Intuitive Transformation Geometry and Frieze Patterns

A thesis presented in partial fulfilment of the requirements for the degree of

MASTER OF SCIENCE

in

Mathematics Education

at

Massey University

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1992

Abstract

This study on intuitive frieze pattern construction and description was set up as an attempt to answer part of a general question: "Do students bring intuitive transformation geometry concepts with them into the classroom and, if so, what is the character of those concepts?" The motivation to explore this topic arose, in part, from the particular relevance that transformation geometry has to New Zealand: *kowhaiwhai* (Maori rafter patterns) are examples of frieze patterns and are suggested by recent mathematics curriculum documents as a way for Form 3 and 4 students to explore transformations.

When very few restrictions were put on the subjects, frieze patterns made by Standard 3 and 4 students displayed evidence of the use of transformations such as translation, vertical reflection, and half-turn. Transformations, such as horizontal reflection and glide reflection, were very rarely used by themselves. However, from the frieze group analysis alone, no strong conclusions could be drawn about the frieze patterns featuring a combination of two or more different symmetry types (besides translation). The Form 4 class surveyed showed similar results, with an increase in the proportion of students using half-turn by itself. Another contrast between the two age groups was the production of disjoint and connected patterns: the Primary students' patterns were mostly disjoint, whereas the Secondary students made almost equal numbers of disjoint and connected designs.

In a restricted frieze construction activity, which required the subjects to use asymmetric objects (right-angled scalene triangles), the use of non-translation transformations reduced considerably from the first exercise, although vertical reflection was still popular amongst 70% of the Primary students. However, the results of a small survey of 10 children suggested that if the strips to be filled in are aligned vertically, the rarer symmetries such as glide reflection may be used more easily than in the horizontal case.

The style analysis revealed that the Primary (pre-formal) and Tertiary (post-formal) groups were quite similar in the patterns they drew under the restricted conditions, and therefore in the probable construction methods used to produce them. The Form 4's patterns differed in several ways, especially by their extensive use of half turn and tilings. It seems that the Fourth Form students were affected by the formal transformation geometry framework to which they had been recently exposed.

Interviews of 10 Primary students provided information about the intentions and methods used to construct the frieze patterns under both restricted and unrestricted conditions. The case studies revealed that several standard approaches to frieze pattern construction were employed, none of which corresponded with the mathematical structure of a symmetry group. It was also found that a number of methods could be used to make the same pattern. The qualitative analysis highlighted some shortfalls of the quantitative approach. For example, some students used transformations not detected by the frieze group analysis, and some symmetries present in the children's patterns were incidental (a spin-off of another motivation) or accidental. Ambiguities in pattern classification also arose.

The Primary children's descriptions of the seven different frieze groups (which were discrete examples) displayed several characteristic features. For instance, they often used a form of simile or metaphor, comparing a pattern part to a real world object with the same set of symmetries. In addition, many children considered a pattern's translation unit to be 'the pattern'. In this case, the interviews suggested that the repetition (translation) was obvious to the students. Also interesting was the tendency of these subjects to write down orientation or direction judgements, omitting the relationships between adjacent congruent figures within a pattern. However, the Primary children did use more explicit transformation terminology when able to describe the patterns orally. A peculiar feature of these explanations was that the symmetry described was often not differentiable from another symmetry. For example, to a child, the phrase "turn upside down" can mean a half-turn or a horizontal reflection or both; the result *is* identical in many cases.

Secondary and Tertiary students tended not to use implicit phrases in their pattern descriptions but were more explicit and precise, using a wider range of criteria in their descriptions. The results from this activity also indicated that the Primary and older students alike did not perceive the patterns to extend infinitely beyond the confines of the the page, highlighting another difference between the mathematical structure of a symmetry group and the intuitive cognitive processes of the students.

An additional matching activity was conducted in the interviews, requiring the subjects to match various pairs of frieze patterns and discuss the similarities they saw. It appeared that transformation criteria were not verbalized predominantly over other criteria such as orientation or direction judgements, although many matches were made between patterns with the same underlying frieze group.

Finally, educational implications for mathematics were indicated and areas for further research were suggested.

Acknowledgements

A number of people have enabled me to complete this thesis through their support.

Firstly, I wish to thank my supervisor, Associate Professor Gordon Knight, from whom I have learnt a great deal. Your advice and encouragement have been very much appreciated.

I am particularly grateful to the schools, teachers, and students who have taken part, or taken an interest, in this project. Without your goodwill, this study would not have eventuated.

A thank-you is also extended to the Mathematics department at Massey University for providing resources and assistance in the production of this thesis. In this respect, the staff at the Massey library interloans office have also been very helpful.

Thank-you to those who shared the study room with me this year, particularly Kim and Patrick. Your humour and our chess games have kept me sane throughout the year. In addition, mention should be made of my flatmates who regularly asked me how my thesis was going - and listened while I explained.

Finally, I would like to thank my parents and family for all your practical and emotional support this year. Similarly, my thanks goes to Cherie, who not only proof read this thesis, but listened to all my moaning. I can't imagine what 1991 would have been like without you.

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1 *Introduction*

1.1 An Explanation of the Topic

Aims

This thesis addresses the question of how students of various ages perceive, or make, frieze patterns. The purpose of this research is to decide whether *intuitive* transformation geometry concepts form a component of either of these processes and, if so, to what extent. Consequently, the main objective of this mathematics education study is to identify and describe the character of the conceptualization and utilization of transformation geometry in students' description or construction of frieze designs. In particular, we consider this conceptualization and utilization as an element of 'geometrical intuition', which is a perceptual function by which a person apprehends spatial relationships *independent of a formal geometry framework*. To outline the nature of this topic, there are three key phrases which probably deserve further explanation: *transformation geometry, frieze patterns*, and *intuition*. These terms are discussed in the three subsections which follow.

1.1.1 Transformation Geometry

Background

Transformation, or 'motion', geometry was secured on firm mathematical ground in the 1870's when Felix Klein and Sophus Lie produced their version of it. In one sense, it can be considered as a refashioning of Euclidean (Sinha, 1986) and other geometries. In hindsight, this progression seems to have been quite natural. For instance, David Hilbert praised Euclid for his foresight in perceiving that 'motion' is a prerequisite for establishing the congruence of two figures (Sinha, 1986).

Rosenfeld (1988) reported that, in 1872, Felix Klein presented a lecture outlining the *Erlangen Program*, entitled (in English) *Comparative Overview of Recent Geometric Investigations*. The types of motions which he considered varied from rigid, affine, and projective transformations to inversive, circular and conformal transformations. Klein noticed that such transformations form groups[1] under composition. His emphasis was therefore on groups of transformations of space or manifolds, and the geometric properties of spatial figures. In this present study, however, the consideration of transformations is generally restricted to the 'rigid' or congruence transformations of the plane, as well as a variety of associated groups. This begs the question: what is a rigid transformation?

Rigid Transformations

Loosely, a rigid transformation is a motion of the plane which doesn't change the size or shape of figures within that plane. However, from a mathematical point of view, a transformation is a *mapping* which describes the relationship of points and their images; the idea of a motion is informal. Martin (1982) showed that there are only four types of rigid 'motions': a *reflection* about a mirror line, a *rotation* about a point, a *translation* in the direction and length of a vector, and a *glide reflection* about a line. The transformation most likely to be unfamiliar to the reader is the glide reflection. This 'motion' can be understood as the composition of a reflection and a translation, although it is a transformation in its own right. If this seems somewhat unexpected or contrived, it may be helpful to remember that a rotation (or a translation) can both be thought of as a product of two reflections. For more formal definitions of these transformations, see section 2.4.

If a transformation maps a set of points onto itself (so that it appears unchanged), it is called a *symmetry* of that set of points. As a consequence, there are four types of symmetry associated with the four types of transformations, which is contrary to a popular view that symmetry is synonymous with reflection symmetry.

1.1.2 Frieze Patterns

Until their own work was published, Grünbaum and Shephard (1987) explained that the term *pattern* had not been defined, even by mathematicians, in a lucid and useful way. For the purposes of this thesis, the word pattern is employed very broadly in its popular use, that is, as some sort of 'regular design'. (A *design* is taken to mean any set of points in the plane). Unlike Grünbaum and Shephard's (1987) definition, tilings (partitions of the plane into regions) are considered to be a special kind of pattern. A *frieze pattern* can be understood, informally , as a set of points in the plane which has translation symmetry in only one direction. An example is shown below.

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[1] The algebraic properties of a group are assumed to be known to the reader.

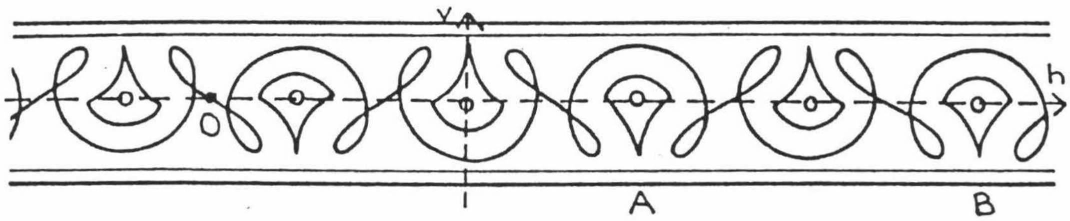


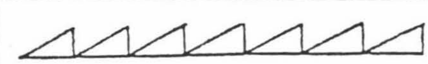
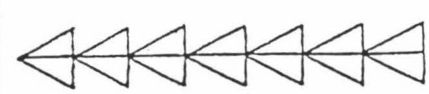
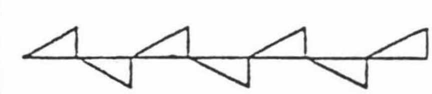
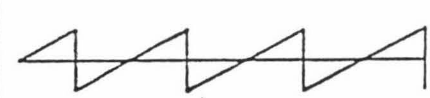
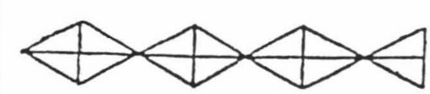
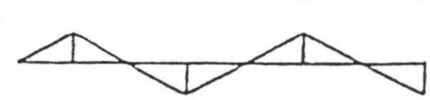
Figure 1.1

Reproduced from Shubnikov and Koptsik (1974), p 90

If we imagine the infinite extension of the pattern shown above (fig. 1.1) and translate it a distance AB (along its 'length'), then it will map onto itself. Of course, this particular example has other types of symmetries as well, such as a half-turn symmetry about the point O, a reflection symmetry about the line v, and a glide reflection symmetry about the line h (with the translation component in the direction of h).

In general, every frieze pattern has an underlying set of symmetries forming a *frieze group*. It was probably first proven by Federov a hundred years ago (Washburn and Crowe, 1988), and was shown again in detail by Martin (1982), that there are only seven different classes of frieze groups. Examples and the corresponding nomenclatures (Coxeter, 1987) are given in the following table:

Table 1

Belov's Crystallographic Notation	*Senechal's Abbreviated Notation*	*Martin's (1982) Notation*	*Examples*
p111	11	F_1	
p1m1	1m	F_1^1	
pm11	m1	F_1^2	
p1a1	1g	F_1^3	
p112	12	F_2	
pmm2	mm	F_2^1	
pma2	mg	F_2^2	

1.1.3 Intuition

Perhaps the most elusive ingredient of this thesis' title is that of *intuition*. Before proposing a working definition, it seems appropriate to consider a number of viewpoints of this concept. For instance, the *Concise Oxford Dictionary* (1990) described intuition as:

> "1. immediate apprehension by the mind without reasoning. 2. immediate apprehension by a sense. 3. immediate insight."

It appears that the term has a similar meaning in psychological circles, but it is viewed somewhat suspiciously by a number of psychologists, as the following extracts from psychological dictionaries indicate:

> "Immediate perception or judgement, usually with some emotional colouring, without any conscious mental steps in preparation; a popular rather than scientific term." (Drever, 1952)

> "1. direct or immediate knowledge without consciousness of having engaged in preliminary thinking. 2. a judgement made without preliminary cogitation. The term is more often used by laymen rather than by scientists." (Chaplin, 1968)

However, some psychologists, such as Carl Jung (1933) have described *intuitive* personality types in detail. Based on Jung's work, a personality type indicator known as *Myers-Briggs* has been developed. It divides perception activities into two categories: sensing and intuition. Jung described both types of perception as *irrational functions*, since neither operation is restricted by "rational direction" (Myers and McCaulley, 1985). Myers and McCaulley gave a description of each of these two perception functions:

> "Sensing ... refers to the perceptions observable by way of the senses. Sensing establishes what exists. Because the senses can bring to awareness only what is occurring in the present moment, persons orientated towards sensing perception tend to focus on the immediate experience and often develop characteristics associated with this awareness such as enjoying the present moment, realism, acute powers of observation, memory for details, and practicality. [In contrast] intuition ... refers to perception of possibilities, meanings, and relationships by way of insight. Jung characterized intuition as perception by way of the unconscious. Intuitions may come to the surface of consciousness suddenly, as a 'hunch', the sudden perception of a pattern in seemingly unrelated events, or as a creative discovery. ... persons orientated toward intuitive perception may become so intent on pursuing possibilities that they may overlook actualities." (p 12)

More recently, interest has increased amongst cognitive psychologists in a related area to intuition; that of explicit and implicit *memory*. Parkin *et al.* (1990) explained that:

> "Explicit memory refers to any test procedure that requires subjects to reflect consciously on a previous learning episode. ... Implicit memory tasks, in contrast, assess subjects' memory for a learning episode without any necessity for a conscious recollection of that episode."

In their experiment, Parkin *et al.* found that explicit memory of an episode was affected by an imposition of secondary processing demands whereas implicit memory was not. Similarly, the spacing of repetitions during initial learning affected explicit memory performance, but not that of implicit memory.

Piaget and Inhelder (1971) were aware of the existence of intuition. After observing that figurative aspects of thought are usually different from operational aspects, they wrote:

> "But there would appear to be an exception to this - the faculty known to mathematicians as geometrical 'intuition'. An adult subject who 'sees in space' ... does not stop at imagining static configurations in three dimensions any more than two. He [or she] is able to imagine movements and even the most complicated transformations thanks to a remarkable adequation of image to operation. This correspondence retains exceptional validity in spite of the well known shortcomings of intuition (such as the difficulty in visualizing curves without tangents, etc.)"
> (p 317)

The description of intuition, or similar notions, has not been restricted to the domain of psychologists. In 1952, for example, the famous mathematician Poincaré related in detail the differences he perceived between two types of mathematical mind, namely, *intuitive* and *logical* (Aiken, 1973). Similarly, Gagatsis and Patronis (1990) reported that:

> "Skemp (1971) draws a distinction between two levels of functioning of intelligence, that is, the intuitive and the reflective. The *intuitive* level involves awareness, through the senses, of data from the external environment which are 'automatically' classified and associated to other data. However, in this activity, the person is not aware of the mental processes involved. In contrast, at the *reflective* level, the mental processes become a focus of introspective awareness."

Of additional interest is Resek and Rupley's (1980) investigation of 'mathophobia'. Using the *Myers-Briggs Type Indicator* and ideas closely related to Skemp's (1979), such as *instrumental* and *relational* understanding, they found that a correlation existed between rule-orientation and sensation, as well as between concept-orientation and intuition.

Drora Booth (1975) has considered the intuitive use of symmetry operations in children's spontaneous pattern painting. In a personal correspondence (1991) with the researcher, she explained her own understanding of intuitive transformation concepts in children's or folk art work:

> "I take the term to mean any symmetry operation that can be identified in a work (painting, carving, weaving, block construction, etc.) that was created without the makers having formal knowledge of the mathematical concept."

This definition is very similar to the one eventually formulated in this thesis. However, in a cultural context, Grünbaum (1985) warned that:

> "Even if we were to believe ... that symmetries can be used to explain the ornaments, that has absolutely no implication on what the creators of these ornaments had in mind. Any of the periodic symmetry groups have as a prerequisite the infinite extent of the ornament; surely no Islamic artist would have dared even to think in such a sacrilegious way about the ornaments he can create. ... [Indeed], up to two centuries ago no artist or craftsman or *mathematician* defined regularity through symmetries. Equal parts - yes; equal position of parts with respect to their neighbours - yes; but equivalence with respect to the whole - never entered the picture."

A suspicion arising from Grünbaum's point is that some, or even all, of the symmetries able to be identified in a frieze pattern may not be intended, even *intuitively*, by the pattern's creator. Such symmetries in a pattern are therefore *accidental*, and labelling them as intuitive may be misleading. Naturally, the mathematical classification of patterns has the benefit of being systematic, but it *may* not provide a great deal of insight into a child's intuitive description or construction of a pattern. Lesh (1976) made a specific cautionary note:

> "...the researcher who begins with the assumption that children think in terms of slides, flips and turns may be just as naïve as the theorist who assumes flips come before turns and slides, just because flips are mathematically the most powerful. It *could* be that children do not conceive of rigid motions as compositions of slides, flips, and turns, but instead use some entirely different system of relations to describe spatial transformations." (p 234)

In conclusion, some of the key facets of the perceptual process of intuition seem to be that it is immediate, non-reflective, informal (independent of a formal framework), and associative (in the non-mathematical sense). While the identification of symmetry or transformations within a pattern may indicate an intuitive use of transformation geometry on the part of the creator, this isn't *necessarily* so. Thus, in this thesis, the working definition of *intuitive transformation geometry* in frieze patterns is the non-accidental presence of transformations or symmetry within a frieze pattern independent of a formal transformation framework. The property that the perception or creation of a design be *immediate* remains a secondary consideration throughout this study. However, in the analysis of survey results, one of the four measures employed to indicate the relative 'intuitive-ness' of the frieze groups addresses this concern also.

One final point: expressions such as 'more intuitive' indicate a comparison of one or more of the facets of intuition discussed above. It is hoped that the context of this phrase will make these facets clear.

1.2 The Motivation for Exploring Intuitive Transformation Geometry and Frieze Designs

"Perhaps more emphasis needs to be devoted to investigations exploring the intuitive (i.e., non-formalized) acquisition of systems of mathematical operations, relations and transformations. There is a popular misconception that concrete and intuitive mathematics is inferior mathematics and that the viability of a mathematical topic is measured solely in terms of its formalization and abstractness. In fact, the situation is often exactly the opposite." (Lesh, 1976, p 203).

"... linear patterns, sometimes called strip or frieze patterns, ... I believe are one of the great untapped geometrical treasure chests." (Williams, 1989)

Grünbaum and Shephard (1987) noted that the art of tiling and pattern-making appears to have begun very early in the history of civilization and, although the cultures emphasized different aspects of design, it seems that:

"Every known human society has made use of tilings and patterns in some form or another." (p 1)

They also claimed that many examples of artifacts from *all* cultures display a high degree of intricacy and complexity. Of particular interest to New Zealand is Knight's (1984a) observation that Maori rafter patterns, *kowhaiwhai*, suggest a well-developed geometrical intuition on the part of their creators.

Figure 1.2

Reproduced from Hamilton (1901).

These designs, when imagined to be infinitely extended along their length, are examples of frieze patterns. The relevance of this present study seems to be supported by Knight's conclusion:

> "The growing awareness of the importance of Maori Culture in New Zealand makes it particularly appropriate for students, both Maori and Pakeha, to relate the mathematics they learn to their cultural heritage."

Not long after Knight's article was published, the New Zealand mathematics syllabus for Forms 1 to 4 (1987) indicated that kowhaiwhai could be used to explore translation symmetry in Forms 3 and 4. However, it appears that little is known about the way in which students *perceive* these patterns, or if intuitive symmetry considerations form a part of this perception. If a teacher is employing a process-orientated approach to this topic (Skovmose, 1985), it may also be of interest to know the character of students' use and understanding of transformation geometry in their *constructions* of strip patterns.

Gagatsis and Patronis (1990) pointed out that intuition (non-reflective information processing) plays an important role in the development of reflective thinking, especially for children. In fact, they maintained that:

> "... intuitive thinking necessarily precedes reflective thinking and can help its evolution."

By implication, it would appear that exploiting a student's informal understanding of a concept may prove to be particularly valuable to mathematics educators. For instance, Bruner (1966), advocated the use of a "child's intuitive level as the starting point for teaching" (Booth, 1984). Booth (1985) herself concluded success in employing children's spontaneous pattern painting as the starting point for teaching art and transformation geometry.

However, only 13 years ago, Shultz (1978) indicated that:

> "Little is yet known about or agreed upon regarding children's cognitive abilities concerning transformation geometry." (p 195)

Today, this still appears to be the case. In addition, Lesh (1976) indicated that difficulties in teaching motion geometry may be a result of the fact that:

> "children make many mathematical judgements using qualitatively different methods than those typically used by adults." (p 186)

He also noted that such differences are not particularly well understood by researchers, particularly in the area of geometry. By focussing on the character of intuitive transformation geometry concepts, this thesis attempts to contribute towards the knowledge in this area.

1.3 An Overview of the Thesis

To explore this topic, we examine, in chapter 2, the mathematics education literature on the role of geometry in the development of spatial sense and consider the merits of the 'transformation approach.' A summary of the relevant psychology and mathematics education literature on the perception and learning of transformations and symmetry is subsequently undertaken. The literature review also considers a study of children's spontaneous pattern painting and its implications to intuitive transformation geometry. The review ends with a summary of some mathematical classifications of designs.

Chapter 3 outlines the design and execution of the surveys and interviews conducted, and describes the analysis methods used to examine the results. Chapters 4 and 5 include the results and discussion of the unrestricted and restricted frieze pattern construction activities. Chapter 6 characterizes the written and oral responses to the frieze pattern description activity. Summaries are given at the end of chapters 4, 5 and 6.

Chapter 7 concludes the study by discussing the implications of this thesis' findings for both researchers and mathematics educators at the Primary and Secondary school levels. To this end, appendix D includes a brief review of some relevant material for use in the learning environment.