POINT AND LIE BÄCKLUND SYMMETRIES OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

A thesis presented in partial fulfilment
of the requirements
for the degree
of MA
in Mathematics at

MASSEY UNIVERSITY

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1994

ACKNOWLEDGEMENTS

I wish to record my gratitude for the support and encouragement given me by Associate Professor Dean Halford in the preparation of this thesis.

I also thank Mrs Gail Tyson for her sterling work in typing this thesis.

CONTENTS

CHAPTER 1: INTRODUCTION

The aim of this thesis is to:

(1) Explore the use of differential forms in obtaining point and contact symmetries of particular partial differential equations (PDEs) and hence their corresponding similarity solutions. [1] and [4].

(2) Explore the generalized or Lie-Bäcklund symmetries of particular PDEs with particular reference to the Korteweg-de Vries-Burgers (KdVB) equation [3].

Finding point symmetries of a PDE $H = 0$ with independent variables (x_1, x_2) which we take to represent space and time and dependent variable (u) means finding the **transformation group**

$$x_1' = x_1 + \varepsilon\, \xi_1\, (x_1, x_2, u) + 0(\varepsilon^2)$$
$$x_2' = x_2 + \varepsilon\, \xi_2\, (x_1, x_2, u) + 0(\varepsilon^2)$$

and

$$u' = u + \varepsilon\eta(x_1, x_2, u) + 0(\varepsilon^2)$$

that takes the variables (x_1, x_2, u) to the system (x_1', x_2', u') and maps solutions of $H = 0$ into solutions of the same equation. The form of $H = 0$ remains invariant. The transformation group is usually expressed in terms of its **infinitesimal generator** (**X**) where

$$\begin{aligned} \mathbf{X} &= \xi_1\, \partial_{x_1} + \xi_2\, \partial_{x_2} + \eta \partial_u \\ &= \xi_i\, \partial_{x_i} + \eta\, \partial_u \qquad\qquad i = 1, 2 \end{aligned}$$

using the tensor summation convention. **X** can be considered as a differential vector operator with components (ξ_1, ξ_2, η) operating in a three dimensional manifold (space) with coordinates (x_1, x_2, u). The invariance of $H = 0$ under the transformation group is expressed in terms of a suitable **prolongation** or **extension** of **X** (denoted by $\mathbf{X}^{(pr)}$) to cover the effect of the transformations on the derivatives of u in $H = 0$.

The **invariance condition** for $H = 0$ under the action of the transformation group is

$$\mathbf{X}^{(pr)}[H] = 0 \text{ whenever } H = 0.$$

We consider x_1, x_2, u and the derivatives of u to be independent variables.

In practical terms, finding point symmetries of $H = 0$ means finding the components (ξ_1, ξ_2, η) of the infinitesimal generator $(\mathbf{X})$. There are two general methods for finding ξ_1, ξ_2 and η.

1 THE CLASSICAL METHOD

The method we follow was developed mainly by Bluman and Cole [2]. Consider for example a k^{th} order PDE $H = 0$. H is regarded as a function of x_1, x_2 and u as well as the partial derivatives of u with respect to x_1 and/or x_2 up to and including the k^{th} order derivatives. The $\mathbf{k^{th}}$ **prolongation** of the infinitesimal generator is

$$\mathbf{X}^{(pr)} = \xi_i \partial_{x_i} + \eta\partial_u + \eta^{(1)}_{i_1}, \partial_{u_{i_1}} + \ldots$$
$$\ldots \; \eta^{(k)}_{i_1 i_2 \ldots i_k} \; \partial_{u_i i_2 \ldots i_k}$$

using again the tensor summation convention where $i_r = 1, 2$ and $r = 1, 2, \ldots k$.

Note: $i_r = 1 \equiv x_1$ and $i_r = 2 \equiv x_2$.

For example:

$$\begin{aligned}\eta^{(2)}_{i_1 i_2} \; \partial_{u_{i_1 i_2}} &= \eta^{(2)}_{11} \; \partial_{u_{11}} + \eta^{(2)}_{12} \; \partial_{u_{12}} + \eta^{(2)}_{22} \; \partial_{u_{22}} \\ &\equiv \eta^{(2)}_{xx}\partial_{u_{xx}} + \eta^{(2)}_{xt}\partial_{u_{xt}} + \eta^{(2)}_{tt}\partial_{u_{tt}}\end{aligned}$$

where x represents space and t time.

The coefficients $\eta^{(k)}_{i_1 i_2 \ldots i_k}$ are given by the expressions

$$\eta^{(1)}_{i_1} = D_{i_1}(\eta) - u_j D_{i_1}(\xi_j)$$

$$\eta^{(2)}_{i_1 i_2} = D_{i_2}\eta^{(1)}_{i_1} - (u_{i_1})_j \; D_{i_2}(\xi_j)$$

$$\eta^{(k)}_{i_1 i_2 \ldots i_k} = D_{i_k}\left(\eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}}\right) - \left(u_{i_1 i_2 \ldots i_{k-1}}\right)_j D_{i_k}\left(\xi_j\right)$$

$$i_r = 1, 2 \text{ or } (x, t) \quad r = 1, 2 \ldots k.$$

$$j = 1, 2 \text{ or } (x, t)$$

and D_{i_k} is the **Total Derivative Operator**

where $$D_\alpha = \partial_\alpha + u_\alpha \partial_u + u_{\alpha_{i_1}} \partial_{u_{i_1}} + u_{\alpha_{i_1 i_2 \ldots i_r}} \partial_{u_{i_1 i_2 \ldots i_r}}$$

$$\alpha = 1, 2 \text{ or } (x, t).$$

For example

$$\eta^{(1)}_1 = \frac{\partial \eta}{\partial x_1} = \left\{\frac{\partial \eta}{\partial u} - \frac{\partial \xi_1}{\partial x_1}\right\}\left(\frac{\partial u}{\partial x_1}\right) - \left(\frac{\partial \xi_2}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right) - \left(\frac{\partial \xi_1}{\partial u}\right)\left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial \xi_2}{\partial u}\right)\left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right)$$

$X^{(k)}[H] = 0$ consists of a polynomial in u and its derivatives and has to hold whenever $H = 0$ for all values of $u = u(x_1 \, x_2)$ that are solutions of the PDE $H = 0$. This implies that the coefficients of u and its derivatives must be identically equal to zero. This gives a set of linear partial differential equations called **determining equations** which can in principle be solved for ξ_1, ξ_2 and η. The general solution of $H = 0$ is a family of surfaces in $(x_1 \, x_2 \, u)$ space.

If $F(x_1, x_2, u) = 0$ defines such a surface then $F(x_1, x_2, u) = 0$ is an invariant of the transformation group

i.e. $$\mathbf{X}[F] = 0$$

or $$\xi_1 \frac{\partial F}{\partial x_1} + \xi_2 \frac{\partial F}{\partial x_2} + \eta \frac{\partial F}{\partial u} = 0$$

This is often referred to as the **invariant surface condition.**

A first step in obtaining similarity solutions is solving the subsidiary equations

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \frac{du}{\eta}$$

The solution involves:

(1) $\zeta(x_1, x_2, u)$ called the **similarity variable** which becomes the independent variable;

(2) then the dependent variable is taken as $v = f(\zeta)$ where f is an arbitrary function of ζ.

The similarity form of the solution of the original PDE is;

$$u = u(x_1, x_2, f(\zeta))$$

Substitution of this form of u into $H = 0$ gives an ordinary differential equation (ODE) for $v = f(\zeta)$ which can in principle be solved for $f(\zeta)$ thus giving the similarity solution for $H = 0$.

2 THE USE OF DIFFERENTIAL FORMS

Harrison and Estabrook [1] used differential forms to formulate systems of partial differential equations and so obtain their point symmetries and similarity solutions.

An Introduction to Differential Forms [4]

We consider two geometrical objects, namely vectors $(\mathbf{V})$ and differential forms (α), which exist in an n-dimensional differentiable manifold with coordinates x_i $(i = 1, 2 \ldots n)$.

A **vector** $(\mathbf{V})$ is a linear differential operator that at each point maps a differentiable, real valued function $f(x_i)$ into a real number. The vector is represented in the coordinate basis as

$$\mathbf{V} = v_a(x_i)\partial_{x_a} \quad (a = 1, 2, \ldots n)$$

using the tensor summation convention.

The functions $v_a(x_i)$ are the components of the vector. An example of a vector is the infinitesimal generator (**X**) of a transformation group which is also referred to as an **isovector**.

We start our consideration of differential forms by defining a **0-form** as a real valued function $f(x_i)$ $(i = 1, 2, \ldots n)$ in the differentiable manifold. A **1-form** is then defined as a linear combination of the basis differentials dx_i $(i = 1, 2, \ldots n)$

$$\text{i.e. } \underset{(1)}{\alpha} = a_i\, dx_i \text{ where the } a_i\text{ s are 0-forms.}$$

1-forms are combined by an operation denoted by $\wedge$ known as the **exterior** or **wedge product.** The exterior product of the 1-forms $\sigma = \sigma_a dx_a$ and $\omega = \omega_b dx_b$, denoted by $\sigma \wedge \omega$, is

$$\sigma \wedge \omega = \sigma_a \omega_b dx_a \wedge dx_b$$
$$= \frac{1}{2}(\sigma_a\, \omega_b - \sigma_b \omega_a)\, dx_a \wedge dx_b$$

$\sigma \wedge \omega$ is a **2-form** and the most general 2-form is a linear superposition of the ${}^nC_2$ basis 2-forms $dx_a \wedge dx_b$, that is

$$\underset{(2)}{\alpha} \leq \alpha_{ab} dx_a \wedge dx_b \quad \text{where} \quad \alpha_{ab} = -\alpha_{ab} \text{ is skew symmetric.}$$

We now generalize. A **p-form** is the exterior product of $p(0 < p \leq n)$ 1-forms

$$\underset{(p)}{\alpha} = \frac{1}{p!}\, \alpha_{a_1 a_2 \ldots a_p} dx_{a_1} \wedge dx_{a_2} \wedge \ldots \wedge dx_{a_p}$$

where the coefficients $\alpha_{a_1 a_2 \ldots a_p}$ are completely skew-symmetric

i.e. $$\alpha_{a_1 a_2 \ldots a_p} = \alpha_{[a_1 a_2 \ldots a_p]}$$

where $[a_1\, a_2 \ldots a_p]$ is an odd permutation of $a_1 a_2 \ldots a_p$.

Differential forms and the operation of exterior product form a Grassmann Algebra in the n-dimensional manifold with the following properties:

(1) Forms of the same degree may be added or subtracted.

(2) $\underset{(p)}{\alpha} \wedge \underset{(q)}{\beta}$ is a p + q-form which is zero if p + q > n.

(3) $\underset{(p)}{\alpha} \wedge \underset{(q)}{\beta} = (-1)^{pq} \underset{(q)}{\beta} \wedge \underset{(p)}{\alpha}.$

This implies that $dx_i \wedge dx_j = -dx_j \wedge dx_i \begin{cases} = 0 & i = j \\ \neq 0 & i \neq j \end{cases}$

(4) The exterior product is distributive, i.e. $(\alpha+\beta)\wedge\gamma = \alpha\wedge\gamma + \beta\wedge\gamma$

(5) The exterior product is associative, i.e. $(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$

To construct a calculus of differential forms we need two differential operators namely

(1) **The exterior derivative** (d), and
(2) **The Lie derivative** ($\pounds_v$) with respect to the vector V.

The exterior derivative acts on a p-form α to produce a (p+1)-form $d\alpha$ and is defined as

$$d\alpha = \frac{1}{p!}\, d\left(\alpha_{a_1 a_2 \ldots a_p}\right) \wedge dx_{a_1} \wedge dx_{a_2} \wedge \ldots \wedge dx_{a_p}$$

where
$$d\left(\alpha_{a_1 a_2 \ldots a_p}\right) = \frac{\partial\left(\alpha_{a_1 a_2 \ldots a_p}\right)}{\partial x_j} dx_j$$

The exterior derivative has the following properties. Let α be a p-form, β a q-form and f a 0-form.

(1) $d(\alpha+\beta) = d\alpha + d\beta$ (linearity)
(2) $d(\alpha\wedge\beta) = d\alpha \wedge\beta + (-1)^p\, \alpha\wedge d\beta$ (Leibniz Rule)
(3) $d(d\alpha) = 0$ (Poincaré Lemma)
(4) $d(f\alpha) = df \wedge\alpha + f d\alpha$

The Lie derivative operator ($\pounds_v$) is a linear differential operator associated with a vector field $V = v_a(x_i)\,\partial_{x_n}$ $(a \leq 1 \ldots n)$ which can be applied to any geometrical object. We will confine it to vectors and differential forms.

The Lie derivative of a vector A with respect to V is the commutator of the two vectors

i.e. $$£_v(A) = [V, A] = - £_A(V)$$

or in coordinate form $$£_v\left(a_i \partial_{x_i}\right) = \left(v_k \frac{\partial a_i}{\partial_{x_k}} - a_k \frac{\partial v_i}{\partial_{x_k}} \right) \partial_{x_i}$$

Before considering the application of the Lie derivative to a differential form it is necessary to consider the **contraction** of a vector (V) and a p-form (α) which gives a (p–1) form β.

Notation contraction $\beta = \langle V, \alpha \rangle$

or $\beta = V \lrcorner \alpha$

In component notation $V \lrcorner \alpha$ is defined as

$$\begin{aligned} V \lrcorner \alpha &= \left(v_b \partial_{x_b}\right) \lrcorner \left(\alpha_{[a_1 a_2 \dots a_p]} dx_{a_1} \wedge dx_{a_2} \wedge \dots \wedge dx_{a_p}\right) \\ &= p!\, v_b \alpha_{[b a_2 \dots a_p]} dx_{a_2} \wedge dx_{a_3} \wedge \dots \wedge dx_{a_p} \end{aligned}$$

Note: The above definition implies that

$$\partial_{x_i} \lrcorner \, dx_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

δ_{ij} is the Kronecker delta.

Properties of the contraction of V and $\underset{(p)}{\alpha}$ include

(1) $V \lrcorner \left(\underset{(p)}{\alpha_1} + \underset{(p)}{\alpha_2} \right) = V \lrcorner \underset{(p)}{\alpha_1} + V \lrcorner \underset{(p)}{\alpha_2}$

(2) $V \lrcorner \left(\underset{(p)}{\alpha} \wedge \underset{(q)}{\beta} \right) = \left(V \lrcorner \underset{(p)}{\alpha} \right) \wedge \underset{(q)}{\beta} + (-1)^p \underset{(p)}{\alpha} \wedge \left(V \lrcorner \underset{(q)}{\beta} \right)$

(3) $Vf = V \lrcorner df$ f = 0-form

For a function f(0-form) the Lie derivative of f with respect to V is a 0-form

$$£_v(f) = V \lrcorner df = V \lrcorner \left(\frac{\partial f}{\partial x_a} dx_a \right) = v_a \frac{\partial f}{\partial x_a}.$$

For a p-form (α) $(0 < p \le n)$ the Lie derivative of α with respect to V is given in terms of the exterior derivative and the contraction as

$$£_v(\alpha) = V \lrcorner\, d\alpha + d(V \lrcorner\, \alpha)$$

which is also a p-form.

Properties of the Lie derivative are

(1) $£_v(d\alpha) = d(£_v(\alpha))$ i.e. Lie derivative and exterior derivative commute

(2) $£_v(\alpha \wedge \beta) = (£_v(\alpha)) \wedge \beta + \alpha \wedge (£_v(\beta))$

(3) $£_v(W \lrcorner\, \alpha) = [V,W] \lrcorner\, \alpha + W \lrcorner\, £_v(\alpha)$, where W is a vector.

The geometric approach of Harrison and Estabrook [1] involves the use of differential forms to find the isovector or infinitesimal generator (X) of the transformation group of a PDE $H = 0$. Here we shall consider a PDE of order k in one dependent and 2 independent variables, u, x_1, x_2 with derivatives of u up to degree k.

The PDE is first represented by a closed set of differential forms α_i $(i = 1, 2, \ldots)$ in an n-dimensional manifold M. This set of forms constitutes a closed differential ideal (I) on the manifold. I being closed means that if $\alpha_i = 0$ then $d\alpha_i = 0$ also.

An **integral manifold** is a submanifold of M on which the differential forms (α_i) are expressed in terms of the independent variables of the PDE and their differentials. The α_i are annulled (take zero values) on the integral manifold to give the following information:

(i) The original partial differential equation

(ii) The definition of the n–3 auxiliary variables in M. These are usually derivatives of u of degree less than k.

(iii) The integrability conditions on u.

Imposing independence of x_1 and x_2 and their differentials dx_1 and dx_2 puts the ideal (I) in involution with x_1 and x_2 (by definition). Cartan's geometric theory of PDEs [6, 7] implies that there exists a general or regular integral manifold that can be considered a solution manifold for the PDE. The Lie groups of point symmetries of PDEs is represented by the infinitesimal group generator, or isovector, X. It is suitably extended or prolonged as a linear differential vector operator $X^{(pr)}$ in the space of the variables x_1, x_2, u and derivatives of u. There are two differential operators that naturally arise in the ring of differential forms. These are

(1) the exterior derivative, and
(2) the Lie derivative.

The Lie derivative, defined in terms of an isovector of the symmetry group, is used in formulating the invariant conditions. Point symmetries of a PDE H = 0 are defined by the action of $X^{(pr)}$ on the PDE,

i.e. $$X^{(pr)}[H] = 0 \text{ whenever } H = 0.$$

With differential forms this is equivalent to saying that the Lie derivative in the direction of X of all differential forms $\alpha_i \in I$ are in the ideal and should vanish if $\alpha_i = 0$. That is $£_x(\alpha_i) = 0$ if $\alpha_k = 0$ or equivalently $£_x(\alpha_i)$ is a linear combination of the differential forms $\underset{(q)}{\alpha_k}$ where $q \le p$ i.e. $£_x(\alpha_i) = \lambda_i^k \wedge \alpha_k$ (sum over k). The λ_i^k are arbitrary differential forms, including in some cases 0-forms or functions. In such cases $\lambda_i^k \wedge \alpha_k$ is usually written as $\lambda_i^k \alpha_k$. After eliminating the λ_i^k forms the symmetry or invariant condition can be reduced to a set of determining equations which can be solved for the components ξ_1, ξ_2 and η of the infinitesimal generator of the invariant transformation group. These components, besides being functions of x_1, x_2 and u also contain a number of arbitrary integration constants.

The finding of **similarity solutions** involves augmenting the ideal of differential forms and imposing the condition that the augmented forms be annulled on the integral manifold as well as the ideal. One way to augment the ideal is by contracting the differential forms (α_i) in the ideal with the isovector (now denoted by V).

That is $$\sigma_i = V \lrcorner \alpha_i .$$

Now $$\begin{aligned} £_v(\sigma_i) &= £_v(V \lrcorner \alpha_i) = V \lrcorner £_v(\alpha_i) = V \lrcorner \left(\lambda_i^k \wedge \alpha_k\right) \\ &= \left(V \lrcorner \lambda_i^k\right)\alpha_k + (-1)^{p-q}\lambda_i^k \wedge \alpha_k \end{aligned}$$

which means the augmented ideal $\{\alpha_i, \sigma_i\}$ is invariant under the action of V. Annulling certain forms in the augmented ideal should then produce similarity solutions of the PDE.

Appendix A shows in detail the use of the classical method of Bluman & Cole for finding similarity solutions of the nonlinear diffusion equation $\phi_{x_2} = \left(K(\phi)\phi_{x_1}\right)_{x_1}$ and the use of differential forms for finding the point symmetries of the Korteweg-deVries (KdV) equation

$$u_{x_2} + uu_{x_1} + \varepsilon u_{x_1x_1x_1} = 0$$

where ε is a constant.

Comparing the two methods as far as hand computation is concerned, I find that they are of comparable difficulty, although some PDEs might be more easily processed by one or other of the two methods. In general the Harrison-Estabrook method using differential forms gives simpler determining equations which are however, usually obtained by more involved manipulations.

The use of differential forms seems to be the preferred method for the various computer packages, for example MACSYMA [8] that are used to find the determining equations. In my opinion the classical method of Bluman & Cole has the characteristics of an algorithm and gives comparatively little insight into the process of finding symmetries and similarity solutions. The use of differential forms on the other hand involves the manipulation of geometric objects in an n dimensional manifold and the process of finding symmetries can be given a geometrical interpretation.

Lie-Bäcklund Symmetries [3, 4, 11 and 12]

We begin by considering a partial differential equation $H(\mathbf{x}, \mathbf{u}^{(n)}) = 0$ with n independent variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and a single dependent variable u.
$\mathbf{u}^{(N)} = (u, \underset{1}{u}, \underset{2}{u}, \ldots \underset{N}{u})$ where $\underset{i}{u}$ are the i^{th} order partial derivatives of u with respect to the components of $\mathbf{x}$. One-parameter Lie point symmetries of such a PDE are transformations of the form

$$x_i' = x_i + \varepsilon\, \xi_i\, (\mathbf{x}, u) + 0(\varepsilon^2) \;\; (i = 1, \ldots, n)$$

$$u' = u + \varepsilon\eta(\mathbf{x}, u) + 0(\varepsilon^2)$$

that leave $H(\mathbf{x}, \mathbf{u}^{(N)}) = 0$ invariant. These point symmetries are usually expressed in terms of their infinitesimal generator $X = \xi_i \partial_{x_i} + \eta\, \partial_u$ suitably prolonged to cover the action of derivatives of u. A possible generalization of this would be to transformations where the coefficients ξ_i and η are also functions of derivatives of u. i.e. $\xi_i = \xi_i(\mathbf{x}, u, \underset{1}{u}, \dots \underset{N}{u})$ and $\eta = \eta(\mathbf{x}, u, \underset{1}{u}, \dots \underset{N}{u})$. If N is finite these so called generalized transformations are either prolonged point transformations or contact transformations [4]. Contact transformations only occur in situations involving a single dependent variable (u) and are characterized by $\xi_i = \xi_i(\mathbf{x}, u, \underset{1}{u})$ and $\eta = \eta(\mathbf{x}, u, \underset{1}{u})$. For broader generalizations we have to consider transformations where the coefficients ξ_i and η contain derivatives of u of arbitrarily high order. The prolongation of $X = \xi_i \partial_{x_i} + \eta\, \partial_u$ to cover the effects of derivatives of u has to be in general an infinite prolongation.

$$X^{(\infty)} = X + \sum_J \eta_J (\mathbf{x}, u, \underset{1}{u}, \dots)\, \partial_{u_J}$$

where $$\eta_J = D_J(\eta - \xi_i u_i) + \xi_i u_{Ji}$$

and $J = j_i \dots j_k$ where j_k is a suitable integer of $\mathbf{x}$ and $k \geq 0$.

In an analogous way to point symmetries X is a **generalized symmetry** of $H(\mathbf{x}, \mathbf{u}^{(N)}) = 0$ if and only if $X^{(\infty)} [H(\mathbf{x}, \mathbf{u}^{(N)})] = 0$ for every smooth* solution $u = f(\mathbf{x})$ of the PDE. In practice $H(\mathbf{x}, \mathbf{u}^{(N)}) = 0$ depends only on a finite number of derivatives of u so only a finite number of terms of $X^{(\infty)}$ are required in any given instance. This means that the question of convergence of $X^{(\infty)}$ does not arise. Generalized symmetries of this type are commonly called **Lie-Bäcklund symmetries.** (Olver [3] uses the term generalized symmetry) and include point and contact symmetries as special cases.

In this thesis we shall deal exclusively with time-evolution equations in two independent variables x and t of the form $H \equiv u_t + K(x, u, u_1, \dots, u_n) = 0$ where $u_i = \dfrac{\partial^i u}{\partial x^i}$ $(i = 0, 1, \dots, n)$ and $u_0 = u$ is the (only) dependent variable.

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* smooth means that u and its derivatives are continuous in the domain of applicability.

Bluman and Kumei [12] and others [3, 4, 10, 11] prove that, for a time evolution equation, a Lie-Bäcklund transformation of the form

$$x' = x + \varepsilon\, \xi_1(x, t, u, u_1 \ldots) + 0(\varepsilon^2)$$
$$t' = t + \varepsilon\, \zeta_2(x, t, u, u_1 \ldots) + 0(\varepsilon^2)$$

and $$u' = u + \varepsilon\, \eta(x, t, u, u_1 \ldots) + 0(\varepsilon^2)$$

acts on a solution surface $F(x, t, u) = 0$ of the PDE in the same manner as

$$x' = x$$
$$t' = t$$

and $$u' = u + \varepsilon Q + 0(\varepsilon^2)$$

where $$Q = \eta - \xi_1 u_1 - \xi_2 u_t$$

This means that the infinitesimal generator can now be expressed in the simpler form $X(Q) = Q(x, t, u, u_1 \ldots)\, \partial_u$. $X(Q)$ is called the **evolutionary infinitesimal generator** and Q is referred to as its **characteristic**. The infinite prolongation of $X(Q)$ now takes the form

$$X^{(\infty)}(Q) = \sum_J D_J[Q]\, \partial_{u_J}$$

From the equivalence of the two Lie-Bäcklund transformations detailed above, the following result can be easily proved [3]; **An infinitesimal generator X is a Lie-Bäcklund symmetry of a PDE if and only if its evolutionary form $X(Q)$ is a Lie-Bäcklund symmetry.**

For a time-evolution equation

$$u_t + K(x, u, u_1 \ldots u_n) = 0$$

the infinite prolongation $X^{(\infty)}(Q)$ of the infinitesimal generator takes the form

$$X^{(\infty)}(Q) = Q\partial_u + (D_t[Q])\partial_{u_t} + \sum_{j=1}^{\infty} D_x^j\, [Q]\partial_{u_j}$$

where $Q = Q(x, t, u, u_1 \dots u_N)$ N arbitrary and D_i is the total derivative operator with respect to x_i. Thus

$$D_x \equiv \partial_x + u_x\partial_u + u_{xt}\partial_{u_t} + u_{xx}\partial_{u_x} + \dots$$

and
$$D_t \equiv \partial_t + u_t\partial_u + u_{tx}\partial_{u_x} + u_{tt}\partial_{u_t} + \dots$$

In considering Lie-Bäcklund symmetries of a PDE it is convenient to use two operators namely:

(i) the Fréchet derivative, and
(ii) the recursion operator.

The **Fréchet derivative** of a smooth differential function $H[u] = H[x, t, u, u_t, u_1, \dots u_n]$ is defined as

$$D_H(Q) = \left|\frac{d}{d\varepsilon} H[u + \varepsilon Q]\right|_{\varepsilon=0}$$

It can be readily shown that this is equivalent to

$$D_H(Q) = \left(\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_t} D_t + \sum_{j=1}^{\infty} \frac{\partial H}{\partial u_j} D_x^j\right)[Q]$$

Comparison with $X^{(\infty)}(Q)[H]$ shows that

$$X^{(\infty)}(Q)[H] = D_H(Q)$$

The invariance condition for the PDE $H = 0$ under the action of $X^{(\infty)}(Q)$ can be written as $D_H(Q) = 0$ whenever $H = 0$. Either form of the invariance condition can be used as an algorithm for finding Q as the solution of a system of determining equations.

Definition

The operator $R = R(u, u_t, u_1, \dots u_n)$ **is a recursion operator of the time evolution equation $H \equiv u_t + K(x, u, u_1, \dots u_n) = 0$ if and only if $[D_H, R]_{H=0} = 0$.** From this definition Fokas [9] and others prove that if R is a recursion operator of $H = 0$ and

$Q = Q(x, t, u, u_1, \ldots u_N)$, N arbitrary, is an Lie-Bäcklund symmetry of $H = 0$, then $X(R^j[Q])_j$ for $j = 1, 2 \ldots$ are also Lie-Bäcklund symmetries of the PDE.

That is, the recursion operator can generate an infinite sequence of Lie-Bäcklund symmetries depending on higher order derivatives of u.

The method of determining all Lie-Bäcklund symmetries of a PDE $H(\mathbf{x}, \mathbf{u}^{(u)}) = 0$ is to start with the evolutionary form $X(Q) = Q\partial_u$ and to decide on some arbitrary order of derivatives for Q. We then use the invariance condition $D_H(Q) = 0$ whenever $H = 0$ to generate an equation involving derivatives of Q and u.

A significant calculational feature is that for time - evolution equations the PDE can be used to substitute for any t derivatives of u which implies that Q involves only x derivatives of u

i.e. $$Q = Q(x, t, u, u_1 \ldots u_N)$$

As the invariance condition holds for any solution $u = u(x, t)$ of $H = 0$ we can equate coefficients of the derivatives of u in descending order to zero and find the general form of Q. Bluman and Kumei [10] use this method to find two finite order Lie-Bäcklund symmetries of the non-linear diffusion equation $\{a(u+b)^{-2}u_x]_x - u_t = 0$ and then obtain the recursion operator by inspection. In this way they can generate the entire sequence of Lie-Bäcklund symmetries for this equation.

To find a Lie-Bäcklund symmetry we must **assume** the order (N) of the highest derivative in Q. **What value of N do we start with?** If a recursion operator exists, then in almost all known cases the point symmetry operator for invariance of $H = 0$ under a t translation is generated (by the recursion operator) from that expressing invariance under a x translation [9].

For a time-evolution equation of the form $u_t + u_n + G(x, u, u_1 \ldots u_{n-1}) = 0$ we have two Lie point symmetries $Q_1 = u_1$ and $Q_2 = u_n + G[u]$. If there is a recursion operator R such that $Q_2 = R[Q_1]$ then $R = D_x^{n-1} + \ldots$. Therefore the first Lie-Bäcklund symmetry is $Q_3 = R[Q_2] = u_{2n-1} + g(x, u, u_1, \ldots, u_{2n-2})$, which implies that $N = 2n - 1$.

Similarity (invariant) solutions

As with point symmetries, similarity or invariant solutions can be found from a given Lie-Bäcklund symmetry [12]. A solution $u = u(x, t)$ of a time-evolution equation $H \equiv u_t + K(x, u, u_1, \ldots, u_n) = 0$ is invariant under the action of a Lie-Bäcklund symmetry if and only if $u = u(x, t)$ satisfies the invariant surface condition $Q(x, t, u, u_1, \ldots, u_N) = 0$. $Q(x, t, u, u_1, \ldots u_N) = 0$ is regarded as an N^{th} order ordinary differential equation in the independent variable x with t as a parameter. The solution of the ODE is a similarity form

$$\phi(x, t, u, c_1(t), \ldots, c_N(t)) = 0$$

with the arbitrary functions $c_1(t), \ldots, c_N(t)$ acting as integration "constants". These integration "constants" can be determined by substitution of the similarity form into the time-evolution equation.

In chapter 3, I intend to study possible Lie-Bäcklund symmetries and similarity solutions of the **Korteweg-de Vries-Burgers** (KdVB) equation

$$u_t + auu_x + bu_{xx} + cu_{xxx} = 0, \text{ where a, b, and c are constants.}$$

This equation is the simplest form of a wave equation that incorporates nonlinearity (the auu_x term), dispersion (cu_{xxx}) and attenuation (bu_{xx}). In a wave equation $u = u(x, t)$ is a perturbation of the medium through which the wave is travelling and can be either perpendicular to (transverse waves) or parallel to (compressional waves) the direction of wave propagation. The KdVB equation has been widely used to model many types of nonlinear wave motion including for example:

(i) the propagation of waves in liquid filled elastic tubes [14]

(ii) tidal bores [21]

(iii) magneto-hydrodynamic shock waves in plasmas [20]

(iv) the propagation of acoustic waves in liquids containing small bubbles [13].

Johnson [14], using phase plane analysis on the steady state (constant wave velocity) form of the KdVB equation, obtained soliton progressive wave, and shock wave solutions by using various values of a, b and c, particularly b (the constant governing the degree of attenuation of the wave by the medium). Exact solutions of the KdVB equation have been obtained by several authors [17 - 19], however Vlieg-Hulstman and Halford [16] demonstrated that these solutions are essentially equivalent to a single exact solution that is a linear combination of particular solutions of the KdV equation and Burgers equation. Lakshmanan and Kaliappan [15] found that the KdVB equation has the following point symmetries

$$\xi_1 = ak_1 t + k_2, \quad \xi_2 = k_3 \quad \text{and} \quad \eta = k_1,$$

where k_1, k_2 and k_3 are integration constants. This implies that the KdVB equation is invariant under the following transformations

$$\begin{aligned} k_1 &= 1 \quad X_1 = at\partial_x + \partial_u \ \text{(Galilean transformation)}, \\ k_2 &= 1 \quad X_2 = \partial_x \ \text{(x-translation)}, \\ k_3 &= 1 \quad X_3 = \partial_t \ \text{(t-translation)}. \end{aligned}$$

These lead to a similarity variable

$$\zeta = \frac{k_1 x}{c} - \frac{ak_2 t^2}{2} + k_3 t$$

and similarity solution

$$u = \left(\frac{k_2 c}{k_1}\right) t + f(\zeta),$$

where $f(\zeta)$ is an arbitrary functional of ζ. Substitution in the KdVB equation gives the ODE

$$c\frac{d^3 f}{d\zeta^3} + b\frac{d^2 f}{d\zeta^2} + \left[\frac{k_3 c}{k_1} + af\right]\frac{df}{d\zeta} - \frac{k_2 c}{k_1} = 0$$

which on integrating gives

$$c\frac{d^2 f}{d\zeta^2} + b\frac{df}{d\zeta} + \frac{a}{2}f^2 + \left(\frac{k_3 c}{k_1}\right)f + \left(\frac{k_2 c}{k_1}\right)\zeta + c_1 = 0,$$

where c_1 is an integration constant.

A suitable Ince transformation [22]

$$z = \left(\frac{-25ac}{12b^2}\right)^{1/2} \exp\left(-\frac{b\zeta}{5c}\right)$$

and $$f = W(z) \exp\left(-\frac{2b\zeta}{5c}\right) + \frac{1}{a}\left(\frac{6b^2}{25c} - \frac{k_3c}{k_1}\right)$$

gives the invariant ODE

$$\frac{d^2W}{dZ^2} = 6W^2 + S(Z)$$

which is free from movable critical points only if $S(Z) = pZ + q$ (p and q are constants) [22]. Hence the invariant ODE has in general movable critical points. Ablowitz and others [23] suggest that this implies that the KdVB equation is not in general exactly solvable. However, Fokas [9] defines exact solvability of a PDE in terms of it admitting a Lax formulation. That is the PDE can be expressed in the form

$$L_t = [A, L]$$

where A is the Fréchet derivative of the t independent part of the PDE and L is the recursion operator.

The motivation to investigate the Lie-Bäcklund symmetries of the KdVB equation is twofold:

(i) to obtain, if possible, more generalized similarity solutions that could extend the use of the KdVB equation to other cases involving nonlinear wave propagation.

(ii) to gain insight into questions of the exact solvability of the KdVB equation.

CHAPTER 2: THE HARRISON-ESTABROOK METHOD

This method is a geometric approach for finding invariance groups and similarity solutions. In this chapter I intend using it on the following partial differential equations:

(1) The one-dimensional nonlinear diffusion equation $\phi_t - (\phi^n \phi_x)_x = 0$ where n is a real constant, $\phi = \phi(x, t)$ the dependent variable and the subscripts denoting differentiating with respect to space (x) and time (t), i.e. $\phi_t = \frac{\partial \phi}{\partial t}$ and $\phi_x = \frac{\partial \phi}{\partial x}$

(2) The variable coefficient Korteweg-de Vries (VcKdV) equation

$$u_t + \alpha t^n u u_x + \beta t^m u_{xxx} = 0$$

where $u = u(x, t)$ is the dependent variable, m and n real numbers, and α and β constant parameters.

The one-dimensional nonlinear diffusion equation

$$\phi_t - (\phi^n \phi_x)_x = 0$$

or

$$\phi_t - n\phi^{n-1}(\phi_x)^2 - \phi^n \phi_{xx} = 0 \tag{2.1}$$

In this section, I intend to check the results obtained by S M Waller in 1990 [5]. The first step in finding invariance transformation groups by the Harrison-Estabrook method [1] is to cast the PDE (2.1) into an equivalent closed ideal of differential forms in a multidimensional space. One suitable ideal for this is

$$\alpha = d\phi - y dx - u dt \tag{2.2}$$

$$d\alpha = -dy \wedge dx - du \wedge dt \tag{2.3}$$

and

$$\beta = (u - n\phi^{n-1} y^2)\, dx \wedge dt - \phi^n\, dy \wedge dt \tag{2.4}$$

Annulling (2.2), (2.3) and (2.4) on the integral manifold $\phi = \phi(x, t)$, where

$$d\phi = \phi_x\,dx + \phi_t dt,$$

$$dy = y_x dx + y_t dt,$$

and
$$du = u_x dx + u_t\,dt,$$

we obtain the following (a tilde indicates restriction of the differential form to the integral manifold):

(i) $\tilde{\alpha} = 0$ implies that $(\phi_x - y)dx + (\phi_t - u)dt = 0$ which gives the definitions of the auxiliary variables $y = \phi_x$ and $u = \phi_t$.

(ii) $(\widetilde{d\alpha}) = 0$ implies that $(y_t - u_x)dx\wedge dt = 0$ which means $y_t = u_x$ or $\phi_{xt} = \phi_{tx}$, the integrability condition on ϕ.

(iii) $\tilde{\beta} = 0$ implies that

$$(u - n\phi^{n-1}y^2 - \phi^n y_x)dx\wedge dt = 0$$

or
$$\phi_t - n\phi^{n-1}(\phi_x)^2 - \phi^n\phi_{xx} = 0,$$

the original PDE (2.1).

The generators of the invariance groups are the isovector

$$V = V^x\partial_x + V^t\partial_t + V^\phi\partial_\phi + V^y\partial_y + V^u\partial_u \tag{2.5}$$

in the 5-dimensional space (x, t, ϕ, y, u). The action of V on the closed ideal is such that the Lie derivatives with respect to V of the forms (2.2) to (2.4) are still in the ideal, i.e.

$$\pounds_V(I) \subset I \tag{2.6}$$

(2.6) is known as the invariant condition. As α is the only 1-form in the ideal (2.6) implies that

$$\pounds_V(\alpha) = \lambda\alpha \tag{2.7}$$

where λ is an arbitrary 0-form. Introducing the 0-form $F = V \lrcorner \alpha$, equation (2.7) on expanding becomes

$$\pounds_V(\alpha) = \lambda\alpha = V \lrcorner d\alpha + d(V \lrcorner \alpha)$$

$$= V \lrcorner d\alpha + dF \tag{2.8}$$

Expanding dF, we have

$$dF = F_x dx + F_t dt + F_\phi d\phi + F_y dy + F_u du \tag{2.9}$$

Also $\quad V \lrcorner\, d\alpha = (V^x\partial_x + V^t\partial_t + V^\phi\partial_\phi + V^y\partial_y + V^u\partial_u) \lrcorner \left(-dy \wedge dx - du \wedge dt\right)$

$$= -V^y dx + V^x dy - V^u dt + V^t du \tag{2.10}$$

Equations (2.8), (2.9) and (2.10) imply that

$$\lambda(d\phi - ydx - udt)$$

$$= -V^y dx + V^x dy - V^u dt + V^t du + F_x dx + F_t dt + F_\phi d\phi + F_y dy + F_u du \tag{2.11}$$

Equating coefficients of the basis 1-forms gives the following set of equations

$$\left.\begin{array}{l} F_\phi = \lambda, \quad V^x = -F_y, \quad V^t = -F_u, \\ V^y = F_x + yF_\phi, \quad V^u = F_t + uF_\phi, \end{array}\right\} \tag{2.12}$$

$$F = V \lrcorner \alpha$$

$$= (V^x\partial_x + V^t\partial_t + V^\phi\partial_\phi + V^y\partial_y + V^u\partial_u) \lrcorner \left(d\phi - ydx - udt\right)$$

$$= V^\phi - yV^x - uV^t$$

or $\quad V^\phi = F - yF_y - uF_u \quad (2.13)$

For the second differential form $d\alpha$ we find

$$\pounds_V(d\alpha) = d(\pounds_V(\alpha))$$

$$= (d\lambda)\wedge\alpha + \lambda d\alpha,$$

which is a 2-form already in the ideal.

For the last differential form β the invariance condition (2.6) implies that $\pounds_V(\beta)$ is a 2-form in the ideal

$$\pounds_v(\beta) = \xi\beta + \zeta d\alpha + \omega\wedge\alpha, \tag{2.14}$$

where ξ and ζ are arbitrary 0-forms and ω is an arbitrary 1-form

Let
$$\omega = Adu + Bdy + Cdx + Ddt + Ed\phi \tag{2.15}$$

where A, B, C, D and E are arbitrary 0-forms.

As
$$\pounds_V(\beta) = V\lrcorner d\beta + d(V\lrcorner\beta)$$

we get, on expanding $V\lrcorner\beta$, $V\lrcorner d\beta$ and ω

$$\begin{aligned}
\pounds_V(\beta) &= V^u dx\wedge dt - n(n-1)\phi^{n-2}y^2\, V^\phi dx\wedge dt \\
&-2ny\phi^{n-1}V^y dx\wedge dt - n\phi^{n-1}V^\phi dy\wedge dt + (u-n\phi^{n-1}y^2)dV^x\wedge dt \\
&-(u-n\phi^{n-1}y^2)dV^t\wedge dx - \phi^n dV^y\wedge dt + \phi^n\, dV^t\wedge dy \\
&= \xi[(u-n\phi^{n-1}y^2)dx\wedge dt - \phi^n dy\wedge dt] \\
&-\zeta(dy\wedge dx+du\wedge dt) + [Adu+Bdy+Cdx+Ddt+Ed\phi]\wedge(d\phi-ydx-udt)
\end{aligned} \tag{2.16}$$

where
$$dV^i = V^i_x\, dx + V^i_x dt + V^i_\phi d\phi + V^i_y dy + V^i_u du$$

and
$i = x, t, \phi, y$ and u in turn.

Equating coefficients of basis 2-forms on both sides of (2.16) gives the following set of equations:

$$\begin{aligned}
&V^u-n(n-1)\phi^{n-2}y^2V^\phi - 2ny\phi^{n-1}V^y + (u-n\phi^{n-1}y^2)V^x_x \\
&+(u-n\phi^{n-1}y^2)V^t_t - \phi^n V^y_x = \xi(u-n\phi^{n-1}y^2) - uC + yD
\end{aligned} \tag{2.17}$$

$$(u-n\phi^{n-1}y^2)\, V^t_\phi = C + yE \tag{2.18}$$

$$(u-n\phi^{n-1}y^2)\, V^t_y + \phi^n V^t_x = \zeta + yB \tag{2.19}$$

$$(u-n\phi^{n-1}y^2)\, V^t_u = yA \tag{2.20}$$

$$-(u-n\phi^{n-1}y^2)\, V^x_\phi + \phi^n V^y_\phi = D + uE \tag{2.21}$$

$$n\phi^{n-1}V^\phi - (u-n\phi^{n-1}y^2)\, V^x_y + \phi^n V^y_y + \phi^n V^t_t = \xi\,\phi^n + uB \tag{2.22}$$

$$-(u-n\phi^{n-1}y^2)V^x_u + \phi^n V^y_u = \zeta + uA \tag{2.23}$$

$$\phi^n V^t_\phi = -B \tag{2.24}$$

$$-\phi_n V^t_u = 0 \tag{2.25}$$

Eliminating the arbitrary 0-forms from (2.17) to (2.25) gives the determining equations for the isovector:

$$V^t_u = 0 \tag{2.26}$$

$$\phi^n(V^y_u - V^t_x - yV^t_\phi) - (u-n\phi^{n-1}y^2)\,(V^x_u + V^t_y) = 0 \tag{2.27}$$

$$\begin{aligned} &y\phi^n V^y_\phi + n(n-1)\phi^{n-2}y^2V^\phi + 2ny\phi^{n-1}V^y - V^u \\ &-(u-n\phi^{n-1}y^2)(V^x_x + yV^x_\phi - V^y_y - n\phi^{-1}V^\phi) \\ &-(u-n\phi^{n-1}y^2)^2\phi^{-n}V^x_y + \phi^n X^y_x = 0 \end{aligned} \tag{2.28}$$

The determining equations and (2.12) and (2.13) can be solved for F to give:

$$F = \frac{1}{n}\left(2\delta_4-\delta_3\right)\phi - (\delta_4 x + \delta_2)y - (\delta_3 t + \delta_1)u \tag{2.29}$$

where $\delta_1, \delta_2, \delta_3$ and δ_4 are arbitrary constants. (2.12) and (2.13) in conjunction with (2.29) give the following set of equations for the components of the isovector

$$\left.\begin{aligned}
V^x &= \delta_4 x + \delta_2 \\
V^t &= \delta_3 t + \delta_1 \\
V^y &= \frac{1}{n}[(2-n)\delta_4 - \delta_3]y \\
V^u &= \frac{1}{n}[2\delta_4 - (n+1)\delta_3]u \\
V^\phi &= \frac{1}{n}[2\delta_4 - \delta_3]\phi
\end{aligned}\right\} \tag{2.30}$$

(2.30) gives a 4 parameter invariance group for the PDE (2.1). Independent generators of the isogroup are found by setting all parameters except one to zero.

The Invariance Group of $\phi_t - (\phi^n\phi_x)_x = 0$

δ	V^x	V^t	V^ϕ	V^u	V^y	Types of transformation
$\delta_1 = 1$	0	1	0	0	0	time dilation
$\delta_2 = 1$	1	0	0	0	0	space dilation
$\delta_3 = 1$	0	t	$-\frac{\phi}{n}$	$-\left(\frac{n+1}{n}\right)u$	$-\frac{y}{n}$	$t-\phi$ scale change
$\delta_4 = 1$	x	0	$\frac{2\phi}{n}$	$\frac{2u}{n}$	$\left(\frac{2-n}{n}\right)y$	$x-\phi$ scale change

This invariance group is the same as that obtained by S M Waller [5].

In (x, t, ϕ) space the infinitesimal generators are:

$$
\left.\begin{aligned}
X_1 &= \partial_x \\
X_2 &= \partial_t \\
X_3 &= t\partial_t - \frac{\phi}{n}\partial_\phi \\
X_4 &= x\partial_x + \frac{2\phi}{n}\partial_\phi
\end{aligned}\right\} \tag{2.31}
$$

The classical method of Bluman and Cole (Appendix A) gives the following invariance group for $\phi_t - (\phi^n \phi_x)_x = 0$

$$
\left.\begin{aligned}
X_1' &= \partial_x \\
X_2' &= \partial_t \\
X_3' &= x\partial_x + 2t\partial_t \\
X_4' &= x\partial_x + \frac{2\phi}{n}\partial_t
\end{aligned}\right\} \tag{2.32}
$$

As $X_3' = 2X_3 + X_4$, the group (2.31) obtained using differential forms is consistent with that of (2.32). The commutator table for the corresponding Lie Algebra for (2.31) is

	X_1	X_2	X_3	X_4
X_1	0	0	X_1	0
X_2	0	0	0	X_2
X_3	$-X_1$	0	0	0
X_4	0	$-X_2$	0	0

The structure constants given by $[X_i, X_j] = C_{ij}^k X_k$ are

(i) $\quad C_{13}^1 = -C_{31}^1 = 1$

(ii) $\quad C_{24}^2 = -C_{42}^2 = 1$

all other structure constants are zero. **Similarity solutions** for (2.1) can be found by augmenting the closed ideal (2.2) to (2.4) by contracting α, $d\alpha$ or β with V and annulling the new differential forms on the integral manifold $\phi = \phi(x, t)$.

For similarity solutions the most useful of the 3 new forms is

$$\begin{aligned} F &= V \lrcorner \alpha \\ &= \left(V^x\partial_x + V^t\partial_t + V^\phi\partial_\phi + V^y\partial_y + V^u\partial_u\right) \lrcorner \left(d\phi - udt - ydx\right) \\ &= V^\phi - uV^t - yV^x \end{aligned} \tag{2.33}$$

Annulling (2.33) on $\phi = \phi(x, t)$, where $u = \phi_t$ and $y = \phi_x$, we obtain

$$V^\phi = \phi_t V^t + \phi_x V^x \tag{2.34}$$

Substituting for V^ϕ, V^t and V^x in (2.34) gives

$$\frac{1}{n}\left(2\delta_4 - \delta_3\right)\phi = \left(\delta_1 + \delta_3 t\right)\phi_t + \left(\delta_2 + \delta_4 x\right)\phi_x$$

or

$$\phi = \frac{n(\delta_1 + \delta_3 t)\phi_t}{2\delta_4 - \delta_3} + \frac{n(\delta_2 + \delta_4 x)\phi_x}{2\delta_4 - \delta_3} \tag{2.35}$$

which is a quasilinear PDE for ϕ and where $2\delta_4 - \delta_3 \neq 0$.

Simplifying (2.35) with $\theta_i = \dfrac{\delta_i}{2\delta_4 - \delta_3}$ $(i = 1, 2, 3, 4)$ gives

$$\phi = n(\theta_1 + \theta_3 t)\,\phi_t + n(\theta_2 + \theta_4 x)\phi_x \tag{2.36}$$

with subsidiary equations

$$\frac{dt}{n(\theta_1 + \theta_3 t)} = \frac{dx}{n(\theta_2 + \theta_4 x)} = \frac{d\phi}{\phi} \tag{2.37}$$

Solving (2.37) by Lagrange's method of characteristics gives

$$\zeta = \frac{\theta_2 + \theta_4 x}{(\theta_1 + \theta_3 t)^{\theta_4/\theta_3}} \tag{2.38}$$

as the similarity variable and

$$\phi(x, t) = G(\zeta)(\theta_1 + \theta_3 t)^{\frac{1}{n\theta_3}} \tag{2.39}$$

as the similarity form for $\phi = \phi(x, t)$. Substituting (2.39) into the original PDE (2.1) we find that the arbitrary function $G(\zeta)$ satisfies the ODE

$$G''(\zeta) + nG(\zeta)^{-1}G'(\zeta)^2 + \frac{\zeta}{\theta_4} G(\zeta)^{-n} G'(\zeta)(\theta_1 + \theta_3 t)^{\frac{2\theta_4-\theta_3-1}{\theta_3}}$$

$$-\frac{1}{n\theta_4^2} G(\zeta)^{1-n}(\theta_1 + \theta_3 t)^{\frac{2\theta_4-\theta_3-1}{\theta_3}} = 0 \tag{2.40}$$

The similarity variable (2.38) and the similarity form (2.39) are the same as Waller's result [5]. However the ODE (2.40) agrees with Waller's version only if $\theta_3 = \theta_4 = 1$.

Waller next considers the special case of $n = -1$ and begins by setting

$$\theta_3 = \theta_4 = 1 \tag{2.41}$$

The condition (2.41) should, in my opinion, have been set prior to his version of the ODE (2.40). Under the condition (2.41) and $n = -1$ the PDE (2.1), the similarity variable (2.38), the similarity form (2.39), and the ODE (2.40) all reduce to:

$$\left.\begin{aligned} &\phi_t + \phi^{-2}(\phi_x)^2 - \phi^{-1}\phi_{xx} = 0 \\ &\zeta = \frac{\theta_2 + x}{\theta_1 + t} \\ &\phi(x,t) = G(\zeta)(\theta_1 + t)^{-1} \\ &\text{and} \\ &G''(\zeta) - G(\zeta)^{-1}G'(\zeta)^2 + \zeta G(\zeta)G'(\zeta) + G(\zeta)^2 = 0 \end{aligned}\right\} (2.42)$$

(2.42) is in exact agreement with Waller's result. We will now consider

$$G(\zeta) = \frac{a}{p + \zeta^2} \tag{2.43}$$

where a and p are constants, as a generating function for a solution to the ODE in (2.42). This implies that

$$\left.\begin{aligned} G'(\zeta) &= -\frac{2a\zeta}{(p+\zeta^2)^2} \\ \text{and} & \\ G''(\zeta) &= \frac{6a\zeta^2-2ap}{(p+\zeta^2)^3} \end{aligned}\right\} \tag{2.44}$$

Substituting (2.43) and (2.44) into the ODE gives

$$\frac{6a\zeta^2 - 2ap}{(p+\zeta^2)^3} - \frac{4a\zeta^2}{(p+\zeta^2)^3} - \frac{2a^2\zeta^2}{(p+\zeta^2)^3} + \frac{a^2}{(p+\zeta^2)^2} = 0$$

or
$$\frac{a(2-a)(\zeta^2-p)}{(p+\zeta^2)^3} = 0 \tag{2.45}$$

(2.45) gives either $a = 0$ implying $\phi = 0$, or $a = 2$ and

$$\begin{aligned} \phi(x,t) &= G(\zeta)(\theta_1 + t)^{-1} \\ &= \left\{\frac{(\theta_1+t)}{2}\left[p + \left(\frac{\theta_2+x}{\theta_1+t}\right)^2\right]\right\}^{-1} \end{aligned} \tag{2.46}$$

The reduction of (2.46) to a one parameter group solution is possible if $\theta_1 = \theta_2 = 0$ which means

$$\phi(x, t) = \left\{\frac{t}{2}\left[p + \left(\frac{x}{t}\right)^2\right]\right\}^{-1} \tag{2.47}$$

The above critique of Waller's 1990 paper [5] basically confirm his results. As a further check I did a second determination of the point symmetries of the nonlinear diffusion equation (2.1) using another closed ideal of differential forms, namely:

$$\left.\begin{aligned}
\alpha_1 &= d\phi\wedge dt - p\,dx\wedge dt \\
\alpha_2 &= d\phi\wedge dx + q\,dx\wedge dt \\
\alpha_3 &= d\phi\wedge dx + \phi^n dp\wedge dt + n\phi^{n-1}p^2 dx\wedge dt \\
\alpha_4 &= -dp\wedge dx - dq\wedge dt
\end{aligned}\right\} \tag{2.48}$$

The details of this computation are in Appendix B, and the results obtained are identical to those of (2.30) giving further confirmation to Waller's result.

The Variable Coefficient KdV Equation

In this section I will consider point symmetries of the variable coefficient Korteweg-deVries equation (VcKdV)

$$u_t + \alpha t^n u u_x + \beta t^m u_{xxx} = 0 \tag{2.49}$$

where α and β are arbitrary constant parameters, and m and n are real numbers. Nirmala, Vedan and Baby in a 1986 paper [24], used the classical method of Bulman and Cole to obtain point symmetries of (2.49). As a check on their results, I intend to attempt to find the point symmetries of (2.49) using the Harrison-Estabrook method involving differential forms. To the best of my knowledge this has not been tried before on this equation. We begin by considering the following ideal of differential forms as a possible representation of the PDE (2.49).

$$\left.\begin{aligned}
\alpha &= dz - w\,dx - y\,dt \\
d\alpha &= -\,dw\wedge dx - dy\wedge dt \\
\beta &= (du - z\,dx)\wedge dt \\
\gamma &= du\wedge dx + \alpha u z t^n dt\wedge dx - \beta t^m dw\wedge dt
\end{aligned}\right\} \tag{2.50}$$

where $z = u_x$, $y = u_{xt}$ and $w = u_{xx}$. As $d\beta = d\gamma = 0$ the ideal (2.50) is closed and forms a basis for a Grassman Algebra of differential forms on the 6-dimensional manifold spanned by (x, t, u, z, y, w).

The annulling of the forms in (2.50) on the solution manifold $u = u(x, t)$ where

$$\left.\begin{aligned} dz &= z_x dx + z_t dt \\ dy &= y_x\, dx + y_t dt \\ &\text{and} \\ dw &= w_x\, dx + w_t dt \end{aligned}\right\} \tag{2.51}$$

gives the following results:

(i) $\tilde{\alpha} = 0$ implies that $w = z_x$ and $y = z_t$

(ii) $d\tilde{\alpha} = 0$ implies that $z_{tx} = z_{xt}$ an integrability condition for z.

(iii) $\tilde{\beta} = 0$ implies that $z = u_x$ and hence that $w = u_{xx}$ and $y = u_{xt}$ the definitions of the necessary prolongation variables.

(iv) $\tilde{\gamma} = 0$ give the PDE (2.49).

These results confirm that the ideal is closed and does represent (2.49). The generators of the invariant transformation groups of the ideal (2.50) are the components of the isovector

$$V = V^x\partial_x + V^t\partial_t + V^u\partial_u + V^z\partial_z + V^y\,\partial_y + V^w\partial_w \tag{2.52}$$

The invariance condition for the transformation groups requires that the Lie derivatives with respect to V for each of the forms in the ideal are linear combinations of members of the ideal. That is:

$$£_v(\alpha) = \lambda\alpha \tag{2.53}$$

where λ is an arbitrary 0-form

$$£_v(\beta) = \xi_1\,\gamma + \zeta_1\beta + \mu_1\, d\alpha + w_1\wedge\alpha \tag{2.54}$$

$$£_v(\gamma) = \xi_2\gamma + \zeta_2\,\beta + \mu_2\, d\alpha + w_2\wedge\alpha \tag{2.55}$$

where ξ_i, ζ_i and μ_i are arbitrary 0-forms and

$w_i = A_i\, dx + B_i\, dt + C_i\, du + D_i dz + E_i dy + G_i dw$
A_i, B_i, C_i, D_i, E_i and G_i are arbitrary 0-forms and $i = 1, 2$.

$\pounds_v(d\alpha) = d\lambda\wedge\alpha + \lambda d\alpha$ is already a linear combination of elements in the ideal and contributes nothing to the determination of the components of the isovector.

We now expand the Lie derivatives of α, β and γ starting with (2.53)

$$\pounds_v(\alpha) = \lambda\alpha = V\lrcorner d\alpha + d(V\lrcorner\alpha) \tag{2.56}$$

Let $F = V\lrcorner\alpha$

$$= \left(V^x\partial_x + V^t\partial_t + V^u\partial_u + V^z\partial_z + V^y\partial_y + V^w\partial_w\right)\lrcorner\,(dz - wdx - ydt)$$

$$= V^z - wV^x - yV^t \tag{2.57}$$

$$V\lrcorner d\alpha = (V^x\partial_x + V^t\partial_t + V^u\partial_u + V^z\partial_z + V^y\partial_y + V^w\partial_w)\lrcorner$$
$$(-dw\wedge dx - dy\wedge dt)$$
$$= -V^w dx + V^x dw - V^y dt + V^t dy \tag{2.58}$$

$$V = \left(V^x\partial_x + V^t\partial_t + V^u\partial_u + V^z\partial_z + V^y\partial_y + V^w\partial_w\right)\lrcorner(dz - wdx - ydt)$$

As $dF = F_x dx + F_t dt + F_u du + F_z dz + F_y dy + F_w dw$

$$= d(V\lrcorner\alpha)$$

then this along with (2.56) and (2.58) mean that

$$(F_x - V^w + \lambda w)dx + (F_t - V^y + \lambda y)dt + F_u du$$
$$+ (F_z - \lambda)dz + (V^t + F_y)dy + (V^x + F_w)dw = 0 \tag{2.59}$$

Equating the coefficients of the basis 1-forms in (2.59) to zero gives the following set of equations for the components of (2.52) in terms of F.

$$\left.\begin{array}{ll} V^x = -F_w & V^t = -F_y \\ F_z = \lambda & F_u = 0 \\ V^w = F_x + wF_z & V^y = F_t + yF_z \\ V^z = F - wF_w - yF_y & \end{array}\right\} \tag{2.60}$$

β and γ both being 2-forms have their Lie derivatives expanded in a similar fashion. See Appendix C for the details. The results are:

$$\begin{aligned} \pounds_v(\beta) &= -V^z\, dx\wedge dt + dV^u\wedge dt - dV^t\wedge du \\ &\quad - zdV^x\wedge dt + zdV^t\wedge dx \\ &= \xi_1\gamma + \zeta_1\beta + \mu_1 d\alpha + w_1\wedge\alpha \end{aligned} \tag{2.61}$$

and

$$\begin{aligned} \pounds_v(\gamma) &= \alpha t^n z V^u dt\wedge dx + \alpha\, t^n u V^z\, dt\wedge dx \\ &\quad + \alpha uznt^{n-1}V^t dt\wedge dx + m\beta t^{m-1}V^t\, dt\wedge dw + dV^n\wedge dx \\ &\quad - dV^x\wedge du + \alpha uzt^n (dV^t\wedge dx - dV^x\wedge dt) \\ &\quad - \beta t^m (dV^w\wedge dt - dV^t\wedge dw) \\ &= \xi_2\gamma + \zeta_2\beta + \mu_2\, d\alpha + w_2\wedge\alpha \end{aligned} \tag{2.62}$$

The determining equations for the coefficients of the isovector (V) are obtained by equating the coefficients of the basis 2-forms on both sides of (2.61) and (2.62) and then eliminating the arbitrary 0-forms. The details of this computation are to be found in Appendix C. The resulting determining equations for the isovectors components are:

$$\begin{aligned} & V^t_y = V^t_w = 0, \quad V^x_y = 0, \quad V^u_y = 0 \\ & V^x_w + \beta t^m V^t_u = 0, \; V^u_w - \beta t^m (V^t_x + wV^t_z + V^w_y) = 0 \\ & \beta t^m (V^t_x + zV^t_u + wV^t_z) = zV^x_w - V^u_w \\ & V^u_x - V^z + wV^u_z + \alpha uzt^n (V^t_x + zV^t_u + wV^t_z) \\ & + z(V^u_u - zV^x_u - V^x_x - wV^x_z) = 0 \\ & m\beta t^{m-1}V^t + \alpha uzt^n (V^x_w - \beta t^m V^t_u) \\ & + \beta t^m (V^w_w + V^t_t - V^u_u - V^x_x - wV^x_z - V^w_y + yV^t_z) = 0 \end{aligned}$$

and $$\alpha t^n(zV^u + uV^z) + \alpha uznt^{n-1}V^t + V^u_t + yV^u_z$$
$$- z V^x_t - zyV^x_z + \beta t^m (V^w_x + zV^w_u + wV^w_z)$$
$$- z V^x_t - zyV^x_z + \beta t^m (V^w_x + zV^w_u + wV^w_z)$$
$$+ \alpha uzt^n (V^t_t - V^u_u - \alpha uzt^n V^t_u + zV^x_u + yV^t_z) = 0$$

Solving the above determining equations with the help of the set (2.60) (the details are found in Appendix C) we get the following components for the isovector. m and n are arbitrary real numbers.

$$\left.\begin{aligned} V^t &= 0 \\ V^x &= \frac{a\alpha t^{n+1}}{n+1} + b \\ V^u &= a \\ V^w &= V^z = 0 \\ V^y &= -a\alpha t^n w \end{aligned}\right\} \tag{2.63}$$

The variable coefficient KdV equation

$$u_t + \alpha t^n u u_x + \beta t^m u_{xxx} = 0$$

where m and n are arbitrary real numbers, has the following 2 parameter invariance group

	V^t	V^x	V^u	V^w	V^z	V^y
$a = 1$	0	$\frac{\alpha t^{n+1}}{n+1}$	1	0	0	$-\alpha t^n w$
$b = 1$	0	1	0	0	0	0

In (x, t, u) space the infinitesimal generators of the transformation group are

$$\left.\begin{aligned} X_1 &= \frac{\alpha t^{n+1}}{n+1}\partial_x + \partial_n \\ X_2 &= \partial_x \end{aligned}\right\} \tag{2.64}$$

(2.64) are the same as the generators found by Nirmala, Vedan, and Baby [24].

The next stage is to determine the invariance transformations for the variable coefficient KdV equation for m and n being linearly related. That is

$$m = k_1 n + k_2 \tag{2.65}$$

where k_1 and k_2 are real constants.

The analysis is identical to that used for arbitrary values of m and n up to the point in Appendix C where the determining equations have been reduced to:

$$\left.\begin{array}{lllllllllll} V^t_x & = & V^t_u & = & V^t_z & = & V^t_y & = & V^t_w & = & 0 \\ & & V^x_u & & & = & V^x_y & = & V^x_w & = & 0 \\ & & V^y_u & & & & & & & = & 0 \\ & & V^z_u & & & = & V^z_y & = & V^z_w & = & 0 \\ & & V^w_u & & & = & V^w_y & & & = & 0 \\ & & & & & & V^u_y & = & V^u_w & = & 0 \end{array}\right\} \tag{C60}$$

$$mV^t + t(V^w_w + V^t_t - V^u_u - V^x_x - wV^x_z) = 0 \tag{C61}$$

$$V^u_x - V^z + wV^u_z + z(V^u_u - V^x_x - wV^x_z) = 0 \tag{C64}$$

$$\begin{aligned} &\alpha t^n (zV^u + uV^z) + \alpha uznt^{n-1}V^t + V^u_t + yV^u_z - zV^x_t \\ &- zyV^x_z + \beta t^m (V^w_x + wV^w_z) + \alpha uzt^n(V^t_t - V^u_u) = 0 \end{aligned} \tag{2.66}$$

and

$$F = a(t)y + g(x,t,z)w + h(x,t,z) \tag{2.67}$$

From (2.60) and (2.67) we obtain the following set of equations

$$\left.\begin{array}{l} V^x = -F_w = -g(x,t,z) \\ V^t = -F_y = -a(t) \\ V^w = F_x + wF_z = g_x w + h_x + g_z w^2 + h_z w \\ V^z = F - wF_w - yF_y = h(x,t,z) \\ V^y = F_t + wF_z = a'(t)y + g_t w + h_t + g_z w^2 + h_z w \\ F_z = \lambda = g_z w + h_z \text{ and } F_u = 0 \end{array}\right\} \tag{2.68}$$

Using the set of equations (2.68) to solve the reduced determining equations (C60) (C61) (C64) and (2.66) we obtain the following components of the isovector

$$\left.\begin{aligned}
V^t &= \left(\frac{a_1 - 2a_2}{k_2}\right)t \\
V^x &= -a_2 x + \frac{a_3 \alpha t^{n+1}}{n+1} + a_4 \\
V^z &= (a_1 + a_2)z \\
V^u &= a_1 u + a_3 \\
V^w &= (2a_2 + a_1)w \\
V^y &= \left(\frac{2a_2 - a_1}{k_2}\right) y + a_3 \alpha t^n w + (a_1 + a_2)w
\end{aligned}\right\} \tag{2.69}$$

where a_1 a_2 a_3 and a_4 are arbitrary integration constants.

The linear relationship (2.65) is now restricted to

$$m = n + k_2 \tag{2.70}$$

with $k_2 \neq 0$.

Details of this computation can be found in Appendix D.

The invariance group of the variable coefficient KdV equation

$$u_t + \alpha t^n u u_x + \beta t^{n+k_2} u_{xxx} = 0$$

is

	V^x	V^t	V^u	V^z	V^w	V^y
$a_1 = 1$	0	$\frac{t}{k_2}$	u	z	w	$w - \frac{y}{k_2}$
$a_2 = 1$	$-x$	$-\frac{2t}{k_2}$	0	z	2w	$\frac{2y}{k_2} + w$
$a_3 = 1$	$\frac{\alpha t^{n+1}}{n+1}$	0	1	0	0	$\alpha t^n w$
$a_4 = 1$	1	0	0	0	0	0

where $z = u_x$ $y = u_{xt}$ and $w = u_{xx}$.

In (x, t, u) space the infinitesimal generators of this isogroup are:

$$\left.\begin{aligned} X_1 &= t\partial_t + u\partial_u \\ X_2 &= \left(\frac{\alpha t^{n+1}}{n+1}\right)\partial_x + \partial_u \\ X_3 &= x\partial_x + 2t\partial_t \\ X_4 &= \partial_x \end{aligned}\right\} \tag{2.71}$$

Its commutator table is:

	X_1	X_2	X_3	X_4
X_1	0	$\left(\frac{n\alpha t^{n+1}}{n+1}\right)X_4 - X_2$	0	0
X_2	$X_2 - \left(\frac{n\alpha t^{n+1}}{n+1}\right)X_4$	0	$-\frac{(2n+1)\alpha t^{n+1}}{n+1}X_4$	0
X_3	0	$\frac{(2n+1)\alpha t^{n+1}}{n+1}X_4$	0	$-X_4$
X_4	0	0	X_4	0

Similarity Solutions of the variable coefficient KdV equation

$$u_t + \alpha t^n u u_x + \beta t^m u_{xxx} = 0 \tag{2.49}$$

(1) For Arbitrary values of m and n

Similarity solutions for (2.49) are found by extending the closed ideal of differential forms (2.50) by contracting the isovector (V) with one of the differential forms in the ideal. The most suitable contraction is $\sigma = V \lrcorner \beta$ which leads to

$$V^u - u_t V^t - u_x V^x = 0 \tag{2.72}$$

(2.72) is a quasi-linear PDE which can be solved using Lagrange's method of characteristics.

For arbitrary values of m and n the infinitesimal generators are (2.64) namely

$$X_1 = \frac{\alpha t^{n+1}}{n+1}\,\partial_x + \partial_u$$

and

$$X_2 = \partial_x$$

For the generator X_2, $V^u = V^t = 0$ and $V^x = 1$ in (2.72) which leads to the similarity variable ζ = constant and the trivial similarity form u = constant. For the generator X_1, $V^x = \frac{\alpha t^{n+1}}{n+1}$ $V^u = 1$ and $V^t = 0$ giving the similarity variable ζ = constant and similarity form

$$u = \left(\frac{\alpha\zeta^{n+1}}{n+1}\right)^{-1} x + f(\zeta) \tag{2.73}$$

On substituting (2.73) into (2.49) we get the solution

$$u = \frac{(n+1)x + \alpha k}{\alpha t^{n+1}} \tag{2.74}$$

where k is an arbitrary constant.

Nirmala, Vedan and Baby obtained the solution

$$u = \frac{a(n+1)x + c}{a\alpha t^{n+1} + b(n+1)} \tag{2.75}$$

where a, b and c are arbitrary constants. (2.75) is the same as (2.74) if $b = 0$. Details of the calculations for (2.74) are found in Appendix H.

(ii) For $m = n + 1$ ($k_2 = 1$)

The variable coefficient KdV equation becomes

$$u_t + \alpha t^n u u_x + \beta t^{n+1} u_{xxx} = 0 \tag{2.76}$$

We obtain similarity solution of (2.76) by using $V^u - u_t V^t - u_x V^x = 0$ in conjunction with the four infinitesimal generators

$$X_1 = t\partial_t + u\partial_u$$

$$X_2 = \left(\frac{\alpha t^{n+1}}{n+1}\right)\partial_x + \partial_u$$

$$X_3 = x\partial_x + 2t\partial_t$$

$$X_4 = \partial_x$$

$X_1 = t\partial_t + u\partial_u$ means that $V^x = 0$ and $V^t = V^u = 1$.
This leads to $x = \zeta =$ constant as the similarity variable and

$$u = tf(\zeta) \tag{2.77}$$

as the similarity form. On substitution of (2.77) into (2.76) we get the ODE

$$\beta t^{n+2} \frac{d^3 f(\zeta)}{d\zeta^3} + \alpha t^{n+2} f(\zeta) \frac{df(\zeta)}{\delta\zeta} + d(\zeta) = 0 \tag{2.78}$$

$$X_2 = \left(\frac{\alpha t^{n+1}}{n+1}\right)\partial_x + \partial_u$$

means that $$V^x = \frac{\alpha t^{n+1}}{n+1}$$

$V^t = 0$ and $V^u = 1$ which lead to the solution

$$u = \frac{(n+1)x + \alpha k}{\alpha t^{n+1}}$$

which is the same as (2.74).

$X_3 = x\partial_x + 2t\partial_t$ means that $V^x = x$, $V^t = 2t$ and $V^u = 0$. This leads to $\frac{x^2}{t} = \zeta$ as the similarity variable and $u - f(\zeta)$ as the similarity form. On substituting the similarity variable into (2.76) we get the ODE

$$4\beta t^{n-2}\left\{2\zeta \frac{d^3 f(\zeta)}{d\zeta} + 3\frac{d^2 f(\zeta)}{d\zeta}\right\}$$

$$+ \left\{2\alpha t^n f(\zeta) - \frac{x}{t}\right\} \frac{df(\zeta)}{d\zeta} = 0 \tag{2.79}$$

Finally $X_4 = \partial_x$ gives the trivial solution u = constant.

In the similarity solutions for the variable coefficient KdV equation, there appeared to be no soliton solution. The reason for this is that a soliton solution requires a similarity solution of the form $u = f(x-ct)$ which in turn implies a similarity variable $\zeta = x - ct$ where c is a constant. This kind of similarity variable is produced by an infinitesimal generator of the form $X = \partial_x + c\partial_t$ which does not appear in any of the invariant transformation groups (2.64) and (2.71).

It has already been noted that the invariance group (2.64), that is

$$X_1 = \left(\frac{\alpha t^{n+1}}{n+1}\right)\partial_u + \partial_u$$

and $$X_2 = \partial_x$$

which were obtained by the Harrison-Estabrook method are the same as those obtained by Nirmala, Vedan and Baby [24] so confirming their results. However, the invariant group (2.71) for m and n related by $m = n + 1$ are substantially different. I intend returning to these in the conclusions in Chapter 4.

CHAPTER 3: LIE-BÄCKLUND SYMMETRIES FOR THE KORTEWEG-deVRIES-BURGERS EQUATION

In the introduction and cited literature, the Korteweg-deVries-Burgers (KdVB) equation is usually expressed as:

$$H = 0 \equiv u_t + auu_x + bu_{xx} + cu_{xxx} = 0 \tag{3.1}$$

where a, b and c are real constants. In this chapter I will use a system of notation where

$$u_0 = u \qquad u_1 = u_x = \frac{\partial u}{\partial x} \quad u_2 = u_{xx} = \frac{\partial^2 u}{\partial x^2} \text{ etc}$$

and (3.1) now becomes

$$H = 0 \equiv u_t + auu_1 + bu_2 + cu_3 = 0 \tag{3.2}$$

This is done to simplify the computation of Lie-Bäcklund (L-B) symmetries. L-B symmetries for time-evolution equations like (3.2) are best expressed in their evolutionary form

$$X(Q) = Q\partial_u \tag{3.3}$$

where $X(Q)$ is a differential operator.

For a time evolution equation like (3.2) the characteristic (Q) of a L-B symmetry is a function of the independent variables (x, t) and the x derivatives of u, so that

$$Q = Q(x, t, u_0\ u_1, \dots u_N) \tag{3.4}$$

for some arbitrary order N.

The existence of L-B symmetries is manifested by the existence of a *recursion operator* (R) that generates higher order symmetries from the usually more easily determined lower order ones. This means the existence of one L-B symmetry implies the existence of infinitely many.

Fokas [9] and Stephani [4] both suggest that for finding recursion operators a starting point is that in *almost all* known cases, the characteristic expressing invariance under a t-translation is generated from that expressing invariance under a x-translation. This means that

$$R[u_1] = u_t = -(cu_3 + bu_2 + auu_1) \tag{3.5}$$

(3.5) implies that the following expressions are possible recursion operators for (3.2)

$$R_1 = -(aD_x^2 + bD_x + mu + nu_1 D_x^{-1}) \tag{3.6}$$

with a = m + n and

$$R_2 = -\left(D_x^2 + \frac{b}{c}D_x + \frac{au_1}{c}D_x^{-1}\right) \tag{3.7}$$

D_x is the total derivative operator with respect to x and $D_x^{-1} = \int_{-\infty}^{x} u_i dx$.

From the definition of R given in the introduction, the necessary and sufficient condition for (3.6) or (3.7) to be a recursion operator is

$$[D_H, R]_{H=0} = 0 \tag{3.8}$$

where
$$D_H = D_t + au_1 + auD_x + bD_x^2 + cD_x^3 \tag{3.9}$$

is the Fréchet derivative of (3.2).

When the condition (3.8) is applied to (3.6) and (3.7), both give the result a = 0 which means that R_1 and R_2 are not recursion operators for the KdVB equation. Details of the computation for (3.7) are found in Appendix E.

The conclusion drawn at this point is either, the recursion operator (R) is not of the form of (3.6) or (3.7) or that it does not exist. I then decided to try and find L-B symmetries of (3.2) using the invariance condition

$$D_H[Q]_{H=0} = 0 \tag{3.10}$$

For a time-evolution equation like (3.2), (3.10) becomes

$$\sum_J D_J[Q]\frac{\partial H}{\partial u_J} = 0 \tag{3.11}$$

As $H = 0 \equiv u_t + auu_1 + bu_2 + cu_3 = 0$, we find that (3.11) takes the form

$$D_t[Q] + au_1Q + auD_x[Q] + bD_x^2\,[Q] + cD_x^3\,[Q] = 0 \qquad (3.12)$$

The required technique is to use (3.12) for $Q = Q(x, t, u, u_1 \ldots u_N)$ with selected values of N to generate at least 2 equations involving the derivatives of Q and u. I will start with $N = 3$ and $N = 5$. As (3.12) holds for all solutions, $u = u(x, t)$ of the KdVB equation, we can equate coefficients of the derivatives of u in descending order to zero and hence determine the form of Q. The reason for selecting two values of N is to try and find a recursion operator by inspection of the two L-B symmetries.

For $$N = 3,\ Q = Q\,(x, t, u, u_1, u_2, u_3\,) \qquad (3.13)$$

In (3.12) D_t and D_x become

$$\left.\begin{aligned} D_t &= \partial_t + u_t\partial_u + u_{t1}\partial u_1 + u_{t2}\partial u_2 + u_{t3}\partial u_3 \\ D_x &= \partial_x + u_1\partial_u - u_2\,\partial u_1 + u_3\partial u_2 + u_4\partial u_3 \end{aligned}\right\} \qquad (3.14)$$

We then substitute (3.13) and (3.14) into the invariance condition (3.12) which is then expanded and simplified. At the same time the t-derivatives of u are replaced by

$$\begin{aligned} u_t &= -(auu_1 + bu_2 + cu_3) \\ u_{t1} &= -(au_1^2 + auu_2 + bu_3 + cu_4) \\ u_{t2} &= -(3au_1u_2 + auu_3 + bu_4 + cu_5) \\ u_{t3} &= -(3au_2^2 + 4au_1u_3 + auu_4 + bu_5 + cu_6) \end{aligned}$$

This leads to the lengthy expression given in Appendix F. The coefficients of descending order derivatives of u in this expression are equated to zero giving a set of determining equations involving the derivatives of Q which are then solved to give the L-B symmetry

$$Q = k_1u_3 + \left(\frac{k_1b}{c}\right)u_2 + \left(\frac{k_1a}{c}\right)uu_1 + (k_3 - ak_2t)u_1 + k_2 \qquad (3.15)$$

(k_1, k_2 and k_3 are arbitrary integration constants) of the KdVB equation

The details of the computation of (3.15) are found in Appendix F.

I then used the same technique with $Q = Q(x, t, u, u_1, u_2, u_3, u_4, u_5)$ to determine any 5th order L-B symmetries of (3.2). **The result was the same as (3.15).** Details of this computation are found in Appendix G. A determination of 2nd order symmetries of (3.2) with $Q = Q(x, t, u, u_1, u_2)$ gave

$$Q = (k_2 - ak_1 t)u_1 + k_1 \tag{3.16}$$

where k_1 and k_2 are arbitrary integration constants.

According to Olver [3] a generalized or L-B symmetry has an infinitesimal generator (X) such that

$$X = \sum_{i=1}^{p} \xi^i [u]\, \partial_{x_i} + \sum_{\alpha=1}^{q} \phi_\alpha [u]\, \partial u_\alpha \tag{3.17}$$

where $\xi^i [u] = \xi^i(x, t, u, u_1 \ldots)$ and $\phi_\alpha[u] = \phi_\alpha(x, t, u, u_1 \ldots)$ are both smooth differential functions.

The associated evolutionary form of (3.17) is

$$X(Q) = \sum_{\alpha=1}^{q} Q_\alpha[u]\, \partial u_\alpha \tag{3.18}$$

where the characteristic (Q_α) is

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \tag{3.19}$$

$$\alpha = 1 \ldots q \text{ and } u_i^\alpha = \frac{\partial u_\alpha}{\partial x^i}$$

In (x, t, u) space a Lie point symmetry transformation has an infinitesimal generator (X) of the form

$$X = \xi \partial_x + \tau \partial_t + \phi\, \partial_u \tag{3.20}$$

where $\xi = \xi(x, t)$ $\tau = \tau(x, t)$ and $\phi = \phi(x, t)$. Its characteristic (Q) is of the form

$$Q = \phi - \xi u_1 - \tau u_t \tag{3.21}$$

Lakshmanun and Kaliappan [15] determined Lie point symmetries for the KdVB equation $u_t + auu_1 + bu_2 + cu_3 = 0$ and found that (3.20) had the form

$$X = (\alpha\beta t+\delta)\partial_x + \alpha\partial_t + \beta\partial_u \tag{3.22}$$

where α, β and δ are arbitrary integration constants. The characteristic (Q) of (3.22) is

$$\begin{aligned} Q &= \beta - (\alpha\beta t + \delta)u_1 - \alpha u_t \\ &= \beta - (\alpha\beta t + \delta)u_1 + \alpha\,(auu_1 + bu_2 + cu_3) \end{aligned} \tag{3.23}$$

Comparing (3.23) with (3.15) we find that they are identical with

$$\alpha = \frac{k_1}{c} \quad \beta = k_2$$

and
$$(\alpha\beta t + \delta) = -(k_3 - ak_2\, t)$$

which implies that $\delta = -k_3$.

This means that the so-called L-B symmetry (3.15) of the KdVB equation is completely equivalent to the characteristic of the Lie point symmetry determined by Lakshmanan and Kaliappan.

4 MAJOR CONCLUSIONS

In Chapter 2 I critiqued two papers that obtained Lie point symmetries for two different PDEs. The papers considered were:

(1) "Isogroup and general similarity solution of a nonlinear diffusion equation" by S M Waller [5]. Waller obtained point symmetries and a similarity solution for the equation

$$\phi_t - (\phi^n \phi_x)_x = 0 \tag{4.1}$$

using differential forms as per the method developed by Harrison and Estabrook [1]; and

(2) "A variable coefficient Korteweg-deVries equation: Similarity analysis and exact solution" by Nirmala and Vedan [24], who obtained various point symmetries and similarity solutions for the equation

$$u_t + \alpha t^n u u_x + \beta t^m u_{xxx} = 0 \tag{4.2}$$

for various values of m and n.

Nirmala and Vedan used the so-called classical method as developed by Bluman and Cole [2].

For Waller's paper, I first checked his analysis using the closed ideal of differential forms

$$\left.\begin{aligned} \alpha &= d\phi - y dx - u dt \\ d\alpha &= -dy \wedge dx - du \wedge dt \\ \beta &= (u - n\phi^{n-1}y^2) dx \wedge dt - \phi^n dy \wedge dt \end{aligned}\right\} \tag{4.3}$$

which he used and confirmed that the symmetry generators

$$\left.\begin{aligned}
V^x &= \delta_4 x + \delta_2 \\
V^t &= \delta_3 t + \delta_1 \\
V^y &= \frac{1}{n}[(2-n)\delta_4 - \delta_3]y \\
V^u &= \frac{1}{n}[2\delta_4 + (n+1)\delta_3]u \\
V^\phi &= \frac{1}{n}[2\delta_4 - \delta_3]\phi
\end{aligned}\right\} \tag{4.4}$$

he obtained were correct.

As a check, I confirmed the results of (4.4) by using as a starting point the closed ideal

$$\left.\begin{aligned}
\alpha_1 &= d\phi\wedge dt - p dx\wedge dt \\
\alpha_2 &= d\phi\wedge dx + q dx\wedge dt \\
\alpha_3 &= d\phi\wedge dx + \phi^n dp\wedge dt + n\phi^{n-1} p dx\wedge dt \\
\alpha_4 &= - dp\wedge dx - dq\wedge dt
\end{aligned}\right\} \tag{4.5}$$

To obtain a similarity solution, Waller solved the quasilinear PDE

$$\frac{1}{n}(2\delta_4 - \delta_3)\phi = (\delta_1+\delta_3 t)\phi_t + (\delta_2 + \delta_4 x)\phi_x \tag{4.6}$$

with the condition $\quad 2\delta_4 - \delta_3 \neq 0 \qquad (4.7)$

By Lagrange's method of characteristics, both Waller and I obtained

$$\zeta = \frac{\theta_2 + \theta_4 x}{(\theta_1 + \theta_3 t)^{\theta_4/\theta_3}} \tag{4.8}$$

as the similarity variable and

$$\phi(x, t) = G(\zeta)(\theta_1 + \theta_3 t)^{\frac{1}{n\theta_3}} \tag{4.9}$$

as the similarity solution for (4.1)

$$\theta_i = \frac{\delta_i}{2\delta_4 - \delta_3} \quad i = 1, 2, 3, 4.$$

When I substituted (4.9) into (4.1) I obtained the ODE in $G(\zeta)$

$$G''(\zeta) + nG(\zeta)^{-1}G'(\zeta)^2 + \frac{\zeta}{\theta_4} G(\zeta)^{-n}G'(\zeta)(\theta_1+\theta_3 t)^{\frac{2\theta_4-\theta_3-1}{\theta_3}}$$

$$-\frac{1}{n\theta_4} G(\zeta)^{n-1}(\theta_1+\theta_3 t)^{\frac{2\theta_4-\theta_3-1}{\theta_3}} = 0 \qquad (4.10)$$

which agrees with Waller's result only if $\theta_3 = \theta_4 = 1$. When Waller considers the special case of $n = -1$ he begins by setting $\theta_3 = \theta_4 = 1$ and obtained the ODE

$$G''(\zeta) - G(\zeta)^{-1}G'(\zeta)^2 + \zeta G(\zeta)G'(\zeta) + G(\zeta)^2 = 0 \qquad (4.11)$$

(4.10) also reduces to (4.11) under the same conditions.

(4.11) is then solved to give Waller's similarity solution to (4.1), that is,

$$\phi(x, t) = \left\{\frac{t}{2}\left[p + \left(\frac{x}{t}\right)^2\right]\right\}^{-1} \qquad (4.12)$$

where $\theta_1 = \theta_2 = 0$ and p is an arbitrary constant.

Waller did not consider the possibility of $2\delta_4 = \delta_3$. If this is done then (4.6) reduces to

$$(\theta_1 + 2t)\phi_t + (\theta_2 + x)\phi_x = 0 \qquad (4.13)$$

where $\theta_i = \frac{\delta_i}{\delta_4}$, $i = 1, 2$ and $\delta_4 \neq 0$

Solving (4.13) by Lagrange's method of characteristics leads to

$$\zeta = \frac{\theta_2 + x}{(\theta_1+2t)^{1/2}} \qquad (4.14)$$

as the similarity variable and

$$\phi(x, t) = G(\zeta) \tag{4.15}$$

as the similarity solution.

For $n = -1$ the substitution of (4.15) into (4.1) gives the ODE for $G(\zeta)$

$$G''(\zeta) - G(\zeta)^{-1}G'(\zeta)^2 + \zeta G(\zeta)G'(\zeta) = 0 \tag{4.16}$$

(4.16) has the solution $G(\zeta) = \dfrac{1}{\zeta^2}$ which means that

$$\phi(x, t) = \frac{\theta_1 + 2t}{\theta_2 + x} \tag{4.17}$$

and $\theta_1 = \theta_2 = 0$ means (4.17) becomes

$$\phi(x, t) = \frac{2t}{x} \tag{4.18}$$

which is the same as (4.12) with the arbitrary constant p set to zero.

For Nirmala and Vedan's paper I first checked their derivation of point symmetries for (4.2) using the Harrison Estabrook method. Starting with the closed ideal

$$\left.\begin{aligned} \alpha &= dz - wdx - ydt \\ d\alpha &= -dw\wedge dx - dy\wedge dt \\ \beta &= (du - zdx)\wedge dt \\ \gamma &= du\wedge dx + \alpha uzt^n dt\wedge dx - \beta t^m dw\wedge dt \end{aligned}\right\} \tag{4.20}$$

and assuming m and n in (4.2) are arbitrary, I obtained the following infinitesimal generators in (x, t, u) space

$$\left.\begin{aligned} X_1 &= \frac{\alpha t^{n+1}}{n+1}\partial_x + \partial_u \\ X_2 &= \partial_x \end{aligned}\right\} \tag{4.21}$$

These turn out to be the same as the generators found by Nirmala and Vedan using the Bluman and Cole method.

Also, my similarity solution determined from (4.21) turned out to be

$$u = \frac{(n+1)x + \alpha k}{a\alpha t^{n+1}} \tag{4.22}$$

where k is an arbitrary constant. (4.22) corresponds to Nirmala and Vedan's

$$u = \frac{a(n+1)x + c}{a\alpha t^{n+1} + b(n+1)} \tag{4.23}$$

a, b and c being arbitrary constants if $b = 0$.

To determine the symmetries for (4.2) if m and n are linearly related I started with

$$m = k_1 n + k_2 \tag{4.24}$$

where k_1 and k_2 are constants to be determined. The Harrison Estabrook method gave as components to the isovector

$$\left.\begin{aligned} V^t &= \left(\frac{a_1 - 2a_2}{k_2}\right)t \\ V^x &= -a_2 x + \frac{a_3 \alpha t^{n+1}}{n+1} + a_4 \\ V^u &= a_1 u + a_3 \end{aligned}\right\} \tag{4.25}$$

where a_1, a_2, a_3 and a_4 are arbitrary constants. (4.24) turned out to have the form

$$m = n + k_2 \ (k_2 \neq 0) \tag{4.26}$$

Nirmala and Vedan obtained the following expressions equivalent to the isovector components

$$\left.\begin{aligned} T(\equiv V^t) &= t \\ X(\equiv V^x) &= (2+n)x + \frac{a\alpha t^{n+1}}{n+1} + b \\ U(\equiv V^u) &= u + a \end{aligned}\right\} \tag{4.27}$$

where a and b are arbitrary constants. Their result for the linear relationship between m and n is

$$m = 3n + 5 \tag{4.28}$$

While (4.27) shows some correspondence with (4.25) the linear relationship (4.28) does not agree with mine (4.26).

I next retraced Nirmala and Vedan's analysis starting with their determining equations

$$\left.\begin{array}{l} -X_t + \alpha t^n[U + u(U_u - X_x)] + n\alpha t^{n-1}uT = 0 \\ U_t + \alpha t^n u\, U_x + \beta t^m U_{xxx} = 0 \\ tU_u - 3tX_x + mT = 0 \\ U_u - T_t = 0 \quad U_{xu} - X_{xx} = 0 \quad U_{uu} - 3X_{ux} = 0 \\ T_x = T_u = X_u = 0 \end{array}\right\} \tag{4.29}$$

The results

$$\left.\begin{array}{l} U = a_1 u + b \\ X = a_2 x + \dfrac{b\alpha t^{n+1}}{n+1} + c \\ T = \left(\dfrac{a_2 - 2a_1}{n}\right)t = \left(\dfrac{3a_2 - a_1}{m}\right)t \end{array}\right\} \tag{4.30}$$

where a_1, a_2, b and c are arbitrary constants, were arrived at without any trouble.

(4.30) shows a close correspondence to my results (4.25).

Continuing on from this point (4.30) implies that $\dfrac{a_2 - 2a_1}{n} = \dfrac{3a_2 - a_1}{m}$ which when combined with $m = k_1 n + k_2$ gives

$$a_2 n(3 - k_1) + n(2k_1 - 1) = (a_2 - 2)k_2 \tag{4.31}$$

if a_1 is *assumed* equal to one.

If it is *assumed* that $k_1 = 3$, then

$$5n = (a_2 - 2)k_2 \tag{4.32}$$

and if it is further *assumed* that $k_2 = 5$ then

$$a_2 = n + 2 \tag{4.33}$$

so that $m = k_1 n + k_2$ becomes $m = 3n + 5$. Also (4.27) follows from (4.30) and (4.33). The components of the isovectors obtained using the Harrison Estabrook method (4.25) show a reasonable correspondence to those obtained using the Bluman and Cole method (4.30). The Harrison Estabrook method gave the relationship between the indices m and n in (4.2) as a condition for one of the determining equations to hold. That is the equation (4.26) $m = n + k_2$ $(k_2 \neq 0)$ arises naturally from the analysis. To obtain Nirmala and Vedan's final result, I had to assign an arbitrary value to the constant a_1 in (4.30). This leads to assigning particular values to k_1 and k_2 in (4.24) to obtain (4.27) and (4.28).

The purpose of Chapter 3 is to find any L-B symmetries of the KdVB equation

$$u_t + auu_1 + bu_2 + cu_3 = 0 \tag{4.34}$$

Attempts to find a recursion operator (R) for the L-B symmetries such that

$$R[u_1] = u_t = -(auu_1 + bu_2 + cu_3) \tag{4.35}$$

using $R_1 = -(cD_x^2 + bD_x + mu + nu_1D_x^{-1})$ with $m + n = a$ and $R_2 = -b(D_x^2 + \frac{b}{c} D_x + \frac{au_1}{c} D_x^{-1})$ showed that R_1 and R_2 are not recursion operators for L-B symmetries of (4.34). As an extension of this approach I tried to find a generalized recursion operator of the form $\alpha D_x^2 + \beta D_x + \gamma u + \delta D_x^{-1}$ using (4.35) where α, β, γ and δ are yet to be determined functions of (x, t, u) and the x derivatives of u. I was unable to solve the set of determining equations for the functions α, β, γ and δ and I suspect that in light of what happened in the subsequent determination of L-B symmetries of (4.34) that these determining equations might not be solvable at all.

A more direct approach to determining L-B symmetries uses the invariance condition

$$D_H[Q]_{H=0} = 0 \qquad 4.36)$$

where $Q = Q(x, t, u, u_1 \ldots u_N)$ is the Nth degree (N arbitrary) evolutionary characteristic.

Using (4.36) with $N = 3$ and $N = 5$ gave, in *both* cases the following form of Q:

$$Q = k_1 u_3 + \left(\frac{k_1 b}{c}\right) u_2 + \left(\frac{k_1 a}{c}\right) u u_1$$

$$+ (k_3 - a k_2 t) u + k_2 \tag{4.37}$$

where k_1, k_2 and k_3 are arbitrary constants. Closer inspection of (4.37) shows it to be the characteristic of the Lie point symmetries

$$X = (a k_2 t - k_3)\partial_x + \left(\frac{k_1}{c}\right)\partial_t + k_2 \partial_u \tag{4.38}$$

This result shows that the 3rd and 5th degree L-B symmetries of (4.34) turn out to be point symmetries. The main conclusion drawn from chapter (3) is that there are probably no generalized or L-B symmetries for the KdVB equation. This would explain why I failed to find a recursion operator using (4.35) for the L-B symmetries. I think it is quite likely that no such recursion operator exists. Fokas [9] defined the exact solvability of a PDE in terms of it admitting a Lax formulation. That is, the PDE $H = 0$ can be expressed in the form

$$H = [D, R]$$

where D is the Fréchet derivative of the time independent part of the PDE and R is of course a recursion operator. The conjectured lack of any L-B symmetries and the consequent non-existent recursion operators mean that the KdVB equation (4.34) is not exactly solvable using symmetry techniques.

REFERENCES

[1] B K Harrison and F B Estabrook
Geometric Approach to Invariance Groups and Solution of Partial Differential Systems J of Math Phys, Vol 12, No 4, 653-666, 1971.

[2] Bluman and Cole
Similarity Methods for Differential Equations [Springer-Verlag 1974].

[3] P J Olver
Applications of Lie Groups to Differential Equations - 2nd Edition, [Springer-Verlag 1993].

[4] H Stephani
Differential Equations. Their Solution using Symmetries [Cambridge University Press 1989].

[5] S M Waller
Isogroup and general similarity solution of a nonlinear diffusion equation, J Phys A: Math Gen 23, 1035-1040, (1990).

[6] E Cartan
Les systémes différentials extérieurs et leurs applications geometriques [Herman-Paris, 1945].

[7] W Slebodzinski
Exterior Forms and their Applications [Polish Scientific, Warsaw, 1970].

[8] P Roseneau and J L Schwarzmeier
Similarity solutions of PDEs using MACSYMA. [Courant Inst of Math Sci, Report No C00-3077 - 160/MF94 1979].

[9] A S Fokas
A symmetry approach to exactly solvable evolution equations, J Math Phys 21 (6) 1318-1325, 1980.

[10] G Bluman and S Kumei
On the remarkable nonlinear diffusion equation J Math Phys 21 (5) 1019-1023, 1980.

[11] N H Ibragimov
Transformation Groups Applied to Mathematical Physics [Reidel, Boston, 1985]

[12] G W Bluman and S Kumei
Symmetries and Differential Equations [Springer, New York, 1989].

[13] L van Wijngaaden
One-dimensional flow of liquids containing small gas bubbles. Ann Rev Fluid Mech 4 369-395, 1972.

[14] R S Johnson
A nonlinear equation incorporating damping and dispersion, J Fluid Mech 42 49-60 (1970).

[15] M Lakshmanan and P Kaliappan
On the invariant solutions of the Korteweg-deVries-Burgers equation, Phys Lett Vol 71A, 166-168. 1979.

[16] M Vlieg-Hulstman and W D Halford
The Korteweg-deVries-Burgers' equation: a reconstruction of exact solutions. Wave Motion (Elsevier) 14, 267-271, 1991.

[17] S L Xiong
An analytic solution of Burgers-KdV equation. Chinese Sci Bull. 34, 1158-1162 (1989).

[18] A Jeffrey and S Xu
Exact solutions of the Korteweg-deVries-Burgers equation, Wave Motion II 559-564 (1989).

[19] I McIntosh
Single phase averaging and travelling wave solutions of the modified Burgers-Korteweg-deVries equation, Phys Lett A 143, 57-61 (1990).

[20] H Grad and P N Hu
Phys Fluids, 10, 2596 (1967).

[21] T B Benjamin and M J Lighthill
Proc Roy Soc A 224, 448 (1954).

[22] E L Ince
Ordinary Differential Equations [Dover, New York, 1956].

[23] M J Ablowitz, A Ramani and H Segur
A connection between nonlinear evolution equations and ODE of the Painlevé type, J Math Phys 21, 715 (1980).

[24] N Nirmala and M J Vedan
A variable coefficient Korteweg-deVries equation: Similarity analysis and exact solution, J Math Phys 27, 2644-2646, (1986).

APPENDIX A

(1) Nonlinear Diffusion Equation

Use of the classical method of Bluman & Cole for finding point symmetries of the non-linear diffusion equation

$$\phi_{x_2} = \left[K(\phi)\phi_{x_1}\right]_{x_1} \tag{A1}$$

Let (A1) be written as

$$\phi_2 = K(\phi)\phi_{11} + K'(\phi)(\phi_1)^2 \tag{A2}$$

As (A2) is invariant under the transformation group $X = \xi_i \partial_{x_i} + \eta\partial_\phi \quad i = 1, 2.$

$$X^{(2)}[\phi_2 - K(\phi)\phi_{11} - K'(\phi)(\phi_1)^2] = 0$$

i.e.

$$\begin{aligned}&(\xi_1\partial_{x_1} + \xi_2\partial_{x_2} + \eta\partial_\phi + \eta_1^{(1)}\partial_{\phi_1} + \eta_2^{(1)}\partial_{\phi_2}\\ &+ \eta_{11}^{(2)}\partial_{\phi_{11}})\,[\phi_2 - K(\phi)\phi_{11} - K'(\phi)(\phi_1)^2] = 0\end{aligned}$$

or

$$\begin{aligned}&\eta_2^{(1)} - \eta K'(\phi)\phi_{11} - \eta_{11}^{(2)}\, K(\phi) - \eta K''(\phi)(\phi_1)^2\\ &- 2\eta_1^{(1)}\, K'(\phi)\phi_1 = 0\end{aligned} \tag{A3}$$

where

$$\begin{aligned}\eta_1^{(1)} = &\frac{\partial\eta}{\partial x_1} + \left(\frac{\partial\eta}{\partial\phi} - \frac{\partial\xi_1}{\partial x_1}\right)\phi_1 - \frac{\partial\xi_2}{\partial x_1}\,\phi_2 - \frac{\partial\xi_1}{\partial\phi}\,(\phi_1)^2\\ &- \frac{\partial\xi_2}{\partial\phi}\,\phi_1\phi_2\end{aligned} \tag{A4}$$

$$\begin{aligned}\eta_2^{(1)} = &\frac{\partial\eta}{\partial x_2} + \left(\frac{\partial\eta}{\partial\phi} - \frac{\partial\xi_2}{\partial x_2}\right)\phi_2 - \frac{\partial\xi_1}{\partial x_2}\,\phi_1 - \frac{\partial\xi_2}{\partial\phi}\,(\phi_2)^2\\ &- \frac{\partial\xi_1}{\partial\phi}\,\phi_1\phi_2\end{aligned} \tag{A5}$$

and $$\eta_{11}^{(2)} = \frac{\partial^2\eta}{\partial x_1^2} + \left(2\frac{\partial^2\eta}{\partial x_1\partial\phi} - \frac{\partial^2\xi_1}{\partial x_1^2}\right)\phi_1 - \frac{\partial^2\xi_2}{\partial x_1^2}\ \phi_2$$

$$+ \left(\frac{\partial^2\eta}{\partial\phi^2} - 2\frac{\partial^2\xi_1}{\partial\phi\partial x_1}\right)(\phi_1)^2 - \frac{\partial^2\xi_1}{\partial\phi^2}\ (\phi_1)^3$$

$$-2\ \frac{\partial^2\xi_2}{\partial\phi\partial x_1}\ \phi_1\phi_2 - \frac{\partial^2\xi_2}{\partial\phi_2}\ (\phi_1)^2\,\phi_2 + \left(\frac{\partial\eta}{\partial\phi} - 2\frac{\partial\xi_1}{\partial x_1}\right)\phi_{11}$$

$$-2\frac{\partial\xi_2}{\partial x_1}\ \phi_{12} - 3\ \frac{\partial\xi_1}{\partial\phi}\ \phi_{11}\phi_1 - \frac{\partial\xi_2}{\partial\phi}\ \phi_{11}\,\phi_2$$

$$-2\ \frac{\partial\xi_2}{\partial\phi}\ \phi_{12}\phi_1 \tag{A6}$$

Substitute (A4) to (A6) in (A3) and equate coefficients of ϕ and its derivatives to zero, we obtain the following determining equations

$$\frac{\partial\xi_1}{\partial\phi} = \frac{\partial\xi_2}{\partial\phi} = \frac{\partial\xi_2}{\partial x_1} = 0 \tag{A7}$$

$$\frac{\partial\zeta_1}{\partial x_2} + K(\phi)\left(2\frac{\partial^2\eta}{\partial x_1\partial\phi} - \frac{\partial^2\xi_1}{\partial x_1^2}\right) + 2K'(\phi)\frac{\partial\eta}{\partial x_1} = 0 \tag{A8}$$

$$K(\phi)\left(\frac{\partial\xi_2}{\partial x_2} - 2\frac{\partial\xi_1}{\partial x_1}\right) + K'(\phi)\eta = 0 \tag{A9}$$

$$K(\phi)\ \frac{\partial^2\eta}{\partial\phi^2} + K'(\phi)\left(\frac{\partial\xi_2}{\partial x_2} - 2\frac{\partial\xi_1}{\partial x_1} + \frac{\partial\eta}{\partial\phi}\right) + K''(\phi)\eta = 0 \tag{A10}$$

$$K(\phi)\frac{\partial^2\eta}{\partial x_1^2} - \frac{\partial\eta}{\partial x_2} = 0 \tag{A11}$$

Solving (A7) to (A11) we obtain the components of X.

$$\xi_1 = \frac{1}{2}\ \beta x_1 + a x_1^2 + b x_1 + \gamma$$
$$\xi_2 = \alpha + \beta x_2$$
$$\eta = (4a x_1 + 2b)\ \frac{K(\phi)}{K'(\phi)}$$

where α_1, β, γ, a and b are arbitrary constants. Three distinct cases arise depending on the form of $K(\phi)$.

Case I If $K(\phi)$ is an arbitrary function of ϕ then (A10) implies that $a = b = 0$. This gives a 3 parameter group of transformations

$$\begin{aligned} \xi_1 &= \frac{1}{2}\beta x_1 + \gamma \\ \zeta_2 &= \alpha + \beta x_2 \\ \eta &= 0 \end{aligned}$$

with the corresponding infinitesimal generators

$$\left.\begin{aligned} X_1 &= \partial_{x_1} = \partial_x \\ X_2 &= \partial_{x_2} = \partial_t \\ &\text{and} \\ X_3 &= x_1\partial_{x_1} + 2x_2\partial_{x_2} = x\partial_x + 2t\partial_t \end{aligned}\right\} \tag{A12}$$

Case II If $K(\phi)$ is not arbitrary, then (A10) implies that $K(\phi) = \lambda\phi^n$ where λ and n are arbitrary constants and $a = 0$. This gives a 4 parameter transformation group

$$\begin{aligned} \xi_1 &= \frac{1}{2}\beta x_1 + bx_1 + \gamma \\ \xi_2 &= \alpha + \beta x_2 \\ \eta &= 2b\left(\frac{\phi}{n}\right) \end{aligned}$$

with the corresponding infinitesimal generators (A12) and

$$X_4 = x\partial_x + \frac{2\phi}{n}\partial_\phi \tag{A13}$$

Case III If $a \neq 0$ then (A10) implies that $\left(3 + \frac{4}{n}\right) = 0$ or $n = -\frac{4}{3}$ so that $K(\phi) = \lambda\phi^{-4/3}$. This gives a 5 parameter group of transformations with corresponding infinitesimal generators (A12), (A13) and

$$X_5 = x^2\partial_x - 3x\phi\partial_\phi \tag{A14}$$

Similarity Solutions

Case I where $K(\phi)$ is an arbitrary function of ϕ

(ii) $X = \partial_x$ has subsidiary equations

$$\frac{dx}{1} = \frac{dt}{0} = \frac{d\phi}{0}$$

which give $t = \zeta$ the similarity variable and $\phi = \phi(\zeta) = \phi(t)$ as the similarity solution. Substitution of $\phi = \phi(t)$ into the PDE (A1) gives the trivial solution $\phi = \text{constant}$.

(ii) $X = \partial_t$ has subsidiary equations

$$\frac{dx}{0} = \frac{dt}{1} = \frac{d\phi}{0}$$

which give $x = \zeta$ and $\phi = \phi(\zeta) = \phi(x)$. When $\phi = \phi(x)$ is substituted into the PDE (A1) we get the solution $\int K(\phi)d\phi = k_1 x + k_2$ where k_1 and k_2 are arbitrary constants.

(iii) $X = x\partial_x + 2t\partial_t$ has subsidiary equations $\frac{dx}{x} = \frac{dt}{2t} = \frac{d\phi}{0}$ which leads to $\frac{x^2}{t} = \zeta$ as the similarity variable and $\phi = \phi(\zeta) = \phi\left(\frac{x^2}{t}\right)$ as the corresponding similarity solution. On substitution of $\phi = \phi\left(\frac{x^2}{t}\right)$ into (A1) we obtain the following ODE in ζ.

$$\frac{d}{d\zeta}\left[4\zeta K(\phi)\frac{d\phi}{d\zeta} - 2\int K(\phi)d\phi\right] = -\zeta\frac{d\phi}{d\zeta}$$

Case II where $K(\phi) = \phi^n$ gives the following similarity solutions

$$\phi = \text{constant}$$

$$\phi = [(n+1)(k_1 x+k_2)]^{\frac{1}{n+1}}$$

and ϕ such that

$$\frac{d}{d\zeta}\left[4\,\zeta\,\phi^n\frac{d\phi}{d\zeta}-\frac{2\phi^{n+1}}{n+1}\right] = -\zeta\frac{d\phi}{d\zeta}$$

The additional infinitesimal generator

$$X = x\partial_x + \frac{\partial\phi}{n}\partial_\phi$$

has subsidiary equations $\frac{dx}{x} = \frac{dt}{0} = \frac{n}{2}\ \frac{d\phi}{\phi}$ which give $t = \zeta$ as the similarity variable and $\phi = x^{\frac{2}{n}}\ \phi(\zeta) = x^{\frac{2}{n}}\ \phi(t)$ which, when substituted into the PDE (A1) gives

$$\phi = \left[\frac{nx^2}{k+2(2+n)t}\right]^{1/n}$$

where k is an arbitrary constant.

Case III where $K(\phi) = \phi^{-4/3}$ gives the following similarity solutions

$$\phi = \text{constant}$$

$$\phi = \left[-\frac{1}{3}\left(k_1\,x + k_2\right)\right]^{-3}$$

ϕ such that

$$\frac{d}{d\zeta}\left[4\zeta\phi^{-4/3}\ \frac{d\phi}{d\zeta} + 6\phi^{-1/3}\right] = \zeta\frac{d\phi}{d\zeta}$$

and $$\phi = x^{-3/2}\,(t+k)^{3/4}$$

The extra symmetry $X = x^2\partial_x - 3x\phi\partial_\phi$ has subsidiary equations $\frac{dx}{x^2} = \frac{dt}{0} = \frac{d\phi}{3x\phi}$ which gives $t = \zeta$ and $\phi = x^{-3}\phi(\zeta) = x^{-3}\phi(t)$ which on substitution in (A1) gives the similarity solution $\phi = kx^{-3}$ where k is an arbitrary constant.

(2) KdV Equation

The use of differential forms for finding point symmetries of the KdV equation $u_t + uu_x + \varepsilon u_{xxx} = 0$ where ε is a constant [1].

The closed ideal of required differential forms is:

$$\left.\begin{aligned} \alpha &= dz - wdx - ydt \\ d\alpha &= -dw \wedge dx - dy \wedge dt \\ \beta &= (du - zdx) \wedge dt \\ &\text{and} \\ \gamma &= du \wedge dx + uzdt \wedge dx - \varepsilon dw \wedge dt \end{aligned}\right\} \tag{A15}$$

$d\beta$ and $d\gamma$ are both equal to zero and are hence not required for the ideal.

Annulling the differential forms (A15) means setting them equal to zero on the surface

$$u = u(x, t)$$

so that $\quad \tilde{\alpha} = 0$ implies that $w = z_x$ and $y = z_t$

$\tilde{\beta} = 0$ implies that $z = u_x$

and hence that $\quad w = u_{xx}$ and $y = u_{xt}$.

w, y, z are the necessary prolongation variables. $d\tilde{\alpha} = 0$ implies that $z_{tx} = z_{xt}$ the integrability condition for z, while $\tilde{\gamma} = 0$ gives the KdV equation. The invariant condition for a group of transforms or isogroup as it is sometimes called, requires that the Lie derivative with respect to the vector operator V for each of the differential forms (A15) be a linear combination of the elements of the ideal (A15). The vector operator V, also called the isovector, is such that

$$V = V^x \partial_x + V^t \partial_t + V^u \partial_u + V^z \partial_z + V^y \partial_y + V^w \partial_w \tag{A16}$$

and is equivalent to a prolongated form of X, the infinitesimal generator of the isogroup. As the Lie derivative does not change the degree of a differential form and α is the only 1-form in (A15) we have

$$\pounds_v(\alpha) = V \lrcorner d\alpha + d(V \lrcorner \alpha) = \lambda \alpha \tag{A17}$$

where λ is an arbitrary 0-form.

If $F(x, t, u, z, y, w) = V \lrcorner \alpha$ then

$$d(V \lrcorner \alpha) = dF$$

$$= F_x dx + F_t dt + F_u du + F_z dz + F_y dy + F_w dw \tag{A18}$$

$$V \lrcorner \alpha = F = V^z - wV^x - yV^t \tag{A19}$$

and

$$V \lrcorner d\alpha = -V^w dx + V^x dw - V^y dt + V^t dy \tag{A20}$$

Substituting (A18) and (A20) into (A17) we obtain the following equations:

$$\left.\begin{array}{l} V^x = -F_w \quad V^t = -F_y \quad V^w = F_x + wF_z \\ V^y = F_t + yF_z \quad \lambda = F_z \text{ and } F_u = 0 \end{array}\right\} \tag{A21}$$

$$\text{(A19) implies that } V^z = F + wV^x + yV^t \tag{A22}$$

For $d\alpha$

$$\pounds_v(d\alpha) = d(\pounds_v(\alpha)) = d(\lambda\alpha) = d\lambda \wedge \alpha + \lambda d\alpha$$

which is already in the ideal.

For β

$$\left.\begin{array}{l} \pounds_v(\beta) = V \lrcorner d\beta + d(V \lrcorner \beta) \\ \qquad = \xi\gamma + \zeta\beta + \mu d\alpha + w \wedge \alpha \end{array}\right\} \tag{A23}$$

where ξ, ζ and μ are arbitrary 0-forms, $w = Adx + Bdt + Cdu + Ddz + Edy + Gdw$ with A, B, C, D, E and G being arbitrary 0-forms.

Equating coefficients on the basis 2-forms in (A23) and eliminating the arbitrary 0-forms, we obtain the equations

$$\left.\begin{array}{l} V^t_y = V^t_w = 0 \quad -V^u_y + zV^x_y = 0 \\ \varepsilon(V^t_x + zV^t_u + wV^t_z) = zV^x_w - V^u_w \\ -V^z + V^u_x - zV^x_x = -uz(V^t_x + zV^t_u + wV^t_z) \\ -z(V^u_u - zV^x_u) + zwV^z_z - wV^u_z \end{array}\right\} \tag{A24}$$

In a similar way

$$\left.\begin{aligned}\pounds_v(\gamma) &= V\lrcorner d\gamma + d(V\lrcorner\gamma)\\ &= \xi\gamma + \zeta\beta + \mu d\alpha + w\wedge\alpha\end{aligned}\right\} \tag{A25}$$

$$= \text{Generalised 2-form on the ideal}$$

Again equating coefficients of the basis 2-forms and eliminating the arbitrary 0-forms we obtain the equations

$$\left.\begin{aligned}&V^x_y = 0 \quad V^u_y = 0 \quad V^x_w + \varepsilon V^t_u = 0\\ &\varepsilon(V^t_x + wV^t_z - V^w_y) = V^u_w\\ &V^u_u + V^x_x + 2uz\ V^t_u - V^w_w - V^t_t + wV^x_z - yV^t_z = 0\\ &-uV^z - zV^u - V^u_t + \varepsilon(zV^w_u - wV^w_z - V^w_x)\\ &+y(V^x_z - V^u_z) + zV^x_t + uz(V^u_u + uzV^t_u - V^t_t\\ &-zV^x_u + yV^t_z) = 0\end{aligned}\right\} \tag{A26}$$

(A21), (A22), (A24) and (A26) are the determining equations for the components V^x V^t V^u V^z V^w and V^y of the isovector V. Solving the determining equations for these components we get

$$\left.\begin{aligned}&V^x = k_1x + k_2t + k_3 \quad &&V^z = -3k_1z\\ &V^t = 3k_1t + k_4 &&V^y = -6k_1y - k_2w\\ &V^u = -2k_1u + k_2 &&V^w = -4k_1w\end{aligned}\right\} \tag{A27}$$

where k_1, k_2 and k_3 are arbitrary constants.

Invariance Transformations of the KdV Equation

$$z = u_x \quad w = u_{xx} \text{ and } y = u_{xt}$$

	V^x	V^t	V^u	V^z	V^y	V^w	Type
$k_1 = 1$	x	3t	-2u	-3z	-6y	-4w	x, t scale change
$k_2 = 1$	t	0	1	0	-w	0	Galilean transformation
$k_3 = 1$	1	0	0	0	0	0	space translation
$k_4 = 1$	0	1	0	0	0	0	time translation

The infinitesimal generators of the Lie Algebra are:

$$\left.\begin{aligned} X_1 &= x\partial_x + 3t\partial_t - 2u\partial_u \\ X_2 &= t\,\partial_x + \partial_u \\ X_3 &= \partial_x \\ X_4 &= \partial_t \end{aligned}\right\} \qquad \text{(A28)}$$

Similarity solutions to the KdV equation are found by augmenting the ideal (A15) with differential forms obtained by contracting α, $d\alpha$, β and γ with the isovector V. This gives the following new forms $F = V \lrcorner \alpha$, $\theta = V \lrcorner\, d\alpha$, $\sigma = V \lrcorner \beta$ and $\tau = V \lrcorner \gamma$ each of which is in the ideal. These new forms are next annulled, i.e. set to zero on the solution surface $u = u(c, t)$.

(1) $F = V \lrcorner \alpha = V^z - wV^x - yV^t$. When F is annulled on $u = u(x\ t)$ we get

$$V^z = u_{xx}\ V^x + u_{xt}\ V^t \qquad \text{(A29)}$$

(ii) $\theta = V \lrcorner d\alpha = -V^w dx + V^x dw - V^y dt + V^t dy.$

On the surface $u = u(x\ t)$

$$dy = y_x dx + y_t\, dt = u_{xtx} dx + u_{xtt} dt$$

and $$dw = w_x dx + w_t dt = u_{xxx} dx + u_{xxt} dt$$

Annulling θ therefore gives

$$V^y = u_{xxt} V^x + u_{xtt} V^t \qquad \text{(A30)}$$

(iii) $\sigma = V \lrcorner \beta = V^u dt - V^t\, du - zV^x dt + zV^t dx$

Annulling σ on $u = u\ (x, t)$ where $z = u_x$ and $du = u_x\, dx + u_t dt$ gives

$$V^u = u_x V^x + u_t V^t \qquad \text{(A31)}$$

Finally

(iv) $\tau = V \lrcorner \gamma$

$$= V^u dx - V^x du + uzV^t dz - uz\, V^x dx - \varepsilon V^w dt + \varepsilon V^t dw$$

which, when annulled on the surface u = u(x, t) gives

$$V^u = u_x V^x + u_t V^t$$

and

$$V^w = u_{xxx} V^x + u_{xxt} V^t \tag{A32}$$

Of the annulled forms (A29) to (A32), only (A31) $V^u = u_x V^x + u_t V^t$ is used to give similarity solutions of the KdV equation. We will begin with $V^x = 1$ $V^u = V^t = 0$ so that $V = \partial_x$ which has subsidiary equations $\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}$. These give $\zeta = t$ as the similarity variable and $u = u(\zeta) = u(t)$ as the similarity solution which, on substitution in the KdV equation, gives the trivial solution $u = k$ an arbitrary constant.

Letting $V^t = 1$ and $V^x = V^u = 0$ gives $V = \partial_t$ with subsidiary equations

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}$$

which give $\zeta = x$ as the similarity variable and $u = u(\zeta) = u(x)$ as the similarity solution. Substitution of $u = u(x)$ in the KdV equation gives the ODE

$$u \frac{du}{dx} + \varepsilon \frac{d^3u}{dx^3} = 0 \tag{A33}$$

Integrating (A33) gives

$$\frac{1}{2} u^2 + \varepsilon \frac{d^2u}{dx^2} = k_1 \tag{A34}$$

where k_1 is an arbitrary constant.

Integrating (A34) gives

$$\varepsilon \left(\frac{du}{dx}\right)^2 + \frac{1}{3} u^3 - 2k_1 u = k_2 \tag{A35}$$

where k_2 is another arbitrary constant.

Assuming that $u \to -c$ (constant) and both $\frac{du}{dx}$ and $\frac{d^2u}{dx^2} \to 0$ as $|x| \to \infty$, we find that (A34) implies that $k_1 = \frac{1}{2}c^2$ and that (A35) implies that $k_2 = \frac{2}{3}c^3$.

(A35) now becomes

$$\varepsilon\left(\frac{du}{dx}\right)^2 + \frac{1}{3}u^3 - c^2 u = \frac{2}{3}c^3$$

or

$$\frac{du}{\left[c^2u + \frac{2}{3}c^3 - \frac{1}{3}u^3\right]^{1/2}} = \frac{dx}{(\varepsilon)^{1/2}} \qquad \text{(A36)}$$

Using the substitution $w^2 = \frac{c+u}{3c}$ and $2wdw = \frac{du}{3c}$ in (A36), we obtain

$$\frac{dw}{w(1-w^2)^{1/2}} = \left(\frac{c}{\varepsilon}\right)^{1/2}\frac{dx}{2} \qquad \text{(A37)}$$

Integration of (A37) gives

$$\operatorname{sech}^{-1} w = \left(\frac{c}{\varepsilon}\right)^{1/2}\frac{x}{2} + k_3 \qquad \text{(A38)}$$

k_3 being an arbitrary constant.

Eliminating w gives the similarity solution

$$u = c\left\{3\operatorname{sech}^2\left[\left(\frac{c}{\varepsilon}\right)^{1/2}\frac{x}{2} + k_3\right] - 1\right\} \qquad \text{(A39)}$$

It is worthwhile to consider the linear combination $V = \partial_t + c\partial_x$ where c is a constant which gives a similarity solution similar to A39, that is:

$$u = 3c\,(\operatorname{sech}^2)\left[\frac{1}{2}\left(\frac{c}{\varepsilon}\right)^{1/2}(x - ct) + k_3\right] \qquad \text{(A40)}$$

(A40) is the soliton or solitary wave solution of the KdV equation with c as the phase velocity and k_3 determined by the initial conditions.

APPENDIX B

On obtaining point symmetries of the nonlinear diffusion equation

$$\phi_t - (\phi^n \phi_x)_x = 0 \tag{B1}$$

by the Harrison-Estabrook method starting with the following closed ideal of differential forms

$$\alpha_1 = d\phi \wedge dt - p dx \wedge dt \tag{B2}$$

$$\alpha_2 = d\phi \wedge dx + q dx \wedge dt \tag{B3}$$

$$\alpha_3 = d\phi \wedge dx + \phi^n dp \wedge dt + n\phi^{n-1} p^2 dx \wedge dt \tag{B4}$$

$$\alpha_3 = - dp \wedge dx - dq \wedge dt \tag{B5}$$

Annulling (B2) to (B5) on the solution space $\phi = \phi(x, t)$ where

$$d\phi = \phi_x dx + \phi_t dt$$

$$dp = p_x dx + p_t dt$$

and

$$dq = q_x dx + q_t dt$$

gives the following results:

(i) $\tilde{\alpha}_1 = 0$ implies that $(\phi_x - p)\, dx \wedge dt = 0$ and $p = \phi_x$ the definition of the first prolongation variable p.

(ii) Similarly $\tilde{\alpha}_2 = 0$ gives $q = \phi_t$ the definition of the second prolongation variable q.

(iii) $\tilde{\alpha}_3 = 0$ implies that

$$(\phi_t - \phi^n p_x - n\phi^{n-1} p^2) dx \wedge dt = 0$$

or

$$\phi_t - \phi^n \phi_{xx} - n\phi^{n-1} (\phi_x)^2 = 0$$

which is the original PDE (B1)

(iv) $\tilde{\alpha}_4 = 0$ implies that $(p_t - q_x)dx\wedge dt = 0$ or $\phi_{xt} = \phi_{tx}$ the integrability condition for ϕ.

The generators of the invariance groups are the coefficients of the isovector

$$V = V^x\partial_x + V^t\partial_t + V^\phi\partial_\phi + V^p\partial_p + V^q\partial_q \tag{B6}$$

in the 5 dimensional space (x, t, ϕ, p, q). The action of V on the closed ideal is such that the Lie derivatives with respect to V of the forms (B2) to (B5) are still on the ideal. As the ideal consists only of 2-forms, this means that

$$\pounds_v(\alpha_i) = A_i\,\alpha_1 + \beta_i\,\alpha_2 + C_i\,\alpha_3 + D_i\alpha_4 \tag{B7}$$

for $i = 1, 2, 3, 4$ and A_i, B_i, C_i and D_i are arbitrary 0-forms. Expanding the invariant condition (B7) according to

$$\pounds_v(\alpha_i) = V\lrcorner\, d\alpha_i + d(V\lrcorner\alpha_i) \tag{B8}$$

for each of the four forms of the ideal we obtain for:

(i) $i = 1$

$$\begin{aligned}
&- V^p dx\wedge dt + dV^\phi\wedge dt - dV^t\wedge d\phi - p dV^x\wedge dt + p dV^t\wedge dx \\
&= A_1\alpha_1 + B_1\alpha_2 + C_1\,\alpha_3 + D_1\,\alpha_4 \\
&= A_1\,(d\phi\wedge dt - p dx\wedge dt) + B_1\,(d\phi\wedge dx + q dx\wedge dt) \\
&+ C_1\,(d\phi\wedge dx + \phi^n dp\wedge dt + n\phi^{n-1}p^2\,dx\wedge dt) \\
&+ D_1\,(- dp\wedge dx - dq\wedge dt)
\end{aligned} \tag{B9}$$

(ii) $i = 2$

$$\begin{aligned}
&V^q dx\wedge dt + dV^\phi\wedge dx - dV^x\wedge d\phi + q dV^x\wedge dt - q dV^t\wedge dx \\
&= A_2\alpha_1 + B_2\,\alpha_2 + C_2\alpha_3 + D_2\alpha_4
\end{aligned} \tag{B10}$$

(iii) $i = 3$

$$\begin{aligned}
&n\phi^{n-1}V^\phi dp\wedge dt + n(n-1)\phi^{n-2}p^2\,V^\phi dx\wedge dt \\
&+ 2np\phi^{n-1}V^p dx\wedge dt + dV^\phi\wedge dx - dV^x\wedge d\phi \\
&+ \phi^n(dV^p\wedge dt - dV^t\wedge dp) + n\phi^{n-1}p^2(dV^x\wedge dt - dV^t\wedge dx) \\
&= A_3\,\alpha_1 + B_3\alpha_2 + C_3\,\alpha_3 + D_3\,\alpha_4
\end{aligned} \tag{B11}$$

(iv) $i = 4$

$$dV^x \wedge dp - dV^p \wedge dx + dV^t \wedge dq - V^q \wedge dt$$
$$= A_4\alpha_1 + B_4\alpha_2 + C_4\alpha_3 + D_4\alpha_4 \qquad \text{(B12)}$$

where $$dV^j = V^j_x\, dx + V^j_t\, dt + V^j_\phi\, d\phi + V^j_p\, dp + V^j_q\, dq \qquad \text{(B13)}$$

$j = x, t, \phi, p$ and q in turn.

Using (B13) and equating coefficients of the basis differential 2-forms in (B9) to (B12), and at the same time eliminating the arbitrary 0-forms A_i B_i C_i D_i for $i = 1, 2, 3, 4$ we obtain the following determining equations for the generators of the isogroups:

$$V^t_p = V^t_q = V^t_\phi = V^t_x = 0 \qquad \text{(B14)}$$

$$V^x_p = V^x_q = V^x_\phi = 0 \qquad \text{(B15)}$$

$$V^\phi_p = V^\phi_q = 0 \qquad \text{(B16)}$$

$$V^p_q = 0 \qquad \text{(B17)}$$

$$V^x_x + V^p_p - V^t_t - V^q_q = 0 \qquad \text{(B18)}$$

$$V^q - V^\phi_t + qV^t_t = -pV^x_t + qV^\phi_\phi \qquad \text{(B19)}$$

$$V^p - V^\phi_x + pV^x_x = pV^\phi_\phi - qV^x_t \qquad \text{(B20)}$$

$$V^p_t - V^q_x = pV^q_\phi + q\{\phi^n(V^x_t - V^q_p) - V^p_\phi\}$$
$$+ n\phi^{-1}p^2(V^x_t + V^q_p) \qquad \text{(B21)}$$

$$n(n-1)\phi^{n-2}p^2 V^\phi + 2np\phi^{n-1}V^p - V^\phi_t - \phi^n V^p_x + n\phi^{n-1}p^2$$
$$(V^x_x + V^t_t) = -p(V^x_t + \phi^n V^p_\phi)$$
$$+ q(V^\phi_\phi + V^x_x - V^p_p - V^t_t - n\phi^{-1}V^\phi) + n\phi^{n-1}p^2(n\phi^{-1}V^\phi$$
$$+ V^p_p - V^t_t) \qquad \text{(B22)}$$

We now solve the determining equations (B14) to (B22) for the components V^j where $j = x, t, \phi, p$ and q in turn of the isovector

$$\text{(B14) implies that } V^t = a(t) \tag{B23}$$

$$\text{(B15) implies that } V^x = g(x, t) \tag{B24}$$

and

$$\text{(B16) implies that } V^\phi = h(x, t, \phi) \tag{B25}$$

where a, g and h are arbitrary functions of the indicated arguments.

From (B20) we get

$$V^p = pV^\phi_\phi - qV^x_t + V^\phi_x - pV^x_x \tag{B26}$$

Then partial differentiating (B26) with respect to q we get

$$V^p_q = pV^\phi_{q\phi} + V^\phi_{qx} - pV^x_{qx} - V^x_t = 0 \tag{B27}$$

As $V^\phi_q = V^x_q = 0$ then (B27) implies that

$$V^x_t = 0 \tag{B28}$$

and with (B24) that

$$V^x = g(x) \tag{B29}$$

Then using (B19) and (B20) we get

$$V^q = V^\phi_t - qV^t_t + qV^\phi_\phi = h_t + q\{h_\phi - a'(t)\} \tag{B30}$$

$$V^p = V^\phi_x - pV^x_x + pV^\phi_\phi = h_x + p\{h_\phi - g'(x)\} \tag{B31}$$

From (B30) we find that

$$V^q_p = 0 \tag{B32}$$

(B23) to (B25) along with (B30) and (B31) mean that only the coefficient V^q contains the variable q. This means that as (B22) is true for all values of q, so that the coefficient of q in (B22) must be zero.

That is $\quad V^{\phi}_{\phi} + V^{x}_{x} - V^{p}_{p} - V^{t}_{t} - n\phi^{-1}V^{\phi} = 0$

or $\quad h_{\phi} + g'(x) - (h_{\phi} - g'(x)) - a'(t) - n\phi^{-1}V^{\phi} = 0$

and $$V^{\phi} = \frac{\phi}{n}\{2g'(x) - a'(t)\} \tag{B33}$$

(B22) now becomes

$$\begin{aligned}&(n-1)\phi^{n-1}p^2\{2g'(x) - a'(t)\} + 2np\phi^{n-1}\{h_x + p(h_{\phi} - g'(x))\}\\&-\frac{\phi a''(t)}{n} - \phi^n\{h_{xx} + p(h_{\phi x} - g''(x))\} + n\phi^{n-1}p^2(g'(x) + a'(t))\\&= -p\phi^n(h_{x\phi} + ph_{\phi\phi}) + n\phi^{n-1}p^2\{g'(x) - 2a'(t) + h_{\phi}\}\end{aligned} \tag{B34}$$

Now (B34) is true for all values of p, which means that the terms in (B34) not involving p must be zero.

That is $$\frac{a''(t)}{n} + \phi^{n-1}h_{xx} = 0 \tag{B35}$$

Now (B35) is true for all values of ϕ which means

(i) $\quad a''(t) = 0$ and

$$a(t) = V^t = \delta_3 t + \delta_1 \tag{B36}$$

where δ_3 and δ_1 are arbitrary constants. Also the coefficient of p in (B34) must also be zero, which means that

$$2n\phi^{n-1}h_x + \phi^n g''(x) = 0 \tag{B37}$$

Now $\quad h_x = V^{\phi}_{x} = \dfrac{2\phi g''(x)}{n}$

from (B33). This, in conjunction with (B37) imply that $g''(x) = 0$ and

$$g(x) = V^x = \delta_4 x + \delta_2 \tag{B38}$$

where δ_4 and δ_2 are arbitrary constants. (B33) along with (B36) and (B38) mean that

$$V^{\phi} = \frac{\phi}{n}\ (2\delta_4 - \delta_3) \tag{B39}$$

(B39) implies that

$$V_x^{\phi} = V_t^{\phi} = 0 \text{ i.e. } h_x = h_t = 0 \tag{B40}$$

(B40) along with (B30) and (B31) mean that

$$V^p = \frac{1}{n}\ \left\{(2-n)\delta_4 - \delta_3\right\}p \tag{B41}$$

and
$$V^q = \frac{1}{n}\ \left\{2\delta_4 - (n+1)\delta_3\right\}q \tag{B42}$$

In summary the components of the isovector are:

$$\begin{aligned}
V^t &= \delta_3 t + \delta_1 \quad V^x = \delta_4 x + \delta_2 \\
V^{\phi} &= \frac{1}{n}\ (2\delta_4 - \delta_3)\phi \\
V^p &= \frac{1}{n}\ \left\{(2-n)\delta_4 - \delta_3\right\}p \\
V^q &= \frac{1}{n}\left\{2\delta_4 - (n+1)\delta_3\right\}q
\end{aligned}$$

APPENDIX C

Details of the use of the Harrison-Estabrook method for finding point symmetries of the variable coefficient KdV equation

$$u_t + \alpha t^n u u_x + \beta t^m u_{xxx} = 0$$

(i) The expansion of the Lie derivatives of the differential forms β and γ with respect to the isovector V.

$$\begin{aligned} \pounds_v(\beta) &= \xi_1 \gamma + \zeta_1 \beta + \mu_1 \, d\alpha + w_1 \wedge \alpha \\ &= V \lrcorner \, d\beta + d(V \lrcorner \, \beta) \end{aligned} \tag{C1}$$

From the ideal (2.50)

$$\beta = du \wedge dt - z dx \wedge dt$$

so that

$$d\beta = - dz \wedge dx \wedge dt \tag{C2}$$

$$\begin{aligned} V \lrcorner \, d\beta &= (V^x \partial_x + V^t \partial_t + V^u \partial_u + V^z \partial_z + V^y \partial_y + V^w \partial_w) \lrcorner \, (-dz \wedge dx \wedge dt) \\ &= - V^z dx \wedge dt + V^x dz \wedge dt - V^t dz \wedge dx \end{aligned} \tag{C3}$$

$$\begin{aligned} V \lrcorner \, \beta = {} & (V^x \partial_x + V^t \partial_t + V^u \partial_u + V^z \partial_z + V^w \partial_w + V^w \partial_w) \lrcorner \, (du \wedge dt \\ & - z dx \wedge dt) \\ = {} & V^u dt - V^t du - z V^x dt + z V^t dx \end{aligned} \tag{C4}$$

$$\begin{aligned} d(V \lrcorner \, \beta) = {} & dV^u \wedge dt - dV^t \wedge du - z dV^x \wedge dt - V^x dz \wedge dt \\ & + z dV^t \wedge dx + V^t dz \wedge dx \end{aligned} \tag{C5}$$

where

$$dV^i = V^i_x dx + V^i_t dt + V^i_u du + V^i_z dz + V^i_y dy + V^i_w dw$$

and

$$i = x, t, u, z, y, w \text{ in turn.}$$

On substituting (C3) and (C5) into (C1), we get (2.61). That is

$$
\begin{aligned}
£_v(\beta) &= -V^z dx\wedge dt + dV^u\wedge dt - dV^t\wedge du - z dV^x\wedge dt \\
&\quad + z dV^t\wedge dx \\
&= \xi_1\gamma + \zeta_1\beta + \mu_1 d\alpha + w_1\wedge\alpha \\
&= \xi_1(du\wedge dx + \alpha u z t^n dt\wedge dx - \beta t^m dw\wedge dt) \\
&\quad + \zeta_1(du\wedge dt - z dx\wedge dt) - \mu_1(dw\wedge dx + dy\wedge dt) \\
&\quad + (A_1 dx + B_1 dt + C_1 du + D_1 dz + E_1 dy + G_1 dw)\wedge \\
&\quad (dz - w dx - y dt) \qquad\qquad (C6)
\end{aligned}
$$

from substituting in the differential forms of the ideal (2.50)

$$
\begin{aligned}
£_v(\gamma) &= \xi_2\gamma + \zeta_2\beta + \mu_2 d\alpha + w_2\wedge\alpha \\
&= V\lrcorner\, d\gamma + d(V\lrcorner\,\gamma) \qquad\qquad (C7)
\end{aligned}
$$

From the ideal (2.50)

$$
\gamma = du\wedge dx + \alpha u z t^n dt\wedge dx - \beta t^m dw\wedge dt
$$

so that

$$
d\gamma = \alpha t^n(z du\wedge dt\wedge dx + u dz\wedge dt\wedge dx) \qquad\qquad (C8)
$$

$$
\begin{aligned}
V\lrcorner\, d\gamma &= (V^x\partial_x + V^t\partial_t + V^u\partial_u + V^z\partial_z + V^y\partial_y + V^w\partial_w)\lrcorner \\
&\quad \alpha t^n(z du\wedge dt\wedge dx + u dz\wedge dt\wedge dx) \\
&= \alpha t^n z(V^u dt\wedge dx - V^t du\wedge dx + V^x du\wedge dt) \\
&\quad + \alpha t^n u(V^z dt\wedge dx - V^t dz\wedge dx + V^x dz\wedge dt) \qquad\qquad (C9)
\end{aligned}
$$

$$
\begin{aligned}
V\lrcorner\,\gamma &= (V^x\partial_x + V^t\partial_t + V^u\partial_u + V^z\partial_z + V^y\partial_y + V^w\partial_w)\lrcorner \\
&\quad (du\wedge dx + \alpha u z t^n dt\wedge dx - \beta t^m dw\wedge dt) \\
&= V^u dx - V^x du + \alpha u z t^n(V^t dx - V^x dt) - \beta t^m(V^w dt - V^t dw) \\
&\qquad\qquad (C10)
\end{aligned}
$$

From (C10) we obtain,

$$\begin{aligned} d(V \lrcorner \gamma) = &\ dV^u \wedge dx - dV^x du + \alpha uzt^n(dV^t \wedge dx - dV^x \wedge dt) \\ &- \beta t^m(dV^w \wedge dt - dV^t dw) + \alpha zt^n(V^t du \wedge dx - V^x du \wedge dt) \\ &+ \alpha ut^n(V^t dz \wedge dx - V^x dz \wedge dt) + \alpha uznt^{n-1} V^t dt \wedge dx \\ &+ m\beta t^{m-1} V^t dt \wedge dw \end{aligned} \tag{C11}$$

On substituting (C9) and (C11) into (C7) we get (2.62). That is,

$$\begin{aligned} £_v(\gamma) = &\ \alpha t^n (zV^u + uV^z) dt \wedge dx + dV^u \wedge dx - dV^x \wedge du \\ &+ \alpha uzt^n(dV^t \wedge dx - dV^x \wedge dt) - \beta t^m(dV^w \wedge dt - dV^t \wedge dw) \\ &+ \alpha uznt^{n-1} V^t dt \wedge dx + m\beta t^{m-1} V^t dt \wedge dw \\ = &\ \xi_2 \gamma + \zeta_2 \beta + \mu_2 d\alpha + w_2 \wedge \alpha \\ = &\ \xi_2 (du \wedge dx + \alpha\, uzt^n dt \wedge dx - \beta t^m dw \wedge dt) \\ &+ \zeta_2 (du \wedge dt - z dx \wedge dt) - \mu_2 (dw \wedge dx + dy \wedge dt) \\ &+ (A_2 dx + B_2 dt + C_2 du + D_2 dz + E_2 dy + G_2 dw) \wedge \\ &\ (dz - w dx - y dt) \end{aligned} \tag{C12}$$

from substituting in the differential forms of the ideal (2.50)

(2) The equating of the coefficients of the basis 2-forms on both sides of (C6) and (C12) and the elimination of the arbitrary 0-forms. We begin by equating coefficients of the basis 2-forms in (C6) to obtain the following set of equations.

$$- V^z + V^u_x - zV^x_x - zV^t_t = - \alpha uzt^n \xi_1 - z\zeta_1 - yA_1 + wB_1 \tag{C13}$$

$$- V^t_x - zV^t_u = - \xi_1 + wC_1 \tag{C14}$$

$$- zV^t_z = A_1 + wD_1 \tag{C15}$$

$$- zV^t_y = wE_1 \tag{C16}$$

$$- zV^t_w = \mu_1 + wG_1 \tag{C17}$$

$$- V^u_u - V^t_t + zV^x_u = - \zeta_1 + yC_1 \tag{C18}$$

$$- V^u_z + zV^x_z = B_1 + yD_1 \tag{C19}$$

$$- V^u_y + zV^x_y = \mu_1 + yE_1 \tag{C20}$$

$$- V^u_w + zV^x_w = \xi_1 \beta t^m + yG_1 \tag{C21}$$

$$V^t_z = C_1 \tag{C22}$$

$$V^t_y = V^t_w = 0 \tag{C23}$$

$$E_1 = G_1 = 0 \tag{C24}$$

We next eliminate the arbitrary 0-forms from the equations (C13) to (C24).

(C23), (C24) and (C17) imply that $\mu_1 = 0$ so therefore by (C20)

$$zV^x_y - V^u_y = 0 \tag{C25}$$

(C18) and (C22) give

$$\zeta_1 = V^u_u + V^t_t - zV^x_u + yV^t_z \tag{C26}$$

while (C14) and (C22) give

$$\xi_1 = V^t_x + zV^t_u + wV^t_z \tag{C27}$$

(C21), (C24) and (C27) imply that

$$\beta t^m \,(V^t_x + zV^t_u + wV^t_z) = zV^x_w - V^u_w \tag{C28}$$

From (C15) and (C19) we obtain

$$-zyV^t_z = yA_1 + ywD_1$$

and $$zwV^x_z - wV^u_z = wB_1 + wyD_1$$

which upon subtracting give

$$wB_1 - yA_1 = zwV^x_z - wV^u_z + zyV^t_z \tag{C29}$$

On substituting (C26), (C27) and (C29) into (C13), we obtain

$$- V^z + V^u_x - zV^x_x = -\alpha uzt^n (V^t_x + zV^t_u + wV^t_z)$$

$$- z(V^u_u - zV^x_u) + w(zV^x_z - V^u_z) \tag{C30}$$

Similarly equating coefficients of the basis 2-forms in (C12), we obtain the following set of equations

$$\begin{aligned}\alpha t^n zV^u + \alpha t^n uV^z + V^u_t + \alpha uzt^n (V^t_t + V^x_x) \\ + \alpha uznt^{n-1}V^t + \beta t^m V^w_x \\ = \xi_2 \alpha uzt^n + \zeta_2 z + yA_2 - wB_2\end{aligned} \tag{C31}$$

$$V^u_u + V^x_x + \alpha uzt^n V^t_u = \xi_2 - wC_2 \tag{C32}$$

$$V^u_z + \alpha uzt^n V^t_z = -A_2 - wD_2 \tag{C33}$$

$$V^u_y + \alpha uzt^n V^t_y = -wE_2 \tag{C34}$$

$$V^u_w + \alpha uzt^n V^t_w - \beta t^m V^t_x = -wG_2 - \mu_2 \tag{C35}$$

$$V^x_t - \alpha uzt^n V^x_u - \beta t^m V^w_u = \zeta_2 - yC_2 \tag{C36}$$

$$-\alpha uzt^n V^x_z - \beta t^m V^w_z = -B_2 - yD_2 \tag{C37}$$

$$-\alpha uzt^n V^x_y - \beta t^m V^w_y = -\mu_2 - yE_2 \tag{C38}$$

$$-\alpha uzt^n V^x_w - \beta t^m (V^w_w + V^t_t) - m\beta t^{m-1}V^t$$

$$= -\xi_2 \beta t^m - yG_2 \tag{C39}$$

$$-V^x_z = -C_2 \tag{C40}$$

$$-V^x_y = 0 \tag{C41}$$

$$-V^x_w - \beta t^m V^t_u = 0 \tag{C42}$$

$$0 = E_2 \tag{C43}$$

$$-\beta t^m V^t_z = G_2 \tag{C44}$$

$$\beta t^m V^t_y = 0 \tag{C45}$$

The next step is to eliminate the arbitrary 0-forms from equations (C31) to (C45). (C23) states that $V^t_y = V^t_w = 0$ which is consistent with (C45), while (C43) and (C34) imply that $V^u_y = 0$. This coupled with (C41) means that

$$V^u_y = V^x_y = 0 \tag{C46}$$

(C46), (C43) and (C38) imply that

$$\mu_2 = \beta t^m V_y^w \tag{C47}$$

Then (C35), (C44), (C47) and (C23) imply that

$$V_w^u - \beta t^m (V_x^t + wV_z^t + V_y^w) = 0 \tag{C48}$$

From (C40) and (C32) we obtain

$$\xi_2 = V_u^u + V_x^x + \alpha uzt^n V_u^t + wV_z^x \tag{C49}$$

while (C40) and (C36) give

$$\zeta_2 = V_t^x - \alpha uzt^n V_u^x - \beta t^m V_u^w + yV_z^x \tag{C50}$$

On substituting (C47) (C49) and (C44) into (C39) we obtain

$$\begin{aligned} &m\beta t^{m-1} V^t + \alpha uzt^n (V_w^x - \beta t^m V_u^t) + \beta t^m \\ &(V_w^w + V_t^t - V_u^u - V_x^x - wV_z^x - V_y^w + yV_z^t) = 0 \end{aligned} \tag{C51}$$

Finally from (C33) and (C37) we get

$$\begin{aligned} -yV_z^u - \alpha uzt^n yV_z^t &= yA_2 + ywD_2 \\ \alpha uzt^n wV_z^x + \beta t^m wV_z^w &= wB_2 + ywD_2 \end{aligned}$$

which on subtracting give

$$yA_2 - wB_2 = -yV_z^u - \alpha uzt^n (yV_z^t + wV_z^x) - \beta t^m wV_z^w \tag{C52}$$

Substitution of (C49), (C50) and (C52) into (C31) gives

$$\begin{aligned} &\alpha t^n (zV^u + uV^z) + \alpha uznt^{n-1} V^t + V_t^u + yV_z^u \\ &- zV_t^x - zyV_z^x + \beta t^m (V_x^w + zV_u^w + wV_z^w) \\ &+ \alpha uzt^n (V_t^t - V_u^u - \alpha uzt^n V_u^t + zV_u^x + yV_z^t) = 0 \end{aligned} \tag{C53}$$

The determining equations for the components of the isovector are:

$$V^t_y = V^t_w = V^x_y = V^u_y = 0 \qquad \text{(C23) and (C46)}$$

$$V^x_w + \beta t^m V^t_u = 0 \qquad \text{(C42)}$$

$$V^u_w - \beta t^m (V^t_x + wV^t_z + V^w_y) = 0 \qquad \text{(C48)}$$

$$\beta t^m (V^t_x + zV^t_u + wV^t_z) = zV^x_w - V^u_w \qquad \text{(C28)}$$

$$V^u_x - V^z + wV^u_z + \alpha\, uzt^n (V^t_x + zV^t_u + wV^t_z)$$

$$+ z(V^u_u - zV^x_u - V^x_x - wV^x_z) = 0 \qquad \text{(C30)}$$

$$m\beta t^{m-1} V^t + \alpha uzt^n (V^x_w - \beta t^m V^t_u) + \beta t^m (V^w_w +$$

$$V^t_t - V^u_u - V^x_x - wV^x_z - V^w_y + yV^t_z) = 0 \qquad \text{(C51)}$$

$$\alpha t^n (zV^u + uV^z) + \alpha uznt^{n-1} V^t + V^u_t + yV^u_z$$

$$- zV^x_t - zyV^x_z + \beta t^m (V^w_x + zV^w_u + wV^w_z)$$

$$+ \alpha uzt^n (V^t_t - V^u_u - \alpha uzt^n V^t_u + zV^x_u + yV^t_z)$$

$$= 0 \qquad \text{(C53)}$$

(3) The solving of the determining equations for the components of the isovector (2.52). This is done with the help of the following set of equations (2.60). That is:

$$V^x = -F_w \quad V^t = -F_y \quad F_z = \lambda \quad F_u = 0$$

$$V^w = F_x + wF_z \quad V^y = F_t + yF_z$$

and $\qquad V^z = F - wF_w - yF_y$

$F_u = 0$ implies that $F = F(x, t, y, z, w)$ and (2.60) shows that the components V^x, V^t, V^w, V^y and V^z are all independent of u so that

$$V^x_u = V^t_u = V^w_u = V^y_u = V^z_u = 0 \qquad \text{(C54)}$$

$V^t = -F_y$ so $V^t_y = -F_{yy} = 0$ by (C23).

Integrating $F_{yy} = 0$ gives

$$F = a(x, t, z, w)y + b(x, t, z, w) \tag{C55}$$

where a and b are arbitrary functions of the arguments shown

(2.60), (C23) and (C55) imply that

$$V^t_w = -F_{yw} = -a_w = 0, \text{ so } a = a(x, t, z).$$

(C28), (C48) and (C54) give

$$2V^u_w - \beta t^m V^w_y = 0 \tag{C56}$$

Differentiating (C56) and (C51) with respect to y gives

$$V^w_{yy} = V^w_{wy} = 0.$$

Now $\qquad V^w_y = F_{xy} + wF_{zy} = a_x + wa_z$

so $\qquad V^w_{wy} = a_z = 0$ implies $a = a(x, t)$.

Then $\qquad V^t_z = -F_{yz} = -a_z = 0.$

(C48) and (C56) then give

$$V^w_y + 2V^t_x = 0. \tag{C57}$$

But $V^t_x = -F_{yx} = -a_x$ and above we deduced $V^w_y = a_x$ so (C57) becomes $a_x = 0$, yielding $a = a(t)$.

(C55) becomes $\qquad F = a(t)y + b(x, t, z, w) \qquad$ (C58)

(2.60) and (C55) mean that

$$V^x = -F_w = -b_w, \text{ while } V^x_w = 0$$

implies that $b_{ww} = 0$ which on integrating gives $b = g(x, t, z)w + h(x, t, z)$ where g and h are two arbitrary functions. (C58) becomes

$$F = a(t)y + g(x,t,z)w + h(x, t, z) \tag{C59}$$

From (2.60) and (C59) we get

$$V^w = F_x + wF_z = g_x w + h_x + w(g_z w + h_z)$$

which is independent of y so that $V^w_y = 0$. Also $V^z = F - wF_w - yF_y = h(x, t, z)$ which means that $V^z_y = V^z_w = 0$.

In summary we have

$$\left.\begin{array}{ccccccccccc} V^t_x & = & V^t_u & = & V^t_z & = & V^t_y & = & V^t_w & = & 0 \\ & & V^x_u & & & = & V^x_y & = & V^x_w & = & 0 \\ & & V^y_u & & & & & & & = & 0 \\ & & V^z_u & & & = & V^z_y & = & V^z_w & = & 0 \\ & & V^w_u & & & = & V^w_y & & & = & 0 \\ & & & & & = & V^u_y & = & V^u_w & = & 0 \end{array}\right\} \tag{C60}$$

(C60) and (C51) give

$$mV^t + t(V^w_w + V^t_t - V^u_u - V^x_x - wV^x_z) = 0 \tag{C61}$$

As m is an arbitrary real number, then (C61) implies that

$$V^t = 0 \tag{C62}$$

and in turn that $V^t_t = 0$ so that (C59) becomes

$$F = g(x, t, z)w + h(x,t,z) \tag{C63}$$

The determining equations now become (C60) and $V^t_t = 0$ while (C30) (C51) and (C53) become

$$V^u_x - V^z + wV^u_z + z(V^u_u - V^x_x - wV^x_z) = 0 \tag{C64}$$

$$V^w_w - V^u_u - V^x_x - wV^x_z = 0 \tag{C65}$$

$$\begin{aligned} &\alpha t^n (zV^u + uV^z) + V^u_t + yV^u_z - zV^x_t - zyV^x_z \\ &+ \beta t^m (V^w_x + wV^w_z) - \alpha uzt^n V^u_u = 0 \end{aligned} \tag{C66}$$

From (C63) and (C260) we get

$$V^w = F_x + wF_z = g_x w + h_x + w(g_z w + h_z)$$
$$V^x = -F_w = -g$$

These two expressions imply that

$$V^w_w = g_x + 2wg_z + h_x$$

$$V^x_z = -g_z \quad V^x_x = -g_x$$

On substitution of the above expressions in (C65) we get

$$V^u_u = 2g_x + 3wg_z + h_z \tag{C67}$$

From (C60), $V^u_w = 0$ which means that V^u and V^u_u are independent of w, so as (C67) is true for all values of w then $g_z = 0$ which means that

$$V^x = -g = -g(x\ t) \text{ and } V^x_z = 0 \tag{C68}$$

(C67) becomes $F = g(x, t)w + h(x, t, z)$ (C69)

while (C64), (C65) and (C66) become

$$V^u_x - V^z + wV^u_z + z(V^u_u - V^x_x) = 0 \tag{C70}$$

$$V^w_w - V^u_u - V^x_x = 0 \tag{C71}$$

and $$\alpha t^n (zV^u + uV^z) + V^u_t + yV^u_z - zV^x_t + \beta t^m (V^w_x + wV^w_z) - \alpha uzt^n V^u_u = 0 \tag{C72}$$

From (2.60) and (C69) we obtain the following results

(1) $$V^z = F - wF_w - yF_y = h(x, t, z) \tag{C73}$$

$V^z_w = V^u_w = V^x_w = 0$ means that V^z, V^u, V^x and their derivatives are independent of w and (C70) is true for all values of w then

$$V^u_z = 0 \tag{C74}$$

(2) $$V^w = F_x + wF_z = g_x w + h_x + wh_z \tag{C75}$$

(3) $$F_z = h_z = \lambda$$

From (C75) we obtain

$$\begin{aligned} V^w_w &= g_x + h_z = g_x + \lambda \\ &= V^u_u + V^x_x \text{ by (C71).} \end{aligned}$$

which means that $V^w_{wz} = h_{zz} = V^u_{uz} + V^x_{xz} = 0$

by (C68) and (C74), so that on integrating $h_{zz} = 0$ we get

$$h(x,t,z) = h(x,t)z + k(x,t) = \lambda(x,t)z + k(x,t)$$

where k is an arbitrary function of (x, t) (C73) now becomes

$$V^z = \lambda z + k \tag{C76}$$

(C67) and $g_z = 0$ means that

$$V^u_u = 2g_x + h_z = 2g_x + \lambda$$

which on integrating gives

$$V^u = (2g_x + \lambda)u + f(x,t) \tag{C77}$$

where f is an arbitrary function of (x, t)

$$V^u_y = V^u_w = V^u_z = 0$$

means that V^u is a function of (x, t, u) only.

(C70) and (C74) mean that

$$V^u_x - V^z + z(V^u_u - V^x_x) = 0 \tag{C78}$$

From (C77) we obtain

$$V^u_x = (2g_{xx} + \lambda_x)u + f_x$$
$$V^u_u = 2g_x + \lambda$$

while (C68) gives

$$V^x_x = -g_x$$

Substitution of the above equations along with (C76) into (C78) gives

$$(2g_{xx} + \lambda_x)u + 3zg_x + f_x - k = 0 \tag{C79}$$

As $f = f(x, t)$ $k = k(x, t)$ and (C79) is true for all values of z and u we obtain $g_x = 0$, $g_{xx} = 0$ and $\lambda_x = 0$ which mean that

$$\left.\begin{array}{l} g = g(t) \;\; V^x_x = 0 \;\; \lambda = \lambda(t) \\ \text{and } f_x = k \end{array}\right\} \tag{C80}$$

The results (C80) when combined with (C62), (C68), (C75), (C76) and (C77) give the following expressions for the coefficients of the isovector

$$\left.\begin{array}{l} V^t = 0 \;\; V^x = -g(t) \;\; V^w = k_x(x, t) + w\lambda(t) \\ V^z = \lambda(t)z + k(x, t) \text{ and } V^u = \lambda(t)u + f(x, t) \end{array}\right\} \tag{C81}$$

Using the expressions of (C81) and the determining equation (C72) we obtain

$$\begin{array}{l} \alpha t^n\{z(\lambda u+f) + u(\lambda z+k)\} + \lambda'(t)u \\ + f_t + zg'(t) + \beta t^m k_{xx} - \alpha uzt^n \lambda = 0 \end{array} \tag{C82}$$

(C82) is true for all values of u and z so the coefficients of u, z and uz must all be identically equal to zero, which gives the following results:

(i) $\alpha t^n(2\lambda - \lambda) = 0$ which means that $\lambda = 0$ and $\lambda'(t) = 0$ so that (C82) becomes

$$\alpha t^n(zf+uk) + f_t + zg'(t) + \beta t^n k_{xx} = 0 \tag{C83}$$

(ii) $\alpha t^n k = 0$ or $k = 0$ and $k_{xx} = 0$ so that (C83) becomes

$$\{\alpha t^n f + g'(t)\}z + f_t = 0 \tag{C84}$$

(iii)
$$\alpha t^n f + g'(t) = 0 \tag{C85}$$

(iv) $f_t = 0$ which, when combined with $f_x = k = 0$ mean that $f = a$ an arbitrary constant, while (C85) implies that

$$a\alpha t^n = -g'(t)$$

or
$$-g(t) = \frac{a\alpha t^{n+1}}{n+1} + b$$

b = an arbitrary constant.

The coefficients of the isovector are therefore

$$\left.\begin{aligned} V^t &= 0 \\ V^x &= \frac{a\alpha t^{n+1}}{n+1} + b \\ V^u &= a \\ V^w &= V^z = 0 \\ V^y &= -a\alpha t^n w \end{aligned}\right\} \tag{2.63}$$

APPENDIX D

On substituting (2.65) and (2.68) into (C61) we get

$$-(k_1 n + k_2)a(t) + t(2g_x + 3g_z w + h_z - a'(t) - V^u_u) = 0 \quad \text{(D1)}$$

$V^u_w = 0$ (C60) means that V^u and its derivatives are independent of w. (D1) is true for all values of w so that

$$g_z = 0 \text{ and } V^x = -g(x, t) \quad \text{(D2)}$$

(D2) implies that $\quad V^x_z = 0 \quad$ (D3)

(D1) now becomes

$$-(k_1 n + k_2)a(t) + t(2g_x + h_z - a'(t) - V^u_u) = 0 \quad \text{(D4)}$$

On substituting $V^x_z = 0$ and $V^z = h(x, t, z)$ into (C64) we get

$$V^u_x - h(x, t, z) + w\,V^u_z + z(V^u_u - V^x_x) = 0 \quad \text{(D5)}$$

(D5) is true for all values of w and V^u and V^x are independent of w so that

$$V^u_z = 0$$

and $\quad V^u_x - h(x, t, z) + z(V^u_u - V^x_x) = 0 \quad$ (D6)

On substituting $V^u_z = V^x_z = 0$ and the appropriate terms from (2.68) into (2.66) we get

$$\begin{aligned}&\alpha t^n\{zV^u + uh(x, t, z)\} - \alpha uznt^{n-1}\,a(t) + V^u_t + \\ &zg_t(x,t) + \beta t^m\{g_{xx}w + h_{xx} + 2wh_{xz} + w^2\,h_{zz}\} \\ &-\alpha uzt^n\,\{a'(t) + V^u_u\} = 0 \end{aligned} \quad \text{(D7)}$$

(D7) is true for all values of w so that on equating the coefficients of w and w^2 to zero, we get

$$g_{xx} + 2h_{xz} = 0 \quad \text{(D8)}$$

$$h_{zz} = 0 \tag{D9}$$

and

$$\alpha t^n\{zV^u + uh(x,t,z)\} - \alpha uznt^{n-1}a(t) + V^u_t + zg_t(x,t) + \beta t^m h_{xx}(x,t,z) - \alpha uzt^n\{a'(t) - V^u_u\} = 0 \tag{D10}$$

$V^z = h(x,t,z)$ and (D9) imply that

$$V^z = h_1(x, t)z + h_2(x, t) \tag{D11}$$

where h_1 and h_2 are arbitrary functions of (x, t). On substituting (D11) and $V^x_x = -g_x(x,t)$ into (D6) we get

$$V^u_x - h_1(x,t)z - h_2(x,t) + z\{V^u_u + g_x(x,t)\} = 0 \tag{D12}$$

$V^u_z = 0$ means that V^u and its derivatives are independent of z so that therefore

$$V^u_x = h_2(x, t) \tag{D13}$$

and

$$V^u_u = h_1(x, t) - g_x(x, t) \tag{D14}$$

Differentiating (D14) with respect to u gives

$$V^u_{uu} = 0. \text{ As } V^u_z = V^u_y = V^u_w = 0$$

then V^u is a function of (x, t, u). So on integrating $V^u_{uu} = 0$ we get

$$V^u = f_1(x, t)u + f_2(x, t) \tag{D15}$$

where f_1 and f_2 are arbitrary functions of (x, t). Differentiating (D15) with respect to x and equating the result to (D13) we get

$$V^u_x = f_1(x,t)_x\, u + f_2(x,t)_x = h_2(x,t) \tag{D16}$$

As (D16) is true for all values of u we obtain the expressions

$$f_1(x,t)_x = 0 \text{ which means } f_1 = f_1(t) \tag{D17}$$

and

$$f_2(x,t)_x = h_2(xt) \tag{D18}$$

From (D14) we get $\quad V^u_u = f_1(t) = h_1(x,t) - g_x(x,t)$ (D19)

(D19) implies that $g_{xx}(x,t) = h_1(x,t)_x$ which, on substituting into (D8), implies in turn that

$$h_1(x,t)_x = 0 \quad \text{and} \quad h_1 = h_1(t) \tag{D20}$$

This also means that $g_{xx}(x,t) = 0$ so that

$$g(x,t) = g_1(t)x + g_2(t) \tag{D21}$$

where g_1 and g_2 are arbitrary functions of t.

At this point we have

$$\left.\begin{aligned}
&V^z = h_1(t)z + f_2(x,t)_x \quad \text{(From D11 and D18)}\\
&V^u = f_1(t)u + f_2(x,t) \quad \text{(From D15 and D17)}\\
&V^x = -g_1(t)x - g_2(t) \quad \text{(From D21)}\\
&\text{also } f_1(t) = h_1(t) - g_1(t) \quad \text{(From D19. D20 and D21)}
\end{aligned}\right\} \tag{D22}$$

Substituting the expressions of (D22) into (D4) gives

$$-(k_1 n + k_2)a(t) + 3tg_1(t) - ta'(t) = 0 \tag{D23}$$

Similarly substituting (D22) into (D10) gives

$$\begin{aligned}
&\alpha t^n \{zf_2(x,t) + u[h_1(t)z + f_2(x,t)_x]\} - \alpha uznt^{n-1} a(t)\\
&+ f_1'(t)u + f_2(x,t)_t + z\{g_1'(t)x + g_2'(t)\}\\
&+ \beta t^m f_2(x,t)_{xxx} - \alpha uzt^n a'(t) = 0
\end{aligned} \tag{D24}$$

(D24) is true for all values of u and z so that on equating the coefficients of u, z and uz to zero we get

$$th_1(t) - na(t) - ta'(t) = 0 \tag{D25}$$

$$\alpha t^n f_2(x,t) + g_1'(t)x + g_2'(t) = 0 \tag{D26}$$

$$\alpha t^n f_2(x,t)_x + f_1'(t) = 0 \tag{D27}$$

and
$$f_2(x,t)_t + \beta t^m f_2(x,t)_{xxx} = 0 \tag{D28}$$

(D27) implies that
$$f_2(x,t)_x = -\frac{f_1'(t)}{\alpha t^n} \tag{D29}$$

so that $f_2(xt)_{xxx} = 0$ and by (D28) $f_2(x,t)_t = 0$ which means that

$$f_2 = f_2(x) \tag{D30}$$

(D29) now becomes $f_2'(x) = -\dfrac{f_1'(t)}{\alpha t^n}$ which on integrating becomes

$$f_2(x) = -\left(\frac{f_1'(t)}{\alpha t^n}\right)x + a_1 \tag{D31}$$

where a_1 is an arbitrary constant.

$f_2(x)_t = 0$ means that $\dfrac{f_1'(t)}{\alpha t^n} = a_2$ where a_2 is an arbitrary constant. $f_1'(t) = a_2\,\alpha t^n$ on integrating gives

$$f_1(t) = \frac{a_2\alpha t^{n+1}}{n+1} + a_3 \tag{D32}$$

where a_3 is an arbitrary constant.

(D22), (D31) and (D32) mean that

$$V^u = \left(\frac{a_2\alpha t^{n+1}}{n+1} + a_3\right)u - a_2 x + a_1 \tag{D33}$$

(D26) and (D31) give

$$\alpha t^n(-a_2 x + a_1) + g_1'(t)x + g_2'(t) = 0 \tag{D34}$$

(D34) is true for all values of x so that

$$g_1'(t) = a_2 \alpha t^n \text{ and } g_2'(t) = -a_1 \alpha t^n$$

which on integrating give

$$g_1(t) = \frac{a_2 \alpha t^{n+1}}{n+1} + a_4 \tag{D35}$$

and $$g_2(t) = -\frac{a_1 \alpha t^{n+1}}{n+1} + a_5 \tag{D36}$$

From (D22) $f_1(t) = h_1(t) - g_1(t)$ we obtain

$$h_1(t) = \frac{2a_2 \alpha t^{n+1}}{n+1} + a_3 + a_4 \tag{D37}$$

This means that

$$V^x = -\left(a_4 + \frac{a_2 \alpha t^{n+1}}{n+1}\right)x + \frac{a_2 \alpha t^{n+1}}{n+1} + a_5 \tag{D38}$$

and $$V^z = \left(\frac{2a_2 \alpha t^{n+1}}{n+1} + a_3 + a_4\right)z - a_2 \tag{D39}$$

(D23) and (D25) now become

$$t\left(\frac{2a_2 \alpha t^{n+1}}{n+1} + a_3 + a_4\right) - na(t) - ta'(t) = 0 \tag{D40}$$

$$-(k_1 n + k_2)a(t) + 3t\left(\frac{a_2 \alpha t^{n+1}}{n+1} + a_4\right) - ta'(t) = 0 \tag{D41}$$

Subtracting (D40) from (D41) gives

$$\{n(1-k_1) - k_2\}\, a(t) + t\left(\frac{a_2 \alpha t^{n+1}}{n+1} + 2a_4 - a_3\right) = 0 \tag{D42}$$

(D42) should be true for all values of n so that

$$k_1 = 1 \text{ and } a_2 = 0 \tag{D43}$$

also $$k_2 a(t) + t(2a_4 - a_3) = 0 \tag{D44}$$

As m and n are linearly related by (2.65) (D43) means that

$$m = n + k_2 \quad k_2 \neq 0 \tag{D45}$$

The coefficients of the isovector are with relabelling of the four nonzero arbitrary constants

$$\left.\begin{aligned}
V^t &= \left(\frac{a_1 - 2a_2}{k_2}\right)t \\
V^x &= -a_2 x - \frac{a_1 \alpha t^{n+1}}{n+1} + a_4 \\
V^z &= (a_1 + a_2)\, z \\
V^u &= a_1 u + a_3 \\
V^w &= (2a_2 + a_1)w \\
V^y &= \left(\frac{2a_2 - a_1}{k_2}\right)y - a_3 \alpha t^n w + (a_1 + a_2)w
\end{aligned}\right\} \tag{2.69}$$

APPENDIX E

To determine if

$$R_2 = -(D_x^2 + \frac{b}{c} D_x + \frac{au_1}{c} D_x^{-1}) \qquad \text{(E.1)}$$

is a recursion operator for the KdVB equation

$$H = 0 \equiv u_t + auu_1 + bu_2 + cu_3 = 0 \qquad \text{(E.2)}$$

The invariance condition for (E.2) under the L-B symmetry given by $X(Q) = Q\partial u$ is $D_H[Q]_{H=0} = 0$ or

$$(D_t + au_1 + auD_x + bD_x^2 + cD_x^3)\,[Q]_{H=0} = 0 \qquad \text{(E.3)}$$

For R_2 to be a recursion operator of (E.2)

$$[D_H R_2] = D_H R_2 - R_2 D_H = 0 \qquad \text{(E.4)}$$

From (E.1) and (E.3)

$$\begin{aligned}
D_H R_2 &= (D_t + au_1 + auD_x + bD_x^2 + cD_x^3)(D_x^2 + \frac{b}{c} D_x + \frac{uu_1}{c} D_x^{-1}) \\
&= D_t D_x^2 + \frac{b}{c} D_t D_x + \frac{a}{c} D_t\,[u_1 D_x^{-1}] + au_1 D_x^2 \\
&+ \frac{abu_1}{c} D_x + \frac{(au_1)^2}{c} D_x^{-1} + au\,D_x^3 + \frac{aub}{c} D_x^2 \\
&+ auD_x\left[\frac{au_1}{c} D_x^{-1}\right] + b\,D_x^4 + \frac{b^2}{c} D_x^3 + b\,D_x^2\left[\frac{au_1}{c} D_x^{-1}\right] \\
&+ c\,D_x^5 + b\,D_x^4 + c\,D_x^3\left[\frac{au_1}{c} D_x^{-1}\right]
\end{aligned} \qquad \text{(E.5)}$$

$$D_t\,[u_1\;D_x^{-1}] = u_{1t}\,D_x^{-1} + u_1\,D_t\,D_x^{-1} \qquad \text{(E.6)}$$

$$D_x\,[u_1\,D_x^{-1}] = u_2\,D_x^{-1} + u_1 \qquad \text{(E.7)}$$

$$\begin{aligned}
D_x^2\,[u_1\,D_x^{-1}] &= D_x\,[u_2 D_x^{-1} + u_1] \\
&= u_3\,D_x^{-1} + 2u_2 + u_1\,D_x
\end{aligned} \qquad \text{(E.8)}$$

$$\begin{aligned}
D_x^3\,[u_1 D_x^{-1}] &= D_x[u_3 D_x^{-1} + 2u_2 + u_1\,D_x\,] \\
&= u_4\,D_x^{-1} + 3u_3 + 3u_2 D_x + u_1\,D_x^2
\end{aligned} \qquad \text{(E.9)}$$

On substituting (E.6) to (E.9) into (E.5), we get

$$
\begin{aligned}
D_H R_2 = {} & D_t D_x^2 + \frac{b}{c} D_t D_x + \frac{au_1}{c} D_t D_x^{-1} + c\, D_x^5 \\
& + 2bD_x^4 + (au + \frac{b^2}{c})\, D_x^3 + \left(2au_1 + \frac{abu}{c}\right) D_x^2 \\
& + \left(\frac{2abu_1}{c} + 3au_2\right) D_x + \frac{a^2\, uu_1}{c} + \frac{2abu_2}{c} + 3au_3
\end{aligned}
\tag{E.10}
$$

Similarly

$$
\begin{aligned}
R_2\, D_H = {} & D_x^2\, D_t + \frac{b}{c} D_x D_t + \frac{au_1}{c} D_x^{-1}\, D_t + cD_x^5 \\
& + 2b\, D_x^4 + \left(au + \frac{b^2}{c}\right) D_x^3 + \left(4au_1 + \frac{abu}{c}\right) D_x^2 \\
& + \left(\frac{3abu_1}{c} + 3au_2\right) D_x + \frac{a^2\, uu_1}{c} + \frac{abu_2}{c} + au_3
\end{aligned}
\tag{E.11}
$$

$$
\begin{aligned}
D_H R_2 - R_2\, D_H &= -\, 2au_1\, D_x^2 - \frac{abu_1}{c}\, D_x \\
&\quad + \frac{abu_2}{c} + 2au_3 \\
&= 0
\end{aligned}
\tag{E.12}
$$

(E.12) implies that $a = 0$.

APPENDIX F

3rd Order L-B Symmetries of the KdVB Equation

The invariance condition for an L-B symmetry of the KdVB equation is

$$D_t[Q] + au_1 Q + auD_x[Q] + bD_x^2[Q] + cD_x^3[Q] = 0 \qquad (F.1)$$

If

$$Q = Q(x, t, u, u_1, u_2, u_3) \qquad (F.2)$$

then D_x and D_t become

$$\left.\begin{aligned} D_x &= \partial_x + u_1\partial_u + u_2\partial_{u_1} + u_3\partial_{u_2} + u_4\partial_{u_3} \\ D_t &= \partial_t + u_t\partial_u + u_{t_1}\partial_{u_1} + u_{t_2}\partial_{u_2} + u_{t_3}\partial_{u_3} \end{aligned}\right\} \qquad (F.3)$$

Incorporating (F.2) and (F.3) into (F.1) gives

$$\begin{aligned} &Q_t + u_tQ_u + u_{t_1}Q_{u_1} + u_{t_2}Q_{u_2} + u_{t_3}Q_{u_3} + au_1 Q \\ &+ au(Q_x + u_1Q_u + u_2Q_{u_1} + u_3Q_{u_2} + u_4 Q_{u_3}) \\ &+ b\, D_x(Q_x + u_1Q_u + u_2Q_{u_1} + u_3 Q_{u_2} + u_4Q_{u_3}) \\ &+ c\, D_x^2 (Q_x + u_1Q_u + u_2Q_{u_1} + u_3Q_{u_2} + u_4Q_{u_3}) \end{aligned} \qquad (F.4)$$

Using the expressions

$$\begin{aligned} u_t &= -(auu_1 + bu_2 + cu_3) \\ u_{t_1} &= -(au_1^2 + auu_2 + bu_3 + cu_4) \\ u_{t_2} &= -(3au_1u_2 + auu_3 + bu_4 + cu_5) \\ u_{t_3} &= -(3au_2^2 + 4au_1u_3 + auu_4 + bu_5 + cu_6) \end{aligned}$$

to eliminate the t-derivatives of u in (F.4) and then by expanding and simplifying we obtain the expression

$$
\begin{aligned}
&Q_t + auQ_x + au_1Q - au_1^2\, Q_{u_1} - 3au_1u_2Q_{u_2} - 3au_2^2\, Q_{u_3} \\
&- 4au_1u_3Q_{u_3} + b[u_4^2\, Q_{u_3u_3} + 2u_4(Q_{xu_3} + u_1Q_{uu_3} \\
&+ u_2Q_{u_1u_3} + u_3Q_{u_2u_3}) + u_3^2\, Q_{u_2u_2} + 2u_3\,(Q_{xu_2} + \\
&u_1Q_{uu_2} + u_2Q_{u_1u_2}) + u_2^2\, Q_{u_1u_1} + 2u_2(Q_{xu_1} \\
&+ u_1Q_{uu_1}) + u_1^2\, Q_{uu} + 2u_1Q_{x_u} + Q_{xx}\,] \\
&+ c[3u_5(Q_{xu_3} + u_1Q_{uu_3} + u_2Q_{u_1u_3} + u_3Q_{u_2u_3} + u_4Q_{u_3u_3}) \\
&+ u_4^3Q_{u_3u_3u_3} + 3u_4^2\,(Q_{xu_3u_3} + u_1Q_{uu_3u_3} + u_2Q_{u_1u_3u_3} + \\
&u_3Q_{u_2u_3u_3} + Q_{u_2u_3}) + 3u_4(Q_{xu_2} + u_1Q_{uu_2} + u_2Q_{u_1u_2} \\
&+ u_3\, Q_{u_2u_2} + u_2Q_{uu_3} + u_3Q_{u_1u_3} + Q_{xxu_3} + 2u_1Q_{xuu_3} \\
&+ 2u_2Q_{xu_1u_3} + 2u_3Q_{xu_2u_3} + u_1^2\, Q_{uuu_3} + u_2^2\, Q_{u_1u_1u_3} \\
&+ u_3^2\, Q_{u_2u_2u_3} + 2u_1u_2Q_{uu_1u_3} + 2u_1u_3Q_{uu_2u_3} \\
&+ 2u_2u_3Q_{u_1u_2u_3}) + u_3^3\, Q_{u_2u_2u_3} + 3u_3^2\,(Q_{xu_2u_2} \\
&+ u_1Q_{uu_2u_2} + u_2\, Q_{u_1u_2u_2} + Q_{u_1u_2}) + 3u_3\,(Q_{xu_1} + \\
&u_1\, Q_{uu_1} + u_2\, Q_{u_1u_1} + u_2\, Q_{uu_2} + 2u_1\, Q_{xuu_2} + u_1^2\, Q_{uuu_2} \\
&+ 2u_2\, Q_{xu_1u_2} + u_2^2\, Q_{u_1u_1u_2} + 2u_1u_2\, Q_{uu_1u_2}) + u_2^3\, Q_{u_1u_1u_1} \\
&+ 3u_2^2(Q_{uu_1} + Q_{xu_1u_1} + u_1Q_{uu_1u_1}) + 3u_2\,(\,Q_{xu} + \\
&u_1\, Q_{uu} + Q_{xxu_1} + 2u_1Q_{xuu_1} + u_1^2\, Q_{uuu_1}) + u_1^3\, Q_{uuu} \\
&+ 3u_1^2\, Q_{xuu} + 3u_1Q_{xxu} + Q_{xxx}] \;=\; 0 \qquad\qquad \text{(F.5)}
\end{aligned}
$$

(F.5) holds for all solutions of the KdVB equation so the coefficients of descending order derivatives of u can be equated to zero. The coefficient of u_5 in (F.5) is

$$3c(Q_{xu_3} + u_1\, Q_{uu_3} + u_2\, Q_{u_1u_3} + u_3\, Q_{u_2u_3} + u_4Q_{u_3u_3}) \qquad \text{(F.6)}$$

(F.6) is true for all values of u_1, u_2, u_3 and u_4 and $c \neq 0$ which means

$$Q_{u_3u_3} = Q_{u_2u_3} = Q_{u_1u_3} = Q_{uu_3} = Q_{xu_3} = 0 \qquad \text{(F.7)}$$

(F.7) means $\quad Q = A(t)u_3 + B(x, t, u, u_1, u_2) \qquad$ (F.8)

where A and B are arbitrary function of the arguments shown.

Also $$Q_{iju_3} = Q_{ju_3} = 0 \quad i, j = x, u, u_1, u_2, u_3 \tag{F.9}$$

(F.9) ensures that the coefficient of u_4^2 is zero and the coefficient of u_4 reduces to

$$3c(B_{xu_2} + u_1 B_{uu_2} + u_2 B_{u_1u_2} + u_3 B_{u_2u_2}) = 0 \tag{F.10}$$

As (Fl0) holds for all values of u_1, u_2 and u_3 then:

$$B_{xu_2} = B_{uu_2} = B_{u_1u_2} = B_{u_2u_2} = 0 \tag{F.11}$$

which means (F.8) can now be written as

$$Q = A(t)u_3 + B(t)u_2 + C(x, t, u, u_1) \tag{F.12}$$

Also $$B_{iju_2} = B_{ju_2} = 0 \quad i, j = x, u, u_1, u_2 \tag{F.13}$$

(F.13) makes the coefficients of u_3^3 and u_3^2 equal to zero and the invariance condition (F.5) reduces to

$$\begin{aligned}
&\left(\frac{dA}{dt}\right)u_3 + \left(\frac{dB}{dt}\right)u_2 + C_t + auC_x - au_1C - au_1^2\, C_{u_1} \\
&- 2au_1u_2B - 3au_2^2\, A - 3au_1u_3\, A + b[u_2^2\, C_{u_1u_1} \\
&+ 2u_2(C_{xu_1} + u_1C_{uu_1}) + u_1^2\, C_{uu} + 2u_1C_{xu} + C_{xx}] \\
&+ c[3u_3(C_{xu_1} + u_1C_{uu_1} + u_2C_{u_1u_1}) + u_2^3\, C_{u_1u_1u_1} \\
&+ 3u_2^2\,(C_{uu_1} + C_{xu_1u_1} + u_1C_{uu_1u_1}) + 3u_2\,(C_{xu} + u_1\, C_{uu} \\
&+ C_{xxu_1} + 2u_1\, C_{xuu_1} + u_1^2\, C_{uuu_1}) + u_1^3\, C_{uuu} \\
&+ 3u_1^2\, C_{xuu} + 3u_1\, C_{xxu} + C_{xxx}] = 0
\end{aligned} \tag{F.14}$$

The coefficient of u_3 in (F.14) is

$$\frac{dA}{dt} - 3au_1A + 3c\,(C_{xu_1} + u_1C_{uu_1} + u_2C_{u_1u_1}) = 0 \tag{F.15}$$

(F.15) is true for all values of u_1 and u_2 so that $C_{u_1u_1} = 0$ which means that

$$C(x, t, u, u_1) = C(x, t, u)u_1 + D(x, t, u) \qquad (F.16)$$

Also $c\, C_{uu_1} = aA$ which on integrating gives

$$C(x, t, u) = \frac{auA}{c} + C(x, t) \qquad (F.17)$$

Finally $\frac{dA}{dt} + 3cC_{xu_1} = 0$ or by (F.16) and (F.17)

$$C(x, t) = -\frac{x}{3c}\frac{dA}{dt} + C(t) \qquad (F.18)$$

Combining (F.12), (F.16), (F.17) and (F.18) we have

$$Q = Au_2 + Bu_2 + \left\{\frac{auA}{c} - \frac{x}{3c}\frac{dA}{dt} + C\right\} u_1 + D \qquad (F.19)$$

and $$C_{iju_1} = C_{u_1u_1} = 0 \quad i, j = x, u, u_1 \qquad (F.20)$$

(F.19) and (F.20) mean that the invariance condition (F.14) reduces to

$$\begin{aligned}
&\left(\frac{dB}{dt}\right)u_2 + \left(\frac{au}{c}\frac{dA}{dt} - \frac{x}{3c}\frac{d^2A}{dt^2} + \frac{dC}{dt}\right)u_1 + D_t \\
&+ au\left(D_x - \frac{u_1}{3c}\frac{dA}{dt}\right) + au_1 D - 2au_1u_2 B \\
&+ b\left[\, 2u_2\left(\frac{au_1A}{ac} - \frac{1}{3c}\frac{dA}{dt}\right) + u_1^2 D_{uu} + 2u_1D_{xu} + D_{xx}\right] \\
&+ c\,[3u_2\left(D_{xu} + u_1D_{uu}\right) + u_1^3 D_{uuu} + 3u_1^2 D_{xuu} + D_{xxx}] = 0
\end{aligned}$$

(F.21)

The coefficient of u_2 in (F.21) is

$$\frac{dB}{dt} - 2au_1B + 2b\left(\frac{au_1A}{c} - \frac{1}{3c}\frac{dA}{dt}\right) + 3c\left(D_{xu} + u_1D_{uu}\right) = 0$$

(F.22)

(F.22) is true for all values of u_1 which means that

$$3c\,D_{uu} + \frac{2abA}{c} - 2aB = 0$$

which on integrating gives

$$D(x, t, u) = \frac{au^2}{3c}\left(B + \frac{bA}{c}\right) + D(x, t)u + E(x, t) \tag{F.23}$$

Also $$3c\,D_{xu} = \frac{2b}{3c}\,\frac{dA}{dt} - \frac{dB}{dt}$$

which on integrating gives

$$D(x, t) = \frac{x}{3c}\left(\frac{2b}{3c}\,\frac{dA}{dt} - \frac{dB}{dt}\right) + D(t) \tag{F.24}$$

(F.19) now becomes

$$\begin{aligned} Q = {} & Au_3 + Bu_2 + \left\{\frac{auA}{c} - \frac{x}{3c}\,\frac{dA}{dt} + C\right\} u_1 \\ & + \frac{au^2}{3c}\,(B - \frac{bA}{c}) + \left\{\left(\frac{2b}{3c}\,\frac{dA}{dt} - \frac{dB}{dt}\right)\frac{x}{3c} + D\right\} u + E \end{aligned} \tag{F.25}$$

(F.25) means the invariance condition (F.21) reduces to

$$\begin{aligned} & \left(\frac{au}{c}\,\frac{dA}{dt} - \frac{x}{3c}\,\frac{dA}{dt} + \frac{dC}{dt}\right)u_1 + \frac{au^2}{3c}\left(\frac{dB}{dt} - \frac{b}{c}\,\frac{dA}{dt}\right) \\ & + \left\{\left(\frac{2b}{3c}\,\frac{d^2A}{dt^2} - \frac{d^2B}{dt^2}\right)\frac{x}{3c} + \frac{dD}{dt}\right\} u + E_t \\ & + au_1\left[\frac{au^2}{3c}\left(B - \frac{bA}{c}\right) + \left\{\left(\frac{2b}{3c}\,\frac{dA}{dt} - \frac{dB}{dt}\right)\frac{x}{3c} + D\right\} u + E\right] \\ & + au\left[\left(\frac{2b}{3c}\,\frac{dA}{dt} - \frac{dB}{dt}\right)\frac{u}{3c} + E_x - \frac{u_1}{3c}\,\frac{dA}{dt}\right] \\ & + b\left[\frac{2au_1^2}{3c}\left(B - \frac{bA}{c}\right) + \frac{2u_1}{3c}\left(\frac{2b}{3c}\,\frac{dA}{dt} - \frac{dB}{dt}\right) + E_{xx}\right] \\ & + c\,E_{xxx} = 0 \end{aligned} \tag{F.26}$$

The coefficient of u_1^2 in (F.26) is

$$\frac{2ab}{3c}\left(B - \frac{bA}{c}\right) = 0$$

as a, b and c are all nonzero then

$$B = \frac{bA}{c} \tag{F.27}$$

The coefficient of u_1 in (F.26) is

$$\frac{au}{c}\frac{dA}{dt} - \frac{x}{3c}\frac{dA}{dt} + \frac{dC}{dt} + \left(\frac{2b}{3c}\frac{dA}{dt} - \frac{dB}{dt}\right)\frac{aux}{3c}$$

$$+ auD + a\,E - \frac{au}{3c}\frac{dA}{dt} + \frac{2b}{3c}\left(\frac{2b}{3c}\frac{dA}{dt} - \frac{dB}{dt}\right) = 0 \tag{F.28}$$

(F.28) is true for all values of u and the functions A, B, C and D are all functions of t only, which means

$$\frac{2a}{3c}\frac{dA}{dt} + \left(\frac{2b}{3c}\frac{dA}{dt} - \frac{dB}{dt}\right)\frac{ax}{3c} + a\,D = 0 \tag{F.29}$$

(F.29) is true for all values of x so that

$$\frac{a}{3c}\left(\frac{2b}{3c}\frac{dA}{dt} - \frac{dB}{dt}\right) = 0 \tag{F.30}$$

(F.27) means that $\frac{dB}{dt} = \frac{b}{c}\frac{dA}{dt}$ which when substituted into (F.30) gives $\frac{dA}{dt} = 0$ or

$$A = k_1 \tag{F.31}$$

where k_1 is an arbitrary constant.

(F.29) reduces to $\quad D = 0 \qquad$ (F.32)

and $\quad B = \frac{bk_1}{c} \qquad$ (F.33)

(F.28) reduces to $\quad \frac{dC}{dt} + a\,E = 0 \qquad$ (F.34) so that $E = E(t)$ and

$$E_x = E_{xx} = E_{xxx} = 0 \qquad \text{(F.35)}$$

The invariance equation (F.26)

$$\frac{dE}{dt} = 0 \text{ and } E = k_2 \qquad \text{(F.36)}$$

where k_2 is an arbitrary constant.

(F.34) implies that

$$C = -ak_2t + k_3 \qquad \text{(F.37)}$$

(F.31) to (F.37) means that (F.25) becomes

$$Q = k_1u_3 + \left(\frac{k_1b}{c}\right)u_2 + \left(\frac{k_1a}{c}\right)uu_1 + (k_3 - ak_2\,t)u_1 + k_2$$

APPENDIX G

5th Order L-B Symmetries of the KdVB Equation

In the determination of the 5th order L-B symmetries we follow the same technique as used in Appendix F for the 3rd order symmetries. In this case

$$Q = Q(x, t, u, u_1, u_2, u_3, u_4, u_5) \qquad \text{(G.1)}$$

and the differential operators D_x and D_t in the invariance condition (F.1) become

$$\left.\begin{aligned} D_x &= \partial_x + u_1 \partial_u + u_2 \partial_{u_1} + u_3 \partial_{u_2} + u_4 \partial_{u_3} + u_5 \partial_{u_4} + u_6 \partial_{u_5} \\ D_t &= \partial_t + u_t \partial_u + u_{t1} \partial_{u_1} + u_{t2} \partial_{u_2} + u_{t3} \partial_{u_3} + u_{t4} \partial_{u_4} + u_{t5} \partial_{u_5} \end{aligned}\right\} \qquad \text{(G.2)}$$

Incorporating (G.1) and (G.2) into (F.1) gives

$$\begin{aligned} &Q_t + u_t Q_u + u_{t1} Q_{u_1} + u_{t2} Q_{u_2} + u_{t3} Q_{u_3} + u_{t4} Q_{u_4} + u_{t5} Q_{u_5} + au_1 Q \\ &+ au(Q_x + u_1 Q_u + u_2 Q_{u_1} + u_3 Q_{u_2} + u_4 Q_{u_3} + u_5 Q_{u_4} + u_6 Q_{u_5}) \\ &+ bD_x(Q_x + u_1 Q_u + u_2 Q_{u_1} + u_3 Q_{u_2} + u_4 Q_{u_3} + u_5 Q_{u_4} + u_6 Q_{u_5}) \\ &+ cD_x^2 (Q_x + u_1 Q_u + u_2 Q_{u_1} + u_3 Q_{u_2} + u_4 Q_{u_3} + u_5 Q_{u_4} + u_6 Q_{u_5}) \end{aligned}$$

$$\text{(G.3)}$$

Using the expressions

$$\begin{aligned} u_t &= -(auu_1 + bu_2 + cu_3) \\ u_{t1} &= -(au_1^2 + auu_2 + bu_3 + cu_4) \\ u_{t2} &= -(3au_1u_2 + auu_3 + bu_4 + cu_5) \\ u_{t3} &= -(3au_2^2 + 4au_1u_3 + auu_4 + bu_5 + cu_6) \\ u_{t4} &= -(10au_2u_3 + 5au_1u_4 + auu_5 + bu_6 + cu_7) \\ u_{t5} &= -(10au_3^2 + 15au_2u_4 + 6au_1u_5 + auu_6 + bu_7 + cu_8) \end{aligned}$$

to eliminate the t-derivatives of u in (G.3) and then by expanding and simplifying we obtain the expression

$- u_5\, 6au_1 Q_{u_5} - u_4\,(5au_1 Q_{u_4} + 15au_2 Q_{u_2})$

$- u_3^2\, 10aQ_{u_5} - u_3(4au_1 Q_{u_3} + 10au_2 Q_{u_4})$

$- u_2^2\, 3aQ_{u_3} - u_2\, 3au_1 Q_{u_2} - u_1^2\, aQ_{u_1} + u_1 aQ$

$+ uaQ_x + Q_t$

$+ b[u_6^2\, Q_{u_5u_5} + 2u_6(Q_{xu_5} + u_1 Q_{uu_5} + u_2 Q_{u_1u_5}$

$+ u_3 Q_{u_2u_5} + u_4 Q_{u_3u_5} + u_5 Q_{u_4u_5}) + u_5^2\, Q_{u_4u_4}$

$+ 2u_5(Q_{xu_4} + u_1 Q_{uu_4} + u_2\, Q_{u_1u_4} + u_3 Q_{u_2u_4}$

$+ u_4 Q_{u_3u_4}) + u_4^2\, Q_{u_3u_3} + 2u_4\,(Q_{xu_3} +$

$u_1 Q_{uu_3} + u_2 Q_{u_1u_3} + u_3 Q_{u_2u_3}) + u_3^2\, Q_{u_2u_2}$

$+ 2u_3(Q_{xu_2} + u_1\, Q_{uu_2} + u_2\, Q_{u_1u_2}) + u_2^2\, Q_{u_1u_1}$

$+ 2u_2\,(Q_{xu_1} + u_1 Q_{uu_1}) + u_1^2\, Q_{uu} + 2u_1 Q_{xu}$

$+ Q_{xx}] + c[3u_7(Q_{xu_5} + u_1 Q_{uu_5} + u_2 Q_{u_1u_5}$

$+ u_3 Q_{u_2u_5} + u_4\, Q_{u_3u_5} + u_5 Q_{u_4u_5} + u_6 Q_{u_5u_5})$

$+ u_6^3\, Q_{u_5u_5u_5} + 3u_6^2\,(Q_{xu_5u_5} + u_1 Q_{uu_5u_5} + u_2\, Q_{u_1u_5u_5}$

$+ u_3 Q_{u_2u_5u_5} + u_4\, Q_{u_3u_5u_5} + u_5\, Q_{u_4u_5u_5} + Q_{u_4u_5})$

$+ 3u_6(Q_{xu_4} + u_1 Q_{uu_4} + u_2\, Q_{u_1u_4} + u_3 Q_{u_2u_4} + u_4\, Q_{u_3u_4}$

$+ u_5\, Q_{u_4u_4} + u_2\, Q_{uu_5} + u_3\, Q_{u_1u_5} + u_4 Q_{u_2u_5} + u_5 Q_{u_3u_5} +$

$Q_{xxu_5} + 2u_1 Q_{xuu_5} + 2u_2 Q_{xu_1u_5} + 2u_3 Q_{xu_2u_5}$

$+ 2u_4\, Q_{xu_3u_5} + 2u_5 Q_{xu_4u_5} + u_1^2\, Q_{uuu_5}$

$+ 2u_1u_2 Q_{uu_1u_5} + 2u_1u_3 Q_{uu_2u_5} + 2u_1u_4 Q_{uu_3u_5}$

$+ 2u_1u_5 Q_{uu_4u_5} + u_2^2\, Q_{u_1u_1u_5} + 2u_2u_3 Q_{u_1u_2u_5}$

$+ 2u_2u_4\, Q_{u_1u_3u_5} + 2u_2u_5 Q_{u_1u_4u_5} + u_3^2\, Q_{u_2u_2u_5}$

$+ 2u_3u_5 Q_{u_2u_3u_5} + 2u_3u_5 Q_{u_2u_4u_5} + u_4^2\, Q_{u_3u_3u_5}$

$+ 2u_4u_5 Q_{u_3u_4u_5} + u_5^2\, Q_{u_4u_4u_5}) + u_5^3\, Q_{u_4u_4u_4}$

$+ 3u_5^2\,(Q_{xu_4u_4} + u_1 Q_{uu_4u_4} + u_2\, Q_{u_1u_4u_4} + u_3\, Q_{u_2u_4u_4}$

$+ u_4\, Q_{u_3u_4u_4} + Q_{u_3u_4}) + 3u_5(Q_{xu_3} + u_1\, Q_{uu_3}$

$+ u_2 Q_{u_1u_3} + u_3 Q_{u_2u_3} + u_4 Q_{u_3u_3} + u_2\, Q_{uu_4}$

$+ u_3 Q_{u_1u_4} + u_4\, Q_{u_2u_4} + Q_{xxu_4} + 2u_1 Q_{xuu_4}$

$2u_2\, Q_{xu_1u_4} + 2u_3 Q_{xu_2u_4} + 2u_4 Q_{xu_3u_4}$

$+ u_1^2\, Q_{uuu_4} + 2u_1u_2 Q_{uu_1u_4} + 2u_1u_3 Q_{uu_2u_4}$

$$
\begin{aligned}
&+ 2u_1u_4Q_{uu_3u_4} + u_2^2\ Q_{u_1u_1u_4} + 2u_2u_3Q_{u_1u_2u_4} \\
&+ 2u_2u_4Q_{u_1u_3u_4} + u_3^2\ Q_{u_2u_2u_4} + 2u_3u_4Q_{u_2u_3u_4} \\
&+ u_4^2\ Q_{u_3u_3u_4}) + u_4^3\ Q_{u_3u_3u_3} + 3u_4^2\ (Q_{xu_3u_3} \\
&+ u_1\ Q_{uu_3u_3} + u_2Q_{u_1u_3u_3} + u_3\ Q_{u_2u_3u_3} + Q_{u_2u_3}) \\
&+ 3u_4(Q_{xu_2} + u_1\ Q_{uu_2} + u_2Q_{u_1u_2} + u_3\ Q_{u_2u_2} \\
&+ u_2\ Q_{uu_3} + u_3\ Q_{u_1u_3} + Q_{xxu_3} + 2u_1Q_{xuu_3} \\
&+ 2u_2Q_{xu_1u_3} + 2u_3Q_{xu_2u_3} + u_1^2\ Q_{uuu_3} \\
&+ 2u_1u_2Q_{uu_1u_3} + 2u_1u_3Q_{uu_2u_3} + u_2^2\ Q_{u_1u_1u_3} \\
&+ 2u_2u_3Q_{u_1u_2u_3} + u_3^2Q_{u_2u_2u_3}) + u_3^3\ Q_{u_2u_2u_2} \\
&+ 3u_3^2\ (Q_{xu_2u_2} + u_1\ Q_{uu_2u_2} + u_2Q_{u_1u_2u_2} + Q_{u_1u_2}) \\
&+ 3u_3\ (Q_{xu_1} + u_1Q_{uu_1} + u_2Q_{u_1u_1} + u_2\ Q_{uu_2} + Q_{xxu_2} \\
&+ 2u_1Q_{xuu_2} + 2u_2Q_{xu_1u_2} + u_1^2\ Q_{uuu_2} + 2u_1u_2Q_{uu_1u_2} \\
&+ u_2^3\ Q_{u_1u_1u_2}) + u_2^3\ Q_{u_1u_1u_1} + 3u_2^2\ (Q_{xu_1u_1} \\
&+ u_1\ Q_{uu_1u_1} + Q_{uu_1}) + 3u_2(Q_{xu} + u_1\ Q_{uu} \\
&+ Q_{xxu_1} + 2u_1Q_{xuu_1} + u_1^2\ Q_{uuu_1}) + u_1^3\ Q_{uuu} \\
&+ 3u_1^2\ Q_{xuu} + 3u_1\ Q_{xxu} + Q_{xxx}] = 0 \qquad \text{(G.4)}
\end{aligned}
$$

(G.4) holds for all solutions of the KdVB equation so the coefficients of descending order derivatives of u can be equated to zero. The coefficient of u_7 in (G.4) is

$$
\begin{aligned}
&3c(Q_{xu_5} + u_1\ Q_{uu_5} + u_2\ Q_{u_1u_5} + u_3\ Q_{u_2u_5} \\
&+ u_4\ Q_{u_3u_5} + u_5Q_{u_4u_5} + u_6\ Q_{u_5u_5}) = 0 \qquad \text{(G.5)}
\end{aligned}
$$

As c is not zero and (G.5) is true for all values of u_1, u_2, u_3, u_4 , u_5 and u_6 then

$$
\begin{aligned}
Q_{xu_5} &= Q_{uu_5} = Q_{u_1u_5} = Q_{u_2u_5} = Q_{u_3u_5} \\
&= Q_{u_4u_5} = Q_{u_5u_5} = 0 \qquad \text{(G.6)}
\end{aligned}
$$

(G.6) implies that

$$
Q = A(t)u_5 + B(x, t, u, u_1, u_2, u_3, u_4) \qquad \text{(G.7)}
$$

where A and B are arbitrary functions of the arguments shown.

Also $$Q_{iju_5} = Q_{ju_5} = 0 \tag{G.8}$$

$$(i, j = x, u, u_1, u_2, u_3, u_4)$$

(G.8) ensures that the coefficients of u_6^3 and u_6^2 are zero and the coefficient of u_6 reduces to

$$3c\,(B_{xuu_4} + u_1 B_{uu_4} + u_2\, B_{u_1u_4} + u_3\, B_{u_2u_4}$$
$$+\ u_4\, B_{u_3u_4} + u_5\, B_{u_4u_4}) = 0 \tag{G.9}$$

As (G.9) holds for all values of u_1, u_2, u_3, u_4 and u_5 then

$$\begin{aligned} B_{xu_4} &= B_{uu_4} = B_{u_1u_4} = B_{u_2u_4} = B_{u_3u_4} \\ &= B_{u_4u_4} = 0 \end{aligned} \tag{G.10}$$

which means (G.7) can now be written as

$$Q = A(t)u_5 + B(t)u_4 + C(x, t, u, u_1, u_2, u_3) \tag{G.11}$$

and $$Q_{iju_4} = Q_{ju_4} = 0 \tag{G.12}$$

$$(i, j = x, u, u_1, u_2, u_3)$$

(G.12) makes the coefficients of u_5^3 and u_5^2 zero. (G.11) and (G.12) mean that the coefficient of u_5 in the invariance condition (G.4) becomes

$$\begin{aligned} &\frac{dA}{dt} - 5au_1A + 3c(C_{xu_3} + u_1\, C_{uu_3} \\ &+\ u_2\, C_{u_1u_3} + u_3\, C_{u_2u_3} + u_4\, C_{u_3u_3}) = 0 \end{aligned} \tag{G.13}$$

(G.13) is true for all values of u_1, u_2, u_3 and u_4 and as c is not zero then

$$C_{u_1u_3} = C_{u_2u_3} = C_{u_3u_3} = 0 \tag{G.14}$$

(G.14) means that (G.11) becomes

$$Q = Au_5 + Bu_4 + C(x,t,u)u_3 + D(x,t,u,u_1,u_2) \tag{G.15}$$

Also $3cC_{uu_3} - 5aA = 0$ or $C_{uu_3} = \frac{5aA}{3c}$ which on integrating gives

$$C(x,t,u) = \frac{5aAu}{3c} + C(x,t) \tag{G.16}$$

While $3cC_{xu_3} + \frac{dA}{dt} = 0$ on integrating gives

$$C(x,t) = -\left(\frac{x}{3c}\right)\frac{dA}{dt} + C(t) \tag{G.17}$$

(G.15) to (G.17) mean that

$$Q = Au_5 + Bu_4 + \left\{\frac{5aAu}{3c} - \left(\frac{x}{3c}\right)\frac{dA}{dt} + C\right\} u_3 + D(x,t,u,u_1,u_2) \tag{G.18}$$

(G.18) implies that the coefficients of u_4^3 and u_4^2 are zero. The coefficient of u_4 in (G.4) is

$$\begin{aligned}\frac{dB}{dt} - 4au_1 B - 15au_2 B + b\left[-\frac{2}{3c}\frac{dA}{dt} + \frac{10au_1 A}{3c}\right] \\ + 3c\,(D_{xu_2} + u_1 D_{u2} + u_2 D_{u_1u_2} \\ + u_3 D_{u_2u_2} + \frac{5au_2 A}{3c}) = 0\end{aligned} \tag{G.19}$$

(G.19) is true for all u_1, u_2 and u_3 which means $D_{u_2u_2} = 0$ and

$$D(x,t,u,u_1,u_2) = D(x,t,u,u_1)u_2 + E(x,t,u,u_1) \tag{G.20}$$

Also $D_{u_1u_2} = \frac{10aA}{3c}$ which on integrating and by (G.20) means that

$$D(x,t,u,u_1) = \frac{10aAu_1}{3c} + D(x,t,u) \tag{G.21}$$

While $D_{uu_2} = \frac{1}{3c}\left(4aB - \frac{10abA}{3c}\right)$ which on integrating with the help of (G.21) gives

$$D(x,t,u) = \frac{2au}{3c}\left(2B - \frac{5bA}{3c}\right) + D(x,t) \qquad (G.22)$$

Finally $D_{xu_2} = \frac{1}{3c}\left(\frac{2b}{3c}\frac{dA}{dt} - \frac{dB}{dt}\right)$ which on integrating with the help of (G.22) gives

$$D(x,t) = \frac{x}{3c}\left(\frac{2b}{3c}\frac{dA}{dt} - \frac{dB}{dt}\right) + D(t) \qquad (G.23)$$

(G.20) to (G.23) means that (G.18) becomes

$$Q = Au_5 + Bu_4 + \left\{\frac{5auA}{3c} - \frac{x}{3c}\frac{dA}{dt} + C\right\} u_3$$

$$+ \left\{\frac{10au_1A}{3c} + \frac{2au}{3c}\left(2B - \frac{5bA}{3c}\right) + \frac{x}{3c}\left(\frac{2b}{3c}\frac{dA}{dt} - \frac{dB}{dt}\right) + D\right\} u_2 + E(x,t,u,u_1) \qquad (G.24)$$

(G.24) ensures that the coefficients of u_3^3 and u_3^2 in the invariance condition (G.4) are zero. The coefficient of u_3 in (G.4) is

$$\begin{aligned} &\frac{4au}{3c}\frac{dA}{dt} - \frac{x}{3c}\frac{d^2A}{dt^2} + \frac{dC}{dt} - 3au_1\left(\frac{5auA}{3c} - \frac{x}{3c}\frac{dA}{dt} + C\right) \\ &- 10au_2B + 2b\left[\frac{1}{3c}\left(\frac{2b}{3c}\frac{dA}{dt} - \frac{dB}{dt}\right) + \frac{2au_1}{3c}\left(2B - \frac{5bA}{3c}\right) + \frac{10au_2A}{3c}\right] \\ &+ 3c\left[E_{xu_1} + u_1E_{uu_1} + u_2E_{u_1u_1} + \frac{2au_2^2}{3c}\left(2B - \frac{5bA}{3c}\right)\right] = 0 \end{aligned}$$

(G.25)

(G.25) is true for all values of x, u, u_1 and u_2 while A, B and C are all arbitrary functions of t which means that

$$E_{u_1u_1} = \frac{2a}{3c}\left(3B - \frac{5bA}{3c}\right) \qquad (G.26)$$

(G.26) implies the $E_{u_1u_1}$ is a function of t only, so that $E_{xu_1u_1} = E_{uu_1u_1} = E_{u_1u_1u_1} = 0$

Integrating (G.26) gives

$$E(x,t,u,u_1) = \frac{au_1^2}{3c}\left(3B - \frac{5bA}{3c}\right) + E(x,t,u)u_1 + F(x,t,u) \qquad (G.27)$$

Also $$E_{uu_1} = \frac{1}{3c}\left[\frac{4ab}{3c}\left(\frac{5bA}{3c} - 2B\right) + \frac{5a^2uA}{c} - \frac{ax}{c}\frac{dA}{dt} + 3aC\right] \qquad (G.28)$$

Integrating (G.28) with the help of (G.27) gives

$$E(x,t,u) = \frac{1}{3c}\left[\frac{4abu}{3c}\left(\frac{5bA}{3c} - 2B\right) + \frac{5a^2u^2A}{2c} - \frac{axu}{c}\frac{dA}{dt} + 3auC + E(x,t)\right]$$

(G.29)

Finally

$$\frac{4au}{3c}\frac{dA}{dt} - \frac{x}{3c}\frac{d^2A}{dt^2} + \frac{dC}{dt} + \frac{2b}{3c}\left(\frac{2b}{3c}\frac{dA}{dt} - \frac{dB}{dt}\right)$$
$$+ 3c\,E_{xu_1} = 0 \qquad (G.30)$$

From (G.29)

$$E_{xu_1} = -\frac{au}{3c^2}\frac{dA}{dt} + E(x,t)_x \qquad (G.31)$$

Substituting (G.31) into (G.30) gives

$$3c\left\{-\frac{au}{3c^2}\frac{dA}{dt} + E(x,t)_x\right\} + \frac{4au}{3c}\frac{dA}{dt} - \frac{x}{3c}\frac{d^2A}{dt^2}$$
$$+ \frac{dC}{dt} + \frac{2b}{3c}\left(\frac{2b}{3c}\frac{dA}{dt} - \frac{dB}{dt}\right) = 0 \qquad (G.32)$$

(G.32) is true for all values of u, which means

$$\frac{a}{3c}\frac{dA}{dt} = 0 \qquad (G.33)$$

(G.33) means $$\frac{dA}{dt} = \frac{d^2A}{dt^2} = 0 \qquad (G.34)$$

as a is nonzero.

Integrating (G.34) gives

$$A = k_1 \tag{G.35}$$

where k_1 is an arbitrary constant.

From (G.32) and (G.34) we obtain

$$E(x,t)_x = \frac{1}{3c}\left(\frac{2b}{3c}\frac{dB}{dt} - \frac{dC}{dt}\right)$$

which on integrating gives

$$E(x,t) = \frac{x}{3c}\left(\frac{2b}{3c}\frac{dB}{dt} - \frac{dC}{dt}\right) + E(t) \tag{G.36}$$

(G.27), (G.29), (G.35) and (G.36) mean that (G.24) becomes

$$\begin{aligned} Q = {} & k_1u_5 + Bu_4 + \left\{\frac{5ak_1u}{3c} + C\right\} u_3 + \\ & \left\{\frac{10ak_1u}{3c} + \frac{2au}{3c}\left(2B - \frac{5bk_1}{3c}\right) - \frac{x}{3c}\frac{dB}{dt} + D\right\} u_2 + \frac{au_1^2}{3c} \\ & \left(3B - \frac{5bk_1}{3c}\right) + \Bigg\{\frac{4abu}{9c^2}\left(\frac{5bk_1}{3c} - 2B\right) + \frac{5a^2k_1u^2}{6c^2} + \frac{auC}{c} \\ & + \frac{x}{3c}\left(\frac{2b}{3c}\frac{dB}{dt} - \frac{dC}{dt}\right) + E\Bigg] u_1 + F(x,t,u) \end{aligned} \tag{G.37}$$

(G.37) ensures that the coefficient of u_2^3 in (G.4) is zero and the coefficient of u_2^2 reduces to

$$\frac{5bk_1}{3c} - B = 0 \tag{G.38}$$

(G.38) means that

$$\frac{dB}{dt} = \frac{d^2B}{dt^2} = 0 \tag{G.39}$$

(G.39) means that the coefficient of u_2 in the invariance condition (G.4) reduces to

$$\frac{dD}{dt} - \frac{10a^2k_1u_1^2}{3c} - 2au_1\left[\frac{10ak_1u_1}{3c} + \frac{2au}{3c}\left(2B - \frac{5bk_1}{3c}\right) + D\right]$$
$$+b\left[-\frac{2}{3c}\frac{dC}{dt} + 2u_1\left\{\frac{4ab}{9c^2}\left(\frac{5bk_1}{3c} - 2B\right) + \frac{10a^2k_1u}{6c^2} + \frac{aC}{c}\right\}\right]$$
$$+c\left[u_1^2\left(\frac{20a^2k_1}{3c}\right) + u_1 \ F_{uu} + F_{xu}\right] = 0 \qquad (G.40)$$

(G.40) is true for all values of u_1^2 which means

$$-\frac{10a^2k_1}{3c} = 0 \qquad (G.41)$$

As a is nonzero (G.41) means that

$$k_1 = 0 \qquad (G.42)$$

and from (G.38) that

$$B = 0 \qquad (G.43)$$

(G.42) and (G.43) reduce (G.37) to

$$Q = Cu_3 + Du_2 + \left\{\frac{auC}{c} - \frac{x}{3c}\frac{dC}{dt} + E\right\} u_1 + F(x,t,u) \qquad (G.44)$$

(G.44) with suitable redesignation of the arbitrary functions C, D, E and F is identical to (F.19). Furthermore, substitution of (G.44) into the invariance condition (G.4) gives

$$\left(\frac{dD}{dt}\right)u_2 + \left\{\frac{au}{c}\frac{dC}{dt} - \frac{x}{3c}\frac{d^2C}{dt^2} + \frac{dE}{dt}\right\}u_1 + F_t$$
$$+ au\left(F_x - \frac{u_1}{3c}\frac{dC}{dt}\right) + au_1 \ F - 2au_1u_2D$$
$$+ b\left[2u_2\left(\frac{au_1C}{c} - \frac{1}{3c}\frac{dC}{dt}\right) + u_1^2 F_{uu} + 2u_1F_{xu} + F_{xx}\right]$$
$$+ c\left[3u_2(F_{xu}+u_1D_{uu}) + u_1^3 F_{uuu} + 3u_1^2 F_{xuu} + F_{xxx}\right] = 0 \qquad (G.45)$$

(G.45) is also identical to the invariance condition (F.21) in Appendix F, again with suitable redesignation of arbitrary functions. (G.44) and (G.45) mean that the final form of the 5th order L-B symmetry reduces to

$$Q = k_1 u_3 + \left(\frac{k_1 b}{c}\right) u_2 + \left(\frac{k_1 a}{c}\right) u u_1 + \left(k_3 - a k_2 t\right) u_1 + k_2$$

where k_1, k_2 and k_3 are arbitrary constants.

APPENDIX H

Similarity solutions for the variable coefficient KdV equation

$$u_t + \alpha t^n u u_x + \beta t^m u_{xxx} = 0 \tag{H1}$$

where m and n have arbitrary values

$$V^u - u_t V^t - u_x V^x = 0 \tag{H2}$$

Using the infinitesimal generator

$$X_1 = \left(\frac{\alpha t^{n+1}}{n+1}\right)\partial_x + \partial_u \tag{H3}$$

(H3) implies that $V^x = \dfrac{\alpha t^{n+1}}{n+1}$ $V^u = 1$ and $V^t = 0$ so the subsidiary equations of (H2) are

$$\frac{dt}{0} = \frac{dx}{\left(\frac{\alpha t^{n+1}}{n+1}\right)} = \frac{du}{1} \tag{H4}$$

$dt = 0$ means the similarity variable

$$t = \zeta = \text{constant} \tag{H5}$$

(H4) and (H5) mean that

$$du = \frac{dx}{\left(\frac{\alpha \zeta^{n+1}}{n+1}\right)}$$

which on integrating gives

$$u = \left(\frac{\alpha \zeta^{n+1}}{n+1}\right)^{-1} x + f(\zeta) \tag{H6}$$

as the similarity solution.

On substituting (H6) into (H1) we get the ODE in ζ

$$\frac{df(\zeta)}{d\zeta} = -\frac{(n+1)f(\zeta)}{\zeta} \tag{H7}$$

which on solving gives

$$f(\zeta) = \frac{k}{\zeta^{n+1}} \tag{H8}$$

(H8) and (H6) give the solution

$$u = \frac{(n+1)x + \alpha k}{\alpha t^{n+1}}$$

Substituting (H9) into H(1) verifies that (H9) is a solution of the variable coefficient KdV equation.