika's (1996) study on teacher reaction to a change in the mathematics curriculum. The New Zealand mathematics curriculum underwent a review in 1992, emerging with a constructivist perspective (Mayers & Britt, 1995). Although the revised curriculum imposed pressure to change on teachers, it appears that many teachers made little or no change to their teaching approach, instead choosing to interpret the curriculum statements in ways that supported their current practice (Buzeika, 1996). Experimental research by Mayers (1994), however, on primary student-teachers' beliefs and attitudes about mathematics and mathematics teaching showed that significant shifts in beliefs and attitudes could occur with appropriate exposure to new information and personal experience. In Mayers' study, the beliefs of a group of student-teachers about mathematics pedagogy significantly shifted during a one semester course from a traditional view towards a constructivist perspective. This was achieved through providing the students with opportunity to experience and discuss constructivist teaching practices. This suggests that where there is limited opportunity for ongoing professional development of teachers, the opportunity to influence teacher beliefs is likely to be limited.

2.5.3 Teacher beliefs and teaching practice

It has been suggested that understanding a teacher's beliefs is the most useful means for determining the decisions that they make in the classroom (Pajares, 1992). Two teachers may possess the same knowledge of mathematics and have the same teaching materials available to them, yet teach in completely different ways due to the independent beliefs that they hold. For example, a teacher who believes mathematics is a procedural tool might place greater emphasis on practicing rules and learning mathematical principles than a teacher who believes mathematical thinking is a disposition (Fennema & Franke, 1992).

Ernest (1988, cited in Thompson, 1992) noted that in addition to teacher knowledge and beliefs concerning mathematics teaching and learning, the social context of the teaching environment was a key influence on teaching practice. Stipek, Givven, Salmon, and MacGyver (2001) in a study on 21 fourth through to sixth grade teachers found a strong association between a teachers beliefs about mathematics and their practice in the classroom. Clark (1993), however, notes that sometimes unavoidable factors present in the classroom prevent teachers from carrying out practices in line with their beliefs. The complexity of school and classroom environments, with teachers experiencing up to 1000 interpersonal contacts within a single day, means that teachers must learn to function on impulse and intuition, and do not have time to reflect on whether their behaviour aligns with their beliefs (Pajares, 1992). Beginning teachers, in particular, have been found to behave differently than they espoused. In one study, beginning teachers who professed to adhere to student-centred philosophies were found to behave in classroom-centred ways (Simmons, Emory, Carter, Coker, Finnegan, Crockett, Richardson, Yager, Craven, Tillotson, Brunkhorst, Twiest, Hossain, Gallagher, Duggan-Haas, Parker, Cajas, Alshannag, McGlamery, Krockover, Adams, Spector, LaPorta, James, Rearden, & Labuda, 1999).

The relationship between teacher beliefs and practice can be explained through understanding the role of attitudes in affecting behaviour. The distinction between beliefs and attitudes has been discussed at length in the field of behavioural psychology, in relation to understanding and predicting behaviour through studying how attitudes are formed. Fishbein and Ajzen (1975) define attitude as a persistent disposition to respond either positively or negatively to a given object or behaviour. An actual behaviour is then the result of combined beliefs and attitudes. Whether a teacher adopts a particular behaviour is the result of the strength of the belief held about the likelihood of that behaviour producing a particular outcome, and the attitude or strength of feeling that the individual possesses towards the desirability of that outcome. The subjectivity applied to a behavioural decision is therefore both cognitive and emotional, and may be the combined result of complex internal and external influences.

The role of beliefs and attitudes in determining behaviour implies that a teacher may not always act in line with their beliefs, as they may consider that the likely outcome is unacceptable or undesirable in a particular context. For example, a teacher may believe that students should learn for understanding, however, if time constraints are high and the assessment tool is designed to favour a procedural approach, the teacher may instead choose to teach to the test with an emphasis on the required procedural rules so as not to disadvantage students. Suggested contextual influences include school practices and ethos, the curriculum and educational policies, and the expectations of both parents and pupils (Askew, Brown, Rhodes, Johnson, & William, 1997; Thompson, 1992).

Buzeika (1996) encourages teachers to be reflective in their practices and open to new challenges in their teaching. The teachers interviewed by Buzeika in her study on teacher's beliefs, however, indicated that trying new ideas was dependent on high levels of confidence and control in the classroom. This can be explained through teachers possessing negative attitudes towards trying new ideas due to their beliefs about the nature of the behavioural outcomes that will result, and the teacher's ability to maintain classroom control. When teachers possess high levels of self-confidence in their teaching, their students also have high self-confidence in learning mathematics (Stipek, Givvin, Salmon, & MacGyvers, 2001). This may be because teacher self-confidence inspires students to perceive that subject mastery is both attainable and desirable.

2.5.3.1 Teacher beliefs about word problems

Teacher beliefs about how real world knowledge influences the way students interpret and attempt to solve word problems have a strong impact on pedagogy and learning outcomes (Verschaffel, Greer, & De Corte, 2000). The use of word problems in the teaching of algebra has generally been justified as the application of mathematics to the real world. A common teacher belief is that the use of word problems both motivates students to see the value of mathematics and curbs student belief that algebra is purely the manipulation of symbols.

However, word problems as real world applications have been criticised by many researchers as artificial and even detrimental to students' understanding of how mathematics is applied by real practitioners (Reed, 1999). Usiskin (1980) considers many word problems to be so ridiculous that they convince students that there are no real applications for algebra.

> The claim that word problems are for practicing real-life problem solving is a weak one, considering that their stories are hypothetical, their referential value is nonexistent, and unlike real-life situational problems, no extraneous information may be introduced. (Gerofsky, 1996, p. 41)

Boaler (1993) regards many contexts used in word problems to be barriers to understanding, distracting rather than stimulating and motivating students. Sierpinska (1995) argues that authentic contexts need to give meaning to the mathematics being learned rather than to everyday life.

Nathan and Koedinger (2000b) undertook a study to compare teacher beliefs about student learning of algebra against actual student performance in solving algebra problems. They found that high school teachers who viewed word problems to be the application of mathematical skill, considered that word problems should be taught after the student has first mastered symbolic problems in context free situations. This teacher belief, called a symbolic precedence view, was related to perceptions that arithmetic problems are generally easier than algebra problems regardless of the whether they are presented as word problems, story problems or in symbolic format, and that story and word problems are more difficult than symbolic equations as they involve the application of "pure mathematics" (Nathan & Koedinger, 2000b). The student results, however, showed that while they tended to have more success solving arithmetic problems than algebraic problems, students actually found the symbolic equations to be the most difficult and the story problems the easiest. The reason found for this apparent anomaly was that teachers were unaware of alternative informal strategies that the students used to solve story problems (Nathan & Koedinger, 2000b).

One potential influence on teacher beliefs about learning algebra is the typical organisation of textbook material following the symbolic precedence view (Nathan & Koedinger, 2000a). The symbolic precedence view is considered by the National Council of Mathematics Teachers (NCTM) to be contrary to the constructivist perspectives promoted in recent curriculum reforms where students' abilities to develop and use their own methods for solving problems is acknowledged (NCTM, 1994). NCTM encourages teachers to familiarise themselves with the strategies that students naturally use in order to build on students' thinking and prior knowledge, rather than telling them how to think.

2.5.4 The influence of Textbooks

Textbooks used in high schools to teach mathematics have been found to greatly influence teacher beliefs and practices. Textbooks are often written for students to learn from, and in many instances are issued to students as a take-home resource. Fauvel (1991, cited in Shield, 1998) views textbooks as a link between author and student where the authors are described as "teachers at a distance". Students, however, do not usually take advantage of the teaching aspects of textbooks, rather they tend to restrict their use to sampling from the extensive exercises presented in textbooks (Shield, 1998). Textbooks have been found to be read as a tool of learning primarily by teachers in order to prepare lessons and make pedagogical decisions (Morris, 1989; Shield, 1991; 1998).

The way in which textbooks are structured reflects the beliefs the authors have about mathematics, teaching and learning. Halliday (1973, cited in Morgan, 1996) elaborated on three major aspects to textbook content and format: *ideational* aspects which consider the beliefs the author has about the nature of mathematics and mathematical activity, as indicated in the content of the textbook; *interpersonal* aspects which indicates to the reader how the textbook should be used, including the roles of teachers and students; *textual* aspects which address the presentation of material as a visual medium, including the use of illustrations.

Due to curriculum reforms in line with a constructivist disposition, many new textbooks have been developed and marketed as reform texts. Even recent textbooks, however, often do not actually reflect the curriculum reforms and tend to be written in a "tell-show-do" approach, where the format is usually a set of notes outlining the procedures and concepts to be learned, followed by example problems, and finally with exercises for students to practice (Shield, 1998). It is often the case that mathematical ideas are presented without any justification. Although some textbooks provide "brief historical snippets, there is no real feeling that the mathematics being learned is a human endeavour" (Shield, 1998, p. 520).

Beginning algebra is often not presented in school textbooks in a manner reflective of curriculum reforms. The symbol precedence view of teaching symbolic manipulation and equation solving prior to word problems has been found to be predominant in most text books that teach beginning algebra (Nathan & Koedinger, 2000a; 2000c). Nathan and Koedinger (2000c) consider that one of the dominant reasons why high school teachers hold to the symbol precedence view of algebra is due to repeated exposure to, and reliance on, textbooks. However, it could also be possible that both teachers and textbook authors, share common beliefs (Nathan & Koedinger, 2000a).

2.6 Summary

This discussion of the literature related to the development, teaching and learning of algebra identifies a range of interrelated issues that are relevant to this study of the use of algebra word problems. The first issue is that algebra is external rather than internal to problems that are presented to students. Algebra is a human construct that involves the abstraction of reality as a means of understanding the way in which relationships are formed and change occurs. Algebra is about forms, the transformation of forms, and equivalence (Pimm, 1995 cited in Nickson, 2000). The development of algebra over many centuries reflects the development of a higher level of thought as one idea has built on another and another and another. From this perspective, algebra as a dispositional view is imposed on problems rather than being inherent within them. Problems exist in their own right and algebraic thought and method is merely one (particularly useful) way of looking at problems in order to better understand them.

If algebra is external to problems then what is the reason for teaching it? From the literature it is clear that the purpose of teaching algebra is both concrete and abstract. As a technique, algebra can be used to solve highly sophisticated problems that otherwise would remain unsolved, and for this reason is explicitly and specifically valuable to society. From a more abstract perspective, however, algebra has inherent value as a higher level of thought, in that an algebraic disposition has the ability to produce further building blocks of knowledge as understandings of complex environments are developed and enhanced through algebraic thought. This provides an argument that the understanding of, and ability to think, algebraically is sufficiently valuable to justify its inclusion in the school curriculum. Moreover, the emphasis on using word problems could be considered to be a reflection of the perceived usefulness of algebra to explain real life contexts.

> When a child learns algebra he or she is enriching his or her problem-solving repertoire considerably and although it may seem to be at a level of abstraction that some may consider inaccessible to all children, the simplest modelling procedures used in solving word problems employ basic ideas of algebra that are immensely powerful. (Nickson, 2000, p. 110)

This raises the question of how to teach algebra as an imposed disposition. The literature indicates that student choice of technique when solving problems is influenced by a variety of factors including: the mathematical structure of those problems, the kind of numbers present, the syntax representing a word problem, and any real world situational context present in a word problem. Student use of informal methods, such as arithmetic, backtracking, and guess-and-check is considered to be predominant when solving problems that contain a mathematical structure with only a single occurrence of an unknown value (Sfard & Linchevski, 1994; Stacey & MacGregor, 1999). Algebraic methods are more likely to be solicited by problems with multiple instances of unknown values.

The mathematical structure of a problem is best represented using symbolic algebra; however, beginning algebra students often do not consider using symbolic equations when faced with a word problem. This brings into question the placement of word problems in the process of teaching algebra. Introducing algebra to students through word problems can utilise students' informal methods as a point of discussion and departure into algebraic thinking and techniques (Reed, 1999). Teachers who adhere to the traditional use of word problems, as the climactic application of algebraic techniques, may find that students prefer to continue using informal techniques over those supposedly just learned. (Nathan & Koedinger, 2000a)

Teacher knowledge and beliefs about the role of word problems in teaching algebra, and how to teach algebra, is a strong determinant of their practice in the classroom. Underlying each teacher's chosen pedagogy is a philosophy of the nature and meaning of mathematics (Thompson, 1992) which in turn shapes their beliefs about teaching and learning mathematics. Therefore, understanding what a teacher believes can be a good indicator of the practices they use in the classroom. Previous studies suggest that actual teaching practice will, however, also be influenced by a range of external factors including: text book presentation of the topic, the content of assessment tools, the complexity of school and classroom environments, teacher confidence and the ability to be both flexible and adaptive in responding to students in real time. Teaching approaches should emphasise the use of an appropriate context when teaching algebra, which is one that demonstrates the use value of algebra as being greater than that of an alternative method.

Teachers also need to be aware of obstacles that students face in learning abstract concepts and the need to build on prior knowledge. Past teaching approaches to algebra have tried to develop students' algebraic thinking inductively through the presentation of a variety of problems to students in succession, progressing from substituting values for variables into algebraic expressions and equations towards more complex symbolic manipulation and solving of equations. With greater teacher knowledge of the obstacles that students face in learning algebra, teachers are more able to explicitly address crucial transitions in thinking to students and monitor development.

3 METHODOLOGY

3.1 Introduction

Investigating the questions posed in this study required the selection of an appropriate research methodology. As this study aims to gain insight into teacher awareness of student learning, and particularly teacher perceptions of how the use of word problems affects the learning of algebra, a qualitative methodology was considered suitable. Qualitative research broadly seeks to make sense of the world through the experiences of individuals. In this study the relevant individuals are teachers and their students. Addressing research questions which focus on teaching and learning, however, requires some unravelling of the complex interdependencies between teachers and students. For this reason a multi-faceted qualitative study was designed to examine how the use of word problems affects the learning of algebra from both teacher and student perspectives. This section of the report discusses the qualitative approach used and describes the research tools and their application.

3.2 The interpretive approach

Nuthall, (cited in Malone & Ireland, 1996, p. 123) suggests "that the study of single variables and the production of simple rules for improving teaching are bound to fail because of the enormous complexity of student experiences in the classroom". This view is based on a constructivist framework where students are viewed as the creators of their own knowledge. Constructivists are interested in the meaning-making activities of groups and individuals because it is the process of sense-making that shapes behaviour (Lincoln & Guba, 2000). If the data gathered through research is unique to individuals, then it is difficult to assume from that data that any underlying truth about learning will be revealed. The implication of adopting a constructivist philosophy for researching aspects of teaching and learning is that an interpretative paradigm must be used rather than a normative one.

A normative paradigm is based on concepts of rule-governed human behaviour and investigation through scientific method. The viewpoint of the observer is imposed on the data through the use of external forms and structures, with the cause of the behaviour attributed to factors in the past (Cohen & Manion, 1994). This positivist approach differs to the interpretive paradigm, where the aim is to maintain the integrity of the phenomena under investigation through studying behaviour from within the subjective experience of the individual. From an interpretive perspective, the simplifications of reality upon which the scientific method is based, result in distortions of reality and impractical conclusions (Burns 1994).

Lincoln and Guba (2000) consider that a defining feature of an interpretive paradigm is the inclusion of values, or axiology (the philosophy of ethics, aesthetics and religion), into the inquiry process. The implication is that ethics are embedded within the paradigm, rather than being "defined out of" the inquiry. "The *way* in which we know is most assuredly tied up with both *what* we know and our *relationships with our research participants*." (Lincoln & Guba, 2000, p. 182).

The focus of the interpretive approach is on intentional behaviour, and actions are therefore attributed to future oriented factors (Cohen & Manion, 1994). Methodologies to support an interpretive approach recognise that although categorisation of data is possible, differences remain between subjects in the interpretation of questions and detail of responses given. In terms of this study this reflects the reality that different people will experience and express the same phenomenon of word problems in different ways. Burns (1994) considers that capturing these distinctive insights is one of the primary advantages of qualitative research, as the development of context-bound conclusions may be of greater practical use in advancing educational policy.

Whereas the positivist paradigm aims to produce generalisations, or "assertions of enduring value that are context free", the interpretivist approach holds that no set of generalisations can be both internally consistent and account for all known phenomena. (Lincoln & Guba, 1985, p. 110). Rather, the general is specific to the individual user. Stake (1979, cited in Lincoln & Guba, 1985) proposed an alternative view of generalisation to the logical approach built on accepted statistical methods of sampling and population. This "naturalistic generalisation" consisted of a psychological approach built on concepts of cognition, abstraction and comprehension, functions which reside within an individual rather than external to them.

In line with the interpretive approach is German philosopher Husserl's (1859 – 1935) theory of phenomenology where behaviour is viewed as being determined by the phenomena of subjective experience. Curtis (1978, cited in Cohen and Manion, 1994) identifies the distinguishing features of the phenomenological viewpoint to be belief in the importance of subjective consciousness and its active role in bestowing meaning, and the ability to gain direct knowledge of the "essential structures to consciousness" through retrospective reflection. Rovio-Johnasson (1999) defines the unit of research in phenomenology to be "a way of experiencing something", where the experience contains both "what" and "how" dimensions related to the phenomenon. Anderson (1998) describes phenomenological research as being less concerned with facts (e.g., How did they learn to do algebra?) and more interested in understanding the nature of human activity (e.g., What was the nature of the learning experience?).

Phenomenology proposes that research should attempt to qualitatively describe the direct experiences of students in learning algebra, and teachers in teaching algebra. The objective is to show how something, that on the surface appears to be the same thing, can be experienced or understood by individuals in a variety of ways (Rovio-Johansson, 1999). Applying this approach in qualitative investigation requires extracting perceptions and interpretations of reality, either from within the context of direct experiences or through reflecting on direct experiences, in order to provide understanding (Burns, 1994).

Extending this paradigm into a methodological approach suggests that the research process should be responsive to the situation (Kreuger, 1994). As Kreuger notes, "The inductive properties of qualitative research assume that the researcher makes decisions and refines the quest for knowledge en route" (p. 141). The researcher must be flexible in defining the methodology as the research questions, literature review, theory and research design form interrelated building blocks that both refine the direction during the course of the study, as well as reveal new research questions (Burns, 2000). While the survey instruments employed here to collect data replicate those used by Nathan (2000b; 2000a), methods to record results and analyse the data developed as the research inquiry proceeded.

3.3 Data collection instruments

Vaughn, Schumm and Sinagub (1996) list the key assumptions of the qualitative paradigm as follows: the nature of reality is phenomenological, the relationship between researcher and respondent has potential to influence survey results, and truth is context or perspective bound. Thus in qualitative research "the goal is to describe findings within a particular situation" (p.16). Instruments should therefore aim to capture the views of individuals within context, while minimizing the impact of the researcher on the individuals studied and the collection of data.

The survey questionnaire is the most common qualitative data collection tool and involves asking questions and assessing responses. Surveys can range from simple descriptive data gathering tools that are designed to provide frequency counts on beliefs about a particular issue (ie., What technique was used to solve a problem?), to complex explanatory tools seeking to determine a correlation between an action or context and a particular result (ie. Was the use of that technique successful?) (Burns, 1994). Along this continuum, surveys can be designed to extract the beliefs of the respondent, as well as some understanding of the nature of those beliefs. In order to capture qualitative responses, surveys can employ mechanisms such as open-ended responses, ranking of strength of belief to reveal attitudes, and requests for thought progression such as showing working through the solving of a problem.

Questionnaires can be administered remotely such as a postal questionnaire, or directly by an interviewer, however, they are commonly used in situations where the researcher needs to collect routine data from a large number of respondents, often in several locations. Key concerns in developing a survey include the need to provide a purposeful and ordered structure, to sequence questions in a logical manner, and to pilot test and consequently review the survey tool to improve effectiveness (Anderson, 1998).

With large numbers of students being surveyed in this study, it was appropriate to use a questionnaire in the form of a test that would solicit the algebraic techniques that individual students use. A key limitation of questionnaires in qualitative research, however, is that responses cannot be explored further. Thus with only a small number of teachers being surveyed in comparison to the number of students, this study sort to create flexibility when exploring teacher beliefs and actions through the additional use of interviews.

Interviewing is a useful means of finding out about the attitudes and behaviour of an individual when those things cannot be directly observed (Merriam, 1998). While interviews can be restricted to the use of question and answer, they offer the opportunity to engage in dialogue and explore the responses of individuals in greater depth through the use of follow-up questions. Considerations in developing interview questions include the use of open and closed questions, the need to minimize ambiguity, the choice of sequencing and the use of cards as visual aids to assist the individual to rank information (Anderson, 1998).

Merriam (1998) considers that the value of an interview will ultimately depend on the relevant knowledge of the interviewer and their ability to ask meaningful questions that are easily understood by the participant. To manage bias, this familiarity with the subject must be balanced with distance from the participant so that the interviewer does not simply share assumptions, but is able to ask real questions that explore assumptions and perceptions (Seidman, 1991, cited in Merriam, 1998).

While, however, individual interviews provide opportunity for clarification and elaboration of perceptions through dialogue, the experience is isolated and doesn't allow the individual to compare these views with those of others (Anderson 1998). Mishler (1986) also discusses the danger of interviews that define meaning from respondents' answers where the research situation is isolated from the contextural grounds of mutual understanding that exist in everyday conversations. Mishler emphasizes that a researcher's coding of a respondent's answers, employs implicit assumptions about the relationship between the meaning of the language used by a respondent and the meaning that is inferred. Moreover, these de-contexualised responses are then re-interpreted based on the contextural understandings of the reader. The result may not be unsimilar to a game of Chinese whispers. To minimize this distortionary effect, Mishler proposes "a view of interviewing as a form of discourse between speakers" (p. 7), that is dependent on the researcher and respondent possessing a shared cultural context, which he calls "language competence", in order to improve the attribution of meaning. This approach is concerned with achieving symmetry of power between researcher and respondents through finding ways to provide respondents with "more control of the processes through which their words are given meaning." (p. 7). In terms of survey instruments, this raises the question as to what type of interviews enable respondents to construct coherent worlds of meaning and make sense of their experiences. Mishler proposes the use of 'narrative analysis' where both the interview structure and meaning are jointly developed by the researcher and respondent.

An increasingly common qualitative data collection technique is the focus group which is designed to find out why people think or behave in the way that they do. The focus group expands on the individual interview through facilitating discussion between a group of individuals who share a common experience in relation to the subject being studied (Anderson, 1990). Vaughn, Schumm and Sinagub (1996) claim that the interaction between the researcher and the group adds depth and dimension to the discussion as the researcher is able to exercise leverage through a set of prepared questions and prior developed hypotheses. They go on to note, however, that this opportunity for dynamic interaction is dependent on researcher self-awareness in order to prevent bias occurring. Application of Mishler's narrative analysis approach to the focus group situation could improve the attribution of meaning as respondents are able to refine and clarify their expression of ideas.

Anderson (1998) considers that much of the success of focus groups is due to group dynamics where a chain of reactions between individuals results in exhaustion of the views on the particular focus topic, thereby producing more insightful discussion. Vaughn, Schumm and Sinagub (1996, p. 4) note that this revelation of understanding is dependent on a "permissive atmosphere that fosters a range of opinions". As focus groups are designed to produce collective results, they are complementary to other data collection tools that concentrate on eliciting responses from individuals. This is particularly relevant given that participants may develop or change their views on a topic during the focus group discussion (Vaughn, Schumm & Sinagub, 1996). Vaughn et al. point out that "as long as information gathered from focus group interviews is treated as everyday knowledge rather than as final generalizable research, it can be very useful. The important goal is to hear the voices and viewpoints of the target individuals" (p. 155).

The choice of approach is also related to practical issues of reliability, time and cost. This study involves both students and teachers to investigate how the teaching of algebra affects the learning of algebra. With a focus on achieving reliability of results a phenomenological approach was adopted to survey both student learning of algebra and the teaching of algebra. Triangulation, using two or more methods of data collection to study an aspect of human behaviour, was also employed in the assessment of teachers.

3.4 Student survey

It is intended here that the learning of algebraic word problems is an object of study within a phenomenological setting – students within the math classroom environment. In following an interpretive approach, the type of data gathered for this study is non-experimental. The data gathering instruments follow closely those of Nathan and Koedinger (2000b, 2000a) to facilitate replication of results so that comparisons can be made.

3.4.1 Student survey design

Using a cross section of students, the study aimed to establish how they approached algebraic word problems through examining the actual process of response to a familiar phenomenon, the math test, within a familiar test environment, the math classroom. From the Rovio-Johnasson (1999) perspective, where the experience of learning algebra contains both "what" and "how" dimensions, the student survey should examine both variations in student experiences of approaching algebra word problems as well as variations in student actions taken in relation to algebra word problems. In approaching test questions, students are expected to reflect on their learning experiences in order to determine strategies for solutions.

The reliability of the survey will be dependent on the nature of the students being tested in relation to the student population. With the need to survey students within their classroom environment, a cluster sampling technique was employed that involved selecting entire classes to be tested. To minimise the homogeneity of results that may occur from any one school, a selection of schools was chosen on a stratified socio-economic basis.

The population in this survey is all year 10 students in New Zealand. The sample was taken from the Wellington region, which has diversity in both ethnicity and socio-economic status. State co-educational schools were selected for the sample, to remove the additional factor of potential differences between single sex and co-educational schools. Exploration of how the use of word problems affects the learning of algebra in single sex schools compared with co-educational schools could be the topic of future research.

In order to reduce socio-economic bias, a stratified sample was selected using four high schools in different parts of the region, representing a range of socio-economic status. In New Zealand a decile rating system is used to rank the socio-economic status of each school from one to ten, with one being the lowest socio-economic status and ten being the highest. This system is based on the income brackets of the families represented within each school according to the most recent census data. With the latest census being conducted in 2001, the decile ratings for schools may be out of alignment and are scheduled to be revised by the Ministry of Education by the end of 2002. This means that there is potential for the decile rating to be a somewhat incorrect reflection of the socio-economic background of the students in a particular school. However, the decile rating system is the best available indicator to gauge socio-economic status of schools.

A total of eight schools were invited to participate in the study via letters (Appendix 1; 2; 3) and follow-up phone calls were made to Principals and Heads of Mathematics Departments. Four schools agreed to participate: two schools had a decile rating above five, and two schools had a rating below five. At their own discretion, each school selected two year-ten classes to participate in the study. Participation in the test was voluntary, with information sheets and consent forms (Appendix 4; 5) administered for both students and parents combined.

3.4.2 Student Data Collection

The purpose of the student survey was to assess the strategies employed to solve algebra problems and the relative success of different strategies. Students were presented with a questionnaire to be completed within a standard class time period. The questions were designed to examine both mathematical structures and presentation formats. To address issues of potential bias the questions contained a variety of contexts, numbers used, and mathematical structure. A set of 27 questions was developed combining three different mathematical structures (result unknown, start unknown, and multiple unknowns) with three presentation formats (equation, word, story) and three different sets of numbers.

For example, a word problem with start unknown would be shown as: *Starting with some number, if I subtract 64 and then divide the result by 3, I get 20.5. What number did I start with?* In comparison, a story problem with multiple unknowns could be expressed as: *To rent a car from Tiger Motors costs $100 per day and 20 cents per km. To rent a car from Kiwi Motors costs $120 per day and 15 cents per km. For what distance is each company the same price?* Finally, an equation problem with result unknown would be shown in the following format: *Solve for x where,* $130 \div 5 + 3 = x$.

From this set, three tests were drawn (A, B and C), each consisting of nine questions; one question of each combination of structure and format, with no number combinations repeated. The different question types were arranged in a different way within each test to reduce any potential influence from question order. The complete set of 27 test questions is provided in Appendix 6. Within each class, students sat one of the three tests, with one third sitting test A, one third sitting test B and one third sitting test C. Each questionnaire was presented to a single class where all students had received the same teaching. To capture variations in student approaches and actions to solving the algebra problems, students were asked to show their working.

In order to control for any potential effects that may result from the use (or non use) of calculators, only one class in each school was permitted to use calculators during the test. The tests were administered by the class mathematics teacher and returned in class sets to the researcher for analysis.

3.5 Teacher Survey

The purpose was to determine the nature of teacher beliefs and attitudes as formed through subjective experiences about the teaching and learning of algebra. A phenomenological approach would seek to achieve this within the teaching and learning environment of the classroom. Assessing teachers within the context of the classroom, however, is subject to bias as the presence of the researcher in the room will impact to some degree on what is taught and/or how it is taught as the teacher attempts to control the views of the researcher about that teacher. It is also likely to affect the behaviour of the students in the class.

Burns (2000) discusses a range of factors that may influence how teachers participate and interact with one another in a survey situation, including priorities, concerns and personal constructs. The use of cluster sampling in this survey meant that the teachers surveyed included both senior managers (Heads of Mathematics Departments) and their teaching staff. For this reason it was important that teachers be examined individually in order to extract actual beliefs and attitudes about teaching and learning algebra.

A range of alternative methods were used to assess the views of teachers on the teaching and learning of algebra: text books used by the teachers for teaching algebra were examined as an influence on how and what may be taught, teachers were requested to complete a questionnaire, and finally teachers were invited to participate in a focus group. Both the questionnaire and the focus group provided opportunities for teachers to retrospectively reflect on the nature of their experiences in teaching algebra.

3.5.1 Teacher questionnaire design

The purpose of the teacher questionnaire was to evaluate beliefs and attitudes towards the teaching and learning of word problems in algebra. While a belief held about teaching algebra can be considered to be a cognitive activity, attitude builds on this belief through applying an affective component related to what the teacher feels about teaching algebra. Measurement of beliefs and attitudes is commonly achieved through the use of attitude scales that allow the individual to indicate the level of favourability they feel toward a particular object (Burns, 2000). The final aspect in assessing beliefs and attitudes is the behavioural component, or how the teacher actually teaches algebra. Burns suggests that the behavioural component may not align with expressed attitudes due to compounding factors such as social constraints. The behavioural component is not directly assessed in this research, however the student test results are likely to reflect to some degree the actual teaching methods employed by the participating mathematics teachers.

Mathematics teachers in each of the sample schools were invited to complete a questionnaire developed by Nathan (2000b) (Appendix 7) consisting of 47 statements. This questionnaire had been tested by Nathan through presentation to a sample set of teachers, and refined in order to improve the accuracy in measuring individual attitude constructs. The questionnaire employed a six point Likhert scale which presented statements in both positive and negative formats to prevent bias occurring. The Likhert scale required the teachers to rate each statement from strongly agree to strongly disagree, in order to create an index of the emotive value of, or attitude toward, the affective component. Burns (2000) considers that one of the advantages of the Likhert scale is the homogeneity in scales and increased likelihood that a unitary attitude is being measured.

Of the 47 statements, 44 could be categorised into one of six constructs that provide insight into that teacher's belief about the most effective way to teach algebra and solve word problems for which algebra could be used. These constructs are:

1. *Product over process.* Four statements were provided to determine the strength of teacher belief that emphasis should be placed on correct answers rather than a student's reasoning processes.
2. *Invented solution.* Eight statements explored teacher beliefs that students are able to learn and invent effective methods for solving problems that may differ from those taught.
3. *Teachers should encourage invented solution methods.* Nine statements examine the construct that students come to class with prior knowledge and should be encouraged to build on that knowledge to develop for themselves effective problem solving approaches.
4. *Algebra is best.* Twelve statements were included to assess teacher beliefs that algebraic procedures are the most effective and best methods for solving algebra level word problems.
5. *Alternative methods indicate knowledge gaps.* The questionnaire included seven statements to examine the construct that informal methods such as arithmetic and guess-and-check demonstrate that a student's knowledge is inadequate.
6. *Symbol precedence view.* Four statements explored teacher beliefs that symbolic questions are easier than word problems, arithmetic problems are easier than algebra problems, and therefore algebra word problems should be pre-empted by symbolic problems and taught at the end of the algebra topic.

The remaining three questions, although not falling directly under any of the above constructs, provide further insight in the beliefs teachers have about teaching in a constructivist manner and the perceived difficulties students have in using algebra to solve problems.

Bias can be generated through how teachers interpret the constructs presented to them. To minimise the effect of this type of bias, the questionnaire contained a variety of similar statements for cross-referencing. This type of redundancy is often used to check for internal consistency (Burns, 2000).

3.5.2 Focus group design

Anderson (1998) notes that although focus groups are informal discussion groups, they need to be carefully planned and moderated, particularly in terms of environment and group composition. Vaughn, Schumm, and Sinagub (1996) suggest that eight to ten participants is the best group size in order to create sufficient synergy in group discussion. While in agreement with this view, Anderson considers that an exception exists where in-depth exploration is required and the participants have substantial experience on the topic. In this situation, Anderson states that mini focus groups consisting of about three to five participants may be preferable.

The group, in this situation, was composed of teachers with the common characteristic of teaching algebra. The levels of teaching experience varied from first year teacher to over ten years. Mathematics teachers at the schools involved in the study were invited to participate in one of two focus groups (Appendix 8; 9), one for the teachers of schools with a decile rating above five and one for schools with a decile rating below five. The environment needed to be natural, ensuring the teachers were relaxed and secure, and therefore able to share both positive and negative comments (Anderson, 1998). For this reason the focus group sessions were held at schools participating in the study. The teachers were not paid to participate in the study, primarily because the research was viewed as potentially being of future assistance to them.

At the focus group sessions, each teacher was presented with a sample of nine questions taken from the student tests. An example of the format used is provided in Table 3.1.

Table 3.1 Sample of student questions presented to teachers

	Equation	Word-equation	Story
Result-unknown	Solve for x where, $(154.50 - 120) \div 0.15 = x$	Starting with 154.50, if I subtract 120 and then divide by 0.15, I get a number. What number do I get?	A rental car costs $154.50. If there was an initial out lay of $120 and then it cost $0.15 per kilometre, how many kilometres was the rental car driven?
Start-unknown	Solve for x where, $0.15x + 120 = 154.50$	Starting with some number, if I multiply it by 0.15 and then add 120, I get 154.50. What number did I start with?	Kiwi rentals charge $0.15 for every kilometre travelled in addition to an initial outlay of $120. If a customer is charged $154.50, how many kilometres did the car travel?
Multiple-unknowns	Solve for x where, $0.20x + 100 = 0.15x + 120$	Starting with some number, if it is multiplied by 0.2 and then added to 100, the result is the same as when it is multiplied by 0.15 and then added to 120. What is the number?	To rent a car from Tiger Motors costs $100 per day and 20 cents per km. To rent a car from Kiwi Motors costs $120 per day and 15 cents per km. For what distance is each company the same price?

The sample used the same numbers and mathematical structure for each sample test question across the constructs of symbolic-equation, word-equation, and story problem to control for the possibility of bias from different interpretations of constructs. The teachers were first individually asked to rank these questions in order of difficulty. Following this exercise, the teachers were asked to participate in a group discussion. Topics for discussion presented by the researcher included: the nature of the difficulty for students in each question, the methods possibly used by students to answer the questions, the techniques used to teach algebra, and in particular the role of word problems in the teaching of algebra.

3.5.3 Teacher data collection

A significant potential limitation on the results of the teacher questionnaire is related to the opportunity for self-selection and the corresponding impact on response rates. The teacher questionnaires were returned voluntarily to the researcher by post in self-addressed stamped envelopes. The response rate was sixty percent. The possible impact of this response rate on results is moderated by the use of triangulation in this study. Each teacher's score was tabulated by assigning numerical values to each answer starting at 1 for strongly agree, up to 6 for strongly disagree. The responses for each question were then collated across all teachers to determine the direction of beliefs and attitudes about teaching and learning algebra.

Each teacher independently categorised student test questions according to the perceived level of difficulty faced by students, before coming together as a group to discuss their viewpoints. This capturing of first impressions from individuals assisted the researcher to identify where changes of opinion occurred during discussion, and counteracted where the discussion may have been dominated by the opinion of one person. It also assisted in limiting the possibility for self-bias of the researcher to affect results due to both the knowledge and beliefs of the researcher about the teaching and learning of algebra, and the accumulated subjective experiences from over ten years of teaching algebra.

Given the inflexibility of teachers' time availability, the focus groups were held in the afternoon following the completion of the teaching day. This is likely to have impacted on the length of the focus groups and participation rates. The focus groups were approximately 45 minutes in length which is half the typical length of 1.5 to 2 hours as stated in Vaughn, Schumm, and Sinagub (1996), however, this may have been linked to the smaller group size of four teachers in each focus group. The focus group sessions were recorded by audiotape and later transcribed by the researcher. The results of both the teacher questionnaire and focus groups are discussed in section 5.5 Teacher Beliefs.

3.5.4 Text book survey design

The purpose of the text book survey was to assess the potential link between the presentation of algebra in text books, the methods used by students to solve algebraic problems and the beliefs held by teachers about teaching algebra. The manner in which algebra was presented in each text book was the object of investigation. To the extent that different schools may use text books with different approaches, evaluating the text books is a means of assessing their potential for influence at each school. It is also a useful tool for triangulation of the teacher survey, in assessing whether teacher beliefs and attitudes align with the text book approach about teaching algebra.

Textbooks used to teach algebra to year 10 students at each of the four participating schools were selected for analysis. Within these schools, only three different textbooks were used, with publication dates of 1989, 1996 and 1998. The books were assessed according to the contents and order of presentation with respect to algebra, the use of symbolic and word problems, single unknowns and multiple unknowns. The results of the textbook survey are discussed in section 4.3.3 Textbook Analysis.

3.6 Analysis

The basic framework used for evaluating the survey data involved separately recording and assessing the results from each survey tool: student questionnaire, teacher questionnaire, teacher focus group and text book survey. This allowed the identification of common themes across the results for further evaluation in conjunction with theories discussed in the literature review. A guiding principle throughout the analysis was to align the depth of analysis of the various result areas, with the purpose of the research questions (Kreuger, 1994).

3.6.1 Student questionnaire data evaluation

Both the strategies and the errors made by students in solving the problems were coded and documented following frameworks presented in the literature and as used by Nathan and Koedinger (2000a; 2001). The results for the classes that used calculators were assessed separately.

The main research question is: are the students using algebraic technique to solve word problems, and if so, are they successful? The strategies used by students to solving the algebraic problems were classified to determine patterns of behaviour and preferential approaches. Four strategies were identified as follows: algebraic, back tracking, arithmetic, and guess and check.

The analysis first calculated overall success rates for each combination of mathematical structure and presentation format. This was followed by examination of which combinations of mathematical structure (result unknown, start unknown, and multiple unknowns) and presentation format (equation, word, story) the students were most successful at. Errors were classified by mathematical structure.

The data analysis examined the strategy used and whether the student was successful. Strategies were assessed separately by mathematical structure and presentation format to see what techniques the students used on different formats. If students weren't successful in their chosen strategy, the researcher investigated whether particular errors occurred in relation to particular strategies. The types of errors identified were classified into: rules based (bad algebra, bad arithmetic), process based (order of operation, inverted operation, missing operation), comprehension, copy slip, give-up, and answer only.

3.6.2 Teacher survey data evaluation

The beliefs and attitudes of mathematics teachers about the teaching and learning of word problems were classified to identify likely behaviour patterns and dominant themes. This was achieved through: calculating mean scores and standard deviations for the six statement constructs in the teacher questionnaire to assess the most strongly held beliefs and attitudes. The aim was to assess whether the teachers had a constructivist perspective to teaching algebra by assessing the extent to which teachers agreed with statements that promoted a constructivist view. The constructs in the teacher questionnaire that are considered to oppose a constructivist view include: product over process, algebra is best, alternative methods indicate knowledge gaps, and symbolic precedence view.

The focus group session built on the teacher questionnaire through discussing which combinations of presentation format and mathematical structure the teachers thought the students would be more successful at, and why. The analysis involved coding both the individual teacher ranking of sample constructs and the group ranking of sample constructs. Krueger's (2000) approach was adopted to assess the reasons given by teachers for the ranking of these constructs during the group discussion. This involved identifying the big ideas – those that cut across the discussion; considering the words used to express meanings, and the frequency, intensity and specificity of comments; evaluating the influence of specific context within teacher responses; and gauging the consistency of responses throughout the interview in terms of the relationship with initial positions as determined by the individual ranking of beliefs.

3.6.3 Textbook survey evaluation

The textbook approaches to presenting algebra were examined to isolate the potential influence on teachers and students. Textbook algebra is usually presented in two main topics: unknowns (collecting like terms, manipulation, factoring, expanding, solving), and variables (patterns and graphing relationships). Within the topic on unknowns, the location of the story and other kinds of word problems was assessed. Story problems have traditionally been considered to be an application of the symbolic, so analysis aimed to determine whether the presentation format was influencing the symbolic precedence view that appears to be predominant among teachers. Investigation sought to determine where the story and other kinds of word problems were located in relation to symbolic manipulation problems. Did the textbook start with symbols or context based situations? Within linguistic situations, did the textbook use story problems or word-equation problems? The researcher also looked for commonalities in both presentation and themes to determine the extent that any common approach was purported by all the textbooks used.

3.6.4 Discussion of results

The survey findings are discussed in section 5 Discussion of Results. This section compares the results from the surveys of students, text books and teachers to look for commonalities and disparities. The discussion of results centres around student's choice of strategies for solving algebraic problems. It is structured around the following themes: the influence of word problems, the influence of problem complexity, the role of calculators in algebra, the effect of teacher beliefs and attitudes.

3.7 Validity and Reliability of Results

The constructivist view holds that criteria for judging validity "are derived from community consensus regarding what is 'real', what is useful, and what has meaning." (Lincoln & Guba, 2000, p. 167). Truth, from this perspective, arises from the relationships and consequent narrative between members of a stakeholding community, where negotiations or dialogue result in agreement about what is valid knowledge. Given, however, the interpretivist goal to influence policy or behaviour through research results, validity is all the more important. The interpretivist approach to validity contrasts with the positivist paradigm, by emphasising the process of interpretation over the belief that method alone can deliver truth (Lincoln & Guba, 2000). The application of "interpretive rigour", therefore, involves employing criteria that focus on the processes and outcomes of the inquiry rather than the application of methods. Such criteria aim to achieve validity and reliability of inquiry results to the extent that the *reader* can assess whether the results are plausible and/or generalisable to a similar situation.

3.7.1 External validity

External validity is concerned with the extent that survey results are generalisable. Given the potential lack of equivalence in the selection of individuals and contexts for this study, measuring external validity through population, sample size and sampling technique is difficult (Merriam, 1998). Merriam discusses a range of alternative concepts of generalization in a qualitative study including: the *working hypothesis* which translates observations in context, considering both controlled and uncontrolled variables; *concrete universals* determined by comparing detailed case studies and transferring the lessons learnt to similar situations encountered subsequently; *naturalistic generalisation* through looking for patterns in particular contexts that can be applied to other situations; and *user generalisability* that allows the reader to determine the relevance and applicability of the findings to other situations. Suggested strategies to enhance generalisability include: providing sufficient description for readers to assess the relevance of findings, describing how typical the situation and individual are, or using several sites to maximise diversity in the context of the object being studied. This study employed a range of these techniques to improve the generalisability of results.

In this survey the sample of four schools represents a range of population demographics of New Zealand secondary schools. The sample was drawn to provide diversity in socio-economic groups through the choice of decile rating, however all the schools were within an urban area. Each of these schools were also coeducational to prevent any gender imbalance. The average size of school roll was 650 students. External factors that may have interfered with student's success in the questionnaire include the ability to comprehend word problems, the context given, and confidence with the kind of numbers used in each question. To the extent that the schools in the sample were not reflective of the population in terms of comprehension and numeracy, the generalisability of the results will be compromised.

3.7.2 Internal validity and reliability

There are three aspects to achieving internal validity and reliability: Do the research findings accurately represent the object studied? Was the object representative of all other similar objects, or, a true representation of reality? Would the same results occur if all or part of the study was repeated? Perhaps the overriding concern is about whether research findings are a true representation of reality. Given the phenomenological view that reality is multidimensional and specific to the individual, there is no one single reality that can be assessed as to its "truth". Validity is instead found by observing multiple sets of individuals' constructions of reality (Lincoln & Guba, 1985, cited in Merriam, 1998). Merriam, (1998) considers the fact that interpretations of reality can be accessed directly through interviews and observations as being one of the strengths of qualitative research. The goal then is to both understand and capture the perspectives of the individuals studied and present a holistic interpretation.

Accurate understanding, interpretation and reporting of results is essential to ensure internal validity. Such recording and reporting is, however, inherently subject to there being a cultural understanding between the researcher and the individuals studied (Mishler 1986). Mishler discusses a process identified by Labov and Fanshel (1997, cited in Mishler, 1986) called "expansion" where the researcher draws on all the available information, including facts, personal experiences, assumptions and shared knowledge, to develop meaning from interview discourse. The achievement of deeper understanding, and identification of linkages across contexts, is likely to reduce bias introduced from subjective interpretations of the researcher.

Traditionally it is assumed that internal validity is increased if replication of a survey, or observations within a survey, produce the same results. The advantage of achieving such reliability is the ability to generalise the research findings beyond the specific inquiry. If, however, an observation is dependent on the context from which it is generated, then reliability is not related to the number of observations or replication of results (Merriam, 1998). Technically, the formal survey instruments used in this study can be assessed in terms of reliability in the same manner as for quantitative research. However, the results of this entire survey are not generalisable in the sense that replication would yield the same results as the links between the teachers, students and text books are to some degree particular to the survey context.

Lincoln and Guba (2000) propose an alternative measure of reliability for qualitative research, that of dependability or consistency of results. They define the term "fittingness" as a measure of the degree of congruence between the contexts of the sender (researcher) and the receiver (reader). If there is sufficient congruence between these contexts, the research results may be transferable to the receiver's context. From this proposition the researcher should provide information about the context of the inquiry, and the survey results should be consistent with the data collected. This process should allow the reader to make an informed judgement as to the transferability of the results to another situation. Methods to facilitate this approach include defining the researcher's position with respect to the subject under investigation, triangulation, and creating an audit trail of the research which explains how the results were arrived at through data collection, evaluation and interpretation (Lincoln & Guba, 1985).

3.7.2.1 Position of the Researcher

This researcher has been teaching secondary school mathematics for over ten years. Personal experience of school environments encompasses five schools and four distinct regions in the north island of New Zealand, ranging from rural to large city, and decile one to decile ten. This broad ranging exposure has contributed to developing an in-depth appreciation of the interdependencies between culture and learning, the various and community specific external pressures on secondary teachers in the classroom, and of the variety of teacher knowledge and skill levels in relation to teaching mathematics and, in particular, algebra. From this basis, the researcher claims a shared cultural knowledge with secondary mathematics teachers and students, from which to understand and interpret teacher and student responses.

3.7.2.2 Triangulation

Triangulation is a commonly used technique to improve internal validity. Burns (1994) notes that triangulation can check both the consistency of results from different data-collection methods, and the consistency of different data sources within the same method. Burns considers that the collection of at least three different viewpoints in a study on teaching situations, such as teacher, student and observer, ensures that the evaluation reflects the multiple realities that exist in the social context.

This study approached the context of teaching and learning algebra from three separate, but interlinked, angles: teachers, their students, and the textbooks used in the classrooms. It was assumed that teachers use textbooks within the classroom in different ways, with some using them as an explicit teaching guide, and others only for student exercises. To control the effect of school resources on teacher views, behaviours and student learning, the mathematics textbooks utilised in the participating schools were reviewed with regard to how they presented algebra. The mathematics teachers were surveyed individually using a questionnaire to elicit their beliefs and attitudes on the role of word problems in algebra, and focus group interviews provided the researcher with an opportunity to develop a deeper understanding of these teacher views. Finally, the student test results provided material against which the researcher was able to compare the beliefs of teachers, and the success of the teaching methods employed, and any linkages between textbook presentation and the teaching and learning of algebra.

3.7.2.3 Audit trail

This study aims to describe the processes used in the survey instrument design, application, data collection and evaluation in order to increase the transparency of the assumptions and decisions made by the researcher as the inquiry progressed.

3.8 Limitations

In general the presence of bias will affect the generalisability of the results. While steps were taken in this study to remove bias wherever possible, there are potential elements of bias that remain, and these will constrain the generalisability of the results obtained to some degree. The extent to which the results are affected is dependent on the nature and scope of the bias unaccounted for.

The key areas where unaccounted bias may lie in relation to this study are potential sources of bias rather than known causes. The sample of schools was taken from a single geographical area, which will create bias to the extent that schools in the Wellington region are not representative of all schools in New Zealand. While there may be differences in schools across the country, there is no evidence that the teaching and learning of algebra is significantly different across New Zealand.

There is potential for differences to lie between the urban nature of the area from which the schools were chosen and rural schools, and variations in the ethnic composition of schools in the sample as compared with schools across New Zealand. The population of New Zealand is becoming increasingly multicultural, with the additional factor of foreign students at the secondary school level in particular schools. The range and diversity in ethnicity, however, is not evenly spread throughout the country, with the Auckland region comprising the maximum diversity and the Southland region the minimum diversity. Given both the central geographical location and moderate size in terms of population, and the fact that Wellington is not overly affected by concentrations of foreign students, it could be argued that the Wellington region provides for ethnical variation but is not necessarily typical of other regions.

In order to meet the ethical requirements of this research, the students and teachers were allowed to self-select. The numbers of student questionnaires received indicated a relatively high response rate. The teachers were able to self-select for both the questionnaire and the focus group. It is not known if the teachers that completed the questionnaire also participated in the focus groups or not.

The presence of the interviewer in the focus group interviews may have affected the teachers and potentially their responses (Krueger, 2000; Lincoln & Guba, 1985; Merriam, 1998; Vaughn, Schumm & Sinagub, 1996). Knowledge that the interviewer was also a colleague, (albeit unknown), is expected to have reduced this potential influence to the extent that while the interviewer was not known to the teachers, the common bond of being a mathematics teacher existed between them. This type of shared understanding assists in creating sufficient rapport so that individuals can share relatively freely.

The extent to which findings were specific to a particular class rather than generalisable are limited by the number of classes and teachers surveyed, and also the consistent use of a dominant text book in all the schools surveyed.

3.9 Ethical Considerations

A number of ethical considerations were addressed during the design and implementation of this study. The key issues involved ensuring that: participation in the study was voluntary, all participants were well informed of the purpose and use of the results from the study, the privacy of individuals was maintained, and participation in the study did not detrimentally impact on individuals. In addition, the values and subjective experiences of the researcher, regarding the teaching and learning of algebra, play a role in this inquiry.

3.9.1 Participation in the study

Participation in this study was voluntary at all levels from school to student involvement. Schools were invited to participate in the study by the researcher. Teachers within the sample were then asked to participate in the survey. Completion of teacher questionnaires was voluntary, with informed consent implied through the completion and return of the questionnaire. Teachers that participated in the focus group interviews completed a consent form.

Within each class selected for the student survey, each student and also their parents/guardians were required to complete a consent form that agreed to participation in the survey and allowed the test results to be released for research purposes. While the test items presented to students were considered to be part of their regular course of study, the students were informed of the specific purpose of the test items in relation to this study.

3.9.2 Privacy of individuals

The researcher was defined as the only person to make contact with the schools and teacher participants during the course of the project, with all materials kept confidential. The names of participating students, teachers, and schools have not been identified in the text of this thesis nor will they be identified in any subsequent papers. Where any direct quotations have been used, the participant has been assigned a code name where appropriate. The information gathered focuses on gleaning insight into how word problems are viewed and used, not identifying any individuals.

3.9.3 Impact on participants

It was important that no harm to students occurred through participation in this study. For this reason it was ensured that the level of difficulty and form of the test problems presented to students was common to year ten courses of study. To minimize any self-esteem issues that may be caused by test anxiety or failure, participating students were instructed that the test questions were voluntary, and the test results did not contribute to any class assessment.

The characteristics of the problems presented in the student tests paralleled those of most year ten text books used in New Zealand schools. While the student test was administered during class time, the flexibility in most year ten programmes, and the potential for the test to provide an appropriate teaching vehicle for algebra at year ten, meant that the impact of displaced class time was minimal.

4 RESULTS

4.1 Introduction

This section discusses the results from each of the survey tools: student responses to the test, teacher responses to the questionnaire, teacher focus groups, and textbook analysis.

4.2 Student performance

The strategies used and the errors made by students in solving the test questions were coded and documented using the strategies outlined in the literature. The terminology used by the researcher is identical to that used by Koedinger, Alibali, and Nathan, (2001). The codes listed below for both strategies and errors are used throughout to document the results.

4.2.1 Student strategies

Each student strategy was coded as follows:

Code	Strategy
A	Algebraic
BT	Back Tracking
+	Arithmetic
GC	Guess and Check

Algebraic strategy

The setting up of equations with symbols for unknowns to show both intermediate steps and the stated solution is classed as algebraic. The solving of equations, by means of transposing or balancing, falls under this category. The predominant algebraic response observed for solving of equations was that of transposing although many responses appeared to employ both approaches.

Eg:

> Starting with some number, if it is multiplied by 0.2 and then added to 100, the result is the same as when it is multiplied by 0.15 and then added to 120. What is the number?

$$x \times 0.2 + 100 = x \times 0.15 + 120$$

$$0.2x + 100 = 0.15x + 120$$

$$0.2x - 0.15x = 120 - 100$$

$$0.05x = 20$$

$$x = \frac{20}{0.05}$$

$$x = 400$$

Back-tracking strategy

Back-tracking defines where transposing takes place mentally and only arithmetic calculations and workings are shown. Back-tracking is sometimes referred to as unwinding or untangling to indicate that calculations are sometimes carried out in an order other than an exact opposite to the correct order of operations.

Eg.

> Betty won some money in a lottery. She kept $64 for herself and gave each of her 3 sons an equal amount of the rest of it. If each son got $20.50, how much did Betty get?

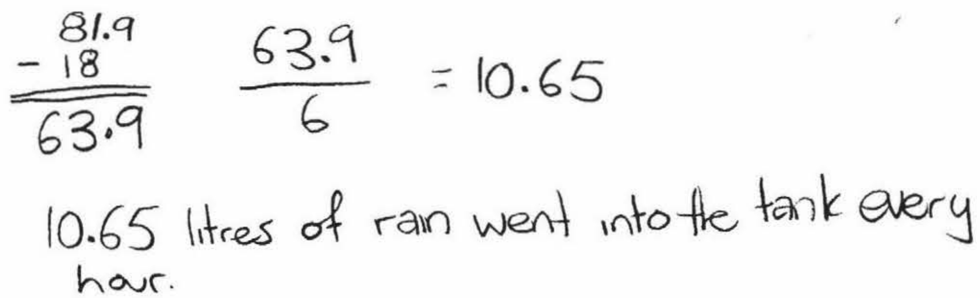
$$\begin{array}{r} 20.50 \\ 20.50 \\ +\ 20.50 \\ \hline 61.50 \end{array} \qquad \begin{array}{r} 61.50 \\ +\ 64.00 \\ \hline 125.50 \end{array}$$

Arithmetic strategy

Arithmetic consists of working forwards using arithmetic operations. Techniques were only coded as arithmetic if it was obvious through the workings shown that this was the method used. Many solutions provided by students in the tests where calculators were permissible may have used arithmetical techniques but did not show any evidence for it.

Eg.

A rain water tank contains 81.9 litres after a rain fall one night. If there was originally 18 litres in the tank and it rained for 6 hours. How much rain per hour went into the tank that night?

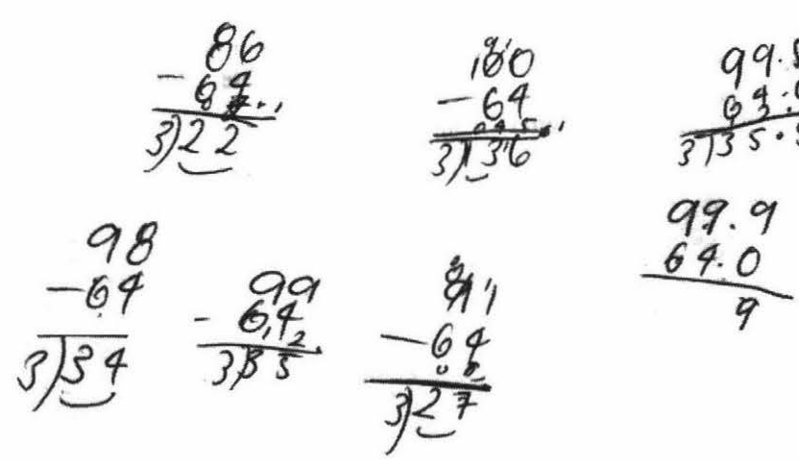


Guess and Check

Guess and check involves the substitution of values either systematically (modelling) or randomly. This strategy manifested in a variety of forms including unsystematic oscillations across the desired result, systematic Tables and graphs. Examples of these are shown below.

Eg. Unsystematic oscillations across the desired result.

Starting with some number, if I subtract 64 and then divide the result by 3, I get 20.5. What number did I start with?



Eg. Systematic Tables.[2]

To rent a car from Tiger Motors costs $100 per day and 20 cents per km.
To rent a car from Kiwi Motors costs $120 per day and 15 cents per km.
For what distance is each company the same price?

	Tiger motors	Kiwi Motors
1 km	$100.20	$120.15
2km	$100.40	$120.30
4km	$100.80	$120.60
6km	$101.20	$120.90
8km	$101.60	$121.20
20km	$104.00	$124.80
22 km	$104.40	$125.10

both companys are athe same price at 400km

working on Back

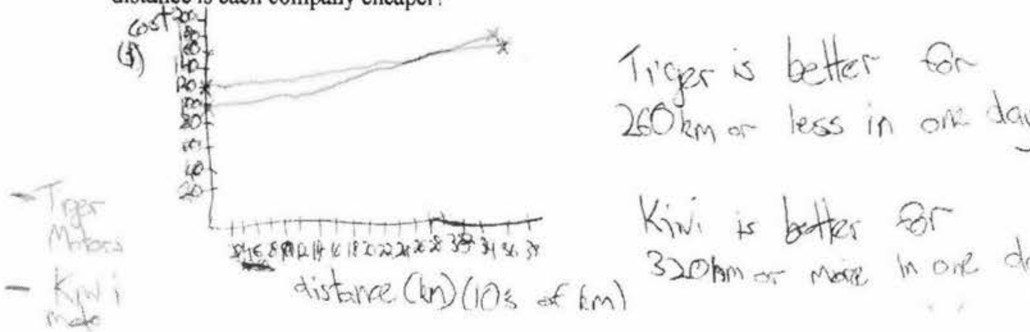
question 4

	Tiger	kiwi
24 km	$104.80	$125.40
26 km	$105.20	$125.70
28 km	$105.60	$126.00
30km	$106.00	$126.30
32 km	$106.40	$126.60
34 km	$106.80	$126.90
36 km	$107.20	$127.20
38 km	$107.60	$127.50
40 km	$108.00	$127.80
80km		

[2] This student response was split over two pages.

Eg. Graphs[3]

To rent a car from Tiger Motors costs $100 per day and 20 cents per km. To rent a car from Kiwi Motors costs $120 per day and 15 cents per km. For what distance is each company cheaper?



4.2.2 Errors

If a response was incorrect then the error made was coded and documented. In the case of multiple errors being made within a single question, the error considered to be the most influential in preventing success was coded. Errors were coded as described in Table 4.1 (Koedinger, Alibali & Nathan, 2001) with examples from the current research presented below.

Table 4.1 Error Codes

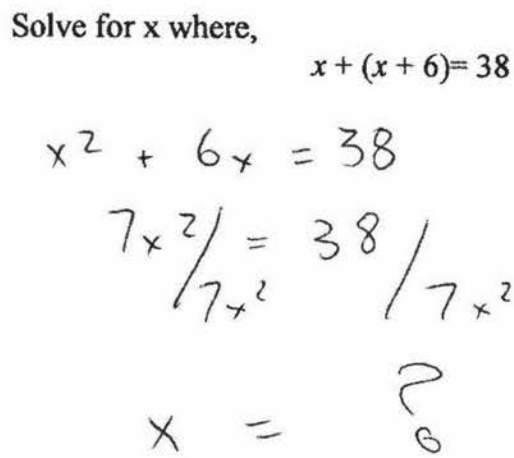
Error		Description
BA	Bad Algebra	Incorrect use of symbols and algebraic rules.
B+	Bad Arithmetic	Arithmetic error (e.g. addition or multiplication)
OO	Order of Operations	Incorrect order of operations carried out
IO	Inverted Operation	Operation is the opposite of that required (e.g. adding instead of subtracting)
MO	Missing Operation	Forgetting to carry out one or more of the required operations, yet giving an answer.
C	Comprehension error	Failing to understand the question and carrying out a series of redefined calculations.
CS	Copy Slip	Incorrect transfer of information from one solution step to another.
GU	Give Up	Working shown but no conclusive answer given
AO	Answer Only (incorrect answer)	A numerical answer with no other markings

Bad Algebra error

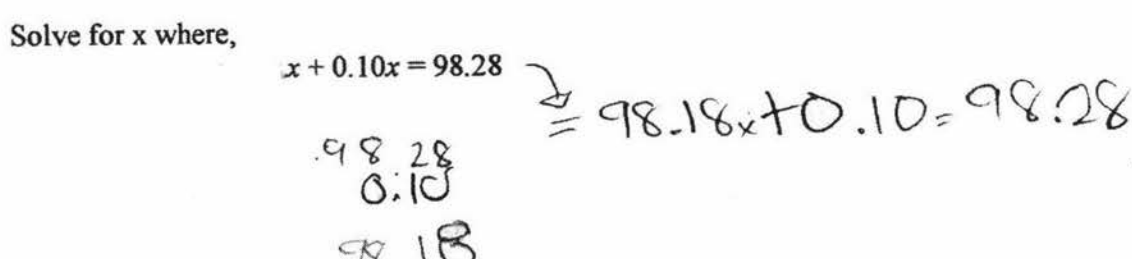
[3] The graphic solution occurred twice during testing of the student survey tool, however did not appear in the final sample evaluated.

There are many errors that a student can make because they do not know or understand the language of algebra. In many cases symbols are manipulated according to half remembered rules or made-up rules without regard to any meaning. Three different examples of bad algebra from the student tests are presented below.

1. In this example the brackets are expanded as though the operation is multiplication between the outside *x* and the brackets. Additionally, the terms x^2 and 6x are conjoined to become $7x^2$.



2. The following example shows where the letter has been ignored.



3. The following error shows the use of the same symbol (x) to indicate different quantities and the misuse of the equals sign. This error appears only in the problem translation phase, and did not influence the student's calculation of the correct solution. Therefore it was not included as an error statistic.

> Starting with some number, if I subtract 64 and then divide the result by 3, I get 20.5. What number did I start with?

$$x - 64 = \frac{x}{3} = 20 \cdot 5$$

$$= 20 \cdot 5 \times 3 = 61 \cdot 5$$

$$= 61 \cdot 5 + 64 = 125 \cdot 5$$

$$= x = 125 \cdot 5$$

Order of Operations

This error relates to when the student carries out operations in the wrong order. In the example given, the student set up their calculation without using brackets. A calculator was then used which carried out the calculations according to the way they have been presented. The student's error is that of an improper set up of the correct order of operations rather than an error in the calculations. An example is provided below.

> **Starting with 81.9, if I subtract 18 and then divide by 6, I get a number. What number do I get?**

$$81.9 - 18 \div 6 = 78.9$$

Comprehension error

Comprehension error involves the student failing to understand the question and carrying out a series of calculations based on an incorrect interpretation of the question. Two examples are provided below.

1. In this example the numbers are operated on without understanding of meaning.

Brian is in a rowboat on a lake. He is 800 metres from the shore. He rows towards the shore at a speed of 30 metres every minute. How far is Brian from the shore after 23 minutes? 8.53mins.

800
30
23
853

2. Part of the text has possibly been poorly read and translated syntactically, reading the 18 litres as 18 litres per hour for 6 hours.

A rain water tank contains 81.9 litres after a rain fall one night. If there was originally 18 litres in the tank and it rained for 6 hours. How much rain per hour went into the tank that night?

^{4}18
× 6
108 litres

Copy Slip

A copy slip is defined as where an incorrect transfer of information from one solution step to another occurs. In the example below the numeral 3 is copied as 5 by the student.

Eg.

Solve for x where,

$$130 \div 5 + 3 = x$$

$$130 \div 5 = 26$$
$$x = 26 + 5$$
$$x = 31$$

4.2.3 Student responses without the use of calculators

Of the 136 students surveyed in total, 71 students sat the test without calculators: 26 students for Test A, 21 students for Test B, and 24 students for Test C. The purpose for not allowing the use of calculators was to solicit greater written responses in order to establish more readily the strategies used. The 71 test papers contained 9 questions each, producing a total of 639 questions. Only 71% of the questions were answered with 29% being left blank. Out of the 71% answered, 40% were result unknown questions, 34% were start unknown questions, and 26% were multiple unknown questions. Of the total responses only 23% of the questions were answered correctly. The overall results for students without calculators are shown in Table 4.2.

Table 4.2 Overall success of student responses without calculators

N = 639	%	count
Correct	23	147
Incorrect	48	306
No Response	29	186
	100	639

Correct responses

The 23% of correct responses relates to 147 questions. Of the 147 correct responses, the percentage of successful responses for each kind of question is shown in Table 4.3[4]. The students found greatest success in solving the result unknown mathematical structure and the story problem presentation format. It is interesting to note that the start unknown story problems is the variation that met with the most success, with 25% of correct solutions.

Table 4.3: Percentage of correct responses for each question type without calculators

N =147	Equations	Word equations	Story problems	Total %
Result unknown	15	20	11	46
Start unknown	7	8	25	40
Multiple unknown	4	4	6	14
Total %	26	32	42	100

Errors without calculators

The 48% of incorrect responses relates to 306 questions. Out of the 306 incorrect responses, the percentage of each kind of error made is shown in Table 4.4. Within this set of results, no copy slip errors were made[5]. A greater number of errors were made by students in answering the result unknown questions (37% of all errors made). However, more result unknown questions were attempted than any other kind. Within the result unknown questions the most common error made was that of bad arithmetic (10% of all errors made). The most common error type overall was that of comprehension (20% of all errors made). More comprehension errors occurred on multiple unknown questions than any other kind.

Of the incorrect responses, 29% came from students who showed no written workings, giving a single numerical answer only. One inverted operation error and one missing operation error were made on the multiple unknown questions, but these count for less than 1% of errors.

---

[4] All values contained within Tables are percentage values.

[5] Some Copy Slip errors (1%) were made in the student tests with calculators available.

Table 4.4 Percentage of error type out of incorrect responses without calculators

N=306	Result unknown	Start unknown	Multiple unknown	Total %
Bad algebra	0	0	1	1
Bad arithmetic	10	5	1	16
Order of operation	8	2	0	10
Inverted operation	0	4	0	4
Missing operation	3	2	0	5
Comprehension	4	3	13	20
Give Up	3	8	5	15
Answer Only	10	8	11	29
Total %	38	31	31	100

Strategies used without calculators

Of the 639 solution responses, only 453 (71%) documented a solution response which could be coded. Of the 453 coded responses, 107 (17%) responses provided an answer only, even though students were requested to show all working. This meant that only 346 (54%) responses were available to assess in terms of student strategy. The percentages of each strategy used out of the 346 responses are shown in Table 4.5.

Back tracking (37% of strategies used) and arithmetic (42% of strategies used) were clearly the most favoured strategies used. Back tracking was predominant with start unknown questions, while working-forwards using arithmetic was predominant with the result unknown questions. The more difficult multiple unknown questions tended to be answered using guess and check.

Table 4.5 Strategy identified from written responses

N =346	Algebraic	Back tracking	Arithmetic	Guess and Check	Total %
Result unknown	1	6	34	1	42
Start unknown	3	28	2	3	36
Multiple unknown	4	3	7	9	23
Total %	8	37	42	13	100

In order to remain consistent with previous research (Nathan & Koedinger, 2000b; 2000a), the multiple unknown problems were separated out into symbolic equation, word equation, and story formats when analysing the strategy used. This further separation was made to determine if the mathematical structure played any part in the use of algebraic techniques. The percentage was calculated for each strategy used within each presentation format, out of the number of questions with intelligible responses for each mathematical structure.

Table 4.6 shows the percentage of each strategy used for problems with a result unknown structure. Working forwards with arithmetic was predominant in the word-equation format (34% of strategies used). Both algebra and guess and check strategies were rarely used.

Table 4.6 Percentage of strategies used for result unknown problems out of all strategies

N=142	Algebraic	Back tracking	Arithmetic	Guess and Check	Total %
Equation	0	1	27	0	28
Word-Equation	1	4	34	0	39
Story	0	10	21	1	32
Total %	1	15	82	1	100

Table 4.7 shows the percentage of each strategy used for problems with a start unknown structure. Back tracking was the predominant strategy used and to a greater degree in the story problem format (41% of strategies used). An increased use of algebra and guess and check methods appear in the equation and word-equation formats compared with responses in the result unknown structure.

Table 4.7 Strategies used for Start unknown problems (Percent of total strategies)

N=124	Algebraic	Back tracking	Arithmetic	Guess and Check	Total %
Equation	4	15	1	2	22
Word-Equation	4	22	1	6	33
Story	1	41	3	0	45
Total %	9	78	5	8	100

Table 4.8 shows the percentage of each strategy used for problems with a multiple unknown structure. There is a noticeable increase in the use of algebraic strategies (17% of strategies used) for this mathematical structure. The increase in the use of guess and check is also noticeable. Guess and check is the predominant strategy used (39% of strategies used) and is used consistently across all presentation formats. An increase in the use of algebra occurs predominantly in attempts at solving symbolic equation formats. The preferred strategy for multiple unknown story problems is arithmetic. This could be due to the high number of comprehension errors made on multiple unknown problems.

Table 4.8 Percentage of strategies used for multiple unknown problems out of total strategies

N=80	Algebraic	Back tracking	Arithmetic	Guess and Check	Total %
Equation	10	5	1	15	31
Word-Equation	6	4	10	14	34
Story	1	6	18	10	35
Total %	17	15	29	39	100

Student strategy proficiency

Table 4.9 shows the percentages for the number of times a response was correct for a given strategy out of the total number of times that strategy was used. Those responses that provided an answer only are included here. Percentage values are indicative of the proficiency students have for each strategy. Back tracking proved to be the strategy most successfully used, with a 51% proficiency level.

Table 4.9 Proficiency of student strategy

	Algebraic	Back tracking	Arithmetic	Guess and Check	Answer Only
Number of responses	27	129	146	44	107
Percentage of correct responses	11%	51%	35%	23%	16%

Table 4.10 classifies the types of errors made within each strategy, as a percentage of the total number of errors made for each strategy. The majority of errors by students

using an arithmetic strategy were that of an incorrect order of operations (40%). Students using algebraic strategies made errors primarily because they did not fully understand what they were doing and either gave up or made invalid algebraic manipulations. It is interesting to note that the majority of arithmetic errors were made by students choosing backtracking strategies. Those students using guess and check strategies who failed to attain correct solutions tended to make comprehension errors or give up.

Table 4.10 Percentage of error types made within each strategy

	Arithmetic	Algebraic	Back-tracking	Guess and check
Bad algebra	0	13	0	0
Bad arithmetic	0	9	26	3
Order of operation	40	4	18	0
Inverted operation	2	9	16	3
Missing operation	12	4	15	0
Comprehension	26	9	11	50
Copy Slip	0	0	0	0
Give Up	20	52	14	44
Total %	100	100	100	100

4.2.4 Student responses with the use of calculators

Of the 136 students surveyed in total, 65 students sat the test with calculators: 20 students for Test A, 23 students for Test B, and 22 students for Test C. The purpose of allowing the use of calculators was to examine whether calculators would affect the choice of strategy. The 65 test papers contained 9 questions each, producing a total of 585 questions. Of the 585 questions 78% were answered with 22% being left blank. Out of the 78% of questions answered, 39% were result unknown questions, 33% were start unknown questions, and 28% were multiple unknown questions. This distribution of questions answered is comparable to the distribution for responses from students who did not use calculators. The questions were identical to those used by students without calculators.

Of the total responses 45% of the questions were answered correctly. The number of correct responses is considerably higher than that of students not able to use a calculator (23%). The overall results for students without calculators are shown in Table 4.11.

Table 4.11 Overall success of student responses relative to use of calculators

	% with calculators N = 585	% without calculators N = 639
Correct	45	23
Incorrect	33	48
No Response	22	29
Total %	100	100

Correct responses using calculators

The 45% of correct responses relates to 264 questions. Out of these correct responses, the percentage of successful responses for each kind of question is shown in Table 4.12. The students found greatest success in solving the result unknown mathematical structure and the symbolic equation presentation format. Result unknown equation problems was the variation that met with the most success (19% of correct responses).

Table 4.12 Percentage of successful student responses

N=264	Equations	Word equations	Story problems	Total %
Result unknown	19	18	12	49
Start unknown	10	11	15	36
Multiple unknown	7	5	3	15
Total %	36	34	30	100

Errors using calculators

The 33% of incorrect responses relates to 191 questions. Out of the incorrect responses, the percentage of each kind of error made is shown in Table 4.13. The most errors were made by students when answering the multiple unknown questions (47% of errors made). As with students not permitted to use a calculator, the most common identifiable error overall was that of *comprehension* (29% of errors made). More comprehension errors were made on multiple unknown questions than any other kind.

Table 4.13 Percentage of errors in questions answered incorrectly using a calculator

N = 191	Result unknown	Start unknown	Multiple unknown	Total %
Bad algebra	0	0	3	3
Bad arithmetic	0	2	2	4
Order of operation	5	1	0	6
Inverted operation	2	3	1	6
Missing operation	3	2	0	5
Comprehension	3	8	18	29
Copy Slip	1	0	0	1
Give Up	0	1	5	6
Answer Only	10	12	18	40
Total %	24	29	47	100

Of the incorrect responses, 40% came from students who showed no written workings, giving a single numerical answer only. Calculators reduced the amount of errors that occurred in all types of error with the exception of answer only error which accounted for 40% of total errors (76 questions). In terms of overall responses with calculators, the answer only errors represent 13% which is comparable to the 14% of answer only errors in the non-calculator sample.

Strategies used with calculators

Of the 585 solution responses, only 455 (78%) documented a solution response which could be coded. Of the 455 codifiable responses, 161 (28%) responses provided an answer-only even though students were requested to show all working. This left only 294 (50%) responses for which a strategy could be determined. The percentage of each strategy used out of the 294 intelligible responses are shown in Table 4.14.

Table 4.14 Percentage of strategy used out of total responses

N=294	Algebraic	Back tracking	Arithmetic	Guess and Check	Total %
Result unknown	4	3	32	0	39
Start unknown	7	25	1	2	35
Multiple unknown	8	6	6	6	26
Total %	19	34	39	8	100

Back tracking (34% of strategies used) and arithmetic (39% of strategies used) are clearly the most favoured strategies used. Backtracking was predominant with start unknown questions (25%), while working forwards using arithmetic was predominant with the result unknown questions (32%). As with results for student responses without the use of calculators, the strategies were separated out to control the variable factor of mathematical structure, and highlight the strategies used for each presentation format.

Table 4.15 Percentage of strategies used for result unknown problems out of total result unknown problems

N=112	Algebrai c	Back tracking	Arithmetic	Guess and Check	Total %
Equation	3	0	27	0	30
Word-Equation	5	0	32	0	37
Story	1	8	24	0	33
Total %	9	8	83	0	100

The percentage of each strategy used for problems with a result unknown structure is shown in Table 4.15. Working forwards with arithmetic was predominant in the word-equation format (34% of strategies used). It is interesting to note that the guess and check strategy is not used at all, and that there is a noticeable increase in the number of students using algebra compared with those students not able to use calculators.

Table 4.16 shows the percentage of each strategy used for problems with a start unknown structure. Back tracking was the predominant strategy used and to a greater degree in the story problem format (37% of strategies used). As with the responses from students not permitted to use calculators, there is an increased use of algebra and guess and check methods in the equation and word-equation presentation formats compared to those in the result unknown structure.

Table 4.16 Percentage of strategies for start unknown problems out of total start unknown problems

N=104	Algebraic	Back tracking	Arithmetic	Guess and Check	Total %
Equation	11	13	0	4	28
Word-Equation	7	20	2	2	31
Story	2	37	2	0	41
Total %	20	70	4	6	100

Table 4.17 shows the percentage of each strategy used for problems with a multiple unknown structure. There is a noticeable increase in the use of algebraic strategies (28% of strategies used) for this mathematical structure. Algebra is the preferred strategy used. The increase in the use of guess and check is noticeable (24% of strategies used) across all presentation formats.

Table 4.17 Percentage of strategies for multiple unknown problems out of total multiple unknown problems

N=78	Algebraic	Back tracking	Arithmetic	Guess and Check	Total %
Equation	14	10	1	9	34
Word-Equation	10	5	8	9	32
Story	4	9	14	6	33
Total %	28	24	23	24	100

As with the increase in algebra and guess and check strategies within start unknown problems, the increases in algebra and guess and check strategies within the multiple unknown problems are also greater in attempts on the equation and word-equation presentation formats. The preferred strategy for multiple unknown story problems is arithmetic. As with responses from students not permitted to use a calculator, the preferred strategy of arithmetic could be due to the high number of comprehension errors made on multiple unknown problems.

Strategy proficiency

Table 4.18 shows the number of times a response was correct for a given strategy out of the number of times that strategy was used. Those responses that provided an answer only are included here. Percentages indicate the proficiency students have in each strategy. There is very little difference in the levels of proficiency between algebra, back tracking and arithmetic. The strategy of guess and check, however, indicates a much poorer level of success. Calculator use resulted in a noticeable increase in proficiency across all strategies except guess and check. In particular, the percentage of successful algebraic responses increased from 11% without calculators to 67% with calculators. The level of success for those students who wrote an unqualified answer is much greater for students using calculators (49% proficiency), than those students not able to use calculators (16% proficiency). This could be due to the greater number of students that used a correct strategy but perceived no need to record the strategy because of calculator support.

Table 4.18 Percentage of success in use of strategies out of total use of strategies

	Algebraic	Back tracking	Arithmetic	Guess and Check	Answer only
Number of responses	52	102	115	25	161
Percentage of correct responses	67%	65%	68%	24%	49%

Table 4.19 shows the number of errors made for each kind of error, as a percentage of the total number of errors made for each strategy. The majority of errors by students using an arithmetic strategy were comprehension errors (62%) where the question was misinterpreted and reinvented. Students using algebraic strategies made errors primarily because they did not fully understand what they were doing and made invalid algebraic manipulations. It is interesting to note that so many arithmetic errors were made by students using algebraic strategies. Students choosing backtracking strategies mainly made comprehension errors. Back tracking strategies tended to incur a high proportion of missing operation (23%) and inverse operation (23 %) errors. As with

students not using calculators, those students using guess and check strategies tended to make comprehension errors or give up.

Table 4.19 Percentage of errors made within each strategy

	Arithmetic	Algebraic	Back-tracking	Guess and check
Bad algebra	0	29	0	0
Bad arithmetic	0	18	0	0
Order of operation	19	6	3	0
Inverted operation	5	12	23	0
Missing operation	3	0	23	0
Comprehension	62	23	40	76
Copy Slip	3	0	0	0
Give Up	8	12	11	24
Total %	100	100	100	100

4.3 Teacher Beliefs and attitudes

Two methods were used to examine teacher beliefs on teaching perspectives as they relate to word problems: Teacher questionnaires and Focus Group discussions. The resulting data is presented in this section.

4.3.1 Teacher responses to questionnaires

From the four participating schools, 25 teachers were given questionnaires (Appendix 7) to complete and return anonymously through the mail. From the 25 teachers, 15 questionnaires were completed and returned, giving a response rate of 60%. In addition to the construct statements designed to explore teachers' beliefs, the questionnaire also gathered demographic information. Responses were received from eight female and seven male teachers, with the average number of years teaching experience being 9.3 years (this included three first year teachers). All except two of the teachers taught both junior and senior mathematics. Two teachers only taught junior classes in mathematics.

The construct statements were presented in a 6-point Likhert scale (1 strongly agree to 6 strongly disagree). The mean rank of responses is presented in Table 4.20. The mean rank for items clustered around the centre (3.5) of the six point scale indicating that the scale was adequate for teachers to express their views. The smaller the mean score the greater the agreement with the construct presented.

Table 4.20 Mean scores for responses to six statement constructs in teacher survey

Construct (n=15)	μ	sd
Product over process	4.53	0.65
Invented solution methods are effective	2.63	0.99
Teachers should encourage invented solution methods	2.43	0.97
Algebra is best	3.45	0.99
Alternative methods indicate knowledge gaps	3.57	0.93
Symbol precedence view	2.73	1.15

The mean responses show that teachers agreed with the pro-constructivist views presented in the statements, *invented solution methods are effective* ($\mu = 2.63$), and *teachers should encourage invented solution methods* ($\mu = 2.43$).

The statements *product over process* ($\mu = 4.53$), *algebra is best* ($\mu = 3.45$), and *alternative methods indicate knowledge gaps* ($\mu = 3.57$) challenge current views expressed in curriculum reform, that students are active constructors of their own knowledge. These statements emphasise correct answers over students' reasoning and solution methods. A strong indication is evident that teachers consider student reasoning to be more important than the final answers students' produce. However, teachers' responses show some indifference in the efficacy of the use of alternative strategies in comparison to algebraic ones, with the constructs *algebra is best* and *alternative methods indicate knowledge gaps* hovering around the middle of the scale.

The majority of teachers held to the *symbol precedence view* ($\mu = 2.73$) where symbolic manipulation skills are prerequisite to word problems. A couple of teachers did not hold to the *symbol precedence view* statements, which caused a skew in the results and a higher standard deviation (sd = 1.15) than other constructs.

Of the three questions lying outside the six main constructs (Appendix 7: questions 8, 19 & 26), teachers indicated more usefulness in allowing students time to explore tasks thoroughly than to adhering strictly to a curriculum timetable. Teachers did not consider that symbol manipulation was the greatest difficulty faced by students learning algebra. However, there was disagreement amongst teachers as to whether general mathematical reasoning skills were the greatest difficulty faced by students.

4.3.2 Teacher ranking of student test questions

In addition to the questionnaire, teachers were invited to participate in a focus group discussion around the assessment of difficulty of the questions presented to students. Two focus groups were held with a total of eight teachers taking part. One focus group consisted of three male and one female teacher with all teachers having more than ten years experience. The other focus group consisted of two male and two female teachers, with one teacher having more than ten years experience, one teacher having between five and ten years experience, one third year teacher and one first year teacher.

After individuals ranked the sample test questions in order of difficulty, cue cards were used to construct a corporate ranking. Teachers manipulated cue cards together such that questions were ordered from least difficult (1) to most difficult (9). The teachers were given the option to give different questions the same difficulty ranking. Each of the sample constructs is coded in Table 4.21 below.

Table 4.21 Coding of sample constructs

	Equation	Word-equation	Story
Result Unknown	RU-E	RU-W	RU-S
Start Unknown	SU-E	SU-W	SU-S
Multiple Unknowns	MU-E	MU-W	MU-S

Results from Focus Group A

The results from the individual rankings are provided in Table 4.22, and the results from the discussion are shown in Table 4.23. Focus group A rated the result unknown equation as the easiest during the individual rankings, but this was amended to result unknown word equation during the discussion. The rankings provided for multiple unknowns did not change.

Table 4.22 Individual ranking of student test questions - Focus Group A

	RU-E	RU-W	SU-E	RU-S	SU-W	SU-S	MU-E	MU-W	MU-S
Average ranking	1.8	2.0	4.0	4.5	4.5	5.3	6.0	8.0	9.0

Table 4.23 Corporate ranking of student test questions - Focus Group A

	RU-W	RU-E	SU-W	RU-S	SU-E	SU-S	MU-E	MU-W	MU-S
ranking	1	2	3	4	5	6	7	8	9

Results from Focus Group B

The results from the individual test rankings are provided in Table 4.24, and the results from the discussion are shown in Table 4.25.

Table 4.24 Individual ranking of student test questions – Focus Group B

	RU-W	RU-E	RU-S	SU-W	SU-E	MU-S	SU-S	MU-W	MU-E
Average ranking	1.0	2.0	3.0	4.3	5.0	6.0	6.5	6.7	8.5

Table 4.25 Corporate ranking of student test questions – Focus Group B

	RU-W	RU-E	RU-S	SU-E	SU-W	SU-S	MU-E	MU-W	MU-S
ranking	1	2	3	4	5	6	7	8	9

General agreement was made, both individually and in discussion, that the result unknown word-equation (RU-W) was the easiest question. The reasons given were: "it's telling you what to do", "You're told exactly what steps to take", and that "there is no context to confuse" the students.

The focus group sessions also showed general consensus that the most difficult questions were the multiple unknown word-equation (MU-W) and the multiple unknown story problem (MU-S). The primary reason given for the difficulty of MU-W and MU-S was the amount of language needing to be interpreted. "This sentence is too big, language is a huge thing…". There was also a realisation by the teachers that students would not see as readily the mathematical structure within the multiple unknown story problems. "Students will not realise [the need] to equate two situations". In not realising the complexity in structure, it was thought that students would probably use informal methods such as *guess and check*. *Guess and check* was not seen as a viable strategy for the multiple unknown problems, but rather one used by students as a last resort stab at a solution.

One teacher in group B considered that the story in MU-S would be an enabling factor for students and ranked this question much easier than all other teachers. "It looks like [the students] would understand the situation and they would have a crack [at it]. The same question with symbols – they wouldn't know where to start". There was a firm belief by this teacher that an honest attempt made by a student would greatly increase the chances for achieving success and therefore any question that motivated the students enough to engage in mathematical activity must be easier.

Greater discussion and disagreement surrounded the questions ranked in the middle with moderate difficulty. The teachers all expressed a common priority when classifying the test questions, in that question interpretation was a major contributor to difficulty; If students could understand the situation and what was being asked of them, then difficulty was greatly reduced. Disagreement amongst teachers occurred when trying to establish what made a question difficult to interpret.

Many of the issues discussed concerned particular groupings of students, such as foreign language students and students with learning difficulties. In general there was a leaning towards a symbolic precedence view. An example of this can be seen in relation to comments regarding mathematical structure. Teachers noted the identical mathematical structure of the problems in the student questionnaire: "It's identical in structure, [you] only have to interpret symbols [here]; ESOL [English as Second or Other Language] students might find the symbolic one easier"; "It's exactly the same as the equation, but I thought it would be easier for [students] to see with all the words taken away". The symbolic precedence view is confirmed in the consensus within both groups that the start unknown story problem be placed after the symbolic and word-equation counterparts in a position of higher difficulty.

The use of informal strategies by students was recognised by teachers and was not considered to be a concern. The placing of the start unknown word equation (SU-W) by group A at such a low level of difficulty was because one teacher convinced the others that students would "...do a straight reversal, kids will understand the backwards working, and then it's the same as [the result unknown word equation]."

When asked if students would also work backwards when solving the same problem with the story format, there was tentative agreement but belief that the greater number of words and less directive nature of sentences would hinder students to the extent that the symbolic equation would be easier. A teacher noted that the story problem (SU-S) "has more sentences which makes it harder. Students will have trouble working out the operations to use. You can just use trial and error with the [symbolic (SU-E)] one". It is interesting to note that this teacher believed year 10 students would use *guess and check* strategies predominantly on the symbolic equation rather than on the story problem. The use of *guess and check* strategies predominantly for symbolic equations is supported by the results from the student tests.

On the issue of using word problems in the classroom, word problems were considered the application of algebra to everyday life, albeit an often oversimplified representation of real life. Word problems were seen to "create a link with reality" and to "show that algebra is used". These views for using word problems, however, appear to be motivated more for the purpose of quelling unrest amidst students when they say "what's the use of that" in reference to equations and equation solving.

Most teachers said that word problems should be used both at the beginning and the conclusion of the algebra topic. One teacher describes the process used in teaching algebra: "I start algebra talking about fish and chips, and I finish algebra talking about those sort of problems [multiple unknown story problems]". When asked about the use of text books, teachers unanimously indicated that they were primarily used as a resource of practice problems rather than a teaching device. Teachers were, however, indifferent about the influence that textbooks had on their teaching and affirmed the possibility that their teaching could be marginally influenced by the order in which problems appeared in the text.

4.3.3 Textbook analysis

The textbooks selected for analysis consisted of those used by the four participating schools and are listed in Table 4.26. To remain consistent with the year level of participating students, the textbooks analysed were designed for year 10 students. All of the participating schools used the more recent Beta Mathematics textbook. The other textbooks analysed were only used by some of the participating schools. The manner in which algebra was presented in each textbook was the object of investigation.

Table 4.26 Textbooks used by participating schools

Title	Author	Publisher	Publication Date
Beta mathematics: A mathematical journey for year 10 students	D. Barton	Longman	1998
National Curriculum Mathematics: Level 5, Book 2	K.Catley, M.Tipler, W.Geldof	The Caxton Press	1996
4 Maths	A. Joyce D. Kibblewhite D. Nightingale	New House Publishers	1989

The *Beta Mathematics* textbook emphasised that mathematics is a "fascinating subject, and one that you learn by doing". Many investigations are presented throughout the textbook, including a spreadsheet investigation in the chapter on algebra. Algebra is presented in a single chapter. The contents and order of presentation include:

1. Formulas and substitution
2. Simplifying algebraic expression
3. Expanding and factorising
4. Solving equations

Each topic area consisted of the "tell-show-do" approach mentioned in the literature. Verbal problems were used throughout each topic, with traditional word problems used in the topic of solving equations. In each instance, the verbal problems were placed at the end of the topic and were preceded by symbolic manipulations. The technique encouraged for the traditional word problems, possessing the mathematical structure of a linear equation with one instance of the unknown, was to "write an equation and solve it". The linear equations with one instant of an unknown were followed by a section on solving linear equations with multiple instances of the unknown, again word problems were preceded by symbolic ones.

The *National Curriculum Mathematics* is part of a series written specifically to incorporate the changes made in the New Zealand curriculum in 1992. The series is designed to be used directly in classrooms as a comprehensive resource for students and teachers. Topics are presented in a "possible teaching order", and encourage mathematics that is "relevant to everyday life" with attention paid to the teaching of mathematics in the "context of problems". Investigations were presented throughout the text, with an investigation on the topic of calendars in the algebra chapters.

The topics of algebra are spread over two separate chapters, which are spaced apart in the text:

- Chapter 6: Expanding and factorising. Expressions
- Chapter 10: Equations. Formulae.

Each chapter begins with a word problem designed to stimulate discussion and give meaning to the kind of mathematics students are likely to encounter within the chapter. After the discussion problem, however, the format followed the "tell-show-do" approach. Word problems were presented at the conclusion of the 'equations' topic entitled "Writing equations". All but one word problem could be written symbolically as a linear equation with only one instance of the unknown. The technique encouraged was to "write and solve equations".

The *4 Maths* textbook was written before changes to the New Zealand curriculum were made. Emphasis within 4 Maths was on "mastery of skills, problem solving, and where appropriate the theme of relevance". The topics of algebra and order of presentation include:

1. Variables
2. Exponents
3. Equations and inequations

The use of word problems is very limited, with only very simplistic story problems of mathematical structure ax + b = c used. The exercises in each chapter were divided into three sections of A, B and C. These divisions were intended by the author to represent a progression of difficulty. The word problems presented were found in exercise 1B, indicating a greater level of difficulty that the symbolic equations presented in exercise 1A. More difficult equations containing multiple instances of an unknown were presented in exercise 1C. There were no word problems, however, within exercise 1C. The technique encouraged for solving the word problems was to "translate each [problem] into a mathematical equation and then solve" it.

The first topic of algebra covered within 4 Maths (variables) contained a series of exercises in simplifying expressions. It is interesting to note that the beginning of this chapter contained two sets of exercises requiring firstly that verbal statements be expressed using symbols and then secondly that symbolic expressions be expressed in words.

5 DISCUSSION

5.1 Introduction

In an attempt to address the key research questions, this discussion adds to comments already made throughout the presentation of results by integrating the linkages between student performance and teacher beliefs in conjunction with the literature reviewed. Three main concerns that influence the teaching and learning of algebra are singled out as prominent features emanating from the results: The nature and influence of the problems students face (in terms of mathematical structure and presentation format) on solution strategy choice, the role of calculators in the teaching and learning of algebra, and teacher beliefs and attitudes about the teaching and learning of algebra. In this discussion the term 'word problems' refers to any problems that contain words, and includes both presentation formats of *word equations* and *story problems*.

5.2 The influence of structural complexity on student performance

A review of the literature indicated that student choice of technique when solving problems is influenced by the mathematical structure of those problems. Student use of informal methods is considered predominant when problems contain a mathematical structure with only a single occurrence of an unknown value (Sfard & Linchevski, 1994; Stacey & MacGregor, 1999), however, algebraic methods are more commonly used when problems possess multiple instances of unknown values. It is interesting to note that previous research indicates that beginning algebra students often do not consider using symbolic equations (algebraic technique) when faced with a word problem.

The structural complexity of the problems used within this research is governed by the position of the unknown within the problem's mathematical structure and the number of unknowns presented within a single problem. In this way structural complexity was presented in three forms: result unknown, start unknown and multiple unknown. The two single unknown forms were those used by Nathan and Koedinger (2000b, 2000a) in earlier research into student strategies with word problems. Nathan and Koedinger (2000a) considered that since result unknown problems can be solved through direct application of arithmetic operations they can be classed as arithmetical, whereas the start unknown problems engender the application of algebraic procedures and so can be classed as algebraic. Solution methods considered to be algebraic include the manipulation of equations by means of balancing or transposing (Kieran, 1992b). Multiple unknowns have since been examined by Koedinger, Alibali and Nathan (2001) and were used in this research to investigate the view that multiple unknown problems are necessary in order to solicit the use of algebraic solution techniques.

The research results show a strong negative correlation between problem complexity and success in that the greater the problem complexity, the less success the students' achieved. In particular, as the problem complexity increased, the errors made by students shifted from predominantly elementary error towards errors that displayed a lack of understanding. Errors made in the structurally less complex problems tended to be primarily arithmetical calculations, especially by those students not permitted to use calculators. The more structurally complex problems solicited a greater number of comprehension errors, incomplete and guessed answers. This suggests that as a problem's structural complexity increased, student understanding diminished in terms of both what the problem was requiring of them and discernment of the approach that would best find a solution.

The results also found that as the structural complexity increased there was a clear shift in the type of strategy being used. Forward working *Arithmetic* was the strategy primarily used for result unknown problems. For start unknown problems, however, the predominant strategy employed was *Back Tracking*, which is also considered an arithmetical technique. Although the multiple unknown problems were poorly attempted by the majority of students, there was a reduction in the use of *Arithmetic* and *Back Tracking* in favour of the strategies *Guess and Check* and *Algebra*. Students were obviously less comfortable with the multiple unknown problems and appeared to recognise that more sophisticated techniques would be required. In research by Sebrechts, Enright, Bennett, and Martin (1996), *Guess and Check* was also found to be preferred by students solving multiple unknown problems. In terms of successfulness, however, Koedinger, Alibali and Nathan (2001) found that students failed to solve the more complex problems intuitively, and those students that used symbolic representations were more successful.

Further investigation is required in order to establish the reasons students would resort to *Guess and Check* for the more difficult problems. One possibility may be that while some students recognised that a more complex technique was required, their knowledge of that technique was inadequate. Students may have been under-exposed to more difficult problems with multiple instances of the unknown. The analysed textbook's under-representation of multiple unknown problems, in particular story problems, may be a contributing factor in inhibiting their use by teachers and therefore student proficiency.

In the absence of proficiency in algebraic technique, students chose to utilise the more intuitive method of *guess-and-check*. Alternatively, it is plausible that students may have considered *Guess and Check* in its more systematic form of modelling to have been an advanced choice of strategy, especially with the real world possibility of using a computer for repetitive iterations towards a solution. Student use of *Guess and Check* in this research, however, was not influenced as a choice of strategy, nor was its proficiency improved, by the use of basic calculators. This finding supports the argument that the majority of *Guess and Check* users were merely muddling towards a solution because they knew of no other means.

5.3 The influence of word problems on student strategies

The presentation format of problems is also considered to affect complexity. This research did not investigate the contributing factors to potentially discriminating effects of differing contexts used in word problems. Linkages have, however, been made to suggest that using words in the presentation of mathematics problems raises the level of problem complexity for some students (Forbes, 2000; Sweller, 1999; Zervenbergen, 2000; MacGregor & Price, 1999; Cooper & Dunne, 1998). The literature suggests that many teachers consider that problems enshrouded in language are more difficult than symbolic equivalents, a finding which reflects a symbolic precedence view (Nathan & Koedinger, 2000b). In contrast to this view, Nathan and Koedinger (2000a) found that students were often more successful with story problems than symbolic counterparts.

The research results presented here are consistent with the research undertaken by Nathan and Koedinger (2000a) on student strategies with result unknown and start unknown problems[6]. The student responses without calculators revealed that the students found the story problems easier than symbolic equations. The success of student results for the *story problem* format (42% of correct responses) was greater than for the symbolic *equation* format (26% of correct responses). These relative success levels of presentation format are even more pronounced when viewed isolated within the start unknown problem structure (60% of the 42% correct responses for story problems compared with 27% of the 26% correct responses for symbolic equations, see Table 4.3).

Most of the teachers who took part in the focus group discussions considered that language reduced problem complexity and assisted students in finding a favourable solution when problems had a lower structural level of result unknown complexity. One teacher noted that "the words can help if they are directive." In particular, the *word equations* were considered to be easy because the language was not symbolic and "told [students] exactly what steps to take."

In contrast, for problems with a high level of structural complexity (multiple unknown problems), some teachers in the focus groups considered that language complicated the problem by introducing additional comprehension requirements. In commenting on one of the problems in the student questionnaire, a teacher commented that "this sentence is too big, language is a huge thing [for students to cope with]". This view proved true for student responses with calculators; the story format proved more difficult for students than the equation format. This result did not hold, however, in the responses without calculators where there was relatively greater success in the story format problems.

In relation to presentation format, results from the student tests conducted with the use of calculators differed markedly from the results without calculator use. Students using calculators showed greater success with problems in an equation format across all levels of structural complexity compared with the responses from students not using calculators. Within the results for responses that used calculators, greater success was achieved for result unknown and multiple unknown structures in the equation format. Whilst student success in start unknown problems remained greater for the *word*

[6] This research by Nathan and Koedinger (2000a) was conducted without the use of basic calculators.

equation and *story problem* format than the symbolic *equation* format, there was considerable improvement in the *equation* format when calculators were used.

The literature suggests that expert problem solvers rather than novice problem solvers possess and use effective problem-solution schemata to assist them in solving problems (Mayer, 1981; Mayer, Larkin, & Kadane, 1984; Reed, 1999; Silver, 1981). Expert problem solvers tend also to predominantly use equations and algebraic techniques rather than intuitive solution strategies (Sebrechts, Enright, Bennett, & Martin 1996; Hall, Kibler, Wenger, & Truxaw, 1989). In contrast to this, novice high school students are inclined to use informal methods rather than equations (Kieran, 1997). This research showed that story problems with result unknown and start unknown complexity solicited a greater proportion of informal strategies than equation problem counterparts. When students chose to use algebra, it was predominantly for problems in an equation format.

This researcher's experience is that students groan when faced with story problems and treat them as activities for which informal means are the norm in search for a favourable solution. Many students believe that some sort of trick is often required to solve story problems and struggle to see the underlying mathematical structures. The majority of teachers in the focus group discussions confirmed this view that the result unknown and start unknown story problems would be solved intuitively. One comment made was: "[students] will solve the word problems before the symbolic equations by whatever means is available to them, before they even know any algebra".

It appears that while the teachers in this study were of the view that simple word problems are easy for students, they also considered that students are unlikely to use algebra as a solution strategy to solve these word problems. In comparison, teachers considered complex story problems to be difficult due to language comprehension requirements. Student success in complex story problems, however, was not entirely consistent with this view. The students appeared to approach word problems differently when equipped with a basic calculator, to the point where the presence of a calculator appeared to have affected student choice of solution strategy.

5.4 The role of calculators in learning algebra

Calculators have been shown to cause changes in the way teachers teach and the way students learn (Waits & Demana, 2000). With the use of a calculator, less class time is spent on the teaching and learning of manual techniques for number manipulation, and greater emphasis can be placed on a deeper level of conceptual understanding (Ministry of Education, 1992; Waits & Demana, 2000).

> What becomes important is not the detail of number work in the sense that the student needs to be able to actually make a calculation using a clay tablet and a stick, but the form and feel of answers that may be being generated and the relation to the underlying principals. (Bullock, Thomas & Tyrell, 1994, p. 73)

Waits and Demana (2000) emphasise that calculators do not degrade the basic arithmetic skills of students, but rather allow for activities that require serious thinking which leads to better mathematical understanding and greater connections made to real-world phenomena in a way that captivates student interest. Stacey (1994) found that students solving word problems with the aid of a calculator were more familiar with the operations of arithmetic and better able to identify the operations required when translating text. Stacey's conclusions revealed that arithmetic is more definite and explicit with the use of a calculator whereas mental techniques of arithmetic are so varied that there can often be a blurring between the differing operators (for example subtraction can be done mentally by using addition in the form of counting-on).

The results show that the use of calculators reduced many inaccuracies prevalent in manual calculations with a marked increase in success for the arithmetic level *result unknown* equation problems. The use of calculators, however, did not significantly benefit students when solving story problems. It is possible that when calculators were not permitted students focused on the procedural rules of manual arithmetic when addressing equation problems, and gave less attention to understanding their answers. Whereas with the word problems, care was taken to understand the problem and its solution set, thus calculations followed a more numerate pathway with greater success achieved.

There was a significant improvement in student performance across both structure and presentation format from attempts without the use of calculators (23% correct) to performance with the use of calculators (45% correct). When calculators were permitted, arithmetic error reduced from 16% to 4% of errors made, and the number of responses where students gave up was reduced from 15% to 6%. This implies that the use of calculators in the student survey reduced problem difficulty.

The numbers used in questions presented to students were difficult enough to reduce the probability of direct mental arithmetic and number facts being used, but easy enough that manual arithmetic algorithms could be used with ease (eg. 5.79 – 0.12 and 5.67 ÷ 7) without the need for rounding (cf. Küchemann, 1983). Thus when calculators were not permitted, the cognitive load on students was increased as they were required to use arithmetic skills that students often considered redundant with the availability of calculators. One teacher, after administering the test questions, commented that they were too difficult for the students without the use of calculators. A few students that gave up on questions annotated their papers to suggest that they had given up because they did not have access to a calculator.

With the use of calculators there was a significant increase in the number of *answer only* (AO) responses (17% to 28%). This increase, however, is evident solely in the responses that were correct. There is no change in the proportion of incorrect responses registering an *answer only,* with 13% of the total number of responses being incorrect for both responses with and without the use calculators. The *answer only* errors can therefore be primarily considered a guess by a student not understanding what to do, rather than a mistake in calculation by a student with some understanding. These responses can be compared to non responses which also show little change when calculators were permitted. It is possible that those students who complained of not being able to use a calculator, and provided an answer only or no answer at all, would not have benefited from the use of a calculator. Those students that provided correct *answer only* responses obviously knew what they were doing and probably worked forwards with arithmetic strategies, particularly those using a calculator. This is consistent with Stacey's (1994) results that students solving word problems with the aid of a calculator tend to be more fluent in the application of arithmetic operations.

Küchemann (1983) notes that the choice of strategy is influenced by the difficulty of the structure, the numbers present and the format of a problem. The results show that when calculators were introduced the choice of strategy changed in two areas: an increase in algebra (from 8% to 19%) and a slight reduction in guess and check (from 13% to 8%). It is plausible that the use of calculators, in reducing the cognitive load of students, allowed students to focus their attention more on algebraic technique. This is in agreement with a review of the literature on calculator use by Kieran, Oates and Thomas (1998, p. 89) in that "when students have calculators available for solving problems on tests, they are able to order their cognitive processes more effectively". Stacey and Macgregor (1999), in their recommendations for teaching algebra, endorse the use of calculators in conjunction with difficult numbers to promote algebraic thinking. The New Zealand Ministry of Education (1992, p. 14) also advocates the use of calculators to "enable students to concentrate on the mathematical ideas rather that on routine mechanical manipulations". The results here support the use of calculators for solving algebraic problems by showing a significant increase in algebraic proficiency, from 11% (Table 4.9) to 67% (Table 4.18) proficient. It appears that the argument that beginning algebra students find it difficult to focus their attention simultaneously on both the algebraic and arithmetical aspects of problems is justified by these findings.

5.5 The effect of teacher beliefs and attitudes on student strategies

The focus group responses were examined to provide insight on a number of underlying issues and propositions in relation to teacher beliefs about the role of word problems in learning algebra. If the order of teaching typically follows the order of perceived difficulty in mathematical structure, we would expect teacher responses to show a positive relationship between 'taught first' and 'perceived to be easy'. For example, a word problem may be considered to be easy if an arithmetic method can be used to solve it. A confounding factor is the possibility that teachers are unaware that students are using non-algebraic technique to solve algebraic problems. If they are, however, aware of this, what is their purpose of teaching non-algebraic methods within the topic of algebra? Conversely, if the goal is for students to learn algebraic method, then why would teachers allow them to use non-algebraic techniques, and why would they present students with problems for which non-algebraic techniques are preferred?

Some insights were highlighted in the responses to the teacher questionnaire and focus group interviews, and agree with the findings of Nathan and Koedinger (2000c). Teachers showed a strong tendency towards a symbolic precedence view, believing that students must first become proficient with symbolic equations in order to be successful with story problems. The philosophical paradigm of *content—centred with a focus on performance (*Kuhs & Ball, 1986, cited in Thompson, 1992), and *transmission* (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997), can be associated with the symbolic precedence view.

Many of the teacher responses to the questionnaire, however, also indicated a *student—centred* philosophical paradigm through the importance they placed on encouraging student driven solution methods. Belief in the importance of process over product suggests that teaching practices emphasise both the techniques and reasoning used by students to solve problems, rather than simply trying to get the right answers. The apparent philosophical conflict in teacher beliefs was also found in the research conducted by Nathan and Koedinger (2000c).

Teachers who participated in the focus groups expressed that they *knew* that students would not use algebra, not just because they considered that the students would prefer to apply simpler arithmetic techniques to solve problems, but also because they considered that student knowledge of algebra was not considered to be developed sufficiently. Teacher sensitivity to the difficulties students encounter when first learning algebra appears to generate considerable motivation to positively influence student attitudes towards algebra. This tendency is linked to the importance placed on student success in algebraic learning and the motivation to provide problems that enable success to be achieved. Student success was considered to be influenced by the ability to maintain student interest and the nature of questions presented to students. Teachers commented that "if interest is high then success is high," and "if the situation is too tough then the students will give up, or if the language is too tough".

Teachers claimed that students often criticise the use of equations and algebra as being too removed from reality, and they therefore demand teachers to justify their efforts in learning algebra. Focus group participants unanimously confirmed that their purpose in presenting word problems to students is to provide applications of mathematics in the real world. Teacher comments included: "word problems represent everyday life", "word problems create a link with reality" and "word problems show that algebra is used". Even though there was a strong suggestion of an underlying teacher belief that the purpose of algebraic technique is to assist in explaining real life situations, teacher awareness that students would not use algebra for the single unknown word problems seems to point more towards the use of word problems as an attempt to increase student interest, as this type of problem could be solved using arithmetic.

Askew et al. (1997) consider the dynamic classroom interactions between teachers and their students to be the most significant influence on student learning. Given the relative time spent in the classroom, such interactions are also likely to be the most significant influence on teacher learning and practice. If, as a form of knowledge, beliefs are developed over time through life experiences and the finding of new evidence, the way in which teachers interpret student learning difficulties and the success of student strategies will influence beliefs and knowledge about how to best teach algebra. The danger of this process is that it is largely internal to the teacher and the classroom and dependent on the entry level of beliefs and knowledge held.

The aggregation of results in this study across four schools dilutes the complexity of the relationship between teaching and learning that occurs in the classroom. Boaler (1999, p. 260) discusses the theory of situated cognition where "the behaviours and practices of students in mathematical situations are not solely mathematical, nor individual, but are emergent as part of the relationship formed between learners and the people and systems of their environments". From this perspective the student's construction of knowledge is not only dependent on cognitive attributes but also on the social context and constraints of the learning environment—the mathematics classroom community. A class-based study that sought to identify the linkages between developing algebraic knowledge and classroom specific social and learning constructs could further enlighten how algebra could be taught most effectively.

6 CONCLUDING REMARKS

This research found that, like other countries where similar research has been undertaken, New Zealand high school students struggle to develop useful algebraic understandings. New Zealand students have a tendency towards using informal arithmetic strategies when solving problems expressly presented to them for the purpose of practising or assessing algebraic knowledge. Students are particularly successful in using informal strategies for elementary word problems that are so commonly used in the beginning algebra classroom.

Some scholars view the class of word problems that can be written as a linear equation with only a single occurrence of an unknown value not to be algebraic at all. Algebra is not, however, inherent within a problem. An algebraic problem is one where the solver perceives algebra to be the best means in finding a solution. The duality of being able to solve some word problems by either arithmetic or algebraic means creates a dilemma in their inclusion for instruction. It has been suggested (Reed, 1999) that elementary word problems be used to introduce algebra in order to use students' informal strategies as a point of departure towards algebraic techniques rather than as a climax to the use of algebraic techniques. Nathan and Koedinger (2000c) recognise that informal strategies are difficult to circumvent and do not view informal methods as replacements for informal methods. Rather, by building on student intuition, informal strategies can act as a bridge to understanding formal representations (Nathan & Koedinger, 2000c).

In order for teachers to be effective scaffolders of student learning, it is important they be aware of how any single task may impact on student learning. The purpose of using word problems when teaching algebra should be to enhance development of students' algebraic knowledge. If teachers anticipate that algebraic techniques should be used to solve word problems, then they first assume that students recognise those problems as algebraic. Establishing that word problems are algebraic is a pre-requisite for students to choose algebraic techniques. Moreover, if students are presented with problems that could be solved using either arithmetic or algebraic techniques, they need to be able to consider that algebra could be used. When students are aware of the option to use algebraic techniques, and word problems begin to be viewed as algebraic, then algebraic learning can occur. If, however, students fail to recognise the possibilities for using algebra, then appropriate algebraic schemata will fail to be formed. Alternatively, students may consider elementary word problems inappropriate for algebraic technique. In either case, in the event that students do not choose algebraic techniques, when this is the sole purpose of presenting word problems to them, then teaching objectives are not being met.

The year 10 students surveyed in this study displayed particularly low levels of algebraic thinking and poor algebraic skills. In conjunction, the prevalence of a symbolic precedence view held by teachers with an emphasis on achieving success in algebraic technique indicates a disparity between what is being taught and what is being learned. A down fall of teachers appears to be the desire to motivate their students through experiences of success and a perceived usefulness of mathematics. Advocating the kinds of word problems presently used in classrooms as being applications of the real world and promoting a goal oriented approach to problem solving condones and encourages the use of informal strategies by students. In order to promote algebraic thinking, teachers should present problems for which algebraic means of finding a solution is both preferred and optimal.

Teacher knowledge of the actual usefulness of algebra in the real world requires greater development. Teachers also need to be aware of the cognitive obstacles students face and the way in which students solve problems. A modification of the tasks presented to students that promote algebraic thinking (such as goal free problems (Sweller, 1999) and the more difficult multiple unknown word problems (MacGregor & Stacey, 1999)), and explicitly addressing common stumbling blocks to learning (such as equivalence (Thomas & Tall, 2001)), may help to discourage students from using informal strategies when algebraic ones would be better. Unless teacher beliefs and attitudes towards algebra word problems change, algebra will continue to be a source of frustration and anxiety for some students. Student inadequacies in algebraic thinking created at the elementary level may not necessarily show up until an academic level is reached when it may be too late to remedy.

A limitation in this study is that student attitudes toward mathematics and algebra were not explored in order to determine whether the pressure teachers feel to keep student interest is such that it overrides decisions on how best to teach algebra. Further research is needed to establish ways in which teachers can get students to 'buy-in' to algebra while keeping student interest levels high.

This discussion suggests that word problems be used with the utmost caution when teaching algebra. Students should be made explicitly aware of the purpose for a particular set task, such as word problems, and monitored carefully in their choice of strategy. Teacher foresight can provide a far greater awareness of the value of using algebra than is appreciable to most students. Therefore, the types of problems presented when teaching beginning algebra need to encourage students to 'buy-in' to the usefulness of algebra, and create student desire to construct algebraic knowledge.

APPENDIX 1: LETTER TO PRINCIPALS

Letterhead

Dear [Principal],

Mathematics Research Project

My name is Philip Bennett, and I am a teacher of mathematics at Hutt International Boys' School. As part of my continuing education, I am completing a thesis for a Master of Educational Studies (Mathematics) through Massey University.

I am writing to invite the Mathematics Department at [school]_to take part in this research. Your school has been selected from a cross section of schools in the Wellington region.

I have enclosed a letter to [HOD mathematics] along with an information sheet outlining the structure and purpose of my research. All of the data gathered will be aggregated so that the identity of your school and all participants will remain anonymous in any reporting.

Your support for this work will be greatly appreciated. The thesis is expected to be finished towards the end of the year. I will forward a summary of the results to the schools that take part in this research.

If you have any questions please phone me on (04) 973 3048 or contact my supervisor Glenda Anthony on (06) 350 5799. Thank you for your co-operation.

Yours sincerely

Philip Bennett

APPENDIX 2: LETTER TO HEADS OF MATHEMATICS DEPARTMENTS

Letterhead

Dear [Head of Mathematics],

My name is Philip Bennett, I am a teacher of mathematics at Hutt International Boys' School. As part of my continuing education, I am completing a thesis for a Master of Educational Studies (Mathematics) through Massey University. I am writing to invite staff in your department to take part in this research.

I have enclosed an information sheet outlining the structure and purpose of my research. I will attempt to contact you by telephone later this week to discuss any questions you may have and your possible involvement in this study. Your assistance in my work will be greatly appreciated. The thesis is expected to be finished towards the end of the year. I will prepare a summary of the results for your department.

Thank you for your co-operation.

Yours sincerely

Philip Bennett
99 Richmond St.
Petone
Phone: 9733048
Email: pbennett@hibs.school.nz

APPENDIX 3: INFORMATION SHEET FOR TEACHERS

Letterhead

The Teaching and Learning of Algebra in New Zealand Secondary Schools

INFORMATION SHEET

My name is Philip Bennett, I am a teacher of mathematics at Hutt International Boys' School. As part of my continuing education, I am completing a thesis for a Master of Educational Studies (Mathematics) through Massey University. My thesis is a qualitative study of the interaction of word problems between teaching and learning within the topic of algebra. Along with other schools, I would like to invite you to be part of my study. Your school has been selected from a cross section of schools in the Wellington region. All of the data gathered will be aggregated so that the identity of your school and all participants will remain anonymous in any reporting.

The research proposed seeks to investigate any differences in the techniques students use to solve algebra problems and those techniques anticipated and taught by teachers. Special focus will be placed on word problems and how they fit into the teaching and learning progression of algebra.

The first part of the data collection involves a teacher questionnaire that examines the beliefs that teachers have about the kinds of problems presented to students in terms of difficulty and beliefs about the teaching/learning of algebra. Teachers who choose to participate will also be invited to participate in a focus group to discuss problems presented to students and issues raised in the questionnaire. Additionally, I would like a teacher to present a set of algebra problems to one of your year 10 classes as a test and collect the responses for analysis.

It is important to note that participation in this study is entirely voluntary and any questions you do not wish to answer can be left blank. All information will be confidential and no identities will be revealed. It will be assumed that filling in the questionnaire implies consent for the use of this unidentified information in the preparation of my thesis and any subsequent papers or presentations relating to this project.

A summary of the results will be prepared for your department. If you have any further questions please contact me or my supervisor.

Researcher:
Philip Bennett
99 Richmond Street
Petone
Phone: 973 3048
Email: pbennett@hibs.school.nz

Supervisor:
Dr. Glenda Anthony
Head of Department
Technology, Science and Mathematics Education
Massey University
Private Bag 11-222
Palmerston North
Phone: (06) 350 5799

APPENDIX 4: INFORMATION SHEET FOR STUDENTS AND PARENTS

Letterhead

Dear Parent/Guardian,

My name is Philip Bennett. I am a teacher of mathematics at Hutt International Boys' School. As part of my continuing education, I am completing a research project on algebra learning and teaching for a Master of Educational Studies (Mathematics) through Massey University. Part of the study investigates the techniques that students use to answer algebra questions and then compares these to the techniques that teachers expect the students to use.

Permission is sought for your son/daughter to complete a short set of algebra questions (similar to standard questions that they will have encountered in class). The test will be administered within the normal classroom lesson and take 20 minutes – those not sitting the test will continue with parallel work set by the teacher. The results of the test do not contribute to any standard assessment results used in the normal classroom programme.

Within the research project all results will be aggregated and the identity of each student (names will be removed by the teachers prior to the teacher copying the papers) and their school will remain anonymous.

The research will provide valuable information for improving the teaching of algebra and anticipating the learning needs of future students. Each participating school will receive a summary of the findings later in the year.

Should you have any concerns or questions about this research project please feel free to contact the class teacher, myself (the researcher), or my supervisor:

Researcher:
Philip Bennett
99 Richmond Street
Petone
Phone: 973 3048
Email: pbennett@hibs.school.nz

Supervisor:
Dr. Glenda Anthony
Head of Department
Technology, Science and Mathematics Education
Massey University,
Private Bag 11-222
Palmerston North,
Phone: (06) 350 5799

If you **and** your son/daughter are happy to participate as outlined above and allow the class teacher to provide the researcher with an anonymous copy of their test responses please both sign the consent form below and return to the class maths teacher.

APPENDIX 5: CONSENT FORM

Letterhead

Student/Parent Consent Form –The Teaching and Learning of Algebra

We have read the information letter regarding the Algebra Research study and have had any queries answered to my satisfaction.

I understand that I have the right to withdraw from the study at any time and to decline to answer any particular questions.

We agree to allow the algebra test responses to be copied in order to be used in the algebra study on the understanding that my name cannot be identified.

I agree to participate in this study under the condition set out in the Information Sheet.

Student Name: ....

Signed (student): ....

Signed (parent/guardian) ....

Date:

APPENDIX 6: STUDENT TEST QUESTIONS

Result unknown questions

A rain water tank contains 81.9 litres after a rain fall one night. If there was originally 18 litres in the tank and it rained for 6 hours. How much rain per hour went into the tank that night?	Starting with 81.9, if I subtract 18 and then divide by 6, I get a number. What number do I get?	Solve for x where, $(81.90 - 18) \div 6 = x$
130 pieces of fudge were divided amongst Emily and four other friends, but Emily had already eaten 3 pieces. How many pieces of fudge did Emily get altogether?	Starting with 130, if I divide by 5 and then add 3, I get a number. What number do I get?	Solve for x where, $130 \div 5 + 3 = x$
Brian is in a rowboat on a lake. He is 800 metres from the shore. He rows towards the shore at a speed of 30 metres every minute. How far is Brian from the shore after 23 minutes?	Starting with 800, if I subtract the result of 30 multiplied by 23, I get a number. What number do I get?	Solve for x where, $800 - 30 \times 23 = x$

Start unknown questions

Betty won some money in a lottery. She kept \$64 for herself and gave each of her 3 sons an equal amount of the rest of it. If each son got \$20.50, how much did Betty get?	Starting with some number, if I subtract 64 and then divide the result by 3, I get 20.5. What number did I start with?	Solve for x where, $(x - 64) \div 3 = 20.5$
Luke bought 7 donuts and paid \$0.12 extra for a box to hold them. If he paid \$5.79 what was the price per donut?	Starting with some number, if I multiply it by 7 and then add 0.12, I get 5.79. What number did I start with?	Solve for x where, $7x + 0.12 = 5.79$
Kim is saving for a mountain bike that costs \$600. She earns \$20 per week by babysitting every Saturday afternoon. She saves all the money she earns. If Kim only needs to save \$260 more, how many weeks has she already been saving?	Starting with some number, if I multiply it by 20 and then subtract the result from 600, I get 260. What number did I start with?	Solve for x where, $600 - 20x = 260$

Multiple unknown questions

Roseanne just paid \$98.28 for a new pair of jeans. She got them at a 10% discount. What was the original price?	Starting with some number, if it is multiplied by 0.1 and then added to itself, the result is 98.28. What number did I start with?	Solve for x where, $x + 0.10x = 98.28$
There are 38 students in a class. If there are 6 more girls than there are boys, how many boys are in the class?	Starting with some number, if it is added to another number that is 6 more than itself, the result is 38. What is the starting number?	Solve for x where, $x + (x + 6) = 38$
To rent a car from Tiger Motors costs \$100 per day and 20 cents per km. To rent a car from Kiwi Motors costs \$120 per day and 15 cents per km. For what distance is each company the same price?	Starting with some number, if it is multiplied by 0.2 and then added to 100, the result is the same as when it is multiplied by 0.15 and then added to 120. What is the number?	Solve for x where, $0.20x + 100 = 0.15x + 120$

APPENDIX 7: TEACHER QUESTIONNAIRE

Cover Page

This questionnaire is designed to provide qualitative information from teachers of mathematics. The objective is to find out how teachers perceive different types of problems when teaching algebra. Please complete as many questions as possible. Note that all responses you give are voluntary and will be recorded anonymously. Some of the demographic information will help in interpreting data.

Are you Male or Female? (circle one)

What year levels of mathematics do you teach? (tick appropriate bo

☐	☐	☐	☐	☐	☐	☐
7	8	9	10	11	12	13

How many years have you been teaching? ________

How many years have you taught mathematics? ________

Do you have a degree in mathematics? [Y / N] Please specify________________

____

Most advanced level of math education____________________________________

____

Select the snapshot closest to your own classroom teaching

A. Tends to be center of classroom, enjoys lecturing and modeling good performance for students
B. Runs a very open-ended classroom, with much student-led discussion

Select the snapshot closest to your own classroom materials

A. Class material is considered traditional, with strong emphasis on skills
B. Class material is considered reform-based, with emphasis on group process and problem solving

Read each statement carefully. Circle the letter that corresponds most accurately with your beliefs about the accompanying statement.

(1) Strongly agree	(2) Agree	(3) Agree more than disagree	(4) Disagree more than agree	(5) Disagree	(6) Strongly disagree

1	Mathematical understanding is more clearly shown in a student's reasoning than in the final answers that student produces.	1	2	3	4	5	6
2	Students must practice the algebraic procedures before they can solve complex story problems.	1	2	3	4	5	6
3	Math problems presented in words are more difficult for students than the identical problems presented symbolically.	1	2	3	4	5	6
4	Expertise in complex story problem solving requires proficiency in symbol manipulation.	1	2	3	4	5	6
5	Students may have a variety of effective methods for problem solving prior to explicit algebraic instruction.	1	2	3	4	5	6
6	When a student uses an arithmetic approach to solve an algebra word problem, that indicates a weakness in that student's math abilities.	1	2	3	4	5	6
7	Given appropriate instructional materials, students can create useful algorithms for algebra story problems.	1	2	3	4	5	6
8	The lack of general mathematical reasoning skills causes students the greatest difficulty in algebra.	1	2	3	4	5	6
9	The most effective method for solving a complex story problem is translating the relevant information into algebraic equations.	1	2	3	4	5	6
10	Students should be encouraged to invent their own methods to solve mathematics problems.	1	2	3	4	5	6
11	Good problem solvers use whatever means are necessary to solve an algebra story problem.	1	2	3	4	5	6
12	Algebra students should be discouraged from providing answers without an algebraic explanation of how they arrived at that solution.	1	2	3	4	5	6
13	Algebra teachers need to encourage mathematics students who can produce correct answers by using untraditional approaches.	1	2	3	4	5	6
14	Students best acquire mathematical knowledge when taught specific methods for solving problems.	1	2	3	4	5	6

15	Individual students learn problem solving best by trying to solve a problem by themselves with little or no help from the teacher.	**1**	**2**	**3**	**4**	**5**	**6**
16	Story problems are more difficult for students to solve than the same problem presented as a mathematical equation.	**1**	**2**	**3**	**4**	**5**	**6**
17	Students' methods of solving problems develop best if they are given some freedom in their mathematical thinking.	**1**	**2**	**3**	**4**	**5**	**6**
18	Use of an arithmetic method to solve an algebra story problem shows an adaptive approach to problem solving.	**1**	**2**	**3**	**4**	**5**	**6**
19	It is more useful to allow students time to explore some tasks thoroughly than to cover all of the curriculum materials.	**1**	**2**	**3**	**4**	**5**	**6**
20	Teachers can learn more from a student's answers than from that student's reasoning.	**1**	**2**	**3**	**4**	**5**	**6**
21	Students should learn most of their methods of solving algebra problems from textbooks and teachers.	**1**	**2**	**3**	**4**	**5**	**6**
22	Students can have successful methods for solving problems even if they have not learned the basic skills.	**1**	**2**	**3**	**4**	**5**	**6**
23	Knowing how to perform a mathematical procedure is more important than understanding why the procedure works.	**1**	**2**	**3**	**4**	**5**	**6**
24	Teachers should demonstrate the right way to do a problem before the students try to work it out.	**1**	**2**	**3**	**4**	**5**	**6**
25	The goals of instruction in mathematics are best achieved when students find their own methods for solving problems.	**1**	**2**	**3**	**4**	**5**	**6**
26	The greatest difficulty most students have with algebra story problems is manipulating symbols.	**1**	**2**	**3**	**4**	**5**	**6**
27	Rewarding right answers and correcting wrong answers is an important part of teaching.	**1**	**2**	**3**	**4**	**5**	**6**
28	Students need explicit instruction in order to tackle complex algebra story problems.	**1**	**2**	**3**	**4**	**5**	**6**
29	Students who apply solution methods that do not use symbol manipulation are at risk of developing bad habits that will inhibit their mastery of algebra skills.	**1**	**2**	**3**	**4**	**5**	**6**
30	Pre-algebra students need explicit instruction on how to solve story problems.	**1**	**2**	**3**	**4**	**5**	**6**
31	There are many effective approaches to solving any algebra story problem, and manipulating symbols is only one method.	**1**	**2**	**3**	**4**	**5**	**6**

32	Key word methods are ideal ways for students to solve story problems.	**1**	**2**	**3**	**4**	**5**	**6**
33	The ability to solve an algebra story problem in multiple ways, including non-algebraic ones, is a sign of mathematical proficiency.	**1**	**2**	**3**	**4**	**5**	**6**
34	Students' uses of alternative problem-solving methods greatly simplifies the solution of mathematical story problems.	**1**	**2**	**3**	**4**	**5**	**6**
35	Intuitive mathematical reasoning can produce valid solutions to mathematical problems even if the solutions differ from what is taught.	**1**	**2**	**3**	**4**	**5**	**6**
36	Teachers should encourage students' own solution approaches to algebra problems even if the solutions are inefficient.	**1**	**2**	**3**	**4**	**5**	**6**
37	Getting the correct answer is a better indicator of learning than is the ability to articulate a good solution approach.	**1**	**2**	**3**	**4**	**5**	**6**
38	Using algebra for story problem solving is the most effective approach there is.	**1**	**2**	**3**	**4**	**5**	**6**
39	Arithmetic story problems are easier for students to solve than algebra story problems.	**1**	**2**	**3**	**4**	**5**	**6**
40	When a student uses a "guess and check" approach to solve an algebra word problem that indicates a weakness in that student's math abilities.	**1**	**2**	**3**	**4**	**5**	**6**
41	Students should always be given algebra problems that require them to manipulate symbols.	**1**	**2**	**3**	**4**	**5**	**6**
42	All algebra story problems can be solved without using algebra.	**1**	**2**	**3**	**4**	**5**	**6**
43	Use of a "guess and check" strategy to solve an algebra story problem shows an adaptive approach to problem solving.	**1**	**2**	**3**	**4**	**5**	**6**
44	Students enter the algebra classroom with intuitive methods for solving algebra story problems.	**1**	**2**	**3**	**4**	**5**	**6**
45	Algebra is more a type of mathematical reasoning rather than a specific notation and set of procedures.	**1**	**2**	**3**	**4**	**5**	**6**
46	Most students cannot figure out for themselves how to solve algebra story problems.	**1**	**2**	**3**	**4**	**5**	**6**
47	How a student works through a problem is a better indicator of understanding than the final answer a student provides.	**1**	**2**	**3**	**4**	**5**	**6**

Thank you for your time and thoughts. Please return the questionnaire in the enclosed envelope.

You are also invited to contribute further by participating in a focus group to discuss the comparative difficulty of the problems presented to students. The purpose of the focus group is to discuss the reasons students find some types of algebra problems more difficult than others. If interested please complete the enclosed 'interest form' and return in the separate envelope—this will ensure the anonymity of this questionnaire. Those interested will be contacted through the school to arrange a convenient time and venue for discussion and I will provide a consent form to complete.

APPENDIX 8: FOCUS GROUP—EXPRESSION OF INTEREST FORM

Letter head

Interest Form for participating in Focus Groups

I am interested in participating in a focus group discussion about students' algebra learning, especially in relation to word problems. I understand that you will contact me – at that stage I can ask further questions about the study and discuss suitable dates and venue. If I am willing to be involved at that point I will complete the consent form for the focus groups interviews.

Name..

School..

Preferred Contact Information (e.g., phone/email/fax)

....

....

....

APPENDIX 9: FOCUS GROUP—CONSENT FORM

Letterhead

The Teaching and Learning of Algebra in New Zealand Secondary Schools

I have read the information sheet and have had details of the study explained to me. My questions have been answered to my satisfaction, and I understand that I may ask further questions at any time.

I understand that I have the right to withdraw from the study at any time and to decline to answer any particular questions.

- I agree to provide information to the researcher on the understanding that my name will not be used without my permission.
- I agree to the interview being audio taped.
- I also understand that I have the right at any time during the focus group discussion to ask for any or all of my individual comments to be removed from the transcript of the session.
- I agree to keep the content and identities of other participants in the focus group confidential.

I agree to participate in this study under the conditions set out in the Information Sheet.

Name: ....

School: ....

Signed: ....

Date: ....

REFERENCES

Anderson, G. (1990). *Fundamentals of educational research.* London: The Falmer Press.

Anderson, G., & Arsenault, N. (1998). *Fundamentals of educational research* (2nd ed.). London: The Falmer Press.

Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy: Report of a study carried out for the Teacher Training Agency 1995-96 by the School of Education King's College London.* London: King's College.

Ball, D., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83-104). Westport, CT: Ablex Publishing.

Baranes, J., Perry, M., & Stigler, J. (1989). Activation of real-world knowledge in the solution of word problems. *Cognition and Instruction, 6*(4), 287-318.

Bell, A. (1996). Problem-solving approach to algebra: Two aspects. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra, perspectives for research and teaching* (pp. 167-185). Dordrecht: Kluwer .

Bell, E. T. (1940). *The development of mathematics*. New York: McGraw-Hill .

Blessing, S., & Ross, B. (1996). Content effects in problem catagorization and problem solving. *Journal of Experimental Psychology: Learning Memory and Cognition, 22*, 792-810.

Boaler, J. (1993). The role of contexts in the mathematics classroom: Do they make mathematics more real? *For the Learning of Mathematics, 13*(2), 12-17.

Boaler, J. (1994). When do girls prefer football to fashion? An analysis of female underachievement in relation to "realistic" mathematics contexts. *British Educational Research Journal, 20*(5), 551-564.

Boaler, J. (1999). Participation, knowledge and beliefs: A community perspective on mathematical learning. *Educational Studies in Mathematics, 40*(3), 259-281.

Bobrow, D. (1968). Natural language input for a computer problem-solving system. In M. Minsky (Ed.), *Semantic information processing* (pp. 146-226). Massachusetts: The MIT Press.

Boote, D. (1998). Physics word problems as exemplars for enculturation. *For the Learning of Mathematics, 18*(2), 28-33.

Booth, L. R. (1988). Childrens difficulties in beginning algebra. In *The ideas of Algebra, K-12, NCTM Yearbook* (pp. 20-32). Reston, VA: NCTM.

Boyer, C. (1968). *A history of mathematics*. New York: John Wiley & Sons, Inc.

Bullock, J., Thomas, M., & Tyrrell. (1994). Technology in the mathematics classroom: The TIME project. In S. Wawrzyniak, & L. Samootin (Eds.), *"Ask me why?": Proceedings of the 10th annual New South Wales computers in education conference* (pp. 69-76). Sydney: New South Wales Computer Education Group.

Burns, R. (1994). *Introduction to research methods* (2nd ed.). Melbourne: Longman Cheshire.

Burns, R. (2000). *Introduction to research methods* (4th ed.). Melbourne: Longman .

Burrell, G., & Morgan, G. *Sociological paradigms and organisational analysis*. London: Heinemann Educational Books.

Buzeika, A. (1996). Teachers' beliefs and practice: The chicken or the egg? *New Zealand Mathematics Magazine, 33*(3), 6-16.

Cajori, F. (1917). *A history of elementary mathematics*. New York: Macmillan.

Cajori, F. (1928). *A history of mathematical notations*. La Salle, Illinois: The Open Court Publishing Company.

Carpenter, T. & Franke, M. (2001). Developing algebraic reasoning in the elementary school: Generalization and proof. *International Congress of Mathematical Instruction (ICMI) Working Group Proceedings - The future of the teaching and learning of algebra,* (pp. 155-162). Melbourne.

Carraher, T., Carraher, D., & Schliemann, A. (1987). Written and oral mathematics. *Journal for Research in Mathematics Education, 18*, 83-97.

Carroll, W. (1994). Using worked examples as an instructional support in the algebra classroom. *Journal of Educational Psychology, 86*(3), 360-367.

Chaiklin, S. (1992). Cognitive studies of algebra problem solving and learning. In S. Wagner, & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 93-114). Reston, VA: NCTM.

Chandler, P., & Sweller, J. (1992). The split-attention effect as a factor in the design of instruction. *British Journal of Educational Psychology, 62*, 233-246.

Clark, D. (1993). Influences on the changing role of the mathematics teacher. In B. Atweh, C. Kanes, M. Carrs, & G. Booker (Eds.), *Contexts in mathematics education* . Brisbane: Mathematics Education Research Group of Australasia.

Clement J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13*, 16-30.

Clement, J., Lochhead, J., & Monk, G. (1981). Translation difficulties in learning mathematics. *American Mathematical Monthly, 88*, 286-289.

Cohen, L., & Manion, L. (1994). *Research methods in education.* London: Routledge.

Cooper, B., & Dunne, M. (1998). Anyone for tennis? Social class differences in children's responses to national curriculum mathematics testing. *The Sociological Review, 88*(January), 115-148.

Cooper, G., & Sweller, J. (1987). Effects of schema acquisition and rule automation on mathematical problem-solving transfer. *Journal of Educational Psychology, 79*(4), 347-362.

Crowley, L., Thomas, M., & Tall, D. (1994). Algebra, symbols and translation of meaning. In J. Du Ponte & J. Matos (Eds.), *Proceedings of the 18th annual conference of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 240-247).* Lisbon, Portugal: PME.

Cummins, D., Kintsch, W., Ruesser, K., & Weimer, R. (1988). The role of understanding in solving word problems. *Cognitive Psychology, 20*, 439-462.

Davis, R. (1984). *Learning mathematics. The cognitive science approach to mathematics education.* London: Croom Helm.

Demby, A. (1997). Algebraic procedures used by 13-to-15-year-olds. *Educational Studies in Mathematics, 33*, 45-70.

Ellerton, N. F., & Clements, M. A. (1992). Implications of Newman research for the issue of "What is basic in school mathematics?". In B. Southwell, B. Perry, & K. Owens (Eds.), *Space: The first and final frontier* (pp. 276-284). Sydney, Australia: Mathematics Education Research Group of Australia.

Falkner, K., & Carpenter, T. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6*(4), 232-236.

Fennema, E., & Franke, M. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: MacMillan.

Fisher, K. (1988). The student-and-professors problem revisited. *Journal for Research in Mathematics Education, 19*(3), 260-262.

Forbes, S. (1999). *Measuring student's education outcomes : Sex and ethnic differences in mathematics* . Unpublished doctoral dissertation, Curtin University of Technology, Perth, WA.

Gerofsky, S. (1996). A linguistic and narrative view of word problems in mathematics education. *For the Learning of Mathematics, 16*(2), 36-45.

Gray, J. (1987). Unit 12, style formalism in the eighteenth centuries. In J. Fauvel, & J. Gray (Eds.), *The seventeenth and eighteenth centuries* . Milton Keynes: Open Universtiy Press.

Greer, B. (1997). Modelling reality in mathematics classrooms: The case of word problems. *Learning and Instruction, 7*(4), 273-307.

Hall, R., Kibler, D., Wenger, E., & Truxaw, C. (1989). Exploring the episodic structure of algebra story problem solving. *Cognition and Instruction, 6*(3), 223-283.

Harper, E. (1987). Ghosts of Diaphantus. *Educational Studies in Mathematics,18*, 75-90.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner, & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (Vol. 4pp. 60-86). Reston, VA: NCTM.

Hewitt, D. (1985). Equations. *Mathematics Teaching, 111*, 15-16.

Hinsley, D., Hayes, J., & Simon, H. (1977). From words to equations meaning and representation in algebra word problems. In M. Just, & P. Carpenter (Eds.), *Cognitive processes in comprehension* (pp. 89-106). Hillsdale, NJ: Lawrence Earlbaum.

Janvier, C. (1996). Modeling and the initiation into algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra, perspectives for research and teaching* (pp. 225-236). Dordrecht: Kluwer .

Kieran, C. (1990). Cognitive processes involved in learning school algebra. In P. Nesher, & Kilpatrick (Eds.), *Mathematics and cognition* (pp. 96-112). Cambridge: Cambridge university press.

Kieran, C. (1992a). The early learning of algebra: A structural perspective. In NCTM *Research issues in the learning and teaching of algebra* (pp. 33-56). Hillsdale, NJ: Lawrence Erlbaum.

Kieran, C. (1992b). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Kieran, C. (1997). Mathematical concepts at the secondary school level: The learning of algebra and functions. In T. Nunes, & P. Bryant (Eds.), *Learning and teaching mathematics, An international perspective* (pp. 133-162). East Sussex: Psychology Press.

Kieran, C., Boileau, A., & Garançon, M. (1996). Introducing algebra by means of a technology-supported, functional approach. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra, perspectives for research and teaching* (pp. 257-293). Dordrecht: Kluwer .

Kieran, C., Oates, G., & Thomas, M. (1998). Graphic calculators and mathematics assessment. *SAMEpapers,* 88-106.

Koedinger, K. R., Alibali, M. W., & Nathan, M. J. (2001). *Trade-offs between grounded and abstract representations: Evidence from algebraic problem solving.*

Kota, S., & Thomas, M. (1998). Students' arithmetic preferences: Effect on problem solving ability. *Hiroshima Journal of Mathematics Education, 6*, 33-47.

Krueger, R. A., & Casey, M. A. (2000). *Focus groups: A practical guide for applied research* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11 - 16* (pp. 102-119). Oxford: John Murray.

Küchemann, D. (1983). Quantitative and formal methods for solving equations. *Mathematics in School, 12*(5), 17-19.

Lave, J. (1993). Word problems: A microcosm of theories of learning. In P. Light, & G. Butterworth (Eds.), *Context and cognition, ways of learning and knowing* (pp. 74-92). Hillsdale, NJ: Lawrence Erlbaum.

Lee, L. (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra, perspectives for research and teaching* (pp. 87-106). Dordrecht: Kluwer .

Leung, M., Low, R., & Sweller, J. (1997). Learning from equations or words. *Instructional Science, 25*(1), 37-70.

Lewis, C. (1981). Skill in algebra. J. R. Anderson (Ed.), *Cognitive skills and their aquisition* (pp. 85-110). Hillsdale, NJ: Lawrence Erlbaum.

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics, 30*, 39-65.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics, 40*, 173-196.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Newbury Park, California: Sage Publications Inc.

Lincoln Yvonna S, & Guba Egon G. (2000). Paradigmatic controversies, contradictions, and emerging confluences. In Denzin Norman K, & Lincoln Yvonna S (Eds.), *Handbook of Qualitative Research* (2nd ed., pp. 163-188). Thousand Oaks, California: Sage Publications Inc.

Lopez-Real, F. (1996). Secondary pupils' translation of algebraic relationships into everyday language: A Hong Kong study. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th annual conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp. 273-280).* Valencia, Spain: PME.

MacGregor, M., & Price, E. (1999). An exploration of aspects of language proficiency and algebra learning. *Journal for Research in Mathematics Education, 30*(4), 449.

MacGregor, M., & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education, 24*(3), 217-232.

MacGregor, M., & Stacey, K. (1996a). Learning to formulate equations for problems. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th annual conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp. 289-296).* Valencia, Spain: PME.

MacGregor, M., & Stacey, K. (1996b). Origins of students' interpretations of algebraic notation. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th annual conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp. 297-304).* Valencia, Spain: PME.

MacGregor, M., & Stacey, K. (1999). A flying start to algebra. *Teaching Children Mathematics, 6*(2), 78-85.

Malone, J., & Ireland, D. (1996). Constructivist research on teaching and learning mathematics. In B. Atweh, K. Owens, & P. Sullivan (Eds.), *Research in mathematics education in Australasia 1992-1995* (pp. 119-134). Sydney: Mathematics Education Research Group of Australasia.

Mayer, R. (1981). Frequency norms and structural analysis of algebra story problems into families, catagories, and templates. *Instructional Science, 10*, 135-175.

Mayer, R. (1982). Memory for algebra story problems. *Journal of Educational Psychology, 74*(2), 199-216.

Mayer, R. (1983). *Thinking, problem solving, cognition.* New York: W.H.Freeman and Company.

Mayer, R. (1992). *Thinking, problem solving, cognition* (2nd ed.). New York: W.H.Freeman and Company.

Mayer, R., Larkin, J., & Kadane, J. (1984). A cognitive analysis of mathematical problem solving ability. In R. Sternberg (Ed.), *Advances in the psychology of human intelligence* (2nd ed., Vol. 2pp. 231-273). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Mayers, C. (1994). Mathematics and mathematics teaching: Changes in pre-service student-teachers' beliefs and attitudes. In G. Bell, B. Wright, N. Leeson, & J. Geake (Eds.), *Challenges in mathematics education: constraints on construction* (Proceedings of the 17th annual conference of the Mathematics Education Research Group of Australasia, pp. 419-428). Lismore: MERGA.

Mayers, C., & Britt, M. (1995). Constructivism in the mathematics classroom. In J. Neyland (Ed.), *Mathematics education: A handbook for teachers* (Vol. 2pp. 60-69). Wellington, NZ: The Wellington College of Education.

McGee, C. & Penlington, C. (2001). *Research on learning, curriculum and teachers' roles. Report 4, Teacher knowledge, beliefs and thinking.* Hamilton, NZ: Waikato Institute for Research in Learning and Curriculum, University of Waikato.

McNamara, O. (1996). Mathematics and the sign. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th annual conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp.369-376).* Valencia, Spain: PME.

Mekhmandarov, I., & Meron, R. (1996). Performance and understanding: A closer look at comparison word problems. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th annual conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp.385-390).* Valencia, Spain: PME.

Merriam, S. (1998). *Qualitative research and case study application in education.* San Francisco,CA: Jossey-Bass Publishers.

Ministry of Education. (1992). *Mathematics in the New Zealand curriculum.* Wellington: Learning Media.

Mishler, E. G. (1986). *Research interviewing: Context and narrative.* Cambridge, Massachusetts: Harvard University Press.

Morgan, C. (1996). "The language of mathematics":Towards a critical analysis of mathematics texts. *For the Learning of Mathematics, 16*(3), 2-10.

Morris, B. (1989). Textbooks: Over-relied on but under-utilised. *Australian Journal of Reading, 12*(4), 312-329.

Nathan, M. J. (2001). Personal correspondence.

Nathan, M. J., Kintsch, W., & Young, E. (1992). A theory of algebra-word-problem comprehension and its implications for the design of learning environments. *Cognition and Instruction, 9*(4), 329-389.

Nathan, M. J., & Koedinger, K. R. (2000a). An investigation of teachers' beliefs of students' algebra development. *Cognition and Instruction, 18*(2), 209-237.

Nathan, M. J., & Koedinger, K. R. (2000b). Moving beyond teachers' intuitive beliefs about algebra learning. *Connecting Research to Learning, 93*(3), 218-222.

Nathan, M. J., & Koedinger, K. R. (2000c). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education, 31*(2), 168-190.

National Research Council. (2002a). *Learning and Understanding: Improving Advanced Study of Mathematics and Science in U.S. High Schools* . Georgetown: National Academy Press.

National Research Council. (2002b). *Learning and Understanding: Improving Advanced Study of Mathematics and Science in U.S. High Schools: Report of the Content Panel for Mathematics* . Georgetown: National Academy Press.

NCTM. (1994). The influences of teachers' beliefs and knowledge on learning environments: Implementing the professional standards for teaching mathematics. *Arithmetic Teacher, 41*(8), 476-480.

NCTM. (2000). Principles and standards for school mathematics. Reston, VA: NCTM.

Nickson, M. (2000). *A teachers guide to recent research and its application*. London: Cassell Education.

NZQA. (1997). *Chief marker's report and marking schedule 1997: School certificate mathematics*. Wellington: NZQA.

NZQA. (1998). *Chief marker's report and marking schedule 1998: School certificate mathematics*. Wellington: NZQA.

NZQA. (1999). *Chief marker's report and marking schedule 1999: School certificate mathematics*. Wellington: NZQA.

NZQA. (2000). *Chief marker's report and marking schedule 2000: School certificate mathematics*. Wellington: NZQA.

Pajares, F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*, 307-332.

Phillips, E., & Lappan, G. (1998). Algebra: The first gate. In L. Leutzinger (Ed.), *Mathematics in the Middle School* . VA: National Council of Teachers of Mathematics.

Pirie, S., & Martin, L. (1997). The equation, the whole equation and nothing but the equation! One approach to the teaching of linear equations. *Educational Studies in Mathematics, 34*, 159-181.

Reed, S. (1999). *Word problems, research and curriculum reform*. Mahwah, NJ: Lawrence Erlbaum.

Rojano, T. (1996). The role of problems and problem solving in the development of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra, perspectives for research and teaching* (pp. 55-62). Dordrecht: Kluwer .

Rovio-Johansson, A. (1999). Constituting different meanings of the content of teaching and learning in higher education. *8th European Conference for Learning and Instruction* Göteborg, Sweden.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: MacMillan.

Schoenfeld, A. (1998). Reflections on a course in mathematical problem solving. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 81-113). Rhode Island: American Mathematical Society.

Sebrechts, M. M., Enright, M., Bennett, R. E., & Martin, K. (1996). Using algebra word problems to assess quantitative ability: Attributes, strategies, and errors. *Cognition and Instruction, 14*(3), 285-343.

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification - The case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Shield, M. (1991). Mathematics textbooks: How are they used? *Australian Journal of Reading, 14*(1), 60-68.

Shield, M. (1998). Mathematics textbooks: Messages to students and teachers. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times* (Proceedings of the 21st Annual Conference of the Mathematics Education Research Group of Australasia Incorporated, pp. 516-523). Brisbane : MERGA.

Sierpinska, A. (1995). Mathematics: "In Context", "pure", or "with applications"?*. *For the Learning of Mathematics, 15*(1), 2-15.

Silver, E. (1979). Student perceptions of relatedness among mathematical verbal problems. *Journal for Research in Mathematics Education, 10*, 195-210.

Silver, E. (1981). Recall of mathematic problem information: Solving related problems. *Journal for Research in Mathematics Education, 12*, 54-64.

Simmons, P., Emory, A., Carter, T., Coker, T., Finnegan, B., Crockett, D., Richardson, L., Yager, R., Craven, J., Tillotson, J., Brunkhorst, H., Twiest, M., Hossain, K., Gallagher, J., Duggan-Haas, D., Parker, J., Cajas, F., Alshannag, Q., McGlamery, S., Krockover, J., Adams, P., Spector, B., LaPorta, T., James, B., Rearden, K., & Labuda, K. (1999). Beginning teachers: Beliefs and classroom actions. *Journal of Research in Science Teaching, 36*(8), 930-953.

Stacey, K. (1994). Arithmetic with a calculator: What do children need to learn? In G Bell, B Wright, N. Lesson, & J. Geake (Eds.), *Challenges in mathematics education: constraints on construction* (Proceedings of the 17th annual conference of the Mathematics Education Research Group of Australasia, pp. 563-570). Lismore: MERGA.

Stacey, K., & MacGregor, M. (1997). Ideas about symbolism that students bring to algebra. *The Mathematics Teacher, 90*(2), 110-113.

Stacey, K., & MacGregor, M. (1999). Taking the algebraic thinking out of algebra. *Mathematics Education Research Journal, 11* (1), 24-38.

Stipek, D., Givvin, K., Salmon, J., & MacGyvers, V. (2001). Teachers' beliefs and practices related to mathematics instruction. *Teaching and Teacher Education, 17*, 213-226.

Sweller, J. (1990). Cognitive processes and instructional procedures. *Australian Journal of Education, 34*(2), 125-130.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction, 4*, 295-312.

Sweller, J. (1999). *Instructional design in technical areas*. Camberwell, Victoria: Acer Press.

Sweller, J., & Chandler, P. (1994). Why some material is difficult to learn. *Cognition and Instruction, 12*(3), 185-233.

Tall, D. (2001). Reflections on early algebra. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th* conference *of the International groups for the Psychology of Mathematics Education* (Vol. 1, pp. 149-152). Utrecht: PME.

Thomas, M. (1994). Process oriented preference in the writing of algebraic equations. In G. Bell, B. Wright, N. Leeson, & J. Geake (Eds.), *Challenges in mathematics education: constraints on construction* (Proceedings of the 17th annual conference of the Mathematics Education Research Group of Australasia, pp. 599-606). Lismore: MERGA.

Thomas, M. (1995). Two major difficulties for secondary school algebra students constructing mathematical thinking. *SAMEpapers,* 239-259.

Thomas, M., & Kota, S. (1996). Problem solving in algebra: The role of arithmetic thinking and affective factors. In P. Clarkson (Ed.), *Technology in mathematics education* (proceedings of the 19th annual conference of the Mathematic Research Group of Australasia, pp. 564-571). Melbourne: MERGA.

Thomas, M., & Tall, D. (2001). The long-term cognitive development of symbolic algebra. In H. Chick & K. Stacey & J. Vincent & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th study conference of the International Commission of Mathematical Instruction, pp. 590-587). Melbourne: The University of Melbourne.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: MacMillan.

Thornton, S. (2001). New approaches to algebra: Have we missed the point? *Mathematics Teaching in the Middle School, 6*(7), 388-392.

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics, 35*, 51-64.

Usiskin, Z. (1980). What should not be in the algebra and geometry curricula of average college bound students. *The Mathematics Teacher, 73*, 413-424.

Vaughn, S., Schumm, J. S., & Sinagub, J. (1996). *Focus group interviews in education and psychology*. Thousand Oaks, California: Sage Publications, Inc.

Verschaffel, L., & De Corte, E. (1997). Word problems: A vehicle for promoting authentic mathematical understanding and problem solving in the primary school. In T. Nunes, & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 69-97). East Sussex: Psychology Press.

Verschaffel, L., De Corte, E., & Lasure, S. (1994). Realistic considerations in mathematical modeling of school arithmetic word problems. *Learning and Instruction, 4*, 273-294.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems*. Netherlands: Swets & Zeitlinger.

Waits, B., & Demana, F. (2000). Calculators in mathematics teaching and learning: Past, present, and future. In M. Burke, & F. Curcio (Eds.), *Learning mathematics for a new century: NCTM 2000 yearbook* (pp. 51-66). Reston, Virginia: National Council of Teachers of Mathematics.

Wheeler, D. (1996). Backwards and forwards: Reflections on different approaches to algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra, perspectives for research and teaching* (pp. 317-325). Dordrecht: Kluwer .

White, A. (1998). Beliefs about the use of calculators in an upper primary school classroom: A partial application of the theory of planned behaviour. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times* (Proceedings of the 21st Annual Conference of the Mathematics Education Research Group of Australasia Incorporated, pp. 685-692). Brisbane: MERGA.

Wilder, R. (1968). *Evolution of mathematical concepts; an elementary study*. New York: Wiley.

Woods, D. (1996). *Teacher cognition in language teaching: Beliefs, decision-making and classroom practice*. Cambridge: Cambridge University Press.

Wright, V. (1998). The learning and teaching of algebra: Patterns, problems and possibilities. *'State of the Art' Maths Education Seminar,* Wellington, NZ.

Wyndhamn, J., & Säljö, R. (1997). Word problems and mathematical reasoning - A study of children's mastery of reference and meaning in textual realities. *Learning and Instruction, 7*(4), 361-382.

Yoshida, H., Verschaffel, L., & De Corte, E. (1997). Realistic considerations in solving problematic word problems: Do Japanese and Belgian children have the same difficulties? *Learning and Instruction, 7*(4), 329-338.

Zevenbergen, R. (2000). "Cracking the code" of mathematics classrooms: School success as a function of linguistic, social, and cultural background. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 201-223). Westport, CT: Ablex Publishing.