DYNAMICS AND NUMERICS OF GENERALISED EULER EQUATIONS

by

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Abstract

This thesis is concerned with the well-posedness, dynamical properties and numerical treatment of the generalised Euler equations on the Bott-Virasoro group with respect to the general H^k metric , $k \geq 2$.

The term "generalised Euler equations" is used to describe geodesic equations on Lie groups, which unifies many differential equations and has found many applications in such as hydrodynamics, medical imaging in the computational anatomy, and many other fields. The generalised Euler equations on the Bott-Virasoro group for $k = 0, 1$ are well-known and intensively studied—the Korteweg-de Vries equation for $k = 0$ and the Camassa-Holm equation for $k = 1$. Unlike these, the equations for $k \geq 2$, which we call the modified Camassa-Holm (mCH) equation, is not known to be integrable. This distinction motivates the study of the mCH equation.

In this thesis, we derive the mCH equation and establish the short time existence of solutions, the well-posedness of the mCH equation, long time existence, the existence of the weak solutions, both on the circle S and $\mathbb{R}$, and three conservation laws, show some quite interesting properties, for example, they do not lead to the blowup in finite time, unlike the Camassa-Holm equation.

We then consider two numerical methods for the modified Camassa-Holm equation: the particle method and the box scheme. We prove the convergence result of the particle method. The numerical simulations indicate another interesting phenomenon: although mCH does not admit blowup in finite time, it admits solutions that *blow up* (which means their maximum value becomes infinity) at *infinite time*, which we call *weak blowup*. We study this novel phenomenon using the method of matched asymptotic expansion. A whole family of self-consistent blowup profiles is obtained. We propose a mechanism by which the actual profile is selected that is consistent with the simulations, but the mechanism is only partly supported by the analysis.

We study the four particle systems for the mCH equation finding numerical evidence both for the non-integrability of the mCH equations and for the existence of the fourth integral. We also study the higher dimensional case and obtain the short time existence and well-posedness for the generalised Euler equation in the two dimension case.

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Xingyou Philip Zhang, July 11, 2008

Contents

List of Spaces

Here is the list of various spaces in Chapter 1 − Chapter 7.

The notation	Its meaning
$B(X,Y)$	The space of all bounded linear operators from X to Y
$C([a,b],X)$	The set of all continuous functions from $[a,b]$ to X
$C^\alpha(\Omega)$	Hölder spaces defined in Section 2.1
$C^{1,c}(\Omega)$	The set of all continuous functions with compact supports in Ω
	and continuous first order derivatives in Ω
$\mathrm{Diff}(S)$	The set of all diffeomorphisms from S to S preserving the orientation
$\mathcal{D}^s(S)$	The set of all H^s diffeomorphisms on S
$\widehat{\mathcal{D}}(S)$	Bott-Virasoro group defined in Section 3.1
$G(X,1,\beta)$	The set of quasi-m-accretive operators in X defined in Section 2.1.2
$GL(V)$	The set of all invertible linear operators from V to V
$H^s(\mathbb{R}^n)$	$W^{s,2}(\mathbb{R}^n)$
$H^s(S)$	The s-th order Sobolev space $W^{s,2}(S)$
$H^\infty(S)$	$\bigcap_{s=1}^{\infty} H^s(S)$
$L^p(\Omega)$	The set of all measurable functions u with $\int_\Omega \lvert u\rvert^p \mathrm{d}x < \infty$
$L^\infty(\Omega)$	The set of essentially bounded measurable functions on Ω
$\mathbb{R}^1$	The standard one dimensional Euclidean space
$\mathbb{R}^n$	The standard n dimensional Euclidean space
S	The unit circle $\mathbb{R}^1/2\pi\mathbb{Z}$
$SO(n)$	The group of special orthogonal transforms in $\mathbb{R}^{n+1}$
$W^{k,p}(\Omega)$	Sobolev spaces defined in Section 2.1