

Copyright is owned by the Author of the thesis. Permission is given for a copy to be downloaded by an individual for the purpose of research and private study only. The thesis may not be reproduced elsewhere without the permission of the Author.

DYNAMICS AND NUMERICS OF GENERALISED EULER EQUATIONS

by

Xingyou (Philip) ZHANG

A Thesis Submitted to Massey University
In Partial Fulfillment of the Requirements
For the Degree of Ph.D in Mathematics

Palmerston North, New Zealand

2008

©Xingyou (Philip) Zhang 2008

Abstract

This thesis is concerned with the well-posedness, dynamical properties and numerical treatment of the generalised Euler equations on the Bott-Virasoro group with respect to the general H^k metric, $k \geq 2$.

The term “generalised Euler equations” is used to describe geodesic equations on Lie groups, which unifies many differential equations and has found many applications in such as hydrodynamics, medical imaging in the computational anatomy, and many other fields. The generalised Euler equations on the Bott-Virasoro group for $k = 0, 1$ are well-known and intensively studied—the Korteweg-de Vries equation for $k = 0$ and the Camassa-Holm equation for $k = 1$. Unlike these, the equations for $k \geq 2$, which we call the modified Camassa-Holm (mCH) equation, is not known to be integrable. This distinction motivates the study of the mCH equation.

In this thesis, we derive the mCH equation and establish the short time existence of solutions, the well-posedness of the mCH equation, long time existence, the existence of the weak solutions, both on the circle S and \mathbb{R} , and three conservation laws, show some quite interesting properties, for example, they do not lead to the blowup in finite time, unlike the Camassa-Holm equation.

We then consider two numerical methods for the modified Camassa-Holm equation: the particle method and the box scheme. We prove the convergence result of the particle method. The numerical simulations indicate another interesting phenomenon: although mCH does not admit blowup in finite time, it admits solutions that *blow up* (which means their maximum value becomes infinity) at *infinite time*, which we call *weak blowup*. We study this novel phenomenon using the method of matched asymptotic expansion. A whole family of self-consistent blowup profiles is obtained. We propose a mechanism by which the actual profile is selected that is consistent with the simulations, but the mechanism is only partly supported by the analysis.

We study the four particle systems for the mCH equation finding numerical evidence both for the non-integrability of the mCH equations and for the existence of the fourth integral. We also study the higher dimensional case

and obtain the short time existence and well-posedness for the generalised Euler equation in the two dimension case.

Acknowledgements

Firstly, I would like to express my sincere thanks to my main supervisor, Prof. Robert McLachlan, who has spent many, many hours over the last four years enthusiastically and patiently teaching me the theory of dynamical systems, geometric integration, how to implement numerically the various mathematics ideas, and how to improve my English! I have always appreciated your friendly manner and encouragement to try new ideas and attend international conferences.

Special thanks also go to my supervisor Dr. Matt Perlmutter for your encouragement, your investment of time and patiently explaining to me when your were bombed with my geometry questions which are quite simple and may be even stupid to you.

I would like to acknowledge all my PhD fellows, especially Dion and Brett, for making my PhD experience at Massey much more rich and pleasant. Many thanks to the IFS staff at Massey who are friendly and helpful, especially to Kee Teo, for helping me to navigate my tutorship and other intricacies of life at Massey.

I am grateful to acknowledge that this work has been supported by NZ-IMA thematic PhD scholarship. Additional travel support was provided by Education Ministry of New Zealand (to enable me to visit Chinese Academy of Sciences, Beijing), NZIMA travel funding and IFSGRF of Massey University (to enable me to visit Newton Institute of Cambridge University and to attend SciCADE07 in France), for which I am thankful.

A special thanks to my family, especially my wife, Wendy, and my son, William, for the support and freedom to pursue my interests. I dedicate this thesis to the memory of my father, who, together with my mother, brought me up under the very hard condition, kept encouraging me to study for my interests but passed away two years ago.

Xingyou Philip Zhang, July 11, 2008

Contents

Abstract	i
Acknowledgements	iii
List of Spaces	vii
1 Introduction	1
1.1 Euler Fluid Equations: A Brief History	2
1.2 Arnold's Viewpoint	3
1.2.1 Shallow Water Equations	4
1.2.2 Abstract Euler-Poincaré Equations	5
1.3 Particle solutions	6
1.4 Numerical Approaches	8
1.4.1 Particle Method	8
1.4.2 Box Scheme	8
1.4.3 Multi-symplectic Methods	9
1.5 Applications	10
1.6 Thesis Preview	10
2 Preliminary Tools	14
2.1 PDE Basics	14
2.1.1 Sobolev Spaces	15
2.1.2 Kato Theory	19
2.2 Riemannian Geometry	21
2.3 Lie Groups	23
2.3.1 Lie Group and Adjoint Representation	24
2.3.2 Co-adjoint Representation of a Lie Group	26
2.3.3 Invariant Metrics of Lie Groups	26
2.3.4 Applications to Hydrodynamics	29

3	Well-posedness	30
3.1	Derivation of the Equations	30
3.1.1	Bott-Virasoro Group	30
3.1.2	Derivation of the Equations	31
3.2	Local Well-posedness	32
3.2.1	Conservation Laws	37
3.3	Global Well-posedness	38
3.3.1	Extra Properties for $a = 0$	40
3.4	Weak Solutions for $a = 0$	44
3.5	The Whole Real Line Case	49
3.6	Remarks on the Generalisations	51
3.7	Conjugate Points and Beyond	51
3.8	Conclusions	55
4	Numerics	57
4.1	Particle methods	57
4.2	Box Scheme	65
4.3	Conclusions	67
5	Asymptotics	69
5.1	Introduction	69
5.2	Asymptotic PDE	70
5.3	Are Steady Solutions Stable?	75
5.3.1	Upstream Boundary Conditions for (5.12)	75
5.3.2	Around the General Steady Solution (5.8)	79
5.3.3	Around the Limit Steady Solution (5.9)	83
5.4	The Family of Steady Solutions	85
5.5	Remarks on the Camassa-Holm Equation	92
5.6	Conclusions	93
6	Four Particle Systems	95
6.1	Motivation	95
6.2	Lyapunov Exponents	96
6.3	How to Compute Them?	96
6.4	Four Particle Systems	98
6.5	Conclusions	100
7	Higher Dimensional Case	105
7.1	Introduction	105
7.2	Local Well-posedness	105

8	Future Work	112
8.1	Stability of the Asymptotic Solutions	112
8.2	Positivity of Solutions	114
8.3	Higher Dimensional Case	114
A	Properties of the Green's Function	115
B	Multi-symplectic Formulation	119
B.1	Multi-symplectic Geometry	119
B.2	Formulation for mCH	129
	Bibliography	133

List of Spaces

Here is the list of various spaces in Chapter 1 – Chapter 7.

The notation	Its meaning
$B(X, Y)$	The space of all bounded linear operators from X to Y
$C([a, b], X)$	The set of all continuous functions from $[a, b]$ to X
$C^\alpha(\Omega)$	Hölder spaces defined in Section 2.1
$C^{1,c}(\Omega)$	The set of all continuous functions with compact supports in Ω and continuous first order derivatives in Ω
$\text{Diff}(S)$	The set of all diffeomorphisms from S to S preserving the orientation
$\mathcal{D}^s(S)$	The set of all H^s diffeomorphisms on S
$\widehat{\mathcal{D}}(S)$	Bott-Virasoro group defined in Section 3.1
$G(X, 1, \beta)$	The set of quasi- m -accretive operators in X defined in Section 2.1.2
$GL(V)$	The set of all invertible linear operators from V to V
$H^s(\mathbb{R}^n)$	$W^{s,2}(\mathbb{R}^n)$
$H^s(S)$	The s -th order Sobolev space $W^{s,2}(S)$
$H^\infty(S)$	$\bigcap_{s=1}^{\infty} H^s(S)$
$L^p(\Omega)$	The set of all measurable functions u with $\int_{\Omega} u ^p dx < \infty$
$L^\infty(\Omega)$	The set of essentially bounded measurable functions on Ω
\mathbb{R}^1	The standard one dimensional Euclidean space
\mathbb{R}^n	The standard n dimensional Euclidean space
S	The unit circle $\mathbb{R}^1/2\pi\mathbb{Z}$
$SO(n)$	The group of special orthogonal transforms in \mathbb{R}^{n+1}
$W^{k,p}(\Omega)$	Sobolev spaces defined in Section 2.1