The Quotient Between Length and Multiplicity

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Abstract

This dissertation examines the finiteness of the algebraic invariants $n_A(M)$ and $\theta_A(M)$. These invariants, based on the ratio of length and multiplicity and the ratio of Loewy length and multiplicity respectively, are studied in general and under certain conditions. The finiteness of $\theta_A(M)$ is established for a large class of algebraic structures. $n_A(M)$ is shown to be finite in the low dimensional case as well as when we restrict our attention to special sets of ideals. Also considered in this dissertation are equivalent conditions for the local case to be bounded by the graded case when evaluating $n_A(M)$.

0.1 Dedication

This dissertation is dedicated to the memory of Professor Wolfgang Vogel. Professor Vogel inspired my leap into the field of mathematics and his constant enthusiasm and encouragement kept me going. During my three and a half years working with him, Professor Vogel was always positive and optimistic about mathematics and life in general. He was always available to talk with me and he introduced me to people from all over the world. I feel truly lucky to have had the opportunity to know Professor Vogel and to work with him. His contribution to mathematics and his ability to inspire his students are just two of the legacies he has left behind. Wolfgang Vogel was a great mathematician and a truly great man, he will be deeply missed.

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Contents

Chapter 1

Introduction

1.1 Motivation

The purpose of this dissertation is to examine the quotient between length and multiplicity. Both of these algebraic concepts have been known about for a long time. The study of the relationship between these two basic notions leads to several interesting theories in Commutative Algebra. For instance, the theory of Cohen-Macaulay local rings stems from this study as does the theory of Buchsbaum rings. Cohen-Macaulay local rings are defined to be local rings, A, such that $\ell(A/\mathbf{q}) = e(\mathbf{q})$ for all parameter ideals $\mathbf{q}$ in A.

During the search for a correct intersection multiplicity for use in Bezout's Theorem, D. Buchsbaum posed the following problem:

Let A be a local ring and let $\mathbf{q}$ be a parameter ideal of A. Is the difference

$$\ell(A/\mathbf{q}) - e(\mathbf{q})$$

independent of $\mathbf{q}$?

W. Vogel gave a counter-example to this problem [29] and starting from this counter-example J. Stückrad and W. Vogel have founded the theory of Buchsbaum modules (see the Introduction of [24]) in the seventies. This theory is considered to be one of the major developments in the field of ring theory in the last 30 years.

Now comes the question of why the study of the quotient between the length and the multiplicity is of interest. The motivation behind this question comes primarily from [9] by again analysing several of the generalisations of Bezout's Theorem given by W. Vogel and his colleagues (Stückrad, Flenner and others). In [30] W. Vogel asked whether the degree of the intersection of two projective varieties can be bounded above using the degree of these varieties. This finally leads to the consideration of the ratio

$$\ell(A/\mathbf{q})/e(\mathbf{q}),$$

where $\mathbf{q}$ is a parameter ideal.

In order to move from the study of a ratio to the study of an algebraic invariant the authors of [16] made the following definition:

$$n(A) = sup\{\ell_A(A/\mathbf{q})/e(\mathbf{q}; A)\}$$

where the supremum is taken over all primary ideals $\mathbf{q}$ of A. Some initial steps towards establishing the finitness of this invariant were made but for the general case no statement could be made regarding the finiteness.

The next steps in the examination of the above ratio and its corresponding algebraic invariant appear in [25]. Here the definition is extended to the module case and the restriction to ideals generated by systems of parameters is also considered. Theorems 1 and 2 of [25] give strong conditions on the finiteness of the invariant and its connection to the quasi-unmixedness of the algebraic structure being considered. It has also been proved that in the graded case the invariant constructed using homogeneous parameter ideals is always finite. While this answers the question of finiteness for a large class of algebraic structures the non-parameter cases and the non-graded parameter case are still open problems. It should be noted here that C. Lech has already obtained a lower bound for $n(A)$ in the local, non-parameter case ([14]).

This brings the consideration of this ratio to the present day and this dissertation. The aim of this piece of work is to answer some of these open problems and to expand the pool of knowledge about the related invariants. While there are still open problems in this area a deeper understanding of the structure of these problems has been attained. This coupled with the introduction and examination of a new invariant based on the ratio of length and multiplicity forms the basis of this dissertation.

1.2 Main Results

Before beginning on these results it should be noted that the reference number for each theorem has the following format: Chapter. Section. Placing within the section.

The main results in this dissertation can be grouped into two main areas. Those dealing with the length of an ideal and $n_A(M)$, and those dealing with the Loewy length of an ideal and $\theta_A(M)$. Both the length and the Loewy length are introduced and discussed in chapter two.

Before describing the results based on $n_A(M)$ and $\theta_A(M)$ there is a more general result that should be mentioned. This result comes from the attempt to bound $n_A(M)$ in the local case by $n_{G_{\mathbf{m}}(A)}(G_{\mathbf{m}}(M))$, where $G_{\mathbf{m}}(M)$ is the associated graded module of M over the associated graded ring $G_{\mathbf{m}}(A)$ of A. We show that under certain conditions the finitness of these invariants are equivalent. For simplicity we only give the result in terms of local rings.

Theorem 1.2.1 *Let A be a local ring. Then $n_A < \infty$ if and only if there exists $\alpha_A < \infty$ such that*

$$\frac{e(in_M(\mathbf{q}); G_{\mathbf{m}}(A))}{e(\mathbf{q}; A)} \leq \alpha_A$$

for all $\mathbf{m}$-primary ideals $\mathbf{q}$ of A, and $n_{G_{\mathbf{m}}(A)} < \infty$

For the case of a quotient of a polynomial ring R modulo a homogeneous ideal I we can characterize the finitness of $n_{R/I}$ in terms of boundness of the ratio between certain multiplicities.

Theorem 1.2.2 *Let $R = k[X_1, \ldots, X_n]$ and let $I \subset R$ be a homogeneous quasi-unmixed ideal of R. Let $lt(I)$ denote the leading term ideal of I. Then*

(i) $n_{R/I} \geq n_{R/lt(I)}$,

(ii) There is $\alpha_{R/I} < \infty$ such that

$$\frac{e(lt(\mathbf{q}+I); R/lt(I))}{e(\mathbf{q}; R/I)} \leq \alpha_{R/I}$$

for all $\mathbf{m}_R$-primary ideals $\mathbf{q}$ if and only if $n_{R/I} < \infty$. In this case

$$n_{R/I} \leq \alpha_{R/I} n_{R/lt(I)}.$$

Under what conditions is the invariant $n_A(M)$ finite? This question provides the motivation for the next two results. The first of these looks at the low dimensional case.

Theorem 1.2.3 *Let A be a local ring such that $A/\mathbf{m}$ is infinite. Let M be a quasi-unmixed A-module such that $dimA = dimM =: d$. Now if $dimA \leq 3$ then $n_A(M)$is finite.*

When $dim A \geq 4$ we still have no answer. However, if we restrict ourselves to the class of parameter ideals $\mathbf{q}$, then we get a positive solution:

Theorem 1.2.4 *Let $(A, \mathbf{m})$ be a local ring. Then*

$$sup\{\ell_A(A/\mathbf{q})/e(\mathbf{q}; A) | \mathbf{q} \in \mathcal{S}\} < \infty$$

where $\mathcal{S} := \{\mathbf{q} = (x_1, \ldots, x_d) \subseteq \mathbf{m}\}$ such that $x_1^*, \ldots, x_d^*$ form a system of parameters of $G_{\mathbf{m}}(A)$.

The theta invariant, $\theta_A(M)$, is bounded above by $n_A(M)$ and strongly connected to $n_A(M)$. Easy examples show that $\theta_A(M) << n_A(M)$. Hence the following result is stronger than Theorem 2 of [25].

Theorem 1.2.5 *When M is a (graded) A-module, where A is a local ring (or a graded k-algebra), then if $\theta_A(M)$ is finite it follows that M is a quasi-unmixed module.*

The finiteness of $\theta_A(M)$ would strongly support the conjecture that $n_A(M)$ is also finite. The main result of Chapter 5 gives a positive answer for this problem for a large class of rings.

Theorem 1.2.6 *Assume that A is a local ring containing an infinite field. Let M be a quasi-unmixed A-module with $dimM = dimA$. Then $\theta_A(M)$ is finite.*

These results go a long way towards establishing a deeper understanding of the nature of the invariants $n_A(M)$ and $\theta_A(M)$. The proofs of all of the above theorems are included, in detail, in the following chapters.

1.3 A conjecture

The motivation for studying this topic stems from trying to prove the following conjecture (see [25], p. 14):

Conjecture 1.3.1 *Let A be a local ring and let M be a quasi-unmixed A-module. Then $n_A(M)$ is finite.*

While this conjecture remains unproven the main results in this dissertation, as well as earlier work on this topic, have shown that this conjecture is true for a large class of algebraic structures. That is, strong support for the above claim has been shown. Consequently we have extended knowledge in this area and provided motivation for ongoing research on this topic.

Chapter two introduces the notation and definitions we will require. It also states some of the well-known results that are needed to prove some of the results in chapters three, four and five.

Chapter three considers the graded case. The quasi-unmixed property is studied with respect to its preservation when passing to the graded case. We also show that we can restrict ourselves to the case of integral domains and find conditions for $n_{G_{\mathbf{m}}(A)}(G_{\mathbf{m}}(M))$ to bound $n_A(M)$. Also considered are initial ideals and the Bayer deformation.

Chapter four looks at $n_A(M)$ and its finiteness. We begin with the low dimensional case. Finally we consider the restriction to parameter ideals and obtain a new proof of Theorem 2 of [25].

Chapter five examines the Loewy length and the invariant $\theta_A(M)$. The finiteness of this invariant is considered and some necessary conditions are constructed. Finally, by inserting additional assumptions, the finiteness of $\theta_A(M)$ is established.

Chapter 2

Preliminary Results

The purpose of this chapter is to establish the notation that will be used in the rest of this dissertation. As well as this there are a number of well known results which are referred to quite frequently in the following chapters. These, along with a reference indicating where a proof may be found, are also included in this chapter. For the sake of completion definitions of most of the key terms are also stated. The reader should note that while most of the terminology used is defined here a certain level of understanding of abstract algebra is assumed. For those interested in the definitions of terms not stated here please refer to [23], [8] or [15].

It should be noted here that the symbols $\subset$,$\supset$, $<$ and $>$ will be used to denote a strict inclusion or a sharp inequality respectively. When equality can occur the symbols $\subseteq$,$\supseteq$,$\leq$ and $\geq$ will be used.

2.1 Notation and Key Definitions

Throughout this document $(A, \mathbf{m})$ will denote a local ring A (usually Noetherian) with maximal ideal $\mathbf{m}$ and residue class field $A/\mathbf{m} \cong k$ unless stated otherwise. While we have some results that are concerned only with such local rings, in general we will be dealing with A-modules which will usually be denoted by M. M will always be a unitary, finitely generated module over A. The main results of the early papers on this topic (eg. [25]) and the results in chapters three, four and five tend to deal with two separate cases: the local case where A is as defined above and the graded case. In the graded case A will denote a graded k-algebra with the maximal homogeneous ideal $\mathbf{m}$. Here a graded k-algebra means a standard graded k-algebra, that is

$$A := A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

where $A_0 = k$ is a field and A is finitely generated by A_1. Whenever the graded case is different to the local case it will be clearly stated. It is also assumed, unless stated otherwise, that $A/\mathbf{m}$ is always an infinite field.

2.1.1 Dimension

Denote by $Spec(A)$ the set of all prime ideals of A. We now make the following definition:

Definition 2.1.1 *The supremum of the lengths r of all strictly decreasing chains*

$$\mathbf{p}_0 \supset \mathbf{p}_1 \supset \cdots \supset \mathbf{p}_r$$

where $\mathbf{p}_i \in Spec(A)$ for all i is called the ***Krull dimension*** *or simply the* ***dimension*** *of A. It is denoted by $dimA$.*

For $\mathbf{p} \in Spec(A)$ the supremum of the lengths of strictly decreasing chains of prime ideals

$$\mathbf{p} = \mathbf{p}_0 \supset \mathbf{p}_1 \supset \cdots \supset \mathbf{p}_r$$

is called the ***height*** *of $\mathbf{p}$ and denoted by $ht\mathbf{p}$. Also the supremum of all the lengths of strictly increasing chains of prime ideals*

$$\mathbf{p} = \mathbf{p}_0 \subset \mathbf{p}_1 \subset \cdots \subset \mathbf{p}_r$$

is called the ***coheight*** *of $\mathbf{p}$ and denoted by $coht\mathbf{p}$.*

Note that

(i) $ht\mathbf{p} = dimA_{\mathbf{p}}$
(ii) $coht\mathbf{p} = dimA/\mathbf{p}$
(iii) $ht\mathbf{p} + coht\mathbf{p} \leq dimA$.

Also we have that if $x \in \mathbf{m}$ and x is a non-zero divisor of $(A, \mathbf{m})$ then

$$dimA/x \cdot A = dimA - 1,$$

(Proposition 20 in Section 4.9 of [19]). Extending this result somewhat gives:

Result 2.1.2 *Let $(A, \mathbf{m})$ be a local ring and let $x_1, \ldots, x_d \in \mathbf{m}$. Then*

$$dimA - d \leq dimA/(x_1, \ldots, x_d) \leq dimA.$$

Also $dimA/(x_1, \ldots, x_d) = dimA - d$ if and only if $x_1, \ldots, x_d$ are all different and form a subset of a system of parameters (see the following notes) for A.

See Proposition 15.22 of [23] for a proof.

Denote by $Ann\, M := \{x \in A | x \cdot M = 0\}$ the **annihilator** of M. Now $dim\, M := dim(A/Ann\, M)$.

This leads us to the consideration of $\mathbf{m}$-primary ideals and systems of parameters. An ideal $\mathbf{q}$ of A is **$\mathbf{m}$-primary** if $\mathbf{m}^n \subseteq \mathbf{q}$ for some $n \in \mathbb{N}^+$. The set of primary ideals of A will play a major role in the following chapters mainly because of their connections with lengths (see Section 2). Now suppose that $dimA = d$. If $x_1, \ldots, x_d$ are d elements of A such that $(x_1, \ldots, x_d)$ is an $\mathbf{m}$-primary ideal of A then $x_1, \ldots, x_d$ are called a **system of parameters** of A and the ideal $(x_1, \ldots, x_d)$ is called a **parameter ideal** of A. Note that for any local ring there always exists a system of parameters.

2.1.2 Blowing-up rings

We have already introduced the concept of a graded k-algebra but there are other graded algebraic objects that we will require later on. The first of these is the **Rees algebra** of A with respect to an ideal I of A. This is defined to be

$$R_I(A) := \bigoplus_{i \geq 0} I^i \cdot T^i$$

for some indeterminate T. $R_I(A)$ is a graded A-algebra. If A is a k-algebra then we can regard $R_I(A)$ as a $k[T]$-algebra.

Another graded structure with close links to the Rees algebra is the **associated graded ring** of A with respect to I. This is defined as

$$G_I(A) := A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

and if $R_I(A)$ is a $k[T]$-algebra then

$$R_I(A)/T \cdot R_I(A) = G_I(A)$$

(Section 6.5 of [8]). Passing from $(A, \mathbf{m})$ to $G_{\mathbf{m}}(A)$ is a good way of getting from the local case to the graded case without losing all the properties of the initial ring A, eg. $dimA = dimG_{\mathbf{m}}(A)$ (Theorem 13.9 [15]). This structure can be extended to the module case by modifying the definition to

$$G_I(M) := M/I \cdot M \oplus I \cdot M/I^2 \cdot M \oplus \cdots.$$

This is the associated graded module of M with respect to I.

Note that for a graded module $M = \oplus M_i$, elements in M_i are called homogeneous elements of degree i. M_i is also denoted by $[M]_i$.

We have already noted that certain properties are preserved when passing to the associated graded ring but a natural question to ask would be what happens to ideals in A or M when we pass to $G_I(A)$ or $G_I(M)$. For this purpose **initial forms** and **initial ideals** are introduced and defined as follows. If $f \in I^i \cdot M \backslash I^{i+1} \cdot M$, then

$$in_I(f) = f \text{ modulo } I^{i+1} \cdot M \in I^i \cdot M/I^{i+1} \cdot M \subset G_I(M)$$

is the initial form of f. If $N \subset M$ is a submodule of M then

$$\begin{aligned} in_I(N) &= (in_I(f) \mid f \in N) \\ &= (N + I \cdot M)/I \cdot M \oplus (N + I^2 \cdot M)/I^2 \cdot M \oplus \cdots \end{aligned}$$

is the initial module of N.

We should note here that in the case where $M = A$ and $f, g \in A$ we have

$$in_I(f) + in_I(g) = in_I(f + g) \text{ or } in_I(f) + in_I(g) = 0$$

when $in_I(f), in_I(g) \in I^i \cdot M \backslash I^{i+1} \cdot M$, and

$$in_I(f) \cdot in_I(g) = in_I(f \cdot g) \text{ or } in_I(f) \cdot in_I(g) = 0.$$

This follows as, if we set $G_i = I^i \cdot M \backslash I^{i+1} \cdot M$, then

$$in_I(f) = \bar{f}, in_I(g) = \bar{g} \in G_i.$$

So $in_I(f) + in_I(g) = \overline{f+g} \in G_i$ by the definition of a quotient module. Clearly, $f + g \in I^i \cdot M$ so if we assume that $f + g \notin I^{i+1} \cdot M$ then

$$in_I(f+g) = \overline{f+g} = in_I(f) + in_I(g) \in G_i.$$

Now suppose that $f + g \in I^{i+1} \cdot M$ so that

$$\overline{f+g} = 0 \in G_i.$$

Hence $in_I(f) + in_I(g) = 0$.

We note that in general $in_I(f+g) \neq 0$.

Now suppose that $f \in I^i \cdot M \backslash I^{i+1} \cdot M, g \in I^j \cdot M \backslash I^{j+1} \cdot M$.

$$in_I(f) \cdot in_I(g) = \overline{f \cdot g} \in G_{i+j}$$

by the definition of quotient modules. Clearly, $f \cdot g \in I^{i+j} \cdot M$, so assume that $f \cdot g \notin I^{i+j+1} \cdot M$ and consequently

$$in_I(f \cdot g) = \overline{f \cdot g} = in_I(f) \cdot in_I(g) \in G_{i+j}.$$

Suppose that $f \cdot g \in I^{i+j+1} \cdot M$ so that $\overline{f \cdot g} = 0 \in G_{i+j}$ and

$$in_I(f) \cdot in_I(g) = 0.$$

(See Exercise 5.1 of [8].)

So if I, J and K are ideals in A then

$$in_I(J) + in_I(K) \subseteq in_I(J+K)$$

and

$$in_I(J) \cdot in_I(K) \subseteq in_I(J \cdot K).$$

The following is well known (see Exercise 5.3 of [8]) however no proof could be found. Hence a proof is included here for the sake of completion.

Lemma 2.1.3 *Suppose $J \subset I$ are ideals in A. Then*

$$G_I(A/J) \cong G_I(A)/in_I(J).$$

Proof: We want to construct a surjection, or an onto map, φ between $G_I(A)$ and $G_I(A/J)$.

$$\varphi : G_I(A) \longrightarrow G_I(A/J).$$

Note that $G_I(A/J) = G_{I/J}(A/J)$.

Also $G_{I/J}(A/J) = (A+J)/(I+J) \oplus (I+J)/(I^2+J) \oplus \cdots$. Define φ as follows:

$$\varphi : \bar{x} = \{x + I^{n+1} \mid x \in I^n\} \mapsto \bar{\bar{x}} = x + (I^{n+1} + J).$$

It is straightforward to check that φ is a surjective homomorphism. We have

$$\begin{aligned} ker(\varphi) &= \{\bar{x} \in I^n/I^{n+1} \mid \varphi(\bar{x}) \in I^{n+1} + J\} \\ &= \{\bar{x} \in I^n/I^{n+1} \mid x \in I^{n+1} + J\} \\ &= (I+J)/I \oplus (I^2+J)/I^2 \oplus \cdots \\ &= in_I(J). \end{aligned}$$

Therefore, applying isomorphism theory, φ is an isomorphism from $G_I(A)/in_I(J)$ to $G_{I/J}(A/J)$. So

$$G_I(A/J) \cong G_I(A)/in_I(J).$$

○

2.1.3 Quasi-unmixedness

The notion of an **exact sequence** will be used extensively in the proofs in chapters three, four and five so it is defined here for the convenience of the reader. If M, M' and M'' are A-modules then the sequence of A-modules and homomorphisms

$$M' \xrightarrow{\psi} M \xrightarrow{\varphi} M''$$

is said to be exact when $Im(\psi) = ker(\varphi)$. The sequence

$$0 \longrightarrow M' \xrightarrow{\psi} M \xrightarrow{\varphi} M'' \longrightarrow 0$$

is called a **short exact sequence** if each pair of consecutive maps is exact. That is, if ψ is an injection, φ is a surjection and $Im(\psi) = ker(\varphi)$.

As mentioned earlier $Spec(A)$ is the set of all prime ideals of A. We now want to define some other sets of prime ideals and show how they relate to one-another. If M is an A-module then

$$Supp(M) := \{\mathbf{p} \in Spec(A) \mid M_{\mathbf{p}} \neq 0\}$$

where $M_{\mathbf{p}} = \{m/p \mid m \in M,\ p \notin \mathbf{p}\}$ is the **localisation** of M at $\mathbf{p}$. The notion of localisation is clearly defined and described in Chapter two of [15]. $Supp(M)$ is called the **support** of M. If we have

$$\mathbf{p} = ann(x) = \{a \in A \mid a \cdot x = 0\}$$

for some $x \in M$ then $\mathbf{p}$ is called an **associated prime ideal** of M. The set of associated prime ideals of M is denoted by $Ass(M)$. Note that if

$$0_M = M_1 \cap \cdots \cap M_r$$

is an **irredundent primary decomposition** of 0 in M then $Ass(M) = \{\mathbf{p}_1, \ldots, \mathbf{p}_r\}$ where $Ass(M/M_i) = \{\mathbf{p}_i\}$ (that is, M_i is $\mathbf{p}_i$-primary) (Theorem 6.8(ii) [15]). Finally we define

$$Assh(M) := \{\mathbf{p} \in Ass(M) \mid dimA/\mathbf{p} = dimM\}.$$

From Theorem 6.5(ii) and (iii) of [15] we see that

$$Assh(M) \subseteq Ass(M) \subseteq Supp(M) \subseteq Spec(A)$$

and the minimal elements of $Ass(M)$ and $Supp(M)$ coincide. With these definitions in place we can now define three classes of A-modules. We follow [18] (pages 124 and 82) for these concepts. An A-module M is said to be **equidimensional** if $dimA/\mathbf{p} = dimM$ for all minimal prime ideals $\mathbf{p} \in Supp(M)$. That is, if $min.Supp(M) = Assh(M)$. M is said to be **quasi-unmixed** if $\widehat{M}$ is equidimensional. Here $\widehat{M}$ is used to denote the **completion** of M with respect to some $\mathbf{m}$. Finally, M is said to be **unmixed** if $dim\hat{A}/\mathbf{p} = dim\widehat{M}$ for all prime ideals $\mathbf{p} \in Ass(\widehat{M})$. That is, if $Ass(\widehat{M}) = Assh(\widehat{M})$.

It turns out that the quasi-unmixed case and the equidimensional case coincide when A is graded. To show this we need to introduce some further concepts.

Definition 2.1.4 *Let A be a ring. A is a* **catenary** *local ring if for any two prime ideals $\mathbf{p}$ and $\mathbf{p}'$ of A with $\mathbf{p} \subset \mathbf{p}'$, there exists a chain of prime ideals*

$$\mathbf{p} = \mathbf{p}_0 \subset \mathbf{p}_1 \subset \cdots \subset \mathbf{p}_n = \mathbf{p}'$$

such that:

(i) there is no prime ideal strictly contained between any two consecutive terms;
(ii) all such chains have the same (finite) length.

Lemma 2.1.5 *Let A be a catenary domain. Now*

$$ht\mathbf{p} + coht\mathbf{p} = dimA$$

for any prime ideal $\mathbf{p}$ of A.

Proof: By definition $coht(0) = dimA/(0) = dimA$ and (0) is prime as A is an integral domain. Let $\mathbf{p}$ be any prime ideal of A. Since A is catenary we can construct chains of primes

$$(0) \subset \mathbf{p}_1 \subset \cdots \subset \mathbf{p}_r \subset \mathbf{p}$$

$$\mathbf{p} \subset \mathbf{p}_{r+1} \subset \cdots \subset \mathbf{p}_{r+s} = \mathbf{m}.$$

So,

$$(0) \subset \mathbf{p}_1 \subset \cdots \subset \mathbf{p}_r \subset \mathbf{p} \subset \mathbf{p}_{r+1} \subset \cdots \subset \mathbf{p}_{r+s} = \mathbf{m}$$

and this chain is maximal. Hence

$$r + s = coht(0) = dimA$$

and

$$r + s = ht\mathbf{p} + coht\mathbf{p}$$

so that

$$dimA = ht\mathbf{p} + coht\mathbf{p}.$$

○

Definition 2.1.6 *A Noetherian ring A is* ***universally catenary*** *if every finitely generated A-algebra is catenary.*

Recall the following:

Lemma 2.1.7 *(i) A quotient ring of a (universally) catenary ring is again (universally) catenary. (ii) A complete Noetherian local ring is the quotient of a regular local ring. In particular it is universally catenary.*

Proof: The first statement follows immediately from Definitions 2.1.4 and 2.1.6. For the proof of the second one see Theorem 29.4(ii) of [15].

○

Lemma 2.1.8 *Let A be a graded k-algebra and let M be a graded A-module. Now M is quasi-unmixed if and only if M is equidimensional.*

Proof: Let A be a standard graded k-algebra. Since $min.Supp(M) = min.Ass(M)$ and $\widehat{Ann_A M} = Ann_{\hat{A}}\hat{M}$, M is quasi-unmixed if and only if $A/Ann_A M$ is also quasi-unmixed. Hence without loss of generality we may assume that $M = A$. Theorem 18.17 of [11] shows that A is quasi-unmixed if and only if A is equidimensional and universally catenary (see Definitions 2.1.4 and 2.1.6).

Refering to page 93 of [15] we see that A is the quotient of a polynomial ring. That is,

$$A \cong k[X_1, \ldots, X_n]/I$$

for some homogeneous ideal I.

Since $k[X_1, \ldots, X_n]$ is universally catenary (by Lemma 2.1.7(ii)) it follows that A is also universally catenary (Lemma 2.1.7(i)).

Therefore the equidimensional and the quasi-unmixed properties coincide in the graded case.

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2.2 A Note on Lengths

To begin with we need to define what is meant by the length of an ideal in a ring. Firstly an A-module M is called **simple** if $M \neq 0$ and it has no submodules other then 0 and itself. It follows that all simple A-modules are isomorphic to $A/\mathbf{m}$. Now construct a chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$$

of submodules of M. This chain is called a **composition series** of M if M_i/M_{i+1} is simple for all $i = 0, \ldots, r-1$. If such a composition series exists then its length r is an invariant of M. That is, any composition series of M will have length r.

Definition 2.2.1 *If M is an A-module with a composition series as above then r is called the **length** of M and is denoted by $\ell_A(M) = r$. If no such composition series exists then $\ell_A(M) = \infty$.*

In order for M to have finite length it is a necessary and sufficient condition that M is both Noetherian and Artinian. That is, M satisfies both the ascending and descending chain conditions for modules (Proposition 7.36 [23]).

In most cases we will only be concerned with the lengths of the quotient modules $M/\mathbf{q}\cdot M$ of M where $\mathbf{q}$ is an $\mathbf{m}$-primary ideal of A. In this case $\ell_A(M/\mathbf{q}\cdot M) < \infty$.

We now need to know some of the properties of lengths.

Lemma 2.2.2 *Suppose $M_1, M_2, \ldots, M_n$ are A-modules of finite length. Then if*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

is exact we have

$$\sum_{i=0}^{n} (-1)^i \cdot \ell_A(M_i) = 0.$$

Proof: See Theorem 20 of Section 1.12 of [19].

○

This result yields the usually more useful following statement. If M_1, M_2 and M_3 are A-modules of finite length and

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is a short exact sequence then

$$\ell_A(M_2) = \ell_A(M_1) + \ell_A(M_3).$$

The following lemma illustrates another reason why passing to the associated graded ring of A is a very useful technique.

Lemma 2.2.3 *Let $(A, \mathbf{m})$ be a local ring and let $G_{\mathbf{m}}(A)$ be the associated graded ring of A with respect to $\mathbf{m}$. Let $\mathbf{q}$ be an $\mathbf{m}$-primary ideal of A and $in_A(\mathbf{q})$ the initial ideal of $\mathbf{q}$ in $G_{\mathbf{m}}(A)$. Then*

$$\ell_A(A/\mathbf{q}\cdot A) = \ell_A(G_{\mathbf{m}}(A)/in_A(\mathbf{q})\cdot G_{\mathbf{m}}(A)).$$

Proof: Since $G_{\mathbf{m}}(A)/in_A(\mathbf{q})\cdot G_{\mathbf{m}}(A) \cong G_{\mathbf{m}}(A/\mathbf{q}\cdot A)$ (Lemma 2.1.3) it follows that

$$\ell_A(G_{\mathbf{m}}(A/\mathbf{q}\cdot A)) = \ell_A(G_{\mathbf{m}}(A)/in_A(\mathbf{q})\cdot G_{\mathbf{m}}(A)).$$

Set $\gamma := A/\mathbf{q}\cdot A$ so that

$$G_{\mathbf{m}}(\gamma) := \gamma/\mathbf{m}\cdot\gamma \oplus \mathbf{m}\cdot\gamma/\mathbf{m}^2\cdot\gamma \oplus \cdots \oplus \mathbf{m}^{k-1}\cdot\gamma/0$$

for some $k >> 0$ as $\mathbf{m}^k\cdot\gamma = 0$ because γ has finite length. This follows from the fact that $\mathbf{q}$ is $\mathbf{m}$-primary in A. $G_{\mathbf{m}}(\gamma)$ is an $A/\mathbf{m}$-vector space so

$$\begin{aligned}
\ell_A(G_{\mathbf{m}}(\gamma)) &= \textstyle\sum_{n=0}^{k} dim(\mathbf{m}^n\cdot\gamma/\mathbf{m}^{n+1}\cdot\gamma)\\
&= \textstyle\sum_{n=0}^{k} \ell_A(\mathbf{m}^n\cdot\gamma/\mathbf{m}^{n+1}\cdot\gamma)\\
&= \ell_A(\gamma/\mathbf{m}\cdot\gamma) + \ell_A(\mathbf{m}\cdot\gamma/\mathbf{m}^2\cdot\gamma) + \cdots + \ell_A(\mathbf{m}^{k+1}\cdot\gamma/0)\\
&= \ell_A(\gamma) - \ell_A(\mathbf{m}\cdot\gamma) + \ell_A(\mathbf{m}\cdot\gamma) - \ell_A(\mathbf{m}^2\cdot\gamma) + \cdots\\
&\quad + \ell_A(\mathbf{m}^{k+1}\cdot\gamma) - \ell_A(0)\\
&= \ell_A(\gamma) - \ell_A(0)\\
&= \ell_A(\gamma)
\end{aligned}$$

Therefore $\ell_A(A/\mathbf{q}\cdot A) = \ell_A(G_{\mathbf{m}}(A)/in_A(\mathbf{q})\cdot G_{\mathbf{m}}(A))$.

○

Note that this result is equally true for an A-module M and $G_{\mathbf{m}}(M)$.

In this section we also want to introduce the notion of the Loewy length.

Definition 2.2.4 *Let M be an A-module and let $\mathbf{q}$ be an $\mathbf{m}$-primary ideal of A. The **Loewy length** of $M/\mathbf{q}\cdot M$ is the smallest positive integer t such that $\mathbf{m}^t\cdot M \subseteq \mathbf{q}\cdot M$. It is denoted by $\ell\ell(M/\mathbf{q}\cdot M)$.*

The name Loewy length was first introduced in [27] (page 162) and some of its properties are examined as well. Applying the definition of $\mathbf{m}$- primary ideals we see that $\ell\ell(M/\mathbf{q}\cdot M) < \infty$.

Remark 2.2.5 Note that if $\ell\ell(M/\mathbf{q}\cdot M) = t$, then we have a chain

$$0 = \mathbf{m}^t\cdot M/\mathbf{q}\cdot M \subset \mathbf{m}^{t-1}\cdot M/\mathbf{q}\cdot M \subset \cdots \subset M/\mathbf{q}\cdot M.$$

Taking a saturation of this chain to get a composition series of $M/\mathbf{q}\cdot M$ we see that

$$\ell\ell(M/\mathbf{q}\cdot M) \leq \ell(M/\mathbf{q}\cdot M).$$

Also, if N is a quotient module of M then $\ell\ell(M/\mathbf{q}\cdot M) \geq \ell\ell(N/\mathbf{q}\cdot N)$.

Example 2.2.6 Suppose $A = M = k[[x,y]]$ and consider $\mathbf{m}^n$ for some $n \in \mathbf{N}^+$. Clearly

$$\mathbf{m}^{n-1} \not\subseteq \mathbf{m}^n$$

and

$$\mathbf{m}^n \subseteq \mathbf{m}^n.$$

Thus $\ell\ell(A/\mathbf{m}^n) = n$.

The chain

$$(x^n, x^{n-1}y, \ldots, y^n) \subset (x^{n-1}, x^{n-2}y, \ldots, y^n) \subset \cdots \subset (x, y)$$

is saturated so $\ell(A/\mathbf{m}^n) = 2n$.

Therefore we can see that

$$\ell\ell(A/\mathbf{m}^n) << \ell(A/\mathbf{m}^n)$$

if $n > 1$.

2.3 A Note on Multiplicities

Following the notion of the length of an ideal in a module our next most important concept is that of the multiplicity of an ideal with respect to a module. Referring to [24] we introduce the notion of multiplicity by examining Hilbert-Samuel functions and their associated polynomials.

Let M be a (graded) Noetherian A-module of dimension $t \geq 0$ where A is a local ring (or a graded k-algebra). Let $\mathbf{q}$ be an $\mathbf{m}$-primary ideal of A so that $\ell_A(M/\mathbf{q} \cdot M) < \infty$ (see section 2). The **Hilbert-Samuel function** is now denoted by $\mathbf{P}_{\mathbf{q},M}(n)$ and defined as follows:

$$\mathbf{P}_{\mathbf{q},M}(n) := \ell_A(M/\mathbf{q}^{n+1} \cdot M)$$

for all integers $n \geq 0$.

It is well known that there exists a polynomial in n, denoted by $\mathbf{p}_{\mathbf{q},M}(n)$, such that

$$\mathbf{P}_{\mathbf{q},M}(n) = \mathbf{p}_{\mathbf{q},M}(n)$$

for all $n \gg 0$ (Theorem 11 on page 320 of [19]). The polynomial $\mathbf{p}_{\mathbf{q},M}(n)$ is the **(characteristic) Hilbert-Samuel polynomial** of the ideal $\mathbf{q}$ with respect to M. There exists integers

$$e_0 := e_0(\mathbf{q}; M)\,(> 0),\ e_1 := e_1(\mathbf{q}; M), \ldots, e_t := e_t(\mathbf{q}; M)$$

such that

$$\mathbf{p}_{\mathbf{q},M}(n) := e_0 \begin{pmatrix} n & +t \\ & t \end{pmatrix} + e_1 \begin{pmatrix} n+ & t-1 \\ t & -1 \end{pmatrix} + \cdots + e_t$$

where $t = dimM$. The leading coefficient $e_0(\mathbf{q}; M) := e(\mathbf{q}; M)$ of $\mathbf{p}_{\mathbf{q},M}(n)$ is called the **multiplicity** of $\mathbf{q}$ with respect to M.

When $\mathbf{q} = \mathbf{m}$ and $M = A$ we have $e(\mathbf{q}; M) = e(\mathbf{m}; A) = e(A)$ and $e(A)$ is called the multiplicity of A.

If A is a graded k-algebra and M is a graded A-module then we define the **degree** of M in A as

$$deg\, M := deg_A M = e(\mathbf{m}_A; M).$$

With the length and the multiplicity now defined we can use these notions to classify different types of local rings.

Definition 2.3.1 *If A is a Noetherian local ring then A will be called a* ***Cohen-Macaulay*** *local ring if for every ideal $\mathbf{q}$ generated by a system of parameters we have*

$$\ell_A(A/\mathbf{q}) = e(\mathbf{q}; A).$$

Definition 2.3.2 *If A is a Noetherian local ring then A will be called a* ***Buchsbaum*** *local ring if the difference of $\ell_A(A/\mathbf{q})$ and $e(\mathbf{q}; A)$ is an integer, say $I(A)$, not depending on the choice of parameter ideal $\mathbf{q}$ of A. That is,*

$$I(A) = \ell_A(A/\mathbf{q}) - e(\mathbf{q}; A)$$

for all parameter ideals $\mathbf{q}$ of A.

A will be called a ***generalized Cohen-Macaulay*** *local ring if the difference of $\ell_A(A/\mathbf{q})$ and $e(\mathbf{q}; A)$ is bounded by an integer, say $I(A)$, not depending on the choice of parameter ideal $\mathbf{q}$ of A. That is,*

$$\ell_A(A/\mathbf{q}) - e(\mathbf{q}; A) \leq I(A) < \infty,$$

for all parameter ideals $\mathbf{q}$ of A.

Note that if A is a Cohen-Macaulay local ring then $I(A) = 0$. So the class of Buchsbaum local rings contains the class of Cohen-Macaulay local rings and the class of generalized Cohen-Macaulay rings contains the class of Buchsbaum rings. The theory of these rings has been extensively developed over the last 30 years, see e.g. the book [24].

We now want to examine some of the properties of multiplicities. The first of these properties is more of an equivalent definition for the multiplicity and can be found on page 107 of [15] (Formula 14.1),

$$e(\mathbf{q}; M) = \lim_{n\to\infty} \frac{t!}{n^t} \cdot \ell_A(M/\mathbf{q}^n \cdot M).$$

It also follows from the definition that if $\mathbf{q}$ and $\mathbf{q}'$ are $\mathbf{m}$-primary ideals and $\mathbf{q}' \subseteq \mathbf{q}$ then

$$e(\mathbf{q}'; M) \geq e(\mathbf{q}; M)$$

(Formula 14.4 [15]).

Lemma 2.3.3 (Associativity formula for multiplicities) *Let $\{\mathbf{p}_1, \ldots, \mathbf{p}_r\}$ be all the minimal prime ideals of A such that $dim A/\mathbf{p}_i = dim M$. Then*

$$e(\mathbf{q}; M) = \sum_{i=1}^{r} e(\mathbf{q}; A/\mathbf{p}_i) \cdot \ell_{A_{\mathbf{p}_i}}(M_{\mathbf{p}_i})$$

for an $\mathbf{m}$-primary ideal $\mathbf{q}$ of A.

Proof: See Theorem 14.7 of [15].

○

We already know how the length behaves with respect to exact sequences so we now wish to know what happens to the multiplicity. In this case we will restrict our attention to short exact sequences. We first need the following notation:

$$e^*(\mathbf{q}; M') := \begin{cases} e(\mathbf{q}; M'), & \text{if } dimM' = dimM \\ 0, & \text{if } dimM' \neq dimM \end{cases}$$

Lemma 2.3.4 *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of A-modules. Then

$$e(\mathbf{q}; M) = e^*(\mathbf{q}; M') + e^*(\mathbf{q}; M'').$$

Proof: See Theorem 14.6 of [15].

○

Note that since the sequence is exact one of the following three cases must hold

$$\begin{array}{ll} \text{(i)} & dimM = dimM' = dimM'', \\ \text{(ii)} & dimM = dimM' > dimM'', \\ \text{(iii)} & dimM = dimM'' > dimM'. \end{array}$$

Note also that it is possible to extend the above result to the case of long exact sequences (see eg. Corollary 1 to Theorem 5 on page 302 of [19]).

In the graded case we have a similar result that we want to reprove here for the sake of completion. First we require a lemma.

Lemma 2.3.5 *Suppose M is an A-module such that $dimM > 0$ and N is an A-module of finite length such that*

$$0 \longrightarrow M' \hookrightarrow M \longrightarrow N \longrightarrow 0$$

is exact. Then $e(\mathbf{q}; G_{\mathbf{m}}(M)) = e^*(\mathbf{q}; G_{\mathbf{m}}(M'))$ where $e^* = 0$ if $dimM' \neq dimM$.

Proof: Apply induction on $\ell_A(N)$. Suppose $\ell_A(N) = 1$ so that it follows that N is a field. Hence $\mathbf{m} \cdot N = 0$. Since $N \cong M/M'$, $\mathbf{m} \cdot M \subseteq M'$. Now $M' \hookrightarrow M$ induces a map $\bigoplus_{i \geq 0} \frac{\mathbf{m}^i \cdot M'}{\mathbf{m}^{i+1} \cdot M'} \xrightarrow{\gamma} \bigoplus_{i \geq 0} \frac{\mathbf{m}^i \cdot M}{\mathbf{m}^{i+1} \cdot M}$. This can be extended to an exact sequence

$$0 \longrightarrow \bigoplus_{i \geq 0} \frac{\mathbf{m}^i \cdot M' \cap \mathbf{m}^{i+1} \cdot M}{\mathbf{m}^{i+1} \cdot M'} \longrightarrow \bigoplus_{i \geq 0} \frac{\mathbf{m}^i \cdot M'}{\mathbf{m}^{i+1} \cdot M'} \xrightarrow{\gamma} \bigoplus_{i \geq 0} \frac{\mathbf{m}^i \cdot M}{\mathbf{m}^{i+1} \cdot M}$$

$$\longrightarrow \bigoplus_{i \geq 0} \frac{\mathbf{m}^i \cdot M}{\mathbf{m}^i \cdot M' + \mathbf{m}^{i+1} \cdot M} \longrightarrow 0.$$

Hence,

$$\bigoplus_{i\geq 0} \frac{\mathbf{m}^i \cdot M' \cap \mathbf{m}^{i+1} \cdot M}{\mathbf{m}^{i+1} \cdot M'} = \bigoplus_{i\geq 0} \frac{\mathbf{m}^{i+1} \cdot M}{\mathbf{m}^{i+1} \cdot M'}$$

and

$$\bigoplus_{i\geq 0} \frac{\mathbf{m}^i \cdot M}{\mathbf{m}^i \cdot M' + \mathbf{m}^{i+1} \cdot M} = \bigoplus_{i\geq 0} \frac{\mathbf{m}^i \cdot M}{\mathbf{m}^i \cdot M'}$$

since $\mathbf{m}^{i+1} \cdot M = \mathbf{m}^i \cdot \mathbf{m} \cdot M \subseteq \mathbf{m}^i \cdot M'$.

Let $\mathcal{L} := \bigoplus_{i\geq 0} \frac{\mathbf{m}^i \cdot M}{\mathbf{m}^i \cdot M'}$ so that $\mathcal{L}(-1) = \bigoplus_{i\geq 0} \frac{\mathbf{m}^{i+1} \cdot M}{\mathbf{m}^{i+1} \cdot M'}$ and hence $e(\mathbf{q}; \mathcal{L}) = e(\mathbf{q}; \mathcal{L}(-1))$ as shifts in the degree do not change the multiplicity.

So we have

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{L}(-1) & \longrightarrow & G_{\mathbf{m}}(M') & & \longrightarrow & & G_{\mathbf{m}}(M) & \longrightarrow \mathcal{L} \longrightarrow 0 \\
 & & & & & \searrow & & \nearrow & & \\
 & & & & & & \mathcal{M} & & & \\
 & & & & & \nearrow & & \searrow & & \\
 & & & & 0 & & & & 0 &
\end{array}$$

for some $\mathcal{M}$.

Now,

$$e(\mathbf{q}; G_{\mathbf{m}}(M')) = e^*(\mathbf{q}; \mathcal{L}(-1)) + e^*(\mathbf{q}; \mathcal{M})$$

$$e(\mathbf{q}; G_{\mathbf{m}}(M)) = e^*(\mathbf{q}; \mathcal{M}) + e^*(\mathbf{q}; \mathcal{L})$$

as $dim\mathcal{L} = dim\mathcal{L}(-1) = 0$. This implies that

$$e^*(\mathbf{q}; \mathcal{M}) = e(\mathbf{q}; G_{\mathbf{m}}(M)) - e^*(\mathbf{q}; \mathcal{L}).$$

Thus,

$$\begin{aligned} e(\mathbf{q}; G_{\mathbf{m}}(M')) &= e^*(\mathbf{q}; \mathcal{L}(-1)) + e(\mathbf{q}; G_{\mathbf{m}}(M)) - e^*(\mathbf{q}; \mathcal{L}) \\ &= e(\mathbf{q}; G_{\mathbf{m}}(M)). \end{aligned}$$

Suppose now that $\ell_A(N) = n$. Take $\widetilde{N}$ such that $\widetilde{N} \subset N$ and $\ell_A(\widetilde{N}) = n - 1$. Let $\widetilde{M}$ denote the preimage of M in $\widetilde{N}$. This produces the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \hookrightarrow & \widetilde{M} & \longrightarrow & \widetilde{N} & \longrightarrow & 0 \\
 & & \| & & \cap & & \cap & & \\
0 & \longrightarrow & M' & \hookrightarrow & M & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Note that $N \cong M/M'$ so if $\widetilde{N} \subset N$ then $\widetilde{N} \cong (N' + M')/M'$ where $N' \subseteq M$. Hence $\widetilde{M}/M' \cong \widetilde{N}$ so that $\widetilde{M} \cong N' + M'$ and thus $M' \subseteq \widetilde{M}$.

By the induction assumption

$$e(\mathbf{q}; G_{\mathbf{m}}(M')) = e(\mathbf{q}; G_{\mathbf{m}}(\widetilde{M})).$$

Also,

$$\begin{array}{ccccccc} 0 \longrightarrow & \widetilde{N} \hookrightarrow & N \longrightarrow & N/\widetilde{N} \longrightarrow 0 \\ & \uparrow & \uparrow & \uparrow \\ 0 \longrightarrow & \widetilde{M} \hookrightarrow & M \longrightarrow & M/\widetilde{M} \longrightarrow 0 \end{array}$$

and $\ell_A(N/\widetilde{N}) = 1$.

This implies that $\ell_A(M/\widetilde{M}) = 1$ as $N/\widetilde{N} \cong M/N' + M' \cong M/\widetilde{M}$ and therefore, from the length one step, we have

$$e(\mathbf{q}; G_{\mathbf{m}}(M)) = e^*(\mathbf{q}; G_{\mathbf{m}}(\widetilde{M})).$$

Thus,

$$e(\mathbf{q}; G_{\mathbf{m}}(M)) = e(\mathbf{q}; G_{\mathbf{m}}(M')).$$

○

We now give a proof of a theorem found in [10]:

Theorem 2.3.6 *If*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of A-modules where $(A, \mathbf{m})$ is a local ring, then for any $\mathbf{m}$-*primary ideal* $\mathbf{q}$ *of A we have*

$$e(\mathbf{q}; G_{\mathbf{m}}(M)) = e^*(\mathbf{q}; G_{\mathbf{m}}(M')) + e^*(\mathbf{q}; G_{\mathbf{m}}(M'')).$$

Proof: Suppose $dimM = 0$ so that $dimM' = dimM'' = 0$ (see eg. Exercise 12.11(a) [8]). Then

$$e(\mathbf{q}; G_{\mathbf{m}}(M)) = e(\mathbf{q}; G_{\mathbf{m}}(M')) + e(\mathbf{q}; G_{\mathbf{m}}(M'')).$$

Now suppose that $dimM > 0$. Assume that $M' \subseteq M$ and $M'' = M/M'$ so that

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

is exact. This can be done without loss of generality as $M' \cong Im(M' \to M) = ker(M \to M'')$ and by the first isomorphism theorem

$$M/ker(M \to M'') \cong Im(M \to M'') = ker(M'' \to 0) = M''$$

so that $M'' \cong M/M'$.

From the above sequence we get the exact sequence

$$0 \longrightarrow G^* \longrightarrow G_{\mathbf{m}}(M) \longrightarrow G_{\mathbf{m}}(M'') \longrightarrow 0,$$

where

$$\begin{aligned} G_{\mathbf{m}}(M) &= \textstyle\bigoplus_{i\geq 0} \frac{\mathbf{m}^i \cdot M}{\mathbf{m}^{i+1}\cdot M} \\ G_{\mathbf{m}}(M'') \cong G_{\mathbf{m}}(M/M') &= \textstyle\bigoplus_{i\geq 0} \frac{\mathbf{m}^i\cdot(M/M')}{\mathbf{m}^{i+1}\cdot(M/M')} \\ &= \textstyle\bigoplus_{i\geq 0} \frac{(\mathbf{m}^i\cdot M+M')/M'}{(\mathbf{m}^{i+1}\cdot M+M')/M'} \\ &\cong \textstyle\bigoplus_{i\geq 0} \frac{\mathbf{m}^i\cdot M+M'}{\mathbf{m}^{i+1}\cdot M+M'} \end{aligned}$$

$$\begin{aligned} G^* &= ker(G_{\mathbf{m}}(M) \to G_{\mathbf{m}}(M'')) \\ &= \bigoplus_{i \geq 0} \frac{\mathbf{m}^i \cdot M \cap (\mathbf{m}^{i+1} \cdot M + M')}{\mathbf{m}^{i+1} \cdot M}. \end{aligned}$$

Now,

$$\begin{aligned} \mathbf{m}^i \cdot M \cap (\mathbf{m}^{i+1} \cdot M + M') &= (\mathbf{m}^i \cdot M \cap \mathbf{m}^{i+1} \cdot M) + (\mathbf{m}^i \cdot M \cap M') \\ &= \mathbf{m}^{i+1} \cdot M + (\mathbf{m}^i \cdot M \cap M') \end{aligned}$$

so

$$\begin{aligned} G^* &= \bigoplus_{i \geq 0} \frac{\mathbf{m}^{i+1} \cdot M + (\mathbf{m}^i \cdot M \cap M')}{\mathbf{m}^{i+1} \cdot M} \\ &= \bigoplus_{i \geq 0} \frac{\mathbf{m}^i \cdot M \cap M'}{(\mathbf{m}^i \cdot M \cap M') \cap \mathbf{m}^{i+1} \cdot M} \\ &= \bigoplus_{i \geq 0} \frac{\mathbf{m}^i \cdot M \cap M'}{\mathbf{m}^{i+1} \cdot M \cap M'}. \end{aligned}$$

By the Artin-Rees Lemma (see eg. Lemma 5.1 [8]) we know that there exists a number s such that

$$\mathbf{m}^i \cdot M \cap M' = \mathbf{m}^{i-s} \cdot (\mathbf{m}^s \cdot M \cap M')$$

for all $i \geq s$. Hence, if $(-s)$ denotes a shift in degrees, then

$$G^* = \bigoplus_{i<s} \frac{\mathbf{m}^i \cdot M \cap M'}{\mathbf{m}^{i+1} \cdot M \cap M'} \bigoplus G_{\mathbf{m}^s \cdot M \cap M'}(-s).$$

Now if $\mathbf{q}$ is an $\mathbf{m}$-primary ideal of A then

$$e(\mathbf{q}; G_{\mathbf{m}}(M)) = e^*(\mathbf{q}; G_{\mathbf{m}}(M'')) + e^*(\mathbf{q}; G^*).$$

Also from the exact sequence

$$0 \longrightarrow G_{\mathbf{m}^s \cdot M \cap M'}(-s) \hookrightarrow G^* \longrightarrow \bigoplus_{i<s} \frac{\mathbf{m}^i \cdot M \cap M'}{\mathbf{m}^{i+1} \cdot M \cap M'} \longrightarrow 0$$

we have

$$e^*(\mathbf{q}; G^*) = e^*(\mathbf{q}; G_{\mathbf{m}^s \cdot M \cap M'})$$

since $dim\left(\bigoplus_{i<s} \frac{\mathbf{m}^i \cdot M \cap M'}{\mathbf{m}^{i+1} \cdot M \cap M'}\right) = 0$. This follows as $\bigoplus_{i<s} \frac{\mathbf{m}^i \cdot M \cap M'}{\mathbf{m}^{i+1} \cdot M \cap M'}$ has only a finite number of degrees and consequently satisfies the descending chain condition. That is, it is Artinian. That the dimension is zero now follows from Corollary 9.1 on page 227 of [8].

The sequence

$$0 \longrightarrow \mathbf{m}^s \cdot M \cap M' \hookrightarrow M' \longrightarrow M'/\mathbf{m}^s \cdot M \cap M' \longrightarrow 0$$

is exact and

$$M' \supset \mathbf{m} \cdot M \cap M' \supset \mathbf{m}^2 \cdot M \cap M' \supset \cdots \supset \mathbf{m}^s \cdot M \cap M'$$

shows that $M'/\mathbf{m}^s \cdot M \cap M'$ has finite length. So applying the previous lemma we have

$$e(\mathbf{q}; G_{\mathbf{m}}(M')) = e^*(\mathbf{q}; G_{\mathbf{m} \cdot M \cap M'})$$

and therefore we can conclude that

$$e(\mathbf{q}; G_{\mathbf{m}}(M)) = e^*(\mathbf{q}; G_{\mathbf{m}}(M')) + e^*(\mathbf{q}; G_{\mathbf{m}}(M'')).$$

○

If we now let $\mathbf{q}$ be a parameter ideal of A then we can state the following property of the multiplicity.

Lemma 2.3.7 *Let A be a d-dimensional Noetherian local ring and let $x_1, \ldots, x_d$ be a system of parameters of A. Set $\mathbf{q} = (x_1, \ldots, x_d)$. Then*

$$\ell_A(A/\mathbf{q}) \geq e(\mathbf{q}; A)$$

and if $x_i \in \mathbf{m}^\alpha$ for all i then

$$\ell_A(A/\mathbf{q}) \geq \alpha^d \cdot e(\mathbf{m}; A).$$

Proof: See Theorem 14.10 of [15].

○

The following notion is important in the study of multiplicities and other subjects in commutative algebra

Definition 2.3.8 *Let A be a Noetherian ring and I be an ideal of A. The ideal $J \subseteq I$ is called a* ***reduction*** *of I if $I^{r+1} = J \cdot I^r$ for all $r \gg 0$. If J does not properly contain a reduction of I, then it is called* ***minimal reduction****.*

If A is a local ring with an infinite residue field then every ideal of A has a minimal reduction. A minimal reduction of an $\mathbf{m}$-primary ideal is a parameter ideal.

Lemma 2.3.9 *Let A be a Noetherian local ring and let $\mathbf{q}$ be an $\mathbf{m}$-primary ideal of A. Let Q be a minimal reduction of $\mathbf{q}$. Then Q is also $\mathbf{m}$-primary and for any finite A-module M we have*

$$e(\mathbf{q}; M) = e(Q; M).$$

Proof: See Theorem 14.13 of [15].

○

Note that if $\mathbf{q}$ is a minimal reduction of $\mathbf{m}$ in A then

$$e(\mathbf{q}; A) = e(\mathbf{m}; A) = e(A).$$

There are many other interesting results based on the multiplicity and the reader is referred to [19] for these.

2.4 $n_A(M)$ and $\theta_A(M)$.

We now have the tools required to introduce the invariants that will be examined in the following chapters. The first pair of invariants were introduced in [16] and then elaborated upon in [25].

Let A be a local ring (or a graded k-algebra) and let M be a (graded) A-module. Then we define the following two invariants:

$$n_A(M) := \sup\{\ell_A(M/\mathbf{q}\cdot M)/e(\mathbf{q}; M)\} \in \mathbf{R}^+ \cup \{\infty\}$$

where the supremum is taken over all (homogeneous) $\mathbf{m}$-primary ideals $\mathbf{q}$ in A;

$$\tilde{n}_A(M) := \sup\{\ell_A(M/\mathbf{q}\cdot M)/e(\mathbf{q}; M)\} \in \mathbf{R}^+ \cup \{\infty\}$$

where the supremum is taken over all (homogeneous) parameter ideals $\mathbf{q}$ of M. When $M = A$ we also denote these invariants simply as n_A and $\tilde{n}_A$.

It should be noted that only homogeneous ideals are considered in the case where A is a graded k-algebra and M is a graded A-module.

Below are some basic properties of these invariants (see [25]):

(i) $1 \leq \tilde{n}_A(M) \leq n_A(M)$;

(ii) $n_A(M) = \tilde{n}_A(M)$ if A is local and $A/\mathbf{m}$ is an infinite field;

(iii) $n_A(M) = \tilde{n}_A(M) = 1$ if and only if M is a (graded) Cohen-Macaulay module.

Stückrad and Vogel gave the following necessary condition for the finiteness of these invariants:

Theorem 2.4.1 (Theorem 1 of [25]) *(1) Let A be a local ring (or a graded k -algebra) and let M be an (graded) A-module. If $n_A(M) < \infty$ then M is quasi-unmixed.*

(2) The following conditions are equivalent

(i) For every local ring A (or graded k-algebra) and every quasi-unmixed (graded) A-module M we have $n_A(M) < \infty$.

(ii) For every complete regular local ring R with infinite residue class field (or for every polynomial ring $R = k[X_0, \ldots, X_n]$ over an infinite field k) we have $n_R(R/\mathbf{p}) < \infty$ for all (homogeneous) prime ideals $\mathbf{p}$ of R.

For the graded case Stückrad and Vogel provided a case where $\tilde{n}_A(M)$ is always finite.

Theorem 2.4.2 (Theorem 2 of [25]) *Let A be a graded k-algebra. Then we have $\tilde{n}_A(M) < \infty$ for all graded A-modules M.*

Besides the above theorem, some other positive results were obtained in [16] and [25]. Namely, if A is a generalized Cohen-Macaulay local ring, or a monomial ring then $n_A(M)$ is finite. These results led to Conjecture 1.3.1 stating that the converse of Theorem 2.4.1 (1) is true. This problem is discussed in Chapters 3, 4 and 5.

We now want to introduce a new invariant and relate it to $n_A(M)$. Let A be a ring and let M be an A-module. Then define

$$\theta_A(M) := \sup\{\ell\ell(M/\mathbf{q}\cdot M)/e(\mathbf{q};M)\},$$

where the supremum is taken over all $\mathbf{m}$-primary ideals $\mathbf{q}$ of A. In general we will only be concerned with the case where $dim\ M = dim\ A$.

If we let J be a minimal reduction of $\mathbf{q}$ in A then

$$\ell\ell(M/J\cdot M) \geq \ell\ell(M/\mathbf{q}\cdot M)$$

and J is a parameter ideal (Remark 2.2.5). By applying Lemma 2.3.9 we see that

$$\frac{\ell\ell(M/J\cdot M)}{e(J;M)} \geq \frac{\ell\ell(M/\mathbf{q}\cdot M)}{e(\mathbf{q};M)}.$$

Hence in order to obtain an upper bound for $\theta_A(M)$ we can restrict our attention to the set of parameter ideals $\mathbf{q}$ of M. By Remark 2.2.5 it follows that

$$\theta_A(M) \leq n_A(M).$$

Very often it happens that $\theta_A(M) << n_A(M)$ since it is often the case that $\ell\ell(A/\mathbf{q}) << \ell(A/\mathbf{q})$ (see Example 2.2.6). Nevertheless it will be shown in Chapter 5 that the finiteness of $\theta_A(M)$ also implies that M satisfies the same conditions as in Theorem 2.4.1 (1). Hence the finiteness of $\theta_A(M)$ would be a further step towards establishing the validity of Conjecture 1.3.1. The study of this invariant will be carried out in Chapter 5.

Chapter 3

Connections with the graded case

This chapter studies the graded case with regards to the invariant $n_A(M)$. We begin by studying the quasi-unmixed property and show that it is preserved when passing to the associated graded ring of a module. The final results in the first section deal with an attempt to bound $n_A(M)$ by $n_{G_{\mathbf{m}}(M)}(G_{\mathbf{m}}(M))$. A general result is, at this stage, still not possible but we do establish some equivalent conditions.

The second section introduces the notion of Bayer deformation (see [8]). This technique is used to study quasi-unmixedness in the case of polynomial rings modulo an initial ideal. In the end of the chapter the relationship between $n_R(R/I)$ and $n_R(R/lt(I))$ is studied. An equivalent condition for $n_R(R/I)$ to be finite is constructed using this relationship.

3.1 Quasi-unmixedness

We begin this section studying the behaviour of the quasi-unmixedness of algebraic structures. The purpose of this section is to examine the possibility of bounding $n_A(A)$ by $n_{G_{\mathbf{m}}(A)}(G_{\mathbf{m}}(A))$. First of all note that the quasi-unmixedness property is preserved by passing to the graded case:

Lemma 3.1.1 *Let $(A, \mathbf{m})$ be a d-dimensional local ring. If A is complete then A and $R_{\mathbf{m}}(A)$ are catenary.*

Proof: The first statement is Lemma 2.1.7(ii)

Further, if $\mathbf{m} = (m_1, \ldots, m_\mu) \cdot A$ then

$$R_{\mathbf{m}}(A) = \bigoplus_{i \geq 0} \mathbf{m}^i \cdot T^i \cong A[y_1, \ldots, y_\mu]/J,$$

for an ideal J of $A[y_1, \ldots, y_\mu]$. Since A is universally catenary and $A[y_1, \ldots, y_\mu]$ and $A[y_1, \ldots, y_\mu]/J$ are finite A-algebras it follows that $R_{\mathbf{m}}(A)$ is catenary (Corollary 18.10 [8]).

$\bigcirc$

The following theorem requires the assertion that the dimension of the Rees algebra of A equals the dimension of A plus one. This is set out in Theorem 15.7 of [15].

Theorem 3.1.2 *Let $(A, \mathbf{m})$ be a d-dimensional local ring and let M be an A-module such that $dimM > 0$. Now if M is quasi-unmixed then $G_{\mathbf{m}}(M)$ is quasi-unmixed.*

Proof: We can assume without loss of generality that $A = \widehat{A}$ as $G_{\mathbf{m}_{\widehat{A}}}(\widehat{A}) = G_{\mathbf{m}}(A)$ (Theorem 7.1(b) [8]).

Let $R_{\mathbf{m}}(A) := \bigoplus_{i \geq 0} \mathbf{m}^i \cdot T^i$ be the Rees algebra of A with respect to $\mathbf{m}$. Take any $\mathbf{p} \in min.SuppG_{\mathbf{m}}(M) \subset SpecR_{\mathbf{m}}(A)$. Note that all primes lie in $SpecR_{\mathbf{m}}(A)$ since $G_{\mathbf{m}}(M)$ is an $R_{\mathbf{m}}(A)$-module and also note that $[\mathbf{p}]_0 = \mathbf{m}$. All the elements of $AssG_{\mathbf{m}}(M)$ are homogeneous and hence so are all the elements of $min.SuppG_{\mathbf{m}}(M)$ (see eg. page 201 [24]). So $\mathbf{p}$ is homogeneous.

Suppose $[\mathbf{p}]_1 = \mathbf{m} \cdot T$ where $[\mathbf{p}]_1 := \mathbf{p} \cap [R_{\mathbf{m}}(M)]_1 = \mathbf{p} \cap \mathbf{m} \cdot T$. Then $\mathbf{p} = \mathbf{m} \oplus_{i>0} \mathbf{m}^i T^i$ is maximal. This contradicts the minimality of $\mathbf{p}$ and the fact that $dimG_{\mathbf{m}}(M) = dimM > 0$. Hence $[\mathbf{p}]_1 \neq \mathbf{m} \cdot T$. So there exists $x \in \mathbf{m}$ such that $x \cdot T \notin \mathbf{p}$.

From page 17 of [8] the sequence

$$0 \longrightarrow R_{\mathbf{m}}(M)_{\mathbf{p}}/(0:x)_{\mathbf{p}} \xrightarrow{x} R_{\mathbf{m}}(M)_{\mathbf{p}} \longrightarrow R_{\mathbf{m}}(M)_{\mathbf{p}}/x \cdot R_{\mathbf{m}}(M)_{\mathbf{p}} \longrightarrow 0$$

is exact and $(0:x)_{\mathbf{p}} = 0$ as x is a non-zero divisor of $R_{\mathbf{m}}(M)_{\mathbf{p}}$ since $xT \notin \mathbf{p}$. By Theorem 4.2 of [15] we have

$$R_{\mathbf{m}}(M)_{\mathbf{p}}/x \cdot R_{\mathbf{m}}(M)_{\mathbf{p}} \cong (R_{\mathbf{m}}(M)/x \cdot R_{\mathbf{m}}(M))_{\mathbf{p}}.$$

Let $y \in \mathbf{m}$ be an arbitrary element. Then $y = x.yT/xT$ in $R_{\mathbf{m}}(M)_{\mathbf{p}}$. Hence $(\mathbf{m}R_{\mathbf{m}}(M))_{\mathbf{p}} = xR_{\mathbf{m}}(M)_{\mathbf{p}}$. Since $R_{\mathbf{m}}(M)/\mathbf{m} \cdot R_{\mathbf{m}}(M) = G_{\mathbf{m}}(M)$, we get the following exact sequence

$$0 \longrightarrow R_{\mathbf{m}}(M)_{\mathbf{p}} \xrightarrow{x} R_{\mathbf{m}}(M)_{\mathbf{p}} \longrightarrow G_{\mathbf{m}}(M)_{\mathbf{p}} \longrightarrow 0.$$

Note that $R_{\mathbf{m}}(M)_{\mathbf{p}} \neq 0$ by Corollary IV.1.8 [24].

As $\mathbf{p} \in min.SuppG_{\mathbf{m}}(M)$ it follows that $\mathbf{p} \in min.AssG_{\mathbf{m}}(M)$ (Theorem 6.5(iii) [15]). Therefore by the definition of dimension, $dimG_{\mathbf{m}}(M)_{\mathbf{p}} = 0$, since a minimal prime of $G_{\mathbf{m}}(M)$ is becoming maximal in $G_{\mathbf{m}}(M)_{\mathbf{p}}$ and there is a one-to-one correspondence in the two sets of primes. So, by Theorem 15.7 [15], we get

$$dimR_{\mathbf{m}}(M)_{\mathbf{p}} = dimG_{\mathbf{m}}(M)_{\mathbf{p}} + 1 = 1.$$

Since $\mathbf{p} \in SuppR_{\mathbf{m}}(M)$ it follows that there exists $P \in min.SuppM$ such that $\widetilde{P} \subseteq \mathbf{p}$ where $\widetilde{P} := \bigoplus_{i \geq 0}(\mathbf{m}^i \cap P) \cdot T^i$ and $\mathbf{m} \subseteq [\mathbf{p}]_0$ (Lemma IV.1.7 [24]).

Now the degree 0 part of $\mathbf{p}$ is $\mathbf{m}$ and $[\widetilde{P}]_0 = P$ which is minimal and hence not $\mathbf{m}$. This implies that $\widetilde{P} \subset \mathbf{p}$.

Since $dimR_{\mathbf{m}}(M)_{\mathbf{p}} = 1$ there cannot exist any prime q such that $\tilde{P} \subset q \subset \mathbf{p}$. As M is quasi-unmixed $dimA/P = dimM$ for all $P \in min.SuppM$. So $dimR_{\mathbf{m}}(A)_{\tilde{P}} = dimA/P + 1$ (Theorem 15.7 [15]).

From the definition of dimension and the fact that A, and hence $R_{\mathbf{m}}(A)$, are catenary (Lemma 3.1.1) it follows that

$$\begin{aligned} dimR_{\mathbf{m}}(A)/\mathbf{p} &= dimR_{\mathbf{m}}(A)/\tilde{P} - 1 \\ &= dimA/P \\ &= dimM \\ &= dimG_{\mathbf{m}}(M). \end{aligned}$$

Therefore $G_{\mathbf{m}}(M)$ is equidimensional and hence quasi-unmixed, as these coincide in the graded case (Lemma 2.1.8).

○

The following theorem gives a situation where we can bound $n_A(M)$ by $n_{G_{\mathbf{m}}(A)}(G_{\mathbf{m}}(M))$.

Theorem 3.1.3 *Let $(A, \mathbf{m})$ be a local ring and let M be an A-module. Let $G_{\mathbf{m}}(M)$ denote the associated graded ring of M with respect to $\mathbf{m}$. Then*

(i) $n_A(M) \geq n_{G_{\mathbf{m}}(A)}(G_{\mathbf{m}}(M))$.

(ii) Assume that

$$\inf_{\mathcal{Q}\ homogeneous,\ \mathbf{m}-primary} \left\{ \frac{\ell_A(G_{\mathbf{m}}(M)/\mathcal{Q} \cdot G_{\mathbf{m}}(M))}{e(\mathcal{Q}; G_{\mathbf{m}}(M))} \right\} > 0.$$

If $n_A(M) < \infty$ then there exists $\alpha_M < \infty$ such that

$$\frac{e(in_M(\mathbf{q}); G_{\mathbf{m}}(M))}{e(\mathbf{q}; M)} \leq \alpha_M$$

for all $\mathbf{m}$-primary ideals $\mathbf{q}$ of A.

(iii) If there exists $\alpha_M < \infty$ such that

$$\frac{e(in_M(\mathbf{q}); G_{\mathbf{m}}(M))}{e(\mathbf{q}; M)} \leq \alpha_M$$

for all $\mathbf{m}$-primary ideals $\mathbf{q}$ of A, then $n_{G_{\mathbf{m}}(A)}(G_{\mathbf{m}}(M)) < \infty$ implies that $n_A(M) < \infty$. In this case $n_A(M) < \alpha_M \cdot n_{G_{\mathbf{m}}(A)}(G_{\mathbf{m}}(M))$.

Proof:(i) By Lemma 2.2.3 $\ell_A(M/\mathbf{q} \cdot M) = \ell_A(G_{\mathbf{m}}(M)/in_M(\mathbf{q}) \cdot G_{\mathbf{m}}(M))$. Further, using the fact $in_A(\mathbf{q})^n \subseteq in_A(\mathbf{q}^n)$, we have

$$\begin{aligned} e(\mathbf{q}; A) &= \lim_{n\to\infty} \tfrac{d!}{n^d} \cdot \ell_A(A/\mathbf{q}^n \cdot A) \\ &= \lim_{n\to\infty} \tfrac{d!}{n^d} \cdot \ell_A(G_{\mathbf{m}}(A)/in_A(\mathbf{q}^n) \cdot G_{\mathbf{m}}(A)) \\ &\leq \lim_{n\to\infty} \tfrac{d!}{n^d} \cdot \ell_A(G_{\mathbf{m}}(A)/in_A(\mathbf{q})^n \cdot G_{\mathbf{m}}(A)) \\ &= e(in_A(\mathbf{q}); G_{\mathbf{m}}(A)). \end{aligned}$$

From this it follows immediately that (i) holds.

(ii) Suppose that

$$\frac{\ell_A(G_{\mathbf{m}}(M)/in_M(\mathbf{q})\cdot G_{\mathbf{m}}(M))}{e(in_M(\mathbf{q});G_{\mathbf{m}}(M))}\geq\beta>0$$

for all $\mathbf{m}$-primary ideals $\mathbf{q}$ of A. Then

$$\frac{\ell_A(G_{\mathbf{m}}(M)/in_M(\mathbf{q})\cdot G_{\mathbf{m}}(M))}{e(\mathbf{q};M)}\geq\beta\cdot\frac{e(in_M(\mathbf{q});G_{\mathbf{m}}(M))}{e(\mathbf{q};M)}.$$

By Lemma 2.2.3 we have

$$\frac{\ell_A(M/\mathbf{q}\cdot M)}{e(\mathbf{q};M)}\geq\beta\cdot\frac{e(in_M(\mathbf{q});G_{\mathbf{m}}(M))}{e(\mathbf{q};M)}.$$

Since $n_A(M)<\infty$ by assumption, we get

$$\frac{e(in_M(\mathbf{q});G_{\mathbf{m}}(M))}{e(\mathbf{q};M)}\leq(1/\beta)\cdot\frac{\ell_A(M/\mathbf{q}\cdot M)}{e(\mathbf{q};M)}\leq(1/\beta)n_A(M)=:\alpha_M<\infty.$$

(iii) Suppose that

$$\frac{e(in_M(\mathbf{q});G_{\mathbf{m}}(M))}{e(\mathbf{q};M)}\leq\alpha_M<\infty.$$

Then, since $\ell_A(M/\mathbf{q}\cdot M)=\ell_A(G_{\mathbf{m}}(M)/in_M(\mathbf{q})\cdot G_{\mathbf{m}}(M))$ (Lemma 2.2.3), we have

$$\frac{\ell_A(M/\mathbf{q}\cdot M)}{e(\mathbf{q};M)}\leq\alpha_M\cdot\frac{\ell_A(G_{\mathbf{m}}(M)/in_M(\mathbf{q})\cdot G_{\mathbf{m}}(M))}{e(in_M(\mathbf{q});G_{\mathbf{m}}(M))}.$$

Hence $n_A(M)<\alpha_M\cdot n_{G_{\mathbf{m}}(M)}(G_{\mathbf{m}}(M))<\infty$.

○

If $n_{G_{\mathbf{m}}(M)}(G_{\mathbf{m}}(M))<\infty$ and α_M of Theorem 3.1.3 (iii) exists, then $n_A(M)<\infty$ if $M=A$ because of the following result by C. Lech.

Theorem 3.1.4 *Let* $\mathbf{q}$ *be an* $\mathbf{m}$*-primary ideal in a local ring* $(A,\mathbf{m})$ *of dimension* d*. Then*

$$\frac{\ell_A(A/\mathbf{q})}{e(\mathbf{q};A)}\geq\frac{1}{d!\cdot e(A)}.$$

Proof: See Theorem 3 of [14].

○

As a consequence of the above two theorems we get

Theorem 3.1.5 *Let A be a local ring. Then $n_A < \infty$ if and only if there exists $\alpha_A < \infty$ such that*

$$\frac{e(in_A(\mathbf{q}); G_{\mathbf{m}}(A))}{e(\mathbf{q}; A)} \leq \alpha_A$$

for all $\mathbf{m}$-primary ideals $\mathbf{q}$ of A, and $n_{G_{\mathbf{m}}(A)} < \infty$

Proof: Assume that $n_A < \infty$. By Theorem 3.1.3(i), $n_{G_{\mathbf{m}}(A)} < \infty$. Hence by Theorem 3.1.4 and Theorem 3.1.3(ii) there exists $\alpha_A < \infty$ such that

$$\frac{e(in_A(\mathbf{q}); G_{\mathbf{m}}(A))}{e(\mathbf{q}; A)} \leq \alpha_A$$

for all $\mathbf{m}$-primary ideals $\mathbf{q}$ of A. The converse follows from Theorem 3.1.3(iii).

○

The following example relies on an exercise given in [8] (Exercise 5.2). This exercise shows that, for a domain $G_{\mathbf{m}}(A)$, a principal ideal $a \cdot A$ is such that $in_A(a \cdot A) = in_A(a) \cdot G_{\mathbf{m}}(A)$.

Suppose that $x = a \cdot y \in a \cdot A$, then if

$$in_A(a) = 0$$

or

$$in_A(y) = 0$$

it follows that $a \in \mathbf{m}^n$ or $y \in \mathbf{m}^n$ for all n.

Hence $in_A(x) = 0$, as $a \cdot y \in \mathbf{m}^n$ for all n, and

$$in_A(x) = in_A(a) \cdot in_A(y).$$

If $in_A(a), in_A(y) \neq 0$ then

$$in_A(a) \cdot in_A(y) \neq 0$$

as $G_{\mathbf{m}}(A)$ is a domain. Now

$$in_A(a) \cdot in_A(y) = in_A(a \cdot y)$$

so it follows that $in_A(x)$ is a multiple of $in_A(a)$. Therefore $in_A(a \cdot A)$ is generated by $in_A(a)$.

Example 3.1.6 Let A be a 1-dimensional local integral domain such that $G_{\mathbf{m}}(A)$ is also an integral domain. Let $\mathbf{q} = a \cdot A$ be a parameter ideal of A. Now $in_A(\mathbf{q}) = in_A(a) \cdot G_{\mathbf{m}}(A)$ (Exercise 5.2 [8]). Since $G_{\mathbf{m}}(A)$ is an integral domain, by Subsection 2.1.2 Blowing-up rings, $in_A(a^n) \cdot G_{\mathbf{m}}(A) = in_A(a)^n \cdot G_{\mathbf{m}}(A)$. Then we have

$$\ell_A(G_{\mathbf{m}}(A)/in_A(\mathbf{q})^n \cdot G_{\mathbf{m}}(A)) = \ell_A(G_{\mathbf{m}}(A)/in_A(\mathbf{q}^n) \cdot G_{\mathbf{m}}(A)).$$

Hence

$$\frac{e(in_A(\mathbf{q}); G_{\mathbf{m}}(A))}{e(\mathbf{q}; A)} = 1 = \alpha_A$$

and $n_A = n_{G_{\mathbf{m}}(A)}$.

Thus, if there exists α_A satisfying Theorem 3.1.5, then the finitness of n_A and $n_{G_{\mathbf{m}}(A)}$ are equivalent. In the rest of this section we provide some conditions for the existence of such an α_A.

Lemma 3.1.7

$$\frac{e(in_A(\mathbf{q}); G_{\mathbf{m}}(A))}{e(\mathbf{q}; A)} \leq 1 + \sup_{n \in N}\{\ell_A\left(\frac{in_A(\mathbf{q}^n) \cdot G_{\mathbf{m}}(A)}{in_A(\mathbf{q})^n \cdot G_{\mathbf{m}}(A)}\right)\}.$$

Proof: Applying Lemma 2.2.3 and the properties of multiplicities, we have

$$\begin{aligned}
\frac{e(in_A(\mathbf{q}); G_{\mathbf{m}}(A))}{e(\mathbf{q}; A)} &= \frac{\lim_{n\to\infty} \frac{d!}{n^d} \cdot \ell_A(G_{\mathbf{m}}(A)/in_A(\mathbf{q})^n \cdot G_{\mathbf{m}}(A))}{\lim_{n\to\infty} \frac{d!}{n^d} \cdot \ell_A(A/\mathbf{q}^n \cdot A)} \\
&= \frac{\lim_{n\to\infty} \frac{d!}{n^d} \cdot \ell_A(G_{\mathbf{m}}(A)/in_A(\mathbf{q})^n \cdot G_{\mathbf{m}}(A))}{\lim_{n\to\infty} \frac{d!}{n^d} \cdot \ell_A(G_{\mathbf{m}}(A)/in_A(\mathbf{q}^n) \cdot G_{\mathbf{m}}(A))} \\
&= 1 + \lim_{n\to\infty} \ell_A\left(\frac{in_A(\mathbf{q}^n) \cdot G_{\mathbf{m}}(A)}{in_A(\mathbf{q})^n \cdot G_{\mathbf{m}}(A)}\right) / e(\mathbf{q}; A) \\
&\leq 1 + \sup_{n \in N}\{\ell_A\left(\frac{in_A(\mathbf{q}^n) \cdot G_{\mathbf{m}}(A)}{in_A(\mathbf{q})^n \cdot G_{\mathbf{m}}(A)}\right)\}.
\end{aligned}$$

○

The following theorem shows that in order to show the existence of such an α_A, we need only consider integral domains.

Theorem 3.1.8 *Let $(A, \mathbf{m})$ be a local ring and let $(G_{\mathbf{m}}(A), \mathbf{m}^*)$ be the associated graded ring of A with respect to $\mathbf{m}$. Let $\mathbf{q}$ be an $\mathbf{m}$-primary ideal of A and $in_A(\mathbf{q})$ the initial ideal of $\mathbf{q}$ in $G_{\mathbf{m}}(A)$. Then we have*

$$\frac{e(in_A(\mathbf{q}); G_{\mathbf{m}}(A))}{e(\mathbf{q}; A)} \leq \max_{\mathbf{p}_i \in Assh\ A} \left\{\frac{e(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p}_i))}{e(\mathbf{q}; A/\mathbf{p}_i)}\right\}.$$

Proof: By Theorem 14.7 of [15] we have

$$e(\mathbf{q}; A) = \sum_{\mathbf{p}_i \in Assh\ A} e(\mathbf{q}; A/\mathbf{p}_i) \cdot \ell_A(A_{\mathbf{p}_i}).$$

Set $L(A) := \sum_{\mathbf{p}_i \in Assh\ A} \ell_A(A_{\mathbf{p}_i})$. Take $\mathbf{p} \in Ass(A)$ such that $dim A/\mathbf{p} = dim A$ and consider the following exact sequence

$$0 \longrightarrow A/\mathbf{p} \longrightarrow A \longrightarrow A/a \cdot A \longrightarrow 0 \qquad (3.1)$$

where $a \in A$ such that $\mathbf{p} = Ann(a)$ by the definition of $Ass(A)$. This sequence is exact as $\mathbf{p} = Ann(a) = 0 :_A a$ (see eg. 2 on page 17 of [8]). From here we get the exact sequence

$$0 \longrightarrow (A/\mathbf{p})_{\mathbf{p}'} \longrightarrow A_{\mathbf{p}'} \longrightarrow (A/a \cdot A)_{\mathbf{p}'} \longrightarrow 0$$

for any $\mathbf{p}' \in Spec A$ such that $dim(A/\mathbf{p}') = dim A$ (Theorem 4.5 [15]). Hence

$$\ell_A(A_{\mathbf{p}'}) = \ell_A((A/\mathbf{p})_{\mathbf{p}'}) + \ell_A((A/a \cdot A)_{\mathbf{p}'}).$$

Now if $(A/\mathbf{p})_{\mathbf{p}'} \neq 0$ then $\mathbf{p} \subseteq Ann(A/\mathbf{p}) \subseteq \mathbf{p}'$ (see the notes on the support of a module at the top of page 26 of [15]). So $\sqrt{\mathbf{p}} = \mathbf{p} \subseteq \mathbf{p}'$ (page 3 [15]). Now $(A/\mathbf{p})_{\mathbf{p}} \cong A_{\mathbf{p}}/\mathbf{p} \cdot A_{\mathbf{p}}$ (Theorem 4.2 [15]) and $\mathbf{p} \cdot A_{\mathbf{p}}$ is maximal so $(A/\mathbf{p})_{\mathbf{p}}$ is a field. Therefore

$$\ell_A((A/\mathbf{p})_{\mathbf{p}'}) = \begin{cases} 0, & \text{if } \mathbf{p}' \neq \mathbf{p} \\ 1, & \text{if } \mathbf{p}' = \mathbf{p} \end{cases}$$

Hence $L(A/a \cdot A) = L(A) - 1$ and we can apply induction on $L(A)$.

Set $L(A) = 1$ so that $L(A/a \cdot A) = 0$. This implies that there are no primes $\bar{\mathbf{p}} \in Spec(A/a{\cdot}A)$ such that $dim(A/a{\cdot}A)/\bar{\mathbf{p}} = dim(A/a{\cdot}A)$ and hence $dim(A/a{\cdot}A) < dimA$. By the Theorem 2.3.6 and the above exact sequence (1) we have

$$e(in_A(\mathbf{q}); G_{\mathbf{m}}(A)) = e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p})) + e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/a \cdot A)).$$

So $e(in_A(\mathbf{q}); G_{\mathbf{m}}(A)) \leq L(A) \cdot e(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p}))$ as $dim(A/a \cdot A) < dimA$ and $L(A) = 1$.

Now suppose $L(A) > 1$. Again by applying Theorem 2.3.6 and the sequence (1) we get

$$e(in_A(\mathbf{q}); G_{\mathbf{m}}(A)) = e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p})) + e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/a \cdot A)).$$

Since $L(A/a \cdot A) < L(A)$ the induction assumption gives

$$e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/a \cdot A)) \leq L(A/a \cdot A) \cdot e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p} + a \cdot A)).$$

Now, the sequence

$$0 \longrightarrow (\mathbf{p} + a \cdot A)/\mathbf{p} \longrightarrow A/\mathbf{p} \longrightarrow A/(\mathbf{p} + a \cdot A) \longrightarrow 0$$

is exact so applying Theorem 2.3.6 again we have

$$e(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p})) = e^*(in_A(\mathbf{q}); G_{\mathbf{m}}((\mathbf{p}+a{\cdot}A)/\mathbf{p})) + e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/(\mathbf{p}+a{\cdot}A))).$$

So $e(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p})) \geq e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/(\mathbf{p} + a \cdot A)))$ and

$$e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/a \cdot A)) \leq L(A/a \cdot A) \cdot e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p})),$$

and hence

$$\begin{aligned} e(in_A(\mathbf{q}); G_{\mathbf{m}}(A)) &\leq e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p})) + L(A/a \cdot A) \cdot e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p})) \\ &= (L(A/a \cdot A) + 1) \cdot e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p})). \end{aligned}$$

Hence

$$e(in_A(\mathbf{q}); G_{\mathbf{m}}(A)) \leq L(A) \cdot e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p})).$$

Now choose $\mathbf{p} = \mathbf{p}_i$ such that $e(\mathbf{q}; A/\mathbf{p}_i) \geq e(\mathbf{q}; A/\mathbf{p}_j)$ for all $\mathbf{p}_i \in Assh\, A$. Then

$$\begin{aligned} \frac{e(in_A(\mathbf{q}); G_{\mathbf{m}}(A))}{e(\mathbf{q}; A)} &\leq \frac{L(A) \cdot e^*(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p}_i))}{L(A) \cdot e^*(\mathbf{q}; A/\mathbf{p}_i)} \\ &\leq \max_{\mathbf{p}_j \in Assh\, A} \left\{ \frac{e(in_A(\mathbf{q}); G_{\mathbf{m}}(A/\mathbf{p}_j))}{e(\mathbf{q}; A/\mathbf{p}_j)} \right\}. \end{aligned}$$

○

3.2 Initial ideals

In this section we are concerned with the graded case, namely with the case $M = R/I$ where $R = k[X_1, \ldots, X_n]$ and I is a homogeneous ideal. If I is a monomial ideal, Stückrad and Vogel [25] have already shown that $n_A(A) < \infty$. In the general case it is known that with each I one can associate monomial ideals, called initial ideals, and denote them by $lt(I)$. These ideals usually have very close connections with each other. Hence in this section we want to study Problem 2 given in [25]:

Problem 3.2.1 *Describe a relationship between $n_R(R/I)$ and $n_R(R/lt(I))$.*

The exact definition of $lt(I)$ is as follows (see [8], Chapter 15):

Set $R := k[X_1, \ldots, X_n]$ where k is a field. Note that by a **monomial** we will mean a product of the variables $X_1, \ldots, X_n$. The **degree** of a monomial m will just be the number of factors, counting repeats, in this product and denoted by $deg(m)$. A monomial order on R is a total well order $>$ on the monomials of R such that if m_1, m_2, $m \neq 1$ are monomials of R then $m_1 > m_2$ implies $m_1 m > m_2 m > m_2$. Usually we consider graded (or homogeneous) monomial orders, i.e. if $deg(m_1) > deg(m_2)$ then $m_1 > m_2$.

If $>$ is a monomial order, then the **initial term** of $f \in I$ is the greatest term of f with respect to $>$. It is denoted by $lt_>(f)$ or $lt(f)$ (when there is no danger of confusion). The **initial ideal** of I is

$$lt(I) = (\{lt(f) | f \in I\}).$$

Note: Initial ideals of homogeneous polynomial ideals and the ideals in local rings given in Chapter 2 have a lot of common properties. Therefore $lt(I)$ is often denoted by $in(I)$ as well. However, in order to avoid confusion we will use, in this dissertation, the notation $lt(I)$.

A very important property of $lt(I)$ is the following result given by Macaulay (see eg. Theorem 15.26 of [8]):

Lemma 3.2.2 *R/I and $R/lt(I)$ have the same Hilbert function.*

There is a connection between this construction and the **weight function**. Recall that a **weight function** λ on R is a linear function $\mathbf{R}^n \to \mathbf{R}$. This function is said to be **integral** if it comes from a linear map $\mathbf{Z}^n \to \mathbf{Z}$. It is easiest now to think of λ as a function on monomials. That is, if $m = x^a$ we would write $\lambda(m) \in \mathbf{Z}$ not $\lambda(a)$. For example the degree defines a weight function: $deg(x^a) = a_1 + \cdots + a_n$. The weight order $>_\lambda$ associated to λ on the monomials of R is defined by the rule $m >_\lambda n$ if and only if $\lambda(m) > \lambda(n)$. This is a partial order. We say that λ is compatible with a given monomial order $>$ if $m >_\lambda n$ implies $m > n$. For example the weight function deg is compatible with graded monomial orders. There are always compatible orders, in fact, it can be shown that every monomial order is the so-called lexicographic product of weight orders (see [20]). The connection is clearly set out in Proposition 15.16 of [8].

Now if $g \in R$ then the **initial term**, $lt_\lambda(g)$, of g is the sum of all the terms of g that are maximal with respect to $>_\lambda$. Also let

$$lt_\lambda(I) := (\{lt_\lambda(g) | g \in I\}) \subseteq R$$

be the **initial ideal** of I with respect to $>_\lambda$.

With these tools we can now construct the Bayer deformation of an ideal I in R.

Take $I \subset R = k[X_1, \ldots, X_n]$ a homogeneous ideal. Let $>_\lambda$ be a weight order that is compatable with the degree where, for any $t \in k \backslash \{0\}$, there is an automorphism of R carrying X_i to $t^{-\lambda(X_i)} X_i$.

Now for $f \in R$ we write

$$f = \sum u_i \cdot m_i$$

where m_i is monomial and $u_i \in k \backslash \{0\}$. Map

$$f \longmapsto f^* := t^b \cdot f(X_1 t^{-\lambda(X_1)}, \ldots, X_n t^{-\lambda(X_n)})$$

where $b = \max\{\lambda(m_i)\}$, so that $f^* \in R[t]$. Set

$$I^* := \{f^* | f \in I\} \subseteq R[t]$$

and call this the **Bayer deformation** of I. It is clear that f^* is $lt_\lambda(f)$ plus t times a polynomial in t and $X_1, \ldots, X_n$.

We now need to define two more properties of algebraic structures:

A finitely generated A-module is said to be **free** if it is isomorphic to a direct sum of n copies of A.

An A-module F is **flat** if for every monomorphism $M' \to M$ of A-modules, the induced map

$$F \otimes_A M' \longrightarrow F \otimes_A M$$

is again a monomorphism.

With these definitions and the above construction we get the following result:

Theorem 3.2.3 *For any ideal $I \subset R = k[X_1, \ldots, X_n]$, the $k[t]$-algebra $R[t]/I^*$ is free, and thus flat, as a $k[t]$-module. Furthermore*

$$(R[t]/I^*)_t \cong (R/I)[t]_t$$

and

$$R/lt_\lambda(I) \cong R[t]/(I^* + t \cdot R[t]).$$

Proof: See Theorem 15.17 of [8].

○

We can now proceed with our investigation of equidimensionality.

Lemma 3.2.4 *Let $(A, \mathbf{m})$ be a local ring and let M be a finitely generated A-module. Suppose $t \notin \mathbf{p}$ for all $\mathbf{p} \in min.SuppM$. Then M is quasi-unmixed if and only if M_t is quasi-unmixed.*

Proof: Since $t \notin \mathbf{p}$ for all $\mathbf{p} \in min.SuppM$ it follows that there is a one-to-one correspondence between these primes and the primes $\mathbf{p} := \mathbf{p} \cdot A_t \in min.SuppM_t$ (Theorem 4.1 [15]). If M is quasi-unmixed, then by Theorem 18.17 (iii) and Lemma 19.4 of [11] we see that

$$dim(A/\mathbf{p})_t = dim(A/\mathbf{p}) - 1.$$

Hence $dim(A_t/\mathbf{p}A_t) = dim(A/\mathbf{p})_t = dimM_t$. Now assume that $dimA_t/\mathbf{p} = dimM_t$ for all $\mathbf{p} \in min.SuppM_t$ so it follows, again from Theorem 4.2 of [15], that

$$\begin{aligned} dim(A/\mathbf{p})_t &= dim(A_t/\mathbf{p} \cdot A_t) \\ &= dim(A_t/\mathbf{p}) \\ &= dimM_t. \end{aligned}$$

It now follows that

$$dim(A/\mathbf{p}) = dimM.$$

○

The following may already be known but we cannot find it in the literature, so we include the full proof here. In the paper [13] this theorem was proved for unmixed ideals (Corollary 1). This theorem shows that by passing to initial ideals the quasi-unmixedness condition is preserved. Again we refer the reader to Proposition 15.16 of [8] for the connection between monomial term orders and weight orders (see the notes after Lemma 3.2.2).

Theorem 3.2.5 *Set $R := k[X_1, \ldots, X_n]$ where k is a field. Let $I \subset R$ be a homogeneous ideal. Then if R/I is quasi-unmixed so is $R/lt(I)$.*

Proof: Since $t \notin \mathbf{p}$ for all $\mathbf{p} \in min.Supp((R/I)[t])$ and R/I is quasi- unmixed if and only if $(R/I)[t]$ is quasi-unmixed it follows that by applying Theorem 3.2.4 we can show that if R/I is quasi-unmixed it follows that $(R/I)[t]_t$ is quasi-unmixed also.

Hence $(R[t]/I^*)_t$ is quasi-unmixed by the properties of the Bayer deformation.

Since $R[t]/I^*$ is a free $k[t]$-module (Theorem 15.17 [8]) it follows that $R[t]/I^*$ has no t-torsion. This then implies that t is a non-zero divisor on $R[t]/I^*$.

Suppose $t^n \in \mathbf{p}$ for some $\mathbf{p} \in min.Supp(R[t]/I^*)$. Then t would be an element of $\mathbf{p}$ also as $\mathbf{p}$ is prime. This contradicts the fact that t is a non-zero divisor on $R[t]/I^*$. Hence $\{1, t, t^2, \ldots\} \not\subseteq \mathbf{p}$ for all $\mathbf{p} \in min.Supp(R[t]/I^*)$.

Theorem 3.2.4 shows that $R[t]/I^*$ is quasi-unmixed and from Theorem 3.2.3 it now follows that $R[t]/I^* + t \cdot R[t]$ is quasi-unmixed. Hence applying the Bayer deformation we get that $R/lt(I)$ is quasi-unmixed.

○

Thus, by the above theorem, passing to the initial ideal the quasi-unmixedness in the necessary condition in Theorem 2.4.1 is preserved. Now we will provide an equivalent condition for the boundness of $n_{R/I}$, and thus, provide a partial answer to Problem 3.2.1 posed above.

Theorem 3.2.6 *Let $R = k[X_1, \ldots, X_n]$ and let $I \subset R$ be a homogeneous quasi-unmixed ideal of R. Let $lt(I)$ denote the leading term ideal of I. Then*

(i) $n_{R/I} \geq n_{R/lt(I)}$,

(ii) There is $\alpha_{R/I} < \infty$ such that

$$\frac{e(lt(\mathbf{q}+I); R/lt(I))}{e(\mathbf{q}; R/I)} \leq \alpha_{R/I}$$

for all $\mathbf{m}_R$-primary ideals $\mathbf{q}$ if and only if $n_{R/I} < \infty$. In this case

$$n_{R/I} \leq \alpha_{R/I} n_{R/lt(I)}.$$

Proof: (i) For an $\mathbf{m}_R$-primary ideal $\mathbf{q}$ we have, by Lemma 3.2.2,

$$\ell_R(R/\mathbf{q}+I) = \ell(R/lt(\mathbf{q}+I)) = \ell(R/lt(\mathbf{q}+I) + lt(I)).$$

In particular $lt(\mathbf{q}+I)$ is an $\mathbf{m}_{R/lt(I)}$-primary ideal of $R/lt(I)$. Since $lt(J)^n + lt(I) \subseteq lt(J^n + I)$ for any ideals $I, J \subseteq R$, by Lemma 3.2.2 we also have

$$\begin{aligned} e(\mathbf{q}; R/I) &= \lim_{n\to\infty} \tfrac{d!}{n^d} \cdot \ell_R(R/\mathbf{q}^n + I) \\ &= \lim_{n\to\infty} \tfrac{d!}{n^d} \cdot \ell_R(R/(\mathbf{q}+I)^n + I) \\ &= \lim_{n\to\infty} \tfrac{d!}{n^d} \cdot \ell_R(R/lt((\mathbf{q}+I)^n + I)) \\ &\leq \lim_{n\to\infty} \tfrac{d!}{n^d} \cdot \ell_R(R/lt((\mathbf{q}+I)^n) + lt(I)) \\ &= e(lt(\mathbf{q}+I); R/lt(I)). \end{aligned}$$

Hence

$$n_{R/I} = \sup\{\frac{\ell_R(R/I+\mathbf{q})}{e(\mathbf{q}; R/I)}\} \geq \sup\{\frac{\ell_R(R/lt(\mathbf{q}+I) + lt(I))}{e(lt(\mathbf{q}+I); R/lt(I))}\} = n_{R/lt(I)}.$$

(ii) Assume that

$$\frac{e(lt(\mathbf{q}+I); R/lt(I))}{e(\mathbf{q}; R/I)} \leq \alpha_{R/I} < \infty.$$

Then the arguments in (i) show that

$$\frac{\ell_R(R/\mathbf{q}+I)}{e(\mathbf{q}; R/I)} \leq \alpha_{R/I} \cdot \frac{\ell_R(R/lt(\mathbf{q}+I) + lt(I))}{e(lt(\mathbf{q}+I); R/lt(I))},$$

by applying the assumption I is quasi-unmixed. Hence by Theorem 3.2.5 $R/lt(I)$ is quasi-unmixed as well. By [25], Theorem 4 we have

$$\frac{\ell_R(R/lt(\mathbf{q}+I)+lt(I))}{e(lt(\mathbf{q}+I);R/lt(I))} \leq n_{R/lt(I)} < \infty.$$

Therefore

$$n_{R/I} \leq \alpha_{R/I} n_{R/lt(I)} < \infty.$$

Conversely, let I be a homogeneous ideal with $n_{R/I} < \infty$. By Theorem 3.1.4 we have

$$\gamma \ell_R(R/lt(\mathbf{q}+I)) = \gamma \ell_R(R/lt(\mathbf{q}+I)+lt(I)) \geq e(lt(\mathbf{q}+I);R/lt(I)),$$

where $\gamma = d!e(R/lt(I)) = d!e(R/I)$. Again by Lemma 3.2.2 we have

$$\begin{aligned} \frac{e(lt(\mathbf{q}+I);R/lt(I))}{e(\mathbf{q};R/I)} &\leq \gamma \frac{\ell_R(R/lt(\mathbf{q}+I))}{e(\mathbf{q};R/I)} \\ &= \gamma \frac{\ell_R(R/\mathbf{q}+I)}{e(\mathbf{q};R/I)} \\ &\leq \gamma n_{R/I} < \infty. \end{aligned}$$

○

In the proof of Theorem 3.2.6 it was observed that $e(\mathbf{q};R/I) \leq e(lt(\mathbf{q}+I);R/lt(I))$. To show that equality does not always occur in this situation we will consider the following example.

Example 3.2.7 Let $R = k[X,Y]$ where k is an infinite field. Also let $f := (X+Y)^2 \in R$. Note that f is homogeneous of degree 2 and neither X or Y divides f. Set $I := f \cdot R$ and $R/I = k[X,Y]/f \cdot R$. Define $\mathbf{q} := (f, Y^3)$. Since $\sqrt{(Y^3)} = (Y)$ and $\sqrt{(f,Y^3)} = (X,Y)$, as $(f,Y) = (X^2,Y)$ so that $\sqrt{(f,Y^3)} = \sqrt{(X^2,Y)} = (X,Y)$, it follows that $\mathbf{q}$ is a parameter ideal of R.

Applying Formula 14.3 of [15] and properties of the multiplicity contained in [19] we have,

$$\begin{aligned} e(\mathbf{q};R/I) &= e(((Y^3), f\cdot R)/f\cdot R; k[X,Y]/f\cdot R) \\ &= 3\cdot e(((Y), f\cdot R)/f\cdot R; R/I) \\ &= 3\cdot 2\cdot e((Y, X+Y)/(X+Y)\cdot R; R/(X+Y)\cdot R) \\ &= 6\cdot e((Y); k[Y]) \\ &= 6. \end{aligned}$$

Assume that $Y > X$. Then $lt(I) = lt(f)\cdot R = Y^2 \cdot R$ and

$$\begin{aligned} lt(\mathbf{q}+I) &= lt((X+Y)^2, Y^3) \\ &= (X^4, X^2Y, Y^2)R. \end{aligned}$$

Hence an analogous calculation gives

$$\begin{aligned} e(lt(\mathbf{q}+I); R/lt(I)) &= e((X^4, X^2Y, Y^2); R/Y^3 \cdot R) \\ &= 3 \cdot \dot{e}((X^4); R/Y \cdot R) \\ &= 3 \cdot e((X^4); R[X]) \\ &= 3 \cdot 4 \cdot \dot{e}((X); R[X]) \\ &= 12. \end{aligned}$$

Therefore

$$e(lt(\mathbf{q}+I); R/lt(I)) = 12 > 6 = e(\mathbf{q}; R/I).$$

Chapter 4

Boundness of $n_A(M)$

This chapter further develops the theory surrounding $n_A(M)$. Specifically, the finiteness of $n_A(M)$ is studied. While a general result is still not possible several positive steps have been taken in that direction.

The first result concerns the low dimensional case. By introducing some well known concepts (see eg. [24] and [4]) we are able to produce a positive result for the case of dimension less than or equal to three.

The next case focuses on parameter ideals. When passing to the associated graded ring certain parameter ideals are preserved. It is these ideals we wish to examine here. We show that if parameter ideals $\mathfrak{q}$ run over this subset of $\mathfrak{m}$-primary ideals in the definition of $\tilde{n}_A(M)$ then we get a bounded quantity. Following this a new proof of Theorem 2.4.2 is produced which gives a better bound for $\tilde{n}_A(M)$ than the one obtained in [25]. We conclude the chapter with two examples that show that this new bound is sharp.

4.1 Low dimensional case

The main purpose of this section is to prove the following theorem:

Theorem 4.1.1 *Let A be a local ring such that $A/\mathfrak{m}$ is infinite. Let M be a quasi-unmixed A-module such that $dimA = dimM =: d$. Now if $dimA \leq 3$ then $n_A(M)$ is finite.*

To prove this theorem we first need some preparation.

If M is an A-module and I is an ideal of A then we define

$$\Gamma_I(M) := 0 :_M (I) = \cup_{k=1}^{\infty}(0 :_M I^k).$$

If $i \geq 0$ is an integer then we denote by $H_I^i(M)$ the i^{th} **local cohomology module** of M with respect to I. Now $H_I^i(M)$ is obtained by applying the right derived functor of Γ_I to an injective resolution of M. Note that

$$H_I^0(M) \cong \Gamma_I(M).$$

If we now consider the local ring $(A, \mathbf{m})$ then

$$H^i_{\mathbf{m}}(M) = 0$$

for all $i > dim\, M$ and the **depth** of M is the least integer i such that $H^i_{\mathbf{m}}(M) \neq 0$. With this definition in place we can characterize generalized Cohen-Macaulay modules as follows:

Lemma 4.1.2 (see Appendix in [24]) If A is a local ring and M is an A-module then M is a generalized Cohen-Macaulay module if and only if

$$\ell_A(H^i_{\mathbf{m}}(M)) < \infty$$

for all i.

Further we introduce the notion of a **canonical module** as given in [4]. If A is a d-dimensional local ring then an A-module K is a canonical module of A if

$$K \otimes_A \widehat{A} \cong Hom_A(H^d_{\mathbf{m}}(A), E_A(A/\mathbf{m}))$$

where $E_A(A/\mathbf{m})$ is the injective envelope of $A/\mathbf{m}$ (for the definition of an injective envelope see [8]).

It is known that, when A is complete then the canonical module of A exists. It is noted in [4] that if

$$h_M : M \longrightarrow Hom_A(Hom_A(M, K), K)$$

is the natural map then if $dim\, M = d$

$$Ker(h_M) = \cap Q$$

where the Q's are primary components of 0 in M such that $dim\, M/Q = dim\, M$.

Finally for a ring A and $x_1, \ldots, x_n \in A$ the **Koszul complex** is a complex K which is defined as

$$\begin{aligned} K_0 &= A \\ K_p &= 0 \text{ if } p \text{ is not in the range } 0 \leq p \leq n \\ K_p &= \oplus A_{e_{i_1 \cdots i_p}}, \end{aligned}$$

where $e_1, \ldots, e_j$ form a basis for K_1 which has rank j. The differential $d : K_p \to K_{p-1}$ is defined as

$$d(e_{i_1 \cdots i_p}) = \sum_{r=1}^{p} (-1)^{r-1} \cdot x_{i_r} e_{i_1 \cdots \hat{i}_r \cdots i_p}.$$

The Koszul complex of M w.r.t. $x_1, ..., x_n$ is written as $K.(x_1, \ldots, x_n)$. For an A-module M

$$H_0(x_1, \ldots, x_n; M) \cong M/(x_1, \ldots, x_n) \cdot M$$

and if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact then so is

$$\begin{aligned}\cdots \longrightarrow\ & H_2(x_1,\ldots,x_n;M) \longrightarrow H_2(x_1,\ldots,x_n;M'') \longrightarrow \\ & H_1(x_1,\ldots,x_n;M') \longrightarrow H_1(x_1,\ldots,x_n;M) \longrightarrow \\ & H_1(x_1,\ldots,x_n;M'') \longrightarrow M'/(x_1,\ldots,x_n)\cdot M' \longrightarrow \\ & M/(x_1,\ldots,x_n)\cdot M \longrightarrow M''/(x_1,\ldots,x_n)\cdot M'' \longrightarrow 0.\end{aligned}$$

Also $(x_1,\ldots,x_n)\cdot H_p(x_1,\ldots,x_n;M) = 0$ for all $p \geq 0$.

We now need an extension of Lech's result (Theorem 3.1.4). A proof of the general extension of Lech's result could not be established but the following lemma restricted to the one-dimensional case is sufficient for our purposes.

Lemma 4.1.3 *Let M be a 1-dimensional module over a local ring $(A,\mathbf{m})$. Then there is a positive number $\beta_A(M)$ such that*

$$\frac{\ell_A(M/\mathbf{q}\cdot M)}{e(\mathbf{q};M)} \geq \beta_A(M)$$

for all $\mathbf{m}$-primary ideals $\mathbf{q}$ of A.

Proof: We may assume without loss of generality that $dimA = 1$. Let $\{\mathbf{p}_1,\ldots,\mathbf{p}_r\}$ be all the minimal primes of A such that $dimM = dimA/\mathbf{p}_i$, $i = 1,\ldots,r$. That is, $\{\mathbf{p}_1,\ldots,\mathbf{p}_r\} = AsshM$.

Theorem 14.7 [15] gives

$$e(\mathbf{q};M) = \sum_{i=1}^{r} e(\mathbf{q};A/\mathbf{p}_i)\cdot \ell_{A_{\mathbf{p}_i}}(M_{\mathbf{p}_i})$$

where $\mathbf{q}$ is considered to be the image of $\mathbf{q}$ in $A/\mathbf{p}_i$ (Proposition 11 on page 341 of [19]).

Now,

$$e(\mathbf{q};M) \leq \left[\sum_{i=1}^{r} \ell_{A_{\mathbf{p}_i}}(M_{\mathbf{p}_i})\right]\cdot max_i\{e(\mathbf{q};A/\mathbf{p}_i)\}.$$

Set $\gamma := \sum_{i=1}^{r} \ell_{A_{\mathbf{p}_i}}(M_{\mathbf{p}_i})$ which depends only on A and M. We note here that since $AsshM \subseteq AssM$ and $AssM$ is finite (Theorem 6.5(i) [15]) it follows that $\gamma < \infty$. Let $\mathbf{p}^* \in \{\mathbf{p}_1,\ldots,\mathbf{p}_r\}$ such that

$$max_i\{e(\mathbf{q};A/\mathbf{p}_i)\} = e(\mathbf{q};A/\mathbf{p}^*).$$

So we have

$$\frac{\ell_A(M/\mathbf{q}\cdot M)}{e(\mathbf{q};M)} \geq \frac{\ell_A(M/\mathbf{q}\cdot M)}{\gamma\cdot e(\mathbf{q};A/\mathbf{p}^*)}$$

Take a primary decomposition $0_M = M_1 \cap \cdots \cap M_\alpha$ of 0_M in M. It follows that $AssM = \{P_1,\ldots,P_\alpha\}$ where M_i is P_i-primary and that $AsshM \subseteq AssM$ (Theorem 6.8(ii) of [15]).

Note that $\mathbf{p}^* \in \{P_1, \ldots, P_\alpha\}$ as $\mathbf{p}^* \in AsshM$. So there exists $M^* \in \{M_1, \ldots, M_\alpha\}$ such that

$$Ass(M/M^*) = \{\mathbf{p}^*\}$$

by the definition of primary decomposition and Theorem 6.8(ii) of [15].

Set $\bar{M} := M/M^*$.

$$0 \longrightarrow M^* \longrightarrow M \longrightarrow \bar{M} \longrightarrow 0$$

is exact so

$$\ell_A(M/\mathbf{q} \cdot M) \geq \ell_A(\bar{M}/\mathbf{q} \cdot \bar{M})$$

by using the exact sequence of Koszul homology modules given before this lemma and properties of the length of a module.

Take a composition series

$$\bar{M} = \bar{M} \supset \bar{M}_1 \supset \cdots \supset \bar{M}_\epsilon = 0$$

such that

$$\bar{M}_i/\bar{M}_{i+1} \cong A/\mathbf{p}_j$$

for some $\mathbf{p}_j \in SpecA$ (Theorem 1 on page 265 of [5]).

Since $\mathbf{p}^* \subseteq \mathbf{p}_j$ and $dimA = 1$ it follows that either $\mathbf{p}^* = \mathbf{p}_j$ or $\mathbf{p}^* = \mathbf{m}$. We will show that

$$\ell_A(\bar{M}/\mathbf{q} \cdot \bar{M}) \geq \ell_A(A/\mathbf{q} + \mathbf{p}^*) - (\epsilon - 1) \cdot e(A).$$

If $\bar{M}_0/\bar{M}_1 \cong A/\mathbf{p}^*$ then

$$\begin{aligned} \ell_A(\bar{M}/\mathbf{q} \cdot \bar{M}) &\geq \ell_A((\bar{M}_0/\bar{M}_1)/\mathbf{q} \cdot (\bar{M}_0/\bar{M}_1)) \\ &= \ell_A(A/\mathbf{q} + \mathbf{p}^*) \\ &\geq \ell_A(A/\mathbf{q} + \mathbf{p}^*) - (\epsilon - 1) \cdot e(A). \end{aligned}$$

If $\bar{M}_0/\bar{M}_1 \cong A/\mathbf{m} \cong k$ we have an exact sequence

$$0 \longrightarrow \bar{M}_1 \longrightarrow \bar{M} \longrightarrow k \longrightarrow 0.$$

Let $\bar{A} = A/Ann\ \bar{M}$ and $\bar{\mathbf{q}} = \mathbf{q} \cdot \bar{A}$. Now

$$\begin{aligned} Ann\ \bar{M} = 0 :_A \bar{M} &= \{a \in A | a \cdot \bar{M} \subset 0\} \\ &= \{a \in A | a \cdot M \subset M^*\} \end{aligned}$$

and

$$\begin{aligned} Ass\bar{A} &= Ass(A/Ann\ \bar{M}) \\ &= \{\mathbf{p} \in SpecA | \mathbf{p} \cdot (a) \subseteq Ann\ \bar{M} \text{ for some } a \in A\} \\ &= \{\mathbf{p} \in SpecA | \mathbf{p} \cdot (a) \cdot M \subseteq M^* \text{ for some } a \in A\} \\ &= AssM/M^* \\ &= \{\mathbf{p}^*\}. \end{aligned}$$

So $Ass\bar{A} = Assh\ M$ and the heights of all the associated primes of $\bar{A}$ are equal. Thus, as $dim\ \bar{A} \leq dim\ A = 1$, it follows that the unmixedness theorem holds for $\bar{A}$.

That is, for all ideals I of $\bar{A}$ generated by r elements the heights of the elements of $Ass(\bar{A}/I)$ are equal.

In particular, by Theorem 17.6 of [15], $\bar{A}$ is a one-dimensional Cohen-Macaulay ring. By [21] Theorem 1.2, $\bar{\mathbf{q}}$ is generated by at most $e(\bar{A})$ elements, say $y_1, \ldots, y_n$ where $n \leq e(\bar{A})$. Applying Koszul homology to the above exact sequence of modules over $\bar{A}$ we get the exact sequence

$$H_1(y_1, \ldots, y_n; k) \longrightarrow \bar{M}_1/\mathbf{q} \cdot \bar{M}_1 \longrightarrow \bar{M}/\mathbf{q} \cdot \bar{M}.$$

So it now follows that

$$\begin{aligned} \ell_A(\bar{M}/\mathbf{q} \cdot \bar{M}) &\geq \ell_A(\bar{M}_1/\mathbf{q} \cdot \bar{M}_1) - n \\ &\geq \ell_A(\bar{M}_1/\mathbf{q} \cdot \bar{M}_1) - e(\bar{A}) \\ &\geq \ell_A(\bar{M}_1/\mathbf{q} \cdot \bar{M}_1) - e(A) \end{aligned}$$

since the exact sequence

$$0 \longrightarrow Ann\ \bar{M} \longrightarrow A \longrightarrow \bar{A} \longrightarrow 0$$

produces $e(A) \geq e(\bar{A})$.

Since $\bar{M}_1 \subset \bar{M}$ it follows that $Ass\bar{M}_1 \subseteq Ass\bar{M}$ (Theorem 6.1(ii) of [15]). From the exact sequence

$$0 \longrightarrow \bar{M}_1 \longrightarrow \bar{M} \longrightarrow k \longrightarrow 0$$

we have

$$\begin{aligned} Ass\bar{M} &\subseteq Ass\bar{M}_1 \cup Ass\, k \\ &= Ass\bar{M}_1 \end{aligned}$$

(Theorem 6.3 of [15]). So $Ass\bar{M}_1 = \{\mathbf{p}^*\}$.

Using induction on ϵ we then get

$$\begin{aligned} \ell_A(\bar{M}/\mathbf{q} \cdot \bar{M}) &\geq \ell_A(\bar{M}_1/\mathbf{q} \cdot \bar{M}_1) - e(A) \\ &\geq \ell_A(A/\mathbf{q} + \mathbf{p}^*) - (\epsilon - 1) \cdot e(A). \end{aligned}$$

Hence

$$\frac{\ell_A(M/\mathbf{q} \cdot M)}{e(\mathbf{q}; M)} \geq \frac{\max\{\ell_A(A/\mathbf{q} + \mathbf{p}^*) - (\epsilon - 1) \cdot e(A),\ 1\}}{\gamma \cdot e(\mathbf{q}; A/\mathbf{p}^*)}.$$

Applying Lech's result (Theorem 3.1.4) to the ring $A/\mathbf{p}^*$ we get

$$\frac{\ell_A(M/\mathbf{q} \cdot M)}{e(\mathbf{q}; M)} \geq \delta(\mathbf{p}^*),$$

for a positive number $\delta(\mathbf{p}^*)$ depending only on A, M and $\mathbf{p}^*$.

○

We are now able to prove Theorem 4.1.1. This requires the equivalent conditions on the finiteness of $\ell_A(H^i_{\mathbf{m}}(M))$ set out in Proposition 16 of the appendix of [24].

Proof of Theorem 4.1.1: Since we are assuming that $A/\mathbf{m}$ is infinite we have $n_A(M) = \tilde{n}_A(M)$, so we can let $\mathbf{q}$ be any parameter ideal of A.

a) If M is a generalized Cohen-Macaulay module then

$$\ell_A(M/\mathbf{q}\cdot M) - e(\mathbf{q};M) \leq \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot \ell_A(H^i_{\mathbf{m}}(M))$$

(Lemma 15 of the appendix of [24]) and $\ell_A(H^i_{\mathbf{m}}(M))$ is finite for all $i = 0, \ldots, d-1$. So

$$\frac{\ell_A(M/\mathbf{q}\cdot M)}{e(\mathbf{q};M)} \leq 1 + \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot \ell_A(H^i_{\mathbf{m}}(M)) < \infty$$

and hence $n_A(M) < \infty$. This fact was first observed in [16]. Now if $dimM \leq 1$ then $n_A(M) < \infty$, as $\ell_A(H^0_{\mathbf{m}}(M)) = \ell_A(M/\mathbf{m}\cdot M) < \infty$, and if $d = 0$ then A is a field and the theorem is trivially true.

b) Now let $d \geq 2$. By Theorem 2.4.1(2) it suffices to prove the statement for the case $M = A = S/\mathbf{p}$, where S is a complete regular ring and $\mathbf{p}$ is a prime ideal of S such that $dimS/\mathbf{p} \leq 3$. If $d = 2$ then A is a generalized Cohen-Macaulay module (applying Theorem 7(iv) and Proposition 16 of the appendix of [24]), and we are done by a) above. Assume that $d = 3$. Since A is complete it has the canonical module K. A also has the A- module $B = Hom_A(K, K)$. Consider the natural map

$$h : A \longrightarrow B \cong Hom_A(Hom_A(A, K), K).$$

As was noted earlier $ker\ h$ is the intersection of all the primary components Q of 0 in A such that $dimA/Q = dimA$. That is, $ker\ h = 0$ in this case. It now follows that the only minimal prime ideal in $Supp_A B$ is 0 as h is one-to-one and A is an integral domain. Applying Theorem 3.2(3) of [4] we see that B satisfies Serre's condition (S_2). That is, $depth(B_{\mathbf{p}}) \geq \min\{2, dimB_{\mathbf{p}}\}$ for all $\mathbf{p} \in Supp_A B$. Hence $depth(B_{\mathbf{p}}) = dimB_{\mathbf{p}}$ (see eg. the proof of Theorem 17.3(iii) in [15]) and, by definition, $B_{\mathbf{p}}$ is a Cohen-Macaulay module for all $\mathbf{p} \in Supp_A B$. As $dimB \leq 3$ it follows from Proposition 16 in the appendix of [24] that B is a generalized Cohen-Macaulay module. Hence $n_A(B) < \infty$.

Let $C = Coker\ h := B/Im\ h \cong B/A$, as h is one-to-one. Since A is quasi-unmixed we can apply (1.11.2) of [4] to show that $dimC \leq 1$. So, we have the exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and from Lemma 2.3.4, $e(\mathbf{q};A) = e(\mathbf{q};B)$. Let $\mathbf{q} + (x, y, z)$ be a parameter ideal of A. Then there is an exact sequence coming from the Koszul homology

$$H_1(x,y,z;C) \longrightarrow A/\mathbf{q}\cdot A \longrightarrow B/\mathbf{q}\cdot B \longrightarrow C/\mathbf{q}\cdot C \longrightarrow 0. \tag{4.1}$$

Lemma 2.2.2 now shows that

$$\ell_A(A/\mathbf{q}\cdot A) \leq \ell_A(B/\mathbf{q}\cdot B) + \ell_A(H_1(x,y,z;C)). \tag{4.2}$$

If $dimC = 0$ then

$$\ell_A((H_1(x,y,z;C)) = 3\ell_A(C),$$

by properties of the Koszul homology. Hence

$$\frac{\ell_A(A/\mathbf{q}\cdot A)}{e(\mathbf{q};A)} \le \frac{\ell_A(B/\mathbf{q}\cdot B)+3\ell_A(C)}{e(\mathbf{q};B)} \le n_A(B)+3\ell_A(C).$$

If $dimC = 1$, we consider C as a module over the one-dimensional ring $\bar{A} = A/Ann\ C$. Denote by $\bar{\mathbf{q}} = (\bar{x},\bar{y},\bar{z})$ the image of $\mathbf{q}$ in $\bar{A}$. Now $\bar{\mathbf{q}}$ has a minimal reduction, say (x_1), in $\bar{A}$. If $x_1 = a\bar{x}+b\bar{y}+c\bar{z}$ then at least one element, say a, among a,b,c must be a unit in $\bar{A}$. This is because x,y,z are a system of parameters of A, and hence are analytically independent, so if $x_1 = a\bar{x}+b\bar{y}+c\bar{z}$ then $0 \ne x_1' = ax+by+cz$, where $x_1 = x_1' + Ann\ C$. This implies that at least one of a,b,c is a unit of A and hence of $\bar{A}$. Therefore $\bar{\mathbf{q}} = (x_1,x_2,x_3)$ where $x_2 = \bar{y}$ and $x_3 = \bar{z}$. We have

$$e(\bar{\mathbf{q}};C) = e((x_1);C)$$

(Lemma 2.3.9) and

$$\ell_A(H_1(x,y,z;C)) = \ell_A(H_1(x_1,x_2,x_3;C)).$$

Consider the following exact sequences of Koszul homology modules:

$$H_1(x_1,x_2;C) \longrightarrow H_1(x_1,x_2,x_3;C) \longrightarrow H_0(x_1,x_2;C),$$

$$H_1(x_1;C) \longrightarrow H_1(x_1,x_2;C) \longrightarrow H_0(x_1;C)$$

and

$$H_0(x_1;C) \longrightarrow H_0(x_1,x_2;C) \longrightarrow 0.$$

These imply that

$$\begin{aligned} \ell_A(H_1(x_1,x_2,x_3;C)) &\le \ell_A(H_1(x_1,x_2;C)) + \ell_A(H_0(x_1,x_2;C)) \\ &\le \ell_A(H_1(x_1;C)) + \ell_A(H_0(x_1;C)) + \ell_A(H_0(x_1,x_2;C)) \\ &\le \ell_A(H_1(x_1;C)) + 2\ell_A(H_0(x_1;C)) \\ &\le 2\ell_A(C/(x_1)\cdot C) + \ell_A(0:_A x_1) \\ &\le 2\ell_A(C/(x_1)\cdot C) + \ell_A(H^0_{\mathbf{m}}(C)), \end{aligned} \tag{4.3}$$

as $H_1(x_1;C) \cong \{\xi \in C | x_1\cdot\xi = 0\} = 0:_C x_1$ and

$$H_0(x_1;C) \longrightarrow H^0_{\mathbf{m}}(C) \longrightarrow 0$$

is exact.

Since $dim\bar{A} = 1$ it follows from case a) above that $n_{\bar{A}}(C) < \infty$. Hence by the definition of $n_{\bar{A}}(C)$ and by Lemma 4.1.3 we have

$$\begin{aligned} \ell_A(C/(x_1)\cdot C) &\le n_{\bar{A}}(C)\cdot e((x_1);C) \\ &= n_{\bar{A}}(C)\cdot e(\bar{\mathbf{q}};C) \\ &\le n_{\bar{A}}(C)\cdot \ell_A(C/\mathbf{q}\cdot C)/\beta_{\bar{A}}(C) \\ &=: u\cdot \ell_A(C/\mathbf{q}\cdot C) \\ &\le u\cdot \ell_A(B/\mathbf{q}\cdot B) \end{aligned}$$

from sequence (2). This, together with (2), (3) and (4) implies that

$$\ell_A(A/\mathbf{q}\cdot A) \leq \ell_A(B/\mathbf{q}\cdot B) + 2u\cdot \ell_A(B/\mathbf{q}\cdot B) + \ell_A(H^0_{\mathbf{m}}(C)).$$

Hence

$$\begin{aligned} \frac{\ell_A(A/\mathbf{q}\cdot A)}{e(\mathbf{q};A)} &\leq (1+2u)\cdot \frac{\ell_A(B/\mathbf{q}\cdot B)}{e(\mathbf{q};B)} + \ell_A(H^0_{\mathbf{m}}(C)) \\ &\leq (1+2u)\cdot n_A(B) + \ell_A(H^0_{\mathbf{m}}(C)) \\ &< \infty. \end{aligned}$$

Therefore

$$n_A(M) < \infty.$$

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4.2 Restriction to parameter ideals

While section 4.1 considered low dimensional local rings this section examines a special class of parameter ideals. We know that the result we are looking for holds for low dimensional local rings (Theorem 4.1.1) and the key result in this section shows that, if we restrict ourselves to certain parameter ideals, then we also get a positive solution.

To begin with we need to establish these special parameter ideals and prove some essential properties of them. To do this we must refer back to Chapter two and the concept of the associated graded ring of a local ring A. Recall that if $(A, \mathbf{m})$ is a local ring then

$$G_{\mathbf{m}}(A) := A/\mathbf{m} \oplus \mathbf{m}/\mathbf{m}^2 \oplus \cdots$$

is the associated graded ring of A with respect to $\mathbf{m}$. The maximal homogeneous ideal $\mathbf{m}^*$ of $G_{\mathbf{m}}(A)$ is given by

$$\mathbf{m}^* := \mathbf{m}/\mathbf{m}^2 \oplus \mathbf{m}^2/\mathbf{m}^3 \oplus \cdots.$$

Now if $x \in A$ then we will denote the initial form of x in $G_{\mathbf{m}}(A)$ by x^*. That is,

$$x^* := in_A(x).$$

We now make the following definition:

Definition 4.2.1 *For an ideal I in $(A, \mathbf{m})$ we will write $\mu(I)$ for the* ***minimal number of generators*** *of I. Also $s(I)$ will denote the* ***analytic spread*** *of I. This is defined to be $dim\, G_I(A) \otimes_A k$ where $k \cong A/\mathbf{m}$.*

Note that if $\mathbf{q}$ is an $\mathbf{m}$-primary ideal of A then $s(\mathbf{q}) = dimA$ and if $\mathbf{q}'$ is a minimal reduction of $\mathbf{q}$ then $\mu(\mathbf{q}') = s(\mathbf{q})$ (Remark 10.11 and Corollary 10.19 of [11]).

With this notation established we can begin with two lemmas.

Lemma 4.2.2 *Let $G_{\mathbf{m}}(A)$ be the associated graded ring of A and let $y_1, \ldots, y_d$ be a system of parameters of $G_{\mathbf{m}}(A)$. Now, deg $(y_i) = n$ for all $i = 1, \ldots, d$ if and only if $(y_1, \ldots, y_d)$ is a minimal reduction of $(\mathbf{m}^*)^n$.*

Proof: First assume that $deg\ (y_i) = n$ for all $i = 1, \ldots, d$ so that

$$(y_1, \ldots, y_d) \subseteq (\mathbf{m}^*)^n$$

as $y_i \in \mathbf{m}^*$ for all $i = 1, \ldots, d$. Now for any $r > 0$ we have

$$(y_1, \ldots, y_d) \cdot ((\mathbf{m}^*)^n)^r \subseteq ((\mathbf{m}^*)^n)^{r+1}.$$

We want to show that

$$((\mathbf{m}^*)^n)^{r+1} \subseteq (y_1, \ldots, y_d) \cdot ((\mathbf{m}^*)^n)^r$$

for some $r > 0$. Since $(y_1, \ldots, y_d)$ is a parameter ideal of $G_{\mathbf{m}}(A)$ we have

$$(\mathbf{m}^*)^p \subseteq (y_1, \ldots, y_d)$$

for some $p > 0$. Take r such that $n(r+1) \geq p$. Now

$$((\mathbf{m}^*)^n)^{r+1} \subseteq (y_1, \ldots, y_d).$$

Let a be a homogeneous element of $((\mathbf{m}^*)^n)^{r+1}$ so that

$$a = \sum_{i=1}^{d} b_i \cdot y_i \text{ where } b_i \in G_{\mathbf{m}}(A).$$

We can assume that the b_i are homogeneous and that $deg(b_i \cdot y_i) = deg(a)$ for all $i = 1 \ldots, d$.

$$\begin{aligned} deg(b_i \cdot y_i) &= deg(b_i) + deg(y_i) \\ &= deg(b_i) + n. \end{aligned}$$

So $deg(a) = deg(b_i) + n$ and as $a \in ((\mathbf{m}^*)^n)^{r+1}$ it follows that

$$deg(a) \geq n(r+1).$$

That is, $deg(b_i) + n \geq n(r+1)$ and hence $deg(b_i) \geq nr$. So $b_i \in ((\mathbf{m}^*)^n)^r$ and

$$((\mathbf{m}^*)^n)^{r+1} \subseteq (y_1, \ldots, y_d) \cdot ((\mathbf{m}^*)^n)^r.$$

Therefore

$$(y_1, \ldots, y_d) \cdot ((\mathbf{m}^*)^n)^r = ((\mathbf{m}^*)^n)^{r+1}$$

and $(y_1, \ldots, y_d)$ is a minimal reduction of $(\mathbf{m}^*)^n$ since $s((\mathbf{m}^*)^n) = s(\mathbf{m}^*) = d$.

Now assume that $(y_1, \ldots, y_d)$ is a minimal reduction of $(\mathbf{m}^*)^n$. So for some $r > 0$ we have

$$(y_1, \ldots, y_d) \cdot ((\mathbf{m}^*)^n)^r = ((\mathbf{m}^*)^n)^{r+1}$$

and $(\mathbf{m}^*)^n = \mathbf{m}^n/\mathbf{m}^{n+1} \oplus \mathbf{m}^{n+1}/\mathbf{m}^{n+2} \oplus \cdots$. Then

$$\begin{aligned}(y_1, \ldots, y_d) &\cdot (\mathbf{m}^n/\mathbf{m}^{n+1} \oplus \mathbf{m}^{n+1}/\mathbf{m}^{n+2} \oplus \cdots)^r \\ &= (\mathbf{m}^{nr+n}/\mathbf{m}^{nr+n+1} \oplus \mathbf{m}^{nr+n+1}/\mathbf{m}^{nr+n+2} \oplus \cdots).\end{aligned}$$

Suppose $deg(y_i) \neq n$, for some $1 \leq i \leq d$, so that $deg(y_i) > n$. Take $a \in \mathbf{m}^{nr+n}/\mathbf{m}^{nr+n+1}$ so we can write

$$a = \sum_{i=1}^{d} y_i \cdot a_i$$

where $a_i \in \mathbf{m}^{nr}/\mathbf{m}^{nr+1} \oplus \mathbf{m}^{nr+1}/\mathbf{m}^{nr+2} \oplus \cdots$ is homogeneous. Note that, $deg(a) = n(r+1)$. Since $deg(a_i) \geq nr$ and $deg(y_i) \geq n+1$ it follows that

$$deg(y_i \cdot a_i) \geq nr + n + 1.$$

Now $a = \sum_{j \neq i} y_j \cdot a_j$ because we can delete any summand which has degree different to the degree of a. So

$$\mathbf{m}^{nr+n}/\mathbf{m}^{nr+n+1} \subseteq \sum_{j \neq i} y_i \cdot \mathbf{m}^{nr}/\mathbf{m}^{nr+1}.$$

Since $\mathbf{m}^i/\mathbf{m}^{i+1}$ generates $(\mathbf{m}^*)^i$ we have

$$((\mathbf{m}^*)^n)^{r+1} \subseteq (y_1, \ldots, \hat{y}_i, \ldots, y_d) \cdot (\mathbf{m}^*)^{nr}.$$

As $(y_1, \ldots, \hat{y}_i, \ldots, y_d) \subseteq (\mathbf{m}^*)^n$ it follows that

$$(y_1, \ldots, \hat{y}_i, \ldots, y_d) \cdot (\mathbf{m}^*)^{nr} \subseteq ((\mathbf{m}^*)^n)^{r+1}$$

and hence

$$(y_1, \ldots, \hat{y}_i, \ldots, y_d) \cdot (\mathbf{m}^*)^{nr} = ((\mathbf{m}^*)^n)^{r+1}.$$

Therefore $(y_1, \ldots, \hat{y}_i, \ldots, y_d)$ is a reduction of $(\mathbf{m}^*)^n$ and it is strictly contained in $(y_1, \ldots, y_d)$. This contradicts the minimality of $(y_1, \ldots, y_d)$. Hence

$$deg(y_i) = n$$

for all $i = 1, \ldots, d$.

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Lemma 4.2.3 *Let $(A, \mathbf{m})$ be a d-dimensional local ring. $(x_1, \ldots, x_d)$ is a minimal reduction of $\mathbf{m}^n$ for some $n \geq 1$, if and only if $(x_1^*, \ldots, x_d^*)$ is a minimal reduction of $(\mathbf{m}^*)^n$ in $G_{\mathbf{m}}(A)$.*

Proof: First assume that $x_1, \ldots, x_d$ is a minimal reduction of $\mathbf{m}^n$. So, for some $r > 0$, we have

$$(x_1, \ldots, x_d) \cdot (\mathbf{m}^n)^r = (\mathbf{m}^n)^{r+1}.$$

By Lemma 4.2.2 we want to show that:

(i) $deg(x_i^*) = n$ for all $i = 1, \ldots, d$;

(ii) $x_1^*, \ldots, x_d^*$ is a system of parameters of $G_{\mathbf{m}}(A)$.

(i) $deg(x_i^*) = n$ if and only if $x_i \in \mathbf{m}^n \backslash \mathbf{m}^{n+1}$. By the reduction assumption $(x_1, \ldots, x_d) \subseteq \mathbf{m}^n$ and hence $x_i \in \mathbf{m}^n$ for all $i = 1, \ldots, d$. Suppose, without loss of generality, that $x_1 \in \mathbf{m}^{n+1}$. Take $a \in (\mathbf{m}^n)^{r+1}$ so that

$$a = \sum_{j=1}^{d} x_j \cdot b_j \text{ where } b_j \in (\mathbf{m}^n)^r.$$

Now, it follows that $x_1 b_1 \in \mathbf{m}^{nr+n+1}$ and

$$x_2 b_2 + \cdots + x_d b_d \in (x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r.$$

So

$$(\mathbf{m}^n)^{r+1} \subseteq \mathbf{m}^{nr+n+1} + (x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r,$$

as $a \in \mathbf{m}^{nr+n+1} + (x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r$ for all $a \in (\mathbf{m}^n)^{r+1}$. Now

$$(\mathbf{m}^n)^{r+1} = \mathbf{m}^{nr+n+1} + (x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r.$$

Fix an r. Take the quotient modulo $(x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r$. That is,

$$B := A/(x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r$$

so that

$$\frac{\mathbf{m}^{nr+n+1} + (x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r}{(x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r} =: \mathbf{m}_B^{nr+n+1},$$

$$\frac{(\mathbf{m}^n)^{r+1} + (x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r}{(x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r} =: (\mathbf{m}_B^n)^{r+1}.$$

Since $(\mathbf{m}_B^n)^{r+1} = \mathbf{m}_B^{nr+n+1} + (x_2, \ldots, x_d) \cdot (\mathbf{m}_B^n)^r$ it follows that

$$\begin{aligned}(\mathbf{m}_B^n)^{r+1} &= \mathbf{m}_B^{nr+n+1} \\ &= (\mathbf{m}_B) \cdot (\mathbf{m}_B^n)^{r+1}.\end{aligned}$$

Therefore by Nakayama's Lemma (Theorem 2.2 of [15]) we get

$$(\mathbf{m}_B^n)^{r+1} = 0.$$

So

$$\frac{(\mathbf{m}^n)^{r+1} + (x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r}{(x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r} = 0$$

and

$$\frac{(\mathbf{m}^n)^{r+1}}{(\mathbf{m}^n)^{r+1} \cap (x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r} = 0.$$

Since $(x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r \subseteq (\mathbf{m}^n)^{r+1}$,

$$\frac{(\mathbf{m}^n)^{r+1}}{(x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r} = 0$$

and hence

$$(\mathbf{m}^n)^{r+1} = (x_2, \ldots, x_d) \cdot (\mathbf{m}^n)^r$$

which contradicts the assumption that $x_1, \ldots x_d$ is a minimal reduction of $\mathbf{m}^n$. Thus $deg(x_i^*) = n$ for all $i = 1, \ldots, d$.

(ii) $x_1^*, \ldots, x_d^*$ is a system of parameters of $G_{\mathbf{m}}(A)$ if and only if $(\mathbf{m}^*)^p \subseteq (x_1^*, \ldots, x_d^*)$ for some $p > 0$. Now,

$$(\mathbf{m}^*)^p = \mathbf{m}^p/\mathbf{m}^{p+1} \oplus \mathbf{m}^{p+1}/\mathbf{m}^{p+2} \oplus \cdots.$$

Take $p \geq n(r+1)$ and let a be a homogeneous element of $(\mathbf{m}^*)^p$. Now

$$a = x + \mathbf{m}^{k+1} = x^*$$

where $x \in \mathbf{m}^k \backslash \mathbf{m}^{k+1}$ and $k \geq p$. So $deg(a) = k$ and from the reduction assumption we have

$$(\mathbf{m}^n)^{r+1} = (x_1, \ldots, x_d) \cdot (\mathbf{m}^n)^r.$$

Since $k \geq p$ it follows that $k \geq n(r+1)$ and

$$\begin{aligned} \mathbf{m}^k &= (\mathbf{m}^n)^{r+1} \cdot \mathbf{m}^{k-n(r+1)} \\ &= (x_1, \ldots, x_d) \cdot (\mathbf{m}^n)^r \cdot \mathbf{m}^{k-n(r+1)} \\ &= (x_1, \ldots, x_d) \cdot \mathbf{m}^{k-n}. \end{aligned}$$

Therefore $x = \sum_{i=1}^d x_i \cdot q_i$ where $q_i \in \mathbf{m}^{k-n}$. Now

$$x^* = x + \mathbf{m}^{k+1} = \sum_{i=1}^{d} x_i \cdot q_i + \mathbf{m}^{k+1}$$

so if we set $y := x - \sum_{x_j q_j \in \mathbf{m}^{k+1}} x_j \cdot q_j$ then, without loss of generality we can write

$$y = \sum_{j=1}^{\delta} x_j \cdot q_j$$

where $\delta \leq d$ since $x_j \in \mathbf{m}^n$, $q_j \in \mathbf{m}^{k-n} \backslash \mathbf{m}^{k-n+1}$ and $deg(q_j^*) = k - n$. In fact

$$\begin{aligned} x^* &= \textstyle\sum_{i=1}^d x_i \cdot q_i + \mathbf{m}^{k+1} \\ &= \textstyle\sum_{j=1}^{\delta} x_j \cdot q_j + \mathbf{m}^{k+1} \\ &= y^*. \end{aligned}$$

From (i) $deg(x_j^*) = n$ and we know that $deg(q_j^*) = k - n$ so

$$\begin{aligned} deg(x_j^* q_j^*) &= deg(x_j^*) + deg(q_j^*) \\ &= n + k - n \\ &= k. \end{aligned}$$

By the choice of j $x_j q_j \in \mathbf{m}^k \backslash \mathbf{m}^{k+1}$ and so

$$x_j^* q_j^* = (x_j q_j)^*.$$

Now

$$\begin{aligned} x^* &= (\textstyle\sum_{j=1}^{\delta} x_j \cdot q_j)^* \\ &= \textstyle\sum_{j=1}^{\delta} (x_j \cdot q_j)^* \\ &= \textstyle\sum_{j=1}^{\delta} x_j^* \cdot q_j^* \\ &= \textstyle\sum_{i=1}^{d} x_i^* \cdot q_i^*. \end{aligned}$$

Hence $a = x^* \in (x_1^*, \dots, x_d^*)$ and so

$$(\mathbf{m}^*)^p \subseteq (x_1^*, \dots, x_d^*).$$

Therefore $x_1^*, \dots, x_d^*$ is a system of parameters of $G_{\mathbf{m}}(A)$ and hence by Lemma 4.2.2 it follows that $x_1^*, \dots, x_d^*$ is a minimal reduction of $(\mathbf{m}^*)^n$ in $G_{\mathbf{m}}(A)$.

Now we will assume that $x_1^*, \dots, x_d^*$ is a minimal reduction of $(\mathbf{m}^*)^n$ in $G_{\mathbf{m}}(A)$. That is, for some $r > 0$ we have

$$(x_1^*, \dots, x_d^*) \cdot ((\mathbf{m}^*)^n)^r = ((\mathbf{m}^*)^n)^{r+1}.$$

Now, by applying Lemma 4.2.2, we know that $deg(x_i^*) = n$ for all $1 \leq i \leq d$. That is $x_i^* = x_i + \mathbf{m}^{n+1}$ and $x_i \in \mathbf{m}^n \backslash \mathbf{m}^{n+1}$. Hence

$$(x_1, \dots, x_d) \subseteq \mathbf{m}^n$$

and

$$(x_1, \dots, x_d) \cdot (\mathbf{m}^n)^s \subseteq (\mathbf{m}^n)^{s+1}$$

for all $s > 0$. Set $s = r$ and take $x \in (\mathbf{m}^n)^{r+1}$. Assume that $x \in \mathbf{m}^p \backslash \mathbf{m}^{p+1}$ so that $p \geq n(r+1)$ and $x^* = x + \mathbf{m}^{p+1}$. Now

$$x^* \in \mathbf{m}^p / \mathbf{m}^{p+1} \subseteq ((\mathbf{m}^*)^n)^{r+1}$$

and it follows that, by the reduction assumption, we have

$$x^* \in (x_1^*, \dots, x_d^*) \cdot ((\mathbf{m}^*)^n)^r.$$

So

$$\begin{aligned} x^* &= \textstyle\sum_{i=1}^{d} x_i^* \cdot g_i^* \\ &= \left(\textstyle\sum_{i=1}^{d} x_i \cdot g_i\right)^* \end{aligned}$$

where $g_i^* \in ((\mathbf{m}^*)^n)^r$ and applying the same process used earlier in the proof. We can conclude that

$$x + \mathbf{m}^{p+1} = \sum_{i=1}^{d} x_i \cdot g_i + \mathbf{m}^{p+1}$$

and by the factor module property we get

$$x \in \sum_{i=1}^{d} x_i \cdot g_i + \mathbf{m}^{p+1}.$$

With regards to the degree we get

$$\begin{aligned} deg(x_i^* g_i^*) &= deg(x_i^*) + deg(g_i^*) \\ &= n + deg(g_i^*) \\ &= p \\ &\geq n(r+1). \end{aligned}$$

So $deg(g_i^*) = p - n \geq nr$ and $g_i \in (\mathbf{m}^n)^r$.

Since $x \in \sum_{i=1}^{d} x_i \cdot g_i + \mathbf{m}^{p+1}$ it follows that

$$x \in (x_1, \ldots, x_d) \cdot (\mathbf{m}^n)^r + \mathbf{m}^{p+1}$$

and, as $p \geq n(r+1)$,

$$(\mathbf{m}^n)^{r+1} \subseteq (x_1, \ldots, x_d) \cdot (\mathbf{m}^n)^r + \mathbf{m}^{nr+n+1}.$$

Therefore

$$(\mathbf{m}^n)^{r+1} = (x_1, \ldots, x_d) \cdot (\mathbf{m}^n)^r + \mathbf{m}^{nr+n+1}$$

and applying Nakayama's Lemma (Theorem 2.2 of [15]) in the same way as before we get

$$(\mathbf{m}^n)^{r+1} = (x_1, \ldots, x_d) \cdot (\mathbf{m}^n)^r.$$

Hence $x_1, \ldots, x_d$ is a reduction of $\mathbf{m}^n$. Since $s(\mathbf{m}) = s(\mathbf{m}^n) = d$ it follows that $x_1 \ldots, x_d$ is a minimal reduction of $\mathbf{m}^n$.

○

With these lemmas in place we can set up the following equivalent conditions:

Lemma 4.2.4 *Let $(A, \mathbf{m})$ be a d-dimensional local ring. Also let $x_1 \ldots, x_d \in \mathbf{m}$ and let n be a positive integer. The following conditions are equivalent:*

(i) $(x_1, \ldots, x_d)$ is a minimal reduction of $\mathbf{m}^n$;

(ii) $(x_1^*, \ldots, x_d^*)$ is a minimal reduction of $(\mathbf{m}^*)^n$;

(iii) $x_1^*, \ldots, x_d^*$ have the same degree n and form a system of parameters of $G_{\mathbf{m}}(A)$.

Proof: Lemma 4.2.3 shows that (i) holds if and only if (ii) holds.

Applying Lemma 4.2.2 it follows that (ii) implies that $deg(x_i^*) = n$ for all $i = 1, \ldots, d$. It also follows that (iii) implies (ii).

All that remains is to show that (ii) implies that $x_1^*, \ldots, x_d^*$ form a system of parameters of $G_{\mathbf{m}}(A)$. However, this is clear as $dim G_{\mathbf{m}}(A) = d$ and $(x_1^*, \ldots, x_d^*)$ is $\mathbf{m}^*$-primary.

○

We now give some notation to our special class of parameter ideals. Set

$$\begin{aligned}\mathcal{S} := \ & \{\mathbf{q} = (x_1, \ldots, x_d) \subseteq \mathbf{m} | \\ & x_1^*, \ldots, x_d^* \text{ form a system of parameters of } G_{\mathbf{m}}(A)\}.\end{aligned}$$

Thus, this subset of parameter ideals of A contains all of the minimal reductions of powers of the maximal ideal $\mathbf{m}$.

We need one more lemma before stating and proving the main result in this section.

Lemma 4.2.5 *With A and $G_{\mathbf{m}}(A)$ as above we have $e(\mathbf{m}^p; A) = e((\mathbf{m}^*)^p; G_{\mathbf{m}}(A))$ for any $p \geq 1$.*

Proof: Since

$$((\mathbf{m}^*)^p)^{n+1} = \mathbf{m}^{pn+p}/\mathbf{m}^{pn+p+1} \oplus \mathbf{m}^{pn+p+1}/\mathbf{m}^{pn+p+2} \oplus \cdots$$

it follows that

$$G_{\mathbf{m}}(A)/((\mathbf{m}^*)^p)^{n+1} = A/\mathbf{m} \oplus \mathbf{m}/\mathbf{m}^2 \oplus \cdots \oplus \mathbf{m}^{pn+p-1}/\mathbf{m}^{pn+p}.$$

Hence

$$\begin{aligned}\ell_A(G_{\mathbf{m}}(A)/((\mathbf{m}^*)^p)^{n+1}) \ & = \ell_A(A/\mathbf{m} \oplus \cdots \oplus \mathbf{m}^{pn+p-1}/\mathbf{m}^{pn+p}) \\ & = \textstyle\sum_{k=0}^{pn+p-1} \ell_A(\mathbf{m}^k/\mathbf{m}^{k+1}) \\ & = \ell_A(A) - \ell_A(\mathbf{m}) + \cdots + \ell_A(\mathbf{m}^{pn+p-1}) - \ell_A(\mathbf{m}^{pn+p}) \\ & = \ell_A(A) - \ell_A(\mathbf{m}^{pn+p}) \\ & = \ell_A(A/\mathbf{m}^{pn+p}).\end{aligned}$$

Therefore we have

$$\ell_A(G_{\mathbf{m}}(A)/((\mathbf{m}^*)^p)^{n+1}) = \ell_A(A/(\mathbf{m}^p)^{n+1})$$

and so, by definition,

$$e(\mathbf{m}^p; A) = e((\mathbf{m}^*)^p; G_{\mathbf{m}}(A))$$

for any $p \geq 1$.

○

In order to prove the main result of this section we need to use Theorem 2.4.2. The proof of part (2) of this theorem, as given in [25], gives an upper bound for $n_A(M)$ based on the local cohomology modules $H^i_{\mathfrak{m}}(M)$ of M. The following proof of this result appears in [1] and is somewhat simpler. It also provides, in general, a much better bound for $\tilde{n}_A(M)$. Part (2) of Theorem 2.4.2 is restated here for the convenience of the reader.

Theorem 4.2.6 (Theorem 2.4.2) *Let A be a graded k-algebra and let M be a graded A-module with $dimM = dimA$. Then $\tilde{n}_A(M)$ is finite.*

Proof: A can be written in the form $A = S/I$ where $S = k[X_1, \dots, X_n]$ and I is a homogeneous ideal of S (see, eg. page 93 of [15]). Assume that I has a minimal system of homogeneous generators $f_1, \dots, f_u$ whose degrees are ordered in a decreasing sequence, $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_u$. Let $\mathfrak{q}$ be a homogeneous parameter ideal of A. Let $g_1, \dots, g_d$ be homogeneous polynomials in S such that their images in S/I generate $\mathfrak{q}$ and $n_1 = deg\ g_1 \geq \cdots \geq n_d = deg\ g_d$. Now consider $f_1, \dots, f_u, g_1, \dots, g_d$ and apply Theorem 1 of [6]. This implies that there is a regular sequence of homogeneous polynomials of $k[X_1, \dots, X_n]$, say $h_1, \dots, h_n$, in $(I, g_1, \dots, g_d)$. Also we have

$$deg\ h_1 \cdots deg\ h_n \leq \delta_1 \cdots \delta_{n-d} n_1 \cdots n_d.$$

Now, since $(h_1, \dots, h_n) \subseteq I + \mathfrak{q}$, it follows that

$$\begin{aligned} \ell_A(A/\mathfrak{q}) &= \ell_A(S/I + \mathfrak{q}) \\ &\leq \ell_A(S/(h_1, \dots, h_n)) \\ &= deg\ h_1 \cdots deg\ h_n \\ &\leq \delta_1 \cdots \delta_{n-d} n_1 \cdots n_d, \end{aligned}$$

as the h_i are homogeneous. Note that the equality in the third line above is a consequence of a Bezout-type theorem (see, e.g., [22]). If M is generated by s_M elements then we can write $M = a_{s_1} \cdot A + \cdots a_{s_M} \cdot A$. Now

$$\begin{aligned} \ell_A(M/\mathfrak{q} \cdot M) &= \ell_A((a_{s_1} \cdot A + \cdots a_{s_M} \cdot A)/\mathfrak{q} \cdot (a_{s_1} \cdot A + \cdots a_{s_M} \cdot A)) \\ &\leq \ell_A(a_{s_1} \cdot A/\mathfrak{q} \cdot (a_{s_1} \cdot A)) + \cdots + \ell_A(a_{s_M}/\mathfrak{q} \cdot (a_{s_M} \cdot A)) \\ &\leq s_M \cdot \max_{1 \leq i \leq M} \{\ell_A(a_{s_i}/\mathfrak{q} \cdot (a_{s_i} \cdot A))\} \\ &\leq s_M \cdot \ell_A(A/\mathfrak{q}) \end{aligned}$$

as $\ell_A(A/\mathfrak{q}) \geq \ell_A(a_{s_i} \cdot A/\mathfrak{q} \cdot (a_{s_i} \cdot A))$ (see also Corollary 23.4 of [18]).

Hence

$$\ell_A(M/\mathfrak{q} \cdot M) \leq s_M \delta_1 \cdots \delta_{n-d} n_1 \cdots n_d. \tag{4.4}$$

Let $\mathfrak{p} \in AsshM$ so that $\mathfrak{q}$ is also a parameter ideal of $A/\mathfrak{p}$. By Theorem 14.9 of [15] we have

$$e(\mathfrak{q}; A/\mathfrak{p}) \geq n_1 \cdots n_d \cdot e(A/\mathfrak{p}).$$

By Lemma 2.3.3 we obtain

$$\begin{aligned} e(\mathfrak{q}; M) &= \textstyle\sum_{\mathfrak{p} \in AsshM} \ell_{A_\mathfrak{p}}(M_\mathfrak{p}) \cdot e(\mathfrak{q}; A/\mathfrak{p}) \\ &\geq \left[\textstyle\sum_{\mathfrak{p} \in AsshM} \ell_{A_\mathfrak{p}}(M_\mathfrak{p}) \cdot e(A/\mathfrak{p})\right] \cdot n_1 \cdots n_d \\ &=: \alpha_M \cdot n_1 \cdots n_d. \end{aligned}$$

Since α_M is independent of $\mathbf{q}$, applying equation (4), we get

$$\tilde{n}_A(M) \leq s_M \cdot \delta_1 \cdots \delta_{n-d}/\alpha_M < \infty.$$

$\bigcirc$

In order to show that the above bound for $\tilde{n}_A(M)$ is sharp we consider the following examples.

Example 4.2.7 Let $R = k[x,y,z]$ and set $I := x \cdot R \cap (y^m, z) \cdot R$ an ideal of R. Consider the graded k-algebra $A := R/I = k[x,y,z]/x \cdot R \cap (y^m, z) \cdot R$.We have $dim\, A = 2$. Now, applying Lemma 2.3.3 to $e(A)$, we get

$$\begin{aligned} \alpha_A &:= \textstyle\sum_{\mathbf{p} \in Assh A} \ell_{A_\mathbf{p}}(A_\mathbf{p}) \cdot e(A/\mathbf{p}) \\ &= e(\mathbf{m}; A) = e(A) = 1, \end{aligned}$$

(Corollary 3 to Theorem 5 on page 303 of [19]).
Further, $\delta_1 = deg\ (xy^m) = m+1$, $d = 2$ and $n - d = 1$. Since $s_A = 1$, as A is generated by exactly one element as an A-module, it follows from Theorem 4.2.6 that

$$\tilde{n}_A(A) \leq m+1.$$

Set $\mathbf{q} = (x+y, z) \cdot A$ so that $\mathbf{q}$ is a parameter ideal of A. We have

$$\begin{aligned} \ell_A(A/\mathbf{q} \cdot A) &= \ell_A(x+y, z, xy^m, xz) \\ &= \ell_A(x+y, xy^m, z) \\ &= \ell_A(x^{m+1}, z) \text{ as } x+y=0 \text{ gives } x=y \\ &= m+1, \end{aligned}$$

and by Lemma 2.3.3,

$$e(\mathbf{q}; A) = e(\mathbf{q} + (x,y,z); A/(x,y,z) \cdot A) = e(0; k) = 1.$$

So we get

$$\tilde{n}_A(A) = m+1$$

and the bound in Theorem 4.2.6 is sharp.

Example 4.2.8 Let $B = k[x,y,u,v]/(x^2, y^2, xu - yv)$ and let $\mathbf{q} = (u,v)$. Also $s_B = 1$ and

$$\delta_1 = deg\ (x^2) = 2,\ \delta_2 = deg\ (y^2) = 2$$

and we have $n - d = 2$ with regards to the proof of Theorem 4.2.6. Since $(x^2, y^2, xu - yv)$ is (x,y)-primary ideal, we have

$$\alpha_B := e(B) = 2.$$

So by Theorem 4.2.6

$$\tilde{n}_B(B) \leq \delta_1 \cdot \delta_2/e(B) = 4/2 = 2.$$

Now,

$$\begin{aligned}\ell_B(B/\mathbf{q}\cdot B) &= \ell(k[x,y,u,v]/(u,v,x^2,y^2,xu-yv))\\ &= \ell(k[x,y,u,v]/(u,v,x^2,y^2))\\ &= 4.\end{aligned}$$

By Lemma 2.3.3we have

$$e(\mathbf{q};B) = e(\mathbf{q}+(u,v);B/(u,v)\cdot B) = e(0;k[x,y]/(x^2,y^2)) = 2.$$

Therefore

$$\tilde{n}_B(B) = 2.$$

So the new proof of Theorem 2.4.2 not only gives us a bound for $\tilde{n}_A(A)$ that is easier to calculate, it also provides a bound that is sharp. That is, it gives us the best possible bound for $\tilde{n}_A(A)$.

The main result of this section is now the following:

Theorem 4.2.9 *Let $(A,\mathbf{m})$ be a local ring. Then*

$$sup\{\ell_A(A/\mathbf{q})/e(\mathbf{q};A) | \mathbf{q}\in\mathcal{S}\} < \infty.$$

Proof: Let $\mathbf{q} = (x_1,\ldots,x_d)\in\mathcal{S}$. Denote by $\mathbf{q}^*$ the initial ideal of $\mathbf{q}$. It is clear that

$$(x_1^*,\ldots,x_d^*)\subseteq\mathbf{q}^*.$$

By Lemma 2.2.3,

$$\begin{aligned}\ell_A(A/\mathbf{q}) &= \ell_A(G_{\mathbf{m}}(A)/\mathbf{q}^*\cdot G_{\mathbf{m}}(A))\\ &\leq \ell_A(G_{\mathbf{m}}(A)/(x_1^*,\ldots,x_d^*)\cdot G_{\mathbf{m}}(A)).\end{aligned}$$

Since $x_1^*,\ldots,x_d^*$ is a system of parameters in $G_{\mathbf{m}}(A)$, by Theorem 4.2.6, we have

$$\frac{\ell_A(G_{\mathbf{m}}(A)/(x_1^*,\ldots,x_d^*)\cdot G_{\mathbf{m}}(A))}{e((x_1^*,\ldots,x_d^*);G_{\mathbf{m}}(A))} \leq \alpha < \infty,$$

for a fixed α which does not depend on $\mathbf{q}$. So it suffices to show that

$$e((x_1,\ldots,x_d);A) = e((x_1^*,\ldots,x_d^*);G_{\mathbf{m}}(A)).$$

For $i=1,\ldots,d$ assume that $x_i\in\mathbf{m}^{r_i}\backslash\mathbf{m}^{r_i+1}$ for some positive integers r_i. Set $r = l.c.m.\{r_1,\ldots,r_d\}$. Also set $s_i = r/r_i$. Let

$$y_1 = x_1^{s_1},\ldots,y_d = x_d^{s_d}.$$

Now $y_i = x_i^{s_i}\in(\mathbf{m}^{r_i})^{s_i}$, as $x_i\in\mathbf{m}^{r_i}$. So

$$y_i\in(\mathbf{m}^{r_i})^{r/r_i} = \mathbf{m}^r.$$

Suppose $y_i\in\mathbf{m}^{r+1}$. By definition $x_i^*\in\mathbf{m}^{r_i}/\mathbf{m}^{r_i+1}$ and if $y_i\in\mathbf{m}^{r+1}$ then

$$y_i = x_i^{s_i}\in\mathbf{m}^{r+1} = \mathbf{m}^{(r_ir/r_i)+1} = \mathbf{m}^{r_is_i+1}.$$

Hence $x_i^{s_i} \in \mathbf{m}^{r_i s_i}$ and $x_i^{s_i} \in \mathbf{m}^{r_i s_i+1}$ so, as $(x_i^*)^{s_i} \in \mathbf{m}^{r_i s_i}/\mathbf{m}^{r_i s_i+1}$, it follows that

$$(x_i^*)^{s_i} = 0$$

in $G_{\mathbf{m}}(A)$. This contradicts the assumption that x_i^*, and hence $(x_i^*)^{s_i}$ (see eg. (b) on page 297 of [19]), is a parameter element of $G_{\mathbf{m}}(A)$. Thus

$$y_i \in \mathbf{m}^r \backslash \mathbf{m}^{r+1}$$

for all $i = 1, \ldots, d$. So it now follows that the $y_i^* = (x_i^*)^{s_i}$ have the same degree r for each $i = 1, \ldots, d$, and $y_1^*, \ldots y_d^*$ form a system of parameters of $G_{\mathbf{m}}(A)$.

By Lemma 4.2.4 $(y_1^*, \ldots y_d^*)$ is a minimal reduction of $(\mathbf{m}^*)^r$ and $(y_1, \ldots, y_d)$ is a minimal reduction of $\mathbf{m}^r$. We have

$$\begin{aligned} e((\mathbf{m}^*)^r; G_{\mathbf{m}}(A)) &= e((y_1^*, \ldots y_d^*); G_{\mathbf{m}}(A)) \\ &= e(((x_1^*)^{s_1}, \ldots, (x_d^*)^{s_d}); G_{\mathbf{m}}(A)) \\ &= s_1 \cdots s_d \cdot e((x_1^*, \ldots, x_d^*); G_{\mathbf{m}}(A)) \end{aligned}$$

by Lemma 2.3.9 and Corollary 1 on page 311 of [19].

On the other hand by Lemma 4.2.5, again by Lemma 2.3.9 and Corollary 1 on page 311 of [19], we get

$$\begin{aligned} e((\mathbf{m}^*)^r; G_{\mathbf{m}}(A)) &= e(\mathbf{m}^r; A) \\ &= e((y_1, \ldots, y_d); A) \\ &= e((x_1^{s_1}, \ldots, x_d^{s_d}); A) \\ &= s_1 \cdots s_d \cdot e((x_1, \ldots, x_d); A). \end{aligned}$$

So

$$e((x_1^*, \ldots, x_d^*); G_{\mathbf{m}}(A)) = e((x_1, \ldots, x_d); A)$$

and hence

$$sup \left\{ \frac{\ell_A(A/\mathbf{q})}{e(\mathbf{q}; A)} \Big| \mathbf{q} \in \mathcal{S} \right\} \leq \tilde{n}_{G_{\mathbf{m}}(A)}(G_{\mathbf{m}}(A)) < \infty.$$

○

Chapter 5

Loewy Length and $\theta_A(M)$

The final chapter in this dissertation is devoted to the Loewy length and $\theta_A(M)$.

The Loewy length will be used to find stronger versions of some of the results found in [25] and in fact a key part of the proof of Corollary 2.1 in [25] is to bound the Loewy length of $M/\mathbf{q}\cdot M$ in the graded case where $\mathbf{q}$ is a homogeneous parameter ideal. In [27] the Loewy length was used to estimate the index of nilpotency for $\mathbf{m}$-primary ideals. In [7] and its references it is shown to be related to a certain invariant of Gorenstein local rings. In the graded case the Loewy length is related to the reduction number.

We begin by examining properties of the Loewy length and studying its connection with other algebraic invariants. Following this we consider the finiteness of $\theta_A(M)$ and give an example to show that $\theta_A(M)$ is not always finite. Necessary conditions for $\theta_A(M)$ to be finite are examined and some positive progress is made in this direction.

Finally, an attempt to bound $\theta_A(M)$ is made. This attempt employs a technique of W. Vasconcelos' based on characteristic polynomials. A positive result is achieved by restricting ourselves to the case of local rings containing an infinite field.

5.1 Properties of the Loewy length

This section recalls the notions of Loewy length and $\theta_A(M)$. We also give some introductory properties of the theta invariant and its connection to the reduction number.

We begin by recalling the definition of the Loewy length. If M is an A-module and $\mathbf{q}$ is an $\mathbf{m}$-primary ideal of A then the Loewy length, $\ell\ell(M/\mathbf{q}\cdot M)$, of $M/\mathbf{q}\cdot M$ is the smallest positive integer t such that $\mathbf{m}^t \cdot M \subseteq \mathbf{q}\cdot M$.

Two basic properties of the Loewy length are given in the following lemmas.

Lemma 5.1.1 *If $\widehat{M}$ is the completion of M with respect to $\mathbf{m}$ then*

$$\ell\ell(M/\mathbf{q}\cdot M) = \ell\ell(\widehat{M}/\mathbf{q}\cdot\widehat{M}).$$

Proof: From Theorem 5 on page 256 of [31] (Vol. II) we see that $\widehat{M} = \hat{A} \cdot M$. Applying this we see that

$$\begin{aligned} \widehat{I \cdot M} &= \hat{A} \cdot I \cdot M \\ &= (\hat{A} \cdot I) \cdot (\hat{A} \cdot M) \\ &= \hat{I} \cdot \widehat{M} \end{aligned}$$

for any ideal I of A. Suppose $\ell\ell(M/\mathbf{q} \cdot M) = n$ so that

$$\mathbf{m}^n \cdot M \subseteq \mathbf{q} \cdot M.$$

Now

$$\begin{aligned} \widehat{\mathbf{m}^n \cdot M} &= \widehat{\mathbf{m}}^n \cdot \widehat{M} \\ \widehat{\mathbf{q} \cdot M} &= \hat{\mathbf{q}} \cdot \widehat{M} \end{aligned}$$

and

$$\widehat{\mathbf{m}}^n \cdot \widehat{M} \subseteq \hat{\mathbf{q}} \cdot \widehat{M}.$$

Hence

$$n \geq \ell\ell(\widehat{M}/\hat{\mathbf{q}} \cdot \widehat{M}).$$

Now suppose that $\widehat{\mathbf{m}}^t \cdot \widehat{M} \subseteq \hat{\mathbf{q}} \cdot \widehat{M}$, for some t, so that

$$\widehat{\mathbf{m}}^t \cdot \widehat{M} \cap M \subseteq \hat{\mathbf{q}} \cdot \widehat{M} \cap M.$$

This implies that

$$\mathbf{m}^t \cdot M \subseteq \mathbf{q} \cdot M$$

(Corollary 1 on page 257 of Volume II of [31]) so that

$$t \geq \ell\ell(M/\mathbf{q} \cdot M).$$

Therefore $\ell\ell(M/\mathbf{q} \cdot M) = \ell\ell(\widehat{M}/\hat{\mathbf{q}} \cdot \widehat{M})$.

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Lemma 5.1.2 *If M, M' and M'' are A-modules such that*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact, then
(i) $\ell\ell(M/\mathbf{q} \cdot M) \geq \ell\ell(M''/\mathbf{q} \cdot M'')$,
(ii) $\ell\ell(M/\mathbf{q} \cdot M) \leq \ell\ell(M'/\mathbf{q} \cdot M') + \ell\ell(M''/\mathbf{q} \cdot M'')$.

Proof: Applying Koszul homology we get the exact sequence

$$M'/\mathbf{q} \cdot M' \xrightarrow{\varphi} M/\mathbf{q} \cdot M \xrightarrow{\psi} M''/\mathbf{q} \cdot M'' \longrightarrow 0.$$

Suppose $\ell\ell(M/\mathbf{q} \cdot M) = n$ so that $\mathbf{m}^n \cdot M \subseteq \mathbf{q} \cdot M$. Now

$$\mathbf{m}^n \cdot M \subseteq 0$$

in $M/\mathbf{q}\cdot M$. Since ψ is an homomorphism it follows that

$$\psi(\mathbf{m}^n\cdot M)=\mathbf{m}^n\cdot\psi(M)\subseteq \mathbf{q}\cdot M''.$$

That is, $\mathbf{m}^n\cdot M''\subseteq \mathbf{q}\cdot M''$, as ψ is an epimorphism. Hence $\ell\ell(M''/\mathbf{q}\cdot M'')\leq n$ and

$$\ell\ell(M/\mathbf{q}\cdot M)\geq \ell\ell(M''/\mathbf{q}\cdot M'').$$

Now let $n=n'+n''$ where

$$n'=\ell\ell(M'/\mathbf{q}\cdot M'),\quad n''=\ell\ell(M''/\mathbf{q}\cdot M'').$$

Take any $x\in M/\mathbf{q}\cdot M$ and $a\in\mathbf{m}^{n'}$, $b\in\mathbf{m}^{n''}$. We have

$$\psi(b\cdot x)=b\cdot\psi(x)=0.$$

Hence

$$b\cdot x=\varphi(y)$$

for some $y\in M'/\mathbf{q}\cdot M'$. Now

$$a\cdot b\cdot x=a\cdot\varphi(y)=\varphi(a\cdot y)=\varphi(0)=0.$$

Since each element of $\mathbf{m}^n$ is a sum of elements of the type ab, $a\in\mathbf{m}^{n'}$, $b\in\mathbf{m}^{n''}$, it follows that

$$\mathbf{m}^n\subseteq\mathbf{q}\cdot M.$$

That is, $n\geq\ell\ell(M/\mathbf{q}\cdot M)$ so that

$$\ell\ell(M/\mathbf{q}\cdot M)\leq\ell\ell(M'/\mathbf{q}\cdot M')+\ell\ell(M''/\mathbf{q}\cdot M'').$$

○

Recall that, if M is an A-module such that $dim\,M=dim\,A$, then

$$\theta_A(M):=sup\{\ell\ell(M/\mathbf{q}\cdot M)/e(\mathbf{q};M)\}$$

where the supremum is taken over all $\mathbf{m}$-primary ideals $\mathbf{q}$ of A.

We can now extend Lemma 5.1.1 to the invariant $\theta_A(M)$.

Lemma 5.1.3 *If M is an A-module and $\widehat{M}$ is the completion of M with respect to* $\mathbf{m}$ *then*

$$\theta_A(M)=\theta_{\hat{A}}(\widehat{M}).$$

Proof: We note that by applying the notes in Section 3 of Chapter 2 we have

$$\begin{aligned} e(\mathbf{q};M) &= \lim_{n\to\infty}\tfrac{d!}{n^d}\cdot\ell(M/\mathbf{q}^n\cdot M)\\ &= \lim_{n\to\infty}\tfrac{d!}{n^d}\cdot\left(\textstyle\sum_{i=0}^{n-1}\ell(\mathbf{q}^i\cdot M/\mathbf{q}^{i+1}\cdot M)\right)\end{aligned}$$

(see page 97 of [15]). Also

$$e(\hat{\mathbf{q}};\widehat{M}) = \lim_{n\to\infty}\frac{d!}{n^d}\cdot\left(\sum_{i=0}^{n-1}\ell(\hat{\mathbf{q}}^i\cdot\widehat{M}/\hat{\mathbf{q}}^{i+1}\cdot\widehat{M})\right)$$

as $dimM = dim\widehat{M}$ (Theorem 30 on page 433 of [19]).

$$0\longrightarrow \mathbf{q}^{i+1}\cdot M\longrightarrow \mathbf{q}^i\cdot M\longrightarrow \mathbf{q}^i\cdot M/\mathbf{q}^{i+1}\cdot M\longrightarrow 0$$

is exact so

$$0\longrightarrow \hat{\mathbf{q}}^{i+1}\cdot \widehat{M}\longrightarrow \hat{\mathbf{q}}^i\cdot \widehat{M}\longrightarrow \hat{\mathbf{q}}^i\cdot \widehat{M}/\hat{\mathbf{q}}^{i+1}\cdot \widehat{M}\longrightarrow 0$$

is exact (Theorem 20 on page 419 of [19]). So

$$\ell(\mathbf{q}^i\cdot M/\mathbf{q}^{i+1}\cdot M) = \ell(\mathbf{q}^i\cdot M) - \ell(\mathbf{q}^{i+1}\cdot M),$$

$$\ell(\hat{\mathbf{q}}^i\cdot \widehat{M}/\hat{\mathbf{q}}^{i+1}\cdot \widehat{M}) = \ell(\hat{\mathbf{q}}^i\cdot \widehat{M}) - \ell(\hat{\mathbf{q}}^{i+1}\cdot \widehat{M}).$$

Now

$$\ell(\mathbf{q}^i\cdot M) = \ell(\hat{\mathbf{q}}^i\cdot\widehat{M})$$

for all i as if

$$\mathbf{q}^i\cdot M\supset M_1\supset M_2\supset\cdots\supset M_n = 0$$

is a composition series then so is

$$\hat{\mathbf{q}}^i\cdot \widehat{M}\supset \widehat{M}_1\supset \widehat{M}_2\supset\cdots\supset \widehat{M}_n = 0$$

(see, eg. the notes on page 424 of [19]). Therefore

$$e(\mathbf{q};M) = e(\hat{\mathbf{q}};\widehat{M})$$

(see also Exercise 3 on page 439 of [19]).

So we now have

$$\frac{\ell\ell(M/\mathbf{q}\cdot M)}{e(\mathbf{q};M)} = \frac{\ell\ell(\widehat{M}/\hat{\mathbf{q}}\cdot\widehat{M})}{e(\hat{\mathbf{q}};\widehat{M})}$$

by Lemma 5.1.1, and thus

$$\theta_A(M)\le\theta_{\hat{A}}(\widehat{M})$$

as there may be parameter ideals of $\widehat{A}$ not accounted for when passing from $\mathbf{q}$ to $\hat{\mathbf{q}}$.

Let $\mathbf{q}' = (x'_1,\ldots,x'_d)\subseteq\hat{\mathbf{m}}$ be a parameter ideal of $\widehat{A}$ such that $\ell\ell(\widehat{A}/\mathbf{q}'\cdot\widehat{A}) = n$. If we consider the chain

$$A\supset\mathbf{m}\supset\mathbf{m}^2\supset\mathbf{m}^3\supset\cdots$$

then we can define $\widehat{M}$ as follows:

$$\widehat{M} := \{g = (g_1, g_2,\ldots)\in\prod_i M/\mathbf{m}^i\cdot M\}$$

where $g_j \equiv g_i \pmod{\mathbf{m}^i \cdot M}$ for all $j > i$ (see, eg. page 181 of [8]). So, by applying this definition we can choose $x_1, \ldots, x_d \in \mathbf{m}$ such that

$$x_i - x_i' \in \mathbf{m}^{n+1}$$

for each $i = 1, \ldots, d$. Set $\mathbf{q} = (x_1, \ldots, x_d) \cdot A$.

Since $x_i - x_i' \in \mathbf{m}^{n+1}$ it follows that $x_i \in \mathbf{m}^p$ for some $p \geq n$. Now, as $\ell\ell(\widehat{A}/\mathbf{q}' \cdot \widehat{A}) = n$, we have $\mathbf{m}^n \subseteq \mathbf{q}'$ and hence $x_i \in \mathbf{q}'$. So $\mathbf{q} \subseteq \mathbf{q}'$ and $\hat{\mathbf{q}} \subseteq \mathbf{q}'$ as $\mathbf{q}'$ is complete.

As $\hat{\mathbf{q}}$ can be defined similarly to the definition given above for $\widehat{M}$, to show that $x_i' \in \hat{\mathbf{q}}$ it will be sufficient to show that for k large enough there is $y_{i_k} \in \mathbf{q}$ such that $x_i' - y_{i_k} \in \mathbf{m}^k$.

We will proceed by induction on k.

For $k = n+1$ let $x_i' = x_i$ so that we can take $y_{i_k} = x_i'$ to get

$$x_i - x_i' \in \mathbf{m}^{n+1}.$$

Now let $k > n+1$. Assume that $y_{i_k} \in \mathbf{q}$ such that $x_i' - y_{i_k} \in \mathbf{m}^k$.

$$x_i' - y_{i_k} \in \mathbf{m}^k = \mathbf{m}^n \cdot \mathbf{m}^{k-n} \subseteq \mathbf{q}' \cdot \mathbf{m}^{k-n}.$$

Hence

$$x_i' - y_{i_k} = x_1'a_1 + \cdots + x_d'a_d$$

where $a_i \in \mathbf{m}^{k-n}$. So

$$x_i' - y_{i_k} = (x_1 + z_1)a_1 + \cdots + (x_d + z_d)a_d$$

where $z_i \in \mathbf{m}^{n+1}$, as $x_i' \in x_i - \mathbf{m}^{n+1}$. Thus

$$x_i' - y_{i_k} = (x_1a_1 + \cdots + x_da_d) + (z_1a_1 + \cdots + z_da_d).$$

Set

$$y_{i_{k+1}} = y_{i_k} + (x_1a_1 + \cdots + x_da_d) \in \mathbf{q}$$

so that

$$\begin{aligned} x_i' - y_{i_{k+1}} &= (z_1a_1 + \cdots + z_da_d) \\ &\in \mathbf{m}^{n+1} \cdot \mathbf{m}^{k-n} \\ &= \mathbf{m}^{k+1}. \end{aligned}$$

Therefore for k large enough we have $y_{i_k} \in \mathbf{q}$ such that $x_i' - y_{i_k} \in \mathbf{m}^k$ for each $i = 1, \ldots, d$.

Thus $x_i' \in \hat{\mathbf{q}}$ and $\hat{\mathbf{q}} = \mathbf{q}'$. From this it follows that

$$\ell\ell(\widehat{M}/\mathbf{q}' \cdot \widehat{M}) = \ell\ell(\widehat{M}/\hat{\mathbf{q}} \cdot \widehat{M}),$$

and

$$\ell\ell(M/\mathbf{q} \cdot M) = \ell\ell(\widehat{M}/\mathbf{q}' \cdot \widehat{M})$$

by Lemma 5.1.1.

So we have
$$\frac{\ell\ell(M/\mathbf{q}\cdot M)}{e(\mathbf{q};M)} = \frac{\ell\ell(\widehat{M}/\mathbf{q}'\cdot\widehat{M})}{e(\mathbf{q}';\widehat{M})}$$
and so $\theta_A(M) \geq \theta_{\hat{A}}(\widehat{M})$ as there may be parameter ideals in A not accounted for by the above construction.

Combining this with the earlier part of the proof we see that
$$\theta_A(M) = \theta_{\hat{A}}(\widehat{M}).$$

○

As we have seen in Chapter 2, this invariant is closely related to the invariant $n_A(M)$. The following results show its connection with the reduction number.

Definition 5.1.4 *Let J be a minimal reduction of I. The number $r_J(I) := \min\{r > 0$; there exists a minimal reduction J such that $I^{r+1} = J\cdot I^r\}$ is called the* ***reduction number*** *of I.*

Let A be a graded k-algebra with maximal ideal $\mathbf{m}$ and let $\mathbf{q}$ be a minimal reduction of $\mathbf{m}$.

Referring to the notation introduced above we get the following result:

Proposition 5.1.5 *With A, $\mathbf{m}$ and $\mathbf{q}$ as above we have*
$$r_{\mathbf{q}}(\mathbf{m}) = \ell\ell(A/\mathbf{q}) - 1.$$

Proof: First suppose that $r_{\mathbf{q}}(\mathbf{m}) = r$ so that
$$\mathbf{m}^{r+1} = \mathbf{q}\cdot\mathbf{m}^r.$$
Now $\mathbf{m}^{r+1} \subseteq \mathbf{q}$ as $\mathbf{q}\cdot\mathbf{m}^r \subseteq \mathbf{q}$ so we have
$$\ell\ell(A/\mathbf{q}) \leq r+1.$$
Therefore
$$r_{\mathbf{q}}(\mathbf{m}) \geq \ell\ell(a/\mathbf{q}) - 1.$$

Now suppose that $\mathbf{m}^r \subseteq \mathbf{q}$ where $\mathbf{m}^{r+1} = \mathbf{q}\cdot\mathbf{m}^r$ and $r_{\mathbf{q}}(\mathbf{m}) = r$. In the graded case we have
$$\mathbf{m}^r = (A_1 \oplus A_2 \oplus \cdots)^r = A_r \oplus A_{r+1} \oplus \cdots.$$
Let $x \in \mathbf{m}^r$ so $x = \gamma_0 a_0 + \gamma_1 a_1 + \cdots \in \mathbf{q}$ where $\gamma_i a_i \in A_{r+i}$ for each $i = 0, 1, \ldots$. That is,
$$\gamma_0 a_0 \in A_r,\ \gamma_1 a_1 \in A_{r+1},\ \ldots.$$
Now
$$\gamma_0 a_0 \in \mathbf{q}_r,\ \gamma_1 a_1 \in \mathbf{q}_{r+1},\ \ldots$$

and

$$\begin{aligned}\gamma_0 a_0 &= \delta_0 b_0\\ \gamma_1 a_1 &= \delta_1 a_1 + \alpha_1 c_1\\ &\vdots\end{aligned}$$

where $\delta_i \in \mathbf{q}_1$, $b_i \in A_{r-1+i}$, $\alpha_i \in \mathbf{q}_2$, $c_i \in A_{r-2+i}$, ... as $x \in \mathbf{m}^r \subseteq \mathbf{q}$. Hence

$$\begin{aligned}x &\in \mathbf{q}_1 \cdot A_{r-1} \oplus \mathbf{q}_1 \cdot A_r \oplus \cdots \oplus \mathbf{q}_2 \cdot A_{r-1} \oplus \mathbf{q}_2 \cdot A_r \oplus \cdots\\ &= \mathbf{q} \cdot A_{r-1} \oplus \mathbf{q} \cdot A_r \oplus \cdots\\ &= \mathbf{q} \cdot (A_{r-1} \oplus A_r \oplus \cdots)\\ &= \mathbf{q} \cdot (A_1 \oplus A_2 \oplus \cdots)^{r-1}\\ &= \mathbf{q} \cdot \mathbf{m}^{r-1}\end{aligned}$$

and

$$\mathbf{m}^r \subseteq \mathbf{q} \cdot \mathbf{m}^{r-1}.$$

So $\mathbf{m}^r = \mathbf{q} \cdot \mathbf{m}^{r-1}$, as $\mathbf{m}^r \supseteq \mathbf{q} \cdot \mathbf{m}^{r-1}$. This contradicts the minimality of $r_{\mathbf{q}}(\mathbf{m})$ so $\mathbf{m}^r \not\subseteq \mathbf{q}$. Hence

$$\ell\ell(A/\mathbf{q}) \geq r + 1 = r_{\mathbf{q}}(\mathbf{m}) + 1.$$

Therefore

$$\ell\ell(A/\mathbf{q}) - 1 = r_{\mathbf{q}}(\mathbf{m}).$$

○

An interesting application of Proposition 5.1.5 is the following (see [12], [26], [28] for other bounds for $r_{\mathbf{q}}(\mathbf{m})$):

Proposition 5.1.6 *Let A be a graded k-algebra with maximal ideal $\mathbf{m}$. For all minimal reductions $\mathbf{q}$ of $\mathbf{m}$ we have*

$$r_{\mathbf{q}}(\mathbf{m}) < \theta_A(A) \cdot e(\mathbf{m}; A).$$

Proof: By the notes following Lemma 2.3.9 $e(\mathbf{q}; A) = e(\mathbf{m}; A)$ so we have

$$\frac{\ell\ell(A/\mathbf{q})}{e(\mathbf{q}; A)} = \frac{\ell\ell(A/\mathbf{q})}{e(\mathbf{m}; A)} \leq \theta_A(A).$$

By Proposition 5.1.5, we have

$$\ell\ell(A/\mathbf{q}) - 1 = r_{\mathbf{q}}(\mathbf{m}).$$

Hence

$$\frac{r_{\mathbf{q}}(\mathbf{m})}{e(\mathbf{m}; A)} < \theta_A(A),$$

which implies

$$r_{\mathbf{q}}(\mathbf{m}) < \theta_A(A) \cdot e(\mathbf{m}; A).$$

○

This result is interesting because W. Vasconcelos recently proved that

$$r_{\mathfrak{q}}(\mathfrak{m}) \leq adeg(A)$$

where the $adeg(A)$ is the **arithmetic degree** of A (see eg. [17], [27] and the references contained within). It is also known that

$$adeg(A) > e(\mathfrak{m}; A).$$

5.2 Necessary conditions

An interesting question to ask here is, is $\theta_A(M) < \infty$ for any A and any M? This section examines this question and gives some necessary conditions for it to hold. We begin with the following example which shows that $\theta_A(M)$ is not always finite.

Example 5.2.1 Let $A = k[x, y, z]$. Set $I = x \cdot A \cap (y, z) \cdot A$ to be an ideal of A. Consider the graded k-algebra A/I.

Let n be an integer such that $n \geq 1$. Set $\mathfrak{q}_n = (x^n, y, z) \cdot A/I$. Now

$$A/I = k[x, y, z]/(xy, xz) \cdot A.$$

We have $dim A/I = 2$ and

$$\mathfrak{m}_{A/I}^{n-1} = (x^{n-1}, y^{n-1}, z^{n-1}, y^{n-2}z, \ldots, yz^{n-2}) \cdot A/I,$$

$$\mathfrak{m}_{A/I}^{n} = (x^n, y^n, z^n, y^{n-1}z, \ldots, yz^{n-1}) \cdot A/I \subseteq \mathfrak{q}_n.$$

If $\mathfrak{m}_{A/I}^{n-1} \subseteq \mathfrak{q}_n$ then $x^{n-1} \in \mathfrak{q}_n$. This is clearly not possible so $\mathfrak{m}_{A/I}^{n-1} \not\subseteq \mathfrak{q}_n$. Hence

$$\ell\ell(A/I + \mathfrak{q}_n) = n.$$

Now, by Lemma 2.3.3, we have

$$e(\mathfrak{q}_n; A/I) = e(\mathfrak{q}_n + (x)I;\ A/(I + (x)I)) = e((y, z);\ k[y, z]) = 1.$$

Therefore

$$\frac{\ell\ell(A/I + \mathfrak{q}_n)}{e(\mathfrak{q}_n; A/I)} = n.$$

So

$$\theta_A(A/I) = \infty.$$

This example shows that $\theta_A(M)$ is not always finite for any local (or graded) A-module M.

So a natural question to ask now is what properties does M have when $\theta_A(M)$ is finite? In the following theorem we show that if $\theta_A(M)$ is finite then M also satisfies the conclusion of Theorem 2.4.1(1). It was mentioned in Section 2.4 that $\theta_A(M) \leq n_A(M)$, and very often $\theta_A(M) << n_A(M)$. Hence this result is stronger than Theorem 2.4.1(1).

Theorem 5.2.2 *Let M be a (graded) A-module, where A is a local ring (or a graded k-algebra). If $\theta_A(M)$ is finite then M is a quasi-unmixed module.*

The proof of Theorem 5.2.2 is essentially the same as the proof of Theorem 2.4.1(1) given in [25]. However there are obviously some differences and for the sake of completion the full proof is given here (see also Proposition 4.4 of [1]).

Before beginning we need some lemmas. Throughout these lemmas let M be an A-module such that there is a minimal prime $\mathbf{p} \in Ass\ M$ with $d' := dim A/\mathbf{p} < dim M =: d$. Let

$$0_M = M_1 \cap M_2 \cap \cdots \cap M_r$$

be a primary decomposition of 0 in M where M_i is $\mathbf{p}_i$-primary for $i = 1, \ldots, r$. Set

$$\begin{aligned} N &= \cap\{M_i | M_i \text{ is } \mathbf{p}'\text{-primary}, \mathbf{p}' \in Ass M, \mathbf{p} \subseteq \mathbf{p}'\} \\ &= \cap\{M_i | \mathbf{p} \subseteq \mathbf{p}_i\}, \\ U &= \cap\{M_i | \mathbf{p} \not\subseteq \mathbf{p}_i\}. \end{aligned}$$

Lemma 5.2.3 $U = 0 :_M \mathbf{p}^t$ *for* $t >> 0$.

Proof:

$$\begin{aligned} 0 :_M \mathbf{p}^t &= (M_1 \cap \cdots \cap M_r) :_M \mathbf{p}^t \\ &= (M_1 :_M \mathbf{p}^t) \cap \cdots \cap (M_r :_M \mathbf{p}^t). \end{aligned}$$

For $i = 1, \ldots, r$ we have

$$M_i :_M \mathbf{p}^t = \begin{cases} M, & \text{if } \mathbf{p} \subseteq \mathbf{p}_i, \\ M_i, & \text{if } \mathbf{p} \not\subseteq \mathbf{p}_i, \end{cases}$$

for $t >> 0$. Therefore we can conclude that

$$0 :_M \mathbf{p}^t = \cap\{M_i | \mathbf{p} \not\subseteq \mathbf{p}_i\} = U$$

for $t >> 0$.

○

Lemma 5.2.4 *Assh $M \subset Ass(M/U)$.*

Proof: First we consider the exact sequence

$$0 \longrightarrow U \longrightarrow M \longrightarrow M/U \longrightarrow 0.$$

From this it follows that

$$Ass\ M \subseteq Ass\ U \cup Ass(M/U).$$

Let $\mathbf{p}' \in Ass\ U$ so that $\mathbf{p}' = Ann(y)$ for some $y \in U$. Now $U = 0 :_M \mathbf{p}^t$ so,

$$y \cdot \mathbf{p}^t = 0.$$

$\mathbf{p}^t \subseteq Ann(y) = \mathbf{p}'$ so $\mathbf{p} \subseteq \mathbf{p}'$ as $\mathbf{p}'$ is prime.

$dimA/\mathbf{p}' \leq dimA/\mathbf{p} = d' < d$, so $dimA/\mathbf{p}' < d$. Hence $\mathbf{p}' \notin Assh\ M$ and thus

$$Ass\ U \cap Assh\ M = \phi.$$

Therefore

$$Assh\ M \subseteq Ass(M/U).$$

○

Lemma 5.2.5 *$dimM/N = d'$.*

Proof: By assumption $dim(A/\mathbf{p}) = d'$ so we want to show that $dim(M/N) = dim(A/\mathbf{p})$. By definition,

$$\begin{aligned} dim(M/N) &= dim(A/Ann(M/N)) \\ &= \max_j\{coht(\mathbf{p}_j)\}, \end{aligned}$$

where $\mathbf{p}_j$ is any prime ideal belonging to the zero submodule (see eg. page 249 of [19]). In this case the zero submodule is

$$N = \cap\{M_i | \mathbf{p} \subseteq \mathbf{p}_i\}.$$

Hence

$$dim(M/N) = \max_{\mathbf{p}\subseteq\mathbf{p}_i}\{coht(\mathbf{p}_i)\} = coht(\mathbf{p}) = d'.$$

○

Lemma 5.2.6 *$Ann(M/U) \not\subseteq \mathbf{p}$.*

Proof: For each i such that $\mathbf{p}_i \not\subseteq \mathbf{p}$ choose $a_i \in \mathbf{p}_i \setminus \mathbf{p}$. Since $d' < d$, the set of such indices is not empty. Let a be the product of all of the a_i. Then $a \in \cap_i \mathbf{p}_i \setminus \mathbf{p}$. For $t \gg 0$ we then obviously get

$$a^t M \subseteq \cap_i M_i = U'$$

Thus $a^t \in Ann(M/U)$, but $a^t \notin \mathbf{p}$.

○

Lemma 5.2.7 *$Ann(M/N) \not\subseteq \bar{\mathbf{p}}$ for all $\bar{\mathbf{p}} \in Ass(M/U)$.*

Proof: Since $\mathbf{p} \subseteq \mathbf{p}_i$ for all $\mathbf{p}_i \in Ass(M/N)$ it follows that $\mathbf{p}$ is the only minimal element of $Ass(M/N)$. Therefore $\sqrt{Ann(M/N)} = \mathbf{p}$ (see eg. page 3 of [15]). So if

$$Ann(M/N) \subseteq \bar{\mathbf{p}} \in Ass(M/U)$$

then $\mathbf{p} \subseteq \bar{\mathbf{p}}$, which contradicts the definition of U.

Thus by Lemma 5.2.6 there exists $x \in Ann(M/U) \setminus \mathbf{p}$ and by Lemma 5.2.7 there exists $y \in Ann(M/N) \setminus \cup\{\bar{\mathbf{p}} | \ \bar{\mathbf{p}} \in Ass(M/U)\}$.

$\bigcirc$

The following is a well known fact (see eg. Exercise 9.19 of [23]):
If $J \subseteq I$ then the exact sequence

$$0 \longrightarrow J \longrightarrow I \longrightarrow I/J \longrightarrow 0$$

gives us

$$Supp(I) = Supp(J) \cup Supp(I/J).$$

So $Supp(I) \supseteq Supp(J)$.

Lemma 5.2.8 *Choose $x, y \in \mathbf{m}$ as above. Then*

$$\begin{aligned} Supp(M/(x+y)\cdot M) &= Supp(M/(x,y)\cdot M) \\ &= Supp(M/(x^n,y)\cdot M) \\ &= Supp(M/(x^n+y)\cdot M), \end{aligned}$$

for all positive integers n.

Proof: Note that if $N \subset M$, then $Supp(N), Supp(M/N) \subseteq Supp(M)$. Since each module in the following range can be considered as a factor module of the successor, we have

$$\begin{aligned} Supp(M/(x,y)\cdot M) \subseteq Supp(M/(x^n,y)\cdot M) &\subseteq Supp(M/(x^n+y)\cdot M) \\ &\subseteq Supp(M/(x+y)\cdot M). \end{aligned}$$

Now we show that

$$Supp(M/(x+y)\cdot M) \subseteq Supp(M/(x,y)\cdot M).$$

Let $a \in Ann(M/(x,y)\cdot M)$. For any $m \in M$ we can write

$$am = xm_1 + ym_2 = (x+y)m_1 + y(m_2 - m_1),$$

where $m_1, m_2 \in M$. Analogously,

$$a(m_2 - m_1) = xm_3 + ym_4 = x(m_3 - m_4) + (x+y)m_4,$$

where $m_3, m_4 \in M$. Hence

$$\begin{aligned} a^2m &= a(x+y)m_1 + ya(m_2 - m_1) \\ &= a(x+y)m_1 + y(x+y)m_4 + xy(m_3 - m_4). \end{aligned}$$

Since $xM \subseteq U$ and $yM \subseteq N$, $xyM \subseteq U \cap N = 0$. Therefore the above equation implies $a^2m \in (x+y)M$, i.e.

$$Ann(M/(x,y)M) \subseteq \sqrt{Ann(M/(x+y)M)}.$$

As $Supp(N)$ is equal to the set of prime ideals that contain $Ann(N)$ for any finite A-module N, we have

$$Supp(M/(x+y)\cdot M) \subseteq Supp(M/(x,y)\cdot M).$$

○

Lemma 5.2.9 *Set* $\mathfrak{q}_n' := (x^n + y, f_1, \ldots, f_{d-1})\cdot A$ *where* $x, y \in \mathfrak{m}$ *such that* $x \cdot M \subseteq U$, $x \notin \mathfrak{p}$ *and* $y \cdot M \subseteq N$, $y \notin \bar{\mathfrak{p}}$ *for all* $\bar{\mathfrak{p}} \in Ass(M/U)$. *Also* $f_1, \ldots, f_{d-1}$ *are homogeneous elements such that* $\mathfrak{q}_n'$ *is a parameter ideal of* M. *Then*

$$e(\mathfrak{q}_n'; M) = e((y, f_1, \ldots, f_{d-1}); M/U).$$

Proof: Since $0 = \cap_{i=1}^r M_i = N \cap U$ we can construct the following exact sequence:

$$0 \longrightarrow M/N\cap U \xrightarrow{\phi} M/U \oplus M/N \xrightarrow{\varphi} M/N + U \longrightarrow 0.$$

Applying Lemma 2.3.4 we get

$$e(\mathfrak{q}_n'; M/U \oplus M/N) = e^*(\mathfrak{q}_n'; M/N\cap U) + e^*(\mathfrak{q}_n'; M/N + U).$$

So,

$$e(\mathfrak{q}_n'; M/U) + e^*(\mathfrak{q}_n'; M/N) = e^*(\mathfrak{q}_n'; M/N\cap U) + e^*(\mathfrak{q}_n'; M/N + U).$$

We know that $dim M/N = d' < d$, so

$$e^*(\mathfrak{q}_n'; M/N + U) = e^*(\mathfrak{q}_n'; M/N) = 0.$$

Also $N \cap U = 0$ so $M/N\cap U = M$ and we have

$$e(\mathfrak{q}_n'; M/U) = e(\mathfrak{q}_n'; M).$$

Since $x \cdot M \subseteq U$ it follows that $x \cdot M/U = 0$ and we get

$$e(\mathfrak{q}_n'; M) = e((y, f_1, \ldots, f_{d-1}); M/U).$$

○

We can now begin the proof of Theorem 5.2.2.

Proof of Theorem 5.2.2: Let $\theta_A(M)$ be finite.

Assume that M is not quasi-unmixed and look for a contradiction. By Lemma 5.1.3 we may assume without loss of generality that A is complete. Now, by the Cohen structure theorem (Theorem 29.4(ii) of [15]), $A = R/I$ where R is a complete regular local ring. Since M is a A-module we can consider the canonical homomorphism $f : R \to R/I$. The mapping $R \times M \to M$ given by $(r, m) \mapsto f(r)\cdot m \in M$ implies that M has R-module structure. Therefore we can assume, without loss of generality, that A is a complete regular local ring or a polynomial ring in finitely many indeterminates over a field k.

By assumption we can consider a minimal prime ideal $\mathbf{p} \in Ass\ M$ such that

$$d' := dim A/\mathbf{p} < dim M =: d.$$

Let $0_M = M_1 \cap M_2 \cap \cdots \cap M_r$ be a primary decomposition of 0 in M where M_i is $\mathbf{p}_i$-primary for $i = 1, \ldots, r$. Set

$$\begin{aligned} N &= \cap\{M_i | M_i \text{ is } \mathbf{p}'\text{-primary}, \mathbf{p}' \in Ass M, \mathbf{p} \subseteq \mathbf{p}'\} \\ &= \cap\{M_i | \mathbf{p} \subseteq \mathbf{p}_i\}, \\ U &= \cap\{M_i | \mathbf{p} \not\subseteq \mathbf{p}_i\}. \end{aligned}$$

We now have the following:

$$\begin{aligned} &U = 0 :_M \mathbf{p}^t, \text{ for } t >> 0; \\ &Assh\ M \subseteq Ass(M/U); \\ &dim M/N = d'; \\ &Ann(M/U) \not\subseteq \mathbf{p}; \\ &Ann(M/N) \not\subseteq \bar{\mathbf{p}}, \text{ for all } \bar{\mathbf{p}} \in Ass(M/U); \end{aligned}$$

(Lemma 5.2.3 - Lemma 5.2.8). The last two statements allow us to take (homogeneous) elements $x, y \in \mathbf{m}$ such that

$$x \cdot M \subseteq U, \quad x \notin \mathbf{p},$$

$$y \cdot M \subseteq N, \quad y \notin \bar{\mathbf{p}} \text{ for all } \bar{\mathbf{p}} \in Ass(M/U).$$

As $(x + y) \cdot M$ reduces the dimension of M by one we have

$$dim(M/(x^n + y) \cdot M) = dim(M/(x^n, y) \cdot M) = d - 1,$$

for all positive integers n. We have $\sqrt{Ann(M/N)} = \mathbf{p}$ and $x \notin \mathbf{p}$. Since $\mathbf{p} + (x) \subseteq Ann(M/\mathbf{p}M + (x, y)M)$, $dim(M/\mathbf{p}M + (x, y)M) \leq dim(A/\mathbf{p} + (x)) = d' - 1$. Hence we can take (homogeneous) elements $f_1, \ldots, f_{d-d'} \in Ann(M/N)$ which form part of a system of parameters for $M/(x, y) \cdot M$.

Consider (homogeneous) elements $f_{d-d'+1}, \ldots, f_{d-1}$ such that $f_1, \ldots, f_{d-1}$ is a system of parameters for $M/(x, y)M$. Then by Lemma 5.2.9, $f_1, \ldots, f_{d-1}$ is a system of parameters for $M/(x^n, y)M$ and for $M/(x^n + y)M)$ for all positive integers n.

Now set

$$\mathbf{q}_n = (x^n, y, f_1, \ldots, f_{d-1}) \cdot A,$$

$$\mathbf{q}'_n = (x^n + y, f_1, \ldots, f_{d-1}) \cdot A,$$

for all positive integers n.

Clearly $\mathbf{q}'_n \subseteq \mathbf{q}_n$ so that

$$\ell\ell(M/\mathbf{q}'_n \cdot M) \geq \ell\ell(M/\mathbf{q}_n \cdot M).$$

Suppose $\ell\ell(M/\mathbf{q}_n \cdot M) = t$ so that $\mathbf{m}^t \subseteq \mathbf{q}_n$. Now $x \in \mathbf{m}$ by construction so $x^t \in \mathbf{q}_n$. Also $y, f_1, \ldots, f_{d-1} \in \mathbf{q}_n$ for all positive integers n so

$$\mathbf{q}_t = (x^t, y, f_1, \ldots, f_{d-1}) \cdot A \subseteq \mathbf{q}_n.$$

Suppose $t < n$, then $x^t \cdot \bar{M} = x^n \cdot \bar{M}$ and we have $x^t \cdot \bar{M} = x^{n-t}(x^t \cdot \bar{M})$, where $\bar{M} = M/(y, f_1, \ldots, f_{d-1}) \cdot M$. By Nakayama's lemma (Theorem 2.2 of [15]) we get $x^t \cdot \bar{M} = 0$. So

$$\begin{aligned} 0 &= dim(M/\mathbf{q}_n \cdot M) \\ &\geq dim(M/(x^t, x^n, y, f_1, \ldots, f_{d-1}) \cdot M) \\ &= dim(M/\mathbf{q}_t \cdot M), \end{aligned}$$

as $t < n$. Now

$$\begin{aligned} M/\mathbf{q}_t \cdot M &= M/(x^t, y, f_1, \ldots, f_{d-1}) \cdot M \\ &= \bar{M}/x^t \cdot \bar{M} \\ &= \bar{M}, \end{aligned}$$

as $x^t \cdot \bar{M} = 0$. So

$$\begin{aligned} dim\bar{M} &= dim(M/(y, f_1, \ldots, f_{d-1}) \cdot M) \\ &= 1 \end{aligned}$$

as $f_1, \ldots, f_{d-1}$ is part of a system of parameters and $(y, f_1, \ldots, f_{d-1}) \cdot M$ is not a parameter ideal of M. Hence

$$1 = dim\bar{M} \leq dim(M/\mathbf{q}_n \cdot M)$$

which is a contradiction.

Therefore

$$\ell\ell(M/\mathbf{q}'_n \cdot M) \geq \ell\ell(M/\mathbf{q}_n \cdot M) \geq n.$$

With regards to the multiplicity we have

$$\begin{aligned} e(\mathbf{q}'_n; M) &= e(\mathbf{q}'_n; M/U) \\ &= e((y, f_1, \ldots, f_{d-1}); M/U) \end{aligned}$$

(Lemma 5.2.9). Hence the multiplicity of $\mathbf{q}'_n$ is independent of n. So,

$$\begin{aligned} \lim_{n\to\infty} \frac{\ell\ell(M/\mathbf{q}'_n \cdot M)}{e(\mathbf{q}'_n; M)} &\geq \lim_{n\to\infty} \frac{n}{e((y, f_1, \ldots, f_{d-1}); M/U)} \\ &= \infty. \end{aligned}$$

Hence, by definition, we have

$$\theta_A(M) = \infty$$

which is a contradiction. Thus M must be quasi-unmixed.

○

The final result for this section is analogous to Theorem 2.4.1(2) but it is given in terms of $\theta_A(M)$. The proof is essentially the same as the one given for Theorem 1(2) in [25] but it is given in full here for the sake of completion.

Theorem 5.2.10 *Given a positive integer d the following conditions are equivalent:*
(i) For every local (or graded) ring A of dimension at most d and every quasi-unmixed (graded) A-module M we have $\theta_A(M) < \infty$*;*
(ii) For every complete regular local ring R with infinite residue class field (or for every polynomial ring $R = k[X_1, \dots, X_n]$ *over an infinite field k) we have* $\theta_R(R/\mathbf{p}) < \infty$ *for all (homogeneous) prime ideals* $\mathbf{p}$ *of R such that* $dim(R/\mathbf{p}) \leq d$*.*

In order to prove the above result we first need a lemma.

Lemma 5.2.11 *Let A be a local ring (or a graded k-algebra) and let M be an (graded) A-module. Suppose*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

is a composition series of M such that

$$M_i/M_{i-1} \cong A/\mathbf{p}_i$$

(up to shift of degrees) for some (homogeneous) prime ideals $\mathbf{p}_i$ *of A,* $i = 1, \dots, r$*, then*

$$\ell\ell(M/\mathbf{q} \cdot M) \leq \sum_{i=1}^{r} \ell\ell(A/\mathbf{q} + \mathbf{p}_i),$$

for any (homogeneous) $\mathbf{m}$*-primary ideal* $\mathbf{q}$*.*

Proof: As $A/\mathbf{p}_i \cong M_i/M_{i-1}$ it follows that

$$\ell\ell(A/\mathbf{q} + \mathbf{p}_i) = \ell\ell((M_i/M_{i-1})/\mathbf{q} \cdot (M_i/M_{i-1})).$$

Applying Lemma 5.1.2(ii) we then get

$$\begin{aligned} \ell\ell(M/\mathbf{p}M) &= \ell\ell(M_r/\mathbf{p}M_r) \\ &\leq \ell\ell(M_{r-1}/\mathbf{p}M_{r-1}) + \ell\ell(A/\mathbf{q} + \mathbf{p}_r) \\ &\leq \ell\ell(M_{r-2}/\mathbf{p}M_{r-2}) + \ell\ell(A/\mathbf{q} + \mathbf{p}_{r-1}) + \ell\ell(A/\mathbf{q} + \mathbf{p}_r) \\ &\dots\dots \\ &\leq \textstyle\sum_{i=1}^{r} \ell\ell(A/\mathbf{q} + \mathbf{p}_i). \end{aligned}$$

○

Proof of Theorem 5.2.10: (i) $\Rightarrow$ (ii) is trivial. For (ii) $\Rightarrow$ (i), we can apply the same method used in the proof of Theorem 5.2.2 so that we can assume, without loss of generality, that A is either a complete regular local ring or a polynomial ring in finitely many indeterminates over a field k.

Let M be a quasi-unmixed (graded) A-module of $dim(M) \leq d$. By Theorem 1 on page 265 of [5] there exists a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of M consisting of (graded) submodules $M_0, \ldots, M_r$ of M such that

$$M_i/M_{i-1} \cong A/\mathbf{p}_i$$

(up to shifts of degree) for some (homogeneous) prime ideals $\mathbf{p}_i$ of A, for $i = 1, \ldots, r$.

By Theorem 2 on page 265 of [5] it follows that $\{\mathbf{p}_1, \ldots, \mathbf{p}_r\} \subseteq Supp\ M$.

If $\mathbf{p}_i$ is not a minimal prime then there exists a minimal prime $\mathbf{p} \subseteq \mathbf{p}_i$ such that

$$\mathbf{p} \in min.Supp\ M = Assh\ M,$$

since M is quasi-unmixed (see Section 1 of Chapter 2). So there exists a prime ideal $\mathbf{p} \in Assh\ M$ such that $\mathbf{p} \subseteq \mathbf{p}_i$ for all $i = 1, \ldots, r$. Note that $dim(A/\mathbf{p}) \leq d$.

Let $\mathbf{q}$ be a (homogeneous) $\mathbf{m}$-primary ideal of A. From Lemma 5.2.11 we have

$$\ell\ell(M/\mathbf{q} \cdot M) \leq \sum_{i=1}^{r} \ell\ell(A/\mathbf{q} + \mathbf{p}_i)$$

considering $A/\mathbf{p}_i$ as an A-module. Now

$$\sum_{i=1}^{r} \ell\ell(A/\mathbf{q} + \mathbf{p}_i) \leq \sum_{\mathbf{p} \in Assh\ M} \left(\sum_{1 \leq i \leq r,\ \mathbf{p} \subseteq \mathbf{p}_i} \ell\ell(A/\mathbf{q} + \mathbf{p}_i) \right)$$

as for each $\mathbf{p}_i$ there exists $\mathbf{p} \in Assh\ M$ such that $\mathbf{p} \subseteq \mathbf{p}_i$ so each $\ell\ell(A/\mathbf{q} + \mathbf{p}_i)$ is included on the right hand side at least once.

$$\sum_{\mathbf{p} \in Assh\ M} \left(\sum_{1 \leq i \leq r,\ \mathbf{p} \subseteq \mathbf{p}_i} \ell\ell(A/\mathbf{q} + \mathbf{p}_i) \right) \leq \sum_{\mathbf{p} \in Assh\ M} \beta_{\mathbf{p}} \cdot \ell\ell(A/\mathbf{q} + \mathbf{p})$$

considering $A/\mathbf{p}$ as an A-module and where

$$\beta_{\mathbf{p}} := \sharp\{i | 1 \leq i \leq r,\ \mathbf{p} \subseteq \mathbf{p}_i\}.$$

This inequality follows from Lemma 5.1.2(i) as $A/\mathbf{p}_i \cong (A/\mathbf{p})/(\mathbf{p}_i/\mathbf{p})$ is a factor module of $A/\mathbf{p}$.

Therefore

$$\ell\ell(M/\mathbf{q} \cdot M) \leq \sum_{\mathbf{p} \in Assh\ M} \beta_{\mathbf{p}} \cdot \ell\ell(A/\mathbf{q} + \mathbf{p}).$$

Define

$$\alpha_{\mathbf{p}} := \ell_{A_{\mathbf{p}}}(M_{\mathbf{p}})$$

for all $\mathbf{p} \in Assh\ M$. It follows by Remark 1 on page 275 of [5] that $1 \leq \alpha_{\mathbf{p}} \leq \beta_{\mathbf{p}}$ for all $\mathbf{p} \in Assh\ M$. Set

$$\rho := \max\{\beta_{\mathbf{p}}/\alpha_{\mathbf{p}} | \mathbf{p} \in Assh\ M\} \geq 1,$$

so that we have

$$\begin{aligned} \ell\ell(M/\mathbf{q} \cdot M) &\leq \textstyle\sum_{\mathbf{p} \in Assh\ M} \beta_{\mathbf{p}} \cdot \ell\ell(A/\mathbf{q} + \mathbf{p}) \\ &= \textstyle\sum_{\mathbf{p} \in Assh\ M} \left(\frac{\beta_{\mathbf{p}}}{\alpha_{\mathbf{p}}}\right) \cdot \alpha_{\mathbf{p}} \cdot \ell\ell(A/\mathbf{q} + \mathbf{p}) \\ &\leq \rho \cdot \textstyle\sum_{\mathbf{p} \in Assh\ M} \alpha_{\mathbf{p}} \cdot \ell\ell(A/\mathbf{q} + \mathbf{p}). \end{aligned}$$

Applying Lemma 2.3.3 we obtain

$$e(\mathbf{q};M) = \sum_{\mathbf{p}\in Assh\ M} \alpha_{\mathbf{p}} \cdot e(\mathbf{q};A/\mathbf{p}).$$

Hence

$$\begin{aligned} \frac{\ell\ell(M/\mathbf{q}\cdot M)}{e(\mathbf{q};M)} &\leq \rho \cdot \frac{\sum_{\mathbf{p}\in Assh\ M} \alpha_{\mathbf{p}}\cdot \ell\ell(A/\mathbf{q}+\mathbf{p})}{\sum_{\mathbf{p}\in Assh\ M} \alpha_{\mathbf{p}}\cdot e(\mathbf{q};A/\mathbf{p})} \\ &\leq \rho \cdot \max\left\{ \frac{\ell\ell(A/\mathbf{q}+\mathbf{p})}{e(\mathbf{q};A/\mathbf{p})} \middle| \mathbf{p} \in Assh\ M \right\} \\ &\leq \rho \cdot \max\left\{ \theta_A(A/\mathbf{p}) | \mathbf{p} \in Assh\ M \right\} \\ &< \infty \end{aligned}$$

by assumption. Therefore

$$\theta_A(M) < \infty.$$

○

5.3 Boundedness of $\theta_A(M)$

The final section in this chapter is concerned with a final result on the Loewy length. In order to prove this result a technique of W. Vasconcelos' (see [27] and [28]) is employed. This technique will be briefly outlined before the main result is proved.

We begin with Vasconcelos' technique. Given a finitely generated standard graded algebra

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

over a field $A_0 = k$ we can use the following construction. Suppose $dim A = d$. There are homogeneous elements $x_1, \ldots, x_d \in A_1$ such that

$$R = k[x_1, \ldots, x_d] = k[\mathbf{z}] \hookrightarrow A = S/I$$

where $S = k[x_1, \ldots, x_d, x_{d+1}, \ldots, x_n]$ and I is chosen so that $A = S/I$. This is a **Noether normalization** of A. That is, the x_i are algebraically independent over k and A is a finite R-module. If $b_1, \ldots, b_s$ is a minimal set of homogeneous generators of A as an R-module then

$$A = \sum_{1\leq i\leq s} R \cdot b_i$$

and $deg(b_i) = r_i$.

We now need some definitions:

Definition 5.3.1 *Let M be an A-module. Define a* ***finite free resolution*** *of M to be an exact sequence*

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

of finite length n such that each F_i is a finite free A-module. That is, each F_i is finitely generated and $F_i \cong A^{m_i}$ for some positive integer m_i, $0 \leq i \leq n$.

Definition 5.3.2 *The* ***rank*** *of a module M over an integral domain A is the maximal number of elements of M linearly independent over A. The rank is also equal to the dimension of the L-vector space $M \otimes_A L$ where L is the field of fractions of A. It is denoted by $[M : A]$.*

With these definitions in place we can construct the following polynomial. Fix a Noether normalization $R = k[\mathbf{z}] \hookrightarrow A = S/I$. Take an endomorphism f of an A-module M,

$$f : M \longrightarrow M.$$

Map a free graded module F over M and lift f to the following:

$$\begin{array}{ccc} F & \xrightarrow{\pi} & M \\ \varphi \downarrow & & \downarrow f \\ F & \xrightarrow{\pi} & M \end{array}$$

Set

$$\begin{aligned} P_\varphi(t) &:= det(t \cdot I + \varphi) \\ &= t^n + \cdots + a_n \end{aligned}$$

where I is the identity matrix and t is an indeterminate over A. This is the **characteristic polynomial** of φ and n is equal to the rank of F.

If we now apply a technique of [3] we can lift f to a mapping from a free resolution of M onto itself,

$$\begin{array}{cccccc} 0 \to F_s & \to \cdots \to F_1 & \to F_0 & \to M & \to 0 \\ \varphi_s \downarrow & \varphi_1 \downarrow & \varphi_0 \downarrow & f \downarrow & \\ 0 \to F_s & \to \cdots \to F_1 & \to F_0 & \to M & \to 0 \end{array}$$

Define

$$P_f(t) := \prod_{i=0}^{s} (P_{\varphi_i}(t))^{(-1)^i}$$

which is now a polynomial in $R[t]$. We note that if M is graded and f is homogeneous then $P_f(t)$ is a homogeneous polynomial with $deg\ P_f(t) = e(M)$.

With this construction in place it is possible to apply the Cayley-Hamilton Theorem (see [27] or [28]) to get some useful properties of this polynomial. Before restating this theorem we need two more definitions.

Definition 5.3.3 *If P is a non-zero polynomial of degree n in R then P is said to be* ***monic*** *precisely when its n-th coefficient is 1_k.*

Definition 5.3.4 *Let A be an integral domain with field of fractions L. An A-module M is* ***torsion-free*** *if every non-zero element of A is a non-zero divisor on M. Equivalently, M is torsion-free if the map $M \to L \otimes_A M$ is a monomorphism.*

Recall that $[M : A] = dim_L(L \otimes_A M)$.

The Cayley-Hamilton Theorem is now as follows:

Theorem 5.3.5 (Cayley-Hamilton) *Let M be a graded module over a ring of polynomials, say R, and let f be an endomorphism of M. If the rank of M over R is n, then $P_f(t)$ is a monic polynomial of degree n. Moreover, if M is torsion-free over R then*

$$P_f(f) \cdot M = 0.$$

We now come to the main result in this section, which also appears in [1] (Theorem 4.6).

Theorem 5.3.6 *Assume that A is a local ring containing an infinite field. Let M be a quasi-unmixed A-module with $dimM = dimA$. Then $\theta_A(M)$ is finite.*

Proof: By Theorem 5.2.10 we may assume that $M = A = k[[X_1, \ldots, X_n]]/\mathbf{p}$, a power series ring modulo $\mathbf{p}$, where $\mathbf{p}$ is a prime ideal. Let $\mathbf{q} = (y_1, \ldots, y_d)$ be a parameter ideal of A. By Theorem 14.5 of [15] we know that the elements $y_1, \ldots, y_d$ are analytically independent over A. That is, if $F(Y_1, \ldots, Y_d)$ is a homogeneous form with coefficients in A, then if $F(y_1, \ldots, y_d) = 0$ it follows that the coefficients of F are in $\mathbf{m}$. So, the subring $S = k[[y_1, \ldots, y_d]]$ of A is a regular ring (Theorem 17.10 of [15]). We can consider A as a finite module over S. Hence A has a finite free resolution over S.

Fix an arbitrary element $x \in A$. Define an S-homomorphism $f : A \to A$ by $f(y) = x \cdot y$. Lift f to a mapping from a free resolution of A into itself:

$$\begin{array}{ccccccccc} 0 \to F_s & \to \cdots \to & F_1 & \to & F_0 & \to & A & \to 0 \\ \varphi_s \downarrow & & \varphi_1 \downarrow & & \varphi_0 \downarrow & & f \downarrow & \\ 0 \to F_s & \to \cdots \to & F_1 & \to & F_0 & \to & A & \to 0 \end{array}$$

For each homomorphism φ_i of the free module F_i let

$$P_{\varphi_i}(t) = det(t \cdot I + \varphi_i),$$

where I is an identity matrix, be the characteristic polynomial of φ_i. Define

$$P_f(t) := \prod_{i=0}^{s} (P_{\varphi_i}(t))^{(-1)^i}.$$

By [2] Section 5 and [3], which set out results on the characteristic polynomial, we know that $P_f(t)$ is a monic polynomial in $S[t]$. Let L be the field of fractions of S. Since A is an integral domain the characteristic polynomial of the vector space mapping

$$f \otimes L : A \otimes L \longrightarrow A \otimes L$$

is precisely $P_f(t)$ (Corollary 5.13 of [2]). Hence $deg\ P_f(t)$ equals the rank of A over S (see Section 3 of [28]), which is given by $dim(A \otimes L)$ by definition.

Again using the fact that A is an integral domain, Corollary 2 on page 300 of Volume II of [31] shows that

$$[A : k[[y_1, \ldots, y_d]]] = e(\mathbf{q}; A) \cdot [(A/\mathbf{m}) : k].$$

That is

$$[A : S] = e(\mathbf{q}; A) \cdot [k : k] = e(\mathbf{q}; A).$$

So the rank of A over S is given by $e := e(\mathbf{q}; A)$. Also, since A is an integral domain, by the Cayley-Hamilton Theorem (Theorem 5.3.5) we get

$$(P_f(t)) \cdot A = 0.$$

Hence there are $c_1, \ldots, c_e \in S$ such that

$$x^e + c_1 x^{e-1} + \cdots + c_e = 0.$$

If no c_i is a unit of S, ie. $c_i \in (y_1, \ldots, y_d) \cdot S$ for all $i = 1, \ldots, e$, then $x^e \in \mathbf{q}$. This follows easily as

$$x^e = -(c_1 x^{e-1} + \cdots + c_e) \in \mathbf{q}.$$

Otherwise, assume that $p \leq e$ is the largest integer such that c_p is a unit of S. Then

$$\begin{aligned} x^p(c_p + c_{p-1}x + \cdots + c_1 x^{e-p-1} + x^{e-p}) &\in (y_1, \ldots, y_d) \cdot S \\ &\subseteq \mathbf{q} \end{aligned}$$

as

$$\begin{aligned} x^p(c_p + c_{p-1}x + \cdots + c_1 x^{e-p-1} + x^{e-p}) &= -(c_{p+1}x^{p+1} + \cdots + c_e) \\ &\in (y_1, \ldots, y_d) \cdot S. \end{aligned}$$

Since

$$c_{p-1}x + \cdots + c_1 x^{e-p-1} + x^{e-p} \in \mathbf{m}$$

it follows that

$$c_p + c_{p-1}x + \cdots + c_1 x^{e-p-1} + x^{e-p}$$

is a unit of A, and hence of S. Therefore $x^p \in \mathbf{q}$ which also implies that $x^e \in \mathbf{q}$. So in both cases we have $x^e \in \mathbf{q}$.

Let $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of degree $u = n(e-1) + 1$ in A. So

$$\alpha_1 + \cdots + \alpha_n = n(e-1) + 1 = ne - n + 1$$

which implies that $\alpha_i \geq e$ for at least one $1 \leq i \leq n$. Thus any such monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ will belong to $\mathbf{q}$. Therefore $\mathbf{m}^u \subseteq \mathbf{q}$ and

$$\ell\ell(A/\mathbf{q}) \leq u.$$

We now have

$$\begin{aligned} \frac{\ell\ell(A/\mathbf{q})}{e(\mathbf{q};A)} &\leq \frac{n(e-1)+1}{e} \\ &= n - \frac{n-1}{e} \\ &\leq n \end{aligned}$$

and this is independent of $\mathbf{q}$. Hence

$$\theta_A(M) < \infty.$$

○

Letting $\theta_A(M)$ be defined in exactly the same way in the graded case as it was in the local case we obtain the following corollary to Theorem 5.3.6.

Corollary 5.3.7 *Let A be a graded k-algebra and let M be a graded quasi-unmixed A-module with $dimA = dimM$. Then $\theta_A(M)$ is finite,*

Proof: Since A is a graded k-algebra it follows that

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots$$

where k is an infinite field. Hence A contains an infinite field and so, by applying a process similar to the proof of Theorem 5.3.6, we have

$$\theta_A(M) < \infty.$$

○

We think that Theorem 5.3.6 will hold for any local ring. Theorem 5.3.6 and Corollary 5.3.7 provide strong support for Conjecture 1.3.1.

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