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# **On the Acoustical Theory of the Trumpet:**

## **Is it Sound?**

A thesis presented in partial fulfilment of the requirements  
for the degree of Master of Science in Mathematics  
at Massey University, New Zealand

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## Abstract

Newton's Second Law of Motion for one-dimensional inviscid flow of an incompressible fluid, in the absence of external forces, is often expressed in a form known as Bernoulli's equation:

$$\int_{x_1}^{x_2} \frac{\partial u}{\partial t} dx + \frac{1}{2} u^2 \Big|_{x_1}^{x_2} + \frac{p}{\rho} \Big|_{x_1}^{x_2} = 0$$

There are two distinct forms of Bernoulli's equation used in the system of equations which is commonly considered to describe sound production in a trumpet.

The flow *between the trumpeter's lips* is, in the literature, assumed to be *quasi-steady*. From this assumption, the first term of the above Bernoulli equation is omitted, since it is then small in comparison to the other two terms.

The flow *within the trumpet* itself is considered to consist of small fluctuations about some mean velocity and pressure. A *linearized* version of Bernoulli's equation (as used in the equations of linear acoustics) is then adequate to describe the flow. In this case it is the second term of the above equation which is neglected, and the first term is retained.

Given that the flow between the trumpeter's lips is that same flow which enters the trumpet itself, a newcomer to the field of trumpet modelling might wonder whether the accepted model is really correct when these two distinct versions of the Bernoulli Equation are used side by side.

This thesis addresses this question, and raises others that arise from a review of the standard theory of trumpet physics. The investigation comprises analytical and experimental components, as well as computational simulations.

No evidence has been found to support the assumption of quasi-steady flow between the lips of a trumpeter. An alternative flow equation is proposed, and conditions given for its applicability.

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I thank God for the occasional divine inspiration, for regular bursts of superhuman strength and fortitude, and for just hanging around the rest of the time.

## Preface

The work discussed in this thesis germinated, in February of 1992, from a project idea for a one-year masters program in Electrical and Electronic Engineering at the University of Canterbury: to synthesise trumpet sounds by implementing, in real-time, a mathematical model of trumpet physics.

After two years at the University of Canterbury, I summarised my findings in a report (Redhead, 1993) and sent copies to four researchers who I thought may be interested. *Chapter One* of this thesis represents (mainly) the findings of my two years at the University of Canterbury.

My report was received favourably by Shigeru Yoshikawa (Japan), who cited it in an article published by the Journal of the Acoustical Society of America (Yoshikawa, 1995), and by Douglas Keefe (U.S.), whose response led to my visit to the University of Washington, Seattle. *Chapter Five* describes the results of an experiment performed when I visited the University of Washington, Seattle, for three weeks of 1994.

*Chapters Two to Four, Chapter Six and Chapter Seven* have originated from my subsequent two years at Massey University.

None of the content of this thesis, in whole or in part, has been presented before for examination at any university.





## Modelling of

### Sound-Production in the Trumpet

The exact means by which sound oscillations are produced and maintained in many wind instruments is, to date, uncertain. A large obstacle to the pursuit of such knowledge has been the difficulty of making detailed physical measurements while the instrument is being played. Many parameters and variables of the system (such as lung pressure and diaphragm force, vocal tract shape and facial muscular tension) are, to a large extent, unmeasurable. But some useful results have been obtained in spite of these difficulties (Martin, 1942; Henderson, 1942; Stubbins, Lillya & Frederick, 1956; Weast, 1963; Bouhuys, 1965; Vivona, 1968; Elliott & Bowsher, 1982; Barbenal, Davies & Kenny, 1986; Yoshikawa & Plitnik, 1993; Copley & Strong, 1994; Yoshikawa, 1995).

It is a far simpler matter to investigate the acoustical response of a particular wind instrument shape in isolation from the musician. By appropriate coupling of an electro-acoustic transducer to the musical instrument a frequency response may be determined for the instrument alone (Webster, 1947; Igarashi & Koyasu, 1953; Benade, 1973; Backus, 1974, 1976; Smith & Daniell, 1976; Pratt, Elliott & Bowsher, 1977; Elliott, Bowsher & Watkinson, 1982). Other less specific studies also have relevance to musical instrument analysis (Jansson & Benade, 1974; Silcox & Lester, 1982; Davies, 1988).

A number of researchers have directed their efforts towards developing apparatus which sounds a wind instrument artificially from a supply of compressed air (Webster, 1919a; Martin, 1942; Backus, 1963, 1964; Backus & Hundley, 1971; Wilson & Beavers, 1974; Fletcher, Silk & Douglas, 1982; Idogawa et al., 1988).

In recent years, the ever-increasing speed, capacity and availability of computational resources has seen mathematical modelling become an invaluable tool in the investigation of wind instrument operation (Pyle, 1969; Stewart & Strong, 1980; Saneyoshi, Teramura & Yoshikawa, 1987; Park & Keefe, 1988; Sommerfeldt & Strong, 1988; Yoshikawa,

1988; Barjau & Agulló, 1989; Keefe, 1990, 1992; Strong, 1990b; Brown & Schumacher, 1990; Strong & Dudley, 1993; Adachi & Sato, 1995). Mathematical modelling offers significant advantages over experimental procedures, particularly in the separate analysis of subsystems of the overall system (e.g. Dudley & Strong, 1990, 1993).

The idea of mathematical modelling of musical instruments has been received eagerly by Electronic Engineers hoping to implement such models using fast and efficient algorithms for real-time synthesis of musical instrument sounds (Smith, 1991a; Valimaki et al., 1992; Cook, 1992; Borin, De Poli & Sarti, 1992; Rodet & Doval, 1992). But while providing additional motivation for the study of musical instrument operation, it is the present author's view that the birth of *physical modelling synthesis* has also brought to the field a sense of urgency which, perhaps, has not been completely advantageous inasmuch as attention to model detail is concerned.

Sound synthesis via a mathematical model of a real musical instrument promises numerous and significant advantages over other synthesis techniques (Smith, 1991a). However, these synthesisers can only yield accurate reproduction of traditional musical instrument sounds if based upon sufficiently accurate mathematical models of those instruments. Some engineers state that "... in general, a serious comparison between reality and simulated results cannot give satisfactory results" (Borin, De Poli & Sarti, 1992). Others have turned their attention away from applying the known models and towards improving the models themselves (Redhead, 1993).

Although this thesis focuses especially upon improving the model of sound-production in the trumpet, some of the conclusions are more broadly applicable to other wind instruments, and particular to other members of the brass family.

## 1.1 Fundamentals of Trumpet-Playing Technique:

To sound a note, the trumpet player draws his or her lips taut and places them against the mouthpiece of the instrument. Air from the lungs is used to set the lips into vibration, as depicted in Figure 1.1. By thus *buzzing* the lips against the trumpet mouthpiece, a stable acoustic oscillation may result - though not necessarily at the same pitch as for the lips alone (Dale, 1965).

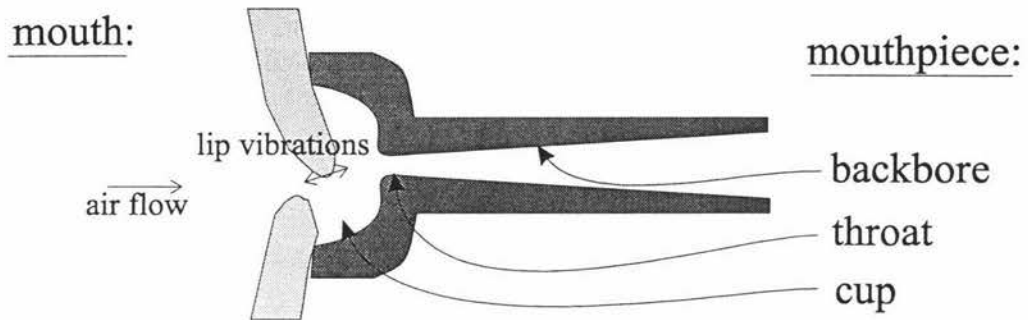


Figure 1.1: Lips Against The Trumpet Mouthpiece

### Valve Action:

The modern trumpet is fitted with three two-way *valves*. Depressing a valve adds a short detour to the path of any travelling sound wave within the instrument. There are a total of  $2^3 = 8$  *fingering positions* possible; the player thus has access to eight different horn geometries from the one instrument. The first, second and third valves lower the pitch by approximately two, one and three semitones respectively. This gives seven possible notes (one of which has alternative fingering) as shown in Figure 1.2.

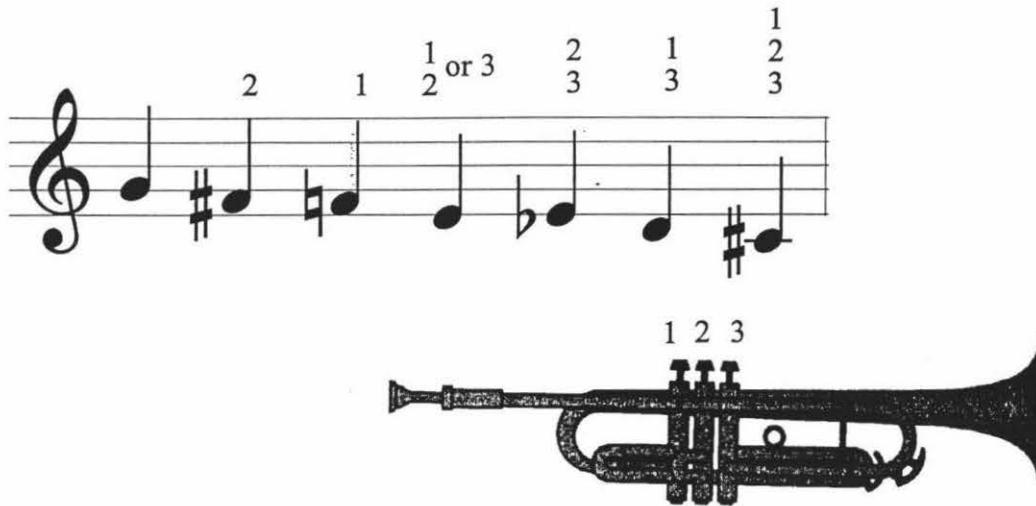


Figure 1.2: Lower Notes Are Produced As More Valves Are Depressed

#### Selection of Higher Harmonics:

Most trumpet tunes contain more than the seven pitches shown in Figure 1.2. The trumpeter is able to extend this range by varying the tautness of the lips. Several stable oscillations are possible for each fingering position, and the lip parameters determine which one of these is excited. Ideally, the geometry of the instrument is such that the fundamentals of these stable oscillations (for any one fingering position) are related to each other through membership of the same harmonic series (Figure 1.3).



Figure 1.3: Some Of The Notes That May Be Sounded With No Valves Depressed

The mouth position and lip parameters of the player define his or her *embouchure*. Using variations of both embouchure and fingering position, the practised trumpeter will often be capable of producing a full complement of notes within a range of around two-and-a-half octaves.

#### Lipping:

The player is also able vary the embouchure to effect small variations to the frequency of a note without swapping to another stable oscillation; the player is then said to be *lipping* the note up. This technique may be used by a skilled player to overcome inherent tuning

deficiencies of an instrument, and also for *bending* a note for musical effect (useful for playing jazz).

#### Dynamic Level:

The trumpet player varies the loudness or *dynamic level* of the music primarily through control of the diaphragm and upper abdominal muscles, thereby causing changes to the flow-rate of air from the lungs. Concurrent adjustment to the embouchure may be necessary to prevent the oscillation from jumping to another harmonic of the system.

#### Tone Control:

The origins of the steady-state *tone* of a trumpet sound are perhaps a little more unclear. A player will often prefer the tone of one instrument over another, and certainly the geometry of the instrument plays a significant role in the tonal quality of the sound it produces. The trumpet, cornet and flugel horn are three brass instruments which all have the same bore length. Their characteristic differences in tone quality are attributed to the different proportions of cylindrical and conical ducting along the length of the instruments.

Significant differences in tonal quality of the instrument can also be afforded by means of a *mute*. Several different shapes of mute are readily available. A mute is designed to sit in the flared end (the *bell*) of the trumpet, thereby affecting its acoustic reflection and transmission properties.

Aside from these more concrete factors, two players will often produce notes of different tonal quality upon playing the same instrument. This indicates that the musician, as well as the instrument, influence the tone. Furthermore, an individual trumpeter is able to deliberately vary the tone of a note while maintaining a constant dynamic level. These effects have been attributed to variations of parameters of the player's oral cavity and vocal tract (Stauffer, 1968). Differences of geometry (determined by tongue position, for example) and perhaps also of the resilience of the muscle walls of the oral cavity and vocal tract (determined by the state of muscular tension) could be responsible for changes in trumpet tone.

Articulation:

The manner in which the player controls the attack of a note is known as *articulation*. If the flow of air into the trumpet is obstructed briefly by the tongue before starting a new note, then that note is said to be *tongued*. Pressure builds behind the tongue and the lips are unable to oscillate. When the tongue is released, there is a sudden burst of air available to aid the initiation of the new oscillation.

The process of tonguing a note is analogous to the pronunciation of unvoiced plosives in speech production - especially /t/. A similar function can be formed further back in the mouth by forming the syllable /k/. The /k/ may be usefully alternated with /t/ in very fast passages; this technique is known as *double-tonguing*.

When a smooth or *slurred* transition between two notes is required, the second note is not tongued. The change of note is effected by the variation of embouchure and/or fingering position only, without interruption of the air supply to the instrument and without a subsequent burst of air to aid the initiation the new oscillation.

## 1.2 The Traditional Trumpet Model:

Any musical instrument which is capable of generating sustained notes of stable loudness is an example of a *self-excited oscillator*.

A self-excited oscillator is characterised by the presence of an energy supply which is either static (i.e. it has constant properties throughout time) or *quasi-static* (i.e. its properties do vary in time, but upon a time scale much longer than the period of the oscillation). It is the motion of the oscillator itself which regulates the rate of transfer of energy from the energy reservoir to the oscillation. The *forcing function* of the oscillator thus depends upon time only through the *state* of the system and its time derivatives.

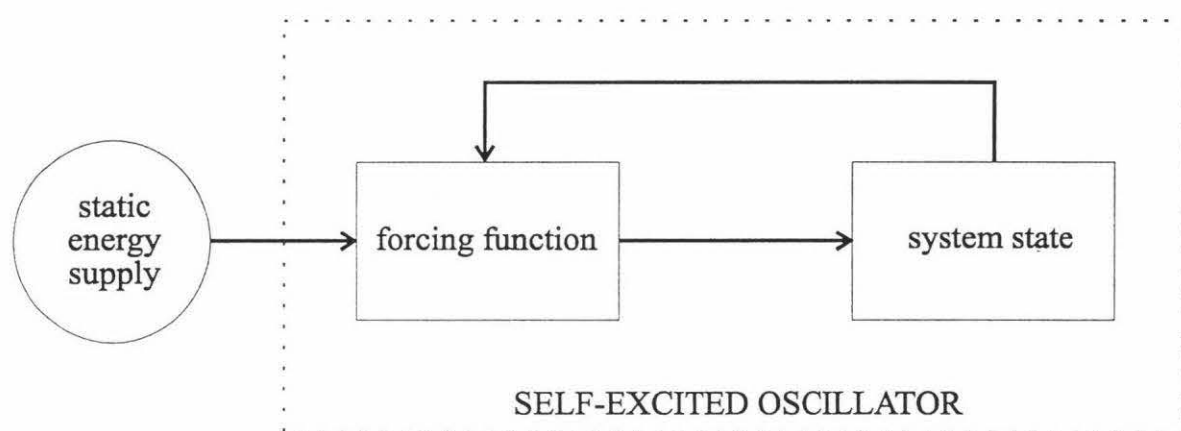


Figure 1.4: Feedback Structure of Self-Excited Oscillator

When, during each cycle of system oscillation, more energy is received from the reservoir than is lost from the oscillator through other means, the amplitude of the oscillation increases. When less energy is received from the reservoir than is lost from the oscillator through other means, the amplitude of the oscillation decreases. An oscillation of stable amplitude results when the energy received per cycle balances that lost from the oscillator.

Wind instruments and bowed string instruments are examples of self-excited musical oscillators. For most wind instruments, a quasi-static energy supply is provided by the force exerted by the musician's diaphragm muscles upon the air held by the lungs. (The pipe organ and the bagpipes are exceptions.) For bowed string instruments, the quasi-static energy supply is in the form of the musician's drawing of the bow steadily across the string(s).

Percussion instruments (e.g. drums, piano) and plucked string instruments are examples of musical oscillators which are not self-excited. These receive energy once, at the beginning of each a note, and the energy of the oscillation begins to decay from that moment.

Energy is lost from a musical oscillator in two important ways. Of most interest is the energy of the system oscillation which is transformed into sound energy. The energy is then propagated away. Secondly, a proportion of the kinetic energy of motion of any kind is always converted irreversibly into heat energy, so that it is no longer useful to the system. This loss of useful energy is known as *damping*. There is damping associated with all moving parts of an oscillator, and there is also damping associated with (both acoustic and non-acoustic) fluid motions.

#### Generator - Resonator Decomposition:

Musically useful self-excited oscillators, though overall strongly non-linear, can often be described in terms of conceptually separate linear and non-linear subsystems which react with each other within a feedback loop (McIntyre, Schumacher & Woodhouse, 1983). This decomposition (see Figure 1.5 below) proves very useful in the analysis of the complicated overall-nonlinear system.

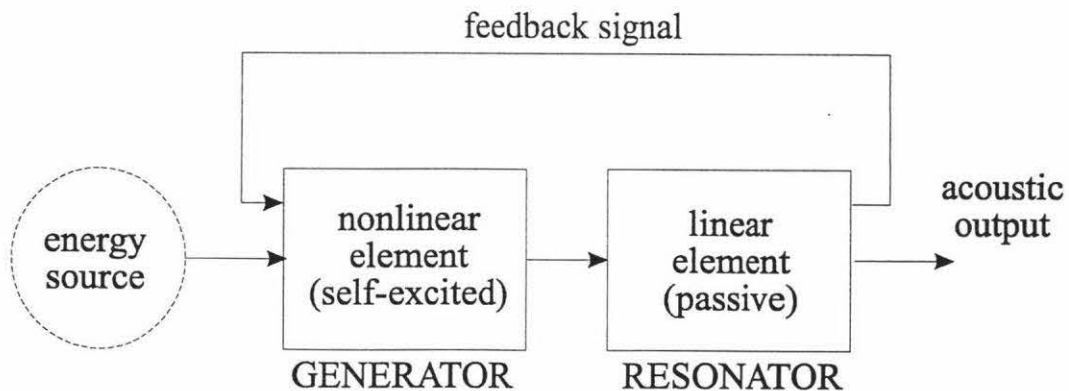


Figure 1.5: Basic Components of a Self-Excited Musical Oscillator

The generator subsystem receives energy from a constant energy source and releases it as an energy oscillation into the resonator. The resonator receives oscillatory energy presented by the generator, and combines this (linearly) with the energy it has already.

Energy oscillations of certain frequencies are favoured by the resonator. Whereas the generator acts as an energy *modulator*, the resonator acts as a linear energy *filter*.

Often feedback between the resonator and the generator provides a means for the generator to “lock in” to a frequency favoured by the resonator. This gives stability to the frequency of the resulting oscillation.

### Wind Instruments

For wind instruments, the source of energy is most often provided by the diaphragm muscles of the musician, exerting a pressure force upon the air contained by the lungs.

The resonator subsystem of a wind instrument is generally considered to be the column of air inside the instrument, in which acoustic standing waves build up. The resonator is provided with acoustic energy by the generator subsystem, and it loses acoustic energy in the form of sound radiated from its open end. In addition to the energy lost in the form of acoustic radiation, there is additional energy loss through the irreversible conversion of kinetic energy to heat energy. This is an inevitable result of motion of any kind; energy is lost from the acoustic motion within the air column via internal fluid friction called *viscosity*. Any wall vibrations of the resonator, in response to the enclosed acoustic oscillations, are responsible for additional radiation and dissipation losses.

The generator subsystem of wind instruments is responsible for the conversion of a steady pressure force, provided by diaphragm of the musician, into acoustic flow energy (which is oscillatory in nature), for provision to the resonator subsystem. Thus it can be said that the generator subsystem, for wind instruments, has a role of *flow modulation*. For all wind instruments, very little is known about the flow modulation performed by the generation subsystem as compared with the amount of knowledge and understanding of the (linear) resonator subsystem.

The vast majority of wind instruments modulate the airflow by means of a vibrating body in a narrow passage somewhere within the system. (Exceptions are the flute and recorder families, whistles and flue organ pipes - these utilise aerodynamic instability to make the conversion from steady to oscillatory flow.) Flow modulation is caused by a vibrating body near a flow constriction in all reed instruments, the brass family, and in the voiced sounds of human speech.

The traditional model of sound-production in the trumpet is based upon a proposition by Webster (1919a) that the trumpeter's lips act as a *pressure-controlled valve*: “A spring of variable tension holds the valve in place and the proper pressure can cause a puff of air, which generates a sound in the horn which on reflection arrives at the valve in the proper phase to maintain vibration.”

### Resonator Equations:

The resonator subsystem of the trumpet is depicted below in Figure 1.6.

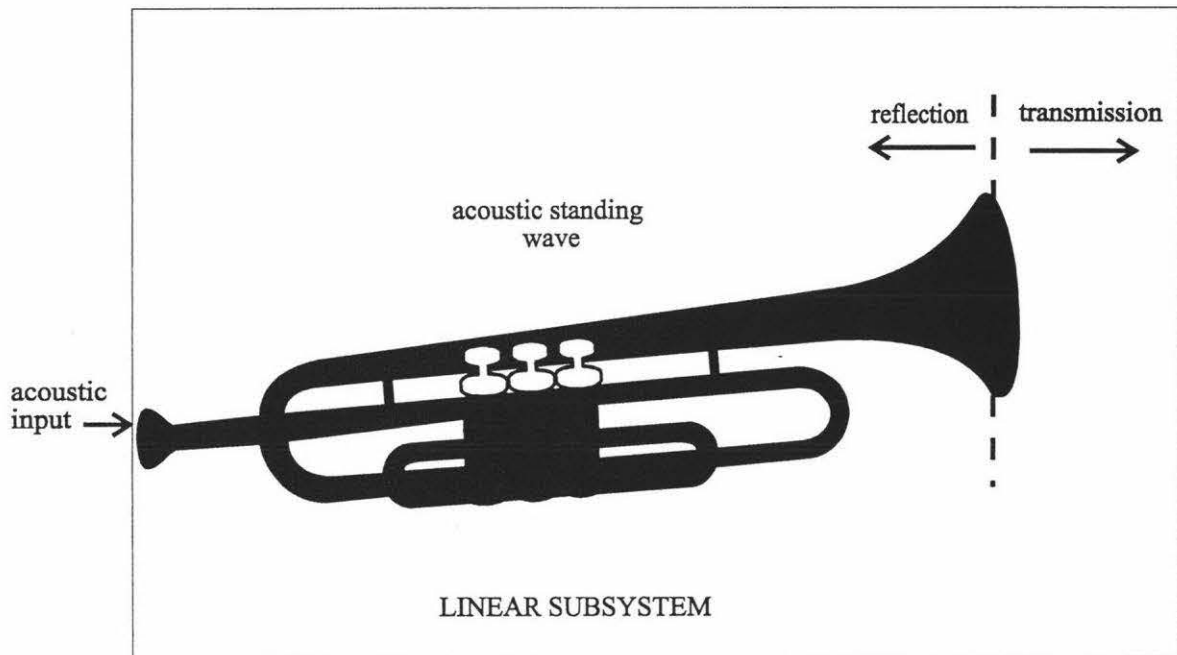


Figure 1.6 Resonator Subsystem for the Trumpeter-Trumpet System

The resonator behaviour is assumed linear and time-invariant, and therefore can be conveniently described in terms of its *acoustical impedance* (Webster, 1919b). The acoustical impedance relates a *pressure difference*  $P(\omega)$  to an *acoustic volume velocity*  $Q(\omega)$  at each frequency  $\omega$ :

$$Z(\omega) = P(\omega) / Q(\omega) .$$

Experimental data are available for the *acoustical input impedance* of trumpets (e.g. Backus, 1976; Elliott, Bowsher & Watkinson, 1982).

Many theoretical results are available for the mathematical description of linear acoustics within the trumpet and also at its open end (Levine & Schwinger, 1948; Young, 1966; Benade, 1968; Benade & Jansson, 1974).

The treatment of sound propagation in a duct, as an analogue to electrical current in a transmission line, is well-known (Morse & Ingard, 1968, §9.1). The basic theory has been refined considerably for application to woodwind instruments especially (Keefe, 1990b).

The idea of discretising the whole duct (transmission line) into a number of short lengths, each having lumped parameters, has found favour among modellers. The method of approximating a duct shape by a concatenation of several short cylindrical tubes has been popular for many years with modellers of the processes of human speech (Flanagan, 1972; Flanagan, Ishikawa & Shipley, 1975; Rabiner & Schafer, 1978; Bonder, 1983). A related method, which gives a better approximation for a given elemental length of duct, but which has not been extensively exploited because of the added numerical complexities, involves the use of truncated cones as duct elements instead (Plitnik & Strong, 1979; Caussé, Kergomard & Lurton, 1984).

When it is desirable to describe the resonator in the time domain the inverse Fourier transform of the acoustical input impedance  $Z(\omega)$  will give the *impulse response*  $z(t)$ . The system response to an arbitrary input signal is then calculated by convolution of the impulse response with that input signal. A more computationally convenient method is to use the *reflection function*, usually denoted  $r(t)$ . This may be thought of as the disturbance found at the mouthpiece after an impulse is sent at  $t=0$  and the tube then terminated there by a perfect absorber. Using the reflection function, the pressure in the mouthpiece is conceptualised as the sum of incoming and outgoing waves (Equation 1.1 below), related through convolution (Equation 1.2 below) (Schumacher, 1978, 1981; McIntyre, Schumacher & Woodhouse, 1983).

$$p_2(t) = p_{in}(t) + p_{out}(t) \quad (1.1)$$

$$p_{in}(t) = r(t) * p_{out}(t) \quad (1.2)$$

The resonator description is then completed by taking the volume velocity between the lips as the input acoustic volume velocity for the trumpet.

$$q = \frac{1}{Z_0} [p_{out}(t) - p_{in}(t)] \quad (1.3)$$

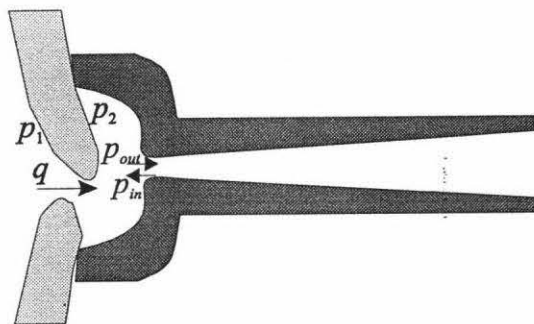


Figure 1.7: Pressures In The Mouthpiece

In Equation (1.3),  $Z_0$  is the *characteristic acoustic impedance* at the entryway to the trumpet, given by  $Z_0 = \rho c / A_{in}$ , where  $A_{in}$  is the cross-sectional area at the entry, and  $\rho$  and  $c$  are the fluid density and sonic speed, respectively, for the fluid at rest conditions.

Alternatively, a *digital waveguide* description of the resonator is also possible (Smith, 1986, 1991b, 1992).

Little further discussion of the resonator description will be made in this thesis. The linearity of the resonator makes its mathematical description straightforward. The choice of the method of description is largely a matter of convenience.

#### Generator Equations:

The following description is of the generator subsystem of a basic trumpet model. Modifications to this basic model structure have been proposed by various authors from time to time, but are not considered in detail here. The model described here has been implemented before for the purpose of physical modelling synthesis of brass instrument sounds (Cook, 1991).

It is common in the literature to consider only the upper lip of the player as a vibrating body in the formulation of the model, the lower lip being treated as a fixed boundary. Experimental evidence seems to support such an approach: trumpet tones can be produced without lower lip vibrations by using a partially filled-in mouthpiece (Henderson, 1942), and some brass players have been observed to play without movement of their lower lip even while using a standard mouthpiece (Weast, 1963).

The lip motion is represented by the displacement  $x$  of a single *mass-spring-damper oscillator*. The mass  $m$  of the vibrating body is assumed constant, the restoring force  $-sx$

is assumed linear and the damping  $-b\dot{x}$  is assumed *viscous* (proportional to the speed of motion). The forcing function for the oscillator is taken as the product of the pressure difference  $p_1 - p_2$  across the lip and its area  $A^T$  normal to the flow (the superscript 'T' being used to denote the *transverse* area), as shown in Figure 1.8 below.

$$m\ddot{x} + b\dot{x} + sx = (p_1 - p_2)A^T \quad (1.4)$$

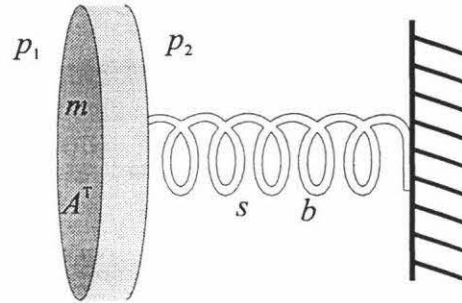


Figure 1.8: The Lip As A Mass-Spring-Damper Oscillator

The pressure in the mouth  $p_1$  is assumed constant and the pressure in the trumpet mouthpiece  $p_2$  is related to the mouth pressure by an equation referred to by some modellers as a *Bernoulli Equation*:

$$p_1 - p_2 = \frac{\rho}{2}u^2 \quad (1.5)$$

Here  $u$  is the speed of the flow between the player's lips, and  $\rho$  is its density.

The area of the orifice formed by the lips is taken to be proportional to the displacement  $x$  of the oscillating body, in accordance with the results of Martin's stroboscopic observations of a trumpeter's lips (Martin, 1942; figs. 4 - 6). Martin's graphs collectively suggest that the horizontal lip displacement and the aperture area are roughly proportional. Thus the *volume velocity* of the flow,  $q$  (cubic metres per second), may be defined by

$$q = k ux \quad (1.6)$$

Equations (1.1) to (1.6) give a system of six equations in six unknowns (assuming that  $r(t)$  is known, as well as the values of the various system parameters). These are repeated here:

$$p_2(t) = p_{in}(t) + p_{out}(t)$$

$$p_{in}(t) = r(t) * p_{out}(t)$$

$$q = \frac{1}{Z_0} [p_{out}(t) - p_{in}(t)]$$

$$m\ddot{x} + b\dot{x} + sx = (p_1 - p_2)A^T$$

$$p_1 - p_2 = \frac{\rho}{2} u^2$$

$$q = k ux$$

### **1.3 Discussion of Some Concerns Regarding the Traditional Model:**

The traditional trumpet model presented in the previous section fails to provide a convincing description of how sound is produced in the trumpet. Although the acoustical theory of the (linear) resonator subsystem is well understood, there is cause for concern in the mathematical description of the generator subsystem.

#### **1.3.1 Concern over the Accepted Bernoulli Equation:**

Equation (1.5), referred to by trumpet modellers as “Bernoulli’s Equation”, is repeated below:

$$p_1 - p_2 = \frac{\rho}{2} u^2$$

It is common in the literature to replace the particle velocity  $u$  using the relation  $q = uA$  where  $q$  is the *volume velocity*, and  $A$  is the duct cross-sectional area at that position. (This replacement assumes that the flow does not separate from the walls of the duct between positions ‘1’ and ‘2’.) Equation (1.5) becomes:

$$p_1 - p_2 = \frac{\rho q^2}{2A^2} \quad (1.7)$$

Actually, this equation is only an approximation to a very special form of Bernoulli’s equation. Equation (1.8) below is that special Bernoulli equation; it is valid for steady, *quasi-1D flow* (flow is quasi-1D when velocity  $u$  and thermodynamic properties of the fluid are constant over any flow cross-section) of a fluid which has zero fluid *viscosity* (i.e. zero internal fluid friction), and which moves with constant density  $\rho$ , as it moves between two places (designated ‘1’ and ‘2’ in Figure 1.9).

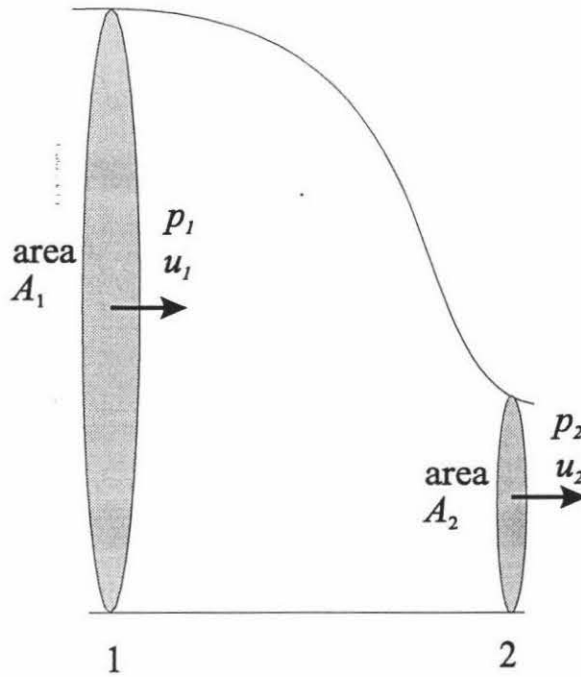


Figure 1.9: Bernoulli Equation Variables at a Constriction

Furthermore, Equation 1.8 requires that there is no momentum transfer between the fluid and its surroundings (including gravitational effects) as it moves.

$$p_1 - p_2 = \frac{1}{2} \rho \left[ \left( \frac{q_2}{A_2} \right)^2 - \left( \frac{q_1}{A_1} \right)^2 \right] \quad (1.8)$$

There is an unwholesome number of unjustified assumptions in the use of even Equation (1.7) to describe the flow between the lips of a trumpet. Comparison between Figures 1.7 and 1.9 shows that  $q$  and  $A$  of Equation (1.7) corresponds to  $q_2$  and  $A_2$  of Equation (1.8). Consequently a further assumption required for this Equation (1.7) to be valid, if Equation (1.8) is already known to be valid, is

$$\left( \frac{q_1}{A_1} \right)^2 \ll \left( \frac{q_2}{A_2} \right)^2$$

It can be concluded that Equation (1.7) will adequately describe the flow between the lips of the trumpeter if:

- the flow is steady (or varies so slowly in time that the steady form of the Bernoulli equation gives an acceptable approximation to the motion),
- the velocity  $u$  and thermodynamic properties of the fluid are constant over any flow cross-section (or such variations as so small as to have no effect upon the motion),
- the fluid viscosity is zero (or is so small that it has negligible effect upon the transport of fluid momentum between the lips),
- the fluid density is constant (or varies so little that these variations have no noticeable effect upon the overall motion of the fluid),
- the fluid is unaffected by gravity (or the effect of gravity upon the fluid is negligible with respect to the overall motion),
- there no momentum transfer to or from the fluid at its boundaries by external forces acting upon the fluid as it moves,
- the flow does not separate from the lips as it flows,
- $\left(\frac{q_1}{A_1}\right)^2 \ll \left(\frac{q_2}{A_2}\right)^2$

### 1.3.2 Further Concern Over the Flow Description Near the Lips:

Somewhere in the vicinity of the trumpeter's vibrating lips, there is some region of space where new acoustic energy is being generated, to be radiated outward as soundwaves. Such a region is said to contain an *acoustic source*. The actions of the source are responsible for the acoustic pressure fluctuations experienced at places far away from that source. But in the *near field* of an acoustic source, there are also other pressure fluctuations present which do not propagate as sound. The traditional model incorrectly assumes that all time-varying pressures at the lips can be regarded as acoustic.

Confusion between *unsteady convective fluid motion* and *acoustic fluid motion* sometimes arises when such fluctuations occur at audio frequencies, on account of both types of motion being accompanied by pressure fluctuations. The term *pseudosound* has been coined by workers in the field of *aero-acoustics* in an attempt to clarify the distinction between fluctuating pressures associated with unsteady convective flow and the sound

proper (Ffowcs Williams, 1969). Pseudosound refers to the pressure fluctuations which exist in the fluid as a result of local fluid accelerations; pseudosound does not propagate at all.

Texts upon the subject of acoustics often refer to non-propagating pressures as being *reactive* (Morse & Ingard, 1968). There are always reactive pressures in the near field of an acoustic source. But, because the reactive pressures do not propagate, only the *radiated* (sound) pressures are significant at places far from the source (i.e. in the *far field*).

In the case of the vibration of the trumpeter's lips, the reader will agree that there is some sort of acoustic source in the vicinity of the lips. Because the propagation medium (the player's breath) is moving relative to this acoustic source, the reactive pressures near the source will be convected away. Thus the convected flow into the trumpet will be characterised by spatial pressure gradients. The pressure fluctuations experienced downstream of the lips will then be a combination of convected pressure gradients and sound propagation.

#### Implications for Experimental Measurements:

The presence of pseudosound fluctuations has implications for the interpretation of pressure measurements made in the vicinity of the lips, since a microphone will respond to all pressure fluctuations, not discriminating between sound and pseudosound (the ear does too - Ffowcs Williams & Lighthill, 1971).

Consider a flow-control valve at  $x = 0$  that opens and closes at a frequency of 680 Hz (which lies within the range of the trumpet). The wavelength of the sound follows from

$$\lambda_s = \frac{c}{f} \approx \frac{340 \text{ ms}^{-1}}{680 \text{ s}^{-1}} = 0.5 \text{ m}$$

A microphone placed at a distance of, say, 10 mm from the flow control will measure a sound pressure which lags the pressure at the control mechanism by one-fiftieth of the period of the oscillation. Such a phase difference is barely significant.

Now consider a pseudosound fluctuation, coincident with the sound fluctuation at the flow-control, but which is convected with a mean velocity of  $10 \text{ ms}^{-1}$ . (Comparable flow speeds have been measured in the throat of a trumpet mouthpiece at approximately this

frequency - Elliott & Bowsher, 1982.) During one period of the flow-control mechanism, the pseudosound is convected by approximately the distance

$$\lambda_{ps} = \frac{\bar{u}}{f} = \frac{10 \text{ ms}^{-1}}{680 \text{ s}^{-1}} \approx 15 \text{ mm}$$

The microphone, 10 mm from the flow-control mechanism, will measure a pseudosound pressure which lags that at the control mechanism by two-thirds of an oscillation period.

The total pressure fluctuation as measured by the microphone cannot be assumed to give a faithful indication of fluctuations that are occurring at the control mechanism since the pressure signal detected by the microphone has an acoustic component and a pseudosound component that correspond to different instants of time in the history of the flow controller. Redhead (1993) has proposed an experiment in which an indication of the relative importance of convected and propagated pressure fluctuations, at a particular location, could be obtained through correlating the microphone output signal with that of another microphone placed a small distance downstream. The results of such an experiment appear in Chapter Five of this thesis.

#### Consequences for the Model:

It is clear that there is no consideration of pseudosound in the traditional trumpet model. The pressure in the player's mouth is assumed static while the pressure within the trumpet mouthpiece is assumed to be acoustic only. The traditional model assumes that all unsteady pressure is sound by equating the whole of the volume flow of air between the trumpeter's lips to the acoustic volume flow into the trumpet. Determination of the relative importance of the pseudosound and acoustical pressures in the vicinity of the lips is integral to the determination of the acoustic energy being produced there, and also to the determination of the fluid forces upon the lips.

Separation of the acoustic contribution to the overall flow field is a problem within the domain of *aero-acoustics*. When the *Mach number* of the flow remains small, i.e.  $u \ll c$ , this separation is simplified by knowledge that the pseudosound is quite uninfluenced by fluid compressibility (Ffowcs Williams, 1969). For this reason, the non-acoustic contribution to an overall flow field is sometimes referred to as the *hydrodynamic* component of the motion.

Little work has surfaced describing fluid-dynamical aspects of flow modulation within wind instruments (but see St. Hilaire, Wilson & Beavers, 1971). Fortunately, this is beginning to change (Hirschberg et al., 1990a, 1990b). Modellers of human speech production seem to have taken the fluid dynamics of the system more seriously (van den Berg, Zantemna & Doomenbal, 1957; Gupta, Wilson & Beavers, 1973; Kaiser, 1983; Teager & Teager, 1983; Pelorson et al., 1994).

### 1.3.3 What Excitation Mechanism for the Flow-Induced Vibration?

Flow modulation is caused by a vibrating body near a flow constriction in all reed instruments, the brass family, and in the voiced sounds of human speech. The action of flow modulation, by the vibrating body upon the passing flow, is coupled to the forcing action of the passing flow upon the oscillating body. The phenomena of *flow modulation* and *flow-induced vibration* are coupled within a feedback loop, as shown in Figure 1.10 below.

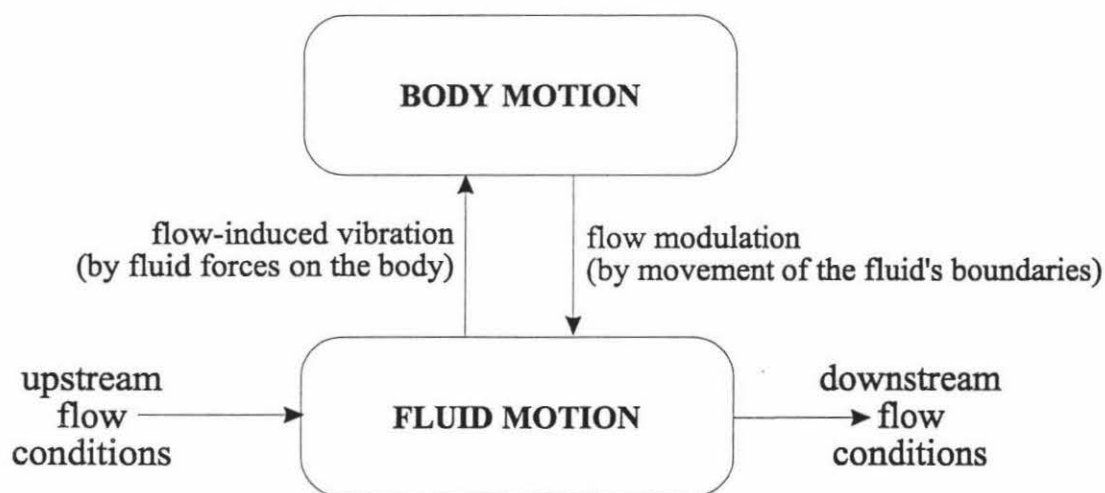


Figure 1.10: Coupling Of Flow Modulation And Flow-Induced Vibration

Many wind instruments (e.g. clarinet, saxophone, oboe, bassoon and some organ pipes) employ a *reed*, which vibrates in a narrow part of the instrument bore as the musician plays. Brass players use their own lips as a reed.

A number of different fluid-dynamic mechanisms may be responsible for the behaviour of a flow-induced vibration, and for many situations it is a complex interaction of several of these which determines the resultant motion (Wambsganss, 1976, 1977). Some oscillators are known to be excited by completely different mechanisms, depending upon

the incident flow velocity (Parkinson & Smith, 1962; Thang & Naudascher, 1986a, 1986b). In this section some fundamental aspects of flow-induced vibration are introduced.

#### Movement-Induced Excitation:

Consider a fluid of infinite extent moving in the  $x$ -direction past a fixed, rigid body. If the body has axial symmetry about the  $x$ -axis, then the fluid force upon the body is in the direction of fluid motion, and can be written (Massey, 1989: §8.8.3)

$$F = C_D \frac{\rho}{2} U^2 A^T$$

Here  $C_D$  is a *drag coefficient*,  $A^T$  is a *characteristic body area* transverse to the flow (the definition of which depends upon the body geometry),  $\rho$  is the fluid density and  $U$  is the *free-stream velocity*; the free-stream velocity is the velocity of the fluid at a distance so far from the body that it is unaffected by the body's presence.

Now consider that the body itself is also moving, but with velocity  $\dot{x}$ , in the same direction as the flow. The resulting drag force will be given by (Morse & Ingard, 1968: Equation 11.3.35)

$$F = C_D \frac{\rho}{2} (U - \dot{x})^2 A^T$$

Notice that the fluid force is now dependent upon the velocity of the body. (The reader may know that, in some cases, a velocity-dependent force upon a simple harmonic oscillator can lead to self-excited oscillations.) Any resulting flow-induced oscillation of the immersed body is said to be due to *movement-induced* excitation (Naudascher & Rockwell, 1980).

Movement-induced vibrations do not require the force upon the body to be axially symmetric as in the above example. More generally, a body moving relative to a fluid will be subject to fluid forces in three orthogonal directions, and moments about their axes. The resulting flow-induced oscillation might have up to six degrees of freedom. The term *galloping* (Blevins, 1977: ch. 4) is usually used to signify an oscillation with only one degree of freedom, while oscillations that rely upon body motions of two or more degrees of freedom for their existence are termed *flutter* (Parkinson, 1971).

### Instability-Induced Excitation:

For a real fluid, the fluid velocity on the surface of a stationary immersed body will be zero. This is known as the *no-slip condition*, and it applies to any body in contact with a *viscous* fluid; all real fluids are viscous. The no-slip condition also applies to bodies in motion; it is the *relative velocity* between the body and the fluid which is then zero upon the body surface. A short distance from the surface of the body, the flow velocity is almost as great as that in the free stream. The region in between, where the fluid velocity changes appreciably (from zero at the body surface to  $U$  a short distance out), is known as a *boundary layer* (Batchelor, 1967; Schlichting, 1968).

If the fluid velocity is sufficiently large (the requisite flow speed depends upon the fluid viscosity and the geometry of the body) then the boundary layer will *separate* from the body surface, and a *wake* will be formed downstream of the body. Downstream of the body, there will continue to be a layer of fluid across which the fluid velocity changes appreciably, and this is known as a *shear layer*. Shear layers are unstable, and when appropriately perturbed may *roll up* into discrete *vortices* which are then convected away with the flow.

If a body in the fluid *sheds vortices* periodically, then the fluid force upon that body also varies periodically, since the fluid force upon the body varies throughout the vortex-shedding process. If the body is not fixed, then the time-varying force upon the body can excite oscillations at the frequency of the vortex-shedding. The resulting flow-induced vibration is said to be a result of *instability-induced excitation* (Naudascher & Rockwell, 1980).

In contrast with the movement-induced excitation described already, the fluid force upon the body does not rely upon the body's motion for its time variation. The fluid force is time-varying even when the body is fixed. Furthermore, when the body is cantilevered, there are two *resonant frequencies* to be considered - the frequency associated with the vortex-shedding from the body when held fixed and the structural resonance frequency of the body when allowed to vibrate in a vacuum. The composite flow-induced vibration is not simple when these two frequencies are close, since the vortex-shedding can *lock on* to the natural vibration frequency of the body: When a body oscillates in response to vortex shedding, the induced motion of the body periodically displaces the point of flow separation from the body, at the frequency of the body vibration. These perturbations consolidate the vortex shedding process, and the vibration amplitude can increase until

such time as the body vibrations and the oscillating flow together extract the maximum amount of energy that can be provided by the incident flow.

A flow-induced vibration caused by instability-induced excitation is sometimes referred to as a *vortex-induced vibration* (Blevins, 1977: ch. 3). However, the shedding of (discrete) vortices is not an essential feature of oscillations caused by instability-induced excitation. The essential feature is the flow instability giving rise to fluid oscillations, and thus an oscillatory force upon the body.

#### Fluid Oscillator Effects:

Two types of flow-induced vibration excitations have been described for a body in a moving fluid of infinite extent. There are important additions that must be considered when the motion of the oscillating body occurs within a confined region of the flow. In such cases the motion of the body can have a drastic effect upon motion of the fluid past it, and thereby, the upstream and the downstream fluid behaviour. Since it is the fluid motion which determines the subsequent forcing function for the body vibrations, the result is a coupling between the *body oscillator* and a *fluid oscillator* (Naudascher & Rockwell, 1980).

A fluid oscillator may appear in conjunction with either of the two excitation mechanisms described above. A fluid oscillator is passive; it does not add energy to the system. It contributes to the behaviour of a self-excited oscillation by modifying (the amplitude and/or phase of) the fluid forcing function upon the body oscillating in the flow.

(The vibration of the trumpeter's lips certainly occurs within a confined region of fluid. Martin (1942) observed that the lips completely obstruct the flow once during each cycle of oscillation.)

As a result of fluid oscillator coupling, the fluid forces upon a structure in a flow are altered due to the fact that an otherwise steady incident flow then has an unsteady component (Blevins, 1977: ch.6). These new effects may sometimes be even more significant than original excitation forces upon the body, even though they rely upon the latter for their existence. It is possible for these fluid oscillator forces to become the major determinants of the flow-induced vibration (Kolkman & Vrijer, 1987).

### Acoustic Resonator Effects:

If a flow-induced vibration occurs in some region where there is a resonant sound field of such magnitude that the acoustic fluid motions contribute significantly to the overall fluid velocity field, then the fluid force upon the body will obviously include a significant contribution from the acoustic fluid motion. But if the flow-induced vibration is caused by instability-induced excitation, there are also other means by which acoustic fluid motion can influence the behaviour of the oscillation:

The instability at a point of flow separation, such as at the *trailing edge* of an immersed body, is known to be receptive to acoustic perturbations (Morkovin & Paranjape, 1971; Ho & Huerre, 1984). At such places, the acoustic and hydrodynamic contributions to the overall fluid motion combine to collectively satisfy a boundary condition known as the *Kutta condition* (Crighton, 1981, 1985). A hydrodynamic flow field which responds to incident sound, in order that the combined acoustic plus hydrodynamic motion might satisfy a Kutta condition, is said to be *receptive* to acoustic perturbations. As the fluid is convected away the shear layer instability leads to amplification of such responses of the hydrodynamic flow field to the incident sound. In this manner the hydrodynamic flow field, and hence the forcing function for the oscillating body, is controlled by the sound field present.

Lock-in of vortex-shedding with the vibrations of a body in a flow has been mentioned already. The shedding of vortices from an immersed body can also lock onto the frequency of a resonant sound field (Graham & Maull, 1971). Vortex-induced vibration of a body which sheds vortices at a frequency dictated by a resonant sound field is thus constrained to motion at that same frequency.

### **1.3.4 On the Mechanism of Sound Production:**

The science of acoustics often classifies acoustic sources as being monopole, dipole or quadrupole in nature (Morse & Ingard, 1968; Lighthill, 1952).

A *monopole* acoustic source is idealised as a point in space where new fluid is introduced, and retracted, in an oscillatory fashion. This source term has no spatial dependence; the generating motion has no preferred direction, but produces a wave which radiates spherically outwards from the centre of the source. The density fluctuation at  $\mathbf{x}$ , a large distance  $r$  from the centre of the source region, will be given by (Lighthill, 1952: Equation 9).

$$\rho(\mathbf{x}, t) - \rho_0 \approx \frac{1}{4\pi r c^2} \int_{\text{source region}} \dot{Q}\left(\mathbf{y}, t - \frac{r}{c}\right) d\mathbf{y}$$

The function  $\dot{Q}$  gives the time-rate of change of the rate of introduction of mass, per unit volume, within the source region. The density at the distance  $r$  is proportional to the rate of change of the flow introduction at a time  $r/c$  earlier. The time  $t - r/c$  is known as the *retarded time*.

A dipole source is characterised by an oscillatory force upon the fluid medium. A dipole might be thought of as two monopole sources side by side, but opposite in sign, so that one expands as the other contracts. The resulting pressure field is directional. When the source region is much smaller than the wavelength of the sound (Lighthill, 1952: Equation 12),

$$\rho(\mathbf{x}, t) - \rho_0 \approx \frac{1}{4\pi r c^3} \frac{x_i}{r} \int_{\text{source region}} \dot{F}_i\left(\mathbf{y}, t - \frac{r}{c}\right) d\mathbf{y}$$

The far field fluctuations are greatest in the direction of the force, since  $x_i = r \cos\theta$ , where  $\theta$  is the angle measured from the force direction.

Quadrupole sources accompany fluctuations in certain types of stresses within a fluid. The specially-formulated *Lighthill stress tensor* is usually denoted  $T_{ij}$ . A quadrupole source might be thought of as comprising four monopole sources. For a *lateral quadrupole*, the four are arranged in a tiny square, and diagonally opposite monopoles have the same sign. The monopoles which make up a *longitudinal quadrupole* lie all along the same line. For both types, the pressure field is directional. When the wavelength of the sound is much larger than the source region, the resulting far-field density fluctuations are given by (Lighthill, 1952: Equation 17)

$$\rho(\mathbf{x}, t) - \rho_0 \approx \frac{1}{4\pi r c^4} \frac{x_i}{r} \frac{x_j}{r} \int_{\text{source region}} \ddot{T}_{ij}\left(\mathbf{y}, t - \frac{r}{c}\right) d\mathbf{y}$$

The previous three equations are approximate solutions for the pressures far away from the source being considered. The *far field* sound pressures radiating from a dipole source are of lower order than those from a monopole source, and those from quadrupole sources are lower again; notice the increasing power of  $c$  in the denominators of the pressure equations (also of  $r$ ).

### Possible Sources at a Trumpeter's Lips:

Some possibilities for the sound-production mechanism at the vibrating lips of the trumpeter will now be examined. Recall that monopole sources result from a changing rate of introduction of new fluid, dipole sources can arise through fluctuating forces applied to the fluid, and quadrupole sources will be due to stresses appearing as a result of the fluid motion. Also, if there are any monopole sources present, these will be the most important, then dipoles, then quadrupoles.

A monopole source would be signalled by any changing rate of introduction of new fluid at any place within the flow. Because the aperture between the lips of the trumpet player is always changing, the rate of introduction of air into the trumpet is time-varying, and so this could be interpreted as a monopole source.

The introduction of air into the trumpet is accompanied by the release of air from the player's mouth, so that there is really a *pair* of monopoles of opposite sign. One monopole represents the changing rate of introduction of fluid into the trumpet mouthpiece; the other represents the changing output rate of fluid from the player's mouth (see Figure 1.11). The combined result of two adjacent, oppositely-signed monopoles is a dipole source.

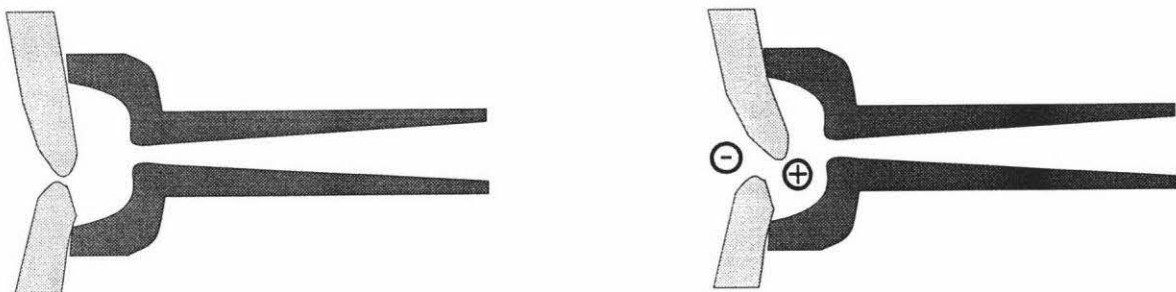


Figure 1.11: Acoustic Sources Due To Changing Flow-Rate Between The Lips

A second dipole source is evident through the motion of the trumpeter's lips. This motion indicates a fluctuating force upon the lips, and this force is necessarily provided by the flow. Since for every action there is an equal and opposite reaction (Newton's First Law), the motion of the lips also indicates the presence of a time-varying force upon the fluid, and this provides a dipole sound source. Radiation is most favoured in the direction of

motion of the lips. In Figure 1.12, the directionality of the motion, and hence the dipole sound field, is indicated by the arrow.

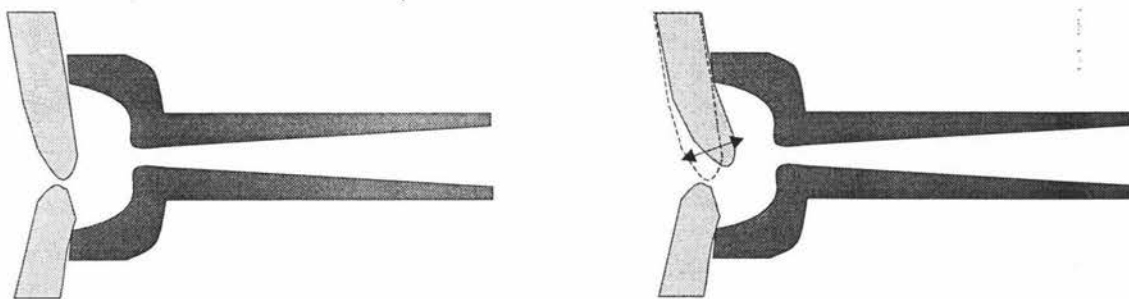


Figure 1.12: Acoustic Source Due To Lip Vibrations

A third dipole source is possible at the back of the mouthpiece cup. Because there is a time-varying flow emerging from between the player's lips (Figure 1.11), there must be a time-varying aerodynamic force upon the inside of the mouthpiece. This time-varying force could be responsible for additional sound generation (Figure 1.13).

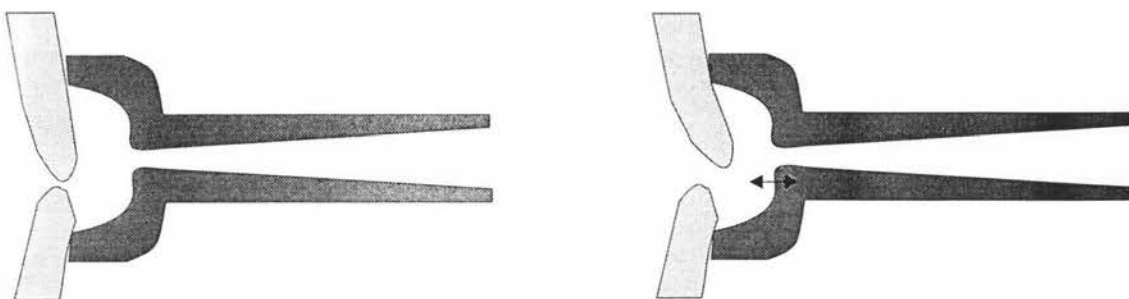


Figure 1.13: Acoustic Source Due To Unsteady Fluid Force Upon Mouthpiece

Any unsteady introduction of fluid between the lips at a time  $t$  will produce acoustic energy at the lips at that same time  $t$ , but will produce sound at the back of the mouthpiece cup at some later time  $t+T$ , where  $T$  is the time taken for the flow unsteadiness to be convected downstream from the lips to the back of the mouthpiece. The mouthpiece source lags the flow-modulation source by a time interval which depends upon the depth of the mouthpiece cup and the speed at which unsteadiness is convected by the flow. Such an acoustic source is thought to be important for the production of *hole tones* (Powell, 1953; Chanaud & Powell, 1965; Wilson et. al., 1971; Ho & Nossier, 1981; Rockwell, 1983).

Three dipole sources have been identified in the vicinity of the trumpeter's lips: the unsteady influx of air from the player's mouth, the time-varying force due to the lip motion directly, and the time-varying fluid force upon the back of the mouthpiece cup.

Since dipole sound sources are of higher order than any quadrupole sources that may be present in the flow (Lighthill, 1952), the contributions of the latter to the radiated sound field are expected to be relatively insignificant.

### 1.3.5 Concern Over the Description of Lip Dynamics:

The function of the generator part of the complete trumpet system is to accept a steady energy influx from the player (air flow from the lungs) and to modulate this in some way to provide acoustic energy for the resonator. Many voice and wind instrument modellers have suggested that the generator subsystem should function as a *self-excited oscillator* in its own right (i.e. without relying upon feedback from the resonator subsystem). However, the generator part of the traditional trumpet model cannot operate alone as a self-excited oscillator.

Self-excited oscillation is a special type of forced oscillation. A self-excited oscillator has the ability to regulate the rate of transfer of energy from a steady (or quasi-steady) energy reserve in such a manner that oscillation is maintained. The forcing function of a self-excited oscillator is not a function of time directly, but instead depends upon the state variables of the oscillator (position, speed and acceleration).

Recall that in the traditional trumpet model the lip vibrations are represented by the motion of a simple mass-spring-damper oscillator under the influence of an applied (pressure) force. This might be written as

$$m\ddot{x} + b\dot{x} + sx = F(t) \quad (1.9)$$

The traditional trumpet model uses the following forcing function for  $F(t)$ :

$$F(t) = (p_1 - p_2)A^T \quad (1.10)$$

Consider what happens to Equation (1.10) when the resonator is decoupled from the generator of the traditional trumpet model. With the trumpet absent, the pressure  $p_2$  becomes the ambient pressure outside the mouth (atmospheric pressure). The force upon the lips then reduces to  $F(t) = \text{constant}$  in the traditional trumpet model. Because the

resulting forcing function does not depend upon time (directly or indirectly), there is no scope for self-excited oscillations.

This is contrary to the experience of trumpet players, suggesting one or more of the following:

- (a)  $p_1$  or  $A^T$  are actually system variables (i.e. they are not constant);
- (b) Equation (1.10) does not accurately describe the fluid force on the lips;
- (c) the assumption of a simple harmonic oscillator is too simplistic, and maybe a modification (such as a nonlinear restoring force) is required.

The third possibility is now considered.

#### Suitability of a One-Mass Model:

The history of voiced-speech generator models shows that self-excited oscillations, though impossible for a single mass-spring-damper description of the vocal folds (Flanagan & Landgraf, 1968; Flanagan & Cherry, 1969), could be produced with a two-mass model (Ishikaza & Flanagan, 1972). Still more complexity has been added by later researchers - by some to give a better representation of physiological reality (Titze, 1973, 1974; Story & Titze 1995) and by others with the aim to produce a more natural-sounding speech synthesiser model (Koizumi, Tanuguchi & Hiromitsu, 1987).

While speech modellers have gone to great lengths to model as closely as possible the anatomy of the vocal folds, the buccolabial musculature has not received any mention in the trumpet-modelling literature. The situation is actually much simpler than for the vocal folds.

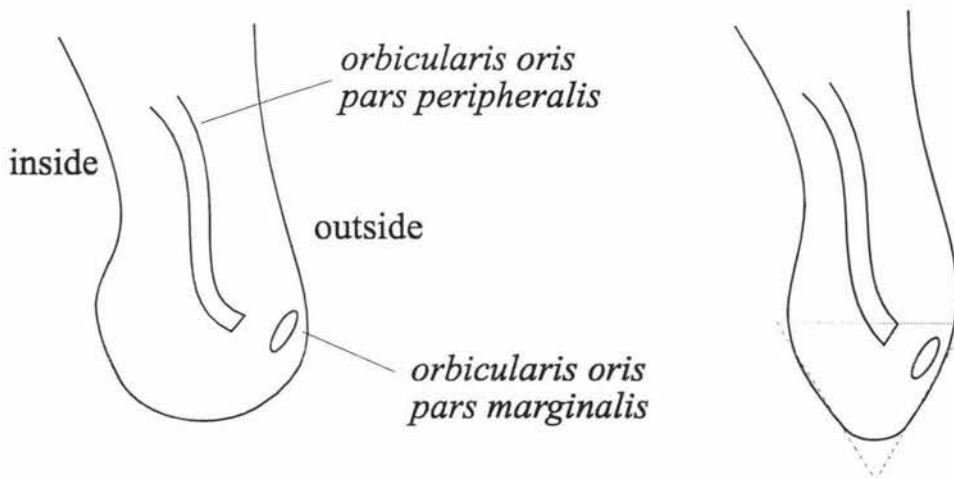


Figure 1.14: Top-Lip Profiles For Different Tensions of *Pars Marginalis*

Contraction of the muscle *orbicularis oris pars marginalis* is considered to alter the sectional profile of the red-lip rim. The upper and lower lips both transform to a narrower shape reminiscent of a truncated isosceles triangle. Both the length and the tension, of the so-called *labial cords* that result, can be delicately controlled (Williams et al., 1989).

The vocal folds are much more difficult to model, as they may entertain oscillations of many degrees of freedom (the term *vocal folds* has gradually replaced *vocal cords* in the literature since this has become known). Titze (1973, 1974) describes a model of the vocal folds that utilises sixteen coupled SDF oscillators.

Stroboscopic and other studies of lip motion during trumpet-playing indicate that only the top lip need be modelled (Martin, 1942; Henderson, 1942; Weast, 1963).

#### Motional Constraint:

In the vertical direction, the distance of separation between the lips (alternatively, the area of the orifice between them) can never take a negative value. In the horizontal direction, the lip vibrations are restricted by the teeth behind them and the trumpet mouthpiece in front. One consequence of these motional constraints has been overlooked in the traditional mathematical description of trumpet operation - that the lips may already be under tension before a note begins. To illustrate this point, consider Figure 1.15 to follow:

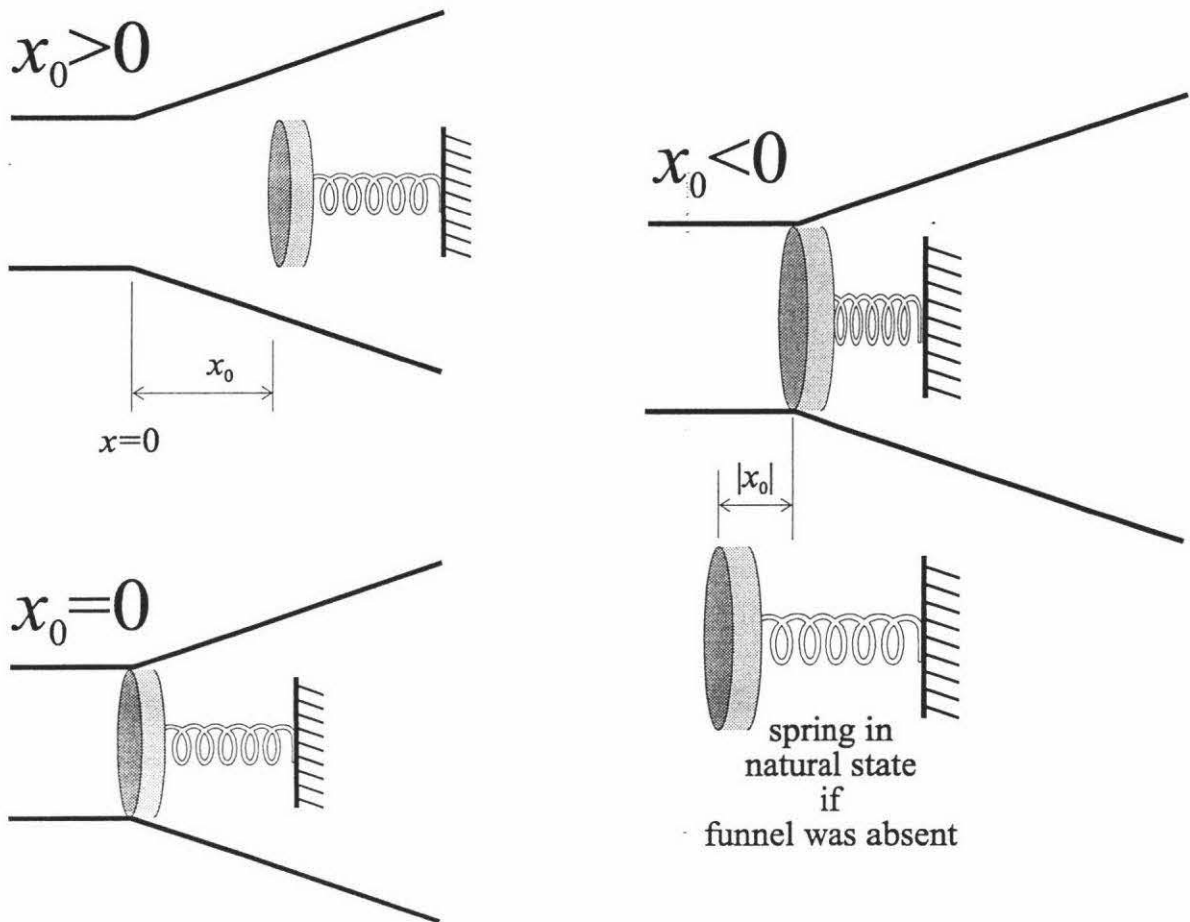


Figure 1.15: Plunger-in-Funnel Model In Absence of Fluid Forcing: “Pre-Tension”

Let  $x_0$  denote the distance from the plunger to the funnel in the absence of fluid forces or motional constraints. Then the equation of motion for the mass, when a forcing function  $F(t)$  is added, is

$$\begin{aligned}
 m\ddot{x} + b\dot{x} + s(x - x_0) &= F(t), \quad x > 0 \\
 \Leftrightarrow m\ddot{x} + b\dot{x} + sx &= F(t) + sx_0, \quad x > 0
 \end{aligned}$$

For the case of  $x_0 < 0$ , the term  $sx_0$  indicates the amount by which the spring is “pre-tensioned” in the absence of any applied fluid forces. When the plunger is first at rest in the funnel (i.e.  $x = 0, \dot{x} = 0$ ) the applied fluid force must reach a threshold value  $|sx_0|$  before any motion of the plunger can commence.

Lack of consideration of motional constraints in the traditional model (and the related possibility of pre-tension forces) has led to some remarkable conclusions regarding the playing of the trumpet. For example: “If the static opening of the lips in the absence of blowing pressure is zero, as is generally the case, then there is no threshold pressure

required...” (Fletcher, 1990). This statement is contrary to the experiences of every brass instrumentalist. (The opposite of Fletcher’s assertion also appears in the literature; see Worman, 1972.)

Similarly disconcerting is a statement that there is no upper limit to the static pressure provided by the musician, above which system oscillation is impossible (Fletcher, 1979a). It is interesting that Fletcher’s two statements, taken in conjunction, infer that any positive pressure gradient across the musician’s lips is suitable for exciting a trumpet into oscillation. It takes very little practice to refute this experimentally.

### Impact

In the above plunger-in-funnel illustration, impact occurs when the plunger returns to the position  $x = 0$ , if it approaches with non-zero velocity.

The question of impact has received rather different treatments in models of some mildly analogous systems. Hirschberg et al. (1990a) studied the behaviour of a reed organ-pipe and modelled the reed impact as an elastic collision. The momentum instantaneously lost from the oscillation through impact was instantaneously returned in the opposite direction: ie.  $\dot{x}(t) \leftarrow -\dot{x}(t)$  at the moment of collision.

Flanagan & Landgraf (1968) consider two different possibilities for the collision of the vocal folds during voiced speech. One scenario involved all momentum being instantaneously sapped from the vibration. For a completely inelastic collision  $\dot{x}(t) \leftarrow 0$  at the instant of impact. The other was termed a “purely viscous contact”, where the boundary constraint  $x \geq 0$  was actually violated, but at such times the damping constant of the motion was substantially increased. The advantage of the latter is that *the closure times* increase with the velocity of the approach at impact.

A more comprehensive model may incorporate a second mass-spring-damper (with possibly different coefficients) to represent any yielding and/or vibration after an impact. Such an arrangement is depicted in Figure 1.16.

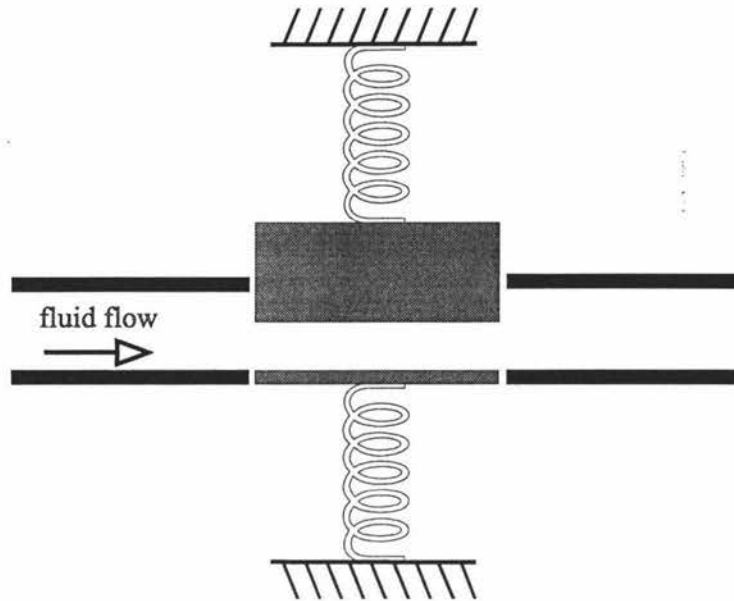


Figure 1.16: Model of Impact of a Transversely Vibrating Piston In A Compliant Duct

The effect of impact upon the behaviour of a single degree-of-freedom oscillator can be marked (Ipanov, 1993, 1994; Budd, Fox & Cliffe, 1995; Narayanan & Sekar, 1995).

### 1.3.6 The System is Incompletely Described:

When a model accurately represents a physical system, the model and the system both function similarly when presented with identical sets of values for the various system inputs. There is no hope for this to be achieved by the traditional trumpet model, since some of the system inputs are not even present in the model description.

Knowledge of trumpet-playing technique (described in Section 1.1) reveals the physical gestures by which the trumpeter initiates and controls the sound produced. Recognition of these gestures can be used to define the controlling variables of the musical oscillator system. Two classes of physical gestures can be distinguished (Cadoz, Luciani & Florens, 1984). These are *excitation gestures* and *modulation gestures*.

An excitation gesture is a means by which the musician effects a transfer of energy to the oscillator. When an instrumentalist has the ability to control the rate at which the energy is supplied to the system, the excitation gesture determines that rate. If this energy supply-rate determines the amount of energy that the system sheds as propagated sound, then the musician becomes part of a feedback loop that maintains the desired level of sound output. The player listens continually to the sound being produced, and makes adjustments using excitation gestures if it is too loud or too soft.

A modulation gesture is a control mechanism used to otherwise modify some characteristic of the oscillation. Modulation gestures may also feature within a feedback loop. For example, if a player hears a note to be out of tune then a modulation gesture may be used to correct this (see Figure 1.17).

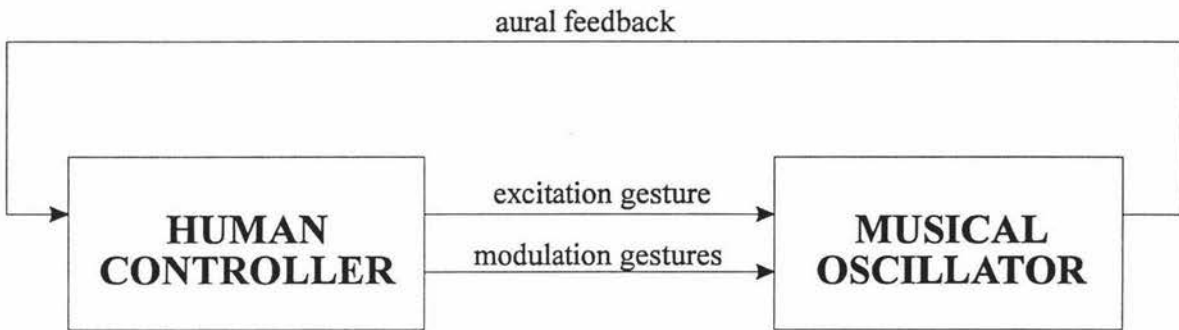


Figure 1.17: Human Control Of A Musical Oscillator

The excitation gesture of the brass player corresponds to the muscular control of his diaphragm, which dictates the flow of air from the lungs; in the case of wind instruments, the influx of energy into the system is coincident with the flow of air from the lungs. Adjustment of the embouchure is a modulation gesture, since such adjustment does not add energy to the system. The same applies to the variation of the vocal tract parameters, and the actions of the fingers in changing the valve positions of the trumpet. A top-level representation of the trumpet-trumpeter system is illustrated in Figure 1.18:

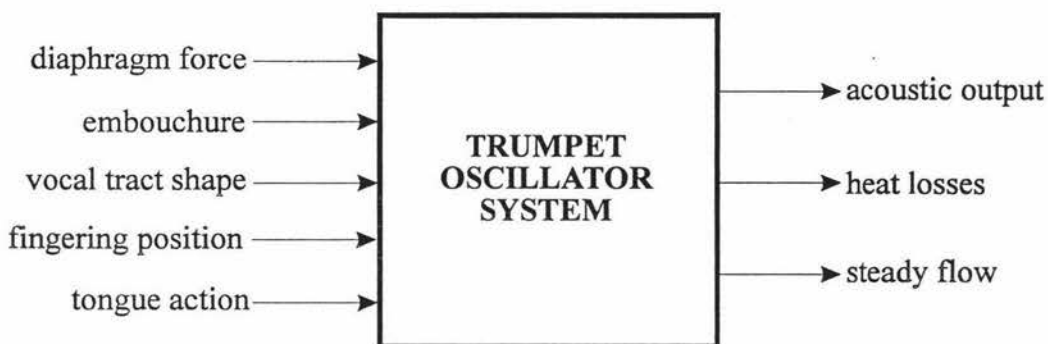


Figure 1.18: System Inputs and Outputs

The trumpet-trumpeter system features an energy conversion mechanism. The energy provided by the excitation gesture of the musician (diaphragm force) is converted to

acoustic, heat and flow energy, and the modulation gestures of the player determine the finer details of that conversion process. The oscillating system thus comprises:

- flow within the lungs, vocal tract and oral cavity of the player,
- flow within the trumpet itself, and
- the player's vibrating lips.

In contrast to system illustrated in Figure 1.18, the system described by the traditional trumpet model begins at the mouth. The resonator subsystem includes only the inside the trumpet itself and the mouth is treated as a constant pressure source. This can give a valid representation of the trumpet oscillator only if the following two conditions are met: firstly, the modulation gestures involving variations of vocal tract characteristics must be shown to be unimportant, and secondly, the mouth pressure must be shown to be a *functional equivalent* of the diaphragm force.

The effects of vocal tract shape in the sounding of wind instruments were examined experimentally many years ago (Hall, 1955; Stauffer, 1968). Another study measured the relative tensions of the facial muscles used in blowing the instrument (Stubbins, Lillya & Frederick, 1956) and the mechanical impedance of the vocal tract walls has been measured directly (Ishizaka, French & Flanagan, 1975). More recently, x-rays of various wind instrumentalists have been used to compare the acoustic resonant frequencies of the vocal tract to the pitch of the notes being sounded (Clinch, Troup & Harris, 1982). The evidence of these experimentalists shows that the effects of the vocal tract parameters *are* important in playing the trumpet. None of these effects can be described using the traditional model.

Consider the respiratory system of the trumpet player, consisting of the lungs, vocal tract and oral cavity. At the lungs, a constant force provided by the diaphragm muscles will produce a steady flow of air from the lungs, while the vibration of the lips dictates that the flow out of the mouth is unsteady. Consequently the amount of air between the lungs and lips of the player must be continually changing. The rate of this change at each instant is determined by the difference between the instantaneous mass outflux at the lips and the (steady) mass influx from the lungs. The mass of air between the lungs and the lips must oscillate in value at the same frequency as the lip vibrations. The vocal tract thus functions as a *fluid oscillator* (Naudascher & Rockwell, 1980; see also Gupta, Wilson & Beavers, 1973).

There are two possible responses to the resulting mass oscillation. Firstly, the *density* of the air within the player's respiratory system may fluctuate in value, and secondly, the walls of the vocal tract may expand in response to the presence of additional air, altering the system's *volume*. Some variation of the density of the air is inevitable and, as a result, the mouth pressure must be oscillating at the frequency of lip vibration. Indeed, actual experimental measurements of a trombonist's mouth pressure have shown periodic fluctuations at the frequency of the lip oscillation (Elliott & Bowsher, 1982).

Since the mouth pressure of the trumpeter is oscillatory, then the force exerted by the player's diaphragm muscles and the pressure in his or her mouth *cannot* be functional equivalents. The traditional trumpet model's exclusion of the lungs and vocal tract of the player is unjustified.

The resulting model is incapable of describing any effects of the player's vocal tract parameters, nor the mass oscillation which occurs in the trumpeter's respiratory system. Some modellers have suggested that consideration of the vocal tract might lead to self-excited oscillation of the model lips even without the trumpet present (Fletcher, 1979a, 1979b; Benade & Hoekje, 1982).

## **1.4 Purpose Of The Current Investigation:**

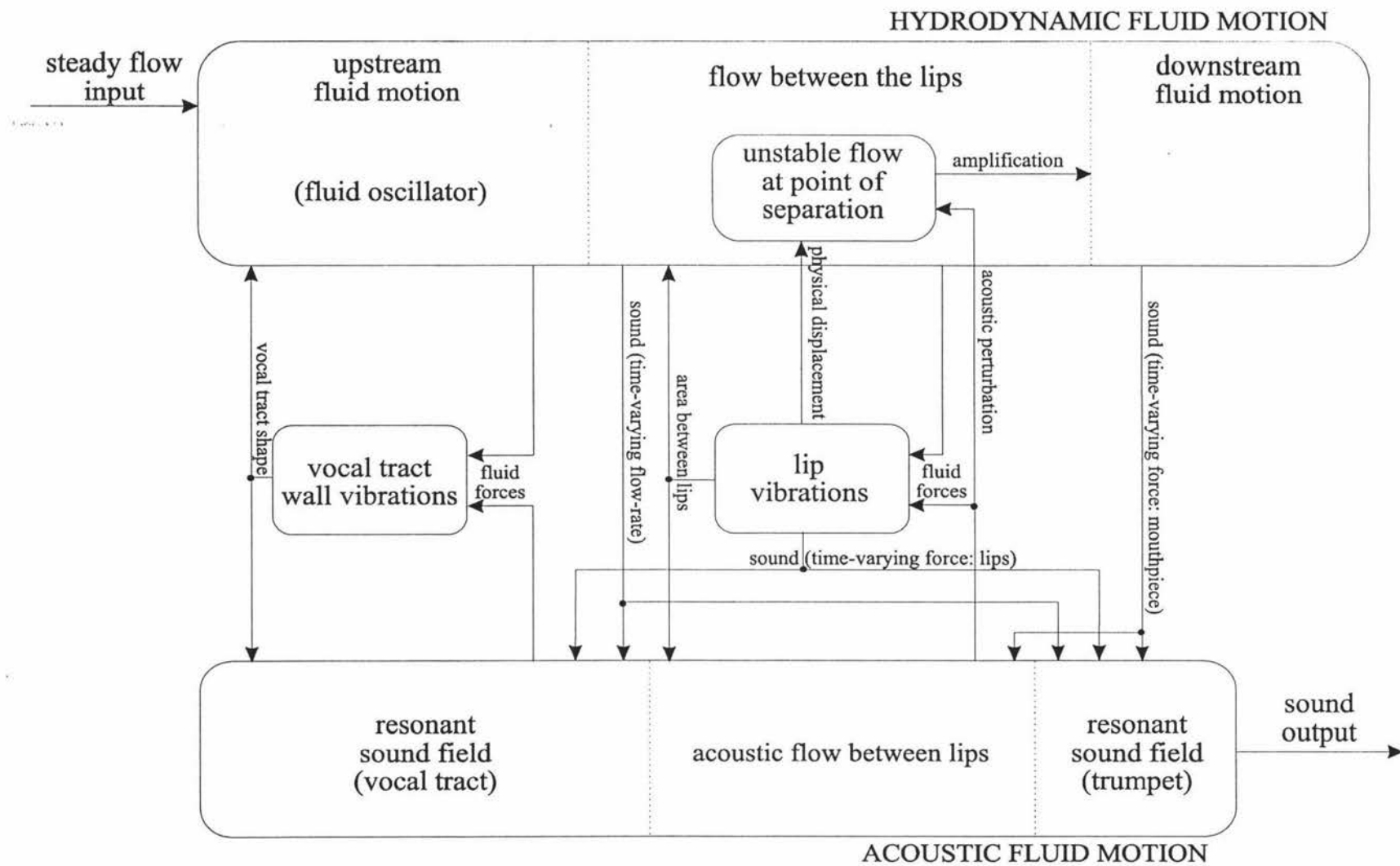
Study of sound production in reed-driven wind instruments has traditionally been from an acoustical standpoint. The operation of the resonator subsystem seems to be adequately described in one spatial dimension by linear acoustic theory. This has presented some incentive to describe the generator subsystem by an extension of the resonator theory. This thesis instead describes investigations into generator operation from a fluid-dynamical perspective. Particular attention is paid to examining the validity of the accepted description of the various fluid phenomena which occur within the generator subsystem, and the viability of various alternative descriptions.

The identification of possible excitation mechanisms, and occurrences of structural and fluid oscillators, is an important first step in the analysis of a complex flow-induced vibration phenomenon. Naudascher & Rockwell (1980) stress that this preliminary analysis should always be carried out, even when it is only a part of the system behaviour which is of direct interest.

A comprehensive model structure is shown in Figure 1.19 which separates the acoustic and hydrodynamic contributions to the overall fluid motion (§1.3.2) and includes the vocal tract of the musician (§1.3.6). The model includes contributions from all three of the possible acoustic sources discussed in §1.3.3. The possibility of instability-induced excitation of the flow-induced vibration is entertained (§1.3.4), as is the influence of incident sound upon this process.

The two resonant sound fields (in the vocal tract and in the trumpet) are coupled by the acoustic flow between the lips; i.e. they share acoustic energy. The effectiveness of this coupling is determined by the *acoustic impedance* of the orifice formed by the lips. The value of this impedance will depend upon the area of the orifice (Sivian, 1935; Ingard & Ising, 1967) and hence it changes with the motion of the lips.

Redhead (1993) has described how a neglected theory of Richardson (1929) fits into the model structure of Figure 1.19. Richardson's treatment is unique in that it does not overlook the existence of unsteady convective flow.



**Figure 1.19: Cause-and-Effect Diagram For Trumpet Operation**

The model structure presented in Figure 1.19 is very complex, and hints at the determination of the relative importance of hydrodynamic and acoustic fluid motions as being the logical next step in trumpet model development.

Chapters Two and Three of this thesis give a mathematical background to the ensuing fluid-dynamical approach to production of sound in the trumpet. The background theory chapters were written for the reader without a prior background in fluid dynamics - for example, an Electrical Engineer wishing to synthesise trumpet sounds from a mathematical model of a trumpet.

Chapter Four extracts the equations of linear acoustics from the equations of general fluid motion, and examines analytically the evolution of an 'ideal' modulated flow. The purpose of this chapter is to show that pseudosound cannot be described using a one-dimensional mathematical model.

Chapter Five describes a simple experimental procedure correlating the output signals of two pressure transducers placed within a modulated flow - one close to a trombonist's lips and the other a small distance downstream. This experiment was designed to reveal the proportion of the fluctuating pressure which is being propagated at the speed of sound, and show how much instead relates to unsteady convected flow.

Chapter Six narrates an investigation into the process of flow modulation by movement of the boundaries of a fluid enclosure as the flow passes by. Numerical simulations are used to illustrate the behaviours of the various models studied.

Chapter Seven uses the results obtained in earlier chapters to simplify the structure of the model shown in Figure 1.19 above.



## General Equations of 3-D Fluid Motion:

In the science of particle mechanics, the motion of discrete bodies is followed as time progresses. The reactions of these bodies to external influences are governed by well-known conservation principles. For example, Newton's Second Law states the principle of conservation of linear momentum for a body of mass  $m$ , whose centre of mass has position  $\mathbf{r}(t)$  at time  $t$ , under the influences of a net force  $\mathbf{F}$ :

$$\mathbf{F} = m \frac{d^2 \mathbf{r}}{dt^2}$$

A fluid may be considered to be composed of a very large number of particles, since all fluids are composed of molecules in constant motion. However, in most practical applications it is only the macroscopic behaviour of many molecules collectively which is of interest. The *continuum* approach to the study of fluids allows a description of macroscopic effects without recourse to the motion of each individual molecule. This approach is valid when the physical dimensions of a fluid problem are all very large compared with molecular distances.

The assumption that a fluid behaves as a continuous substance is the basis of classical fluid dynamics. In the continuum mechanics approach it is assumed that each fluid property (such as fluid density, temperature, velocity) has a definite value at each point in space, and that all of the fluid properties are continuous functions of position. The continuum assumption leads directly to a field representation of fluid properties. Scalar, vector and tensor fields are all used in the study of continuum mechanics.

The same conservation principles which are applied to individual bodies in the science of particle mechanics can also be applied to any appropriately chosen *system* in continuum mechanics. A system is defined as an arbitrary quantity of mass of fixed identity. The constituent matter of the system may move as a whole, and different elements may move relative to each other, but throughout any motion the conservation principles are obeyed by the system. It is possible to write the conservation principles mathematically with respect to a frame of reference which moves with the local velocity of a system. But when

knowledge of fluid properties is required at a certain location in space (as is more usually the case in practical problems of fluid mechanics) the fluid motion is referred to co-ordinates which are fixed in space.

In this chapter, elementary *control volume analysis* is used to develop expressions describing general fluid motion of a fluid with respect to fixed co-ordinates. A *control volume* is a finite region with open boundaries, chosen carefully to allow meaningful expressions to be written down relating the fluid properties within the region to the properties of the fluid passing through its boundaries (White, 1979). The boundaries of the control volume are collectively known as the *control surface*.

The derivations in this chapter will make use of a control volume chosen at a fixed location in space and having rigid boundaries; the choices of size and location are arbitrary so long as the control volume lies wholly within the fluid. A system will be chosen as that matter which at time  $t$  is coincident with the control volume. After a short time  $\delta t$ , the fluid constituents of the system will generally have moved to new locations in space, so that the system and control volume are no longer coincident. The nature of this motion is governed by the various conservation principles, which relate the system behaviour to the effects of outside influences.

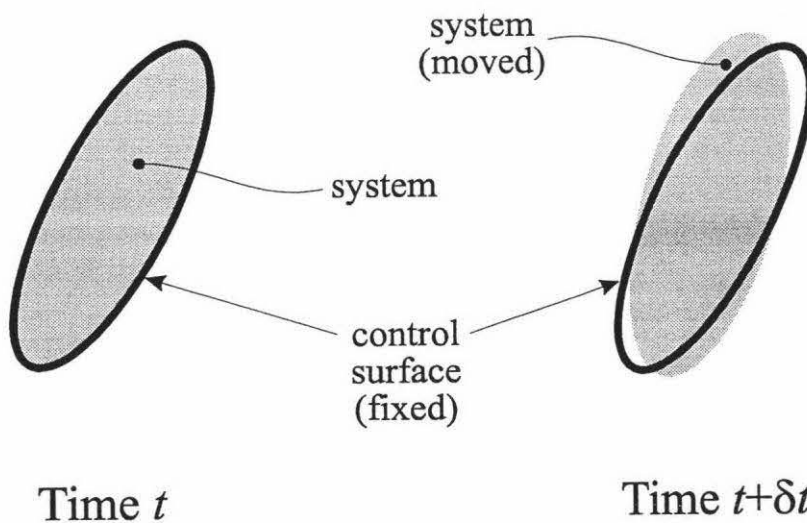


Figure 2.1 System and Control Volume at Two Time Instants

Because the system and the control volume are coincident at time  $t$ , the following equation will hold for any arbitrary fluid property  $\chi$ :

$$\iiint_{system(t)} \chi dV = \iiint_{CV} \chi dV$$

By incorporating knowledge of the motion of a system relative to a fixed control volume, it is possible to adapt conservation principles, (which apply to a system) to describe instead the fluid behaviour within the fixed control volume. *Reynold's transport theorem*, developed in Section 2.1, facilitates this description.

In Section 2.2 the principle of conservation of mass is applied to yield a *continuity equation* valid in any general fluid motion.

In Section 2.3 the linear momentum of the fluid is considered. Since linear momentum is a vector quantity, it may be resolved into three orthogonal components, and a conservation equation may be written for each of these directions.

In Section 2.4 conservation of fluid energy is described.

Constitutive, thermodynamic and heat transfer properties of fluids are discussed in Section 2.5, and related to the transport equations already derived. The number of system equations then matches the number of unknown quantities. A meaningful solution exists when an appropriate set of initial and boundary conditions is specified.

## 2.1 Reynold's Transport Theorem:

Consider an arbitrary control volume which lies wholly within a fluid. A system is chosen as being those fluid particles which are resident within the control volume at a certain time  $t$ , i.e. the fixed control volume coincides with the system volume at time  $t$ . At a second time instant  $t + \delta t$ , the fluid (generally) will have moved and the system defined will no longer lie completely within the fixed control volume.

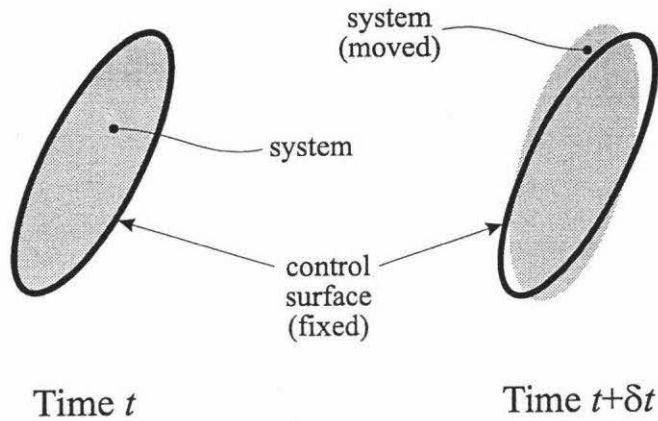


Figure 2.1 (repeated): System and Control Volume at Two Time Instants

It is possible to represent the system total of any fluid property by adding the contributions from all fluid elements within the system. Any small element of the system has its own fixed mass, and so the system total may be obtained by integrating the (*mass*) *specific* value for the property (i.e. the amount per unit mass) over all mass elements.

$$\begin{aligned}\Theta_{\text{system}} &= \int_{\text{system}} \theta dm \\ &= \iiint_{\text{system}} \theta \frac{dm}{dV} dV \\ &= \iiint_{\text{system}} \rho \theta dV\end{aligned}$$

The property  $\Theta_{\text{system}}$  is known as an *extensive property*, since its value depends upon the amount of fluid under consideration (ie. upon the size of the system chosen). The fluid properties  $\rho$  (fluid density) and  $\theta$  (the amount of  $\Theta$  per unit mass) are instead *intensive properties*; they have values defined at every point within the flow, and these values are independent of the amount of fluid under study.

It is always the time-rate of change of an extensive property, i.e.  $\frac{d\Theta_{system}}{dt}$ , which appears in each of the conservation laws. This rate of change for an arbitrary property  $\Theta$  can be developed as follows:

$$\begin{aligned}\frac{d\Theta_{system}}{dt} &= \frac{d}{dt} \iiint_{system} \rho \theta dV \\ &= \lim_{\delta t \rightarrow 0} \left\{ \frac{1}{\delta t} \left[ \iiint_{system(t+\delta t)} \rho(t+\delta t) \theta(t+\delta t) dV - \iiint_{system(t)} \rho(t) \theta(t) dV \right] \right\}\end{aligned}$$

It is helpful at this point to add a new term to the first integral and subtract the same amount from the second:

$$\begin{aligned}\frac{d\Theta}{dt} &= \lim_{\delta t \rightarrow 0} \left\{ \frac{1}{\delta t} \left[ \iiint_{system(t+\delta t)} \rho(t+\delta t) \theta(t+\delta t) dV - \iiint_{system(t)} \rho(t+\delta t) \theta(t+\delta t) dV \right] \right\} \\ &\quad + \lim_{\delta t \rightarrow 0} \left\{ \frac{1}{\delta t} \left[ \iiint_{system(t)} \rho(t+\delta t) \theta(t+\delta t) dV - \iiint_{system(t)} \rho(t) \theta(t) dV \right] \right\} \\ &= \lim_{\delta t \rightarrow 0} \left\{ \frac{1}{\delta t} \left[ \iiint_{\substack{system(t+\delta t) \\ -system(t)}} \rho(t+\delta t) \theta(t+\delta t) dV \right] \right\} + \iiint_{system(t)} \frac{\partial}{\partial t} (\rho \theta) dV\end{aligned}\tag{2.1}$$

The remaining limit in Equation (2.1) can be simplified by considering that a boundary element of the system,  $dS(t)$ , evolves into the boundary element  $dS(t+\delta t)$  during the intervening time interval  $\delta t$ . The perpendicular distance from  $dS(t)$  to  $dS(t+\delta t)$  is  $\mathbf{u} \cdot \mathbf{n} \delta t$ , where  $\mathbf{n}$  denotes the unit normal vector to the surface  $dS$  and  $\mathbf{u}$  is the fluid velocity. Consequently, to first order the elemental volume that lies between the two elemental surfaces has size given by  $\delta V = \mathbf{u} \cdot \mathbf{n} \delta t dS$ .

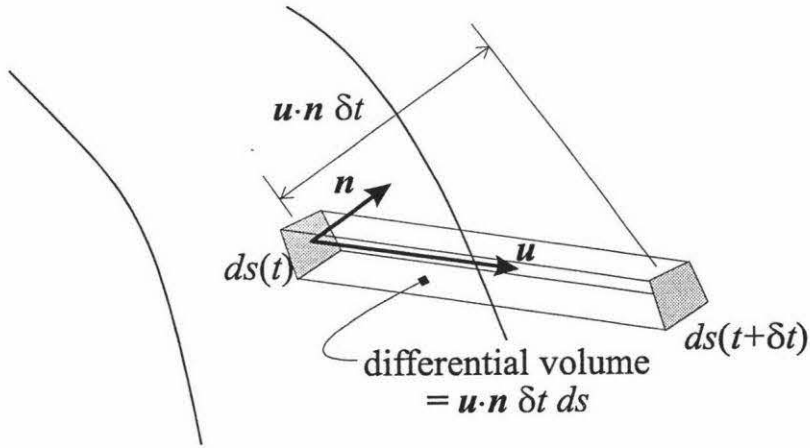


Figure 2.2 Elemental Change of System Volume in Time Interval  $\delta t$

The limit in Equation (2.1) can now be rewritten:

$$\lim_{\delta t \rightarrow 0} \left\{ \frac{1}{\delta t} \left[ \iiint_{\substack{\text{system}(t+\delta t) \\ -\text{system}(t)}} \rho(t+\delta t)\theta(t+\delta t)dV \right] \right\} = \lim_{\delta t \rightarrow 0} \left\{ \frac{1}{\delta t} \left[ \iint_{\text{system surface}(t)} \rho(t+\delta t)\theta(t+\delta t)\mathbf{u} \cdot \mathbf{n} \delta t dS \right] \right\}$$

$$= \iint_{\substack{\text{system} \\ \text{surface}(t)}} \rho(t)\theta(t)\mathbf{u} \cdot \mathbf{n} dS$$

The divergence theorem can be used to convert this to a volume integral:

$$\iint_{\substack{\text{system} \\ \text{surface}(t)}} \rho\theta\mathbf{u} \cdot \mathbf{n} dS = \iiint_{\text{system}(t)} \nabla \cdot (\rho\theta\mathbf{u})dV$$

Equation (2.1) becomes:

$$\frac{d\Theta_{\text{system}}}{dt} = \frac{d}{dt} \left[ \iiint_{\text{system}} \rho\theta dV \right] = \iiint_{\text{system}(t)} \frac{\partial}{\partial t} (\rho\theta) dV + \iiint_{\text{system}(t)} \nabla \cdot (\rho\theta\mathbf{u}) dV$$

$$= \iiint_{\text{system}(t)} \left[ \frac{\partial}{\partial t} (\rho\theta) + \nabla \cdot (\rho\theta\mathbf{u}) \right] dV$$

Recall that the fixed control volume is coincident with the system at time  $t$ . Consequently the above equation can be rewritten in terms of the fluid properties within the fixed control volume.

$$\frac{d}{dt} \left[ \iiint_{\text{system}} \rho\theta dV \right] = \iiint_{CV} \left[ \frac{\partial}{\partial t} (\rho\theta) + \nabla \cdot (\rho\theta\mathbf{u}) \right] dV \quad (2.2)$$

Equation (2.2), which is known as *Reynold's transport theorem* (Owczarek, 1968: Equation 3-15), expresses a system derivative in terms of happenings inside a fixed region which the system occupies at the time  $t$ . Using this theorem, integral equations expressing laws of mass, linear momentum and energy conservation within a chosen control volume can be written down.

## 2.2 Mass Transport - The Continuity Equation:

One important property of a system is that, since it always consists of the same constituent particles, its total mass is conserved as it moves.

$$\frac{dM_{system}}{dt} = 0$$

The fluid within an arbitrary, fixed control volume must also obey the principle of conservation of mass. Fortunately, Reynold's transport theorem is available to express  $M_{system}$  in terms of the fluid mass that lies within the fixed control volume at time  $t$  and that which is passing through the control surface. The *specific* quantity of Equation (2.2) corresponding to the mass of the system is 'mass per unit mass', which equals unity, and so for mass conservation  $\theta = 1$  in Equation (2.2):

$$\frac{dM_{system}}{dt} = 0 = \iiint_{CV} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] dV$$

Because this equation holds true for any arbitrary control volume, the integrand in the above equation must itself be equal to zero:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (2.3)$$

Equation (2.3) is known as the *equation of continuity* for a general fluid motion. It is a partial differential equation which states that fluid mass is conserved at every place within the flow.

### Alternative form of Reynold's transport theorem:

Using the equation of continuity it is possible to write down a simpler version of Reynold's transport theorem which is valid for any scalar fluid property. Recall Equation (2.2), which describes the rate of change of a system property  $\Theta$  in terms of the distribution of its specific counterpart  $\theta$  throughout a given control volume  $V$ . If  $\theta$  is a scalar quantity then the following development is possible:

$$\begin{aligned} \frac{d\Theta}{dt} &= \iiint_{CV} \left[ \frac{\partial}{\partial t} (\rho \theta) + \nabla \cdot (\rho \theta \mathbf{u}) \right] dV \\ &= \iiint_{CV} \left[ \rho \frac{\partial \theta}{\partial t} + \theta \frac{\partial \rho}{\partial t} + \theta \nabla \cdot (\rho \mathbf{u}) + \rho \mathbf{u} \cdot \nabla \theta \right] dV \end{aligned}$$

The continuity equation can be used to remove the second and third terms:

$$\begin{aligned}
 \frac{d\Theta}{dt} &= \iiint_{CV} \left\{ \rho \frac{\partial \theta}{\partial t} + \theta \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] + \rho \mathbf{u} \cdot \nabla \theta \right\} dV \\
 &= \iiint_{CV} \rho \left( \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta \right) dV \\
 &= \iiint_{CV} \rho \frac{D\theta}{Dt} dV
 \end{aligned} \tag{2.4}$$

Equation (2.4) employs the *material derivative* of the intensive fluid property  $\theta$ . Physically, the material derivative represents the rate of change experienced by a material element of the fluid as it moves. This operator is defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$$

Although the above derivation specified that  $\theta$  was a scalar quantity, the positioning of the brackets in the above definition of the material derivative allows general application of Equation (2.4) for vectors as well as scalars.

### 2.3 Momentum Transport:

The rate of increase of fluid momentum of a system is equal to the net result of all applied forces acting upon the constituent fluid. This is a statement of Newton's second law of motion, and maybe represented mathematically by the following equation:

$$\frac{d}{dt} \left[ \iiint_{\text{system}} \rho \mathbf{u} dV \right] = \sum \mathbf{F}$$

This vector equation represents three scalar equations, one in each co-ordinate direction.

In the  $i^{\text{th}}$  direction, the equation is:

$$\frac{d}{dt} \left[ \iiint_{\text{system}} \rho u_i dV \right] = \sum F_i \quad (2.5)$$

It is possible to apply this principle to fluid within an arbitrary, fixed control volume by defining a system as that fluid which is resident in the control volume at the time  $t$ , and then using Reynold's transport theorem to convert the system derivative of Equation (2.5) to derivatives of fluid properties within the fixed control volume.

#### Fluid Forces:

Forces experienced by fluids can be divided into two broad categories: *Body forces* are the result of a *force field* upon the fluid system; they act upon every particle which makes up the fluid. It is possible to represent the total effect of a body force upon a system by adding the contributions acting upon each infinitesimal fluid element within that system. Any constituent element of the system has its own fixed mass, and so the system total may be obtained by integrating the *specific force*  $f_{(body)}$  over all mass elements. The force field is a vector field, and a *conservative* one if it can be written as the gradient of a scalar field. (In the expression  $f_{(body)} = -\nabla\Psi$  the quantity  $\Psi$  is called the [*scalar*] *potential* of the force field.)

$$\begin{aligned} F_{i (body)} &= \int_{\text{system}} f_{i (body)} dm \\ &= \iiint_{\text{system}} f_{i (body)} \frac{dm}{dV} dV \\ &= \iiint_{\text{system}} \rho f_{i (body)} dV \end{aligned}$$

Because the system and the control volume are coincident at time  $t$ , the body forces acting upon the system at time  $t$  can be described by the following equation:

$$F_{i (body)} = \iiint_{CV} \rho f_{i (body)} dV \quad (2.6)$$

*Surface forces* originate from molecular interactions, and so act only between fluid elements which are in actual mechanical contact. The surface force, per unit area, which acts upon any given surface is known as the *stress* or *traction force* upon that surface.

An elemental surface may be represented by a vector in the direction of the outward normal  $\mathbf{n}$  of the surface element, with magnitude equal to the surface element's area  $\delta S$ . Any stress on such a surface element, denoted  $\mathbf{T}^{(n)}$ , can be decomposed into a *tensile* (i.e. normal) component and a *shear* (i.e. tangential) component.

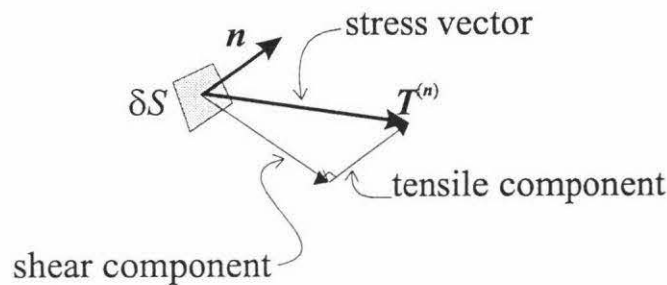


Figure 2.3 Stresses on a Surface Element

The stress on a given surface depends upon the orientation of that surface, i.e. the stress on a surface is a function of  $\mathbf{n}$ .

The *stress at a particular point*, (rather than that on a given surface) is completely specified by choosing any three elemental surfaces which are all orthogonal to one another at the given point, and listing the stresses upon those three elemental surfaces. In three dimensions a force has three orthogonal components, and since three orthogonal choices are possible for the orientation of the elemental surface at any given point, a total of nine scalar quantities completely specifies the stress at a given point in the flow.

It is convenient to write the stress at a point as a second-order tensor. In Cartesian coordinates:

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

The primary subscript for each element of this tensor indicates the direction of the normal of the surface upon which the stress acts; the secondary subscript indicates the direction of the stress component itself.

The stress *on a given surface* (as depicted in Figure 2.3 above) is a vector quantity, and the stress upon a surface whose normal is  $\mathbf{n}$  is denoted in this thesis by the vector  $\mathbf{T}^{(n)}$ .

In contrast, the stress *at a given point* (in three dimensions) is a second-order tensor quantity. Using standard tensor notation, the element which is in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of a stress tensor  $\sigma$  is specified as  $\sigma_{ij}$ , where each subscript can take the value 1, 2 or 3. This thesis employs the usual *summation convention* whenever tensor expressions are used: when an index appears twice in a single term, then summation over all possible values of that index is assumed.

The stress upon a given surface is related to the stress at a point by *Cauchy's formula*: When  $\mathbf{n} = (n_1, n_2, n_3)$  and the stress tensor elements are written in terms of the three coordinate directions ( $x_1, x_2$  and  $x_3$ ), then the  $i^{\text{th}}$  component of the total stress on a elemental surface perpendicular to  $\mathbf{n}$  is given by

$$[\mathbf{T}^{(n)}]_i = \sum_{j=1}^3 \sigma_{ji} n_j$$

Using the summation convention, Cauchy's formula is written as

$$[\mathbf{T}^{(n)}]_i = \sigma_{ji} n_j$$

Using standard tensor notation, the surface force over an entire fluid system may be written as follows:

$$F_{i(\text{surface})} = \iint_{\substack{\text{system} \\ \text{surface}}} \sigma_{ji} n_j dS$$

Because the system and the control volume are coincident at time  $t$ , the following equation will then hold

$$F_{i \text{ (surface)}} = \oint\!\!\!\oint_{CS} \sigma_{ji} n_j dS$$

The divergence theorem may be used to convert the surface integral to a volume integral. In tensor notation this theorem is written for an arbitrary vector  $\chi$  as:

$$\oint\!\!\!\oint_{CS} \chi_j n_j dS = \iiint_{CV} \frac{\partial \chi_j}{\partial x_j} dV$$

The expression for the surface forces upon the system, at time  $t$ , becomes:

$$F_{i \text{ (surface)}} = \iiint_{CV} \frac{\partial \sigma_{ji}}{\partial x_j} dV \quad (2.7)$$

### Momentum Changes:

Newton's second law of motion relates the time-rate of change of momentum of a system to the sum of the applied forces experienced by that system. Recall Reynold's transport theorem in the form of Equation (2.4): this equation can be used to relate the time-rate of change of system momentum to the fluid behaviour within a fixed control volume.

$$\frac{d\Theta}{dt} = \iiint_{CV} \rho \frac{D\theta}{Dt} dV$$

The appropriate choice for the specific property  $\theta$  is fluid velocity vector  $\mathbf{u}$ ; the product  $\rho \mathbf{u}$  which appears in the resulting equation is given the name *momentum density*, being a measure of fluid momentum per unit volume.

$$\frac{d}{dt} \left[ \iiint_{\text{system}} \rho u_i dV \right] = \iiint_{CV} \rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) dV$$

### Newton's Law Revisited:

Recall Newton's second law of motion for a fluid system, Equation (2.5):

$$\frac{d}{dt} \left[ \iiint_{\text{system}} \rho u_i dV \right] = \sum F_i$$

The right-hand side of Equation (2.5) can be simplified by employing Equations (2.6) and (2.7) obtained already. With the left-hand side developed as above using Reynold's transport theorem, Equation (2.5) can be rewritten as follows:

$$\begin{aligned} \iiint_{CV} \rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) dV &= \iiint_{CV} \rho f_{i(body)} dV + \iiint_{CV} \frac{\partial \sigma_{ji}}{\partial x_j} dV \\ \Rightarrow \iiint_{CV} \left[ \rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) - \rho f_{i(body)} - \frac{\partial \sigma_{ji}}{\partial x_j} \right] dV &= 0 \end{aligned}$$

Since this equation is valid for an arbitrary control volume, the integrand itself is zero, and may be extracted to form an equation describing momentum transport at any point within the flow:

$$\begin{aligned} \rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) - \rho f_{i(body)} - \frac{\partial \sigma_{ji}}{\partial x_j} &= 0 \\ \Rightarrow \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} &= f_{i(body)} + \frac{1}{\rho} \frac{\partial \sigma_{ji}}{\partial x_j} \end{aligned} \quad (2.8)$$

Alternatively, the three component equations represented by Equation (2.8) may be written as a single vector equation:

$$\frac{D\mathbf{u}}{Dt} = \mathbf{f}_{body} + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} \quad (2.9)$$

The last term of Equation (2.9) employs a *tensor divergence*, the interpretation of which is obvious by comparing this equation with Equation (2.8). The left-hand side of Equation (2.9) employs the *material derivative* of the velocity. Recall that this operator is defined by  $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$ . Equations (2.8) and (2.9) are alternative forms of a partial differential equation which expresses that linear momentum of the fluid, a vector quantity, is conserved at every place within the flow.

### Transport of Angular Momentum:

Resolution of torques and consideration of the principle of conservation of angular momentum about the three cartesian axes produces the result that the stress tensor  $\boldsymbol{\sigma}$  is in fact symmetric, i.e.  $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$  or  $\sigma_{ij} = \sigma_{ji}$ . Many standard fluid dynamic texts derive this result (e.g. White, 1979: §4.4).

## 2.4 Energy Transport:

The first law of thermodynamics expresses the principle of conservation of energy for a thermodynamic system. This law applies to a system originally at rest, and at rest again after some event: the change in system energy due to the event is equal to the sum of the total work done upon the system in the course of the event and any heat which was added. Mathematically, this is written as  $dE = dW + dQ$ . With care, this law can be applied to a moving element of a continuum by instead calculating the rates of change of energy, work and heat (Batchelor, 1967; Lewis & Randall, 1961):

$$\frac{dE}{dt} = \frac{dW}{dt} + \frac{dQ}{dt} \quad (2.10)$$

### Rate of Change of System Energy

The total energy  $E$  of a system comprises both *kinetic energy* and *internal energy* components. In the science of particle mechanics the *kinetic energy* of a body of mass  $m$  and travelling at speed  $u$  is well-known as  $E_{kinetic} = \frac{1}{2}mu^2$ . For a fluid system the expression is analogous:

$$E_{kinetic} = \iiint_{system} \frac{1}{2}\rho(\mathbf{u} \cdot \mathbf{u})dV$$

The system's *internal energy* can be written in terms of the *specific internal energy*, denoted  $e$ , which is related through thermodynamic relationships to other thermodynamic properties of a fluid. (More will be said of thermodynamic properties and their interrelationships in Sections 2.5, 3.3 and 3.4.)

$$E_{internal} = \iiint_{system} \rho e dV$$

The kinetic and internal energy components may be summed to give the total energy of the system:

$$\begin{aligned} E &= E_{kinetic} + E_{internal} \\ &= \iiint_{system} \left[ \frac{1}{2}\rho(\mathbf{u} \cdot \mathbf{u}) + \rho e \right] dV \\ &= \iiint_{system} \rho \left[ \frac{1}{2}(\mathbf{u} \cdot \mathbf{u}) + e \right] dV \end{aligned}$$

The above integrand represents the total *energy density* at a particular location in the fluid.

Consider that the system is chosen to be coincident at a certain time  $t$  with an arbitrary, fixed control volume. Reynold's transport theorem, in the form of Equation (2.4), may be applied as follows to relate the rate of change of energy of the fluid system to the behaviour of the fluid within the arbitrary, fixed control volume:

$$\begin{aligned} \frac{d}{dt} \iiint_{system} \rho \theta dV &= \iiint_{CV} \rho \left( \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta \right) dV \\ \Rightarrow \frac{dE}{dt} &= \iiint_{CV} \rho \left[ \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + e \right) + \mathbf{u} \cdot \nabla \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + e \right) \right] dV \end{aligned}$$

The following simplifications are possible:

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) &= \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} \\ \nabla \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) &= (\mathbf{u} \cdot \nabla) \mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}) \\ \mathbf{u} \cdot (\mathbf{u} \times \boldsymbol{\omega}) &= 0, \text{ for arbitrary } \boldsymbol{\omega} \end{aligned}$$

These lead to, in both vector and Cartesian tensor notation:

$$\begin{aligned} \frac{dE}{dt} &= \iiint_{CV} \rho \left[ \frac{\partial e}{\partial t} + \mathbf{u} \cdot \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla e \right) \right] dV \\ &= \iiint_{CV} \rho \left[ \frac{\partial e}{\partial t} + \mathbf{u} \cdot \left( \frac{D\mathbf{u}}{Dt} + \nabla e \right) \right] dV \\ \text{or } \frac{dE}{dt} &= \iiint_{CV} \rho \left[ \frac{\partial e}{\partial t} + u_i \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial e}{\partial x_i} \right) \right] dV \end{aligned} \quad (2.11)$$

### Rate of Work Done upon the System

The rate of change of work which appears in Equation (2.10) may include contributions from both body forces and surface forces. The rate of work done by body forces may be calculated as the sum of contributions of work done upon each infinitesimal element of the system. Using Equation (2.6) and noting that  $dW = \mathbf{F} \cdot d\mathbf{x}$ , the rate of work upon the system at a time  $t$  may be written as

$$\begin{aligned} \frac{dW}{dt} \text{ body forces} &= \iiint_{system(t)} \rho (\mathbf{f}_{body} \cdot \mathbf{u}) dV \\ &= \iiint_{system(t)} \rho f_i^{(body)} u_i dV \end{aligned}$$

Recall that the system is chosen to be coincident with the arbitrary, fixed control volume at time  $t$ . Then the above equation can also be written as

$$\frac{dW}{dt} \text{ body forces} = \iiint_{CV} \rho f_i \text{ (body)} u_i dV$$

Using a Cartesian co-ordinate system, the contribution to the rate of work from *surface forces*, at time  $t$ , can be developed as follows:

$$\frac{dW}{dt} \text{ surface forces} = \iint_{\substack{\text{system} \\ \text{surface}(t)}} \sigma_{ji} n_j u_i dS$$

Because the system and the control volume are coincident at this time, the above equation can be rewritten as

$$\frac{dW}{dt} \text{ surface forces} = \iint_{CS} \sigma_{ji} n_j u_i dS$$

Application of the divergence theorem gives:

$$\begin{aligned} \frac{dW}{dt} \text{ surface forces} &= \iiint_{CV} \left[ \frac{\partial}{\partial x_j} (\sigma_{ji} u_i) \right] dV \\ &= \iiint_{CV} \left[ \sigma_{ji} \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial \sigma_{ji}}{\partial x_j} \right] dV \end{aligned}$$

The total rate of work done upon the system is the sum of the body force and surface force contributions. At time  $t$  it obeys:

$$\frac{dW}{dt} = \iiint_{CV} \left[ \rho f_i \text{ (body)} u_i + \sigma_{ji} \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial \sigma_{ji}}{\partial x_j} \right] dV \quad (2.12)$$

#### Rate of Heat Transferred to the System:

The transfer of heat to or from the system may include both *heat conduction* and *heat radiation*. Since any chosen system consists of always the same constituent fluid, there is no *heat convection* possible between a system and its surroundings. (Note that the possibility of heat production by means of chemical or nuclear reactions within the system is not considered here.) By defining  $\mathbf{q}$  as the heat flow per unit area, the rate of heat transferred to the system can be written as:

$$\frac{dQ}{dt} = \iint_{\substack{\text{system} \\ \text{boundary}}} \mathbf{q} \cdot (-\mathbf{n}) dS$$

At time  $t$ :

$$\frac{dQ}{dt} = \iint_{CS} \mathbf{q} \cdot (-\mathbf{n}) dS$$

Using the divergence theorem, this can be converted to a volume integral:

$$\begin{aligned} \frac{dQ}{dt} &= - \iiint_{CV} \nabla \cdot \mathbf{q} dV \\ &= - \iiint_{CV} \frac{\partial q_i}{\partial x_i} dV \end{aligned} \quad (2.13)$$

### First Law Revisited:

Equation (2.10) can now be written out in full incorporating the above developments of the various terms as given by Equations (2.11) to (2.13):

$$\begin{aligned} \frac{dE}{dt} &= \frac{dW}{dt} + \frac{dQ}{dt} \\ \iiint_{CV} \rho \left[ \frac{\partial e}{\partial t} + u_i \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial e}{\partial x_i} \right) \right] dV &= \iiint_{CV} \left[ \rho f_{i(\text{body})} u_i + \sigma_{ij} \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial \sigma_{ji}}{\partial x_j} \right] dV - \iiint_{CV} \frac{\partial q_i}{\partial x_i} dV \\ \Rightarrow \iiint_{CV} \left\{ \rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) - \sigma_{ji} \frac{\partial u_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} + u_i \left[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - \rho f_{i(\text{body})} - \frac{\partial \sigma_{ji}}{\partial x_j} \right] \right\} dV &= 0 \end{aligned}$$

A multiple of the momentum transport equation (Equation 2.8) is embedded in the above equation and this may be removed to yield an equation delineating the transport of the specific internal energy  $e$ . Because the control volume may be chosen arbitrarily, the integrand itself is zero.

$$\begin{aligned} \iiint_{CV} \left[ \rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) - \sigma_{ji} \frac{\partial u_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} \right] dV &= 0 \\ \Rightarrow \rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) - \sigma_{ji} \frac{\partial u_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} &= 0 \\ \Rightarrow \rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) &= \sigma_{ji} \frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} \end{aligned} \quad (2.14)$$

Equation (2.14) describes energy transport during a general fluid motion. It is a partial differential equation which expresses that fluid energy is conserved at every place within the flow.

## 2.5 Auxiliary Equations:

Equations (2.3), (2.8) and (2.14) (all repeated below) describe the transport of fluid mass, linear momentum and energy, respectively, during a general fluid motion. These equations are based upon conservation principles for these flow attributes. Chemical or nuclear reactions have not been included:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_i^{(body)} + \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j}$$

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = \sigma_{ij} \frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

The vector quantities represented in the above equations ( $\mathbf{u}$ ,  $\mathbf{f}$  and  $\mathbf{q}$ ) have three independent components each and the symmetric stress tensor  $\sigma$  has six. With the two scalar quantities  $\rho$  and  $e$ , there is a total of seventeen interdependent scalar unknowns. Consequently the above system of five scalar equations is underdetermined and more information is necessary before a solution is possible.

### Specification of Body Forces:

When the force field represented by  $f_{body}$  is unaffected by the motion of the fluid, as is the case for gravitational body forces, then this force field may be considered in isolation from the resulting fluid motion. The three components of  $f_{body}$  can then be treated as known quantities at all places in the flow. (The situation is more complicated when the flow consists of charged particles, since the motion of each particle alters the electromagnetic force field experienced by neighbouring charged particles.)

### Specification of Stress Components:

It has been mentioned in Section 2.3 that consideration of the principle of conservation of angular momentum during a general fluid motion produces the result that the stress tensor  $\sigma$  is in fact symmetric, i.e.  $\sigma = \sigma^T$  or  $\sigma_{ij} = \sigma_{ji}$ .

The six different unknown components of the stress tensor  $\sigma$  may be determined through their relation to the actual fluid motion which causes the stress. When the stress-

motion relationship is consistent with the following conditions, the fluid is said to be Newtonian (Currie, 1974):

- (1) When the fluid is at rest the stress is *hydrostatic*, and the pressure exerted by the fluid is the same as the thermodynamic pressure.
- (2) The nine components of the stress tensor are linearly related to the components of the *rate-of-strain tensor* (where  $\partial u_k / \partial x_l$  gives the  $kl^{\text{th}}$  component of the rate-of-strain tensor).
- (3) Since there is no shearing action in a solid-body rotation of the fluid, no shear stresses will act during such a motion.
- (4) There is no preferred direction in the flow, so that the fluid properties are point functions.

For a *Newtonian fluid* the relationship between the stress at a point and the local rate of strain is well-known (proof: Feistauer, 1993; § 1.8.7):

$$\sigma_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} \quad (2.15)$$

Equation (2.15) utilises the *Kronecker delta* tensor  $\delta$  which is defined as having value unity when its indices are equal and value zero when they differ, ie.:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The three variables  $p$ ,  $\mu$  and  $\lambda$  introduced by Equation (2.15) are the fluid *pressure*, (*dynamic*) *viscosity* and *second coefficient of viscosity* respectively. These are all intensive properties of a fluid. The terms involving  $\mu$  and  $\lambda$  are called *viscous stresses*. Values for the two viscosity coefficients have been obtained by experimentalists and tabulated for many fluids under a wide variety of conditions.

Utilising Equation (2.15), conservation of linear momentum for a Newtonian fluid can be written as follows:

$$\begin{aligned} \rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) &= \rho f_{i \text{ (body)}} + \frac{\partial}{\partial x_j} \left[ -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] \\ &= \rho f_{i \text{ (body)}} - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + \frac{\partial}{\partial x_i} \left( \lambda \frac{\partial u_k}{\partial x_k} \right) \end{aligned} \quad (2.16)$$

Further simplification is possible if the two viscosity coefficients are essentially constant over the range of conditions under consideration:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_i^{(body)} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\mu + \lambda}{\rho} \frac{\partial^2 u_j}{\partial x_i \partial x_j} \quad (2.17)$$

The ratio of the dynamic viscosity to the fluid density,  $\mu/\rho$ , which appears on the right-hand side of Equation 2.17, is given the name *kinematic viscosity* and is often denoted by  $\nu$ .

#### Specification of Heat Flow:

The heat flow per unit area  $q$  can be decomposed into a *radiation* component and a *conduction* component. According to *Fourier's law*, the conduction component of the heat transfer can be written as  $-\kappa \nabla T$  where  $\kappa$  represents the *coefficient of thermal conductivity* of the fluid and  $T$  is the local temperature; these are both intensive properties of a fluid. Very often the heat flow due to radiation is insignificant compared with that due to conduction, so that the following equation is considered to be an acceptable approximation to the total heat flow:

$$q = -\kappa \nabla T$$

The behaviour of the thermal conductivity of various fluids has been studied experimentally; for some fluid flow situations a sufficiently accurate approximation is obtained when  $\kappa$  is assumed to have a constant value, but generally  $\kappa$  varies a little with the temperature of the fluid. When  $\kappa$  is known from experimental data then Fourier's law introduces just one new unknown, namely the fluid temperature  $T$ , while at the same time yielding three new scalar equations.

By substituting Fourier's law of heat conduction, and also writing out the stress tensor in full using Equation (2.15), the energy equation becomes:

$$\begin{aligned} \rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) &= \left[ -p \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] \frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial x_i} \left( -\kappa \frac{\partial T}{\partial x_i} \right) \\ &= -p \frac{\partial u_i}{\partial x_i} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \lambda \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \frac{\partial}{\partial x_i} \left( \kappa \frac{\partial T}{\partial x_i} \right) \end{aligned} \quad (2.18)$$

If the coefficient of thermal conductivity remains essentially constant over the range of conditions under consideration, then simplification of the last term is possible:

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = -p \frac{\partial u_i}{\partial x_i} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \lambda \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \kappa \frac{\partial^2 T}{\partial x_i \partial x_i} \quad (2.19)$$

### Equations of State:

Most thermodynamic relationships apply to a system which is originally at rest, and then at rest again after some event. Quantities whose values describe the thermodynamic system at rest (before and after such an event) are called *parameters of state* of the system. If at least one parameter of state of a system changes, then the *state* of the system is said to have changed; a thermodynamic *process* has taken place. A process may be thought of as a succession of various states of the given system (Sychev, 1991).

When no external forces act upon the system, the state of a substance is uniquely determined if any two independent intensive parameters are given. Any other parameter is a function of two given parameters. Hence any three parameters of state (e.g. temperature, pressure and density) of a homogeneous substance are uniquely related to each other. The equation that connects any three parameters is called an *equation of state* for a given substance. Although an equation of state applies to a system in thermodynamic equilibrium, at rest and under the influence of no external forces, with care it is possible to adapt such an equation for more general application.

The most widely used parameters of state are temperature, pressure and density. These and several other parameters have appeared already in this chapter. The continuity equation (2.3) mentions the fluid density  $\rho$ ; the momentum transport equations for Newtonian fluids (2.16) also incorporate the pressure  $p$  and two coefficients of viscosity,  $\mu$  and  $\lambda$ ; the energy equation when radiation is negligible (2.18, repeated below) further includes the specific internal energy of the fluid  $e$ , the coefficient of thermal conductivity  $\kappa$  and the temperature  $T$ :

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = -p \frac{\partial u_i}{\partial x_i} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \lambda \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \frac{\partial}{\partial x_i} \left( \kappa \frac{\partial T}{\partial x_i} \right)$$

In many circumstances the coefficients  $\kappa$ ,  $\mu$  and  $\lambda$  can be treated as constants over the range of thermodynamic states which occur during a given flow problem. When this is appropriate, only four parameters of state ( $\rho$ ,  $p$ ,  $e$  and  $T$ ) remain as variables in the system of equations established thus far. Since an equation of state relates any three parameters, two different equations are necessary to relate four parameters.

The System Well-Defined:

Consider that the coefficients  $\kappa$ ,  $\mu$  and  $\lambda$  can be treated as constants. Then the continuity equation (2.3), the constant-viscosity momentum equation (2.17), the energy equation (2.18) and two equations of state (a total of one vector equation and four scalar equations) together are sufficient to solve for  $\mathbf{u}$  (the velocity vector) and the four thermodynamic properties  $\rho, p, e$  and  $T$ .

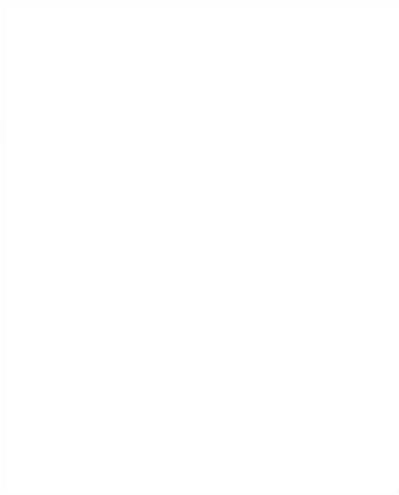
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \quad (2.3)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_i^{(body)} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\mu + \lambda}{\rho} \frac{\partial^2 u_j}{\partial x_i \partial x_j} \quad (2.17)$$

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = -p \frac{\partial u_i}{\partial x_i} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \lambda \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \kappa \frac{\partial^2 T}{\partial x_i \partial x_i} \quad (2.19)$$

$$e = e(\rho, p)$$

$$T = T(\rho, p)$$



### Approximations Valid in Certain Circumstances:

The collection of equations which concludes Chapter Two provides a practical starting point for this chapter, in which a selection of common approximations and simplifications is examined. The equations from Section 2.5 are repeated below:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \quad (2.3)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_i^{(body)} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\mu + \lambda}{\rho} \frac{\partial^2 u_j}{\partial x_i \partial x_j} \quad (2.17)$$

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = -p \frac{\partial u_i}{\partial x_i} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \lambda \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \kappa \frac{\partial^2 T}{\partial x_i \partial x_i} \quad (2.19)$$

$$e = e(\rho, p)$$

$$T = T(\rho, p)$$

It should be remembered that a level of approximation is inherent in these equations already, so that these equations cannot be regarded as the most general equations of fluid dynamics possible. In particular, the above momentum equation (Equation 2.17) is restrictive in that it assumes that the fluid behaves as a *Newtonian fluid*. It also assumes that both coefficients of viscosity remain essentially constant throughout the range of thermodynamic states that occur during the motion under consideration - although it is possible to cater for variable viscosity coefficients by employing Equation (2.16) instead of Equation (2.17) and by adding two more equations of state to the system of equations (refer Section 2.5). Body forces are assumed known; if any are coupled with the actual fluid motion (for example when the fluid consists of electrically charged particles) then additional system equations are required.

The applicability of the energy equation (Equation 2.19) is restricted by the assumption of Newtonian fluid motion. Furthermore, heat transfer via radiation is regarded as negligible in comparison to that due to conduction; the total heat transfer is assumed

adequately described by Fourier's law of heat conduction. Equation (2.19) also assumes that the coefficient of thermal conductivity  $\kappa$  remains essentially constant throughout the range of states under consideration - although it is possible to cater for a variable coefficient of thermal conductivity by employing Equation (2.18) instead and including another equation of state (refer to Section 2.5). No consideration has been given to energy changes arising from chemical or nuclear reactions.

Notwithstanding these limitations, the system of equations comprising Equations (2.3), (2.17), (2.19) and two equations of state is useful for a great number of practical fluid flow problems. Furthermore, it will be shown that many different types of fluid motion rely heavily upon only a few features of these equations. It is worthwhile to discover whether any of these equations can be simplified further for a particular flow situation without compromising the detail of the important characteristics of the overall flow.

Sections 3.1, 3.2 and 3.3 to follow examine some common approximations to the equations of continuity, momentum transport and energy transport respectively. In addition, *Bernoulli equations* (Section 3.2.1) and *vorticity transport equations* (Section 3.2.2) are developed from the momentum transport equation. An *entropy transport equation* (Section 3.3.1) is developed from the energy equation.

Section 3.4 focuses upon the implications of some common thermodynamic approximations and assertions. Of particular importance is the fact that certain equations of state decouple the energy equation from the rest of the system.

Section 3.5 describes simplifications of the general three-dimensional flow equations that can be made by appealing to geometrical symmetries of particular flows.

### 3.1 Approximate Continuity Equations:

The equation of continuity for a general fluid motion, Equation (2.3) derived in Section 2.2, may be expanded using tensor notation as follows:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) &= 0 \\ \Rightarrow \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} &= 0 \end{aligned}$$

In different flow situations, one of the three terms in the above equation may be unimportant in comparison to the others, and a approximate continuity equation may be written by omitting the smaller term. Because the three terms in the above equation sum to zero, then if one term is much smaller than another term, then it is also much smaller than the remaining term.

#### (Quasi-) Steady Flow:

When the flow is steady, all terms which refer explicitly to time may be removed. Derived from Equation (2.3), the following equation expresses conservation of fluid mass in a steady flow:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} &= 0 \\ \frac{\partial \rho}{\partial t} = 0 \Rightarrow u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} &= 0 \end{aligned}$$

If the flow is not strictly steady, the steady flow continuity equation still gives an acceptable approximation when the following remain true:

$$\frac{\partial \rho}{\partial t} \ll u_i \frac{\partial \rho}{\partial x_i} \quad \text{or} \quad \frac{1}{\rho} \frac{\partial \rho}{\partial t} \ll \frac{\partial u_i}{\partial x_i}$$

Only one of these conditions need be tested for any particular flow situation, since the general continuity equation implies that these two conditions are equivalent. (Recall that if one term is much smaller than another term, then it is also much smaller than the remaining term.) If these conditions are met, then the flow might be called *quasi-steady*.

#### Density-Invariant Flow:

When the density of a fluid is uniform and unchanging (i.e.  $\rho = \text{const.}$ ) the equation of continuity (Equation 2.3) takes a particularly simple form:

$$\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

$$\rho = \text{const.} \Rightarrow \frac{\partial u_i}{\partial x_i} = 0$$

This states that the velocity field associated with such a flow has zero divergence. Such a vector field is sometimes termed *solenoidal* (Batchelor, 1967).

By rewriting the general equation of continuity, it can be seen that the condition for a solenoidal velocity field can be relaxed a little:

$$\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

$$\Leftrightarrow \frac{1}{\rho} \frac{D\rho}{Dt} + \frac{\partial u_i}{\partial x_i} = 0$$

$$\therefore \frac{\partial u_i}{\partial x_i} = 0 \quad \text{if} \quad \frac{D\rho}{Dt} = 0$$

This implies that the velocity is solenoidal whenever a chosen material element does not experience any variation in fluid density as it moves. This condition is less restrictive than  $\rho = \text{const.}$ , since it is possible for each material element to have a different value of  $\rho$ .

#### Irrotational Flow:

A velocity potential  $\phi$  can be defined by  $\mathbf{u} = \nabla\phi$  if the velocity field is irrotational (i.e. if  $\nabla \times \mathbf{u} = \mathbf{0}$ ). The equation of continuity (Equation 2.3) for such a flow can be written as

$$\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

$$u_i = \frac{\partial \phi}{\partial x_i} \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x_i} \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial^2 \phi}{\partial x_i \partial x_i} = 0$$

The advantage of this description is that the three unknown scalar components of the velocity vector  $\mathbf{u}$  are replaced by the spatial derivatives of a single unknown scalar function  $\phi$ .

#### Irrotational and Solenoidal Velocity Field:

When the velocity field during a fluid motion is both irrotational (so that  $\mathbf{u} = \nabla\phi$ ) and solenoidal (so that  $\nabla \cdot \mathbf{u} = 0$ ), then the equation of continuity becomes Laplace's equation for the velocity potential:

$$u_i = \frac{\partial \phi}{\partial x_i} \quad \Rightarrow \quad \frac{\partial u_i}{\partial x_i} = 0$$
$$\frac{\partial^2 \phi}{\partial x_i \partial x_i} = 0$$

#### Linearised Continuity Equation:

It is possible to consider fluid motions to be perturbations about some state where the fluid is at rest with uniform fluid density,  $\bar{\rho}$  say. When such density perturbations are sufficiently small in magnitude, the terms of the continuity equation which are of second (or higher) order in fluctuating quantities may be neglected to yield a first-order (i.e. *linear*) approximation to the full continuity equation. This procedure is demonstrated in Section 4.1.

### 3.2 Approximate Momentum Transport:

For Newtonian fluids with constant coefficients of viscosity, Equation (2.17) below describes the transport of fluid momentum that accompanies any general fluid motion:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_i^{(body)} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\mu + \lambda}{\rho} \frac{\partial^2 u_j}{\partial x_i \partial x_j}$$

The left-hand side of this equation describes the rate of change of fluid momentum at position  $\mathbf{x}$  and time  $t$ , while the right-hand side lists the applied forces which act upon the fluid at that position. A natural starting point for consideration of possible simplifications to this equation is to compare the magnitudes of the applied forces. If one or more of the forces is always of insignificant size compared to another, then the smaller can often be removed to yield an approximate momentum transport equation describing essentially the same fluid behaviour.

#### Removal of Body Forces:

It is possible to neglect the effects of electromagnetic body forces when the fluid under consideration does not consist of any charged particles and the motion itself does not generate any electrostatic potential.

A little more care is required to eliminate gravitational forces from the above momentum equation, since all fluid has mass which is subject to the gravitational influence of surrounding matter. According to Newton's Universal Law of Gravitation, the specific force of gravitational attraction experienced by a body at a position  $\mathbf{r}$  in space, due to another body of mass  $m_1$  centred at  $\mathbf{r}_1$ , has magnitude given by:

$$f(\mathbf{r}) = \frac{Gm_1}{|\mathbf{r} - \mathbf{r}_1|^2}$$

The value of the specific body force is calculable directly from this equation, and so comparisons can be made between typical magnitudes of this body force and the surface forces which are in effect during the motion.

When the variation of depth of the fluid somewhere upon the surface of some celestial body is small compared with the radius of the body itself, then the magnitude of the specific gravitational body force vector varies very little throughout the fluid. Its components have dimensions of acceleration.

When all body forces are known to be insignificant in comparison to the other forces which act upon the fluid during its motion, Equation (2.17) reduces to

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\mu + \lambda}{\rho} \frac{\partial^2 u_j}{\partial x_i \partial x_j}$$

### Zero Bulk Viscosity:

When a fluid is at rest, the pressure at a point is measured as (minus) the *mean normal stress* at that place. When the fluid is moving, however, the thermodynamic equilibrium pressure and the mean normal stress are not always coincident in value. To distinguish the two, the mean normal stress is often termed the *mechanical pressure*. The difference between thermodynamic and mechanical pressures can be quantified by utilising Equation (2.15) from Section 2.5:

$$\begin{aligned} \sigma_{ij} &= -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} \\ \Rightarrow \quad \sigma_{ii} &= -3p + (2\mu + 3\lambda) \frac{\partial u_k}{\partial x_k} \\ \Rightarrow \quad \left( -\frac{1}{3}\sigma_{ii} \right) - p &= \left( \frac{2}{3}\mu + \lambda \right) \frac{\partial u_k}{\partial x_k} \end{aligned}$$

The property  $\frac{2}{3}\mu + \lambda$  is known as the *bulk viscosity* of the fluid. For a fluid which has zero bulk viscosity, the thermodynamic and mechanical pressures are always coincident. There is then only one independent viscosity coefficient, since the second coefficient of viscosity is fixed by  $\lambda = -\frac{2}{3}\mu$ ; such a fluid is called a *Stokesian* fluid. For monatomic gases, aptness of this relationship is supported by consideration of the motion at the molecular level. For other real fluids whose bulk viscosity is not zero, the assumption of such is seen to provide an error which depends upon the magnitude of the divergence of the fluid velocity.

For a fluid with zero bulk viscosity, the momentum equation (Equation 2.17) becomes:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_i^{(body)} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{3} \frac{\mu}{\rho} \frac{\partial^2 u_j}{\partial x_i \partial x_j}$$

Zero-Divergence Flow:

For a non-Stokesian fluid, the difference between thermodynamic and mechanical pressures is in general dependent upon the divergence of the fluid velocity. If the velocity field has zero divergence always, then the value of the second coefficient of viscosity is

unimportant to the transport of fluid momentum, since  $\frac{\partial^2 u_j}{\partial x_i \partial x_j}$  is identically zero. The

momentum equation for zero-divergence flow is:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_{i \text{ (body)}} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho} \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Inviscid Fluid:

An inviscid fluid is one whose viscosity  $\mu$  and second coefficient of viscosity  $\lambda$  are both zero. Fluid viscosity contributes to the momentum equation (Equation 2.17) through the following terms:

$$\frac{\mu}{\rho} \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad \text{and} \quad \frac{\mu + \lambda}{\rho} \frac{\partial^2 u_j}{\partial x_i \partial x_j}$$

This suggests that the effects of viscosity upon the momentum transport of the fluid are predominant in places where the fluid velocity changes significantly over small distances. The inviscid momentum equation below is useful for describing the motion away from such places:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_{i \text{ (body)}} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \quad (3.1)$$

Irrotational Flow:

Converting to vector notation, the general equation of momentum transport in a Newtonian fluid with constant coefficients of viscosity (Equation 2.17) becomes

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = \rho \mathbf{f}_{\text{(body)}} - \nabla p + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla(\nabla \cdot \mathbf{u}) \quad (3.2)$$

The following vector identities are useful:

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})$$

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$$

Substitution of these gives:

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}) \right] = \rho \mathbf{f}_{(body)} - \nabla p - \mu \nabla \times (\nabla \times \mathbf{u}) + (2\mu + \lambda) \nabla(\nabla \cdot \mathbf{u})$$

The curl of the velocity, written  $\nabla \times \mathbf{u}$ , is known as the fluid *vorticity* and is denoted  $\boldsymbol{\omega}$ . (The behaviour of the vorticity field is examined further in Section 3.2.2.) When the vorticity of the fluid is everywhere zero, the flow is called *irrotational* and the above momentum equation reduces to:

$$\begin{aligned} \rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) \right] &= \rho \mathbf{f}_{(body)} - \nabla p + (2\mu + \lambda) \nabla(\nabla \cdot \mathbf{u}) \\ &= \rho \mathbf{f}_{(body)} - \nabla p + (2\mu + \lambda) \nabla^2 \mathbf{u} \end{aligned} \quad (3.3)$$

#### (Quasi-) Steady Flow:

When a flow is steady, then the terms explicitly involving the temporal variable  $t$  may be removed from any of the various forms of the momentum transport equation discussed already. A flow might be termed *quasi-steady* when, although not strictly time-invariant, removal of the time-dependent term from the momentum equation nonetheless produces an acceptable description of the fluid motion. If the condition  $\frac{\partial \mathbf{u}}{\partial t} \ll (\mathbf{u} \cdot \nabla) \mathbf{u}$  is satisfied, then a steady description of momentum transport is adequate. The momentum transport equation (Equation 2.17) then takes the form

$$\rho(\mathbf{u} \cdot \nabla) \mathbf{u} = \sum \mathbf{F}$$

### 3.2.1 Bernoulli Equations:

Momentum transport for an inviscid fluid (Equation 3.1) is written in vector notation as follows:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mathbf{f}_{(body)} - \frac{1}{\rho} \nabla p$$

The vector identity  $(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})$  is useful, leading to:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}) - \mathbf{f}_{(body)} + \frac{1}{\rho} \nabla p = 0$$

The scalar product of the above equation with an arbitrary displacement vector  $d\mathbf{r}$  gives:

$$\left[ \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}) - \mathbf{f}_{(body)} + \frac{1}{\rho} \nabla p \right] \cdot d\mathbf{r} = 0$$

If the specific body force is conservative, so that it may be written in terms of the gradient of a scalar potential (e.g.  $f_{(body)} = -\nabla\Psi$ ), then three of the above terms may be simplified by noting that, for a general scalar function  $\zeta$ , the identity  $\nabla\zeta \cdot d\mathbf{r} = d\zeta$  holds. The above equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} \cdot d\mathbf{r} + \frac{1}{2} d(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}) \cdot d\mathbf{r} - d\Psi + \frac{1}{\rho} dp = 0$$

The term  $\mathbf{u} \times (\nabla \times \mathbf{u}) \cdot d\mathbf{r}$  in this equation is equal to zero whenever the displacement vector  $d\mathbf{r}$  is parallel to the velocity vector  $\mathbf{u}$  (since  $\mathbf{u} \cdot \mathbf{u} \times \boldsymbol{\omega} = 0$  for arbitrary  $\mathbf{u}$  and  $\boldsymbol{\omega}$ ).

A locus of points which is everywhere parallel to  $\mathbf{u}$  is called a *streamline* of the flow. By integrating the above equation along such a streamline, the following *Bernoulli equation* results:

$$\int_A^B \frac{\partial u}{\partial t} dl + \frac{1}{2} [u^2]_A^B + [\Psi]_A^B + \int_A^B \frac{dp}{\rho} = 0 \text{ along a streamline,} \quad (3.4)$$

where  $u = |\mathbf{u}|$  and  $dl$  is the arc-length along the streamline.

Streamlines can be calculated from knowledge of the velocity field: since every vector arc-length  $d\mathbf{r}$  along a streamline must be tangent to  $\mathbf{u}$ , their respective components must be in exact proportion:

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3} = \frac{dl}{u} \quad (3.5)$$

#### Steady Bernoulli Equation:

For (quasi-)steady flows, the first term in Equation (3.4) is identically zero.

#### Constant-Density Bernoulli Equation:

When the fluid density is constant along a streamline, it may be taken outside the integral in the final term of Equation (3.4):

$$\int_A^B \frac{\partial u}{\partial t} dl + \frac{1}{2} [u^2]_A^B + [\Psi]_A^B + \frac{1}{\rho} [p]_A^B = 0$$

### Constant-Density, Steady Bernoulli Equation:

A particularly simple version of the Bernoulli equation is valid for inviscid flows, under the influence of conservative body forces, when the fluid density remains invariant along a streamline and the flow is steady. Equation (3.4) becomes

$$\frac{1}{2}[u^2]_A^B + [\Psi]_A^B + \frac{1}{\rho}[p]_A^B = 0$$

$$\text{or } \frac{1}{2}u^2 + \Psi + \frac{p}{\rho} = \text{const. along a streamline}$$

### Bernoulli Equations for Irrotational Flows:

Recall the general momentum transport equation for an irrotational fluid motion (Equation 3.3):

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) \right] = \rho \mathbf{f}_{(body)} - \nabla p + (2\mu + \lambda) \nabla(\nabla \cdot \mathbf{u})$$

For an irrotational motion it is possible to write the velocity field as the gradient of some scalar potential  $\phi$  so that  $\mathbf{u} = \nabla\phi$ . From the above equation, it is seen that if the fluid is either inviscid or solenoidal, and any body forces conservative, then momentum transport during irrotational fluid motion obeys

$$\frac{\partial}{\partial t}(\nabla\phi) + \frac{1}{2} \nabla(\nabla\phi \cdot \nabla\phi) + \nabla\Psi + \frac{1}{\rho} \nabla p = 0$$

Taking the scalar product of the above equation with an arbitrary displacement vector  $d\mathbf{r}$ , and invoking the identity  $\nabla\zeta \cdot d\mathbf{r} = d\zeta$  for general  $\zeta$ , gives

$$d\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}d(\nabla\phi \cdot \nabla\phi) + d\Psi + \frac{dp}{\rho} = 0 \quad (3.6)$$

This is the (unsteady) irrotational Bernoulli equation. There is no longer a restriction that the equation be applied along a streamline of the flow; the above equation is valid along any path in an irrotational flow which is either inviscid or solenoidal. Integration of the above equation gives

$$\left[ \frac{\partial\phi}{\partial t} \right]_A^B + \left[ \frac{1}{2}(\nabla\phi \cdot \nabla\phi) \right]_A^B + [\Psi]_A^B + \int_A^B \frac{dp}{\rho} = 0$$

If, along any path, the fluid density is invariant, then density may be taken outside the integral of the final term in the above equation. Along such a path the following equation holds:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi \cdot \nabla \phi) + \Psi + \frac{p}{\rho} = \text{const.}$$

For (quasi-)steady flows, the first of the terms in Equation (3.6) is identically zero, so that, along any path in the fluid

$$\frac{1}{2}[\mathbf{u} \cdot \mathbf{u}]_A^B + [\Psi]_A^B + \int_A^B \frac{dp}{\rho} = 0$$

### 3.2.2 Vorticity Transport:

Using vector notation the momentum equation for fluids under the influence of only conservative body forces can be written as (from Equation 3.2)

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\rho \nabla \Psi - \nabla p + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla(\nabla \cdot \mathbf{u})$$

Substitution of the vector identity  $(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) + (\nabla \times \mathbf{u}) \times \mathbf{u}$  gives

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla \Psi + \frac{1}{\rho} [-\nabla p + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla(\nabla \cdot \mathbf{u})]$$

Consider now the curl of the above equation, while noting the vector identities  $\nabla \times \nabla \zeta = \mathbf{0}$  and  $\nabla \times (\zeta \mathbf{a}) = \zeta \nabla \times \mathbf{a} + \nabla \zeta \times \mathbf{a}$  for an arbitrary scalar  $\zeta$  and arbitrary vector  $\mathbf{a}$ .

$$\begin{aligned} \nabla \times \frac{\partial \mathbf{u}}{\partial t} + \nabla \times [(\nabla \times \mathbf{u}) \times \mathbf{u}] &= \frac{1}{\rho} \nabla \times [-\nabla p + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla(\nabla \cdot \mathbf{u})] \\ &\quad - \frac{1}{\rho^2} \nabla \rho \times [-\nabla p + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla(\nabla \cdot \mathbf{u})] \\ &= \frac{\mu}{\rho} \nabla \times \nabla^2 \mathbf{u} - \frac{1}{\rho^2} \nabla \rho \times [-\nabla p + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla(\nabla \cdot \mathbf{u})] \end{aligned}$$

The curl of the velocity field is the vorticity field, denoted  $\boldsymbol{\omega}$ . The first three terms in the above equation can be expressed more conveniently in terms of the vorticity as follows:

$$\begin{aligned} \nabla \times \frac{\partial \mathbf{u}}{\partial t} &= \frac{\partial}{\partial t} (\nabla \times \mathbf{u}) = \frac{\partial \boldsymbol{\omega}}{\partial t} \\ \nabla \times [(\nabla \times \mathbf{u}) \times \mathbf{u}] &= \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) \\ &= \boldsymbol{\omega}(\nabla \cdot \mathbf{u}) - \mathbf{u}(\nabla \cdot \boldsymbol{\omega}) + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} \\ \frac{\mu}{\rho} \nabla \times (\nabla^2 \mathbf{u}) &= \frac{\mu}{\rho} \nabla^2 (\nabla \times \mathbf{u}) = \frac{\mu}{\rho} \nabla^2 \boldsymbol{\omega} \end{aligned}$$

Substitution of these results in a general vorticity transport equation valid whenever all body forces acting upon the fluid are conservative

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} - \boldsymbol{\omega} (\nabla \cdot \mathbf{u}) + \frac{\mu}{\rho} \nabla^2 \boldsymbol{\omega} - \frac{1}{\rho^2} \nabla \rho \times \left[ -\nabla p + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla (\nabla \cdot \mathbf{u}) \right] \quad (3.7)$$

#### Constant-Density Fluid:

When the density is constant  $\nabla \rho = 0$ . Furthermore, the equation of continuity (Equation 2.3) stipulates that a constant value for the fluid density demands solenoidality of the velocity field, i.e.  $\nabla \cdot \mathbf{u} = 0$ . For fluids of constant density then, Equation (3.7) becomes

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + \frac{\mu}{\rho} \nabla^2 \boldsymbol{\omega}$$

#### Inviscid Fluid:

When both coefficients of viscosity,  $\mu$  and  $\nu$ , of a given fluid are zero Equation (3.7) becomes

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} - \boldsymbol{\omega} (\nabla \cdot \mathbf{u}) + \frac{1}{\rho^2} \nabla \rho \times \nabla p$$

A further simplification is available if a thermodynamic relationship exists between pressure and density alone, i.e.  $p = p(\rho)$ . Such a fluid motion is called *barotropic*. (Barotropy is discussed more fully in Section 3.4.) In a barotropic fluid motion, the vectors  $\nabla p$  and  $\nabla \rho$  are in the same direction always, so that  $\nabla \rho \times \nabla p = \mathbf{0}$  and the final term in the above equation can also be removed:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} - \boldsymbol{\omega} (\nabla \cdot \mathbf{u})$$

### 3.3 Approximate Energy Equations:

In the absence of chemical reactions, and when heat radiation is negligible, the specific internal energy within a Newtonian fluid with constant coefficients of viscosity is transported according to the Equation (2.19) repeated here:

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = -p \frac{\partial u_i}{\partial x_i} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \lambda \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \kappa \frac{\partial^2 T}{\partial x_i \partial x_i}$$

#### Zero Bulk Viscosity:

If a fluid has zero *bulk viscosity* (Section 3.2) then  $\lambda = -\frac{2}{3}\mu$  and the energy equation becomes

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = -p \frac{\partial u_i}{\partial x_i} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} - \frac{2}{3}\mu \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \kappa \frac{\partial^2 T}{\partial x_i \partial x_i}$$

If the fluid is not Stokesian, then the difference between  $\lambda$  and  $-\frac{2}{3}\mu$  is proportional to the divergence of the velocity field.

#### Inviscid Fluid:

When both coefficients of viscosity are zero, then the energy equation (Equation 2.19) becomes

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = -p \frac{\partial u_i}{\partial x_i} + \kappa \frac{\partial^2 T}{\partial x_i \partial x_i}$$

#### Solenoidal Velocity Field:

If the velocity field during certain fluid motion has zero divergence throughout, then the following energy equation follows from Equation (2.19):

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \kappa \frac{\partial^2 T}{\partial x_i \partial x_i}$$

#### Adiabatic Motion:

A thermodynamic process is said to be *adiabatic* when there is no exchange of heat between the system and its surroundings. When the motion of an infinitesimal material

element of the fluid is adiabatic the coefficient of thermal conductivity is zero, and the heat conduction term of the energy equation is identically zero. Equation (2.19) becomes

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = -p \frac{\partial u_i}{\partial x_i} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \lambda \left( \frac{\partial u_i}{\partial x_i} \right)^2$$

A similar energy equation applies if the Laplacian of the fluid temperature is zero, i.e.  $\nabla^2 T = 0$ . Since  $\nabla^2 T = \nabla \cdot (\nabla T)$  this condition is assured, for example, if the temperature has zero gradient everywhere, i.e. if the fluid motion is *isothermal*.

### 3.3.1 Entropy Transport:

Recall that a thermodynamic *process* may be thought of as a succession of various *states* of a given system, describing the evolution of the system from *initial state* until some later *final state*. If this succession of states can also be traversed in the opposite order, i.e. from the final state to the initial state, then the process is said to be *reversible* (Lewis & Randall, 1961).

Thermodynamic quantities can be divided into two categories: *process functions* and *state functions*. The change in the value of a process function due to some event depends upon the path of the process from the initial to the final states; heat and work are both process functions. The change in value of any state function after some event depends only upon the initial and final states of the system; all parameters of state are state functions (Sychev, 1991).

The concept of fluid *entropy*, denoted  $S$ , which is an extensive parameter of state of a system, may be introduced via the second law of thermodynamics. For a reversible process, this law is represented by the equality  $T dS = dQ$ , where  $Q$  represents heat added to the system.

During a process which is adiabatic  $dQ = 0$ , since no heat is exchanged between the system and its surroundings. So for a process which is both reversible and adiabatic, the equation  $dS = 0$  holds; such a process is said to be *isentropic*. By regarding each material element of fluid as an infinitesimal system, it is possible to describe isentropic fluid motion as motion which obeys the equation  $\frac{Ds}{Dt} = 0$ , where  $s$  represents the *specific entropy* of the fluid at a point.

In a general isentropic motion, each material element may have its own constant value of entropy; if the value is the same for all fluid elements then the motion is said to be *homentropic* (Batchelor, 1967).

The first law of thermodynamics states that any change in energy of a system is equal to the sum of the work done upon that system and the heat added to it:

$$dE = dW + dQ$$

If a reversible process is studied the second law may be used to replace the process function  $Q$  by state functions  $T$  and  $S$ . (The equality  $T dS = dQ$  is only true for reversible processes.) When work of compression is concurrently performed upon the system, and when this work is also carried out reversibly, then the amount of this work is given by  $dW = -p dV$  where  $dV$  represents the change in system volume during the compression. For such a reversible process, comprising work  $dW$  and heat transfer  $dQ$ , the first and second laws combine to give

$$dE = T dS - p dV$$

All of the quantities in this expression are now state functions; there are no process functions remaining. This indicates that the above equation is valid for any general process, reversible or irreversible.

The above equation can be rewritten as follows, using specific quantities now instead, for a material element of fluid, noting that *specific volume* is the same as the reciprocal of mass density.

$$\frac{De}{Dt} = T \frac{Ds}{Dt} - p \frac{D}{Dt} \left( \frac{1}{\rho} \right) \quad (3.8)$$

The equation of continuity (Equation 2.3) leads to

$$\frac{D}{Dt} \left( \frac{1}{\rho} \right) = \frac{1}{\rho} \nabla \cdot \mathbf{u}$$

These two results, substituted into the energy transport equation (Equation 2.19) lead to the *entropy transport equation*:

$$\rho \left( \frac{\partial s}{\partial t} + u_i \frac{\partial s}{\partial x_i} \right) = \frac{\mu}{T} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \frac{\lambda}{T} \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \frac{\kappa}{T} \frac{\partial^2 T}{\partial x_i \partial x_i}$$

As with the energy equation discussed above, simpler variations of the entropy equation can also be written. In particular, isentropic motion, which obeys  $\frac{Ds}{Dt} = 0$ , is seen from the above equation to be equivalent to adiabatic or isothermal motion of a inviscid fluid.

### 3.4 Approximate Thermodynamic Relations:

Recall that when no external forces act upon a system in thermodynamic equilibrium, its state is uniquely defined if values of two independent intensive parameters are given; it is accepted that equilibrium thermodynamics (of a homogeneous system) is a science of two independent variables (Sychev, 1991). Each equation of state thus relates a dependent variable to two independent variables. The two independent variables may be chosen arbitrarily. If fluid pressure  $p$  and entropy  $s$  are chosen as the two independent parameters, for example, then any changes in the fluid density  $\rho$  obey

$$d\rho = \left( \frac{\partial \rho}{\partial p} \right)_s dp + \left( \frac{\partial \rho}{\partial s} \right)_p ds \quad (3.9)$$

The subscript following a bracketed derivative here designates which parameter is to be held constant during the evaluation of the partial derivative. This shows which two of all possible parameters of state have been chosen as the two independent variables.

#### Isentropic and Homentropic Flows:

Consider the Equation (3.9) for the special case of an isentropic process. Substituting

$ds = 0$ , the equation becomes  $d\rho = \left( \frac{\partial \rho}{\partial p} \right)_s dp$ . It is common to adopt the notation

$$c^{-1} = \sqrt{\left( \frac{\partial \rho}{\partial p} \right)_s} \quad (3.10)$$

The quantity  $c$  is another thermodynamic parameter of state; for example,  $c = c(p, s)$  when pressure and entropy are chosen as the two independent variables. Discussion of the physical interpretation of  $c$  is deferred until Chapter Four. In terms of  $c$ , an isentropic process is seen to obey (from Equation 3.9)

$$d\rho = \frac{1}{c^2} dp$$

If the thermodynamic system under study is a material element moving within a fluid, then the following two equations are equivalent:

$$\frac{Ds}{Dt} = 0 \qquad \frac{D\rho}{Dt} = \frac{1}{c^2} \frac{Dp}{Dt}$$

Recall that, in general, the quantity  $c$  is dependent upon the local values of two thermodynamic parameters of state, for example fluid pressure and entropy. If the flow

has a constant value of entropy throughout, it is said to be *homentropic*. So if  $c = c(p, s)$  in general, then  $c$  for a homentropic motion depends only upon  $p$ .

### Barotropic Motion:

For a homentropic flow, the relationship  $d\rho = \frac{1}{c^2} dp$  relates only two parameters of state, since, for such a flow, the value of  $c$  depends only upon  $\rho$ . As such, a homentropic flow is an example of a *barotropic flow*: a flow in which pressure can be expressed as a function of density only, without reference to any other parameter of state.

Barotropy of a fluid motion makes solution considerably easier, since the velocity field may then be solved without recourse to the energy equation: the following system of equations - comprising a continuity equation (Equation 2.3), momentum transport equations (Equation 2.17) and a barotropic equation of state - is well-defined when  $\mu$  and  $\lambda$  can be regarded as known constants:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_i^{(body)} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\mu + \lambda}{\rho} \frac{\partial^2 u_j}{\partial x_i \partial x_j}$$

$$p = p(\rho)$$

### Linearised Barotropic Relation:

An approximation to a barotropic relation  $p = p(\rho)$  for a particular flow may be obtained by expanding as a single-variable Taylor series about some reference density  $\bar{\rho}$  and truncating this series after a finite number of terms. This procedure is demonstrated in Section 4.1.

### Incompressible Flow:

Strictly speaking, a fluid is *incompressible* when the density of an element of fluid is not affected by changes in fluid pressure. This may be written mathematically as  $\frac{\partial \rho}{\partial p} = 0$ .

If the two independent parameters of state of a thermodynamic system are chosen as fluid pressure  $p$  and entropy  $s$ , for example, then

$$\frac{d\rho}{dp} = \left( \frac{\partial \rho}{\partial p} \right)_s + \left( \frac{\partial \rho}{\partial s} \right)_p \frac{ds}{dp}$$

Multiplying by  $dp$  and recalling the definition given already for  $c$  (Equation 3.10) leads to

$$d\rho = \frac{1}{c^2} dp + \left( \frac{\partial \rho}{\partial s} \right)_p ds$$

This equation implies that a fluid will behave as incompressible if the value of  $c$  is infinite throughout the motion. If in addition the motion is isentropic, then  $ds = 0$ . (Note that the term “incompressible” is often used more loosely to infer such density invariance without the additional requirement of isentropy.)

When  $ds = 0$ , any material element moving with the fluid obeys  $\frac{D\rho}{Dt} = 0$ . The equation of continuity (Equation 2.3) shows that this material derivative varies proportionally with the divergence of the velocity field. Consequently an incompressible, isentropic fluid motion has a solenoidal velocity field.

#### Constant Viscosities, Thermal Conductivity, Specific Heats:

Sometimes a given parameter of state is regarded as constant and yet an independent equation of state relating three other state parameters is still presumed available. The apparent anomaly serves as a reminder that it is important to establish which parameters are to be regarded as strictly constant (such as entropy in homentropic flow, and temperature in isothermal flow) and which are merely approximately constant over some specified range of conditions. When too many strict conditions are imposed upon the thermodynamic parameters of a fluid system, the resulting collection of equations may become over-determined and impossible to solve.

In the momentum and energy transport equations (Equations 2.17 and 2.19, respectively), parameters  $\mu$  and  $\lambda$  (viscosity coefficients) and  $\kappa$  (thermal conductivity) have been regarded as constants.

The use of the constant viscosity version of the momentum equation (Equation 2.17 rather than Equation 2.16) does not actually require that the coefficients of viscosity are strictly constant (i.e. that  $d\mu = 0$ ,  $d\lambda = 0$ ), but rather that the viscosity variations are so small over some range of conditions that the effect of these variations makes very little difference to the overall momentum transport.

Similarly, the thermal conductivity of the fluid need not be strictly constant in order to choose Equation (2.19) over (2.18), since some small variations may still have negligible effect upon transport of energy.

Thus Equations (2.17) and (2.19) can still be used alongside an equation of state relating three other state parameters..

The *specific heats* of a fluid are also often regarded as being approximately constant over a range of conditions. A specific heat is the amount of heat given to unit mass of a fluid per unit rise in temperature in a small reversible change. It is not determined uniquely until the conditions are further specified under which the reversible change occurs. The *principal* specific heats are the *specific heat at constant volume*  $c_v$  and the *specific heat at constant pressure*  $c_p$ . These are defined by

$$c_v = \left( \frac{\partial q}{\partial T} \right)_v \quad c_p = \left( \frac{\partial q}{\partial T} \right)_p$$

Since  $dq = de + p dv$  for a small reversible change (Section 3.3.1), these principal specific heats can also be written as

$$c_v = \left( \frac{\partial e}{\partial T} \right)_v \quad c_p = \left( \frac{\partial e}{\partial T} \right)_p + p \left( \frac{\partial v}{\partial T} \right)_p \quad (3.11)$$

### Perfect Gas:

The most famous equation of state is probably that written for a *perfect gas*. The equation relates the fluid pressure  $p$ , density  $\rho$  and temperature  $T$ .

$$p = R\rho T \quad (3.12)$$

In this equation,  $R$  is a universal gas constant divided by the mean molecular weight of the gas or gas mixture.

The molecules of a perfect gas exert no force upon each other except at occasional collisions, and have negligible volume. Thus a perfect gas is a material for which the internal energy is the sum of the separate energies of the molecules, i.e. independent of  $\rho$ . Consequently the specific heat at constant volume for a perfect gas (cf. Equation 3.11) can be written in terms of the following ordinary derivative:

$$c_v = \frac{de}{dT}$$

### 3.5 Simplifications for Certain Flow Geometries:

Recall the general continuity equation (Equation 2.3) as derived in Section 2.2:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

In three dimensions, this relation equates the sum of four terms to zero. Much of the following discussion centres upon the concept of a *streamfunction*, which is available when the flow is such that two of these terms are identically zero.

Several different definitions of streamfunction are employed in the following discussion, but the same symbol  $\psi$  will be used for each case.

#### Two-Dimensional (Planar) Flow:

Consider motion which, in a Cartesian co-ordinate system, is independent of the  $x_3$  co-ordinate and perpendicular to the  $x_3$ -axis. The velocity of the fluid is then given by

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$$

A common shorthand is to write  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$  since the third component is always zero. The

corresponding vorticity vector, defined as  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ , is given by

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \end{pmatrix} \quad (3.12)$$

The single non-zero component of the vorticity vector  $\boldsymbol{\omega}$  is often denoted  $\omega$ .

For two-dimensional motion, the scalar product  $\mathbf{u} \cdot \boldsymbol{\omega}$  is identically zero, and so is the term  $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$  which appears in the vorticity transport equations of Section 3.2.2. For a general two-dimensional motion, vorticity transport is described by (cf. Equation 3.7):

$$\frac{D\omega}{Dt} = -\omega(\nabla \cdot \mathbf{u}) + \frac{\mu}{\rho} \nabla^2 \omega - \frac{1}{\rho^2} \nabla \rho \times [-\nabla p + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla(\nabla \cdot \mathbf{u})]$$

#### Case of Steady 2-D Flow:

For a steady 2-D fluid motion, the equation of continuity (Equation 2.3) simplifies to

$$\frac{\partial}{\partial x_1}(\rho u_1) + \frac{\partial}{\partial x_2}(\rho u_2) = 0$$

A *streamfunction*  $\psi$  may be defined such that

$$\rho u_1 = \frac{\partial \psi}{\partial x_2} \quad \text{and} \quad \rho u_2 = -\frac{\partial \psi}{\partial x_1}$$

Substitution reveals that such a streamfunction satisfies the continuity equation automatically. An arbitrary constant may be added to  $\psi$  without effect.

#### Case of Solenoidal 2-D Flows:

For (steady or unsteady) solenoidal 2-D fluid motion, Equation (2.3) becomes

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \quad (3.13)$$

A streamfunction can again be defined - this time such that

$$u_1 = \frac{\partial \psi}{\partial x_2} \quad \text{and} \quad u_2 = -\frac{\partial \psi}{\partial x_1} \quad (3.14)$$

Any such streamfunction satisfies the continuity equation (Equation 3.13) automatically. An arbitrary function of time may be added to  $\psi$  without effect. In this case, it is the two velocity components  $u_1$  and  $u_2$  which have been replaced by the spatial derivatives of a scalar function  $\psi$ .

It is possible to show that the value of the streamfunction  $\psi$  remains constant along a *streamline* of the flow field (defined in Section 3.2.1). From Equation (3.5), the equation of a streamline in a two-dimensional flow is

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2}$$

or  $u_1 dx_2 - u_2 dx_1 = 0$

Substitution of Equation (3.14) gives

$$\frac{\partial \psi}{\partial x_2} dx_2 + \frac{\partial \psi}{\partial x_1} dx_1 = 0$$

The left-hand side of this equation is simply the total derivative  $d\psi$ . Thus  $d\psi = 0$ , or  $\psi$  is constant, along a streamline.

Case of Solenoidal and Irrotational 2-D Flows:

A streamfunction  $\psi$  has been defined for solenoidal 2-D fluid motions by Equations (3.14) above. If the velocity field is also irrotational, then  $\nabla \times \mathbf{u} = \mathbf{0}$ . From Equation (3.12), irrotationality implies that

$$\begin{aligned} \frac{\partial}{\partial x_2} \left( \frac{\partial \psi}{\partial x_2} \right) - \frac{\partial}{\partial x_1} \left( -\frac{\partial \psi}{\partial x_1} \right) &= 0 \\ \Rightarrow \nabla^2 \psi &= 0 \end{aligned}$$

It is apparent that the streamfunction of a solenoidal, irrotational 2-D motion satisfies Laplace's equation; such a function is said to be *harmonic*.

It has been mentioned already that a velocity potential  $\phi$  can be defined for any velocity field which is irrotational (i.e. if  $\nabla \times \mathbf{u} = \mathbf{0}$ ). For two-dimensional flows, the velocity potential is defined by

$$u_1 = \frac{\partial \phi}{\partial x_1} \quad \text{and} \quad u_2 = \frac{\partial \phi}{\partial x_2} \quad (3.15)$$

Like  $\psi$ ,  $\phi$  is determined only to within an arbitrary function of time.

When the flow is solenoidal as well as irrotational, so that Equation (3.13) applies, a second Laplace's equation results, this time for the velocity potential:

$$\begin{aligned} \frac{\partial}{\partial x_1} \left( \frac{\partial \phi}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \frac{\partial \phi}{\partial x_2} \right) &= 0 \\ \Rightarrow \nabla^2 \phi &= 0 \end{aligned}$$

By comparing Equations (3.14) and (3.15), it is seen that

$$\frac{\partial \phi}{\partial x_1} = \frac{\partial \psi}{\partial x_2} \quad \text{and} \quad \frac{\partial \phi}{\partial x_2} = -\frac{\partial \psi}{\partial x_1}$$

These are the well-known *Cauchy-Riemann equations* of complex variable theory. Since they apply, it follows that  $\phi + i\psi$  is an *analytic function* of a complex variable  $z = x_1 + ix_2$ . The quantity  $w(z) = \phi + i\psi$  is called the *complex potential* of the fluid motion.

The solution method of so-called *potential flows* centres upon finding the complex potential for the flow in question; the fluid velocity can be recovered from knowledge of the complex potential through differentiation and subsequent separation into real and imaginary parts:

$$\frac{dw}{dz} = \frac{\partial\phi}{\partial x_1} + i \frac{\partial\psi}{\partial x_1} = -i \frac{\partial\phi}{\partial x_2} + \frac{\partial\psi}{\partial x_2} = u_1 - i u_2$$

The quantity  $u_1 - i u_2$  is known as the *complex velocity* of the flow.

Conformal mappings of analytic functions may allow the transformation of a complicated flow pattern to yield a simpler flow pattern in the transformed complex plane.

#### Case of Planar 2-D Flow in Polar Co-ordinates:

In a cylindrical  $(r, \theta, z)$  co-ordinate system, the continuity equation (Equation 2.3) for a general fluid motion is

$$\begin{aligned} & \frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0 \\ \Rightarrow & \frac{\partial\rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r\rho u_r) + \frac{1}{r} \frac{\partial}{\partial\theta} (\rho u_\theta) + \frac{\partial}{\partial z} (\rho u_z) = 0 \end{aligned} \quad (3.16)$$

When the motion is independent of the  $z$  co-ordinate and perpendicular to the  $z$ -axis, then the final term of the above equation is not required. When the velocity field is also solenoidal, Equation (3.16) becomes

$$\frac{\partial}{\partial r} (r u_r) + \frac{\partial u_\theta}{\partial\theta} = 0$$

A suitable streamfunction can be defined by

$$u_r = -\frac{1}{r} \frac{\partial\psi}{\partial\theta} \quad \text{and} \quad u_\theta = \frac{\partial\psi}{\partial r}$$

If the velocity field is irrotational, then the velocity potential is defined by

$$u_r = \frac{\partial\phi}{\partial r} \quad \text{and} \quad u_\theta = \frac{1}{r} \frac{\partial\phi}{\partial\theta}$$

#### Axially Symmetrical 3-D Flows:

The continuity equation for a cylindrical co-ordinate system has been stated already (Equation 3.16) and a streamfunction defined for the case of solenoidal flows perpendicular to the  $z$ -axis. Another useful streamfunction can be developed in cylindrical co-ordinates, this time when the flow is three-dimensional but axially symmetric (about the  $z$ -axis) such that the fluid velocity is everywhere independent of the  $\theta$  co-ordinate. When such a flow is also solenoidal, then a simplified continuity equation follows from Equation (3.16):

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial u_z}{\partial z} = 0$$

The appropriate streamfunction is defined by

$$u_r = \frac{1}{r} \frac{\partial \Psi}{\partial z} \quad \text{and} \quad u_z = -\frac{1}{r} \frac{\partial \Psi}{\partial r}$$

If the velocity field is irrotational, then the velocity potential is defined by

$$u_r = \frac{\partial \phi}{\partial r} \quad \text{and} \quad u_z = \frac{\partial \phi}{\partial z}$$

Axially symmetric fluid motions can also be conveniently described in a spherical co-ordinate system.

### Unidirectional Solenoidal Flow:

Consider that the velocity vector of a fluid motion (in a long cylindrical region, perhaps) has the same direction everywhere, and that the  $x_1$  axis of a Cartesian co-ordinate system is aligned with the flow direction so that the velocity of the fluid can be written as

$$\mathbf{u} = \begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix}$$

If the flow is solenoidal, so that  $\nabla \cdot \mathbf{u} = 0$ , then the value of  $u_1$  is independent of the distance  $x_1$  in the flow direction, i.e.  $u_1 = u_1(x_2, x_3, t)$ . The vorticity is given by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \begin{pmatrix} 0 \\ \frac{\partial u_1}{\partial x_3} \\ -\frac{\partial u_1}{\partial x_2} \end{pmatrix}$$

As was the case for two-dimensional fluid motions, the term  $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$ , which features in the vorticity transport equation (Equation 3.5), is identically zero.

The term  $(\mathbf{u} \cdot \nabla)\mathbf{u}$  which features in the momentum transport equation (Equation 2.17) is also identically zero. The resulting momentum transport equation for a unidirectional solenoidal flow is linear:

$$\begin{aligned}\rho \frac{\partial u_1}{\partial t} &= \rho f_{1 \text{ (body)}} - \frac{\partial p}{\partial x_1} + \mu \left( \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \right) \\ \rho f_{2 \text{ (body)}} &= \frac{\partial p}{\partial x_2} \\ \rho f_{3 \text{ (body)}} &= \frac{\partial p}{\partial x_3}\end{aligned}$$

### One-Dimensional Flow:

Consider motion which, in a Cartesian co-ordinate system, is independent of the  $x_2$  and  $x_3$  co-ordinates and always parallel to the  $x_1$  axis. Then the velocity of the fluid is

$$\mathbf{u} = \begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where } u_1 = u_1(x_1, t).$$

The curl of the velocity field is identically zero, i.e. the flow is irrotational.

Using  $u$  and  $x$  as shorthand for  $u_1$  and  $x_1$ , respectively, the continuity equation simplifies to (cf. Equation 2.3)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \quad (3.17)$$

In the absence of body forces, the appropriate momentum transport equations (cf. Equation 2.17) become

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x} + (2\mu + \lambda) \frac{\partial^2 u}{\partial x^2} \quad (3.18)$$

The energy transport equation (Equation 2.19) becomes

$$\rho \left( \frac{\partial e}{\partial t} + u \frac{\partial e}{\partial x} \right) = -p \frac{\partial u}{\partial x} + (2\mu + \lambda) \left( \frac{\partial u}{\partial x} \right)^2 + \kappa \frac{\partial^2 T}{\partial x^2} \quad (3.19)$$

### Radial Flow:

Consider motion which, in a spherical co-ordinate system, has velocity that is at every point in a radial direction, with magnitude independent of both angular co-ordinates. The resulting equations are almost as simple as those given above for one-dimensional flows. The velocity may be written as

$$\mathbf{u} = \begin{pmatrix} u_r \\ 0 \\ 0 \end{pmatrix}, \quad \text{where } u_r = u_r(r, t)$$

Again the curl of the velocity field is identically zero, i.e. the flow is irrotational.

## Study of

### an “Ideal” Modulated Flow in 1-D

This chapter describes the application of several standard mathematical techniques to the investigation of an “ideal” modulated flow in one dimension.

The motion is considered to be inviscid, adiabatic and unaffected by body forces (such as gravity). The investigation is of a specific flow problem defined as follows:

The flow is considered to be one-dimensional in nature, and solution is sought for the domain of  $x \geq 0, t \geq 0$ . A flow perturbation is introduced at the position  $x = 0$ , and behaviour of this perturbation is known at all times  $t \geq 0$ . The fluid is considered to be initially in a state of constant uniform motion, with velocity  $u(x,0) = \bar{u}, \forall x > 0$ , when the disturbance is introduced at  $x = 0$ .

The special case of  $\bar{u} = 0$  corresponds to the action of the disturbance upon quiescent fluid.

In Section 4.1, consideration is given to the case of zero mean flow, and to small-amplitude excitation. The problem is then one of linear acoustic propagation in one dimension. The equations that describe such a motion are derived from first principles using the equations of general 3-D motion developed in Chapter Two.

Section 4.2 extends the solution of Section 4.1 to cater for non-zero mean flow, although small-amplitude excitation is again assumed.

Section 4.3 examines the effect of lifting the restriction upon the excitation amplitude, through a study of the *characteristics* of the governing equations.

Section 4.4 concludes the chapter by examining the nature of energy transport by the “ideal” modulated flow.

## **4.1 Equations of Linear Acoustics:**

The equations of linear acoustics are obtained by considering all fluid motions to be small perturbations about some state in which the fluid is at rest with uniform density  $\bar{\rho}$ . The entropy of the fluid is therefore also uniform in this rest state, and it is assumed that this constant value of entropy is maintained as the fluid is perturbed; i.e. the motion is homentropic. Consequently a well-defined system of equations can be constructed using only the equation of continuity, momentum equations and the barotropic equation of state (see Section 3.4).

When the magnitudes of the perturbations about the rest state are sufficiently small, linearised approximations to the describing equations can be combined to give a first-order estimate of the nature of the motion. This is the foundation of the Equations of Linear Acoustics.

### **4.1.1 Derivation from Equations of General Fluid Motion:**

Consider that all density variations of a given three-dimensional fluid motion can be written as the sum of the constant rest-state value of  $\bar{\rho}$  and a small fluctuating perturbation  $\tilde{\rho}$ , i.e.

$$\rho(\mathbf{x}, t) = \bar{\rho} + \tilde{\rho}(\mathbf{x}, t) \quad (4.1)$$

In this section,  $\mathbf{x}$  represents the position vector (in any an arbitrary co-ordinate system) and  $t$  represents time.

The linear approximations to follow require that the magnitude of the density perturbations is always considerably smaller than the value of undisturbed density, i.e.  $|\tilde{\rho}| \ll \bar{\rho}$ .

Variations of the fluid velocity  $\mathbf{u}$  might generally be considered to be small perturbations  $\tilde{\mathbf{u}}$  about a constant velocity  $\bar{\mathbf{u}}$ . In this section a special case scenario is considered where perturbations are considered to be about a state of rest (i.e. in this case  $\bar{\mathbf{u}} = \mathbf{0}$ ):

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \bar{\mathbf{u}} + \tilde{\mathbf{u}}(\mathbf{x}, t) \\ \bar{\mathbf{u}} = \mathbf{0} &\Rightarrow \mathbf{u}(\mathbf{x}, t) = \tilde{\mathbf{u}}(\mathbf{x}, t) \end{aligned} \quad (4.2)$$

Note that since  $\bar{\mathbf{u}} = \mathbf{0}$ , it is not possible at this point to write a condition similar to  $|\tilde{\rho}| \ll \bar{\rho}$  in order to define exactly what is meant by the “smallness” of the velocity perturbations.

#### Linearised Barotropic Equation of State:

When fluid pressure  $p$  and entropy  $s$  are chosen as the two independent variables in a general thermodynamic system, any elemental change of fluid density obeys Equation (3.13), repeated below:

$$d\rho = \left( \frac{\partial \rho}{\partial p} \right)_s dp + \left( \frac{\partial \rho}{\partial s} \right)_p ds$$

When a thermodynamic process occurs isentropically, such that  $ds = 0$ , this equation simplifies to

$$d\rho = \frac{1}{c^2} dp \quad \text{where} \quad c = \sqrt{\left( \frac{\partial p}{\partial \rho} \right)_s}$$

The quantity  $c$  is, in general a function of both of the independent variables:  $c = c(p, s)$ . If, however, the motion is homentropic (i.e.  $s$  is constant) then  $c$  can be written as a function of pressure only:  $c = c(p)$ . The above equation can then be written as

$$\frac{dp}{d\rho} = c^2(p) \tag{4.3}$$

This relationship, involving fluid pressure as the only independent variable, demonstrates that the flow is barotropic.

A linear approximation to a barotropic relation  $p = p(\rho)$  for a particular flow may be obtained by expanding  $p$  as a single-variable Taylor series about the reference density  $\bar{\rho}$  and truncating this series after a finite number of terms. The Taylor series is written

$$p = p(\bar{\rho}) + (\rho - \bar{\rho}) \left. \frac{dp}{d\rho} \right|_{\rho=\bar{\rho}} + \frac{(\rho - \bar{\rho})^2}{2!} \left. \frac{d^2 p}{d\rho^2} \right|_{\rho=\bar{\rho}} + \dots$$

It is possible to define a reference pressure  $\bar{p}$  and a pressure fluctuation  $\tilde{p}$  as follows:

$$p(\mathbf{x}, t) = \bar{p} + \tilde{p}(\mathbf{x}, t) \tag{4.4}$$

Since a functional relationship exists between the fluid pressure and density alone,  $p = \bar{p}$  whenever  $\rho = \bar{\rho}$  so that  $p(\bar{\rho}) = \bar{p}$ .

Substitution of Equations (4.1) and (4.4) into the above Taylor series gives:

$$\tilde{p} = \tilde{\rho} \left. \frac{dp}{d\rho} \right|_{\rho=\bar{\rho}} + O(\tilde{\rho}^2)$$

Defining the constant value  $\bar{c}$  by  $\bar{c} = c(\bar{\rho})$ , it is possible to write a linearised barotropic relation by substituting Equation (4.3) above.

$$\tilde{p} \cong \tilde{\rho} \bar{c}^2 \quad (4.5)$$

The error introduced by this approximation depends critically upon the (second and higher-order derivatives) of the function  $p(\rho)$ . Consider, for example, the following infinitely-differentiable function:

$$p(\rho) = \frac{\bar{\rho} \bar{c}^2}{2\pi\chi} \sin\left(\frac{2\pi\chi\rho}{\bar{\rho}}\right), \text{ where } \chi \text{ has a constant value.}$$

The  $(i+1)^{\text{th}}$  derivative of this function, evaluated at  $\rho = \bar{\rho}$ , is always factor of  $\frac{2\pi\chi}{\bar{\rho}}$  larger than the derivative of order  $i$ . Consequently, an error bound for the approximation in Equation (4.5) will, for this example, depend upon the value of  $\chi$  as well as  $\tilde{\rho}$ .

#### Linearised Continuity Equation:

In terms of steady and fluctuating components of density and velocity, the continuity equation may be developed as follows:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho + \rho \nabla \cdot \mathbf{u} &= 0 \\ \Rightarrow \frac{\partial}{\partial t}(\bar{\rho} + \tilde{\rho}) + \tilde{\mathbf{u}} \cdot \nabla(\bar{\rho} + \tilde{\rho}) + (\bar{\rho} + \tilde{\rho})\nabla \cdot \tilde{\mathbf{u}} &= 0 \end{aligned}$$

This is simplified by noting that, since  $\bar{\rho}$  is a constant, its partial derivatives are zero:

$$\frac{\partial \tilde{\rho}}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\rho} + (\bar{\rho} + \tilde{\rho})\nabla \cdot \tilde{\mathbf{u}} = 0$$

If the perturbations of  $\rho$  around  $\bar{\rho}$  are sufficiently small in value, so that  $|\tilde{\rho}| \ll \bar{\rho}$ , then the third term above is negligibly affected by the unsteady component of the fluid density:

$$\frac{\partial \tilde{\rho}}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\rho} + \bar{\rho} \nabla \cdot \tilde{\mathbf{u}} \cong 0 \quad (4.6)$$

The second term in the above expression is often neglected at this point, as it involves a product of two fluctuating quantities while the first and third terms each contain only one fluctuating quantity (e.g. Lighthill, 1978: Equation 5). This is a little mischievous, since the assumption that the density fluctuations are small (i.e. that  $|\tilde{\rho}| \ll \bar{\rho}$ ) does not in itself impose any bound upon the size of its partial derivatives.

Instead suppose that in a particular flow the spatial distributions of  $\tilde{\rho}$  and  $\tilde{\mathbf{u}}$  are characterised by a length scale  $L$  (meaning that in general  $\tilde{\rho}$  and  $\tilde{\mathbf{u}}$  vary only slightly over distances small compared with  $L$ ), and that the temporal distribution of  $\tilde{\rho}$  is characterised by a time scale  $T$ . Consider further that the variations of  $\tilde{\rho}$  with respect to both space and time have the magnitude  $R$ , and the variations of  $\tilde{\mathbf{u}}$  have the magnitude  $U$ . Then in general the orders of magnitude of the three terms in Equation (4.6) are:

$$\frac{\partial \tilde{\rho}}{\partial t} \approx \frac{R}{T} \quad \tilde{\mathbf{u}} \cdot \nabla \tilde{\rho} \approx \frac{UR}{L} \quad \bar{\rho} \nabla \cdot \tilde{\mathbf{u}} \approx \bar{\rho} \frac{U}{L}$$

The second term in Equation (4.6) is insignificant compared to the first term whenever  $U \ll \frac{L}{T}$ , and insignificant compared to the third whenever  $R \ll \bar{\rho}$ .

The condition  $R \ll \bar{\rho}$  is satisfied for all the small perturbations under current consideration by virtue of the earlier assumption that  $|\tilde{\rho}| \ll \bar{\rho}$ . Because the three terms of Equation (4.6) sum to zero, this means that if the second term is insignificant in comparison to the third then it is necessarily insignificant compared with the first term also. This implies that the condition  $U \ll \frac{L}{T}$  is also fulfilled.

This is a more proper justification for the neglect of the second term in Equation (4.6). Without its second term, Equation (4.6) becomes

$$\frac{\partial \tilde{\rho}}{\partial t} + \bar{\rho} \nabla \cdot \tilde{\mathbf{u}} \cong 0 \quad (4.7)$$

It should be noted that implicit in the foregoing discussion is the assumption that the spatial variation of both the density  $\tilde{\rho}$  and the velocity  $\tilde{\mathbf{u}}$  can be characterised by the same length  $L$ . In the following linearisation of the momentum transport equation, it will

be further assumed that the temporal variations of  $\tilde{\rho}$  and  $\tilde{\mathbf{u}}$  can be characterised by the same time scale  $T$ .

### Linearised Momentum Transport:

Consider motion of an inviscid fluid in the absence of any body forces. In terms of steady and fluctuating components of density, pressure and velocity, the appropriate momentum transport equation may be developed as follows:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{\rho} \nabla p &= \mathbf{0} \\ \Rightarrow \frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}} + \frac{1}{\bar{\rho} + \tilde{\rho}} \nabla(\bar{p} + \tilde{p}) &= \mathbf{0} \end{aligned}$$

Because  $\bar{p}$  is constant, its partial derivatives are zero, so that

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}} + \frac{1}{\bar{\rho} + \tilde{\rho}} \nabla \tilde{p} = \mathbf{0}$$

It is possible to write the quotient  $\frac{1}{\bar{\rho} + \tilde{\rho}}$  as the sum of an infinite geometric series:

$$\frac{1}{\bar{\rho} + \tilde{\rho}} = \frac{1}{\bar{\rho}} \left[ 1 - \frac{\tilde{\rho}}{\bar{\rho}} + \frac{\tilde{\rho}^2}{\bar{\rho}^2} - \dots \right]$$

Consequently, the error in approximating  $\frac{1}{\bar{\rho} + \tilde{\rho}}$  by  $\frac{1}{\bar{\rho}}$  is of order  $\frac{\tilde{\rho}}{\bar{\rho}}$ . This approximation leads to:

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}} + \frac{1}{\bar{\rho}} \nabla \tilde{p} \cong \mathbf{0} \quad (4.8)$$

The linearised barotropic equation of state (Equation 4.5) becomes useful at this point. The unknown pressure fluctuation  $\tilde{p}$  can be replaced by the corresponding density fluctuation:

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}} + \frac{\bar{c}^2}{\bar{\rho}} \nabla \tilde{\rho} \cong \mathbf{0} \quad (4.9)$$

Suppose that in a particular flow the spatial distributions of  $\tilde{\rho}$  and  $\tilde{\mathbf{u}}$  are both characterised by a length scale  $L$  and that the temporal distribution of  $\tilde{\mathbf{u}}$  is characterised by the same time scale  $T$  that characterises the temporal variations of  $\tilde{\rho}$ . Consider further that the variations of  $\tilde{\mathbf{u}}$  with respect to both space and time have the magnitude  $U$ , and

the variations of  $\tilde{\rho}$  have the magnitude  $R$ . Then in general the orders of magnitude of the three terms in Equation (4.9) are:

$$\left| \frac{\partial \tilde{\mathbf{u}}}{\partial t} \right| \approx \frac{U}{T} \quad |(\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}}| \approx \frac{U^2}{L} \quad \left| \frac{\bar{c}^2}{\bar{\rho}} \nabla \tilde{\rho} \right| \approx \frac{\bar{c}^2}{\bar{\rho}} \frac{R}{L}$$

Consequently, the second term of Equation (4.9) is negligible compared with the first if  $U \ll \frac{L}{T}$ , and negligible compared with the third when  $\frac{U^2}{\bar{c}^2} \ll \frac{R}{\bar{\rho}}$ .

Because the three terms of Equation (4.9) sum to zero, fulfilment of either of these two conditions implies fulfilment of both. The preceding discussion upon the linearisation of the continuity equation showed that the condition  $U \ll \frac{L}{T}$  is automatic whenever

$|\tilde{\rho}| \ll \bar{\rho}$ ; this has been assumed already. Consequently, the condition  $\frac{U^2}{\bar{c}^2} \ll \frac{R}{\bar{\rho}}$  is also

fulfilled. Without its second term, Equation (4.9) reduces to

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \frac{\bar{c}^2}{\bar{\rho}} \nabla \tilde{\rho} \equiv 0 \quad (4.10)$$

Equations (4.7) and (4.10) together form a well-defined system of (linear) equations for the unknown quantities  $\tilde{\rho}$  and  $\tilde{\mathbf{u}}$ . (The rest-state values of  $\bar{\rho}$ ,  $\bar{p}$  and  $\bar{c}$  are assumed known from experimental data.) An alternative formulation replaces the fluctuating density  $\tilde{\rho}$  by fluctuating pressure  $\tilde{p}$ ; pressure fluctuations are easily measured by electromechanical transducers. Utilising Equation (4.5), Equations (4.7) and (4.10) become

$$\frac{\partial \tilde{p}}{\partial t} + \bar{\rho} \bar{c}^2 \nabla \cdot \tilde{\mathbf{u}} \equiv 0 \quad (4.11)$$

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \frac{1}{\bar{\rho}} \nabla \tilde{p} \equiv 0 \quad (4.12)$$

### 4.1.2 Acoustic Wave Equations in 3-D:

Consider the partial derivative of Equation (4.11) with respect to time:

$$\begin{aligned} \frac{\partial \tilde{p}}{\partial t} + \bar{\rho} \bar{c}^2 \nabla \cdot \tilde{\mathbf{u}} &\equiv 0 \\ \Rightarrow \frac{\partial^2 \tilde{p}}{\partial t^2} + \bar{\rho} \bar{c}^2 \frac{\partial}{\partial t} (\nabla \cdot \tilde{\mathbf{u}}) &\equiv 0 \end{aligned} \quad (4.13)$$

Note that the validity of this step, where the boundedness of a function is assumed to concur with boundedness of its derivative, is not automatic. For example, the function  $\sin(t) + \sin(1000t) \leq 2$ , whereas its derivative  $\cos(t) + 1000\cos(1000t)$  can reach values as large as 1001. Implicit in Equation (4.13) - and later in Equations (4.14), (4.19) and (4.20) - is an assumption that variations of  $\tilde{p}$  and  $|\tilde{\mathbf{u}}|$  have similar length scales (e.g. wavelengths) and similar time-scales (periods).

Consider now the divergence of Equation (4.12):

$$\begin{aligned} \frac{\partial \tilde{\mathbf{u}}}{\partial t} + \frac{1}{\bar{\rho}} \nabla \tilde{p} &\equiv 0 \\ \Rightarrow \nabla \cdot \frac{\partial \tilde{\mathbf{u}}}{\partial t} + \frac{1}{\bar{\rho}} \nabla^2 \tilde{p} &\equiv 0 \end{aligned} \quad (4.14)$$

By multiplying Equation (4.14) by the constant amount  $\bar{\rho} \bar{c}^2$ , and subtracting the result from Equation (4.13), a *linear wave equation* results:

$$\begin{aligned} \frac{\partial^2 \tilde{p}}{\partial t^2} + \bar{\rho} \bar{c}^2 \frac{\partial}{\partial t} (\nabla \cdot \tilde{\mathbf{u}}) - \bar{\rho} \bar{c}^2 \left[ \nabla \cdot \frac{\partial \tilde{\mathbf{u}}}{\partial t} + \frac{1}{\bar{\rho}} \nabla^2 \tilde{p} \right] &\equiv 0 \\ \Rightarrow \frac{\partial^2 \tilde{p}}{\partial t^2} - \bar{c}^2 \nabla^2 \tilde{p} &\equiv 0 \end{aligned} \quad (4.15)$$

Linear wave equations have been studied extensively in the past. A plane travelling wave is known to be a solution of such an equation. A general solution of Equation (4.15), in a Cartesian co-ordinate system, is given by

$$\tilde{p}(\mathbf{x}, t) = \tilde{p}_+(\xi x_1 + \eta x_2 + \zeta x_3 - \bar{c}t) + \tilde{p}_-(\xi x_1 + \eta x_2 + \zeta x_3 + \bar{c}t), \quad (4.16)$$

$$\text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } \xi^2 + \eta^2 + \zeta^2 = 1$$

The arbitrary function  $\tilde{p}_+$  determines the wave-shape of a pressure wave travelling in the direction of the vector  $(\xi, \eta, \zeta)$  with a constant speed of propagation  $\bar{c}$ . The function  $\tilde{p}_-$  determines the wave-shape of a pressure wave travelling with the same speed  $\bar{c}$  but in the

direction of the vector  $(-\xi, -\eta, -\zeta)$ . Such pressure waves are *acoustic* or *sound waves*, and the thermodynamic parameter of state  $\bar{c}$  is given the name *speed of sound* or *sonic speed* of the fluid.

An analogous procedure using Equations (4.7) and (4.10) - alternatively, substitution of Equation (4.5) into Equation (4.15) - reveals that the fluid density also satisfies approximately a linear wave equation:

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - \bar{c}^2 \nabla^2 \tilde{\rho} \cong 0 \quad (4.17)$$

A general solution to Equation (4.17) is

$$\tilde{\rho}(\mathbf{x}, t) = \tilde{\rho}_+(\xi x_1 + \eta x_2 + \zeta x_3 - \bar{c}t) + \tilde{\rho}_-(\xi x_1 + \eta x_2 + \zeta x_3 + \bar{c}t), \quad (4.18)$$

$$\text{where } \xi^2 + \eta^2 + \zeta^2 = 1.$$

The solutions to Equations (4.15) and (4.17) are related through

$$\tilde{\rho}_+ = \bar{c}^2 \tilde{\rho}_+ \quad \text{and} \quad \tilde{\rho}_- = \bar{c}^2 \tilde{\rho}_-$$

#### An Expression for Fluid Velocity:

Consider now the partial derivative of Equation (4.12) with respect to time:

$$\begin{aligned} \frac{\partial \tilde{\mathbf{u}}}{\partial t} + \frac{1}{\bar{\rho}} \nabla \tilde{p} &\cong \mathbf{0} \\ \Rightarrow \frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2} + \frac{1}{\bar{\rho}} \frac{\partial}{\partial t} (\nabla \tilde{p}) &\cong \mathbf{0} \end{aligned} \quad (4.19)$$

Consider also the gradient of Equation (4.11):

$$\begin{aligned} \frac{\partial \tilde{p}}{\partial t} + \bar{\rho} \bar{c}^2 \nabla \cdot \tilde{\mathbf{u}} &\cong 0 \\ \Rightarrow \nabla \left( \frac{\partial \tilde{p}}{\partial t} \right) + \bar{\rho} \bar{c}^2 \nabla (\nabla \cdot \tilde{\mathbf{u}}) &\cong \mathbf{0} \end{aligned} \quad (4.20)$$

Equation (4.20) is now divided by the constant amount  $\bar{\rho}$ , and the result subtracted from Equation (4.19):

$$\begin{aligned} \frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2} + \frac{1}{\bar{\rho}} \frac{\partial}{\partial t} (\nabla \tilde{p}) - \frac{1}{\bar{\rho}} \left[ \nabla \left( \frac{\partial \tilde{p}}{\partial t} \right) + \bar{\rho} \bar{c}^2 \nabla (\nabla \cdot \tilde{\mathbf{u}}) \right] &= \mathbf{0} \\ \Rightarrow \frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2} - \bar{c}^2 \nabla (\nabla \cdot \tilde{\mathbf{u}}) &= \mathbf{0} \end{aligned}$$

By invoking the vector identity  $\nabla^2 \tilde{\mathbf{u}} = \nabla(\nabla \cdot \tilde{\mathbf{u}}) - \nabla \times (\nabla \times \tilde{\mathbf{u}})$  for any arbitrary  $\tilde{\mathbf{u}}$ , this equation can be rewritten as

$$\frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2} - \bar{c}^2 \nabla^2 \tilde{\mathbf{u}} = \bar{c}^2 \nabla \times (\nabla \times \tilde{\mathbf{u}}) \quad (4.21)$$

Equation (4.21) is not a linear wave equation. The term on the right-hand side can be identified with the curl of the fluid vorticity.

Recall that the fluid under study is considered to be inviscid, barotropic and uninfluenced by body forces (Section 4.1.1). Kelvin's circulation theorem (Currie, 1974) can be applied to the motion of such a fluid: this theorem states that any motion which starts out as irrotational will remain irrotational for all time, so that vorticity is conserved as the fluid moves about. Consequently, if the fluid motion has zero vorticity at some specified time, then it has zero vorticity always.

Consider now that an inviscid fluid motion is known to be irrotational at a certain time; such is the case when the fluid is everywhere at rest. Then, by Kelvin's circulation theorem, it is irrotational always. The fact that  $\nabla \times \tilde{\mathbf{u}} = \mathbf{0}$  identically implies that a velocity potential  $\tilde{\phi}$  exists such that  $\tilde{\mathbf{u}} = \nabla \tilde{\phi}$ . Equation (4.21) becomes

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} (\nabla \tilde{\phi}) - \bar{c}^2 \nabla^2 (\nabla \tilde{\phi}) = \mathbf{0} \\ \Rightarrow & \nabla \left[ \frac{\partial^2 \tilde{\phi}}{\partial t^2} - \bar{c}^2 \nabla^2 \tilde{\phi} \right] = \mathbf{0} \\ \Rightarrow & \frac{\partial^2 \tilde{\phi}}{\partial t^2} - \bar{c}^2 \nabla^2 \tilde{\phi} = \vartheta(t) \end{aligned}$$

A velocity potential is defined only to within an arbitrary function of time; since  $\tilde{\mathbf{u}} = \nabla \phi(x, t)$ , an arbitrary function of time can be added to  $\phi$  without having any effect upon  $\tilde{\mathbf{u}}$ . Consequently there is no loss of generality by setting  $\vartheta(t) = 0$  in the above equation. Another linear wave equation results:

$$\frac{\partial^2 \tilde{\phi}}{\partial t^2} - \bar{c}^2 \nabla^2 \tilde{\phi} = 0 \quad (4.22)$$

A solution for Equation (4.22) is given by

$$\tilde{\phi}(\mathbf{x}, t) = \tilde{\phi}_+(\xi x_1 + \eta x_2 + \zeta x_3 - \bar{c}t) + \tilde{\phi}_-(\xi x_1 + \eta x_2 + \zeta x_3 + \bar{c}t), \quad (4.23)$$

$$\text{where } \xi^2 + \eta^2 + \zeta^2 = 1.$$

To see how the solutions for the fluid pressure and velocity potential are related, consider Equation (4.12) which is repeated below in terms of the velocity potential  $\tilde{\phi}$  :

$$\begin{aligned} \frac{\partial \tilde{\mathbf{u}}}{\partial t} + \frac{1}{\bar{\rho}} \nabla \tilde{p} &= \mathbf{0} \\ \Rightarrow \frac{\partial}{\partial t} (\nabla \tilde{\phi}) + \frac{1}{\bar{\rho}} \nabla \tilde{p} &= \mathbf{0} \\ \Rightarrow \nabla \left[ \frac{\partial \tilde{\phi}}{\partial t} + \frac{\tilde{p}}{\bar{\rho}} \right] &= \mathbf{0} \\ \Rightarrow \frac{\partial \tilde{\phi}}{\partial t} + \frac{\tilde{p}}{\bar{\rho}} &= \vartheta(t) \end{aligned}$$

By again setting  $\vartheta(t) = 0$ , it is apparent that  $\tilde{p}$  can be obtained by taking the partial derivative of the velocity potential (from Equation 22) with respect to time and multiplying the result by  $-\bar{\rho}$ , i.e.,

$$\tilde{p} = -\bar{\rho} \frac{\partial \tilde{\phi}}{\partial t} \quad (4.24)$$

This gives  $\tilde{p}_+ = -\bar{\rho} \bar{c} \tilde{\phi}'_+$  and  $\tilde{p}_- = \bar{\rho} \bar{c} \tilde{\phi}'_-$  where a prime designates differentiation of the function with reference to its subject.

The velocity can be recovered from its potential as follows:

$$\begin{aligned} \tilde{\mathbf{u}} &= \nabla \tilde{\phi}(\mathbf{x}, t) \\ &= \nabla \tilde{\phi}_+(\xi x_1 + \eta x_2 + \zeta x_3 - \bar{c}t) + \nabla \tilde{\phi}_-(\xi x_1 + \eta x_2 + \zeta x_3 + \bar{c}t) \\ &= \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tilde{\phi}'_+ + \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tilde{\phi}'_- \end{aligned}$$

### 4.1.3 Simpler 1-D Equations for Sound in a Duct:

Consider linear acoustic motion which has non-zero velocity in only one particular direction, the value of the velocity being uniform within any plane perpendicular to this direction. If the fluid density and pressure are also uniform within any such plane, then the motion can be written entirely in terms of time and just one spatial variable. Denoting this spatial variable by  $x$ , and the velocity in the  $x$  direction by  $u$ , Equations (4.11) and (4.12) become

$$\frac{\partial \tilde{u}}{\partial x} + \frac{1}{\bar{\rho} \bar{c}^2} \frac{\partial \tilde{p}}{\partial t} = 0 \quad \frac{\partial \tilde{p}}{\partial x} + \bar{\rho} \frac{\partial \tilde{u}}{\partial t} = 0 \quad (4.25)$$

Because the fluid velocity is dependent upon only one spatial co-ordinate, the flow is always irrotational, and so a velocity potential  $\tilde{\phi}$  may be defined such that  $\tilde{u} = \frac{\partial \tilde{\phi}}{\partial x}$ . This velocity potential satisfies a linear wave equation (Equation 4.22):

$$\frac{\partial^2 \tilde{\phi}}{\partial t^2} - \bar{c}^2 \frac{\partial^2 \tilde{\phi}}{\partial x^2} \equiv 0$$

A solution to this wave equation can be written in the form

$$\tilde{\phi}(x, t) = \tilde{\phi}_+(x - \bar{c}t) + \tilde{\phi}_-(x + \bar{c}t) \quad (4.26)$$

The pressure and velocity variations can be recovered from knowledge of  $\tilde{\phi}$  through (cf. Equation 4.23)

$$\tilde{p} = -\bar{\rho} \frac{\partial \tilde{\phi}}{\partial t} \quad \text{and} \quad \tilde{u} = \frac{\partial \tilde{\phi}}{\partial x} \quad (4.27)$$

#### Sound in a Duct of Constant Cross-Section:

Consider now that the 1-D fluid motion described above is actually occurring along a duct, of constant cross-sectional area  $A$ , which is aligned with the  $x$ -axis. The acoustic *volume velocity*  $\tilde{q}$  in the duct at position  $x$  and time  $t$  is related to the fluctuating fluid velocity  $\tilde{u}$  by

$$\tilde{q}(x, t) = \tilde{u}(x, t) A$$

For a duct of constant cross-section, an elemental change in the volume velocity satisfies

$$d\tilde{q} = d(\tilde{u}A) = A d\tilde{u}$$

In terms of the volume velocity, Equations (4.24) become

$$\frac{\partial \tilde{q}}{\partial x} + \frac{A}{\bar{\rho} \bar{c}^2} \frac{\partial \tilde{p}}{\partial t} = 0 \quad \frac{\partial \tilde{p}}{\partial x} + \frac{\bar{\rho}}{A} \frac{\partial \tilde{q}}{\partial t} = 0 \quad (4.28)$$

In terms of the velocity potential, the volume velocity is written as

$$\tilde{q} = A \frac{\partial \tilde{\phi}}{\partial x} \quad (4.29)$$

Equations (4.28) are directly comparable to the so-called *transmission-line equations* for the case of a electrical transmission line with negligible resistance losses. In Equations

(4.30) to follow,  $v$  represents voltage and  $i$  represents current;  $L$  and  $C$  represent the inductance and capacitance, respectively, per unit length of the transmission line.

$$\frac{\partial i}{\partial x} + C \frac{\partial v}{\partial t} = 0 \quad \frac{\partial v}{\partial x} + L \frac{\partial i}{\partial t} = 0 \quad (4.30)$$

A direct analogy is possible between transmission-line theory and the linear acoustic propagation along a tube of constant cross-section:

<u>Acoustical Quantity:</u>	<u>Analogous Electrical Quantity:</u>
$\tilde{u}$ - pressure	$v$ - voltage
$\tilde{q}$ - volume velocity	$i$ - current
$\frac{\bar{\rho}}{A}$ - acoustic inductance	$L$ - inductance
$\frac{A}{\bar{\rho}c^2}$ - acoustic capacitance	$C$ - capacitance

The fact that the governing equations are linear allows the application of some powerful analytical techniques to the study of sound propagation along a cylindrical tube. In particular, linearity of the governing equations means that a sinusoidal input of a certain frequency will yield a sinusoidal output of that same frequency. Consequently the relationship between input and output signals at a given frequency can be described in terms of the ratio of signal amplitudes and in terms of their phase differences.

The response to an arbitrary input signal can be determined as the superposition of the system's individual responses to the various Fourier components of that signal.

## **4.2 Sound Propagation in a Uniform Mean Flow:**

The three-dimensional equations of linear acoustics were obtained earlier in Section 4.1.1 by considering the motion of an inviscid fluid in terms of small perturbations about some state in which the fluid was at rest with a uniform value of density  $\bar{\rho}$ . It was assumed that the constant value of entropy associated with this rest state was preserved as the fluid was perturbed, i.e. the fluid motion was assumed to be homentropic, and thus barotropic.

When the magnitudes of such perturbations about the rest state are sufficiently small, linearised approximations to the describing equations can be combined to give a first-order estimate of the nature of the motion. A well-defined system of equations was constructed using linearised versions of the equation of continuity, the momentum equations and the barotropic equation of state.

In this section, fluid motions are instead treated as being small perturbations about a state of uniform motion in which the fluid everywhere has a constant value of velocity  $\bar{\mathbf{u}}$ . The ensuing fluid motion is again assumed to be homentropic. When the magnitudes of perturbations about the state of uniform motion are sufficiently small, linearised approximations to the describing equations can be combined to give a first-order estimate of the nature of the motion.

### **4.2.1 Modified Equations for Additional Mean Flow:**

The question of a fluid medium initially in uniform motion (Section 4.2) or initially at rest (Section 4.1) is mainly a matter of coordinate transformation. However, when a problem is considered which includes boundaries and sources of sound, relative motion between the fluid and the various material bodies involved cannot be eliminated by a coordinate transformation; a transformation that brings the fluid medium to rest sets the sources and the boundaries in motion.

In the following discussion, two frames of reference will be used: A coordinate system  $S$  is considered to be fixed with respect to the material boundaries and acoustic sources in the problem under study. Quantities referred to the  $S$  coordinate system will have values as would be measured by an observer at rest. If a second coordinate system  $S'$  is also defined, which travels at the same uniform velocity as the undisturbed medium, then quantities referred to the  $S'$  coordinate system will have values as would be measured by an observer moving with the constant velocity of the medium.

In Section 4.1 the perturbations of the fluid velocity were considered to be about a state of rest. That theory can be extended by considering the situation in Section 4.1 to be a special case of a more general scenario in which the fluctuations of fluid velocity  $\mathbf{u}$  are considered to be small perturbations  $\tilde{\mathbf{u}}$  of a medium moving with constant velocity  $\bar{\mathbf{u}}$ , i.e.  $\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}} + \tilde{\mathbf{u}}(\mathbf{x}, t)$ . In Section 4.1 the special case of  $\bar{\mathbf{u}} = \mathbf{0}$  was considered.

The coordinate system  $S'$  is defined to be moving with the fluid medium. Using primes to indicate the independent variables of the moving coordinate system, this means that

$$\mathbf{u}(\mathbf{x}', t') = \mathbf{0} + \tilde{\mathbf{u}}(\mathbf{x}', t')$$

A sound field in a moving medium can be referred to the co-moving coordinate system  $S'$  using the equations for mass and momentum transport derived already in Section 4.1. A prime is used in the following versions of Equations (4.11) and (4.12) to indicate that the variables here are associated with the  $S'$  coordinate system:

$$\frac{\partial \tilde{p}}{\partial t'} + \bar{\rho} \bar{c}^2 \frac{\partial \tilde{u}_i}{\partial x'_i} \equiv 0 \quad (4.31)$$

$$\frac{\partial \tilde{u}_i}{\partial t'} + \frac{1}{\bar{\rho}} \frac{\partial \tilde{p}}{\partial x'_i} \equiv 0 \quad (4.32)$$

The constant velocity of the fluid medium is  $\bar{\mathbf{u}}$ , and so this is also the velocity of the coordinate system  $S'$  with respect to the fixed coordinate system  $S$ . The transformation of the coordinates from co-moving system to the  $S$  coordinate system is given by

$$x_i = x'_i + \bar{u}_i t' \quad t = t' \quad (4.33)$$

The partial derivatives of any function of space and time, since  $x'_i$  and  $t'$  are functions of  $x_i$  and  $t$ , satisfy:

$$\begin{aligned} \frac{\partial}{\partial x'_i} &= \frac{\partial x_j}{\partial x'_i} \frac{\partial}{\partial x_j} + \frac{\partial t}{\partial x'_i} \frac{\partial}{\partial t} = \frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial t'} &= \frac{\partial t}{\partial t'} \frac{\partial}{\partial t} + \frac{\partial x_i}{\partial t'} \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + \bar{u}_i \frac{\partial}{\partial x_i} \end{aligned}$$

Using these transformations in Equations (4.31) and (4.32) leads to:

$$\left( \frac{\partial}{\partial t} + \bar{u}_i \frac{\partial}{\partial x_i} \right) \tilde{p} + \bar{\rho} \bar{c}^2 \frac{\partial \tilde{u}_i}{\partial x_i} \equiv 0 \quad (4.34)$$

$$\left( \frac{\partial}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j} \right) \tilde{u}_i + \frac{1}{\bar{\rho}} \frac{\partial \tilde{p}}{\partial x_i} \cong 0 \quad (4.35)$$

By subtracting the divergence of Equations (4.35) from the derivative  $\frac{\partial}{\partial t} + \bar{u}_i \frac{\partial}{\partial x_i}$  of

Equation (4.34), the following Modified Wave Equation results:

$$\left( \frac{\partial}{\partial t} + \bar{u}_i \frac{\partial}{\partial x_i} \right)^2 \tilde{p} - \bar{c}^2 \frac{\partial^2 \tilde{p}}{\partial x_i \partial x_i} \cong 0 \quad (4.36)$$

The nonlinear derivative upon the left of Equation (4.36) can be developed as follows (noting that all partial derivatives of the uniform velocity  $\bar{u}$  are identically zero):

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \bar{u}_i \frac{\partial}{\partial x_i} \right)^2 &= \left( \frac{\partial}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j} \right) \left( \frac{\partial}{\partial t} + \bar{u}_i \frac{\partial}{\partial x_i} \right) \\ &= \frac{\partial^2}{\partial t^2} + 2\bar{u}_i \frac{\partial^2}{\partial x_i \partial t} + \bar{u}_i \bar{u}_j \frac{\partial^2}{\partial x_i \partial x_j} \end{aligned}$$

#### 4.2.2 Series Solution of the Modified Wave Equation:

Acoustic motion will be considered which has non-zero velocity in only one particular direction; the value of the velocity is uniform within any plane perpendicular to this direction. The superimposed uniform mean flow will also be considered to be in the same direction as the propagated sound.

The resulting motion can be written entirely in terms of time and just one spatial variable. Denoting this spatial variable by  $x$ , and the velocity in the  $x$  direction by  $u(x, t) = \bar{u} + \tilde{u}(x, t)$ , Equation (4.36) becomes

$$\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \tilde{p} - \bar{c}^2 \frac{\partial^2 \tilde{p}}{\partial x^2} \cong 0 \quad (4.37)$$

##### Non-Dimensionalised Version:

A suitable scheme for removing physical dimensions from Equation (4.37) employs the following change of variables:

$$x = L X \quad t = \frac{L}{\bar{c}} T \quad \tilde{p} = \bar{p} P \quad (4.38)$$

In such a description, the variables  $X$ ,  $T$  and  $P$  are dimensionless; i.e. they have no units of measurement associated with them.

The parameter  $L$  is a 'characteristic length' chosen in consideration of the problem under study. For example, if the behaviour of Equation (4.37) is to be studied along a long cylindrical pipe which is aligned with the  $x$ -axis, then a suitable choice for the parameter  $L$  would be the length of the pipe. The ratio  $L/\bar{c}$  has been used above to non-dimensionalise time. Such a 'characteristic time' may be interpreted as the time taken for an acoustic wave, propagating at sonic speed  $\bar{c}$ , to travel the aforementioned characteristic length  $L$  in the absence of any mean flow. The relative pressure has been transformed by Equations (4.38); the absolute pressure can later be recovered from

$$p = \bar{p} + \tilde{p} = \bar{p}(1 + P)$$

The change of variables stipulated by Equations (4.38) transforms Equation (4.37) as follows:

$$\left( \frac{\bar{c}}{L} \frac{\partial}{\partial T} + \frac{\bar{u}}{L} \frac{\partial}{\partial X} \right)^2 \bar{p}P - \bar{c}^2 \frac{\bar{p}}{L^2} \frac{\partial^2 P}{\partial X^2} = 0$$

To aid in simplification, consider the above equation in terms of a *Mach number* defined as  $M = \frac{\bar{u}}{\bar{c}}$ :

$$\left( \frac{\partial}{\partial T} + M \frac{\partial}{\partial X} \right)^2 P - \frac{\partial^2 P}{\partial X^2} = 0$$

Expanding the non-linear differential operator leads to

$$\frac{\partial^2 P}{\partial T^2} + 2M \frac{\partial^2 P}{\partial X \partial T} + (M^2 - 1) \frac{\partial^2 P}{\partial X^2} = 0 \quad (4.39)$$

This equation reduces to a regular linear wave equation when  $M = 0$ . This is required for consistency with the results of Section 4.1, since a fluid everywhere at rest is a special case of the more general scenario of a fluid in uniform motion.

#### Series Solution in Powers of $M$ :

A series solution for Equation (4.39) is now proposed, in which  $P$  has the form

$$P(X, T) = P_0(X, T) + MP_1(X, T) + M^2 P_2(X, T) + \dots \quad (4.40)$$

where  $M < 1$  is required for convergence of the series.

The fluid motion, as described by  $P(X, T)$ , will be considered known at  $X = 0$  for all values of time  $T \geq 0$ . Let  $P(0, T) = f_0(T)$  where the value of  $f_0$  is known for all non-negative values of  $T$ . At time  $T = 0$  the fluid is considered to be in its undisturbed state, i.e.  $P(X, 0) = 0$ ,  $X > 0$ . Solution is sought for  $0 < X \leq 1$  and  $T \geq 0$ .

Substitution of Equation (4.40) into Equation (4.39) leads to the following:

$$\left( \frac{\partial^2}{\partial T^2} + 2M \frac{\partial^2}{\partial X \partial T} + [M^2 - 1] \frac{\partial^2}{\partial X^2} \right) [P_0(X, T) + MP_1(X, T) + M^2 P_2(X, T) + \dots] = 0$$

This expression may be regrouped as follows, in terms of ascending powers of  $M$ :

$$\begin{aligned} \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_0 + M \left[ \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_1 + 2 \frac{\partial^2 P_0}{\partial X \partial T} \right] \\ + \sum_{i=2}^{\infty} \left\{ M^i \left[ \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_i + 2 \frac{\partial^2 P_{i-1}}{\partial X \partial T} + \frac{\partial^2 P_{i-2}}{\partial X^2} \right] \right\} = 0 \end{aligned} \quad (4.41)$$

In order for Equation (4.40) to describe the solution to Equation (4.41), it is necessary for each term in Equation (4.41) to equal zero identically, since the ascending powers of  $M$  ensure that all terms in this series are independent of each other.

It is convenient to use the function  $P_0(X, T)$  alone to satisfy the given boundary conditions at the position  $X = 0$ . In this manner, every higher-order function  $P_i(X, T)$ ,  $i > 0$  will have the property  $P_i(0, T) = 0$ .

#### Order Zero Terms:

By disregarding in Equation (4.41) all terms whose degree in  $M$  is greater than zero, it is seen that the function  $P_0(X, T)$  is actually a solution to a linear wave equation:

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_0 = 0$$

Any function of the form  $P_0 = P_0(X - T)$  is known to be a solution to the above equation. This can be demonstrated as follows:

$$\begin{aligned} P_0 &= P_0(X - T) \\ \Rightarrow \quad \frac{\partial P_0}{\partial X} &= P_0'(X - T) \quad \text{and} \quad \frac{\partial P_0}{\partial T} = -P_0'(X - T) \end{aligned}$$

(Here a prime denotes ordinary differentiation of the function with respect to its subject.)

$$\begin{aligned} \Rightarrow \quad \frac{\partial^2 P_0}{\partial X^2} &= P_0''(X-T) \quad \text{and} \quad \frac{\partial^2 P_0}{\partial T^2} = P_0''(X-T) \\ \Leftrightarrow \quad \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_0(X-T) &= P_0''(X-T) - P_0''(X-T) = 0 \end{aligned}$$

(Any function  $P_0 = P_0(X+T)$  is also known to be a solution but since, for the current flow problem, the fluid is initially quiescent for  $X > 0$ , any function of this form must be identically zero to satisfy the initial conditions.)

Imposing the boundary condition that  $P(0, T) = f_0(T)$  requires that

$$P_0(-T) = f_0(T)$$

This implies that the function  $P_0$  is a reflection in the  $X$ -axis of the function  $f_0$ . If the function  $f_0(t)$  is assumed, then the function  $P_0$  is known also, since the two are related according to

$$P_0(X-T) = f_0(T-X) \quad (4.42)$$

#### Order $M$ Terms:

By retaining in Equation (4.41) only those terms which are of order  $M$ , the following equation results:

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_1 + 2 \frac{\partial^2 P_0}{\partial X \partial T} = 0$$

Rearranging this equation reveals that it has the form of a non-homogeneous linear wave equation:

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_1 = -2 \frac{\partial^2 P_0}{\partial X \partial T}$$

Utilising from above that  $P_0 = f_0(T-X)$  leads to:

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_1 = 2 f_0''(T-X) \quad (4.43)$$

Consider a solution to Equation (4.43) which has the form

$$P_1(X, T) = G(X, T) f_0'(T-X) \quad (4.44)$$

If such a solution is available, then its partial derivatives must satisfy Equation (4.43):

$$\frac{\partial P_1}{\partial X} = \frac{\partial G}{\partial X} f_0'(T-X) - G(X, T) f_0''(T-X)$$

$$\begin{aligned}\frac{\partial P_1}{\partial T} &= \frac{\partial G}{\partial T} f_0'(T-X) + G(X,T) f_0''(T-X) \\ \frac{\partial^2 P_1}{\partial X^2} &= \frac{\partial^2 G}{\partial X^2} f_0'(T-X) - 2 \frac{\partial G}{\partial X} f_0''(T-X) + G(X,T) f_0'''(T-X) \\ \frac{\partial^2 P_1}{\partial T^2} &= \frac{\partial^2 G}{\partial T^2} f_0'(T-X) + 2 \frac{\partial G}{\partial T} f_0''(T-X) + G(X,T) f_0'''(T-X) \\ \therefore \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_1 &= \left( \frac{\partial^2 G}{\partial T^2} - \frac{\partial^2 G}{\partial X^2} \right) f_0' + 2 \left( \frac{\partial G}{\partial T} + \frac{\partial G}{\partial X} \right) f_0''\end{aligned}\quad (4.45)$$

Comparison of Equations (4.43) and (4.45) shows that  $G(X,T)$  can be any function which simultaneously satisfies both of the following partial differential equations:

$$\begin{aligned}\frac{\partial^2 G}{\partial T^2} - \frac{\partial^2 G}{\partial X^2} &= 0 \\ \frac{\partial G}{\partial T} + \frac{\partial G}{\partial X} &= 1\end{aligned}\quad (4.46)$$

The second of these conditions is stronger than the first, as can be demonstrated as follows:

$$\begin{aligned}\frac{\partial^2 G}{\partial T^2} - \frac{\partial^2 G}{\partial X^2} &= 0 \\ \Leftrightarrow \left( \frac{\partial}{\partial T} - \frac{\partial}{\partial X} \right) \left( \frac{\partial G}{\partial T} + \frac{\partial G}{\partial X} \right) &= 0\end{aligned}$$

Substitution of Equation (4.46) now leads to a condition which is always true:

$$\left( \frac{\partial}{\partial T} - \frac{\partial}{\partial X} \right) (1) = 0$$

The homogeneous version of Equation (4.46), for disturbances introduced at  $X = 0$  and travelling rightward into undisturbed fluid, is known to have the general solution  $G(X,T) = f_1(T-X)$ . A particular solution for Equation (4.46) itself (non-homogeneous) is  $G(X,T) = X$ . Consequently a general solution for Equation (4.46) has the form

$$G(X,T) = f_1(T-X) + X$$

Substitution into Equation (4.44) gives

$$P_1(X,T) = [f_1(T-X) + X] f_0'(T-X)$$

Finally, it is noted that addition of this first-order solution to the previous solution of order zero must not be allowed to affect the result at  $X = 0$ , since the boundary condition there is satisfied already by  $P_0$ . Consequently,  $P_1(0, T) = 0$ , giving

$$\begin{aligned} & [f_1(T-0) + 0] f_0'(T-0) = 0 \\ \Rightarrow & f_1(T) P_0'(T) = 0 \\ \Rightarrow & f_1 = 0 \end{aligned}$$

Thus the appropriate solution for the order  $M$  terms in Equation (4.41) is

$$P_1(X, T) = X f_0'(T - X) \quad (4.47)$$

#### Squared $M$ Terms:

When retaining only those terms of Equation (4.41) which are of second order in  $M$ , the following equation results:

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_2 + 2 \frac{\partial^2 P_1}{\partial X \partial T} + \frac{\partial^2 P_0}{\partial X^2} = 0$$

Substitution of the expressions obtained already for  $P_0$  and  $P_1$  (Equations 4.42 and 4.47) leads to another inhomogeneous wave equation:

$$\begin{aligned} & P_0 = f_0(T - X) \\ \Rightarrow & \frac{\partial^2 P_0}{\partial X^2} = f_0''(T - X) \\ & P_1(X, T) = X f_0'(T - X) \\ \Rightarrow & \frac{\partial^2 P_1}{\partial X \partial T} = f_0''(T - X) - X f_0'''(T - X) \\ \therefore & \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_2 = -2 \frac{\partial^2 P_1}{\partial X \partial T} - \frac{\partial^2 P_0}{\partial X^2} \\ & = -3f_0'' + 2X f_0''' \end{aligned} \quad (4.48)$$

Consider a solution of Equation (4.48) which has the form

$$P_2(X, T) = G_{21}(X, T) f_0'(T - X) + G_{22}(X, T) f_0''(T - X) \quad (4.49)$$

If such a solution is available, then its partial derivatives must satisfy Equation (4.48):

$$\frac{\partial P_2}{\partial X} = \frac{\partial G_{21}}{\partial X} f_0'(T - X) + \left( \frac{\partial G_{22}}{\partial X} - G_{21} \right) f_0''(T - X) - G_{22} f_0'''(T - X)$$

$$\frac{\partial P_2}{\partial T} = \frac{\partial G_{21}}{\partial T} f_0'(T-X) + \left( \frac{\partial G_{22}}{\partial T} + G_{21} \right) f_0''(T-X) + G_{22} f_0'''(T-X)$$

$$\begin{aligned} \frac{\partial^2 P_2}{\partial X^2} &= \frac{\partial^2 G_{21}}{\partial X^2} f_0'(T-X) + \left( \frac{\partial^2 G_{22}}{\partial X^2} - 2 \frac{\partial G_{21}}{\partial X} \right) f_0''(T-X) \\ &\quad + \left( G_{21} - 2 \frac{\partial G_{22}}{\partial X} \right) f_0'''(T-X) + G_{22} f_0^{(iv)}(T-X) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 P_2}{\partial T^2} &= \frac{\partial^2 G_{21}}{\partial T^2} f_0'(T-X) + \left( \frac{\partial^2 G_{22}}{\partial T^2} + 2 \frac{\partial G_{21}}{\partial T} \right) f_0''(T-X) \\ &\quad + \left( G_{21} + 2 \frac{\partial G_{22}}{\partial T} \right) f_0'''(T-X) + G_{22} f_0^{(iv)}(T-X) \end{aligned}$$

$$\begin{aligned} \therefore \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_2 &= \left( \frac{\partial^2 G_{21}}{\partial T^2} - \frac{\partial^2 G_{21}}{\partial X^2} \right) f_0' \\ &\quad + \left[ \left( \frac{\partial^2 G_{22}}{\partial T^2} - \frac{\partial^2 G_{22}}{\partial X^2} \right) + 2 \left( \frac{\partial G_{21}}{\partial T} + \frac{\partial G_{21}}{\partial X} \right) \right] f_0'' \\ &\quad + 2 \left( \frac{\partial G_{22}}{\partial T} + \frac{\partial G_{22}}{\partial X} \right) f_0''' \end{aligned}$$

This can now be compared to Equation (4.48), which is repeated below:

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_2 = -3f_0'' + 2X f_0'''$$

Comparison reveals that the following system of partial differential equations must be solved simultaneously if a solution of the proposed form is to be found:

$$\frac{\partial^2 G_{21}}{\partial T^2} - \frac{\partial^2 G_{21}}{\partial X^2} = 0 \quad (4.50)$$

$$\left( \frac{\partial^2 G_{22}}{\partial T^2} - \frac{\partial^2 G_{22}}{\partial X^2} \right) + 2 \left( \frac{\partial G_{21}}{\partial T} + \frac{\partial G_{21}}{\partial X} \right) = -3 \quad (4.51)$$

$$\frac{\partial G_{22}}{\partial T} + \frac{\partial G_{22}}{\partial X} = X \quad (4.52)$$

Equation (4.51) can be simplified using Equation (4.52) as follows:

$$\left( \frac{\partial^2 G_{22}}{\partial T^2} - \frac{\partial^2 G_{22}}{\partial X^2} \right) + 2 \left( \frac{\partial G_{21}}{\partial T} + \frac{\partial G_{21}}{\partial X} \right) = -3$$

$$\begin{aligned}
&\Rightarrow \left(\frac{\partial}{\partial T} - \frac{\partial}{\partial X}\right)\left(\frac{\partial G_{22}}{\partial T} + \frac{\partial G_{22}}{\partial X}\right) + 2\left(\frac{\partial G_{21}}{\partial T} + \frac{\partial G_{21}}{\partial X}\right) = -3 \\
&\Rightarrow \left(\frac{\partial}{\partial T} - \frac{\partial}{\partial X}\right)X + 2\left(\frac{\partial G_{21}}{\partial T} + \frac{\partial G_{21}}{\partial X}\right) = -3 \\
&\Rightarrow \frac{\partial G_{21}}{\partial T} + \frac{\partial G_{21}}{\partial X} = -1 \quad (4.53)
\end{aligned}$$

Equation (4.50) is now seen to be unnecessary, since it is always true when Equation (4.53) is satisfied already (c.f. Equation 4.46).

The homogeneous version of Equation (4.52) is known to have the general solution  $G_{22}(X, T) = f_{22}(T - X)$  for disturbances travelling rightward into undisturbed fluid. A particular solution for (the non-homogeneous) Equation (4.52) is  $G_{22}(X, T) = \frac{1}{2}X^2$ . Consequently a general solution for Equation (4.52) has the form

$$G_{22}(X, T) = f_{22}(T - X) + \frac{1}{2}X^2$$

The homogeneous version of Equation (4.53) is known to have the general solution  $G_{21}(X, T) = f_{21}(T - X)$ . A particular solution for (the non-homogeneous) Equation (4.53) itself is  $G_{21}(X, T) = -X$  and so a general solution for Equation (4.53) has the form

$$G_{21}(X, T) = f_{21}(T - X) - X$$

Substituting both of these expressions into Equation (4.49) gives

$$P_2(X, T) = [f_{21}(T - X) - X]f_0'(T - X) + [f_{22}(T - X) + \frac{1}{2}X^2]f_0''(T - X)$$

The appropriate boundary condition to be imposed upon  $P_2(X, T)$  is that  $P_2(0, T) = 0$ .

Substituting  $X = 0$  into the above expression for  $P_2(X, T)$  gives:

$$\begin{aligned}
&[f_{21}(T - 0) - 0]f_0'(T - 0) + [f_{22}(T - 0) + \frac{1}{2}0^2]f_0''(T - 0) = 0 \\
\Rightarrow &f_{21}(T)f_0'(T) + f_{22}(T)f_0''(T) = 0 \\
\Rightarrow &f_{21} = 0 \quad \text{and} \quad f_{22} = 0
\end{aligned}$$

Thus the appropriate solution for the order  $M^2$  terms in Equation (4.41) is

$$P_2(X, T) = -X f_0'(T - X) + \frac{1}{2}X^2 f_0''(T - X) \quad (4.54)$$

#### General Solution for Order $i$ Terms in $M$ :

When the index  $i$  in Equation (4.41) is two or greater, the equation to be solved after collecting terms in  $M^i$  has the form

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_i + 2 \frac{\partial^2 P_{i-1}}{\partial X \partial T} + \frac{\partial^2 P_{i-2}}{\partial X^2} = 0, \quad i \geq 2 \quad (4.55)$$

The previous solutions for terms of order  $M^0$  (Equation 4.42),  $M^1$  (Equation 4.47) and  $M^2$  (Equation 4.54) seem to suggest that the sequence of solutions for higher orders in  $M$  may have the following form (where the  $g_{ij}$  are constant coefficients):

$$P_i(X, T) = \sum_{j=1}^i g_{ij} X^j f_0^{(j)}(T - X) \quad (4.56)$$

Any function of the form of Equation (4.56) will certainly satisfy the requisite boundary condition  $P_i(0, T) = 0, \forall T$ .

The following proof for the hypothesis that a function with the form of Equation (4.56) can always be found to solve Equation (4.55) for any  $i \geq 2$  utilises the Principle of Mathematical Induction. Equation (4.54) demonstrates validity of Equation (4.56) for the case  $i = 2$ , so it remains only to show that validity of the  $i^{\text{th}}$  term is assured if the expression is known to be valid for all terms before the  $i^{\text{th}}$ .

Assume it is known that Equation (4.56) is valid for  $i - 1$  and  $i - 2$ , i.e.:

$$P_{i-1}(X, T) = \sum_{j=1}^{i-1} g_{(i-1)j} X^j f_0^{(j)}(T - X) \text{ and } P_{i-2}(X, T) = \sum_{j=1}^{i-2} g_{(i-2)j} X^j f_0^{(j)}(T - X).$$

Proof is now obtained in a straightforward manner by substitution of Equation (4.56) into Equation (4.55):

$$\begin{aligned} & \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_i + 2 \frac{\partial^2 P_{i-1}}{\partial X \partial T} + \frac{\partial^2 P_{i-2}}{\partial X^2} = 0 \\ \Rightarrow & \sum_{j=1}^i g_{ij} X^j P_0^{(j+2)} - \sum_{j=1}^i g_{ij} \left[ j(j-1) X^{j-2} P_0^{(j)} - 2jX^{j-1} P_0^{(j+1)} + X^j P_0^{(j+2)} \right] \\ & + 2 \sum_{j=1}^{i-1} g_{(i-1)j} \left[ jX^{j-1} P_0^{(j+1)} - X^j P_0^{(j+2)} \right] \\ & + \sum_{j=1}^{i-2} g_{(i-2)j} \left[ j(j-1) X^{j-2} P_0^{(j)} - 2jX^{j-1} P_0^{(j+1)} + X^j P_0^{(j+2)} \right] = 0 \end{aligned}$$

$$\begin{aligned}
\Rightarrow & -\sum_{j=1}^i g_{ij} j(j-1) X^{j-2} P_0^{(j)} + 2\sum_{j=1}^i g_{ij} j X^{j-1} P_0^{(j+1)} \\
& + 2\sum_{j=1}^{i-1} g_{(i-1)j} j X^{j-1} P_0^{(j+1)} - 2\sum_{j=1}^{i-1} g_{(i-1)j} X^j P_0^{(j+2)} \\
& + \sum_{j=1}^{i-2} g_{(i-2)j} j(j-1) X^{j-2} P_0^{(j)} - 2\sum_{j=1}^{i-2} g_{(i-2)j} j X^{j-1} P_0^{(j+1)} + \sum_{j=1}^{i-2} g_{(i-2)j} X^j P_0^{(j+2)} = 0
\end{aligned}$$

It is helpful to change the indices of some summations to aid the collection of like terms:

$$\begin{aligned}
\Rightarrow & -\sum_{j=1}^i g_{ij} j(j-1) X^{j-2} P_0^{(j)} + 2\sum_{j=2}^{i+1} g_{i(j-1)} (j-1) X^{j-2} P_0^{(j)} \\
& + 2\sum_{j=2}^i g_{(i-1)(j-1)} (j-1) X^{j-2} P_0^{(j)} - 2\sum_{j=3}^{i+1} g_{(i-1)(j-2)} X^{j-2} P_0^{(j)} \\
& + \sum_{j=1}^{i-2} g_{(i-2)j} j(j-1) X^{j-2} P_0^{(j)} - 2\sum_{j=2}^{i-1} g_{(i-2)(j-1)} (j-1) X^{j-2} P_0^{(j)} + \sum_{j=3}^i g_{(i-2)(j-2)} X^{j-2} P_0^{(j)} = 0
\end{aligned}$$

Both terms for which  $j=1$  are zero, since they each have a factor of  $j-1$  contained within them. Terms for  $j=2$  give:

$$-2g_{i2} + 2g_{i1} + 2g_{(i-1)1} + 2g_{(i-2)2} - 2g_{(i-2)1} = 0 \quad (4.57)$$

(Strictly, the fourth of the terms in Equation (4.57) should be excluded for case of  $i=3$ .)

Terms for values of  $j$  ranging from 3 to  $i-2$  give, for  $i > 4$ :

$$\begin{aligned}
& -j(j-1)g_{ij} + 2(j-1)g_{i(j-1)} + 2(j-1)g_{(i-1)(j-1)} - 2g_{(i-1)(j-2)} \\
& + j(j-1)g_{(i-2)j} - 2(j-1)g_{(i-2)(j-1)} + g_{(i-2)(j-2)} = 0
\end{aligned} \quad (4.58)$$

Terms for  $j=i-1$  are, for  $i > 3$ :

$$\begin{aligned}
& -(i-1)(i-2)g_{i(i-1)} + 2(i-2)g_{i(i-2)} + 2(i-2)g_{(i-1)(i-2)} - 2g_{(i-1)(i-3)} \\
& - 2(i-2)g_{(i-2)(i-2)} + g_{(i-2)(i-3)} = 0
\end{aligned} \quad (4.59)$$

Terms for  $j=i$  are, for  $i > 2$ :

$$-i(i-1)g_{ii} + 2(i-1)g_{i(i-1)} + 2(i-1)g_{(i-1)(i-1)} - 2g_{(i-1)(i-2)} + g_{(i-2)(i-2)} = 0 \quad (4.60)$$

Terms for  $j=i+1$  are:

$$2ig_{ii} - 2g_{(i-1)(i-1)} = 0 \quad (4.61)$$

The proof by the principle of induction requires that the  $g_{ij}$  coefficients be determined uniquely by Equations (4.57) through (4.61) if the  $g_{(i-1)j}$  and  $g_{(i-2)j}$  coefficients are known already. This is indeed the case, as Equation (4.61) determines  $g_{ii}$  uniquely, and

the other  $g_{ij}$  coefficients are then available by back-substitution into Equations (4.60) and (4.59), then Equation (4.58) as many times as necessary and finally Equation (4.57).

The proof of the hypothesis that the solution of Equation (4.55) for general values of  $i \geq 2$  (and the given boundary condition) has the form of Equation (4.56) is now complete.

Determination of the Matrix of Coefficients:

A  $g$ -matrix of triangular structure may be generated using an algorithm based upon Equations (4.57) through (4.61). The initial value of  $g_{11} = 1$  comes from Equation (4.47), while the values  $g_{21} = -1$  and  $g_{22} = \frac{1}{2}$  are derived from Equation (4.54). It is a straightforward matter to program such an algorithm using MATLAB. The  $g$ -matrix is found to be related to Pascal's Triangle:

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \\ 1 & -2 & 1 & 0 & 0 & 0 & \\ -1 & 3 & -3 & 1 & 0 & 0 & \\ 1 & -4 & 6 & -4 & 1 & 0 & \\ -1 & 5 & -10 & 10 & -5 & 1 & \\ \vdots & & & & & & \ddots \end{bmatrix} \begin{pmatrix} 1 \\ \frac{1}{2!} \\ \frac{1}{3!} \\ \frac{1}{4!} \\ \frac{1}{5!} \\ \frac{1}{6!} \\ \vdots \end{pmatrix}$$

Recognition of the binomial coefficients within the structure of the  $g$ -matrix leads to an explicit relation for each of the elements, viz.

$$g_{ij} = \frac{(-1)^{(i+j)}}{j!} \frac{(i-1)!}{(i-j)!(j-1)!} \tag{4.62}$$

This expression may now be substituted into Equation (4.56), which is repeated below:

$$\begin{aligned} P_i(X,T) &= \sum_{j=1}^i g_{ij} X^j f_0^{(j)}(T-X) \\ &= \sum_{j=1}^i \frac{(-1)^{(i+j)}}{j!} \frac{(i-1)!}{(i-j)!(j-1)!} X^j f_0^{(j)}(T-X) \end{aligned}$$

Final Power Series Solution for  $P$  in terms of  $M$ :

The above general solution for the terms of order  $M^i$  is seen to be valid for any natural number  $i$ . The final solution of Equation (4.40) can thus be written as the following power series:

$$\begin{aligned}
 P(X, T) &= \sum_{i=0}^{\infty} P_i(X, T) M^i \\
 &= f_0(T - X) + \sum_{i=1}^{\infty} \sum_{j=1}^i \left\{ \frac{(-1)^{(i+j)}}{j!} \frac{(i-1)!}{(i-j)!(j-1)!} M^i X^j f_0^{(j)}(T - X) \right\}
 \end{aligned}$$

Assuming uniform convergence of the infinite series, the order of the summations can be reversed:

$$\begin{aligned}
 \Leftrightarrow P(X, T) &= f_0(T - X) + \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} \left\{ \frac{(-1)^{(i+j)}}{j!} \frac{(i-1)!}{(i-j)!(j-1)!} M^i X^j f_0^{(j)}(T - X) \right\} \\
 \Leftrightarrow P(X, T) &= f_0(T - X) + \sum_{j=1}^{\infty} \left\{ \frac{1}{j!(j-1)!} X^j f_0^{(j)}(T - X) \sum_{i=j}^{\infty} \left[ (-1)^{(i+j)} \frac{(i-1)!}{(i-j)!} M^i \right] \right\} \\
 &= f_0(T - X) + \sum_{j=1}^{\infty} \left\{ \frac{1}{j!(j-1)!} X^j f_0^{(j)}(T - X) \sum_{k=0}^{\infty} \left[ (-1)^{(2j+k)} \frac{(j+k-1)!}{k!} M^{j+k} \right] \right\} \\
 &= f_0(T - X) + \sum_{j=1}^{\infty} \left\{ \frac{1}{j!(j-1)!} (MX)^j f_0^{(j)}(T - X) \sum_{k=0}^{\infty} \left[ \frac{(j+k-1)!}{k!} (-M)^k \right] \right\}
 \end{aligned}$$

Consider now the infinite series with index  $k$ ; it may be considered to be a function of the natural number  $j$ , leading to:

$$P(X, T) = f_0(T - X) + \sum_{j=1}^{\infty} \left\{ \frac{S(j)}{j!(j-1)!} (MX)^j f_0^{(j)}(T - X) \right\} \quad (4.63)$$

$$\text{where } S(j) = \sum_{k=0}^{\infty} \left[ \frac{(j+k-1)!}{k!} (-M)^k \right] \quad (4.64)$$

Setting  $j = 1$  gives:

$$S(1) = \sum_{k=0}^{\infty} (-M)^k = 1 - M + M^2 - M^3 + \dots$$

This is a geometric series; since  $M < 1$  in the current application, the infinite sum has the

$$\text{value } S(1) = \frac{1}{1+M}.$$

By setting  $j = 2$  instead, Equation (4.64) gives:

$$\begin{aligned}
S(2) &= \sum_{k=0}^{\infty} [(k+1)(-M)^k] \\
&= \sum_{k=1}^{\infty} [k(-M)^{k-1}] \\
&= \sum_{k=0}^{\infty} [k(-M)^{k-1}] \\
&= -\frac{d}{dM} S(1) \\
&= \frac{1}{(1+M)^2}
\end{aligned}$$

Similarly  $j = 3$  in Equation (4.64) gives:

$$\begin{aligned}
S(3) &= \sum_{k=0}^{\infty} [(k+1)(k+2)(-M)^k] \\
&= \sum_{k=2}^{\infty} [k(k-1)(-M)^{k-2}] \\
&= \sum_{k=0}^{\infty} [k(k-1)(-M)^{k-1}] \\
&= -\frac{d}{dM} S(2) \\
&= \frac{2}{(1+M)^3}
\end{aligned}$$

An expression for  $S(j)$  for arbitrary values of  $j$  is now quite straightforward:

$$S(j) = \frac{(j-1)!}{(1+M)^j}$$

This expression may be used to simplify Equation (4.63) above:

$$P(X, T) = f_0(T - X) + \sum_{j=1}^{\infty} \left\{ \frac{1}{j!} \left[ \frac{MX}{1+M} \right]^j f_0^{(j)}(T - X) \right\} \quad (4.65)$$

The form of this solution suggests that  $\frac{M}{M+1}$  may have been a more convenient expansion parameter. This is confirmed in the Appendix. The solution in terms of the new expansion parameter also serves as a check, confirming the correctness of Equation (4.65).

### 4.2.3 Alternative Solution via a Change of Variables:

In Section 4.1.2 linearised equations describing conservation of fluid mass and momentum (Equations 4.11 and 4.12) were combined to yield a linear wave equation for the fluctuating fluid pressure  $\tilde{p}$ . These equations are repeated below for convenience

$$\frac{\partial \tilde{p}}{\partial t} + \bar{\rho} \bar{c}^2 \nabla \cdot \tilde{\mathbf{u}} \equiv 0 \qquad \frac{\partial \tilde{\mathbf{u}}}{\partial t} + \frac{1}{\bar{\rho}} \nabla \tilde{p} \equiv \mathbf{0}$$

The appropriate wave equation in one dimension only is (c.f. Equation 4.15)

$$\frac{\partial^2 \tilde{p}}{\partial t^2} - \bar{c}^2 \frac{\partial^2 \tilde{p}}{\partial x^2} = 0 \qquad (4.66)$$

The general solution to this wave equation is well-known:

$$\tilde{p}(x, t) = \tilde{p}_+(x - \bar{c}t) + \tilde{p}_-(x + \bar{c}t)$$

The following auxiliary restrictions shall now be imposed: Firstly, only fluctuations travelling in the positive  $x$  direction will be considered. Secondly, the fluid motion will be considered known at the position  $x=0$  for all values of time  $t \geq 0$ . Let  $\tilde{p}(0, t) = f(t)$  where the value of  $f$  is known for all non-negative values of  $t$ . With these two restrictions enforced, the solution to Equation (4.66) has the form (c.f. Equation 4.42):

$$\tilde{p}(x, t) = f\left(t - \frac{x}{\bar{c}}\right) \qquad (4.67)$$

In Section 4.2.1 the standard equations of linear acoustics were modified to take into account additional mean flow of the fluid medium. The resulting modified wave equation (Equation 4.36) was obtained from the standard formulation (Equations 4.31 and 4.32) via a simple change of variables (Equation 4.33). These equations are repeated below :

$$\frac{\partial \tilde{p}}{\partial t'} + \bar{\rho} \bar{c}^2 \frac{\partial \tilde{u}_i}{\partial x'_i} \equiv 0 \qquad \frac{\partial \tilde{u}_i}{\partial t'} + \frac{1}{\bar{\rho}} \frac{\partial \tilde{p}}{\partial x'_i} \equiv 0 \qquad (4.31; 4.32)$$

$$x_i = x'_i + \bar{u}_i t' \qquad t = t' \qquad (4.33)$$

$$\therefore \left( \frac{\partial}{\partial t} + \bar{u}_i \frac{\partial}{\partial x_i} \right)^2 \tilde{p} - \bar{c}^2 \frac{\partial^2 \tilde{p}}{\partial x_i \partial x_i} \equiv 0 \qquad (4.36)$$

Now since the change of variables described by Equation (4.33) converts Equation (4.66) into Equation (4.36), that same transformation should also convert the solution of

Equation (4.66) (which is Equation 4.67 above) into the required solution of Equation (4.36). Writing Equation (4.67) using primed quantities first, the substitutions given by Equation (4.33) gives:

$$\begin{aligned}\tilde{p}(x', t') &= f\left(t' - \frac{x'}{\bar{c}}\right) \\ \Rightarrow \tilde{p}(x, t) &= f(\bar{c}t - x + \bar{u}t)\end{aligned}\quad (4.68)$$

The same scheme (Equations 3.38) used to non-dimensionalise the modified wave equation, can also be applied to Equation (4.68):

$$\begin{aligned}\tilde{p}(X, T) &= f\left(\bar{c} \frac{L}{\bar{c}} T - LX + \bar{u} \frac{L}{\bar{c}} T\right) \\ &= f\left(\left[\frac{\bar{u}}{\bar{c}} + 1\right] T - X\right)\end{aligned}$$

In terms of the Mach number defined in Section 4.2.2 as  $M = \frac{\bar{u}}{\bar{c}}$ , this equation becomes

$$\tilde{p}(X, T) = f((M + 1)T - X) \quad (4.69)$$

Alternatively, in terms of the dimensionless parameter N defined in the Appendix by Equation (A1):

$$\tilde{p}(X, T) = f\left(\frac{T}{1 - N} - X\right) \quad (4.70)$$

Necessary in the reconciliation of Equation (4.69) with the previous power series solution Equation (4.65) - also Equation (A16) in the Appendix - is recognition that the power series solution is actually in the form of a Taylor Series. Compare Equation (4.65), repeated here, with Taylor's formula for  $f_0(z)$  with centre at  $a$  (Equation 4.71 below) :

$$\begin{aligned}p(X, T) &= \sum_{i=0}^{\infty} \frac{\left[\frac{M}{M+1} X\right]^i}{i!} f_0^{(i)}(T - X) \\ f_0(z) &= \sum_{i=0}^{\infty} \frac{(z - a)^i}{i!} f_0^{(i)}(a)\end{aligned}\quad (4.71)$$

Comparison yields that  $a = T - X$  and  $z - a = \frac{M}{M+1} X$ . Consequently,

$$p(X, T) = f_0\left(\frac{-1}{M+1} X + T\right) \quad (4.72)$$

The relationship between this equation and Equation (4.69) is obvious.

### 4.3 Study of the Characteristics of 1-D Inviscid Flow Equations:

Recall from Section 4.2.3 that the solution for the modified wave equation in one dimension (Equation 4.37, repeated below) is given by Equation (4.68) (also repeated below).

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x}\right)^2 \tilde{p} - \bar{c}^2 \frac{\partial^2 \tilde{p}}{\partial x^2} \equiv 0 \quad (4.37)$$

$$\tilde{p}(x, t) = f((\bar{u} + \bar{c})t - x) \quad (4.68)$$

The function  $f$  is determined by the boundary condition specified at  $x = 0$ . It can be seen from the form of Equation (4.68) that the solution for  $\tilde{p}$  has a constant value along any curve for which the value of  $(\bar{u} + \bar{c})t - x$  remains constant. This statement holds true regardless of the specifics of the function  $f$ . In fact,  $f$  is not even required to have continuous derivatives in order for  $f$  to be constant along the curve described by

$$(\bar{u} + \bar{c})t - x = \text{const} \quad (4.73)$$

The Equation (4.73) is known as a *characteristic* of Equation (4.37). Information propagates through the solution domain along the characteristic curves of a partial differential equation. If discontinuities in the derivative of the dependent variable do exist, then these can propagate through the solution domain only along the characteristic curves. (If no real characteristics exist for a particular PDE, then there are no preferred paths of information propagation.)

#### 4.3.1 Characteristics of the Linear Acoustic Equations:

Because discontinuities in the derivatives of the solution, if they exist, must propagate along the characteristic curves, one approach to finding the characteristics is to determine whether there are any paths in the solution domain along which the derivatives of the solution are multi-valued or discontinuous.

In one spatial dimension, acoustic pressure fluctuations in a fluid otherwise at rest propagate according to the equation

$$\tilde{p}_{tt} - \bar{c}^2 \tilde{p}_{xx} = 0 \quad (4.74)$$

(A subscript is here used to denote partial differentiation).

Two more equations relating the second partial derivatives of  $\tilde{p}$  are obtained by applying the chain rule to determine the total derivatives of  $f_x$  and  $f_t$ , since these are themselves functions of  $x$  and  $t$  :

$$d(\tilde{p}_x) = \tilde{p}_{xx} dx + \tilde{p}_{xt} dt \quad (4.75)$$

$$d(\tilde{p}_t) = \tilde{p}_{tx} dx + \tilde{p}_{tt} dt$$

The preceding two equations can be combined with Equation (4.74) and written in matrix form as follows :

$$\begin{bmatrix} 1 & 0 & -\bar{c}^2 \\ dt & dx & 0 \\ 0 & dt & dx \end{bmatrix} \begin{pmatrix} \tilde{p}_{tt} \\ \tilde{p}_{xt} \\ \tilde{p}_{xx} \end{pmatrix} = \begin{pmatrix} 0 \\ d(\tilde{p}_t) \\ d(\tilde{p}_x) \end{pmatrix} \quad (4.76)$$

Equation (4.76) can be solved to yield unique finite values of  $\tilde{p}_{tt}$ ,  $\tilde{p}_{xt}$  and  $\tilde{p}_{xx}$  unless the determinant of the coefficient matrix vanishes. In that case, the second derivatives of  $\tilde{p}(x,t)$  are either infinite or they are indeterminate, and thus multi-valued or discontinuous.

Setting the determinant of the coefficient matrix of Equation (4.76) to zero yields

$$\begin{aligned} 1(dx)^2 - \bar{c}^2(dt)^2 &= 0 \\ \Rightarrow (dx)^2 &= (\bar{c} dt)^2 \\ \Rightarrow dx &= \pm \bar{c} dt \\ \Rightarrow \frac{dx}{dt} &= \pm \bar{c} \end{aligned} \quad (4.77)$$

Equation (4.77) shows that the linear wave equation (Equation 4.74) has two families of real characteristics. The *forward-running* characteristics have a slope of  $+\bar{c}$ . Information propagates along these in the positive  $x$  direction as time progresses. The forward-running characteristics are described by equations of the form

$$x = \bar{c}t + const$$

The backward-running characteristics have equations of the form  $x = -\bar{c}t + const$ .

Determination of the characteristics of a partial differential equation is very useful, since along the characteristic curves of a PDE an ordinary differential equation instead describes the behaviour of the solutions. The ODE valid along a certain characteristic curve is called the *compatibility equation* for that characteristic.

Compatibility equations for the governing equations of linear acoustics can be determined by substituting the characteristic equation into the original PDEs as follows. First note that Equation (4.77) shows that the value of  $x$  along the characteristic depends upon the single variable  $t$ , so that it is possible to make the following substitutions along such characteristics:

$$\frac{\partial}{\partial t} \rightarrow \frac{d}{dt} \qquad \frac{\partial}{\partial x} \rightarrow \frac{dt}{dx} \frac{d}{dt} = \frac{1}{\pm \bar{c}} \frac{d}{dt}$$

Consequently

$$\begin{aligned} \frac{\partial \tilde{p}}{\partial t} + \bar{\rho} \bar{c}^2 \frac{\partial \tilde{u}}{\partial x} &= 0 \\ \Rightarrow \frac{d\tilde{p}}{dt} + \bar{\rho} \bar{c} \left( \frac{1}{\pm \bar{c}} \frac{d\tilde{u}}{dt} \right) &= 0 \\ \Rightarrow \frac{d}{dt} (\tilde{p} \pm \bar{\rho} \bar{c} \tilde{u}) &= 0 \\ \Rightarrow \tilde{p} \pm \bar{\rho} \bar{c} \tilde{u} &= \text{const.} \end{aligned} \tag{4.78}$$

The compatibility equation  $\tilde{p} + \bar{\rho} \bar{c} \tilde{u} = \text{const.}$  is valid along forward-running characteristic curves of the form  $\frac{dx}{dt} = +\bar{c}$ . The compatibility equation  $\tilde{p} - \bar{\rho} \bar{c} \tilde{u} = \text{const.}$  is valid along backward-running characteristics, described by  $\frac{dx}{dt} = -\bar{c}$ . In general, the value of the constant of Equation (4.78) will be different for each characteristic.

In this subsection, interest is focused upon propagation into quiescent fluid (fluid at rest) of a known disturbance introduced at the position  $x = 0$ . The behaviour of fluid for positive values of  $x$  and  $t$  only is to be studied.

Figure 4.1 below depicts four forward-running characteristics that originate from  $x = 0$  at various times  $t \geq 0$ . The value of the constant in the compatibility equation  $\tilde{p} + \bar{\rho} \bar{c} \tilde{u} = \text{const.}$  depends, at each of these four instants, upon the flow perturbation introduced at  $x = 0$  (which is assumed known).

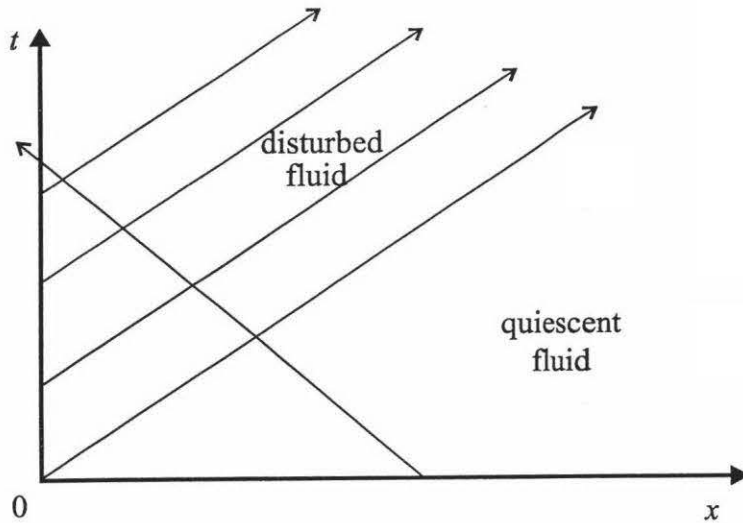


Figure 4.1 Characteristics in the  $xt$ -plane for disturbance at  $x = 0$

The case of a backward-running characteristic originating in quiescent fluid is especially interesting. Recall that along any such characteristic, the appropriate compatibility equation has the form  $\tilde{p} - \bar{\rho}\bar{c}\tilde{u} = \text{const.}$  Because each backward-running characteristic originates in quiescent fluid - where the fluid velocity,  $u = \tilde{u}$  (Equation 4.2), is zero and the fluid pressure,  $p = \bar{p} + \tilde{p}$  (Equation 4.4), has the constant value  $\bar{p}$  - the compatibility equation along every backward-running characteristic is the same:

$$\begin{aligned} \tilde{p} - \bar{\rho}\bar{c}\tilde{u} &= 0 \\ \Rightarrow \tilde{p} &= \bar{\rho}\bar{c}\tilde{u} \end{aligned} \quad (4.79)$$

Every point in the first quadrant of the  $xt$ -plane can be associated with a backward-running characteristic that originates in quiescent fluid. Consequently Equation (4.79) is valid for all positive values of  $x$  and  $t$ .

This new information can be used to update the compatibility equation for the forward-running characteristics also. Substituting Equation (4.79) into Equation (4.78) gives

$$\begin{aligned} \tilde{p} + \bar{\rho}\bar{c}\tilde{u} &= \text{const.} \\ \Rightarrow 2\tilde{p} &= 2\bar{\rho}\bar{c}\tilde{u} = \text{const.} \end{aligned}$$

This leads to the conclusion that  $\tilde{p}$  and  $\tilde{u}$  both take constant values along each forward-running characteristic.

### 4.3.2 Modifications due to Superimposed Uniform Flow:

While Section 4.3.1 examined the propagation of small disturbances at  $x = 0$  into quiescent fluid, this subsection examines perturbations (introduced at  $x = 0$ ) about a state of uniform fluid motion; the velocity of the undisturbed fluid medium has everywhere the value  $\bar{u}$  in the positive  $x$ -direction, and the uniform value of the pressure of the unperturbed fluid is  $\bar{p}$ .

If the acoustic pressure fluctuations described in Section 4.3.1 are superimposed upon a uniform mean flow, with constant fluid velocity aligned with the direction of propagation, then the appropriately modified wave equation was found to be (Equation 4.37)

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x}\right)^2 \tilde{p} - \bar{c}^2 \frac{\partial^2 \tilde{p}}{\partial x^2} \equiv 0$$

or

$$\frac{\partial^2 \tilde{p}}{\partial t^2} - 2\bar{u} \frac{\partial^2 \tilde{p}}{\partial x \partial t} + (\bar{u}^2 - \bar{c}^2) \frac{\partial^2 \tilde{p}}{\partial x^2} = 0$$

This may be combined with Equations (4.75) which are repeated below

$$d(\tilde{p}_x) = \tilde{p}_{xx} dx + \tilde{p}_{xt} dt$$

$$d(\tilde{p}_t) = \tilde{p}_{tx} dx + \tilde{p}_{tt} dt$$

The following matrix equation results :

$$\begin{bmatrix} 1 & -2\bar{u} & (\bar{u}^2 - \bar{c}^2) \\ dt & dx & 0 \\ 0 & dt & dx \end{bmatrix} \begin{bmatrix} \tilde{p}_{tt} \\ \tilde{p}_{xt} \\ \tilde{p}_{xx} \end{bmatrix} = \begin{bmatrix} 0 \\ d(\tilde{p}_t) \\ d(\tilde{p}_x) \end{bmatrix} \quad (4.80)$$

Setting the determinant of the coefficient matrix to zero this time yields :

$$\begin{aligned} & 1(dx)^2 - (-2\bar{u})dxdt + (\bar{u}^2 - \bar{c}^2)(dt)^2 = 0 \\ \Rightarrow & (dx + \bar{u}dt)^2 = (\bar{c}dt)^2 \\ \Rightarrow & dx + \bar{u}dt = \pm \bar{c}dt \\ \Rightarrow & \frac{dx}{dt} = \bar{u} \pm \bar{c} \end{aligned} \quad (4.81)$$

This equation is identical to Equation (4.73).

Compatibility equations for this problem can be developed by substituting the characteristic equation into one of the original governing equations (either Equation 4.34 or 4.35). First note that Equation (4.81) shows that the value of  $x$  along the characteristic

depends upon the single variable  $t$ , so that it is possible to make the following substitutions along such characteristics:

$$\frac{\partial}{\partial t} \rightarrow \frac{d}{dt} \quad \frac{\partial}{\partial x} \rightarrow \frac{dt}{dx} \frac{d}{dt} = \frac{1}{\bar{u} \pm \bar{c}} \frac{d}{dt}$$

This leads to

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \tilde{p} + \bar{\rho} \bar{c}^2 \frac{\partial \tilde{u}}{\partial x} = 0 \\ \Rightarrow & \frac{d\tilde{p}}{dt} + \bar{u} \left( \frac{1}{\bar{u} \pm \bar{c}} \frac{d\tilde{p}}{dt} \right) + \bar{\rho} \bar{c}^2 \left( \frac{1}{\bar{u} \pm \bar{c}} \frac{d\tilde{u}}{dt} \right) = 0 \\ \Rightarrow & (2\bar{u} \pm \bar{c}) \frac{d\tilde{p}}{dt} + \bar{\rho} \bar{c}^2 \frac{d\tilde{u}}{dt} = 0 \\ \Rightarrow & \tilde{p} + \frac{\bar{\rho} \bar{c}^2}{2\bar{u} \pm \bar{c}} \tilde{u} = \text{const.} \end{aligned}$$

The equation  $\tilde{p} + \frac{\bar{\rho} \bar{c}^2}{2\bar{u} - \bar{c}} \tilde{u} = \text{const}$  is valid along backward-running characteristics. All backward-running characteristics originate in fluid where the fluid pressure is the uniform value  $\bar{p}$  and the fluid velocity has the value  $\bar{u}$ . Consequently the following equation holds true within the interior of the whole first quadrant of the  $xt$ -plane (c.f. Figure 4.1)

$$\begin{aligned} & \tilde{p} + \frac{\bar{\rho} \bar{c}^2}{2\bar{u} - \bar{c}} \tilde{u} = 0 \\ \Rightarrow & \tilde{p} = \frac{\bar{\rho} \bar{c}^2}{\bar{c} - 2\bar{u}} \tilde{u} \end{aligned} \quad (4.82)$$

The equation  $\tilde{p} + \frac{\bar{\rho} \bar{c}^2}{2\bar{u} + \bar{c}} \tilde{u} = \text{const.}$  is valid along forward-running characteristics, and each characteristic will generally have its own value for the constant in this equation. Equation (4.82) may profitably be substituted into this compatibility equation as follows:

$$\begin{aligned} & \tilde{p} + \frac{\bar{\rho} \bar{c}^2 \tilde{u}}{2\bar{u} + \bar{c}} = \text{const.} \\ \Rightarrow & \frac{\bar{\rho} \bar{c}^2}{\bar{c} - 2\bar{u}} \tilde{u} + \frac{\bar{\rho} \bar{c}^2}{2\bar{u} + \bar{c}} \tilde{u} = \text{const.} \\ \Rightarrow & \tilde{u} = \text{const.} \end{aligned}$$

As with the case of linear acoustics studied in Section 4.3.1, the fluctuating component of the velocity  $\tilde{u}$  remains constant along each of the forward-running characteristic curves.

Reconsideration of the forward-running compatibility equation shows that the fluctuating component of the pressure  $\tilde{p}$  also remains constant along such characteristics.

### 4.3.3 Propagation of Larger Amplitude Fluctuations:

Characteristic equations will now be found for general one-dimensional inviscid fluid motions in the absence of significant body forces; the governing equations will no longer be linearised, but barotropic fluid motion will again be assumed. Using subscripts to denote partial differentiation, the equation of continuity for general one-dimensional flows is (c.f. Equation 3.17)

$$\rho_t + u\rho_x + \rho u_x = 0 \quad (4.83)$$

The 1-D momentum equation for an inviscid fluid is (c.f. Equation 3.18)

$$u_t + uu_x + \frac{1}{\rho} p_x = 0 \quad (4.84)$$

If the fluid motion is barotropic, then it is possible to write (c.f. Equation 4.3)

$$dp = c^2(p)d\rho$$

Notice that the variable  $c$ , unlike the constant  $\bar{c}$ , is a function of the fluid pressure  $p$ . The above equation may be rewritten in terms of the material derivatives following the fluid motion:

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt}$$

$$\text{or} \quad p_t + up_x = c^2(\rho_t + u\rho_x)$$

This may be substituted into Equation (4.83) to remove the partial derivatives of  $\rho$ :

$$p_t + up_x + \rho c^2 u_x = 0 \quad (4.85)$$

Two more relationships are provided by the total derivatives of  $u$  and  $p$ :

$$du = u_t dt + u_x dx$$

$$dp = p_t dt + p_x dx$$

The following matrix equation combines these with Equations (4.84) and (4.85)

$$\begin{bmatrix} \rho & \rho u & 0 & 1 \\ 0 & \rho c^2 & 1 & u \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} \begin{pmatrix} u_t \\ u_x \\ p_t \\ p_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ du \\ dp \end{pmatrix}$$

The characteristic equation is determined by setting the determinant of the coefficient matrix equal to zero. The result is

$$-\rho(dx)(dx - udt) + (dt)[\rho u(dx - udt) + \rho c^2 dt] = 0$$

This is a quadratic equation for  $\frac{dx}{dt}$ ; rearranging yields

$$(dx)^2 - 2(dx)(dt) + (u^2 - c^2)(dt)^2 = 0$$

Solving for  $\frac{dx}{dt}$  by the quadratic formula gives

$$\frac{dx}{dt} = u \pm c \quad (4.86)$$

Equation (4.86) shows that there are two distinct real roots associated with the characteristic equation. However, in this non-linear fluid flow case,  $u$  and  $c$  are not constants (c.f. Equation 4.81) so the characteristics are curved lines that depend upon the solution itself. At each point in the flow, signals propagate at the local acoustic speed  $c$  with respect to the local fluid velocity  $u$ , and at the absolute local speeds  $u \pm c$  with respect to the fixed co-ordinate system.

Equation (4.86) also shows that the characteristic equation is no longer of the simple form  $x = x(t)$ . Instead  $x = x(t, u, c)$  and this means that simple substitutions such as  $\frac{\partial}{\partial t} \rightarrow \frac{d}{dt}$  and  $\frac{\partial}{\partial x} \rightarrow \frac{1}{u \pm c} \frac{d}{dt}$  are not able to be used to convert the partial derivatives of Equations (4.84) or (4.85) to ordinary derivatives. This makes determination of the compatibility equation for these characteristics somewhat more of a challenge.

The classical derivation of the compatibility equations valid along the characteristics of Equations (4.84) and (4.85) is due to Riemann (Lighthill, 1978). The following integral is very important in the derivation, and so it is given its own symbol. The function  $\wp$  is defined by

$$\wp = \int_0^p \frac{dp}{\rho c} \quad (4.86)$$

Note that because the motion is assumed barotropic, the variables  $\rho$  and  $c$  appearing within this integral are functions of the fluid pressure alone. The function  $\wp$  is also a function of the fluid pressure.

In terms of  $\wp$ , Equations (4.84) and (4.85) can be rewritten as follows (noting that  $d\wp = \frac{dp}{\rho c}$ ):

$$u_t + uu_x + c\wp_x = 0$$

$$\wp_t + u\wp_x + cu_x = 0$$

The usefulness of the transformation described by Equation (4.87) is shown by taking the sum of these two equations to give Equation (4.88) below, and taking the difference to the Equation (4.89):

$$(u + \wp)_t + (u + c)(u + \wp)_x = 0 \quad (4.88)$$

$$(u - \wp)_t + (u - c)(u - \wp)_x = 0 \quad (4.89)$$

Equation (4.88) states that the quantity  $u + \wp$  is constant along any curve which obeys  $dx = (u + c)dt$ , (forward-running characteristic) while Equation (4.89) states that the quantity  $u - \wp$  is constant along curves obeying  $dx = (u - c)dt$  (backward-running characteristics). The fluid pressure may be recovered from the quantity  $\wp$  via Equation (4.86).

Consider now the value of  $u - \wp$  along a backward-running characteristic which originates in fluid moving with a uniform mean flow  $\bar{u}$ . If the uniform pressure of the fluid in the mean flow has value  $\bar{p}$ , then

$$\begin{aligned} u - \wp &= \bar{u} - \int_0^{\bar{p}} \frac{dp}{\rho c} \\ &= \bar{u} - \bar{\wp}, \text{ say.} \end{aligned} \quad (4.90)$$

Equation (4.90) is valid in the interior of the whole first quadrant of the  $xt$ -plane, and so the compatibility equation along forward-running characteristics may be simplified accordingly from Equation (4.88):

$$\begin{aligned}u + \wp &= \text{const.} \\ \Rightarrow (u - \wp) + 2\wp &= \text{const.} \\ \Rightarrow \bar{u} - \bar{\wp} + 2\wp &= \text{const.} \\ \Rightarrow \wp &= \text{const.}\end{aligned}$$

Recall that  $d\wp = \frac{1}{\rho c} dp$ , so that

$$\begin{aligned}d\wp = 0 &\Rightarrow \frac{1}{\rho c} dp = 0 \\ \Rightarrow dp &= 0\end{aligned}$$

This equation dictates that the fluid pressure is constant along a characteristic running forward into uniformly moving fluid. Consequently the fluid velocity is also constant along such a characteristic.

#### **4.4 Energy Transported by a 1-D Modulated Flow:**

The energy content at a particular place in a fluid flow is often described in terms of either the *specific energy* or the *energy density* of the fluid. The specific energy is that amount of energy that the fluid has per unit mass, while the energy density is that amount it has per unit volume. The value of the energy density of a flow at a certain place and time may be obtained from knowledge of the specific energy by simply multiplying by the fluid density  $\rho(x,t)$ .

The energy of a fluid will in general consist of kinetic energy and internal energy components. These quantities were introduced in Section 2.4. The combined energy density at place  $x$  and time  $t$  is here denoted  $\varepsilon$ :

$$\varepsilon = \frac{1}{2}\rho(\mathbf{u} \cdot \mathbf{u}) + \rho e$$

In one-dimensional flows this simplifies to:

$$\varepsilon = \frac{1}{2}\rho u^2 + \rho e \quad (4.91)$$

In Equation (4.91),  $e$  represents the specific internal energy of the fluid, which is related to any general 3-D motion of a Newtonian fluid through Equation (2.19), repeated here:

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = -p \frac{\partial u_i}{\partial x_i} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \lambda \left( \frac{\partial u_i}{\partial x_i} \right)^2 + \kappa \frac{\partial^2 T}{\partial x_i \partial x_i}$$

In one dimensional flows this simplifies to

$$\rho \left( \frac{\partial e}{\partial t} + u \frac{\partial e}{\partial x} \right) = -p \frac{\partial u}{\partial x} + (2\mu + \lambda) \left( \frac{\partial u}{\partial x} \right)^2 + \kappa \frac{\partial^2 T}{\partial x^2}$$

If the fluid motion is inviscid then  $\mu = 0$  and  $\lambda = 0$ . If it is adiabatic then  $\kappa = 0$ . Thus for inviscid adiabatic motion in one dimension, the specific internal energy obeys

$$e_t + ue_x + \frac{p}{\rho} u_x = 0 \quad (4.92)$$

Comparison with Equation (4.85) (repeated below), which is valid for any general barotropic motion of a one-dimensional flow, gives

$$\begin{aligned}
 p_t + up_x + \rho c^2 u_x &= 0 \\
 \Rightarrow e_t + ue_t &= \frac{p}{\rho^2 c^2} (p_t + up_x) \\
 \Rightarrow \frac{De}{Dt} &= \frac{p}{\rho^2 c^2} \frac{Dp}{Dt} \\
 \Rightarrow de &= \frac{p}{\rho^2 c^2} dp
 \end{aligned}$$

It is known from Section 4.3.3 that  $dp = 0$  along a characteristic running forward into fluid in uniform motion. Consequently  $de = 0$  also along such a characteristic.

Any small change in the energy density  $\varepsilon$  obeys the following equation, obtained by differentiation of Equation (4.91)

$$\begin{aligned}
 d\varepsilon &= d\left(\frac{1}{2}\rho u^2 + \rho e\right) \\
 &= \left(\frac{1}{2}u^2 + e\right)d\rho + \rho u du + \rho de
 \end{aligned} \tag{4.93}$$

It has been shown above that  $de = 0$  for any characteristic running forward into quiescent fluid. In Section 4.3.3 it was shown that  $dp = 0$  and  $du = 0$  also along such a characteristic. If the fluid motion is barotropic then  $dp = 0 \Rightarrow d\rho = 0$ . Consequently Equation (4.93), describing any small change in the energy density, becomes, for the case of changes along characteristic curve running forward into uniformly moving fluid:

$$d\varepsilon = 0 \tag{4.94}$$

Equation (4.94) shows that fluid energy is conserved as the disturbance propagates into the quiescent fluid. All fluctuations in the energy density  $\varepsilon(x, t)$  propagate unattenuated in the positive  $x$ -direction with a speed of propagation given by the local slope of the forward-running characteristic curve

$$\frac{dx}{dt} = u(x, t) + c(x, t)$$

## **Experimental Observations of a Real Modulated Flow**

This chapter describes an experimental investigation into the nature of air-flow issuing from between the vibrating lips of trombonist Jay Bulen. The experiment was designed specifically to determine the nature of energy transport within the resulting fluid motions downstream of the lips. The experiment was conducted by the author in November of 1994, under the oversight of Associate-Professor Douglas Keefe at the University of Washington, Seattle.

### **5.1 Experimental Procedure:**

A considerable amount of information about the nature of a fluctuating flow along a duct can be determined from pressure measurements that are recorded simultaneously by two or more transducers at different locations along the duct.

The duct used was a 60 cm length of 19 mm pipe, of circular cross-section. Holes were drilled and threaded so that pressure transducers could be installed at four positions along the duct. These positions were 10 mm, 20 mm and 30 mm from one end of the duct (designated the 'top' end) and 20 mm from the other ('bottom') end of the duct.

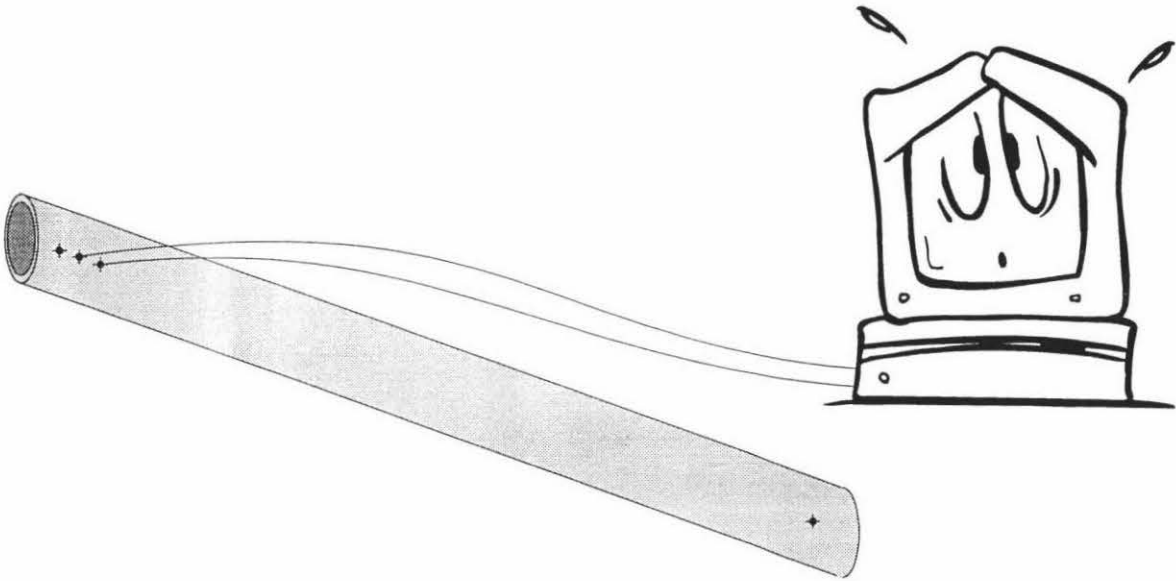


Figure 5.1 Experimental Apparatus

Two pressure transducers were available, and these were shifted between experiments to allow different sets of measurements. The vacant transducer holes were plugged, and then sealed with wax.

Transducers and plugs were inserted, as closely as possible, to be flush with the inner surface of the pipe.

The analogue output signals of the two pressure transducers were, for each experiment, sampled at 44.1 kHz (16-bit resolution, linear scaling) and stored by a *NeXT* computer in the form of stereo *.snd* (sound) files. The resulting data files were later imported into MATLAB for analysis.

## 5.2 Calibration:

No two pressure transducers can ever have exactly the same response characteristics. In an experiment such as the one described in this chapter (in which the output of two transducers are compared) some indication of the differences in the performance of transducers themselves is desirable.

The procedure used to compare the transfer functions of the two pressure transducers is known as *calibration by reciprocity*. The two transducers were installed in the pipe as indicated in Figure 5.2 below. The transducer locations in the pipe are designated *A* and *B*. When the pipe is excited with a known acoustical disturbance, the resulting pressure at position *A* is denoted  $p_A$  and the pressure at position *B* is denoted  $p_B$  (these are the *actual pressures*, not the transducer outputs).

The transducers are designated *L* and *R*. The *transfer functions* of the transducers (defined by the relationship between pressure input signal and electrical output signal) are denoted  $H_L$  and  $H_R$ . The *signal output* of each transducer is written as *S*, with two subscripts indicating which transducer is used (*L* or *R*) and its placement in the duct (*A* or *B*).

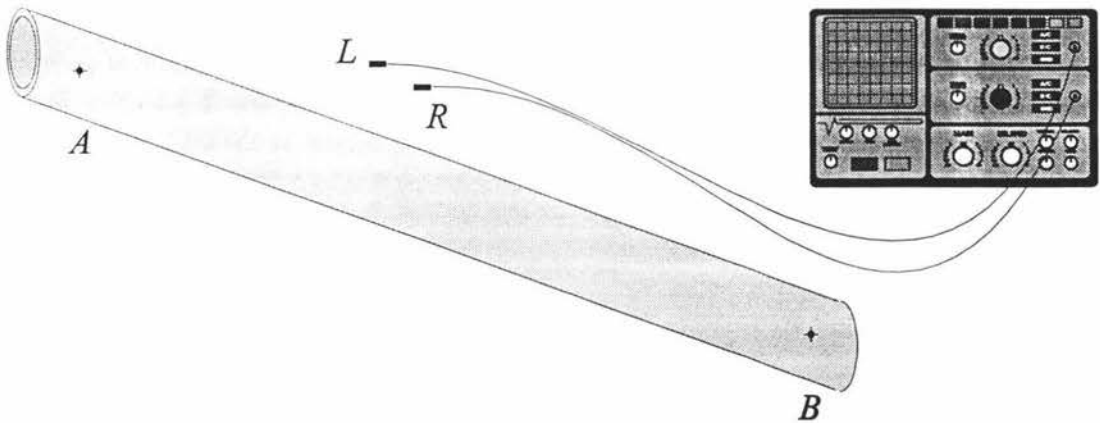


Figure 5.2 Calibration by Reciprocity

The object of the calibration procedure is to determine the ratio  $H_R:H_L$ . The following development shows how this can be achieved:

$$\begin{aligned}
S_{LA} &= H_L P_A & S_{RB} &= H_R P_B \\
S_{RA} &= H_R P_A & S_{LB} &= H_L P_B \\
\Rightarrow \frac{S_{RB}}{S_{LA}} &= \frac{H_R P_B}{H_L P_A} & \text{and} & \frac{S_{LB}}{S_{RA}} = \frac{H_L P_B}{H_R P_A} \\
\Rightarrow \frac{\frac{S_{RB}}{S_{LA}}}{\frac{S_{LB}}{S_{RA}}} &= \frac{H_R P_B}{H_L P_A} \frac{H_R P_A}{H_L P_B} = \left( \frac{H_R}{H_L} \right)^2 \\
\Rightarrow \frac{H_R}{H_L} &= \sqrt{\frac{\frac{S_{RB}}{S_{LA}}}{\frac{S_{LB}}{S_{RA}}}} & & (5.1)
\end{aligned}$$

A signal generator with lock-in amplifier was set to produce a sine-wave. The signal generator was used to drive a piezoelectric transducer affixed to one end of the pipe; the other end of the pipe was stopped (to give a pressure anti-node). The magnitude and phase characteristics of each transducer were recorded as the signal generator performed a frequency sweep of the audio spectrum. The resulting magnitude plot of the ratio in Equation (5.1) is shown in Figure 5.3 to follow. The average magnitude is 1.10 db.

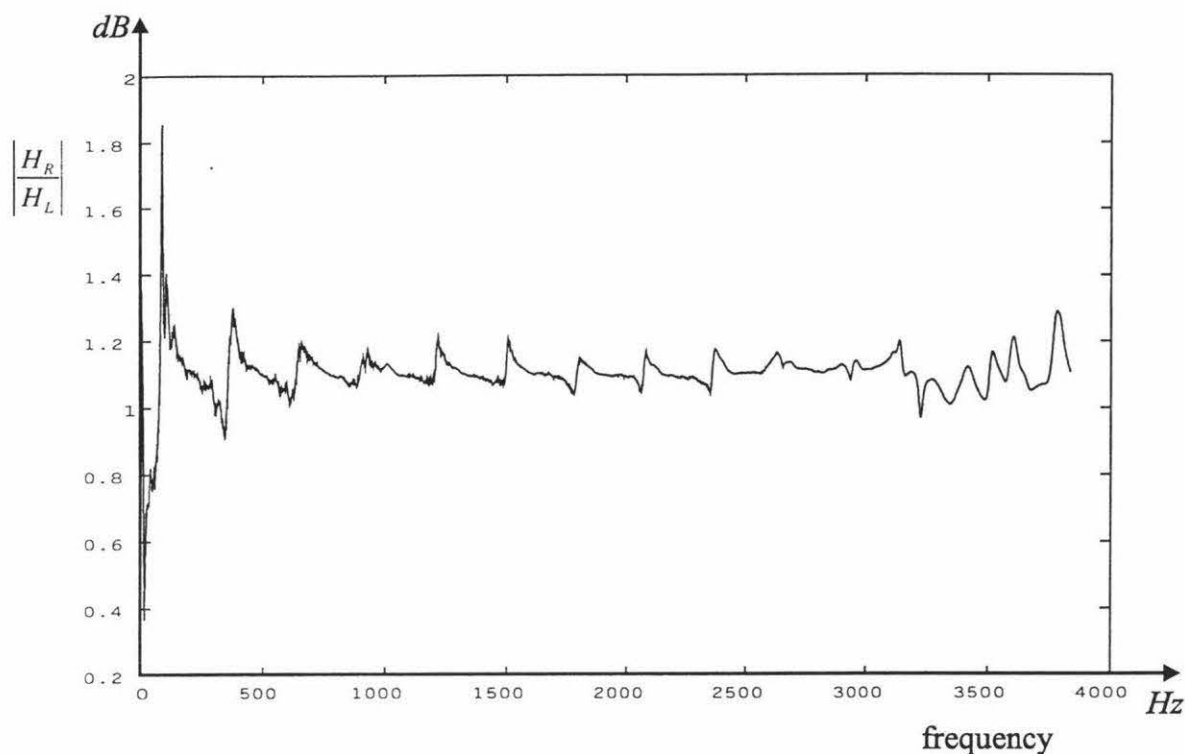


Figure 5.3 Comparison Between the Transducer Transfer Functions

#### Calibration of Dual-Channel ADC

The analogue-to-digital converter used had gain control for each channel. In order to check the effect of difference between the gain-settings of the two channels, a signal generator was used to input a sinusoid into both channels. The sine wave used had a frequency of 1 kilohertz, and an amplitude of 1 volt (RMS).

The measured outputs of the dual channel ADC showed a difference of 2.01dB between the channels. This difference in gain was taken as being representative throughout the audio range.

### 5.3 Results:

Three characteristic times are important in this experiment:

- (i) The time required for a pressure disturbance to be propagated a distance of 10mm from one transducer to the next is obtained by dividing this distance by the speed of sound propagation.

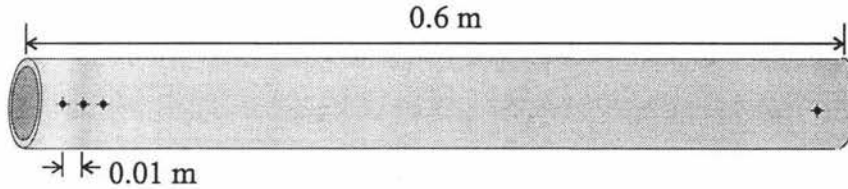


Figure 5.4: Characteristic Lengths for Calculation of Characteristic Times

Estimating the velocity of sound propagation at conditions inside the pipe to be  $345\text{ms}^{-1}$ ,

the *time delay between transducers* is  $\frac{0.01\text{ m}}{345\text{ ms}^{-1}} = 29\text{ }\mu\text{s}$ .

- (ii) A first approximation to the *round-trip propagation time* (the time required for disturbances to propagate the distance of the tube, and then return after any reflection at the other tube end) may be calculated in a similar manner:

$$\frac{2 \times 0.6\text{ m}}{345\text{ ms}^{-1}} = 3478\text{ }\mu\text{s}$$

A better estimate may be obtained by taking also into account the convection speed of the flow through the pipe. Consider that the pipe flow may approximated reasonably accurately as an acoustic perturbation of a uniform mean flow of speed  $U$ , measured in  $\text{m}\cdot\text{s}^{-1}$ . Then the round-trip propagation time is more accurately represented by

$$\begin{aligned} & \frac{0.6\text{ m}}{345\text{ ms}^{-1} + U} + \frac{0.6\text{ m}}{345\text{ ms}^{-1} - U} \\ &= \frac{0.6\text{ m}(345\text{ ms}^{-1} - U) + 0.6\text{ m}(345\text{ ms}^{-1} + U)}{(345\text{ ms}^{-1} + U)(345\text{ ms}^{-1} - U)} \\ &= \frac{2 \times 0.6\text{ m} \times 345\text{ ms}^{-1}}{(345\text{ ms}^{-1})^2 - U^2} \end{aligned}$$

- (iii) The time between samples of the analogue-to-digital converter (called the *sampling period*) is equal to the reciprocal of the sampling rate:

$$\frac{1}{44.1\text{ kHz}} = 23\text{ }\mu\text{s}$$

Comparing these three time intervals, it is seen that the propagation delay between transducers (spaced apart by 10 mm) is slightly longer than one sample period. It takes between 153 and 154 sample periods for an acoustic disturbance to be propagated the length of the pipe and back.

### High Notes

Notes of three different stable pitches were measured for the experiment. Figure 5.5 below shows the pressure measured 10 mm from the blown end of the pipe, during the start-up transient of one of the high-pitched notes.

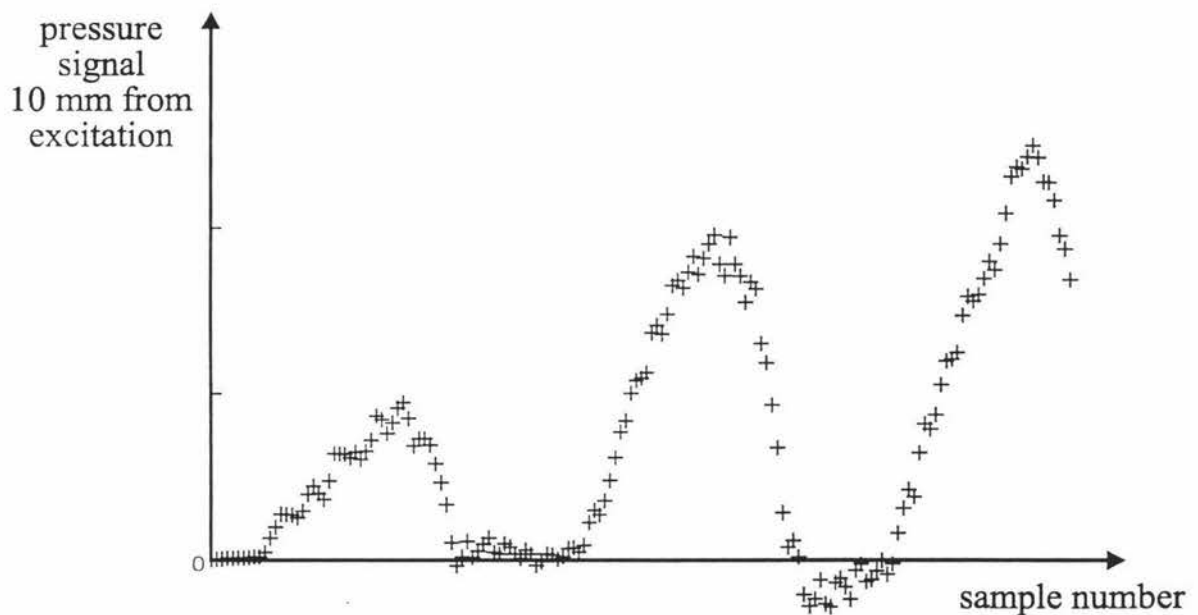


Figure 5.5: Pressure Measured During Start-Up of 'High' Note

There appears to be a stochastic component to the measured pressure waveform. However, the pressure measured simultaneously at the second transducer correlates well. Recall that the time required for a pressure fluctuation to propagate from one transducer to the next is slightly longer than one sample period. Figure 5.6 to follow shows that the whole of the pressure fluctuation measured at the upstream transducer is propagated acoustically to the downstream transducer. The round-trip time of 154 sample periods is illustrated also in Figure 5.6 for comparison.

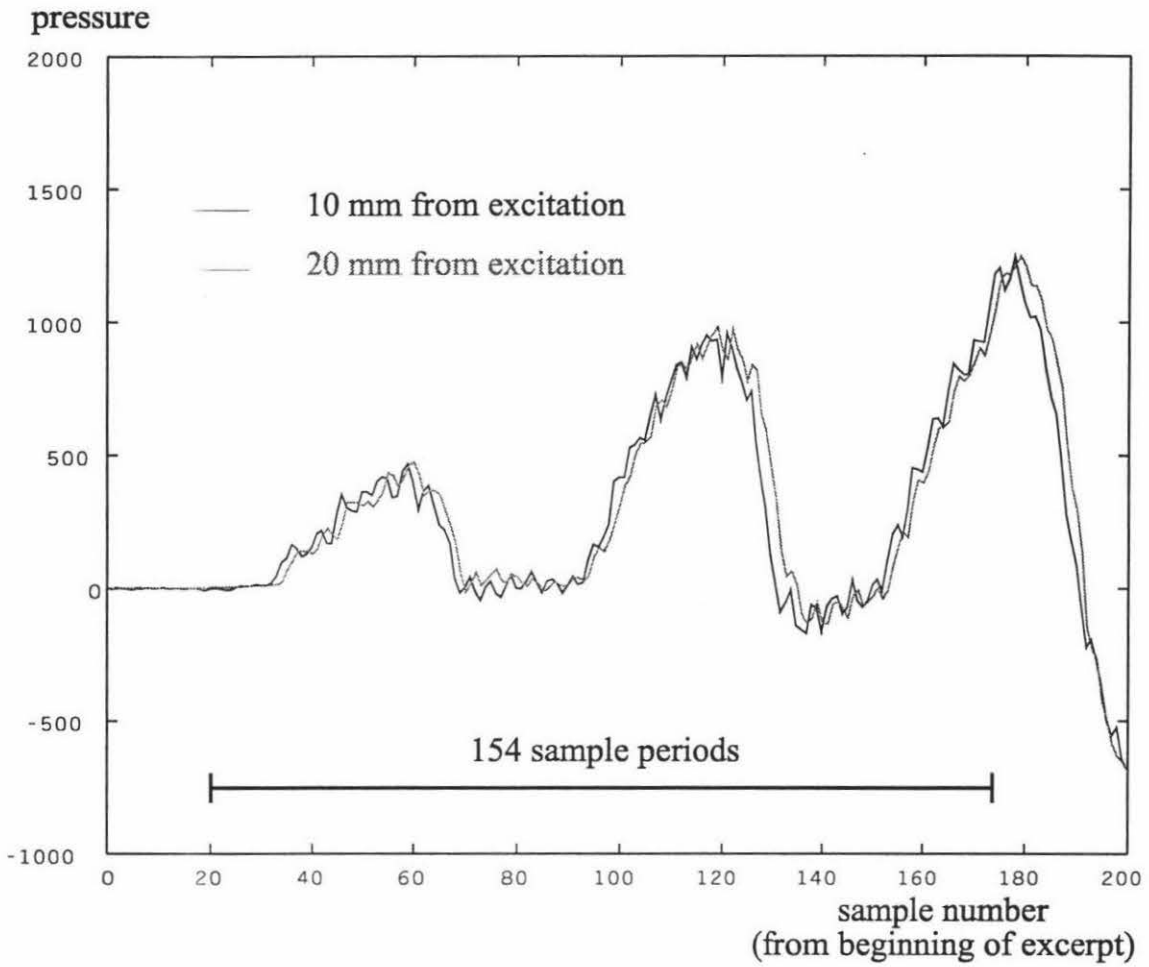


Figure 5.6: Scaled Pressures at 10 mm and 20 mm for a High Note

### Low Note

Figure 5.7 to follow shows pressures measured at distances 10 mm and 20 mm from the top of the pipe during the start-up transient of one of the low notes played.

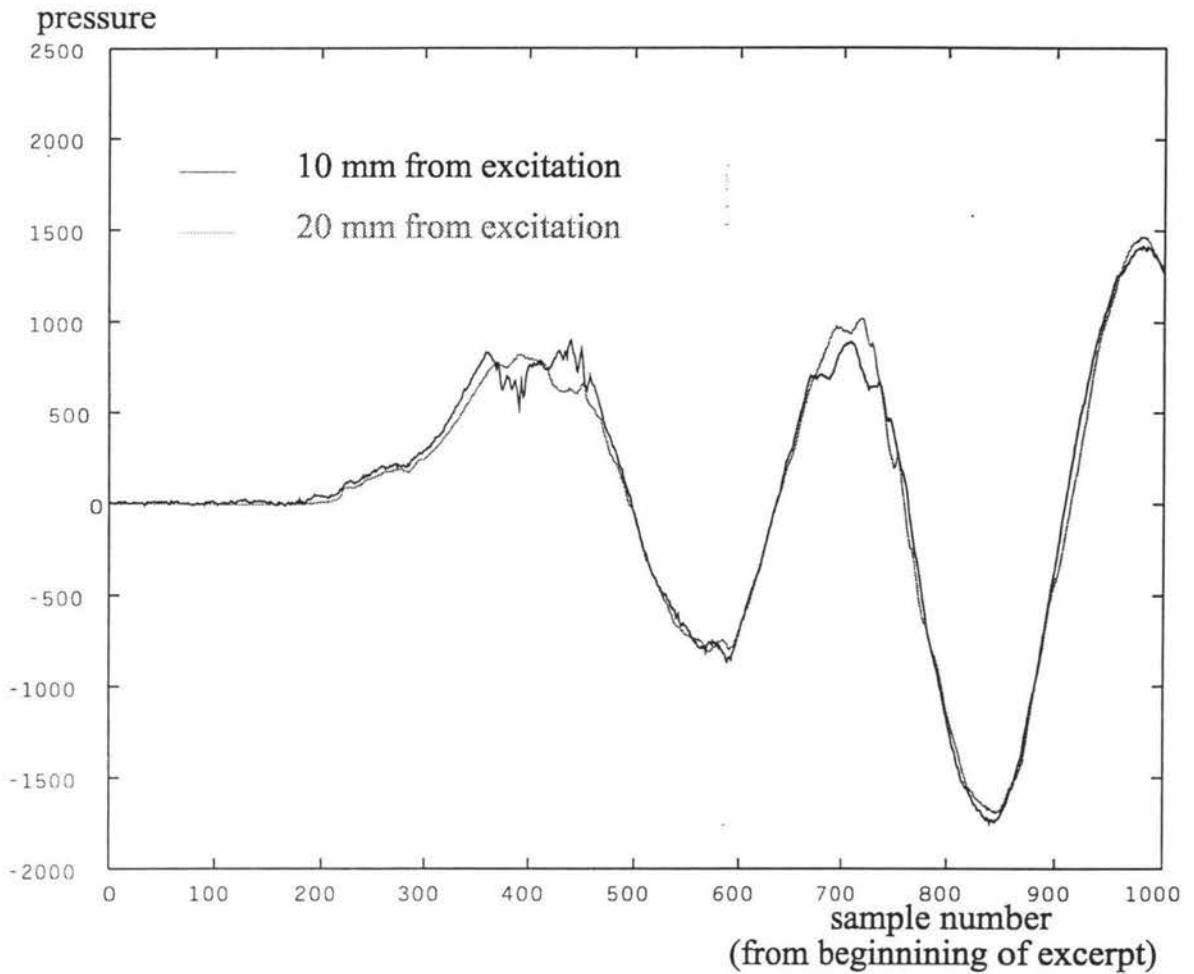


Figure 5.7: Scaled Pressures for a 'Low' Note at 10mm and 20 mm

In contrast to the case of the high note, the downstream pressures measured for the low note start-up does not mimic the upstream pressure consistently. This indicates the presence of pressure fluctuations which are not propagated at the speed of sound from one transducer to the next. Being non-acoustic in nature, but being pressure fluctuations nonetheless, this phenomenon is sometimes given the name *pseudosound*.

The pseudosound measured does not increase as the amplitude of the oscillation increases. In fact, the difference between the transducer outputs is minimal after the second peak of Figure 5.7. The next figure shows the evolution of the waveforms for the next few cycles.

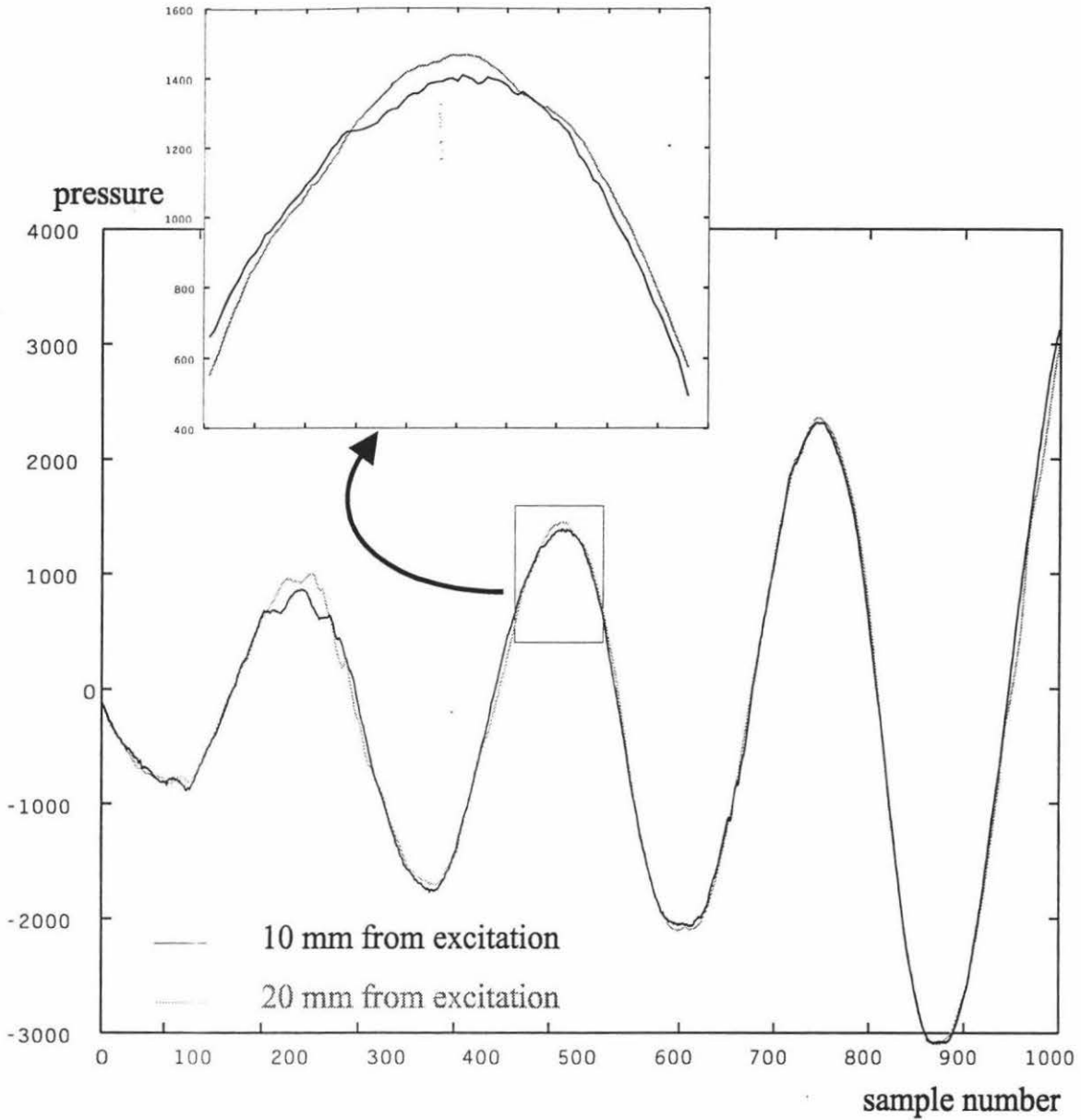


Figure 5.8: Decay of Pseudosound with Time

The presence of pseudosound is apparent during the start-up transient of all of the low notes played during the course of the experiment. Figures 5.9 and 5.10 show the pressures measure at the 10 mm and 20 mm positions for two other low notes.

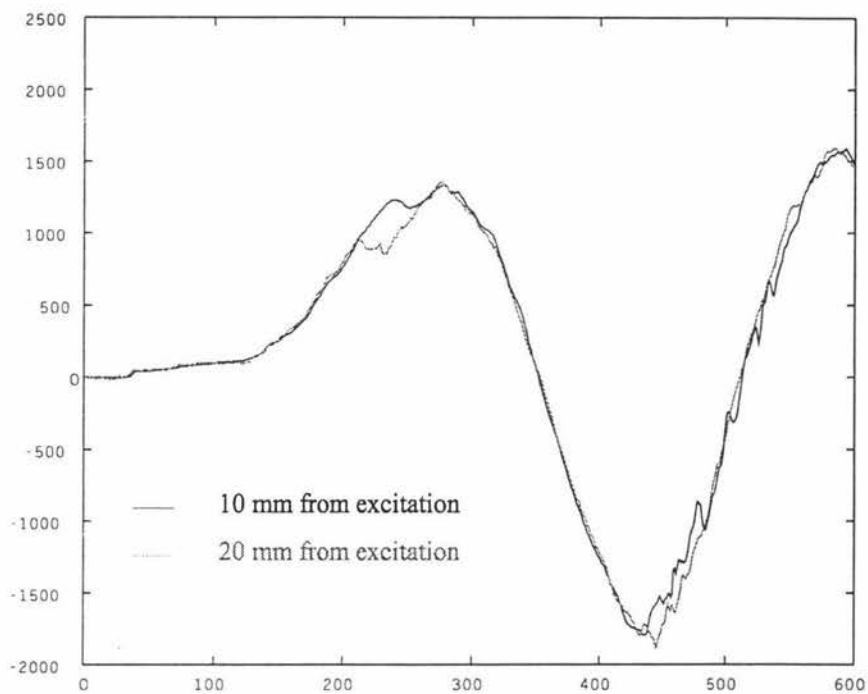


Figure 5.9: Start-Up Transient of Another 'Low' Note

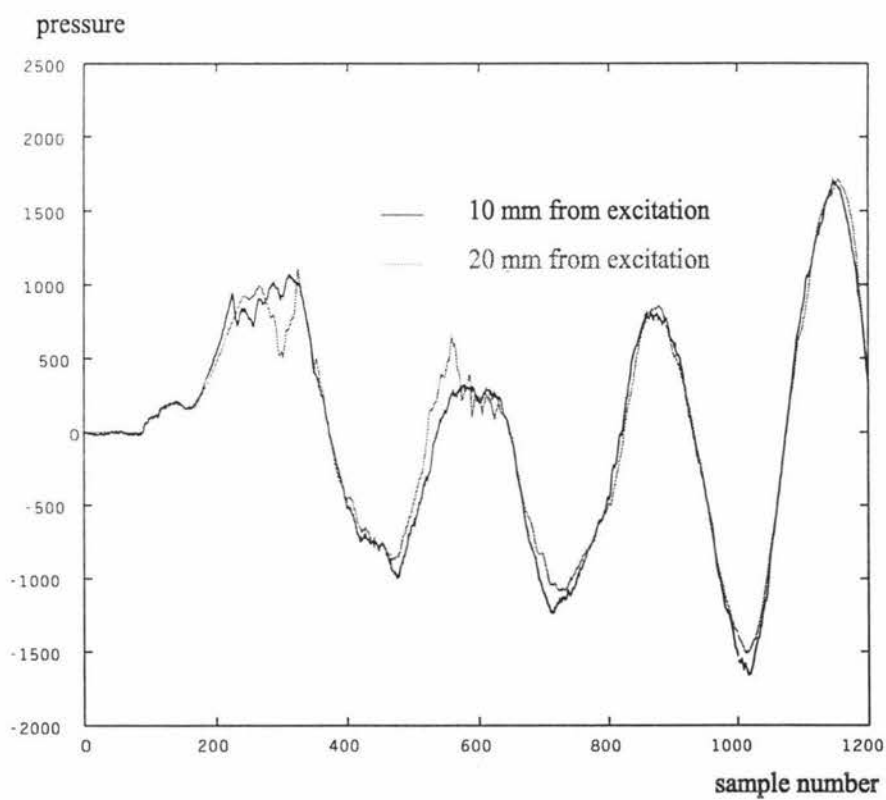


Figure 5.10: Start-Up Transient of One More 'Low' Note

The pseudosound was also apparent at the transducer position 30 mm from the blown end of the pipe. Figure 5.11 shows the pressures measured simultaneously at positions 20 mm and 30 mm from the top of the pipe during the start of a low note.

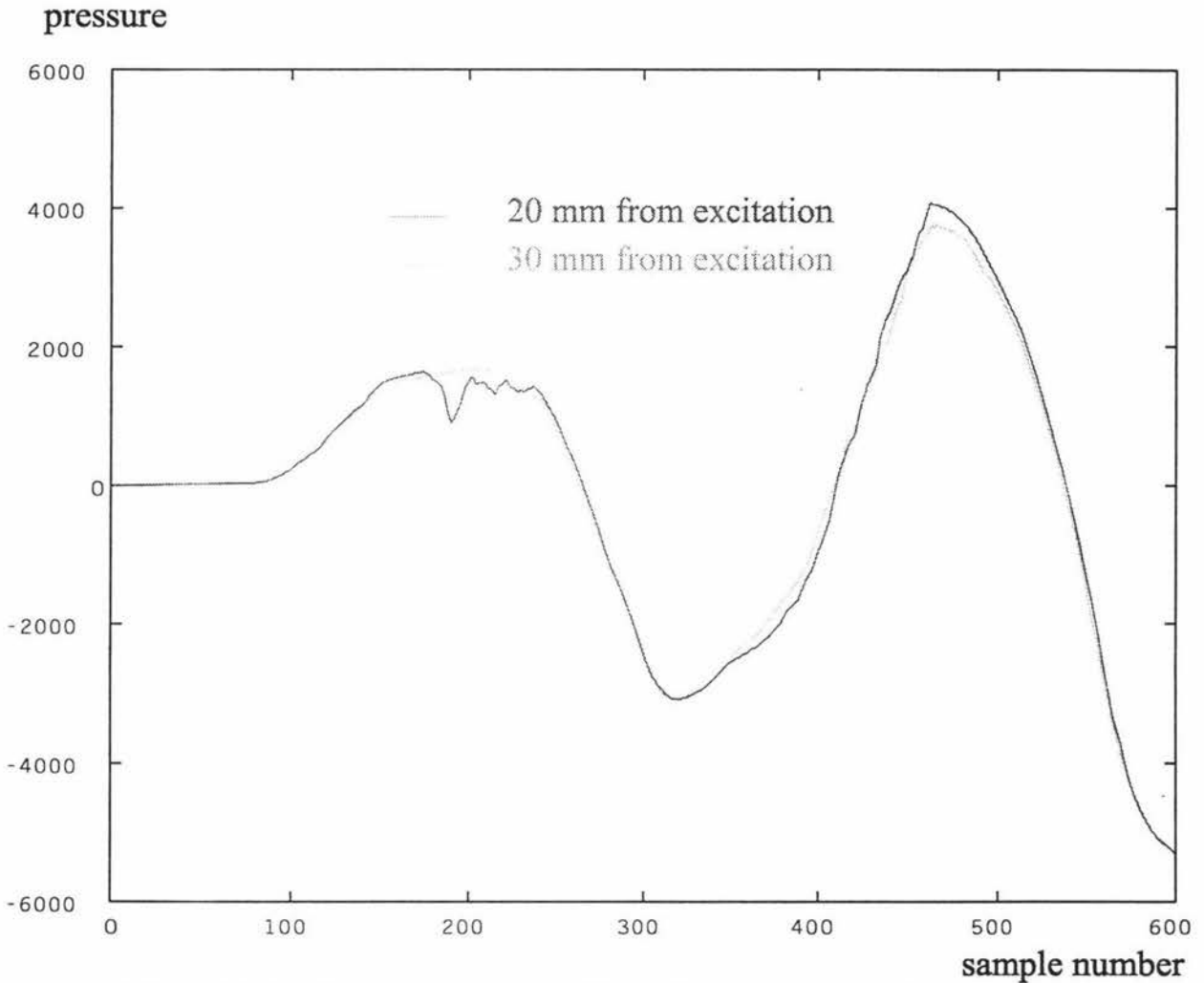


Figure 5.11: Start-Up Transient of 'Low' Note at 20 mm and 30 mm

It is apparent that the level of pseudosound decreases with the distance from the excitation. There was no measurable pseudosound at the opposite end of the tube; when the tube was blown from the 'bottom' end, the output signals of two transducers (spaced 10 mm apart at the top end) mimicked each other consistently.

## Flow Modulation

### by an Oscillating Constriction in a Duct

This chapter narrates an investigation into one simple method by which a fluid flow might be expected to be modulated. No attempt is made here to describe specifically the flow-modulation phenomenon which occurs at the lips of a trumpeter, or within any other musical instrument. Instead a study of the characteristic behaviour of one specific flow modulation mechanism is reported.

In the next three sections, the flow is modulated by a movable obstruction within a stationary rigid fluid-filled duct, as shown in Figure 6.1 below:

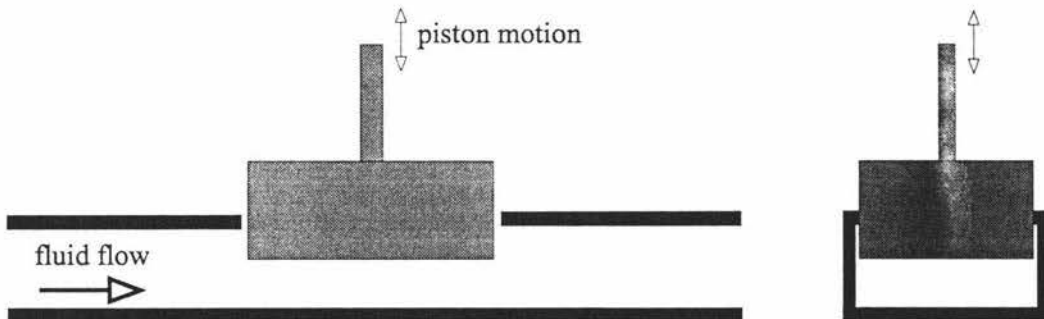


Figure 6.1: Side View and Front View of one type of Flow Modulator

The cross-sectional area of the aperture formed by the obstruction in the duct is dependent upon the position of the movable obstruction. Consequently, motion of this obstructing body regulates the flow through the channel, and body oscillations produce a *modulated flow*.

For each situation studied, the response of the flow to forced body motion is studied first. This is followed by an examination of the fluid forces exerted upon the body by the flow. Of fundamental interest is whether such forces act to reinforce or to oppose the body motion. Evolution of the system behaviour may then be predicted in situations

where the body is set into motion and then left to move freely except for the influence of the fluid reaction.

When the fluid's response to some prescribed initial body motion produces forces upon the body which act to develop a stable system oscillation, then the system is said to be *self-excited*. While the moving body performs the function of *flow modulation* it also causes an oscillatory fluid reaction force, resulting in continued *flow-induced vibration* of the body. These two actions are coupled via a feedback loop, as depicted below in Figure 6.2.

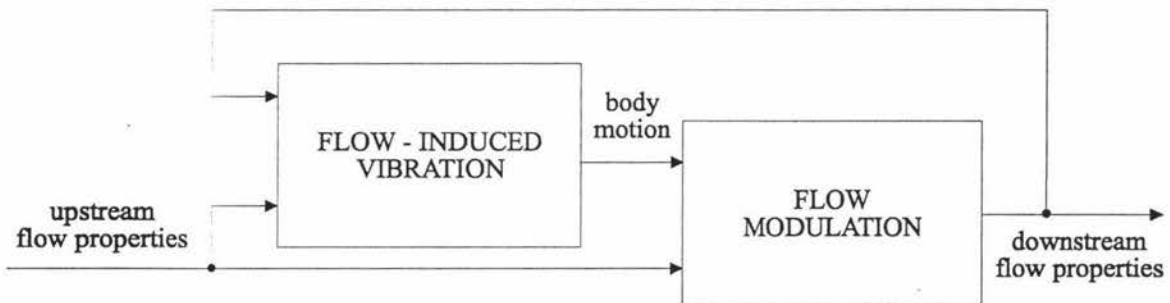


Figure 6.2: Feedback between Flow Modulation and Flow-Induced Vibration

### 6.1 Modulation by Transverse Motion of a Wall-Mounted Piston:

The particular flow-modulating mechanism under consideration consists of a piston capable of motion only perpendicular to the flow in a duct to which it is fitted. The duct is of rectangular cross-section, width  $w$ . The piston extends distance  $L$  along the duct. The cross-sectional area, at time  $t$ , of the aperture formed by the piston in the tube is denoted  $A(t)$ .

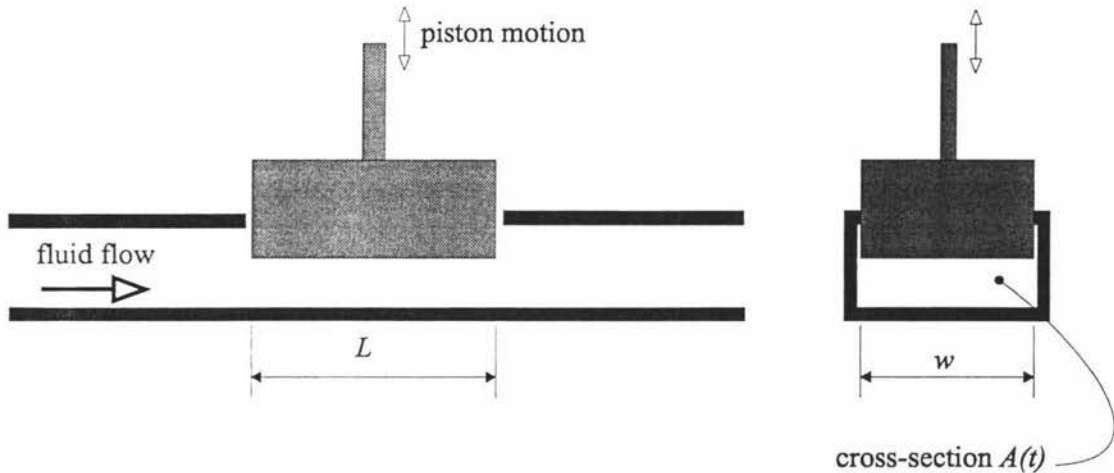


Figure 6.3: Piston-in-Tube Geometry, showing Dimensions

The study of the flow modulation performed by the oscillation of the piston is facilitated by certain simplifying assumptions regarding the nature of the fluid motion.

The fluid is assumed Newtonian (refer to Chapter Two). All gases and many liquids (including water) behave as Newtonian fluids (Batchelor, 1967).

The motion is assumed to be unaffected by external force fields so that all body force terms are absent from the equations of momentum transfer (refer to Section 3.2). Neglect of gravitational effects is common for gas motions, except for some very large scale movements such as those of meteorology (Lighthill, 1978).

The viscosity of the fluid will be assumed negligible so that the only forces acting upon the fluid are those due to the distribution of pressure throughout the fluid. No real fluid is ever inviscid. However, in some situations a fluid may behave as if it is inviscid if the forces associated with fluid viscosity (internal fluid friction) are insignificant compared with other forces that are influencing the motion of the fluid. The assumption of inviscid fluid motion, then, is situation dependent. The constraint of inviscid fluid motion is relaxed later in the chapter.

The fluid motion is assumed adiabatic. A fluid motion is adiabatic when there is no transfer of heat between any part of the fluid and its surroundings (refer to Section 3.3). This assumption is common in modelling of most acoustic phenomena, where fluid changes involve high-speed oscillation of flow-property values about some equilibrium state. The argument is that temperature changes are too quick to allow time for heat transfer (Morse & Ingard, 1968).

Conditions of adiabatic and inviscid motion together indicate that the only changes in the internal energy of the fluid are those resulting from work performed (reversibly) upon the fluid by forces of compression. This is equivalent to saying that there is no mechanism by which the entropy of the fluid can change, or the fluid motion is isentropic. A barotropic relationship is presumed, relating the fluid pressure to the density alone at any particular place and time (refer to Section 3.4)

The fluid motion is assumed to be quasi-one dimensional: A quasi-one dimensional description of fluid motion inside a duct or channel is possible in certain circumstances. Consider that the  $x$ -axis of a Cartesian co-ordinate system is aligned to coincide with the longitudinal direction of the duct or channel. The cross-sectional area perpendicular to the  $x$ -axis, at any place along the duct, can be written independently of  $y$  and  $z$ , i.e.  $A = A(x, t)$ , where  $A$  represents the cross-sectional area at any position along the duct. The fluid motion inside the duct can also be described without reference to  $y$  and  $z$  if flow properties are uniform over the flow cross-section. If this is only approximately true, then a reasonable approximation to the value of  $\psi(x, y, z, t)$  for any arbitrary flow property  $\psi$  is given by  $\psi^*(x, t)$  defined as

$$\psi^*(x, t) = \frac{1}{A(x)} \int_{A(x)} \psi(x, y, z, t) dA$$

The validity of this assumption depends upon the proportion of the flow cross-section which makes up the *boundary layer*, which is generally related to the *Reynold's number* of the flow. If the flow does not *separate* from the duct interior at any place then the flow cross-sectional area is everywhere the same as the duct cross-section. *Free jets* are not modelled by a quasi-1D description unless catered for specifically by extension to the basic formulation.

### 6.1.1 Procedure for Numerical Solution:

The flow modulation mechanisms to be studied in Sections 6.1 and 6.2 are cast into a form which may be described mathematically by a system of first-order ordinary differential equations. This section describes the Runge-Kutta-Fehlberg method for solving first-order ordinary differential equations and systems of first-order ordinary equations. As a prelude, Runge-Kutta methods are discussed in general.

#### One-Step vs Multi-Step Methods:

The *Runge-Kutta methods* for solving ordinary differential equations all belong to a class known as *one-step methods*. One-step methods use knowledge of the function  $y$  only at the time instant  $t_n$  to estimate the value of  $y$  at a new time  $t_{n+1}$ . The elapsed time between  $t_n$  and  $t_{n+1}$  is known as the *step-size*, denoted  $h$ .

Methods which are not one-step methods are known as *multi-step methods*; these also require knowledge of  $y$  at some previous time(s), eg.  $t_{n-1}, t_{n-2}, \dots$  to estimate  $y$  at time  $t_{n+1}$ . A multi-step method will generally require less computational effort than a one-step method to produce results to a comparable order of accuracy.

However, multi-step methods cannot simply march forward in time from a given set of initial conditions. When information is only known at a single time instant, viz. initial time  $t_0$  a one-step method must first be invoked to estimate  $y$  at  $t_1, t_2, \dots$ . When  $y$ -values are obtained at enough time instants, the multi-step method can then be used to continue. (In contrast, one-step methods are said to be *self-starting* because the same procedure is used at time  $t_0$  that is used at any other time  $t_n$ .)

Multi-step methods are also more complex when, at some time instant  $t_n$ , a change of the step-size  $h$  is required. (No difficulty is encountered when the step-size of a one-step method is altered.) For example, if the step-size of a three-step method changes from  $h_1$  to  $h_2$  at time instant  $t_n$ , then knowledge of  $y$  at time instants  $t_n, t_n - h_2$  and  $t_n - 2h_2$  is required to calculate  $y$  at  $t_n + h_2$ , but only knowledge of  $y$  at times  $t_n, t_n - h_1$  and  $t_n - 2h_1$  is available. The required  $y$ -values must first be estimated via interpolation before the three-step algorithm can continue. As well as involving extra computation, this introduces interpolation errors which are then carried through all subsequent time-steps. (Alternatively, a single-step method may be implemented until enough solution points are known for the multi-step method to be used again.)

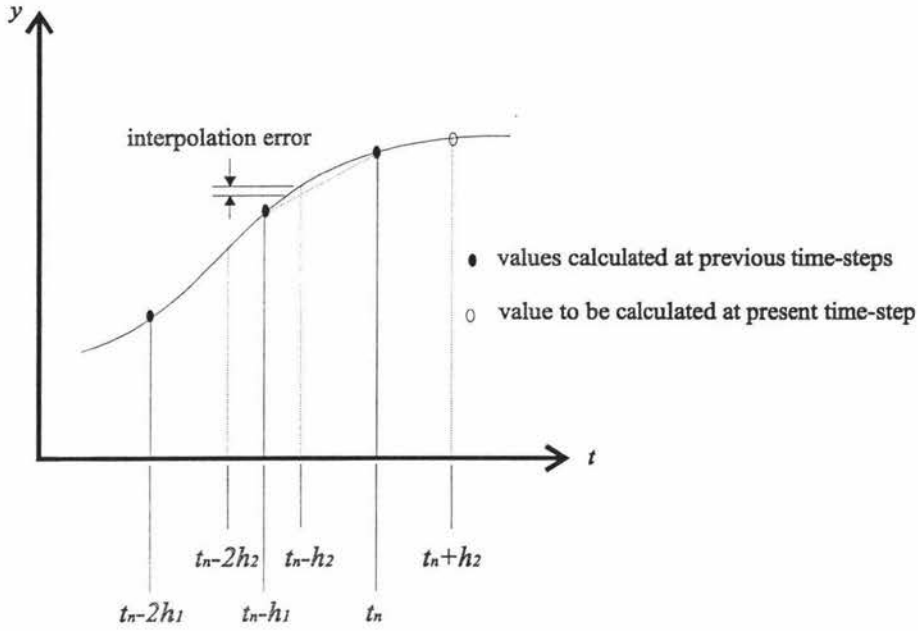


Figure 6.4: Change of Step-Size for Three-Step Method, with Linear Interpolation

A one-step method was chosen for the current application because, with the system variables involved, the gain in computing speed afforded by using a multi-step method is not so critically important, and does not seem to justify the added complexities described above.

#### Runge-Kutta Family of Methods:

The aim of a one-step method for solving the ordinary differential equation  $\dot{y} = f(t, y)$  is as follows:

"Given a time instant  $t_n$  and knowing the value of  $y$  at that time instant (denoted  $y_n$ ), estimate a value for  $y$  at the advanced time  $t_{n+1} = t_n + h$  (denoted  $y_{n+1}$ )."

The *Euler method* is a useful example (Hoffman, 1991). The derivative in the ordinary differential equation is replaced by a first-order *finite difference approximation*; the result is a *finite difference equation* expressing  $y_{n+1}$  explicitly in terms of known quantities.

$$\begin{aligned} \dot{y}_n &= f(t_n, y_n) \\ \Rightarrow \frac{y_{n+1} - y_n}{h} &\cong f(t_n, y_n) \\ \Rightarrow y_{n+1} &\cong y_n + hf(t_n, y_n) \end{aligned}$$

The Euler method can be thought of as the *one-stage* member of the Runge-Kutta family of numerical methods. The *two-stage* Runge-Kutta methods use the slope estimated by the Euler method to obtain a second estimate of the actual slope. The two estimates are then combined to give an improved value for  $y_{n+1}$ .

More stages can be added, each incorporating the estimates yielded by the previous stages. Each stage requires one evaluation of the function  $f(t,y)$ , where the values of  $y$  to be used in these function evaluations are determined from the results of previous stages. Thus an implementation of an  $r$ -stage Runge-Kutta method requires  $r$  evaluations of the function  $f(t,y)$ . The estimate at the  $i^{\text{th}}$  stage is normally denoted  $k_i$ . Finally, the new value used for  $y_{n+1}$  is taken as a weighted sum of the  $k_i$  values. The general structure of Runge-Kutta methods can be written as follows:

$$\begin{aligned} k_1 &= h f(t_n, y_n) \\ k_2 &= h f(t_n + \alpha_2 h, y_n + h\beta_{21}k_1) \\ k_3 &= h f(t_n + \alpha_3 h, y_n + h[\beta_{31}k_1 + \beta_{32}k_2]) \\ &\vdots \\ k_r &= h f\left(t_n + \alpha_r h, y_n + h \sum_{j=1}^{r-1} \beta_{rj}k_j\right) \end{aligned} \tag{6.1}$$

$$\text{and } y_{n+1} = y(t_n + h) = y_n + \gamma_1 k_1 + \gamma_2 k_2 + \dots + \gamma_r k_r.$$

A Runge-Kutta method is said to be of order  $m$  when the  $\alpha$ ,  $\beta$  and  $\gamma$  coefficients of the method are chosen in such a way that expansion for  $y_{n+1}$  matches, to the  $m^{\text{th}}$  order, a Taylor Series expansion of  $y(t_n + h)$  centred at  $t_n$ . There is a limit to the maximum order which can be achieved by a Runge-Kutta method with a specified number of stages. For example, when  $r = 1, 2, 3, 4$  the maximum  $m$  possible is equal to  $r$ ; for  $r = 5, 6, 7$  the maximum order is  $r-1$ . Runge-Kutta methods with more than seven stages are seldom used, because the number of function evaluations required at each time-step (equal to the number of stages,  $r$ ) becomes prohibitively large (Hoffman, 1991).

#### The Runge-Kutta-Fehlberg Method:

The *Runge-Kutta-Fehlberg method* combines two Runge-Kutta methods, but the  $\alpha$ ,  $\beta$  and  $\gamma$  coefficients are chosen in such a way as to allow the same  $k_i$  to be used for both. Six function evaluations are required to calculate the six  $k_i$ . The  $k_i$  are summed twice, each time with different  $\gamma_i$  coefficients. The first set of  $\gamma_i$  are chosen to yield a

$4^{\text{th}}$ -order approximation to  $y_{n+1}$ ; the second set are chosen to give a  $5^{\text{th}}$ -order approximation. The purpose of making two approximations of different order is that the difference between them gives a good indication of the error in  $y_{n+1}$ .

The *RKF-45* formulae are:

$$\begin{aligned}
 k_1 &= hf(t_n, y_n) \\
 k_2 &= hf\left(t_n + \frac{h}{4}, y_n + \frac{h}{4}k_1\right) \\
 k_3 &= hf\left(t_n + \frac{3h}{8}, y_n + h\left[\frac{3}{32}k_1 + \frac{9}{32}k_2\right]\right) \\
 k_4 &= hf\left(t_n + \frac{12h}{13}, y_n + h\left[\frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right]\right) \\
 k_5 &= hf\left(t_n + h, y_n + h\left[\frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right]\right) \\
 k_6 &= hf\left(t_n + \frac{h}{2}, y_n + h\left[-\frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right]\right)
 \end{aligned} \tag{6.2}$$

with  $4^{\text{th}}$ -order approximation to  $y_{n+1}$  given by

$$y_{n+1} = y_n + h\left[\frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5\right], \tag{6.3}$$

the  $5^{\text{th}}$ -order approximation by

$$y_{n+1} = y_n + h\left[\frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56437}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6\right] \tag{6.4}$$

and the error estimate obtained as the difference:

$$error = h\left[\frac{1}{135}k_1 - \frac{128}{4275}k_3 - \frac{2197}{75240}k_4 + \frac{1}{50}k_5 + \frac{2}{55}k_6\right]. \tag{6.5}$$

By monitoring the size of the error estimate, decisions can be made as to the appropriateness of the step-size  $h$  being used for the calculations. If at a particular time  $t_n$  the error estimated becomes undesirably large, the value obtained for  $y(t_n + h)$  can be discarded and the method re-applied from  $t_n$  with a smaller step-size  $h_{new}$  to yield instead a value for  $y(t_n + h_{new})$  with improved accuracy. Because Runge-Kutta methods are one-step methods, changes in step-size present no difficulty.

### 6.1.2 Operation of MATLAB program *myRKF.m*

This section describes the development of the MATLAB program *myRKF.m* which applies the Runge-Kutta-Fehlberg method to solve a system of ODEs. Section 6.1.1 outlined how this method can yield an approximate solution to a differential equation of the form  $\dot{y} = f(t, y)$ . A straightforward extension allows solution also of systems of ordinary differential equations in several variables, of the form  $\dot{y} = f(t, y)$ .

### Universal Applicability:

The program *myRKF.m* was designed to be universally applicable for all simulations described in Sections 6.1 and 6.2 without any need for alterations. Consequently, it was required to solve for an arbitrary number of system variables. This is achieved simply in MATLAB by passing *vectors* of system variables as input arguments for the function *myRKF*, rather than passing each system variable individually. For example, in the command line

$$[t, y] = \text{myRKF}('piston', 0, 1, y0);$$

the input argument  $y0$  is a vector containing the initial values for each of the system variables in the simulation.

Furthermore, the program *myRKF.m* solves a system of equations that is specified externally in another MATLAB function file. For example, the above command line employs *myRKF* to solve the system of equations described in a file named *piston.m*. At each time-step, the MATLAB command `feval` is used to call whatever MATLAB function file is named as an input argument for *myRKF.m*. The MATLAB function file which contains these system equations accepts as input arguments a time  $t$  and the state of the system,  $y$ , at that time. The function returns a value of  $\dot{y}$ , calculated as  $\dot{y} = f(t, y)$ .

By using the string-name of the file containing the system equations as an input variable, and by passing initial conditions in vector form as another input variable (as discussed above), the function *myRKF* is able to solve an arbitrary system of ODEs having the form  $\dot{y} = f(t, y)$ .

### Implementation of RKF-45 Algorithm:

Recall the structure of a Runge-Kutta method as outlined in Equation (6.1) of Section 6.1.1. The  $i^{\text{th}}$  stage of an  $r$ -stage method can be written as

$$k_i = hf \left( t_n + \alpha_i h, y_n + h \sum_{j=1}^{i-1} \beta_{ij} k_j \right) \quad (6.6)$$

Now that the method is solving for a whole system of ODEs, each instance of  $k$  and  $y$  is now a vector.

Arrays may be used to store the coefficients of the method, which were given previously in Equations (6.2).

$$\alpha = \begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{3}{8} \\ \frac{12}{13} \\ 1 \\ \frac{1}{2} \end{pmatrix}, \quad \beta = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{4} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{3}{32} & \frac{9}{32} & \cdot & \cdot & \cdot & \cdot \\ \frac{1932}{2197} & -\frac{7200}{2197} & \frac{7296}{2197} & \cdot & \cdot & \cdot \\ \frac{439}{216} & -8 & \frac{3680}{513} & -\frac{845}{4104} & \cdot & \cdot \\ -\frac{8}{27} & 2 & -\frac{3544}{2565} & \frac{1859}{4104} & -\frac{11}{40} & \cdot \end{bmatrix}$$

In the above vector and matrix, row number  $i$  is used for the  $i^{\text{th}}$  stage. A dot is used to denote an element which is not accessed by the instruction which implements Equation (6.6). By padding the matrix with zeroes instead, the upper limit of  $i-1$  used in the summation in Equation (6.6) may be replaced by  $r$ . i.e:

$$k_i = hf \left( t_n + \alpha_i h, y_n + h \sum_{j=1}^r \beta_{ij} k_j \right)$$

$$\text{with } \alpha = \begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{3}{8} \\ \frac{12}{13} \\ 1 \\ \frac{1}{2} \end{pmatrix}, \quad \beta = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{32} & \frac{9}{32} & 0 & 0 & 0 & 0 \\ \frac{1932}{2197} & -\frac{7200}{2197} & \frac{7296}{2197} & 0 & 0 & 0 \\ \frac{439}{216} & -8 & \frac{3680}{513} & -\frac{845}{4104} & 0 & 0 \\ -\frac{8}{27} & 2 & -\frac{3544}{2565} & \frac{1859}{4104} & -\frac{11}{40} & 0 \end{bmatrix}$$

In Equation (6.6) the  $k$  obtained at each stage is a column vector, the number of rows being equal to the number of system variables. By now constructing  $K$  as a matrix containing all of the  $k$ -vectors, i.e.  $K = [k_1 \ k_2 \ k_3 \ k_4 \ k_5 \ k_6]$ , the above summation can be expressed more economically in terms of a single matrix multiplication,  $k_i = hf(t_n + \alpha_i h, y_n + hK \cdot \beta_i^T)$ , where  $\beta_i^T$  denotes the  $i^{\text{th}}$  column vector of matrix  $\beta^T$ .

In MATLAB, calculation of all of the  $k$ -vectors can be programmed simply by the following code:

```
for i = 1:6
    ki = feval(yprime, t + alpha(i)*h, y + h*K*betaT(:,i));
    K(:,i) = ki;
end
```

In the above, `yprime` is a variable holding the string-name of the file where the system equations can be found (which is earlier passed to `myRKF.m` as an input argument). The

`feval` command above evaluates each derivative by invoking the system equations at the specified time  $t$  and system state  $\mathbf{y}$ .

The fifth-order estimate for the new value of  $\mathbf{y}_{n+1}$  is given by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \sum_{i=1}^6 \gamma_i \mathbf{k}_i = \mathbf{y}_n + h\mathbf{K} \cdot \boldsymbol{\gamma}$$

and the error estimate is:

$$\text{error} = h \sum_{i=1}^6 \delta_i \mathbf{k}_i = h\mathbf{K} \cdot \boldsymbol{\delta}$$

where

$$\boldsymbol{\gamma} = \begin{pmatrix} \frac{16}{135} \\ 0 \\ \frac{6656}{12825} \\ \frac{28561}{56430} \\ -\frac{9}{50} \\ \frac{2}{55} \end{pmatrix}, \quad \boldsymbol{\delta} = \begin{pmatrix} \frac{1}{360} \\ 0 \\ -\frac{128}{4275} \\ -\frac{2197}{75240} \\ \frac{1}{50} \\ \frac{2}{55} \end{pmatrix}$$

#### Error Control:

The error estimate is used for step-size control in *myRKF.m*. Since  $\mathbf{y}$  is a vector with one or more elements, it is necessary to check if the error is sufficiently small for every system variable contained in  $\mathbf{y}$ . In *myRKF*, the relative error of each system variable is calculated separately, and the maximum of these estimates is used to decide whether alteration of the step-size is warranted.

Only if the error is deemed to be acceptable is the solution updated and the process can be moved along to the next time-step. If the error is considered to be too large, then the RKF algorithm is applied to the same time instant once again, but this time with a reduced value of  $h$ . (Since the error is fifth-order in  $h$ , halving of the step-size results in error reduction by a factor of  $2^5 = 32$ .) When the calculations at a particular time-step are approved, then  $t_n + h$  is appended to a vector containing all of the other time instants, and the vector obtained for  $\mathbf{y}(t_n + h)$  is added to the information collected about the state of the system at each of these time instants.

Program Outputs:

Records of time instants, and the state of the system at each of these, are updated as *myRKF.m* progresses. Occasionally, it is desirable that some other information about the system be accessible to the user for later scrutiny and perhaps plotting. To keep *myRKF* as a multi-purpose program, any additional calculations are performed within the external program which calculates the system derivatives (eg. within *piston.m*). Since the number and nature of these auxiliary quantities is dependent upon the particular simulation, it was decided to pass these to *myRKF* as a single vector.

For example, imagine that a derivative described in *piston.m* is the sum of three terms, and plots are required of each component to illustrate their relative importance. This can be implemented by the following code in the MATLAB function *piston.m*:

```
function [y_dot, components] = piston(t,y)
...
y_dot = first_term + second_term + third_term;
components = [first_term; second_term; third_term];
```

The function *myRKF.m* can use the value of *y\_dot* to take the solution forward to a new time-step, but now the three different terms used in the calculation are also available to the user. Each time the calculations at a particular time-step are approved, the vector of *components* is used to update a record of this auxiliary information which has been collected at each of the previous time-steps. This is achieved with the following lines in *myRKF.m*:

```
[y_dot, aux] = feval(yprime,t,y);
aux_out = [aux_out, aux];
```

When *myRKF* completes a simulation, all of *t\_out*, *y\_out* and *aux\_out* are available to the user. The matrix *y\_out* contains one column vector for each of the values of time contained in the row vector *t\_out*. There is also one column vector in the *aux\_out* matrix for each *t\_out* entry.

### 6.1.3 Equations of Flow Beneath the Piston:

A technique known as *control volume analysis* is here used to derive first an equation of continuity and then a momentum equation for the fluid passing beneath the moving piston. From Figure 6.5 below it will be seen that the chosen control volume at time *t* has volume  $A(t)L$ .

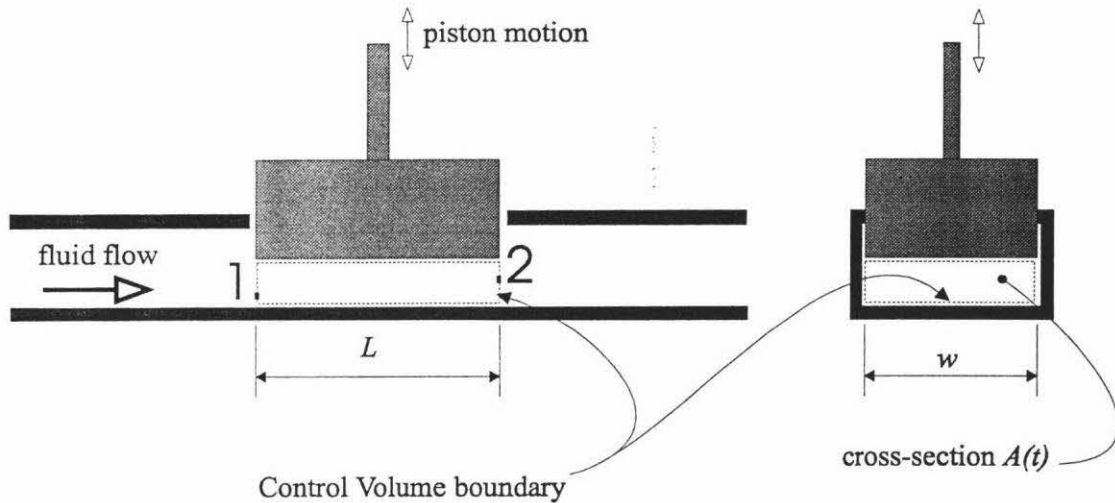


Figure 6.5: Piston-in-Tube Geometry, showing Control Volume

In the analysis which follows, a subscript 1 indicates a flow quantity measured at the upstream edge of the piston, while subscript 2 designates a flow quantity at the downstream edge. For example, fluid entering the control volume has pressure  $p_1$  and fluid leaving has pressure  $p_2$ .

#### Continuity Equation:

A continuity equation is derived by applying the Principle of Conservation of Mass to the fluid in the aperture formed by the piston in the tube. The rate of change of fluid mass in the control volume is equal to the difference between the rate at which fluid enters the upstream side of the region and the rate at which fluid leaves the downstream side. Recalling that the assumption of quasi-one dimensional fluid motion implies uniform fluid properties over the flow cross-section, this may be written as follows:

$$\begin{aligned} \frac{\partial}{\partial t} \left[ \int_{x_1}^{x_2} \rho A \, dx \right] &= (\rho_1 u_1 - \rho_2 u_2) A \\ \Rightarrow \frac{\partial}{\partial t} \left[ A \int_{x_1}^{x_2} \rho \, dx \right] &= (\rho_1 u_1 - \rho_2 u_2) A \\ \Rightarrow \frac{\partial}{\partial t} \left[ \int_{x_1}^{x_2} \rho \, dx \right] + \frac{\dot{A}}{A} \int_{x_1}^{x_2} \rho \, dx &= \rho_1 u_1 - \rho_2 u_2 \quad \text{where } \dot{A} = \frac{dA}{dt} \end{aligned}$$

The derivation cannot be taken further without knowledge of the spatial variation of density in the aperture. However it is possible to obtain an approximate description of the

real behaviour by employing the well-known Trapezoidal Rule to replace the two integrals.

A Taylor Series expansion shows that accuracy of this approximation is to second order in the step-length  $L$ . Consider a series centred at  $x_1$ :

$$\rho(x, t) = \rho(x_1, t) + (x - x_1) \frac{\partial \rho}{\partial x}(x_1, t) + \frac{(x - x_1)^2}{2!} \frac{\partial^2 \rho}{\partial x^2}(x_1, t) + \dots$$

(In the above expansion, it is understood that the partial derivative of the density indicates differentiation with the temporal variable  $t$  regarded as a constant. Although density is a property of a thermodynamic system, and is therefore dependent upon two other system properties through an Equation of State, only dependence upon  $t$  and  $x$  is relevant here.)

Expanding for the case of  $x = x_2$ :

$$\begin{aligned} \rho(x_2, t) &= \rho(x_1, t) + L \frac{\partial \rho}{\partial x}(x_1, t) + \frac{L^2}{2!} \frac{\partial^2 \rho}{\partial x^2}(x_1, t) + \dots \\ \Rightarrow \frac{\partial \rho}{\partial x}(x_1, t) &= \frac{\rho(x_2, t) - \rho(x_1, t)}{L} + \frac{L}{2} \frac{\partial^2 \rho}{\partial x^2}(x_1, t) + \dots \end{aligned}$$

This expression can now be substituted into the original Taylor Series centred at  $x_1$ :

$$\rho(x, t) = \rho(x_1, t) + (x - x_1) \left[ \frac{\rho(x_2, t) - \rho(x_1, t)}{L} + \frac{L}{2} \frac{\partial^2 \rho}{\partial x^2}(x_1, t) + \dots \right] + \frac{(x - x_1)^2}{2!} \frac{\partial^2 \rho}{\partial x^2}(x_1, t) + \dots$$

Integrating between the limits of  $x_1$  and  $x_2$  gives:

$$\begin{aligned} \int_{x_1}^{x_2} \rho(x, t) dx &= \rho(x_1, t) \int_{x_1}^{x_2} dx \\ &+ \left[ \frac{\rho(x_2, t) - \rho(x_1, t)}{L} + \frac{L}{2} \frac{\partial^2 \rho}{\partial x^2}(x_1, t) + \dots \right] \int_{x_1}^{x_2} (x - x_1) dx \\ &+ \frac{\partial^2 \rho}{\partial x^2}(x_1, t) \int_{x_1}^{x_2} \frac{(x - x_2)^2}{2!} dx + \dots \\ &= L \rho(x_1, t) + \frac{L^2}{2!} \left[ \frac{\rho(x_2, t) - \rho(x_1, t)}{L} + \frac{L}{2} \frac{\partial^2 \rho}{\partial x^2}(x_1, t) + \dots \right] + \frac{L^3}{3!} \frac{\partial^2 \rho}{\partial x^2}(x_1, t) + \dots \\ &= \frac{\rho(x_1, t) + \rho(x_2, t)}{2} L + \mathcal{O}(L^3) \end{aligned}$$

This result establishes the second-order accuracy of the Trapezoidal Rule; the nature of the approximation is shown pictorially in Figure 6.6 to follow:

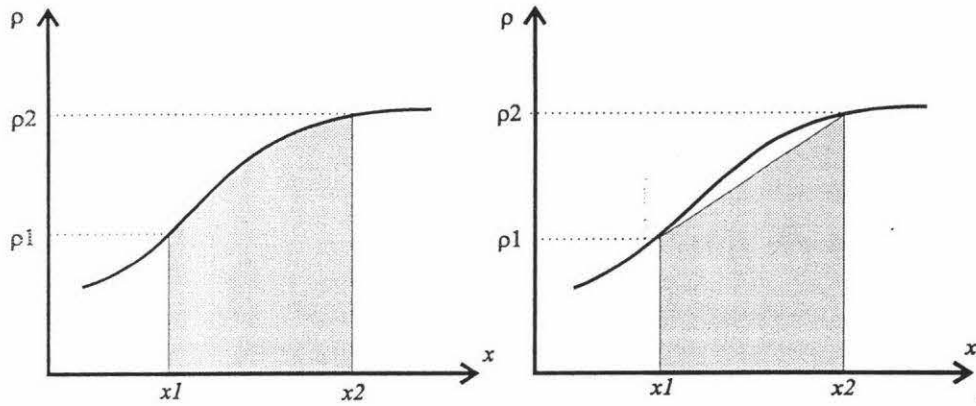


Figure 6.6: Trapezoidal Rule Approximation to an Integral

To second order in  $L$ , then, and using an over-dot to signify the time-derivative of any function of time only:

$$\begin{aligned} \frac{\dot{\rho}_1 + \dot{\rho}_2}{2} L + \frac{\dot{A}}{A} \frac{\rho_1 + \rho_2}{2} L &= \rho_1 u_1 - \rho_2 u_2 \\ \Rightarrow \frac{\dot{\rho}_1 + \dot{\rho}_2}{2} + \frac{\dot{A}}{A} \frac{\rho_1 + \rho_2}{2} &= \frac{\rho_1 u_1 - \rho_2 u_2}{L} \end{aligned} \quad (6.7)$$

#### Momentum Equation:

Newton's Second Law of Motion states that any change of the amount of fluid momentum contained by the control volume, other than that due to the advection of momentum across the region's boundaries, is due to the net effect of external forces acting upon the fluid in the region. For an inviscid fluid with properties uniform over a flow cross-section, in the absence of external force field effects such as gravity, this can be expressed mathematically as follows:

$$\begin{aligned} \frac{\partial}{\partial t} \left[ \int_{x_1}^{x_2} \rho u A \, dx \right] + (\rho_2 u_2^2 - \rho_1 u_1^2) A &= (p_1 - p_2) A \\ \Rightarrow \frac{\partial}{\partial t} \left[ A \int_{x_1}^{x_2} \rho u \, dx \right] + (\rho_2 u_2^2 - \rho_1 u_1^2) A &= (p_1 - p_2) A \\ \Rightarrow \frac{\partial}{\partial t} \left[ \int_{x_1}^{x_2} \rho u \, dx \right] + \frac{\dot{A}}{A} \int_{x_1}^{x_2} \rho u \, dx + \rho_2 u_2^2 - \rho_1 u_1^2 &= p_1 - p_2 \end{aligned} \quad (6.8)$$

The Trapezoidal Rule can again be invoked at this point as it was previously. To second order in  $L$ :

$$\int_{x_1}^{x_2} \rho u dx = \frac{\rho_1 u_1 + \rho_2 u_2}{2} L \quad (6.9)$$

The mean-of-products in this expression may be replaced by a product-of-means to give an alternative second-order approximation:

$$\int_{x_1}^{x_2} \rho u dx = \frac{\rho_1 + \rho_2}{2} \frac{u_1 + u_2}{2} L \quad (6.10)$$

The second-order equivalence of Equations (6.9) and (6.10) can be demonstrated as follows. First the velocity and density at position  $x_2$  are written as Taylor Series expansions centred at  $x_1$

$$\begin{aligned} \rho_2(t) &= \rho_1(t) + L \frac{\partial \rho}{\partial x}(x_1, t) + O(L^2) \\ u_2(t) &= u_1(t) + L \frac{\partial u}{\partial x}(x_1, t) + O(L^2) \end{aligned}$$

The mean-of-products and product-of means are formed as follows:

$$\begin{aligned} \frac{\rho_1 u_1 + \rho_2 u_2}{2} L &= \frac{L}{2} \left\{ \rho_1 u_1 + \left[ \rho_1(t) + L \frac{\partial \rho}{\partial x}(x_1, t) + O(L^2) \right] \left[ u_1(t) + L \frac{\partial u}{\partial x}(x_1, t) + O(L^2) \right] \right\} \\ &= \rho_1 u_1 L + \frac{L^2}{2} \left[ \rho_1(t) \frac{\partial u}{\partial x}(x_1, t) + u_1(t) \frac{\partial \rho}{\partial x}(x_1, t) \right] + O(L^3) \\ \frac{\rho_1 + \rho_2}{2} \frac{u_1 + u_2}{2} L &= \frac{L}{4} \left\{ \left[ 2\rho_1(t) + L \frac{\partial \rho}{\partial x}(x_1, t) + O(L^2) \right] \left[ 2u_1(t) + L \frac{\partial u}{\partial x}(x_1, t) + O(L^2) \right] \right\} \\ &= \rho_1 u_1 L + \frac{L^2}{2} \left[ \rho_1(t) \frac{\partial u}{\partial x}(x_1, t) + u_1(t) \frac{\partial \rho}{\partial x}(x_1, t) \right] + O(L^3) \end{aligned}$$

This demonstrates the second order equivalence of Equations (6.9) and (6.10).

The product-of-means is chosen here to then allow a straightforward substitution of the continuity equation (Equation 6.7) derived earlier. Substituting Equation (6.10) into Equation (6.8) gives, to second order in  $L$

$$\frac{\rho_1 + \rho_2}{2} \frac{\dot{u}_1 + \dot{u}_2}{2} + \frac{\dot{\rho}_1 + \dot{\rho}_2}{2} \frac{u_1 + u_2}{2} + \frac{A}{L} \frac{\rho_1 + \rho_2}{2} \frac{u_1 + u_2}{2} + \frac{\rho_2 u_2^2 - \rho_1 u_1^2}{L} = \frac{p_1 - p_2}{L}$$

A multiple of Equation (6.7) can be removed to give

$$\frac{\rho_1 + \rho_2}{2} \frac{\dot{u}_1 + \dot{u}_2}{2} + \frac{\rho_1 u_1 - \rho_2 u_2}{L} \frac{u_1 + u_2}{2} + \frac{\rho_2 u_2^2 - \rho_1 u_1^2}{L} = \frac{p_1 - p_2}{L}$$

$$\begin{aligned} \Rightarrow & \frac{\rho_1 + \rho_2}{2} \frac{\dot{u}_1 + \dot{u}_2}{2} + \frac{\rho_1 u_1^2 + u_1 u_2 (\rho_1 - \rho_2) - \rho_2 u_2^2}{2L} + \frac{\rho_2 u_2^2 - \rho_1 u_1^2}{L} = \frac{p_1 - p_2}{L} \\ \Rightarrow & \frac{\rho_1 + \rho_2}{2} \frac{\dot{u}_1 + \dot{u}_2}{2} + \frac{\rho_2 u_2^2 + u_1 u_2 (\rho_1 - \rho_2) - \rho_1 u_1^2}{2L} = \frac{p_1 - p_2}{L} \\ \Rightarrow & \frac{\rho_1 + \rho_2}{2} \frac{\dot{u}_1 + \dot{u}_2}{2} + \frac{\rho_1 u_1 + \rho_2 u_2}{2} \frac{u_1 - u_2}{L} = \frac{p_1 - p_2}{L} \end{aligned}$$

The derivation is continued further by utilising again the second-order equivalence of Equations (6.9) and (6.10):

$$\begin{aligned} & \frac{\rho_1 + \rho_2}{2} \frac{\dot{u}_1 + \dot{u}_2}{2} + \frac{\rho_1 + \rho_2}{2} \frac{u_1 + u_2}{2} \frac{u_2 - u_1}{L} = \frac{p_1 - p_2}{L} \\ \Rightarrow & \frac{\dot{u}_1 + \dot{u}_2}{2} + \frac{u_1 + u_2}{2} \frac{u_2 - u_1}{L} = \frac{2}{\rho_1 + \rho_2} \frac{p_1 - p_2}{L} \end{aligned} \quad (6.11)$$

This expression is directly comparable to the partial differential equation expressing conservation of fluid momentum for one-dimensional inviscid flows in the absence of body forces:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

#### Conversion to Acoustic Variables:

It has been established that, to second order in  $L$ , the flow modulation mechanism depicted in Figure 6.5 can be described mathematically by the following pair of differential equations (Equations 6.7 and 6.11):

$$\begin{aligned} \frac{\dot{\rho}_1 + \dot{\rho}_2}{2} + \frac{\rho_1 + \rho_2}{2} \frac{\dot{A}}{A} + \frac{\rho_2 u_2 - \rho_1 u_1}{L} &= 0 \\ \frac{\dot{u}_1 + \dot{u}_2}{2} + \frac{u_1 + u_2}{2} \frac{u_2 - u_1}{L} + \frac{2}{\rho_1 + \rho_2} \frac{p_2 - p_1}{L} &= 0 \end{aligned}$$

These equations rely upon the assumptions of inviscid and quasi-one dimensional motion of the fluid, and insignificance of body forces (such as gravity).

If the motion of the fluid is barotropic, an equation of state can be written for the fluid in the form  $p = p(\rho)$ . For variations of  $p$  sufficiently small, a linear approximation to this relation will be satisfactory. This can be written  $p - p_{ref} \cong \bar{c}^2 (\rho - \rho_{ref})$ . The quantities  $p_{ref}$  and  $\rho_{ref}$  denote reference conditions for the flow;  $\bar{c}^2$  is equal to the slope of the pressure-density curve at these reference conditions.

The following equivalences result:

$$\begin{aligned}
 p_1 - p_2 &= (p_1 - p_{ref}) - (p_2 - p_{ref}) \\
 &= \bar{c}^2(\rho_1 - \rho_{ref}) - \bar{c}^2(\rho_2 - \rho_{ref}) \\
 &= \bar{c}^2(\rho_1 - \rho_2) \\
 p_1 + p_2 &= (p_1 - p_{ref}) + (p_2 - p_{ref}) + 2p_{ref} \\
 &= \bar{c}^2(\rho_1 - \rho_{ref}) + \bar{c}^2(\rho_2 - \rho_{ref}) + 2p_{ref} \\
 &= \bar{c}^2(\rho_1 + \rho_2) - 2\bar{c}^2\rho_{ref} + 2p_{ref}
 \end{aligned}$$

If a tangent drawn on the pressure-density graph through the reference point passes also through the origin, then  $p_{ref} = \bar{c}^2\rho_{ref}$  and the above relations become:

$$\begin{aligned}
 p_1 - p_2 &= \bar{c}^2(\rho_1 - \rho_2) \\
 p_1 + p_2 &= \bar{c}^2(\rho_1 + \rho_2)
 \end{aligned}$$

This will be assumed of the fluid. The fluid is not a real fluid already (since its motion has been assumed inviscid) so it is not a remarkable extension to dictate a linear pressure-density relationship for this model fluid. This choice allows for simpler system equations:

$$\begin{aligned}
 \frac{\dot{p}_1 + \dot{p}_2}{2} + \frac{p_1 + p_2}{2} \frac{\dot{A}}{A} + \frac{p_2 u_2 - p_1 u_1}{L} &= 0 \\
 \frac{\dot{u}_1 + \dot{u}_2}{2} + \frac{u_1 + u_2}{2} \frac{u_2 - u_1}{L} + \frac{2\bar{c}^2}{p_1 + p_2} \frac{p_2 - p_1}{L} &= 0
 \end{aligned}$$

Next, the system is rewritten using flow-rates to replace velocities. When the fluid velocity  $u$  in a duct is uniform over the flow cross-section  $A$  at any position, the flow-rate at any point is equal to the product of the fluid velocity with the cross-sectional area. In acoustics literature the flow-rate is referred to as the *volume velocity*, and usually denoted by  $q$ .

The volume velocity is the preferred system variable, rather than particle velocity  $u$ , in descriptions of quasi-1D flow in tubes of varying cross-sectional area. This is because the volume velocity is continuous even when the flow encounters an abrupt change in cross-sectional area. Two or more system elements can then be concatenated with ease, since the output flow properties of one element are always the input properties to the next. This accommodates a useful analogy between pipe flows and electrical networks, allowing tools and theories of electrical circuit analysis to be modified to conveniently describe the behaviour of quasi-1D flows in pipes.

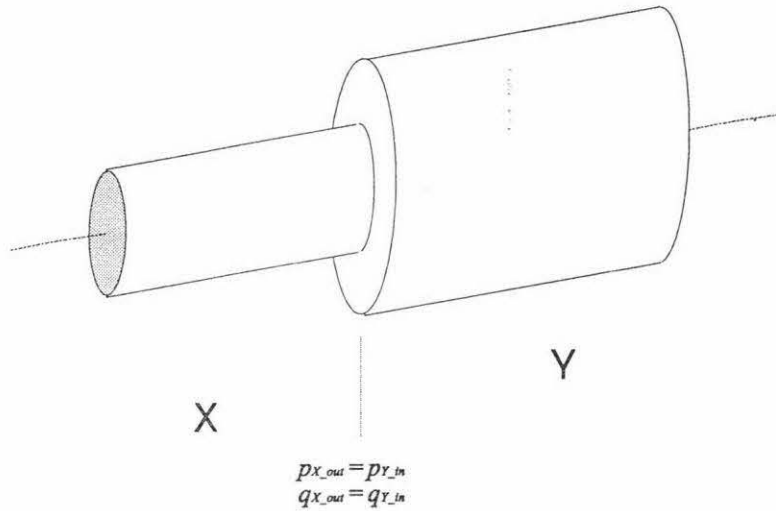


Figure 6.7: Concatenation of Two Tube Elements of Different Cross-Section

Since  $u = \frac{q}{A}$ , its derivative is written as  $\dot{u} = \frac{\dot{q}}{A} - \frac{q\dot{A}}{A^2}$ . Substitution, to eliminate instances of  $u$  and  $\dot{u}$  from the above system equations, gives:

$$\frac{\dot{p}_1 + \dot{p}_2}{2} A + \frac{p_1 + p_2}{2} \dot{A} + \frac{p_2 q_2 - p_1 q_1}{L} = 0$$

$$\frac{\dot{q}_1 + \dot{q}_2}{2} A + \frac{q_1 + q_2}{2} \left[ \frac{q_2 - q_1}{L} - \dot{A} \right] + \frac{2\bar{c}^2}{p_1 + p_2} \frac{p_2 - p_1}{L} A^2 = 0$$

Finally, recognition that  $p_2 q_2 - p_1 q_1 = \frac{1}{2} [(p_1 + p_2)(q_2 - q_1) + (p_2 - p_1)(q_1 + q_2)]$  leads to equations of the form:

$$\frac{\dot{p}_1 + \dot{p}_2}{2} A + \frac{p_1 + p_2}{2} \left[ \frac{q_2 - q_1}{L} + \dot{A} \right] + \frac{p_2 - p_1}{L} \frac{q_1 + q_2}{2} = 0 \quad (6.12)$$

$$\frac{\dot{q}_1 + \dot{q}_2}{2} A + \frac{q_1 + q_2}{2} \left[ \frac{q_2 - q_1}{L} - \dot{A} \right] + \frac{2\bar{c}^2}{p_1 + p_2} \frac{p_2 - p_1}{L} A^2 = 0 \quad (6.13)$$

#### 6.1.4 Case Study of Static Pressure Gradient, Forced Motion:

This subsection details the study of the transversely-oscillating piston in situations where the upstream and downstream fluid pressures are somehow maintained at constant values. The reaction of the flow beneath the piston is examined as the piston is forced up and down with some specified oscillatory motion.

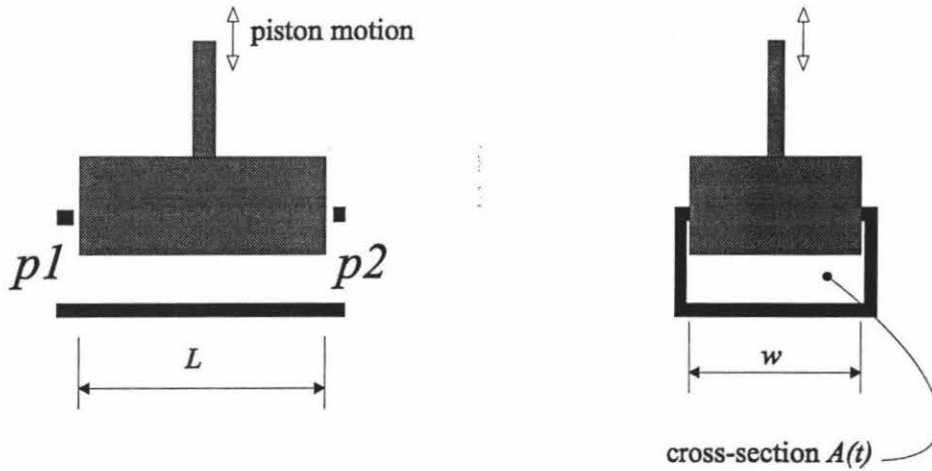


Figure 6.8 Constant Pressure across Transversely-Oscillating Piston

Model Implementation:

Recall Equations (6.12) and (6.13) (repeated below) which express mass and momentum conservation, respectively, in the control volume depicted earlier in Figure 6.5:

$$\frac{\dot{p}_1 + \dot{p}_2}{2} A + \frac{p_1 + p_2}{2} \left[ \frac{q_2 - q_1}{L} + \dot{A} \right] + \frac{p_2 - p_1}{L} \frac{q_1 + q_2}{2} = 0$$

$$\frac{\dot{q}_1 + \dot{q}_2}{2} A + \frac{q_1 + q_2}{2} \left[ \frac{q_2 - q_1}{L} - \dot{A} \right] + \frac{2\bar{c}^2}{p_1 + p_2} \frac{p_2 - p_1}{L} A^2 = 0$$

When the upstream and downstream pressures,  $p_1$  and  $p_2$  respectively, are held constant the first term of the continuity equation can be omitted, since then  $\dot{p}_1 = \dot{p}_2 = 0$ , giving

$$\frac{p_1 + p_2}{2} \left[ \frac{q_2 - q_1}{L} + \dot{A} \right] + \frac{p_2 - p_1}{L} \frac{q_1 + q_2}{2} = 0 \quad (6.14)$$

In order to solve this system of equations numerically using *myRKF.m* (see Section 6.1.2), it must first be cast into the form  $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ , where  $\mathbf{f}$  is a vector-valued function containing expressions for the derivatives of all system variables contained in  $\mathbf{y}$  (see Section 6.1.1). In this subsection, the motion of the piston is to be specified by the user. Consequently  $A(t)$  and its derivatives can be assumed known, and  $A$  needn't feature as an unknown in the vector of system variables.

This leaves only two unknown quantities whose derivatives appear in the system equations, viz.  $q_1$  and  $q_2$ , and their derivatives both appear in the momentum equation. The continuity equation is an algebraic equation which relates  $q_1$  and  $q_2$  in terms of known quantities. Consequently, the continuity equation may be differentiated with

respect to time and the result used to eliminate occurrences of either  $\dot{q}_1$  or  $\dot{q}_2$  from the momentum equation.

An alternative approach, which is followed below, is to adopt the quantity  $\frac{q_1 + q_2}{2}$  as the system variable. The momentum equation then simplifies to:

$$\dot{y}A + y \left[ \frac{q_2 - q_1}{L} - \dot{A} \right] + \frac{2\bar{c}^2}{p_1 + p_2} \frac{p_2 - p_1}{L} A^2 = 0 \quad \text{where } y = \frac{q_1 + q_2}{2}$$

The steady-pressure continuity equation (Equation 6.14) can then be used to express the remaining appearances of  $q_1$  and  $q_2$  in terms of the new system variable  $y$ :

$$\begin{aligned} \frac{p_1 + p_2}{2} \left[ \frac{q_2 - q_1}{L} + \dot{A} \right] + \frac{p_2 - p_1}{L} y = 0 \\ \Rightarrow \quad \frac{q_2 - q_1}{L} - \dot{A} = -\frac{2}{p_1 + p_2} \frac{p_2 - p_1}{L} y - 2\dot{A} \end{aligned}$$

Finally substitution into the momentum equation yields:

$$\begin{aligned} \dot{y}A - 2y\dot{A} + \frac{2}{p_1 + p_2} \frac{p_2 - p_1}{L} [\bar{c}^2 A^2 - y^2] = 0 \\ \text{or, } \quad \dot{y} = 2y \frac{\dot{A}}{A} + \frac{2}{p_1 + p_2} \frac{p_1 - p_2}{L} \left[ \bar{c}^2 - \left( \frac{y}{A} \right)^2 \right] A \end{aligned} \quad (6.15)$$

This equation now has the form  $\dot{y} = f(t, y)$ . It has been found that only one system variable is necessary to specify the state of the system for any given piston motion. This system variable, chosen as  $\frac{q_1 + q_2}{2}$ , represents the average of the upstream and downstream volume velocities for the flow region under the piston.

Equation (6.15) was implemented in a MATLAB function file, and the function *myRKF* was used to study numerically the behaviour of the system under different ambient conditions and with different piston trajectories. The results of these investigations are described in the remainder of this subsection.

#### Special Case of Piston Held Fixed:

When the piston is held fixed,  $\dot{A} = 0$  and Equation (6.15) describing the flow beneath the piston becomes:

$$\frac{\dot{y}}{A} = \frac{2}{p_1 + p_2} \frac{p_1 - p_2}{L} \left[ \bar{c}^2 - \left( \frac{y}{A} \right)^2 \right]$$

Since  $A$  is unvarying for this particular situation, it is possible to reformulate the problem using the quantity  $y/A$  as the system variable. Since  $y$  currently represents the average volume velocity ( $q$ ) of the fluid in the region beneath the piston,  $y/A$  represents the average (particle) velocity there. Consequently, the letter  $u$  will be used to denote this new system variable. The following ODE for  $u$  is equivalent to the equation above:

$$\dot{u} = k[\bar{c}^2 - u^2], \quad \text{where } k = \frac{2}{p_1 + p_2} \frac{p_1 - p_2}{L} \quad (6.16)$$

Before proceeding to an analytical solution for this equation, some preliminary observations may be used to gain a physical understanding of the behaviour of  $u$ . If  $u$  begins at zero, then Equation (6.16) dictates that  $u$  will increase at an initial rate of  $\dot{u} = k\bar{c}^2$ . As it does so, the value of  $\bar{c}^2 - u^2$  decreases. The continuing effect is that  $u$  approaches, asymptotically, the constant value  $\bar{c}$ . (The same occurs if  $u$  begins at a value larger than  $\bar{c}$ . Equation (6.16) then prescribes that  $u$  will decrease. As it does so, the value of  $\bar{c}^2 - u^2$  also decreases.) The line  $u(t) = \bar{c}$  itself is actually a solution of Equation (6.16), but requires an initial condition of  $u(0) = \bar{c}$  to be realised in practice.

The following graphs show the result of numerical solution of Equation (6.15) for constant  $A$ , using *myRKF*:

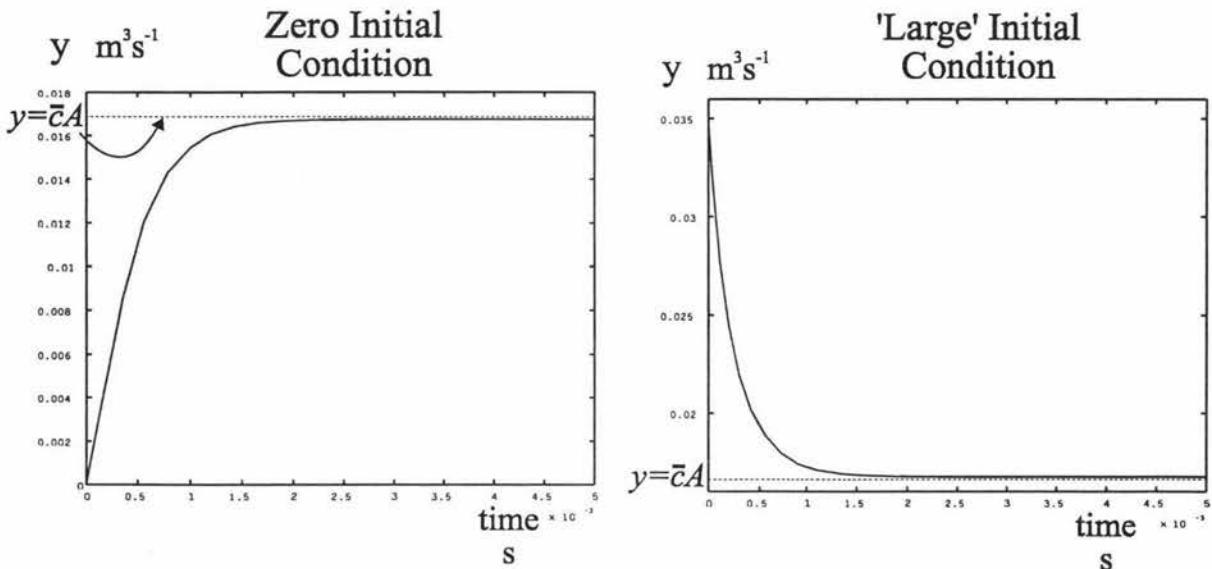


Figure 6.9: Solution for  $y$  for two different initial conditions

The above differential equation for  $u$  (Equation 6.16) is first-order and, because the right-hand side is quadratic in  $u$ , it belongs to the category of non-linear ODEs known as the *Riccati* equations. The solution found already, viz.  $u(t) = \bar{c}$ , can be used to find other members in the family of solution. The first step in the procedure is to postulate a solution of the form  $u(t) = v(t) + \bar{c}$ , where  $v$  is some other function of time yet to be determined. Substitution into Equation (6.16) yields:

$$\begin{aligned}\dot{v} &= k[\bar{c}^2 - (v + \bar{c})^2] \\ &= -k[2\bar{c}v + v^2]\end{aligned}$$

This equation for  $v$ , while still quadratic, does not have a constant term on the right-hand side. The new form allows for further simplification by another change of variables: Let  $v = \frac{1}{w}$ , then  $\dot{v} = -\frac{\dot{w}}{w^2}$ . Substitution yields:

$$\begin{aligned}-\frac{\dot{w}}{w^2} &= -k\left[\frac{2\bar{c}}{w} + \frac{1}{w^2}\right] \\ \Rightarrow \dot{w} &= k[2\bar{c}w + 1]\end{aligned}$$

This differential equation for  $w$  is first-order, autonomous and linear, and hence its solution is straightforward. The general solution of the homogeneous equation  $\dot{w} - 2\bar{c}kw = 0$  is  $w(t) = \eta \exp(2\bar{c}kt)$ , and a particular solution for  $\dot{w} - 2\bar{c}kw = k$  is  $w(t) = -\frac{1}{2\bar{c}}$ . Combining these gives:

$$\begin{aligned}w(t) &= \eta \exp(2\bar{c}kt) - \frac{1}{2\bar{c}} \\ &= -\frac{1}{2\bar{c}}[1 - 2\eta\bar{c} \exp(2\bar{c}kt)]\end{aligned}$$

The solution for  $v(t)$  may now be obtained by inverting  $w$ , since  $v = \frac{1}{w}$ :

$$v(t) = -\frac{2\bar{c}}{1 - 2\eta\bar{c} \exp(2\bar{c}kt)}$$

Finally,  $u$  is provided by the transformation  $u(t) = v(t) + \bar{c}$ , leading to:

$$\begin{aligned}
u(t) &= \bar{c} - \frac{2\bar{c}}{1 - 2\eta\bar{c}\exp(2\bar{c}kt)} \\
&= \frac{\bar{c}(1 - 2\eta\bar{c}\exp(2\bar{c}kt)) - 2\bar{c}}{1 - 2\eta\bar{c}\exp(2\bar{c}kt)} \\
&= -\bar{c} \frac{1 + 2\eta\bar{c}\exp(2\bar{c}kt)}{1 - 2\eta\bar{c}\exp(2\bar{c}kt)} \tag{6.17}
\end{aligned}$$

This is the general form of the solution for  $u$ . The initial condition determines which member of the family of solution curves is applicable; different members have different values for the parameter  $\eta$ . For example, the previously known solution  $u(t) = \bar{c}$  requires that  $\eta = 0$ , while the solution which passes through the initial condition  $u(0) = 0$  has the parameter value  $\eta = -\frac{1}{2\bar{c}}$ .

The findings of the above qualitative and quantitative analyses of the differential equation for  $u$  (Equation 6.16) are not borne out in real-life fluid flow situations. The general solution (Equation 6.17) dictates that a constant pressure difference applied between the two ends of a fluid duct will cause fluid motion, the velocity of the resulting flow increasing until (assuming that the constant pressure difference is maintained) the flow reaches the speed of acoustic propagation in the fluid,  $\bar{c}$ . In reality, a terminal fluid velocity can be reached when  $u$  is still much smaller than  $\bar{c}$ .

The discrepancy arises because the model equations describe the response of a fluid restrained to motion that is inviscid. In Section 6.3, fluid viscosity is incorporated into the model description.

#### Flow Reaction to Sinusoidal Piston Motion:

This subsection concludes with an investigation into the response of the fluid to oscillation of the piston in which the displacement varies sinusoidally with time. Three parameters are chosen to completely specify the effect of the motion upon the cross-sectional area of the gap under the piston. These are:

- the aperture cross-sectional area ( $A_0$ ) at the central position of the piston,
- the ratio of the area oscillation amplitude to the area at the midpoint ( $\alpha$ ), and
- the angular frequency ( $\omega$ ) of piston oscillation.

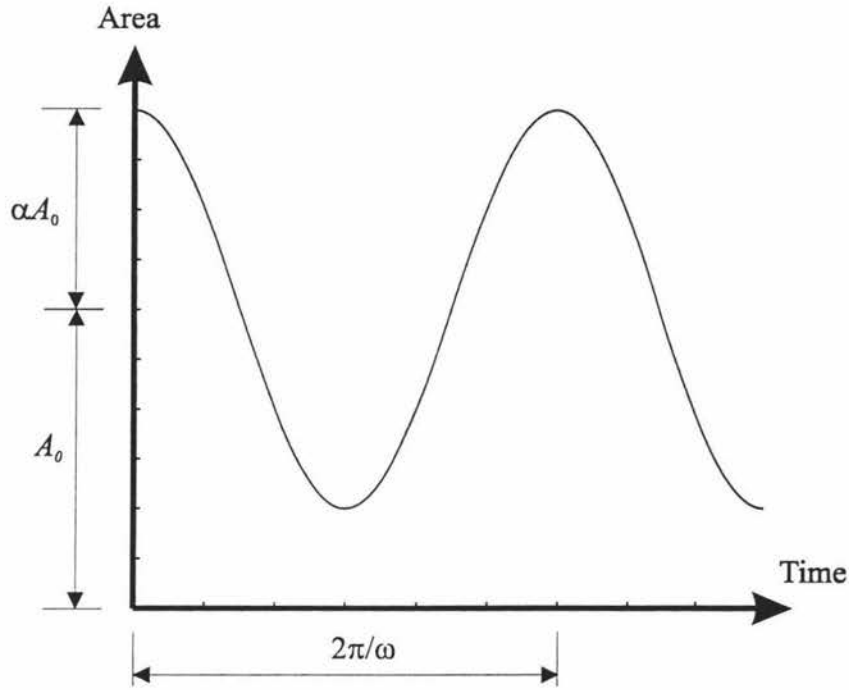


Figure 6.10 Sinusoidal Aperture Area Variation showing Parameters

Mathematically, these parameters are combined as follows to describe the time-variation of the aperture cross-section:

$$A(t) = A_0(1 + \alpha \cos(\omega t)) \quad (6.18)$$

This expression has physical meaning when the chosen parameters fall within the following ranges:

$$A_0 \geq 0 \quad |\alpha| < 1$$

Equation (6.15), derived earlier to describe the reaction of the fluid to arbitrary changes in channel cross-section, is repeated below:

$$\dot{y} = \frac{1}{A} \left[ 2y\dot{A} + \frac{2}{p_1 + p_2} \frac{p_1 - p_2}{L} (\bar{c}^2 A^2 - y^2) \right]$$

Taking the derivative of Equation (6.18) gives  $\dot{A}(t) = -A_0\alpha\omega \sin(\omega t)$ , and substitution for  $A$  and its derivative into Equation (6.15) above leads to the following expression:

$$\begin{aligned} \dot{y} &= \frac{1}{A_0(1 + \alpha \cos(\omega t))} \left\{ -2y\alpha A_0\omega \sin(\omega t) + k \left[ \bar{c}^2 A_0^2 (1 + \alpha \cos(\omega t))^2 - y^2 \right] \right\} \\ &= \frac{-k}{A_0(1 + \alpha \cos(\omega t))} y^2 - \frac{2\alpha\omega \sin(\omega t)}{1 + \alpha \cos(\omega t)} y + k \bar{c}^2 A_0^2 (1 + \alpha \cos(\omega t))^2 \end{aligned}$$

In this equation  $k = \frac{2}{p_1 + p_2} \frac{p_1 - p_2}{L}$  as it did for the case of a static piston above.

Like the equation obtained for the static piston scenario, this is a Riccati equation, since the right-hand side is again quadratic in  $y$ . However, in this case the equation is not autonomous. The trigonometrical coefficient functions turn even the search for a particular solution into a formidable task.

#### Transformation of System Equation to a 2<sup>nd</sup>-Order Linear ODE:

It is known that any Riccati equation can be transformed into an equivalent second-order ordinary differential equation by a suitable change of variable (Bender & Orszag, 1978). Furthermore, the new differential equation will be linear with respect to the new system variable. Such a transformation is below applied to Equation (6.15), which describes the response of the average volume velocity of the fluid in the aperture to any arbitrary cross-section variations of the form  $A(t)$ .

Recall Equation (6.15):

$$\dot{y} = 2y \frac{\dot{A}}{A} + kA \left[ \bar{c}^2 - \left( \frac{y}{A} \right)^2 \right]$$

Put  $y = \frac{Az}{kz} \Rightarrow \dot{y} = \frac{\dot{A}z}{kz} + \frac{A\dot{z}}{kz} - \frac{Az^2}{kz^2}$ . Substituting into Equation (6.15) gives:

$$\begin{aligned} \frac{\dot{A}z}{kz} + \frac{A\dot{z}}{kz} - \frac{Az^2}{kz^2} &= 2 \frac{\dot{A}}{A} \frac{Az}{kz} + kA \left[ \bar{c}^2 - \left( \frac{1}{A} \frac{Az}{kz} \right)^2 \right] \\ \Rightarrow \ddot{z} - \frac{\dot{A}}{A} \dot{z} - k^2 \bar{c}^2 z &= 0 \end{aligned} \quad (6.19)$$

Equation (6.19) is linear for any choice of  $A(t)$ . It is autonomous for the special case where  $\dot{A}(t) = 0, \forall t$  and the solution is then given by:

$$z(t) = \eta_1 \exp(-k\bar{c}t) + \eta_2 \exp(k\bar{c}t), \text{ where } \eta_1, \eta_2 \text{ are constants.} \quad (6.20)$$

The condition that  $\dot{A}(t) = 0, \forall t$  implies that the piston is held fixed. This solution is comparable to that obtained earlier for the case of the static piston, but there are now two parameters in the general solution, while the previous solution (Equation 6.17, repeated below) of the equivalent differential equation for  $u$  (Equation 6.16) had only one.

$$u(t) = -\bar{c} \frac{1 + 2\eta\bar{c} \exp(2\bar{c}kt)}{1 - 2\eta\bar{c} \exp(2\bar{c}kt)}$$

The secondary parameter of Equation (6.20) has its origins in the change of variables  $y = \frac{A\dot{z}}{kz}$  above, since this transformation expresses  $y$  in terms of a derivative of  $z$ . (A constant of integration is introduced by this transformation.) Equation (6.20) above written for  $z(t)$  is equivalent to Equation (6.17) for  $u$  when  $\eta_2 = -2\eta\bar{c}\eta_1$ , as can be demonstrated as follows:

$$\begin{aligned} u &= \frac{y}{A} = \frac{\dot{z}}{kz} = \frac{-\eta_1 k\bar{c} \exp(-k\bar{c}t) + \eta_2 k\bar{c} \exp(k\bar{c}t)}{k[\eta_1 \exp(-k\bar{c}t) + \eta_2 \exp(k\bar{c}t)]} \\ &= \frac{-\eta_1 k\bar{c} \exp(-k\bar{c}t) + (-2\eta\bar{c}\eta_1)k\bar{c} \exp(k\bar{c}t)}{k[\eta_1 \exp(-k\bar{c}t) + (-2\eta\bar{c}\eta_1) \exp(k\bar{c}t)]} \\ &= -\bar{c} \frac{\exp(-k\bar{c}t) + 2\eta\bar{c} \exp(k\bar{c}t)}{\exp(-k\bar{c}t) - 2\eta\bar{c} \exp(k\bar{c}t)} \\ &= -\bar{c} \frac{1 + 2\eta\bar{c} \exp(2k\bar{c}t)}{1 - 2\eta\bar{c} \exp(2k\bar{c}t)} \end{aligned}$$

Equation (6.20) then becomes  $z(t) = \eta_1 [\exp(-k\bar{c}t) - 2\eta\bar{c} \exp(k\bar{c}t)]$ . In this equation the parameter  $\eta$  is determined by the initial conditions specified (as it was before when it appeared in Equation (6.17)), whereas any arbitrary non-zero value may be chosen for  $\eta_1$ .

The form of Equation (6.19) and of its solution, for the case of  $\dot{A}(t) = 0$ , indicate that there is no natural frequency associated with the flow in the absence of cross-sectional variation.

Equation (6.19), repeated below, is also autonomous for the special case where the piston motion obeys the equation  $\dot{A}(t) = \gamma A(t)$ , where  $\gamma$  has any constant value.

$$\ddot{z} - \frac{\dot{A}}{A} \dot{z} - k^2 \bar{c}^2 z = 0$$

Whenever  $A(t) = \zeta \exp(\gamma t)$ , this equation takes the form of

$$\ddot{z} - \gamma \dot{z} - k^2 \bar{c}^2 z = 0 \quad (6.21)$$

Equation (6.21) has the following characteristic equation

$$m^2 - \gamma m - k^2 \bar{c}^2 = 0$$

Its roots are, by the quadratic formula:

$$m = \frac{\gamma \pm \sqrt{\gamma^2 + 4k^2 \bar{c}^2}}{2}$$

Both roots of the characteristic equation are necessarily real, since the discriminant is always positive. By the triangle inequality, one of the roots is necessarily negative and has magnitude smaller than  $k\bar{c}$ , while the other is positive with magnitude greater than  $\gamma + k\bar{c}$ . The solution of Equation (6.21) thus has the form of an exponentially decaying solution superimposed upon an exponentially growing solution.

The form of Equation (6.21), for the case of  $\dot{A}(t) = \gamma A(t)$ , indicates that there is no natural frequency associated with the flow for this special case of exponential time-variance of the aperture cross-sectional area. As a consequence, any arbitrary piston motion which can be formed by the superposition of a finite or infinite number of functions having the form  $A(t) = \zeta \exp(\gamma t)$  cannot produce a solution of an oscillatory nature.

If a piston motion itself has oscillatory component, then any steady state oscillations of the full system will have frequencies determined only by those frequencies present in the piston motion. The above discussion indicates that transients of the system will not produce any system oscillations which are system-dependent, rather than input-dependent. There are no resonant frequencies associated with the piston configuration itself.

For the harmonic variation of cross-section as described in Equation (6.18) above, Equation (6.19) becomes:

$$\ddot{z} - \frac{\alpha\omega \sin(\omega t)}{1 + \alpha \cos(\omega t)} \dot{z} - k^2 \bar{c}^2 z = 0$$

#### Retreat to Numerical Solution:

Although expression of the previous Riccati equation (Equation 6.15) as an equivalent second-order linear ODE has provided further insight into the behaviour of the system, the complexity of both forms of the equation indicate that numerical solution could be a more viable option. This was accomplished in MATLAB using *myRKF*, together with a secondary program which expressed Equations (6.15) and (6.18) above.

Typical operation of the system is represented in Figure 6.11 below. The spatially-averaged flow-rate in the aperture region,  $y$ , is seen to be the highest when the cross-

sectional area is the largest, i.e.  $y$  and  $A$  are in phase. The phase plot for  $y$  shows the system trajectory converging rapidly from the initial conditions to a periodic orbit. This is not a surprising phase plot for a forced oscillation. The steady state oscillation of  $y$  contains only the frequency of the variation of  $A$ .

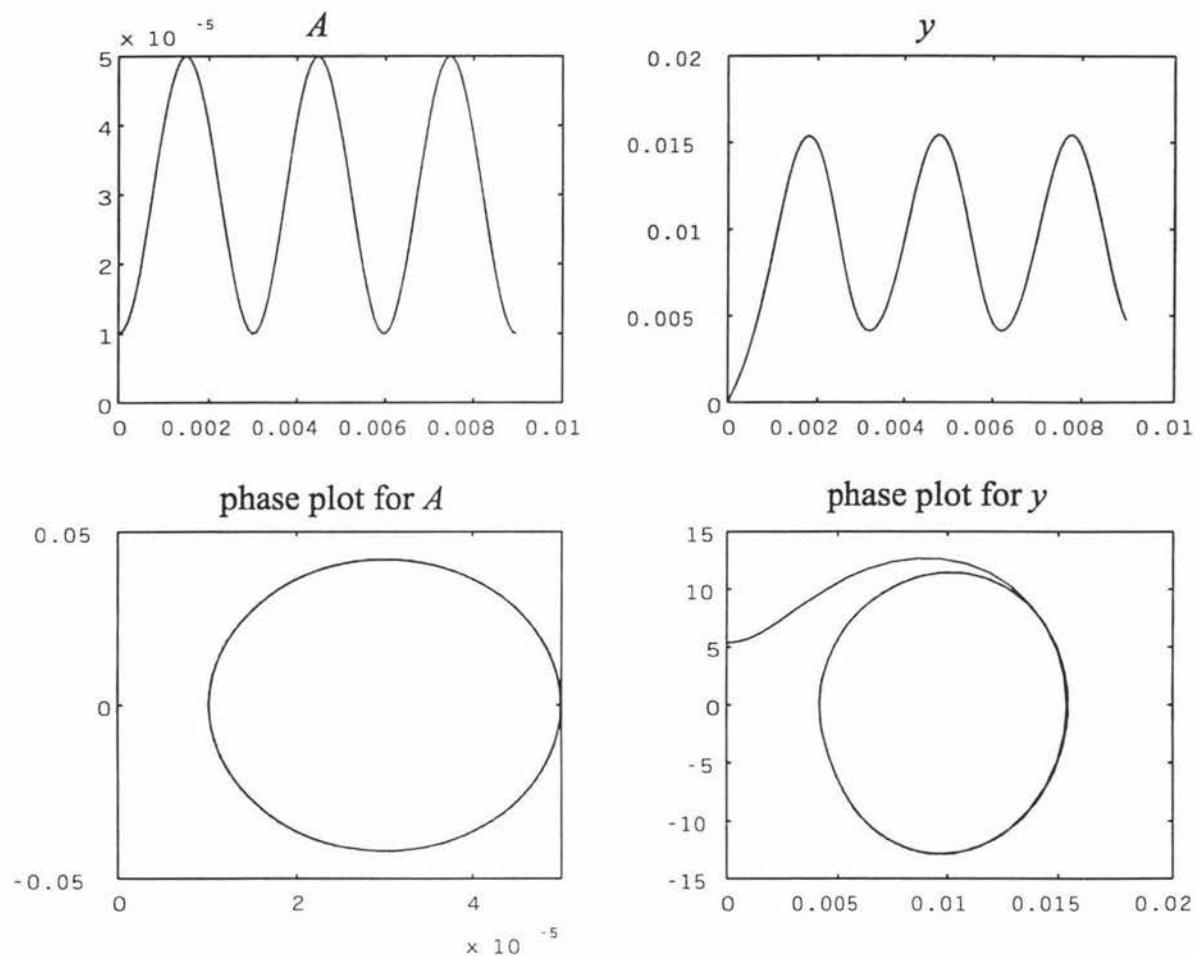


Figure 6.11: Sample Numerical Solution for Equation (6.15)

### 6.1.5 Revised Equations for Additional Upstream Reservoir:

Now coupled to the oscillating piston arrangement is an upstream fluid reservoir; the input to the reservoir is either a constant pressure force or a constant volume flow.

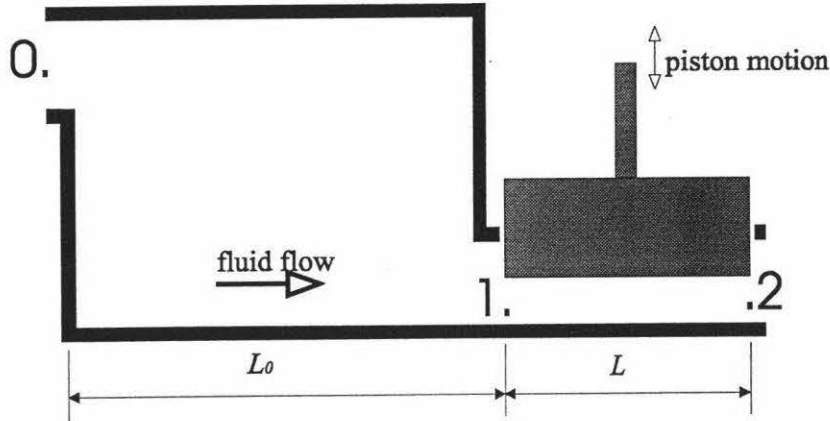


Figure 6.12: Piston-In-Tube Geometry, With Additional Upstream Reservoir

In Section 6.1.3, equations of continuity and fluid momentum for the aperture region beneath the piston were derived by means of a control volume analysis. Integrals in the resulting equations were approximated by the well-known Trapezoidal Rule, leading to Equations (6.12) and (6.13), repeated below, which relate upstream and downstream pressures and volume velocities to the motion of the piston.

$$\frac{\dot{p}_1 + \dot{p}_2}{2} A + \frac{p_1 + p_2}{2} \left[ \frac{q_2 - q_1}{L} + \dot{A} \right] + \frac{p_2 - p_1}{L} \frac{q_1 + q_2}{2} = 0$$

$$\frac{\dot{q}_1 + \dot{q}_2}{2} A + \frac{q_1 + q_2}{2} \left[ \frac{q_2 - q_1}{L} - \dot{A} \right] + \frac{2\bar{c}^2}{p_1 + p_2} \frac{p_2 - p_1}{L} A^2 = 0$$

In this subsection an alternative approach is demonstrated. The equations of continuity and momentum conservation are retained in the form of partial differential equations. When in the desired format, *finite difference approximations* are used to convert the PDEs to *finite difference equations*.

#### Continuity Equation:

The motion of the fluid within the duct can be related to the motion of the walls of the duct using a control volume analysis as follows. A deformable control volume is employed which is, at every instant of time, coincident with the aperture beneath the piston:

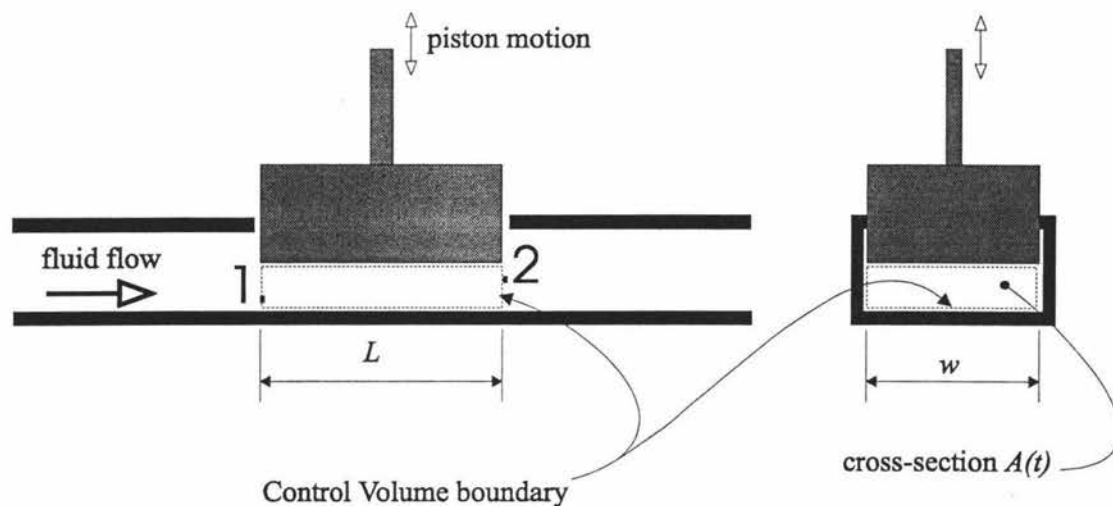


Figure 6.13 Deformable Control Volume of Length  $L$

A continuity equation is derived by applying the principle of conservation of mass to the fluid in the aperture formed by the piston in the tube. The rate of change of fluid mass in the control volume is equal to the difference between the rate at which fluid enters the upstream edge of the region and the rate at which fluid leaves the downstream edge. Recalling that the assumption of quasi-one dimensional fluid motion implies uniform fluid properties over the flow cross-section, this may be written as follows:

$$\begin{aligned} \frac{\partial}{\partial t} \left[ \int_{x_1}^{x_2} \rho A \, dx \right] &= (\rho_1 u_1 - \rho_2 u_2) A \\ \Rightarrow \frac{\partial}{\partial t} \left[ A \int_{x_1}^{x_2} \rho \, dx \right] &= (\rho_1 u_1 - \rho_2 u_2) A \\ \Rightarrow \int_{x_1}^{x_2} \frac{\partial \rho}{\partial t} \, dx + \frac{\dot{A}}{A} \int_{x_1}^{x_2} \rho \, dx &= [-\rho u]_{x_1}^{x_2} = - \int_{x_1}^{x_2} \frac{\partial}{\partial x} (\rho u) \, dx \end{aligned}$$

Within the aperture region, the integrands themselves may be equated:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\dot{A}}{A} = - \frac{\partial}{\partial x} (\rho u)$$

Consequently, the continuity equation valid in the aperture beneath the piston becomes:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\dot{A}}{A} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0 \quad (6.22)$$

Momentum Equation:

The Bernoulli Equation is independent of area variations (Lighthill, 1975). This can be demonstrated as follows:

Newton's Second Law of Motion states that any change of the amount of fluid momentum contained by the deformable control volume (refer to Figure 6.13), other than that due to the advection of momentum across the region's boundaries, is due to the net effect of external forces acting upon the fluid in the region. For an inviscid fluid with properties uniform over a flow cross-section, in the absence of external force field effects such as gravity, this can be expressed mathematically as follows:

$$\begin{aligned} & \frac{\partial}{\partial t} \left[ \int_{x_1}^{x_2} \rho u A \, dx \right] + (\rho_2 u_2^2 - \rho_1 u_1^2) A = (p_1 - p_2) A \\ \Rightarrow & \frac{\partial}{\partial t} \left[ A \int_{x_1}^{x_2} \rho u \, dx \right] + (\rho_2 u_2^2 - \rho_1 u_1^2) A = (p_1 - p_2) A \\ \Rightarrow & \int_{x_1}^{x_2} \frac{\partial}{\partial t} (\rho u) \, dx + \frac{\dot{A}}{A} \int_{x_1}^{x_2} \rho u \, dx + [\rho u]_{x_1}^{x_2} + [p]_{x_1}^{x_2} = 0 \\ \Rightarrow & \int_{x_1}^{x_2} \frac{\partial}{\partial t} (\rho u) \, dx + \frac{\dot{A}}{A} \int_{x_1}^{x_2} \rho u \, dx + \int_{x_1}^{x_2} \frac{\partial}{\partial x} (\rho u) \, dx + \int_{x_1}^{x_2} \frac{\partial p}{\partial x} \, dx = 0 \end{aligned}$$

Equating the sum of the integrands themselves to zero (valid within the aperture region) gives

$$\frac{\partial}{\partial t} (\rho u) + \frac{\rho u \dot{A}}{A} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial p}{\partial x} = 0$$

An exact multiple of the continuity equation (Equation 6.22) can be extracted as follows

$$\begin{aligned} & \frac{\partial}{\partial t} (\rho u) + \frac{\rho u \dot{A}}{A} + \frac{\partial}{\partial x} (\rho u^2) + \frac{\partial p}{\partial x} = 0 \\ \Rightarrow & u \frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial t} + \frac{\rho u \dot{A}}{A} + u^2 \frac{\partial \rho}{\partial x} + 2\rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0 \\ \Rightarrow & u \left( \frac{\partial \rho}{\partial t} + \frac{\rho \dot{A}}{A} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} \right) + \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \frac{\partial p}{\partial x} = 0 \end{aligned}$$

The Bernoulli Equation remains:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \quad (6.23)$$

Conversion to Acoustic Variables:

Proportionality between  $p$  and  $\rho$  is assumed appropriate, as it was earlier in Section 6.1.3.

Equations (6.22) and (6.23) are modified accordingly:

$$\frac{\partial p}{\partial t} + p \frac{\dot{A}}{A} + p \frac{\partial u}{\partial x} + u \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\bar{c}^2}{p} \frac{\partial p}{\partial x} = 0$$

Volume velocity  $q$  is again used to replace particle velocity  $u$ . The following substitutions are used:

$$u = \frac{q}{A} \quad \frac{\partial u}{\partial x} = \frac{1}{A} \frac{\partial q}{\partial x} \quad \frac{\partial u}{\partial t} = \frac{1}{A} \frac{\partial q}{\partial t} - \frac{q \dot{A}}{A^2}$$

The resulting equations are:

$$\frac{\partial p}{\partial t} + \frac{p}{A} \left[ \frac{\partial q}{\partial x} - \dot{A} \right] + \frac{q}{A} \frac{\partial p}{\partial x} = 0 \quad (6.24)$$

$$\frac{\partial q}{\partial t} + \frac{q}{A} \left[ \frac{\partial q}{\partial A} + \dot{A} \right] + \frac{\bar{c}^2}{p} \frac{\partial p}{\partial x} A = 0 \quad (6.25)$$

Finite Difference Equations:

Spatial derivatives of  $p$  and  $q$  are now replaced by finite difference approximations. Assume that the properties of the fluid are known at two positions;  $x_1$  denotes a point at the upstream edge of the aperture region and  $x_2$  is a point at the downstream edge.

There are two different schemes possible by which the available information can be utilised to form finite difference approximations to Equations (6.24) and (6.25). In the first scheme, the differential equation is applied at the point  $x_1$ , and the spatial derivative of any fluid property at  $x_1$  is approximated by Newton's first-order forward-difference formula (Hoffman, 1991). For any fluid property  $\psi$ :

$$\frac{\partial \psi}{\partial x}(x_1, t) \cong \frac{\psi(x_2, t) - \psi(x_1, t)}{x_2 - x_1} = \frac{\psi_2 - \psi_1}{L}$$

The equations below are finite difference equations which are approximations to Equations (6.24) and (6.25) to first order in  $L$ :

$$\dot{p}_1 + \frac{p_1}{A} \left[ \frac{q_2 - q_1}{L} + \dot{A} \right] + \frac{q_1}{A} \frac{p_2 - p_1}{L} = 0$$

$$\dot{q}_1 + \frac{q_1}{A} \left[ \frac{q_2 - q_1}{L} - \dot{A} \right] + \frac{\bar{c}^2}{p_1} \frac{p_2 - p_1}{L} A = 0$$

In the second scheme, the differential Equations (6.24) and (6.25) are applied at the point midway along the piston, ie. at  $x = \frac{x_1 + x_2}{2} = x_1 + \frac{L}{2}$ . The spatial derivatives there are approximated using a second-order centred-difference formula:

$$\frac{\partial \psi}{\partial x} \left( \frac{x_1 + x_2}{2}, t \right) \equiv \frac{\psi(x_2, t) - \psi(x_1, t)}{x_2 - x_1} = \frac{\psi_2 - \psi_1}{L}$$

Although this scheme yields approximations to spatial derivatives which are accurate to higher order in  $L$ , there is a difficulty in that the values of  $\psi$  and  $\frac{\partial \psi}{\partial t}$  are unknown at  $x = \frac{x_1 + x_2}{2}$ ; these must be estimated by interpolation. Since the data are only known at the two positions, linear interpolation is required to estimate  $p$ ,  $\dot{p}$ ,  $q$  and  $\dot{q}$  at the central position. Each of these estimates introduces interpolation errors. The resulting finite difference equations are:

$$\frac{\dot{p}_1 + \dot{p}_2}{2} + \frac{p_1 + p_2}{2A} \left[ \frac{q_2 - q_1}{L} + \dot{A} \right] + \frac{q_1 + q_2}{2A} \frac{p_2 - p_1}{L} = 0 \quad (6.26)$$

$$\frac{\dot{q}_1 + \dot{q}_2}{2} + \frac{q_1 + q_2}{2A} \left[ \frac{q_2 - q_1}{L} - \dot{A} \right] + \frac{2\bar{c}^2}{p_1 + p_2} \frac{p_2 - p_1}{L} A = 0 \quad (6.27)$$

It will be noticed that Equations (6.26) and (6.27), which use central differences and linear interpolations to approximate Equations (6.24) and (6.25) at the position  $x = \frac{x_1 + x_2}{2}$ , are identical to Equations (6.12) and (6.13) which re-appeared at the beginning of this subsection.

### Reservoir Flow Equations:

Section 6.3.1 described further simplification of the above two equations for situations where the upstream and downstream pressures were somehow maintained at constant values. In this subsection, the pressure gradient between the upstream and downstream

faces of the piston is instead subject to time-variation which depends upon the behaviour of the fluid properties in the upstream reservoir. Consequently, further simplification of the aperture flow equations is not possible; more equations are instead required to couple the aperture flow properties to those within the upstream reservoir.

Equations describing the behaviour of the upstream fluid properties may be written down by comparison with the situation described by Equations (6.26) and (6.27). A subscript 0 is now used to denote a flow property measured at the input of the reservoir (refer back to Figure 6.12). The physical dimensions of the reservoir are considered to remain constant. (The walls of the upstream vessel do not flex in response to internal pressure fluctuations, for example.) The appropriate equations are:

$$\frac{\dot{p}_0 + \dot{p}_1}{2} + \frac{p_0 + p_1}{2A_0} \frac{q_1 - q_0}{L_0} + \frac{q_0 + q_1}{2A_0} \frac{p_1 - p_0}{L_0} = 0 \quad (6.28)$$

$$\frac{\dot{q}_0 + \dot{q}_1}{2} + \frac{q_0 + q_1}{2A_0} \frac{q_1 - q_0}{L_0} + \frac{2\bar{c}^2}{p_0 + p_1} \frac{p_1 - p_0}{L_0} A_0 = 0 \quad (6.29)$$

Together, Equations (6.26) to (6.29) are four ordinary differential equations describing the response of the six flow properties,  $p_0$ ,  $p_1$ ,  $p_2$ ,  $q_0$ ,  $q_1$  and  $q_2$ , to motion of the piston  $A(t)$ . Two of these six properties can be fixed by imposing suitable boundary conditions. For example, numerical implementation is simplest when  $\dot{p} = 0$  is specified at one end of the system and  $\dot{q} = 0$  is specified at the other.

The case of  $\dot{p}_2 = 0$ ,  $\dot{q}_0 = 0$  is also a practical choice. The constant downstream pressure may be chosen as atmospheric. By setting  $\dot{q}_0 = 0$ , a constant volume influx is prescribed at the upstream end of the reservoir. This may be realised in practice, for example, by a pump arrangement introducing new fluid into the reservoir at a constant rate.

#### Case of Sinusoidal Piston Motion:

Equations (6.26) to (6.29), together with boundary conditions  $\dot{p}_2 = 0$  and  $\dot{q}_0 = 0$ , are sufficient to describe the response of the fluid to time-variations of the aperture cross-sectional area  $A(t)$ . The four unknowns of the flow ( $p_0$ ,  $p_1$ ,  $q_1$  and  $q_2$ ) are intricately related to each other, and to the specified boundary conditions and system input  $A(t)$ ,

through the four system equations. The complicated nature of the dependence of each system quantity upon the others is shown pictorially in Figure 6.14 below.

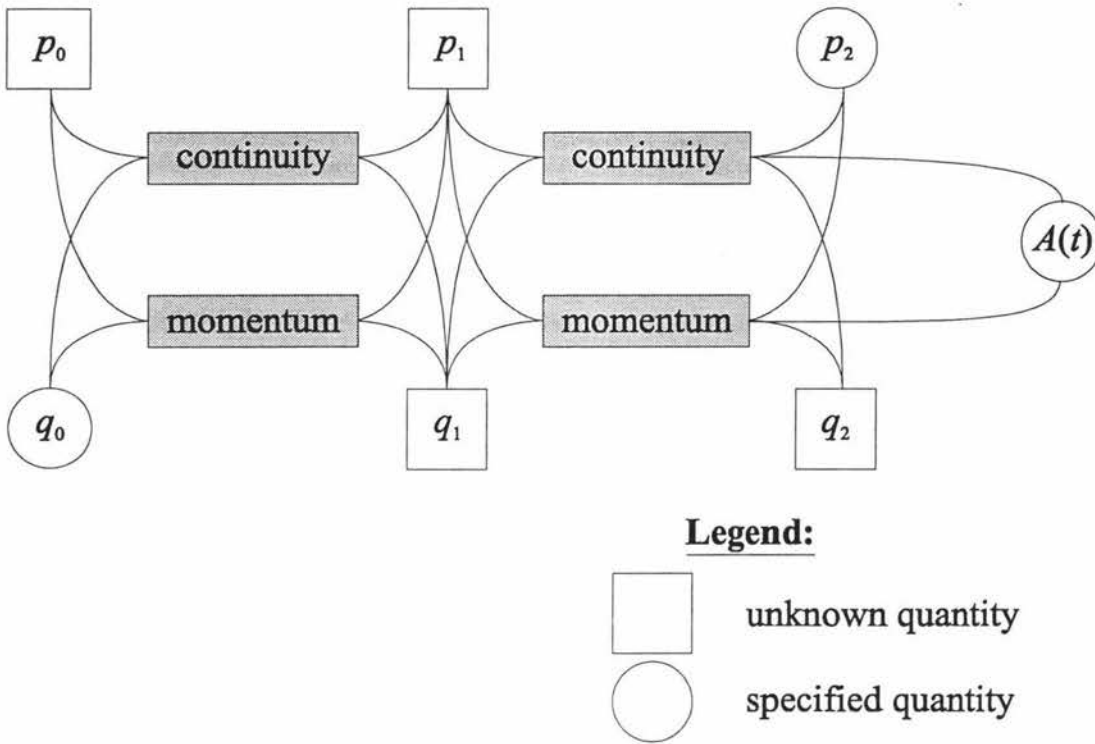


Figure 6.14: Interdependence of System Quantities

The flow's response to sinusoidal piston motion is investigated below. The three parameters used to describe the details of the accompanying aperture area variations are the same as those used previously in Section 6.1.4. These are:

- the aperture cross-sectional area ( $A_0$ ) at the central position of the piston,
- the ratio of the area oscillation amplitude to the area at the midpoint ( $\alpha$ ), and
- the angular frequency ( $\omega$ ) of piston oscillation.

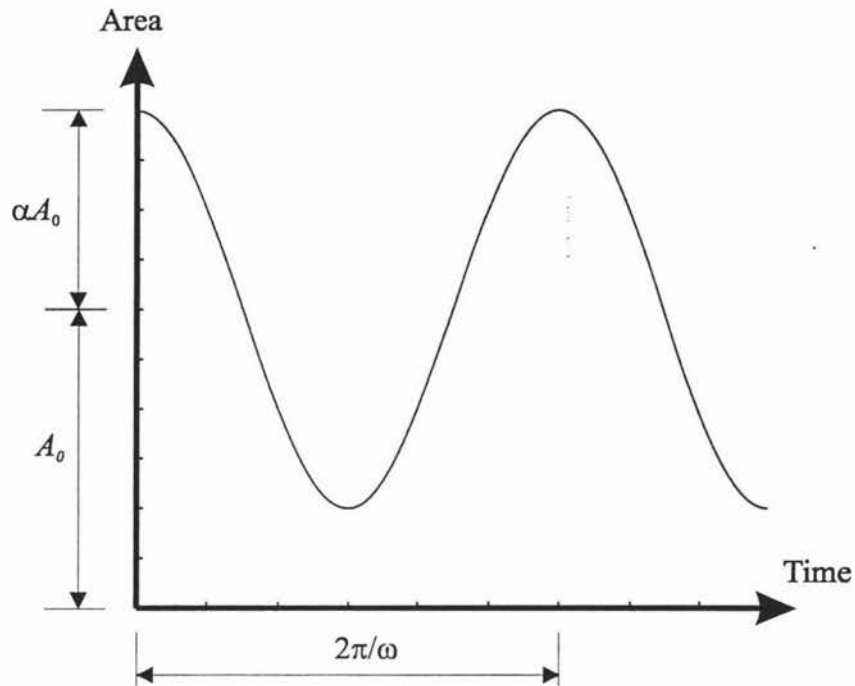


Figure 6.15 Sinusoidal Aperture Area Variation showing Parameters

Again, the time-variation of the aperture cross-section is described by Equation (6.18):

$$A(t) = A_0(1 + \alpha \cos(\omega t))$$

Equations (6.26) to (6.29) were implemented in a MATLAB function file, to investigate the flow response to the sinusoidal piston motion. The program *myRKF.m* was used to solve the system for the values of the four flow unknowns. Figure 6.16 below shows the solution for  $p_1$  while the oscillation of  $A(t)$  completes a tenth of one cycle (at 335 Hz). The graph exhibits a secondary oscillation, whose frequency is approximately 200 times that of the piston frequency.

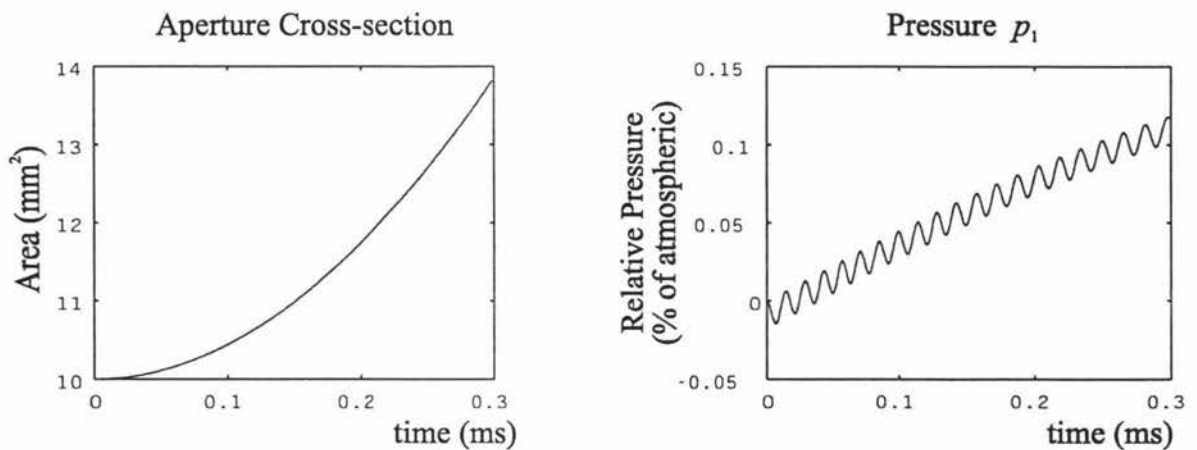


Figure 6.16: Small-Time Variation of Pressure  $p_1$

Although the secondary oscillation is interesting, its presence is a hindrance to the investigation of longer-term effects of the upstream reservoir. This is because a very small time-step is required to calculate the secondary oscillation to within the prescribed limits of accuracy. With such a small time-step, calculation of the long-time behaviour of the system requires an excessive amount of computation time. In situations such as this, the system is often said to be *stiff*.

Figure 6.17 below shows the solution for  $p_1$  while  $A(t)$  complete three cycles of oscillation. The graph depicts larger-scale pressure fluctuations which seem closely reliant upon the motion of the piston, in addition to the smaller-amplitude high-frequency fluctuations that were shown in more detail in Figure 6.16.

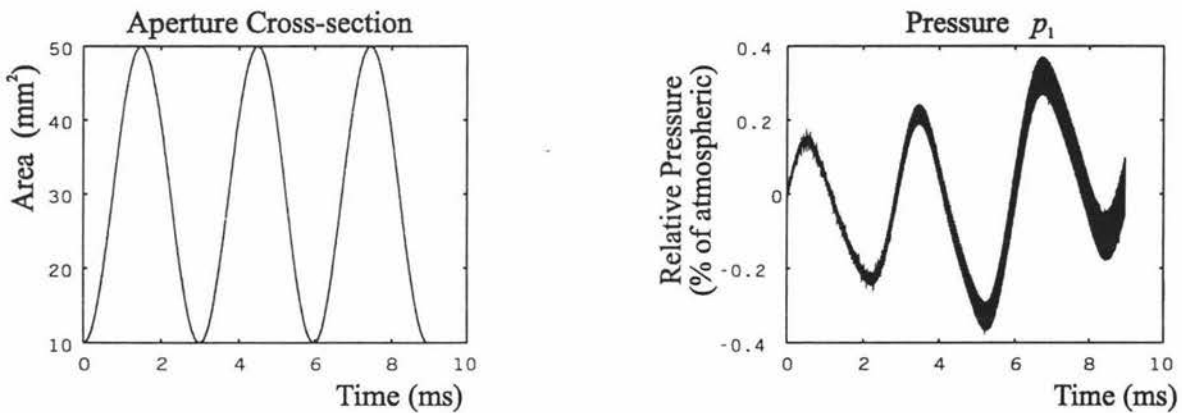


Figure 6.17: Medium-Term Variation of Pressure  $p_1$

The high-frequency oscillation does not appear to be part of a transient response; rather, it seems likely that it will remain present throughout time. To study the long-term behaviour of the system two approaches are possible: Firstly, computationally intensive long-time simulations can be programmed to run themselves (for example, overnight) and the results saved to files for later perusal. These results files may be very large if the data calculated at all time-steps is archived. However, statistics (*viz.* moving average and standard deviation) may be used to reduce this memory requirement. The record of the mean pressure can later be plotted to show the large-scale pressure fluctuations, and the standard deviation can be plotted to show the long-time variation of the amplitude of the high-frequency pressure oscillation.

The second approach available is to modify the equations of the system so as to eliminate the high-frequency oscillation. The next section describes this development in detail. The high-frequency oscillation is eliminated by assuming that the fluid behaves as if incompressible in the aperture region. Success in this respect suggests that the higher frequency oscillation of Figure 6.17 is the result of acoustic reflections back and forth within the aperture region; such reflections are absent from an incompressible description.

## 6.2 Removal of Fluid Compressibility:

In this section the fluid in the aperture region beneath the piston is treated as incompressible.

In a compressible fluid, the influence of a flow disturbance introduced at a given point will not be seen by an observer at a second place in the fluid until after a finite delay time (Section 4.3). The value of the delay depends upon the slope of the characteristic curves originating from the disturbance, and upon the distance between the disturbance and the observer. When this distance is small enough (how small depends upon the slope of the characteristics) then the delay time is negligible; the disturbances are seen practically instantaneously by the observer. The flow is then described reasonably well by an incompressible description.

It is postulated that the dimensions of the piston, of the flow modulator mechanism under study, are sufficiently small that the assumption of incompressible flow will give a satisfactory description. This is considered to be a reasonable approximation when the dimensions of the aperture region are all very small compared with the wavelength of any flow disturbances.

### 6.2.1 Revised Equations of Flow Beneath the Piston:

This subsection follows the same approach as Section 6.1.5 to derive flow equations that are applicable in the aperture region beneath the piston for the case of an incompressible fluid.

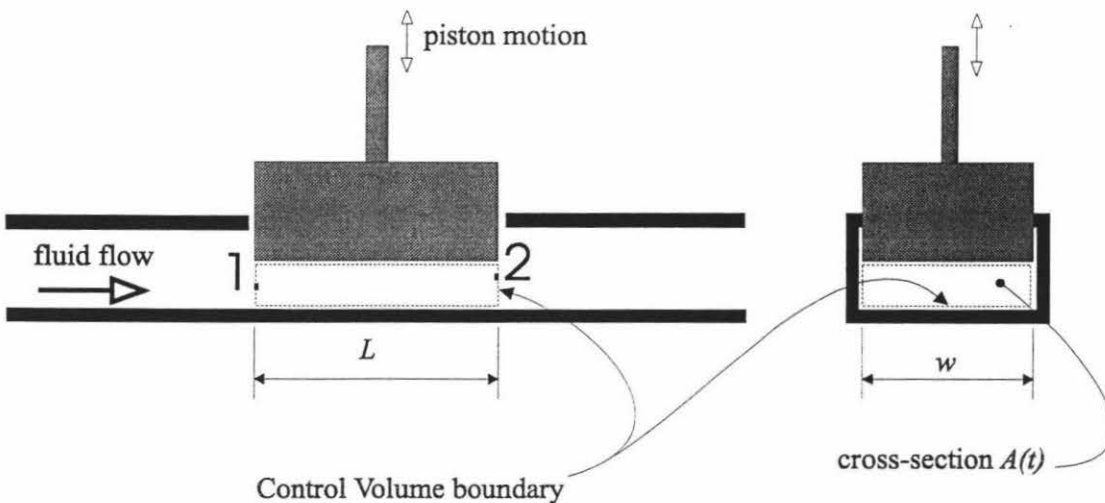


Figure 6.18 Deformable Control Volume of Length  $L$

A continuity equation applicable in the control volume depicted above in Figure 6.18 was derived in Subsection 6.1.5. Recall Equation (6.22) repeated below:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\dot{A}}{A} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0$$

For the case of a fluid of constant density, the first and the last terms of the above equation are identically zero. The resulting continuity equation valid in the aperture region is simply

$$\frac{\dot{A}}{A} + \frac{\partial u}{\partial x} = 0 \quad (6.30)$$

In Section 6.1.5, the Bernoulli Equation (Equation 6.23) was shown to be independent of area variations. For the case of a fluid of constant density  $\bar{\rho}$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\bar{\rho}} \frac{\partial p}{\partial x} = 0$$

Substitution of Equation (6.30) into the Bernoulli equation gives:

$$\frac{\partial u}{\partial t} - u \frac{\dot{A}}{A} + \frac{1}{\bar{\rho}} \frac{\partial p}{\partial x} = 0 \quad (6.31)$$

Converting to acoustic variables, volume velocity  $q$  is used to replace particle velocity  $u$ .

The following substitutions are used:

$$u = \frac{q}{A}, \quad \frac{\partial u}{\partial x} = \frac{1}{A} \frac{\partial q}{\partial x}, \quad \frac{\partial u}{\partial t} = \frac{1}{A} \frac{\partial q}{\partial t} - \frac{q \dot{A}}{A^2}$$

The resulting continuity equation equivalent to Equation (6.30) is:

$$\frac{\partial q}{\partial x} = -\dot{A} \quad (6.32)$$

A momentum equation equivalent to Equation (6.31) is:

$$\frac{1}{A} \frac{\partial q}{\partial t} - 2 \frac{q \dot{A}}{A^2} + \frac{1}{\bar{\rho}} \frac{\partial p}{\partial x} = 0 \quad (6.33)$$

The spatial derivatives of  $p$  and  $q$  can be approximated by applying a centred-difference approximation at the midpoint between  $x_1$  and  $x_2$ . Values for  $q$  and  $\frac{\partial q}{\partial t}$  at the midpoint are estimated by linear interpolations. The following equations approximate the behaviour of Equations (6.32) and (6.33):

$$\frac{q^2 - q_1}{L} = -\dot{A} \quad (6.34)$$

$$\frac{\dot{q}_1 + \dot{q}_2}{2} - 2 \frac{q_1 + q_2}{2} \frac{\dot{A}}{A} = - \frac{A}{\bar{\rho}} \frac{p_2 - p_1}{L} \quad (6.35)$$

### 6.2.2 Case of Static Pressure Gradient, Forced Motion:

Sections 6.2.2 and 6.2.3 detail the study of the transversely-oscillating piston in situations where the upstream and downstream fluid pressures are somehow maintained at constant values. (This is not possible to implement exactly in practice; a more realistic upstream boundary condition can be introduced with the addition of an upstream reservoir). The reaction of the flow beneath the piston is examined as the piston is forced up and down with some prescribed oscillatory motion.

Equations (6.34) and (6.35) which express mass and momentum conservation respectively in the aperture beneath the piston. The continuity equation is an algebraic equation which relates  $q_1$  and  $q_2$  in terms of known quantities. Consequently, the continuity equation may be differentiated with respect to time and the result used to eliminate occurrences of either  $\dot{q}_1$  or  $\dot{q}_2$  from the momentum equation.

An alternative approach, which is followed below, is to adopt the quantity  $\frac{q_1 + q_2}{2}$  as the system variable and solve the momentum equation for this new variable, and then use the continuity equation to recover  $q_1$  and  $q_2$ . The quantity  $\frac{q_1 + q_2}{2}$  represents the space-averaged volume velocity in the aperture region; it is denoted henceforth by  $y(t)$  and its time-derivative is written as  $\dot{y}(t)$ .

The momentum equation then simplifies to:

$$\dot{y} - 2y \frac{\dot{A}}{A} = \frac{A}{\bar{\rho}} \frac{p_1 - p_2}{L}$$

This equation can be simplified further following recognition that

$$\dot{y} - 2y \frac{\dot{A}}{A} = A^2 \frac{d}{dt} \left( \frac{y}{A^2} \right)$$

Substitution gives:

$$\begin{aligned} A^2 \frac{d}{dt} \left( \frac{y}{A^2} \right) &= \frac{A}{\bar{\rho}} \frac{p_1 - p_2}{L} \\ \Rightarrow \frac{d}{dt} \left( \frac{y}{A^2} \right) &= \frac{1}{\bar{\rho}A} \frac{p_1 - p_2}{L} \end{aligned}$$

This momentum equation can be solved by integration. Recalling that upstream and downstream pressures are regarded as constants for the present investigation, a general solution can be written for  $y(t)$ :

$$\begin{aligned} \frac{y}{A^2} &= \frac{1}{\bar{\rho}} \frac{p_1 - p_2}{L} \int \frac{dt}{A} \\ \Rightarrow y(t) &= \frac{1}{\bar{\rho}} \frac{p_1 - p_2}{L} A^2(t) \int \frac{dt}{A(t)} \end{aligned} \quad (6.36)$$

Equation (6.36) describes the time-variation of the spatially-averaged volume velocity through the aperture when the aperture area varies as  $A(t)$ . The value of  $y(t)$  at the time instant  $t = 0$  fixes the constant of integration in the above equation. Equation (6.34) may be used to recover  $q_1$  and  $q_2$ ; recalling that  $y(t)$  is defined as  $\frac{q_1 + q_2}{2}$ , the equation of continuity leads to:

$$\begin{aligned} q_1 &= y + \frac{1}{2} L \dot{A} \\ q_2 &= y - \frac{1}{2} L \dot{A} \end{aligned} \quad (6.37)$$

#### Special Case of Piston Held Fixed:

When the piston is held fixed,  $\dot{A} = 0$  and the integration in Equation (6.36) is straightforward:

$$\begin{aligned} y(t) &= \frac{1}{\bar{\rho}} \frac{p_1 - p_2}{L} A^2 \int \frac{dt}{A} \\ &= \frac{1}{\bar{\rho}} \frac{p_1 - p_2}{L} A \int dt \\ &= \frac{A}{\bar{\rho}} \frac{p_1 - p_2}{L} t + \text{const.} \end{aligned}$$

Alternatively, this equation can be expressed in terms of the spatially averaged particle velocity in the aperture, here denoted by  $u = \frac{y}{A}$ :

$$\begin{aligned} u &= \frac{1}{\bar{\rho}} \frac{p_1 - p_2}{L} t + \text{const.} \\ &= \bar{k} t + \text{const.} \quad \text{where } \bar{k} = \frac{1}{\bar{\rho}} \frac{p_1 - p_2}{L} \end{aligned} \quad (6.38)$$

This result indicates a velocity in the aperture which increases linearly with time. The growth rate depends upon the pressure difference across the aperture as well as its longitudinal dimension  $L$  and cross-sectional area  $A$ .

It is interesting to note the influence of compressibility upon the solution for the special case of a static piston. When the initial condition  $u(0) = 0$  is specified, Equation (6.17) describing the solution for compressible flow is

$$u = -c_0 \frac{1 - \exp(2c_0 kt)}{1 + \exp(2c_0 kt)}$$

Equation (6.38) describing the solution for a fluid of constant density is

$$u = \bar{k} t$$

While the compressible flow solution indicates a spatially-averaged flow velocity which increases asymptotically to a value  $u = c_0$ , the incompressible flow solution increases without bound as time continues.

The fact that the volume velocity increases without bound does not signal an error in the model equations or the calculations to this point. Rather, it serves as a reminder that the model equations are built upon a set of simplifying assumptions which collectively define a 'domain of applicability' for the resulting model; the lack of an upper bound indicates that the volume velocity grows out of the domain of applicability of the above equation within a finite time.

#### Flow Reaction to Sinusoidal Piston Motion:

This subsection concludes with an investigation into the response of the fluid to sinusoidal oscillation of the piston about some central position. The area of the aperture beneath the piston also varies sinusoidally. Three parameters may be chosen to completely specify the piston motion, as illustrated below in Figure 6.19. These are:

- the aperture cross-sectional area ( $A_0$ ) at the central position of the piston,
- the ratio of the area oscillation amplitude to the area at the midpoint ( $\alpha$ ), and
- the angular frequency ( $\omega$ ) of piston oscillation.

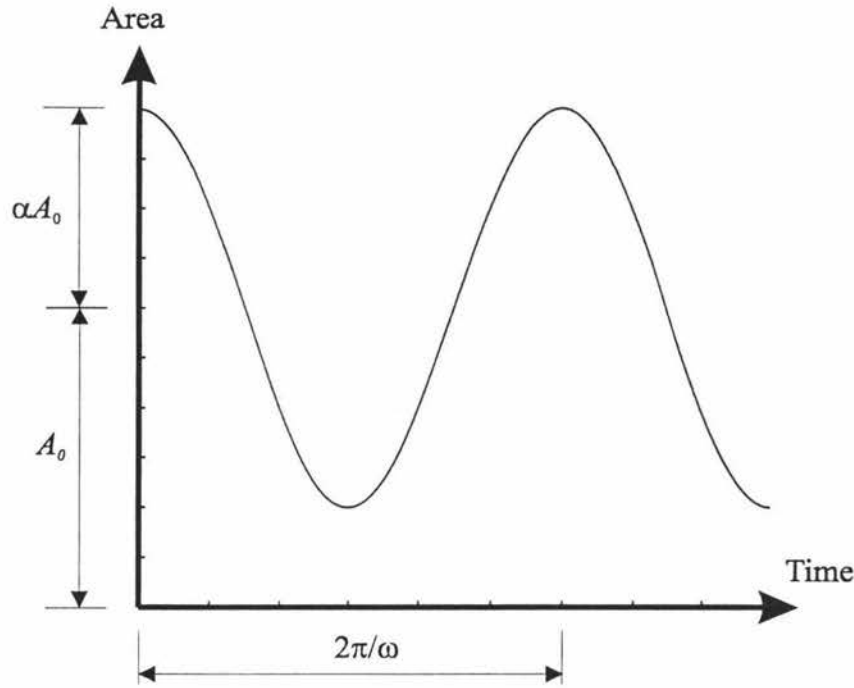


Figure 6.19 Sinusoidal Aperture Area Variation showing Parameters

The aperture cross-sectional area varies according to:

$$A(t) = A_0(1 + \alpha \cos(\omega t)) \quad (6.39)$$

This expression has physical meaning when  $A_0 \geq 0$  and  $|\alpha| < 1$ .

Recall Equation (6.36) (repeated below) which describes the flow response to arbitrary variation of the area of the aperture beneath the piston.

$$y = \frac{1}{\bar{\rho}} \frac{p_1 - p_2}{L} A^2(t) \int \frac{dt}{A(t)}$$

Equation (6.39) can be substituted into this equation to reveal the reaction of the volume velocity to sinusoidal area variations.

First consider the integral to be evaluated. Substituting for  $A$  gives

$$\int \frac{dt}{A(t)} = \int \frac{dt}{A_0(1 + \alpha \cos(\omega t))}$$

It is possible to replace  $\alpha$  by a new parameter  $\theta$  defined by  $\alpha = \cos(\theta)$ . This substitution does not restrict the choice of  $\alpha$  since  $-1 < \alpha < 1$  already. Geometrically, the new parameter may be interpreted as shown below in Figure 6.20:

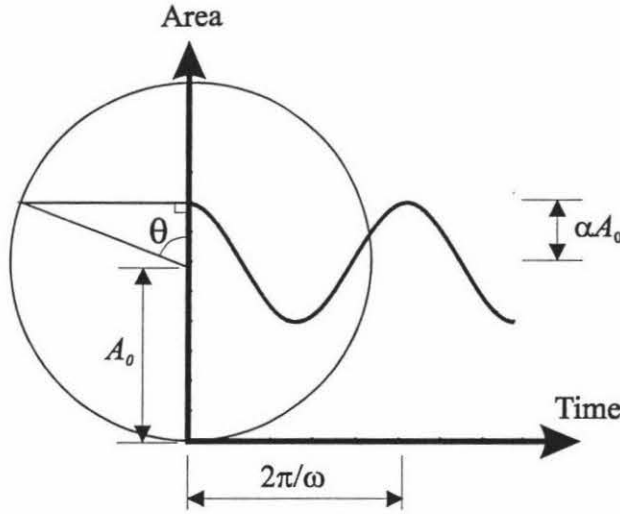


Figure 6.20 Physical Interpretation of the Parameter  $\theta$

The resulting integral appears as an entry in a standard Table of Integrals.

$$\begin{aligned} \int \frac{dt}{A(t)} &= \frac{1}{A_0} \int \frac{dt}{1 + \cos(\theta) \cos(\omega t)} \\ &= \frac{1}{\omega A_0} 2 \csc(\theta) \tan^{-1} \left[ \tan\left(\frac{\theta}{2}\right) \tan\left(\frac{\omega t}{2}\right) \right] + const. \end{aligned} \quad (6.40)$$

Figure 6.21 to follow shows geometrically how the factor  $\frac{2}{\omega} \tan^{-1} \left[ \tan\left(\frac{\theta}{2}\right) \tan\left(\frac{\omega t}{2}\right) \right]$  in

Equation (6.40) performs a non-linear scaling operation upon the time ordinate. Although the embedded tangent function has singularities at every odd multiple of  $\pi/\omega$ , the inverse tangent operation removes these; odd and even multiples of  $\pi/\omega$  are all invariants of the transformation.

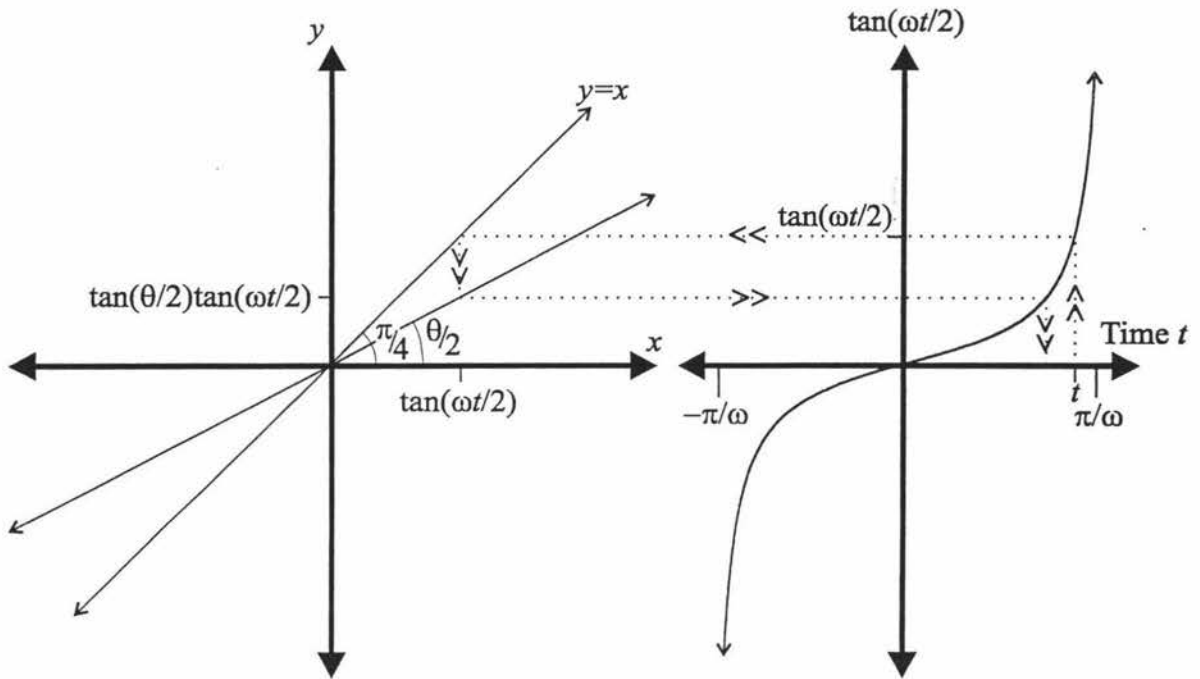


Figure 6.21 Non-Linear Scaling Operation of  $\tan^{-1}(\tan(\theta/2)\tan(\omega t/2))$

Figure 6.21 above also illustrates the effect of changes of the parameter  $\theta$ . The non-linearity of the time-scaling operation is more severe as the value of the parameter  $\theta$  moves further from  $\pi/2$ . When  $\theta = \pi/2$  the transformation is a linear one.

Recall that  $\theta = \cos^{-1}(\alpha)$ . The inverse cosine operation is multi-valued but  $\theta$  may be restricted to lie within the range  $0 \leq \theta \leq \pi$  without loss of generality, as shown below in Figure 6.22 below. The value of  $\theta = \pi/2$  is seen to correspond to  $\alpha = 0$ .

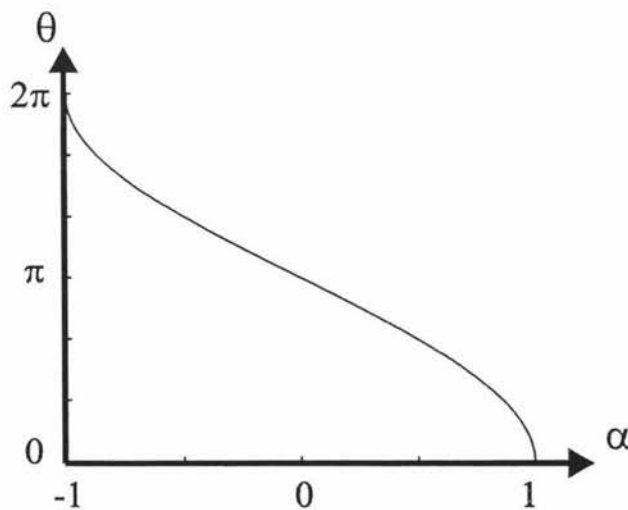


Figure 6.22 The Principal Value for  $\cos^{-1}(\alpha)$  is Chosen to Define  $\theta$

Since the inverse tangent operation upon an argument is multi-valued, it is important that consideration be given to which of the possible values is appropriate. (The MATLAB inverse tangent function `atan` returns only the principal value.) The correct multiple of  $2\pi/\omega$  to be added to the principal value is that which generates a function that increases monotonically and is continuous, since the inverse tangent operation arises within the solution of  $\int \frac{dt}{A(t)}$  and  $A > 0, \forall t$ .

Figure 6.23 below compares the resulting transformations for cases of  $\alpha = 0$  and  $\alpha = \frac{1}{2}$ . As  $\alpha$  increases further, the function looks less like a straight line and more like an ascending staircase. For any value of  $\alpha$ , the scaled value of  $t$  is always bounded above by the smallest integer multiple of  $2\pi/\omega$  which is greater than  $t$ , and bounded below by the largest integer multiple of  $2\pi/\omega$  which is smaller than  $t$ .

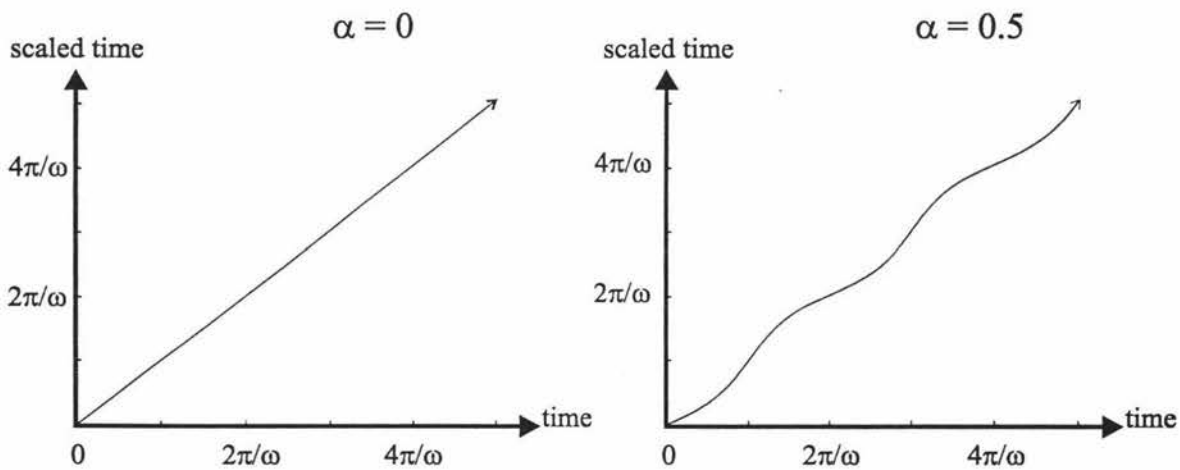


Figure 6.23 Time-Warp Effect for Two Values of  $\alpha$

Figure 6.23 illustrates the time-scaling effect of the factor  $\frac{2}{\omega} \tan^{-1} \left[ \tan\left(\frac{\theta}{2}\right) \tan\left(\frac{\omega t}{2}\right) \right]$  within the solution of  $\int \frac{dt}{A(t)}$ . Recall that Equation (6.40) shows a further (linear) scaling by the amount  $\frac{\csc(\theta)}{A_0}$  to complete the solution. The complete solution of Equation (6.36) may be developed as follows:

$$\begin{aligned}
y(t) &= \frac{1}{\bar{\rho}} \frac{p_1 - p_2}{L} A^2(t) \int \frac{dt}{A(t)} \\
&= \frac{1}{\bar{\rho}} \frac{p_1 - p_2}{L} A^2(t) \frac{\csc(\theta)}{A_0} \left\{ \frac{2}{\omega} \tan^{-1} \left[ \tan\left(\frac{\theta}{2}\right) \tan\left(\frac{\omega t}{2}\right) \right] + \text{const.} \right\} \quad (\text{from 6.40}) \\
&= \frac{1}{\bar{\rho}} \frac{p_1 - p_2}{L} A_0^2 [1 + \alpha \cos(\omega t)]^2 \frac{\csc(\theta)}{A_0} \left\{ \frac{2}{\omega} \tan^{-1} \left[ \tan\left(\frac{\theta}{2}\right) \tan\left(\frac{\omega t}{2}\right) \right] + \text{const.} \right\} \\
&= \frac{\csc(\theta) A_0}{\bar{\rho}} \frac{p_1 - p_2}{L} [1 + \cos(\theta) \cos(\omega t)]^2 \left\{ \frac{2}{\omega} \tan^{-1} \left[ \tan\left(\frac{\theta}{2}\right) \tan\left(\frac{\omega t}{2}\right) \right] + \text{const.} \right\} \quad (6.41)
\end{aligned}$$

Alternatively, this equation can be expressed in terms of the spatially averaged particle velocity  $u$ ; recall that  $y = uA = uA_0[1 + \cos\theta \cos(\omega t)]$ . Equation (6.41) becomes

$$\begin{aligned}
u &= \bar{k} \csc(\theta) [1 + \cos(\theta) \cos(\omega t)] \left\{ \frac{2}{\omega} \tan^{-1} \left[ \tan\left(\frac{\theta}{2}\right) \tan\left(\frac{\omega t}{2}\right) \right] + \text{const.} \right\} \\
&\quad \text{where } \bar{k} = \frac{p_1 - p_2}{\bar{\rho} L} \quad (6.42)
\end{aligned}$$

This may be compared with the solution for the special case of a fixed piston position (Equation 6.38, repeated below):

$$u = \bar{k} t + \text{const.}$$

The differences are:

- The aperture area  $A$  is replaced by  $\frac{A^2}{A_0}$  which is equal to  $A[1 + \cos(\theta) \cos(\omega t)]$

Note that  $1 + \cos(\theta) \cos(\omega t)$  oscillates between  $1 + \cos(\theta)$  and  $1 - \cos(\theta)$  with angular frequency  $\omega$ .

- Time  $t$  is transformed non-linearly, by  $t \rightarrow \frac{2}{\omega} \tan^{-1} \left[ \tan\left(\frac{\theta}{2}\right) \tan\left(\frac{\omega t}{2}\right) \right]$

Note that this transformation is also periodic with angular frequency  $\omega$ .

- The solution is scaled by the factor  $\csc(\theta)$ .

It is possible to recover Equation (6.38) from Equation (6.42) by setting  $\alpha = 0$ , i.e.  $\theta = \frac{\pi}{2}$  as follows.

$$u = \bar{k} \csc\left(\frac{\pi}{2}\right) \left[ 1 + \cos\left(\frac{\pi}{2}\right) \cos(\omega t) \right] \left\{ \frac{2}{\omega} \tan^{-1} \left[ \tan\left(\frac{\pi}{2}\right) \tan\left(\frac{\omega t}{2}\right) \right] + \text{const.} \right\}$$

$$= \bar{k} t + \text{const.}$$

A sample solution of Equation (6.42) is shown in the following figure:

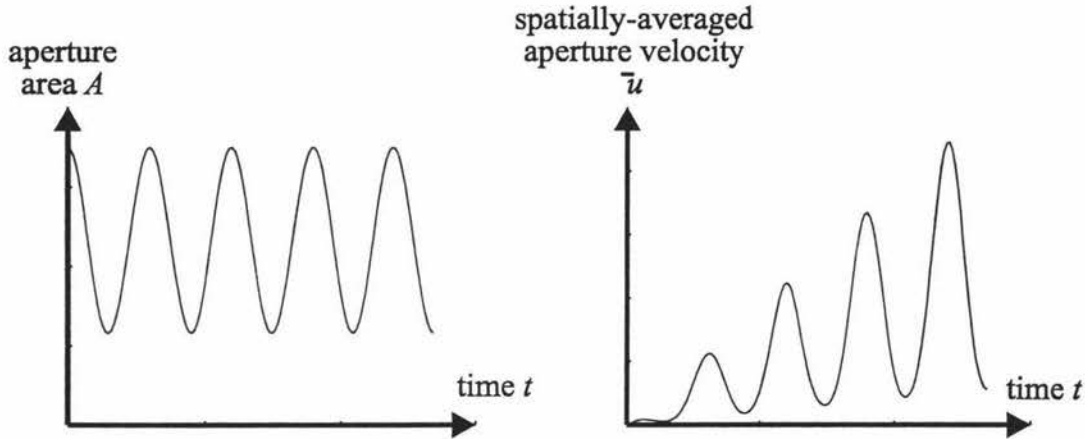


Figure 6.24 Sample Velocity Fluctuation - Zero Initial Value

### 6.2.3 Case of Static Pressure Gradient, Self-Determined Motion:

In this subsection the piston motion is not specified *a priori* as it was in Section 6.2.2. Instead the properties of the fluid are used to determine the fluid force supplied by the flow as it passes beneath the piston. An equation of motion is derived below for the motion of the piston and combined with equations derived in Section 6.2.1 for the motion of the fluid within the aperture. This produces a system of equations which describes the interaction between the fluid motion and the piston motion.

Of fundamental interest is whether the resulting system is capable of self-sustained oscillations when provided with suitable values for initial conditions and system parameters.

#### An Equation of Motion for the Piston:

The well-known arrangement of mass, spring and damper is used to model the response of the piston to the applied fluid force. The piston is modelled as a point mass, of fixed value  $m$ . The piston is connected by a massless spring, which has a linear restoring force, to a fixed point of reference. The *stiffness constant* of the spring is denoted  $s$ .

Any motion, in reality, is accompanied by unrecoverable dissipation of energy. The amount of energy lost by the mass-spring-damper oscillator at a time  $t$  is considered to vary in linear proportion to the instantaneous velocity of the piston at that time; the constant of proportionality, called the *damping constant*, is denoted  $b$ .

The equation of motion for this mass-spring-damper scenario can be written down directly: the displacement  $h(t)$  of the piston while under the influence of a force  $F(t)$  is given by

$$m\ddot{h} + b\dot{h} + sh = F \quad (6.43)$$

Equation (6.43) describes the displacement of the piston from its equilibrium position. By a simple change of variables it is possible to instead use the bottom edge of the duct as a datum, as shown below in Figure 6.24:

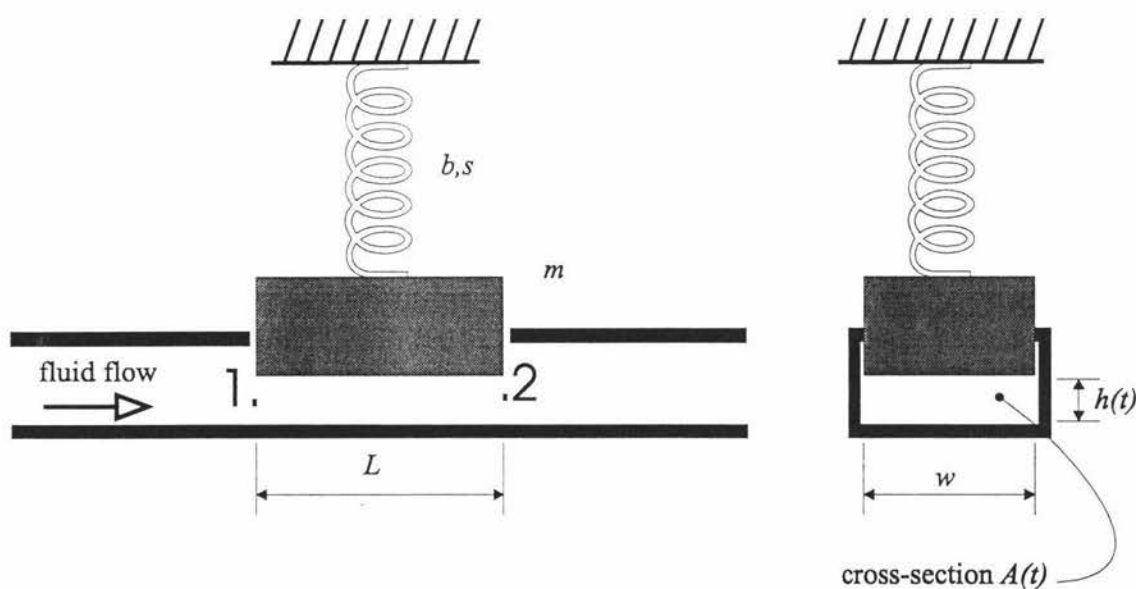


Figure 6.23 Piston-in-Pipe Arrangement Showing Height  $h$

In the following equation,  $h_0$  denotes the equilibrium position of the unforced piston, measured from the bottom edge of the duct.

$$\begin{aligned} m\ddot{h} + b\dot{h} + s(h - h_0) &= F \\ \Rightarrow m\ddot{h} + b\dot{h} + sh &= F + sh_0 \end{aligned}$$

A positive value for  $h_0$  means that the piston rests a distance  $h_0$  above the duct bottom when in an unforced state. A negative value for  $h_0$  is used if the piston is already partially compressed when resting on the bottom of the duct. In this case the term  $sh_0$  in

the above equation may be thought of as a *pre-tension* force; the spring must be compressed even further to raise the piston above the bottom of the duct (c.f. Figure 1.15).

The fluid force acting upon the piston may be found by integrating the pressure distribution along its underside:

$$F = \int_{\substack{\text{piston} \\ \text{underside}}} p dS = w \int_{x_1}^{x_2} p dx$$

The Trapezoidal Rule can be used to approximate this integral to second order in  $L$  (Figure 6.6 of Section 6.1.3) giving  $F \cong w \frac{p_1 + p_2}{2} L$ , so that:

$$m\ddot{h} + b\dot{h} + sh = w \frac{p_1 + p_2}{2} L + sh_0$$

By writing  $A(t) = wh(t)$ , (refer to Figure 6.23 above) occurrences of  $h$  and its derivatives may be removed:

$$m\ddot{A} + b\dot{A} + sA = \frac{1}{2} Lw^2 (p_1 + p_2) + sw h_0 \quad (6.44)$$

This is the equation of motion for the piston.

#### Analytical Solution for the Piston Motion:

The equation of motion for the piston can be combined with the equations of continuity and momentum conservation for the fluid beneath the piston (Equations 6.34 and 6.35 respectively) to yield a three-equation system which describes the interaction between the fluid and piston motions. These equations are repeated below:

$$m\ddot{A} + b\dot{A} + sA = \frac{1}{2} Lw^2 (p_1 + p_2) + sw h_0$$

$$\frac{q_2 - q_1}{L} = -\dot{A}$$

$$\frac{\dot{q}_1 + \dot{q}_2}{2} - 2 \frac{q_1 + q_2}{2} \frac{\dot{A}}{A} = -\frac{A}{\bar{\rho}} \frac{p_2 - p_1}{L}$$

This subsection studies the fluid and piston response within an environment of constant upstream and downstream pressures. When these pressures are held constant, the first of these three equations becomes self-determinant; the solution for  $A(t)$  is not influenced in any way by either of the flow variables  $q_1$  and  $q_2$ . Consequently this equation may be studied in isolation. The solution found for  $A(t)$  can then be regarded as an input for a

forced-piston treatment of the flow response as described by the second and third equations above.

For the case of a constant applied pressure, Equation (6.44) takes the form:

$$m\ddot{A} + b\dot{A} + sA = \text{const.} \quad (6.45)$$

This equation has a classical structure which has been studied extensively in the past. A characteristic equation may be written down for the homogeneous version of this equation:

$$ma^2 + ba + s = 0$$

By the quadratic formula, the roots of the characteristic equation are

$$a = \frac{-b \pm \sqrt{b^2 - 4ms}}{2m}$$

Consequently the solution of the homogeneous version of Equation (6.45) has the form

$$\begin{aligned} A(t) &= \eta_1 \exp\left[\frac{-b - \sqrt{b^2 - 4ms}}{2m} t\right] + \eta_2 \exp\left[\frac{-b + \sqrt{b^2 - 4ms}}{2m} t\right] \\ &= \left[ \eta_1 \exp\left(\frac{-\sqrt{b^2 - 4ms}}{2m} t\right) + \eta_2 \exp\left(\frac{\sqrt{b^2 - 4ms}}{2m} t\right) \right] \exp\left(-\frac{b}{2m} t\right) \end{aligned}$$

When the discriminant of the characteristic equation is non-negative the solution is an exponential decay. For a discriminant less than zero the solution is a damped harmonic motion, since  $\exp(i\phi) = \cos(\phi) + i \sin(\phi)$ . In such cases the solution may be rewritten as:

$$\begin{aligned} A(t) &= \left[ \eta_3 \cos\left(\frac{\sqrt{4ms - b^2}}{2m} t\right) + \eta_4 \sin\left(\frac{\sqrt{4ms - b^2}}{2m} t\right) \right] \exp\left(-\frac{b}{2m} t\right) \\ &\quad \text{where } \eta_3 = \eta_1 + \eta_2 \text{ and } \eta_4 = i(\eta_2 - \eta_1) \end{aligned}$$

A particular solution for Equation (6.45) is  $A = \frac{\text{const.}}{s}$  and comparison between Equations (6.44) and (6.45) shows that  $\text{const.} = \frac{1}{2} Lw^2(p_1 + p_2) + sw h_0$ . The complete solution of Equation (6.44) is:

$$A(t) = wh_0 + \frac{Lw^2}{2s}(p_1 + p_2) + \left[ \eta_1 \exp\left(\frac{-\sqrt{b^2 - 4ms}}{2m} t\right) + \eta_2 \exp\left(\frac{\sqrt{b^2 - 4ms}}{2m} t\right) \right] \exp\left(-\frac{b}{2m} t\right)$$

This equation represents either an exponential decay to, or a damped harmonic oscillation around, the constant value  $wh_0 + \frac{Lw^2}{2s}(p_1 + p_2)$  as shown in Figure 6.26 below:

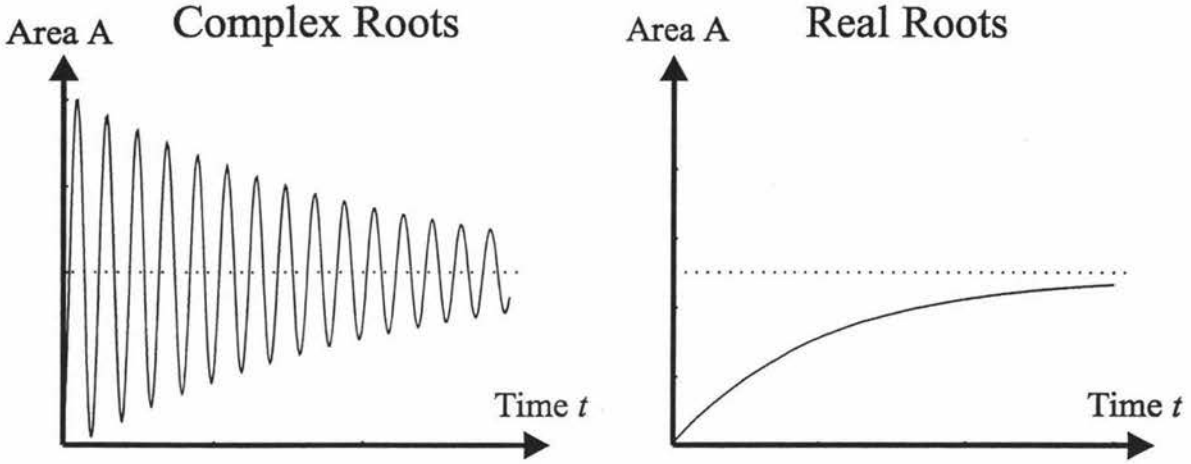


Figure 6.26: Dependence of Solution on Roots of Characteristic Equation

If the discriminant takes negative value, then the complete solution of Equation (6.44) may be rewritten as:

$$A(t) = wh_0 + \frac{Lw^2}{2s}(p_1 + p_2) + \left[ \eta_3 \cos\left(\frac{\sqrt{4ms - b^2}}{2m} t\right) + \eta_4 \sin\left(\frac{\sqrt{4ms - b^2}}{2m} t\right) \right] \exp\left(-\frac{b}{2m} t\right) \quad (6.46)$$

The values of the parameters  $\eta_3$  and  $\eta_4$  depend upon the initial values of the aperture area  $A(t)$  and its time-rate of change  $\dot{A}(t)$ . Substituting  $t=0$  in Equation (6.46) and its derivative yields:

$$\begin{aligned} A(t) &= wh_0 + \frac{Lw^2}{2s}(p_1 + p_2) + \left[ \eta_3 \cos\left(\frac{\sqrt{4ms - b^2}}{2m} t\right) + \eta_4 \sin\left(\frac{\sqrt{4ms - b^2}}{2m} t\right) \right] \exp\left(-\frac{b}{2m} t\right) \\ \Rightarrow A(0) &= wh_0 + \frac{Lw^2}{2s}(p_1 + p_2) + \eta_3 \\ \Rightarrow \eta_3 &= A(0) - wh_0 - \frac{Lw^2}{2s}(p_1 + p_2) \end{aligned}$$

$$\begin{aligned} \dot{A}(t) &= \left\{ \frac{\sqrt{4ms-b^2}}{2m} \left[ -\eta_3 \sin\left(\frac{\sqrt{4ms-b^2}}{2m}t\right) + \eta_4 \cos\left(\frac{\sqrt{4ms-b^2}}{2m}t\right) \right] \right. \\ &\quad \left. - \frac{b}{2m} \left[ \eta_3 \cos\left(\frac{\sqrt{4ms-b^2}}{2m}t\right) + \eta_4 \sin\left(\frac{\sqrt{4ms-b^2}}{2m}t\right) \right] \right\} \exp\left(-\frac{b}{2m}t\right) \\ \Rightarrow \dot{A}(0) &= \eta_4 \frac{\sqrt{4ms-b^2}}{2m} - \eta_3 \frac{b}{2m} \\ \Rightarrow \eta_4 &= -\frac{2m\dot{A}(0) + b\eta_3}{\sqrt{4ms-b^2}} \end{aligned}$$

The decay of amplitude of the motion described by Equation (6.46) is caused by the exponential term. Since the parameters  $b$  and  $m$  are necessarily positive for any real-life oscillation, this decay is unavoidable regardless of the values chosen for any other parameters, and energy is necessarily lost by the oscillation. When  $b = 0$  the oscillation is harmonic and amplitude is maintained - but in reality this damping constant, however small, is always non-zero.

#### Question of Impact:

Finally, it is important to note that the solution of Equation (6.44) is not complete without additionally specifying the condition that  $A(t) \geq 0$ , since a negative value for the aperture's cross-sectional area is not physically sensible. Certain combinations of parameter values may cause the solution of  $A(t)$  to intersect the time axis, as shown below in Figure 6.27. In such a situation, the transversely oscillating piston becomes an *impact oscillator*.

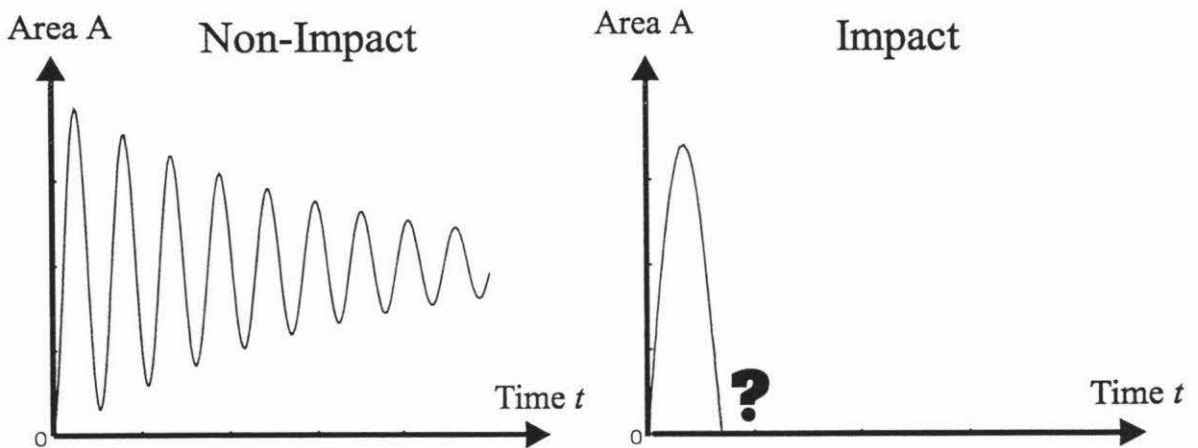


Figure 6.27 The Piston Motion is Unknown after the Time of Impact

To model this phenomenon, it is necessary to specify the effect that the collision has upon the piston motion. Possible piston behaviour as an impact oscillator is not studied here. In addition to the work required to model the impact phenomenon itself, consideration must also be given to the behaviour of the flow at those moments when the aperture under the piston is just closing and just opening.

**6.2.4 Constant Downstream Pressure, Constant Upstream Flow-Rate:**

This subsection considers a piston arrangement identical to that studied in Sections 6.2.2 and 6.2.3. However, different boundary conditions are here imposed upon the fluid passing beneath the piston. The downstream pressure is held constant as before; it is convenient, for example, to chose atmospheric pressure for  $p_2$ . The earlier condition of static upstream pressure  $p_1$  is in this section replaced by the requirement that the flow-rate into the piston region remains constant, ie. that  $\dot{q}_1 = 0$ . It is the upstream pressure  $p_1$  and downstream volume velocity  $q_2$  which here take the role of system variables.

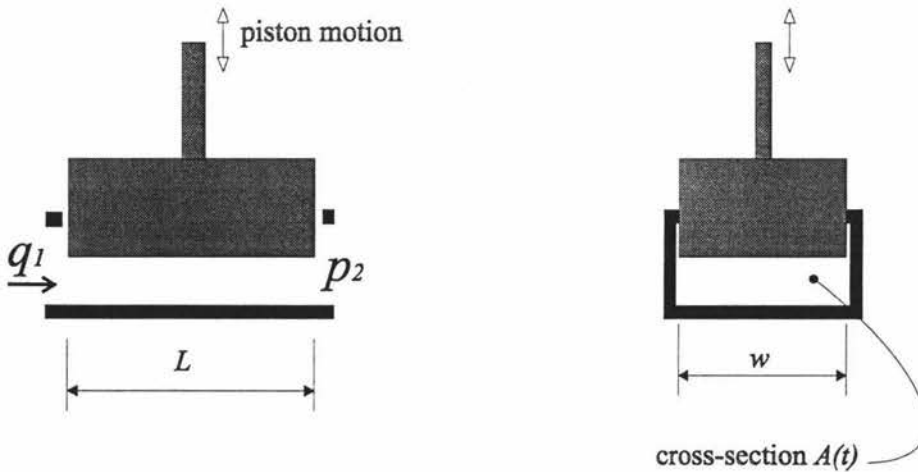


Figure 6.28: New Upstream Boundary Condition

Recall Equations 6.34 and 6.35 from Section 6.2.1 (repeated below). These express mass and momentum conservation respectively in the aperture beneath the piston:

$$\frac{q_2 - q_1}{L} = -\dot{A}$$

$$\frac{\dot{q}_1 + \dot{q}_2}{2} - 2 \frac{q_1 + q_2}{2} \frac{\dot{A}}{A} = -\frac{A}{\rho} \frac{p_2 - p_1}{L}$$

Of the two new system variables (viz.  $q_2$  and  $p_1$ ), only  $q_2$  appears in the continuity equation. Consequently it is possible to use the continuity equation to eliminate  $q_2$  from the momentum equation and thereby reduce the system to a single equation for the other system variable  $p_1$ .

The momentum equation can be rearranged to give:

$$p_1 = p_2 + \frac{\rho L}{A} \left[ \frac{\dot{q}_1 + \dot{q}_2}{2} - 2 \frac{q_1 + q_2}{2} \frac{\dot{A}}{A} \right]$$

The continuity equation is equivalent to the statement that  $q_2 = q_1 - L\dot{A}$ . Differentiating this equation with respect to time yields  $\dot{q}_2 = \dot{q}_1 - L\ddot{A}$ . Substituting these two expressions into the momentum equation leads to:

$$p_1 = p_2 + \frac{\rho L}{A} \left[ \left( \dot{q}_1 - \frac{1}{2} L\ddot{A} \right) - 2 \left( q_1 - \frac{1}{2} L\dot{A} \right) \frac{\dot{A}}{A} \right] \quad (6.47)$$

It is possible to omit the  $\dot{q}_1$  term since  $q_1$  is now considered to remain constant.

$$p_1 = p_2 - 2\rho L q_1 \frac{\dot{A}}{A^2} + \rho L^2 \frac{\dot{A}^2 - \frac{1}{2} A\ddot{A}}{A^2} \quad (6.48)$$

This equation describes the upstream pressure variations which accompany an arbitrary piston motion when conditions of a constant upstream flow-rate and constant downstream fluid pressure are imposed.

#### Case of Piston Held Fixed:

When the piston does not move, so that  $A(t) = \text{const}$  and all time derivatives of  $A$  are identically zero, then Equation (6.48) states that  $p_1 = p_2$  irrespective of the value of  $q_1$ . Further, the continuity equation indicates that  $q_1 = q_2$  when  $A$  is held fixed. It appears that, for an incompressible inviscid flow, the case of a stationary piston is analogous to an electrical short-circuit.

#### Flow Reaction to Sinusoidal Piston Motion:

The response of the fluid to sinusoidal oscillation of the piston about some central position is now considered. The three parameters chosen in Subsection 6.1.4 to specify the piston motion will again be used. Consequently, the time-variation of the aperture cross-section is again described by:

$$A(t) = A_0(1 + \alpha \cos(\omega t)) \quad A_0 \geq 0, \quad -1 < \alpha < 1$$

The first and second time-derivatives of  $A(t)$  are:

$$\begin{aligned}\dot{A}(t) &= -A_0\alpha\omega \sin(\omega t) \\ \ddot{A}(t) &= -A_0\alpha\omega^2 \cos(\omega t)\end{aligned}$$

These expressions may be substituted into Equation (6.48) to reveal the pressure response to sinusoidal aperture area variation:

$$\begin{aligned}p_1 &= p_2 - 2\rho Lq_1 \frac{\dot{A}}{A^2} + \rho L^2 \frac{\dot{A}^2 - \frac{1}{2} A\ddot{A}}{A^2} \\ &= p_2 - 2\rho Lq_1 \frac{-A_0\alpha\omega \sin(\omega t)}{[A_0(1 + \alpha \cos(\omega t))]^2} \\ &\quad + \rho L^2 \frac{[-A_0\alpha\omega \sin(\omega t)]^2 - \frac{1}{2} A_0(1 + \alpha \cos(\omega t))[-A_0\alpha\omega^2 \cos(\omega t)]}{[A_0(1 + \alpha \cos(\omega t))]^2} \\ &= p_2 + \frac{2\alpha\rho Lq_1\omega}{A_0} \frac{\sin(\omega t)}{[1 + \alpha \cos(\omega t)]^2} \\ &\quad + \frac{\alpha\rho L^2\omega^2}{2} \frac{\alpha[2\sin^2(\omega t) + \cos^2(\omega t)] + \cos(\omega t)}{[1 + \alpha \cos(\omega t)]^2} \\ &= p_2 + \frac{2\alpha\rho Lq_1\omega}{A_0} \frac{\sin(\omega t)}{[1 + \alpha \cos(\omega t)]^2} \\ &\quad + \frac{\alpha\rho L^2\omega^2}{4} \frac{3\alpha - \alpha \cos(2\omega t) + 2\cos(\omega t)}{[1 + \alpha \cos(\omega t)]^2}\end{aligned}\tag{6.49}$$

The oscillation of the third term does not rely upon the influx of new fluid (ie. it is independent of the value of  $q_1$ ). It describes the suction effect upon the fluid as the piston moves upwards, and the expulsion of fluid from space that is reclaimed during the piston's descent.

Consider now that the piston motion is required to modulate a flow, and that the amplitude of the resulting pressure fluctuations is required to increase in response to an increase of the constant upstream flow-rate  $q_1$ . Then parameters must be chosen which make the second term of Equation (6.49) dominant over the third, i.e.

$$\begin{aligned}\frac{2\alpha\rho Lq_1\omega}{A_0} &\gg \frac{(4\alpha + 2)\alpha\rho L^2\omega^2}{4} \\ \Rightarrow q_1 &\gg \frac{3}{4} LA_0\omega\end{aligned}$$

When the system parameters are chosen so that  $q_1 \gg \frac{3}{4} LA_0\omega$ , Equation (6.49) may be approximated as follows:

$$p^1 - p^2 \cong \frac{2\rho L\alpha\omega q_1}{A_0} \frac{\sin(\omega t)}{[1 + \alpha \cos(\omega t)]^2} \quad (6.50)$$

This shows how the amplitude of the excess pressure variations (for example, measured peak-to-peak, or the root-mean-square pressure) responds to changes in piston parameters  $L$ ,  $\omega$  and  $A_0$  and upstream flow-rate  $q_1$ . As long as the parameter  $\alpha$  is not varied, the shape of the fluctuations described by Equation (6.50) will remain the same.

Consider now the square of  $1 + \alpha \cos(\omega t)$  which appears in the denominator of Equation (6.50). Since this varies in value between  $1 - \alpha$  and  $1 + \alpha$  during each cycle, the parameter  $\alpha$  can have a dramatic effect upon the shape of the pressure waveform, especially if  $|\alpha| \approx 1$ . The ratio of the largest to the smallest values of excess pressure that arise during each cycle of the piston motion is:

$$ratio = \left( \frac{1 + \alpha}{1 - \alpha} \right)^2$$

For the four examples in Figure 6.29 to follow, this ratio has values of 1.5, 16, 361 and 39601.

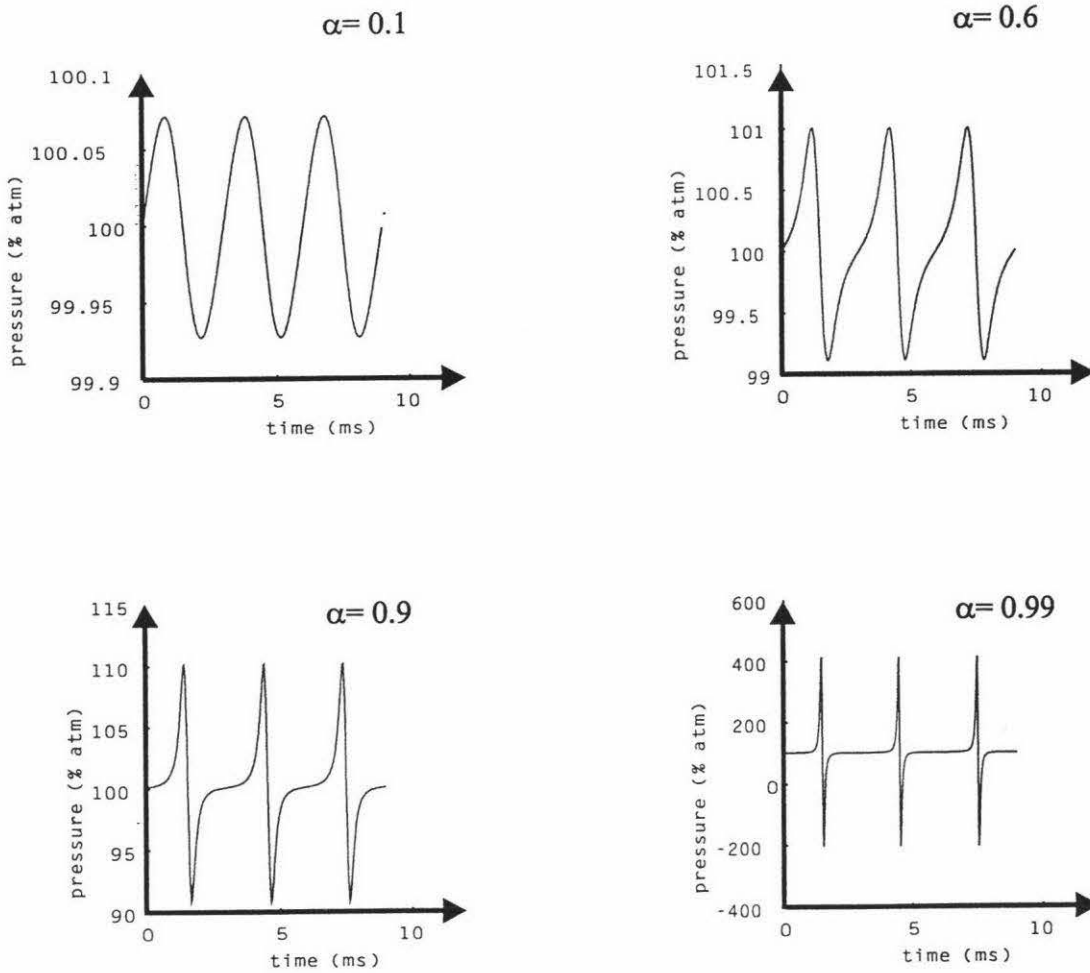


Figure 6.29: Upstream Pressure for Various  $\alpha$ , Other Parameters Fixed

The solutions expressed by Equations (6.49) and (6.50) vary considerably during each period as the ratio  $\alpha$  approaches unity. In such circumstances the validity of the model is doubtful, since when the aperture is almost completely closed off by the piston, the neglect of viscous effects is unreasonable, as is the assumption of uniform fluid properties over the flow cross-section.

When all three terms of Equation (6.49) are retained, the shape of the pressure waveform depends upon the angular frequency  $\omega$  as well as the ratio  $\alpha$ , since the second term of Equation (6.49) varies in proportion to  $\omega$  while the third term (absent from Equation 6.50 and Figure 6.29 above) varies as  $\omega^2$ . Figure 6.30 below shows the waveshape much less affected by frequency changes than for changes in  $\alpha$ .

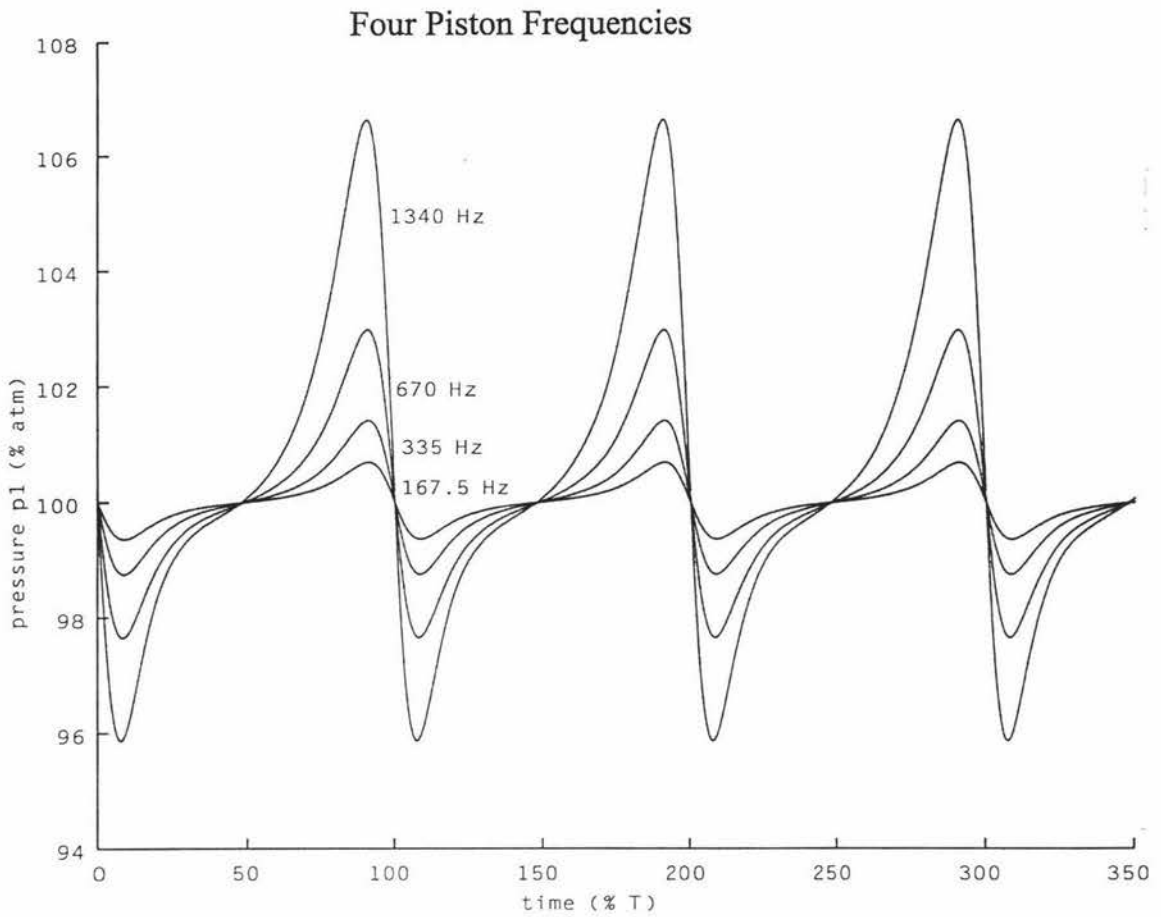


Figure 6.30: Effect Of Piston Frequency On Upstream Pressure

In Figure 6.31 to follow, the differences in waveshape are illustrated more clearly by scaling the curves in the preceding figure against one another using the maximum pressures attained, and by again non-dimensionalising time using the period of each oscillation.

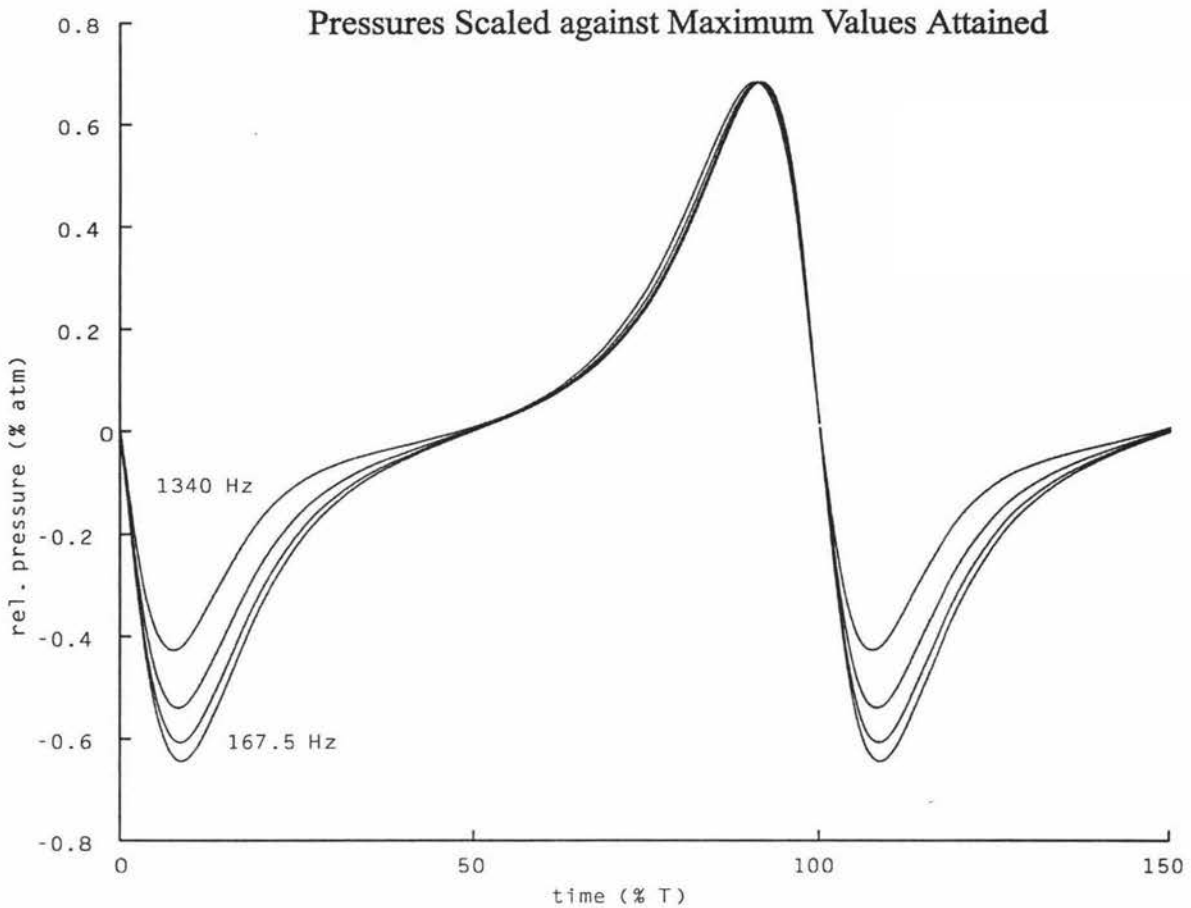


Figure 6.31: Pressures Scaled For Comparison Of Waveshapes

### 6.2.5 Potential for Self-Excited Oscillation:

This subsection describes the results of an examination into the question of whether, with appropriate choices of system parameter values and initial conditions, it is possible for the fluid forces upon the piston to cause evolution of the system response into a state of stable oscillation. Following the similar analysis in Section 6.2.3, the piston is regarded as a single degree-of-freedom mass-spring-damper oscillator. The four parameters associated with the piston motion are the piston mass,  $m$ ; a coefficient of viscous damping,  $b$ ; the stiffness constant,  $s$ , for the spring; and  $h_0$ , the height of the piston above the bottom of the duct when it is not subject to any external forces. The following equation derived earlier (Equation 6.44) is a suitable equation of motion for the piston motion:

$$m\ddot{A} + b\dot{A} + sA = \frac{1}{2}Lw^2(p_1 + p_2) + sw h_0$$

The upstream pressure may be eliminated from the above expression using Equation (6.48), which is repeated here:

$$p_1 = p_2 - 2\rho L q_1 \frac{\dot{A}}{A^2} + \rho L^2 \left[ \frac{\dot{A}^2 - \frac{1}{2} A \ddot{A}}{A^2} \right]$$

Substitution gives:

$$\begin{aligned} m\ddot{A} + b\dot{A} + sA &= \frac{1}{2} L w^2 \left\{ p_2 - 2\rho L q_1 \frac{\dot{A}}{A^2} + \rho L^2 \left[ \frac{\dot{A}^2 - \frac{1}{2} A \ddot{A}}{A^2} \right] + p_2 \right\} + s w h_0 \\ \Rightarrow [m\ddot{A} + b\dot{A} + sA] A^2 &= L w^2 \left[ p_2 A^2 - \rho L q_1 \dot{A} + \frac{1}{2} \rho L^2 (\dot{A}^2 - \frac{1}{2} A \ddot{A}) \right] + s w h_0 A^2 \\ \Rightarrow \ddot{A} + \frac{bA^2 - \frac{1}{2} \rho L^3 w^2 \dot{A} + \rho L^2 w^2 q_1}{mA^2 + \frac{1}{4} \rho L^2 w^2 A} \dot{A} + \frac{s}{mA^2 + \frac{1}{4} \rho L^2 w^2 A} A &= \frac{(L w^2 p_2 + s w h_0) A^2}{mA^2 + \frac{1}{4} \rho L^2 w^2 A} \end{aligned} \quad (6.51)$$

Now the behaviour described by this equation will be considered under the condition that

$A \gg \frac{1}{4} \frac{\rho w^2 L^3}{m}$ , so that  $mA^2 + \frac{1}{4} \rho L^2 w^2 A \cong mA^2$  is a valid approximation. This leads to

simplification of the denominator of the three fractions in Equation (6.51). Since  $A = wh$ , this is equivalent to a condition upon the height  $h$  of the piston above the datum,

viz.  $h \gg \frac{1}{4} \frac{\rho w L^3}{m}$  (e.g. for atmospheric air and piston geometry of  $w = 10$  mm and

$L = 5$  mm a piston mass of one gram requires that  $h \gg 32$  nm.) In such circumstances, Equation (6.51) can be approximated by

$$\ddot{A} + \left( \frac{b}{m} - \frac{\frac{1}{2} \rho L^3 w^2 \dot{A} - \rho L^2 w^2 q_1}{mA^2} \right) \dot{A} + \frac{s}{m} A = \frac{L w^2 p_2 + s w h_0}{m} \quad (6.52)$$

### Criteria for Self-Excited Oscillation of a Mass-Spring-Damper System:

A *self-excited oscillation* is a special type of forced oscillation. The distinguishing feature of a self-excited oscillator is an ability to draw from an energy reservoir which has a constant level of energy available throughout time. It is the motion of the oscillator itself which co-ordinates the timing and amount of energy transfer from the reservoir to the oscillation. It follows that the forcing function of a self-excited oscillator does not depend upon time explicitly, but instead indirectly through dependence upon other variables of the oscillation.

Not all forcing functions have the ability to produce self-excited oscillations. To determine some criteria for predicting stable oscillation, consider a simple mass-spring-damper system under the influence of a general forcing function whose value depends in some way upon the variable of the oscillation and its time derivatives.

$$m\ddot{h} + b\dot{h} + sh = f(h, \dot{h}, \ddot{h}) \quad (6.53)$$

The energy of the oscillation at any moment in time is composed of both the kinetic energy of the travelling mass and the potential energy stored by the compressed or stretched spring. The total system energy changes at a rate given by

$$\begin{aligned} \frac{d}{dt}(E_k + E_p) &= \frac{d}{dt}\left(\frac{1}{2}m\dot{h}^2\right) + \frac{d}{dt}\left(\frac{1}{2}sh^2\right) \\ &= m\dot{h}\ddot{h} + sh\dot{h} \\ &= \dot{h}(m\ddot{h} + sh) \end{aligned}$$

Combining this with Equation (6.53) reveals that the total oscillation energy varies according to  $\dot{E} = \dot{h}(f - b\dot{h})$ . In one complete cycle the energy of oscillation increases by an amount

$$\Delta E_{\text{cycle}} = \int_{\text{cycle}} \dot{h}(f - b\dot{h}) dt \quad (6.54)$$

If the value of the integral is negative, then the system energy decreases during a cycle; if it is positive, then the oscillation will gain energy during that cycle. For the amplitude of the oscillation to be maintained steady, the value of the integral must be zero.

Equation (6.54) reveals the condition under which the energy provided by the negative damping force is sufficient to balance energy lost through (positive) viscous damping. The simplest form of the function  $f$  that will maintain a steady amplitude of oscillation is the function  $f = b\dot{h}$ . The resulting motion (from Equation 6.53) is purely sinusoidal, being the solution to  $m\ddot{h} + sh = 0$ .

The integrand of Equation (6.54) is equal to the difference between the rate at which energy is contributed by the driving force, equal to  $\dot{h}f$ , and the rate at which energy is lost through damping, equal to  $b\dot{h}^2$ . The latter is necessarily positive; the loss of some energy through damping is inevitable for any real-life oscillation. The term  $\dot{h}f$ , however, is positive only when  $\dot{h}$  and  $f$  are both positive or both negative. When only one is positive, and the other negative, the product  $\dot{h}f$  is negative, and the driving force is then working against the oscillation, taking energy from it. It can be said that the forcing function contributes energy at those times when it is *in phase* with the velocity.

It is often convenient to write the forcing function as the product of the velocity and some other function which, like  $f$ , depends upon the system variables, e.g.

$$f(h, \dot{h}, \ddot{h}) = g(h, \dot{h}, \ddot{h})\dot{h}$$

The forcing function will then be in phase with the velocity whenever the function  $g$  is positive. At these times the forcing function contributes energy to the oscillation. Substituting this equation into Equation (6.53) leads to

$$m\ddot{h} + [b - g(h, \dot{h}, \ddot{h})]\dot{h} + sh = 0 \quad (6.55)$$

Such a self-exciting force is known as a *negative damping* force.

#### Bias Force:

Although an appropriate negative damping force is sufficient to ensure self-excited oscillations in a simple mass-spring-damper system, there is an added complication in the case of the piston-in-tube oscillation due to the motional constraint that  $h \geq 0$ . Additional to the negative resistance term of the forcing function, there must be some force available to shift the midpoint of the oscillation so that impact may be avoided.

Consider that the piston motion is decomposed into DC and AC contributions, and let the midpoint of a certain stable self-excited oscillation be denoted  $h_{DC}$ . Let the position of the piston when under the influence of no external forces (and no motional constraints), be denoted  $h_0$ . The applied force which shifted the time-average of the piston displacement from the position  $h = h_0$  to the position  $h = h_{DC}$  must work against the restoring force of the spring to do so. Hence the applied force itself must also have a DC component (i.e. its time-averaged value is not zero). The force which moves the oscillation midpoint in this manner acts as a *bias force*. A nonzero bias force is required to produce any oscillation for which  $h_{DC} \neq h_0$ .

Self-excited oscillation of the piston-in-duct oscillator (without impact) will only occur if the forcing function for the piston provides a suitable bias force for the oscillation, as well as a negative damping force.

#### Potential for Self-Excited Piston Motion:

Recall Equation (6.52), repeated below, which gives a good approximation to the motion

$$\text{when } h \gg \frac{1}{4} \frac{\rho w L^3}{m}.$$

$$\ddot{A} + \left( \frac{b}{m} - \frac{\frac{1}{2}\rho L^3 w^2 \dot{A} - \rho L^2 w^2 q_1}{mA^2} \right) \dot{A} + \frac{s}{m} A = \frac{Lw^2 p_2 + sw h_0}{m}$$

Comparing this to Equation (6.55) leads to

$$\begin{aligned} g &= \frac{\frac{1}{2}\rho L^3 w^2 \dot{A} - \rho L^2 w^2 q_1}{A^2} \\ &= \frac{\rho L^2 w^2}{2A^2} (L\dot{A} - 2q_1) \end{aligned}$$

The function  $g(t)$  produces negative damping whenever it has a positive value, i.e. whenever  $L\dot{A} > 2q_1$ . If it is specified that  $q_1 > 0$  (i.e. that the flow never reverses at the upstream edge of the aperture region) then negative damping can only happen on an upstroke of the piston, i.e. when  $\dot{A} > 0$ . Furthermore, the condition  $L\dot{A} > 2q_1$  is fulfilled only when the piston velocity is so large that the rate of volume released by the upwards motion of the piston exceeds double the rate of volume flow into the aperture region from upstream.

Equation 6.34, derived in Section 6.2.1, expresses mass conservation in the aperture beneath the piston. This equation can be written in the form  $q_2 + L\dot{A} = q_1$ . It can be seen that the condition  $L\dot{A} > q_1$  is equivalent to a condition that  $q_1 + q_2 < 0$ . In other words, the piston motion only receives energy from the flow at times when the piston is rising so quickly that there is fluid being drawn back into the aperture region from its downstream edge, and further that the rate of fluid entering the aperture from the downstream side exceeds the rate entering from the upstream side.

A bias force is seen to be in operation in this system. The right-hand side of Equation (6.52) implies that the piston is subject to a constant force which moves the midpoint of the piston oscillation (irrespective of whether such an oscillation is self-sustained or damped). The time averaged cross-sectional area of the aperture over each cycle of an underdamped or undamped piston motion, or the asymptotic value of cross-sectional aperture area for critically or overdamped motion, is given by

$$A_{DC} = \frac{Lw^2 p_2 + sw h_0}{m}$$

### Addition of an Upstream Reservoir:

The above analysis (for piston motion under the constraint of constant flow-rate at the upstream edge of the aperture) can be extended quite simply to cater for arbitrary upstream conditions. The same equation of motion (Equation 6.44, repeated below) can be used again to describe the conservation of momentum of the piston as it oscillates.

$$m\ddot{A} + b\dot{A} + sA = \frac{1}{2}Lw^2(p_1 + p_2) + sw h_0$$

The appropriate flow equation is Equation (6.47) derived earlier in Section 6.4.2 and repeated below:

$$p_1 = p_2 + \frac{\rho L}{A} \left[ \left( \dot{q}_1 - \frac{1}{2}L\ddot{A} \right) - 2 \left( q_1 - \frac{1}{2}L\dot{A} \right) \frac{\dot{A}}{A} \right]$$

Substitution leads to the following equation which is very similar in form to Equation (6.51):

$$\ddot{A} + \frac{bA^2 - \frac{1}{2}\rho L^3 w^2 \dot{A} + \rho L^2 w^2 q_1}{mA^2 + \frac{1}{4}\rho L^2 w^2 A} \dot{A} + \frac{sA^2 - \frac{1}{2}\rho L^2 w^2 \dot{q}_1}{mA^2 + \frac{1}{4}\rho L^2 w^2 A} A = \frac{(Lw^2 p_2 + sw h_0)A^2}{mA^2 + \frac{1}{4}\rho L^2 w^2 A}$$

The behaviour described by this equation will be considered under the condition that  $A \gg \frac{1}{4} \frac{\rho w^2 L^3}{m}$ , so that  $mA^2 + \frac{1}{4}\rho L^2 w^2 A \cong mA^2$  is a valid approximation. In such circumstances, the above equation can be approximated by

$$\ddot{A} + \left( \frac{b}{m} - \frac{\frac{1}{2}\rho L^3 w^2 \dot{A} - \rho L^2 w^2 q_1}{mA^2} \right) \dot{A} + \left( \frac{s}{m} - \frac{\frac{1}{2}\rho L^2 w^2 \dot{q}_1}{mA^2} \right) A = \frac{Lw^2 p_2 + sw h_0}{m} \quad (6.56)$$

This equation can be compared to Equation (6.52), repeated below, which is valid for the case of constant upstream flow-rate  $q_1$

$$\ddot{A} + \left( \frac{b}{m} - \frac{\frac{1}{2}\rho L^3 w^2 \dot{A} - \rho L^2 w^2 q_1}{mA^2} \right) \dot{A} + \frac{s}{m} A = \frac{Lw^2 p_2 + sw h_0}{m}$$

The difference between Equations (6.52) and (6.56) shows that the effect of the varying upstream flow-rate is to modify the restoring force experienced by the piston. The restoring force is no longer dependent only upon the properties of the mass-spring-damper parameters, but it is now dynamic, and affected by any fluid oscillations which may be occurring upstream of the aperture region.

It is possible to rewrite Equation (6.56) as follows, to give an equation of the same form as Equation (6.53):

$$\begin{aligned}
 m\ddot{A} + b\dot{A} + sA &= Lw^2 p_2 + sw h_0 + \left( \frac{\frac{1}{2}\rho L^3 w^2 \dot{A} - \rho L^2 w^2 q_1}{A^2} \right) \dot{A} + \frac{\frac{1}{2}\rho L^2 w^2 \dot{q}_1}{A^2} A \\
 &= Lw^2 p_2 + sw h_0 + \frac{1}{2}\rho L^2 w^2 \left[ \frac{L\dot{A}^2 - 2q_1\dot{A} + \dot{q}_1 A}{A^2} \right] \quad (6.57)
 \end{aligned}$$

From Equation (6.54), the change in oscillator energy, per cycle, is proportional to

$$\int_{\text{cycle}} \dot{A} \left( Lw^2 p_2 + sw h_0 + \frac{1}{2}\rho L^2 w^2 \left[ \frac{L\dot{A}^2 - 2q_1\dot{A} + \dot{q}_1 A}{A^2} \right] - b\dot{A} \right) dt$$

### **6.3 Incorporation of Fluid Viscosity in the Aperture Region:**

In this section, the restriction of inviscid fluid motion is lifted. The remaining assumptions are that the fluid is Newtonian and unaffected by body forces such as gravity.

The equation of momentum transport for a viscous Newtonian fluid unaffected by body forces differs from the inviscid momentum equation by the addition of terms proportional in value to the second spatial derivatives of the fluid velocity (cf. Equation 2.17).

$$\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\mu + \lambda) \frac{\partial^2 u_j}{\partial x_i \partial x_j} \quad (6.58)$$

In the preceding studies of transverse motion of a piston in a duct, all spatial derivatives were replaced by two-point finite difference approximations; knowledge of any arbitrary fluid property (denoted  $\psi$  say) at any two positions  $x_1$  and  $x_2$  was assumed sufficient to estimate the spatial derivatives of  $\psi$  at a position midway between  $x_1$  and  $x_2$ , by the formula

$$\frac{\partial \psi}{\partial x} \left( \frac{x_1 + x_2}{2} \right) \cong \frac{\psi(x_2) - \psi(x_1)}{x_2 - x_1}$$

In Section 6.2.1 such an approximation was shown to be accurate to second order in the small distance  $x_2 - x_1$ .

Unfortunately, a two-point finite difference approximation is not possible for spatial derivatives of higher than the first order. Equation (6.58), which possesses second order spatial derivatives of the fluid velocity, shows that the earlier luxury of a simple piston-in-duct model of the flow-modulation mechanism (utilising two-point finite difference approximations to spatial derivatives of flow quantities within the aperture) is no longer possible. In order to model viscosity, a more complicated piston geometry is therefore required, as well as the obvious additional complications of the appropriate momentum transport equations.

#### **6.3.1 Modulation by a Smoothly-Varying Constriction in a Duct:**

Consider the flow of a fluid along a duct which is aligned with the  $x$ -axis in an  $xyz$  (Cartesian) co-ordinate system. Consider that the cross-sectional area  $A$  at a distance  $x$  along the duct varies in some prescribed manner upon time, so that  $A = A(x, t)$  in general. Consider also that the thermodynamic properties of the fluid, such as fluid

pressure and density, have uniform values across any cross-section taken perpendicular to the  $x$ -axis; thermodynamic properties are everywhere independent of the  $y$  and  $z$  coordinates so that, for example,  $p = p(x, t)$  and  $\rho = \rho(x, t)$ . Finally, consider that the fluid velocity component aligned with the  $x$ -axis also maintains a uniform value over the flow cross-section at any time, so that  $u = u(x, t)$ . Any fluid motion which adheres to these aforementioned specifications will be described as a *quasi-1D* fluid motion.

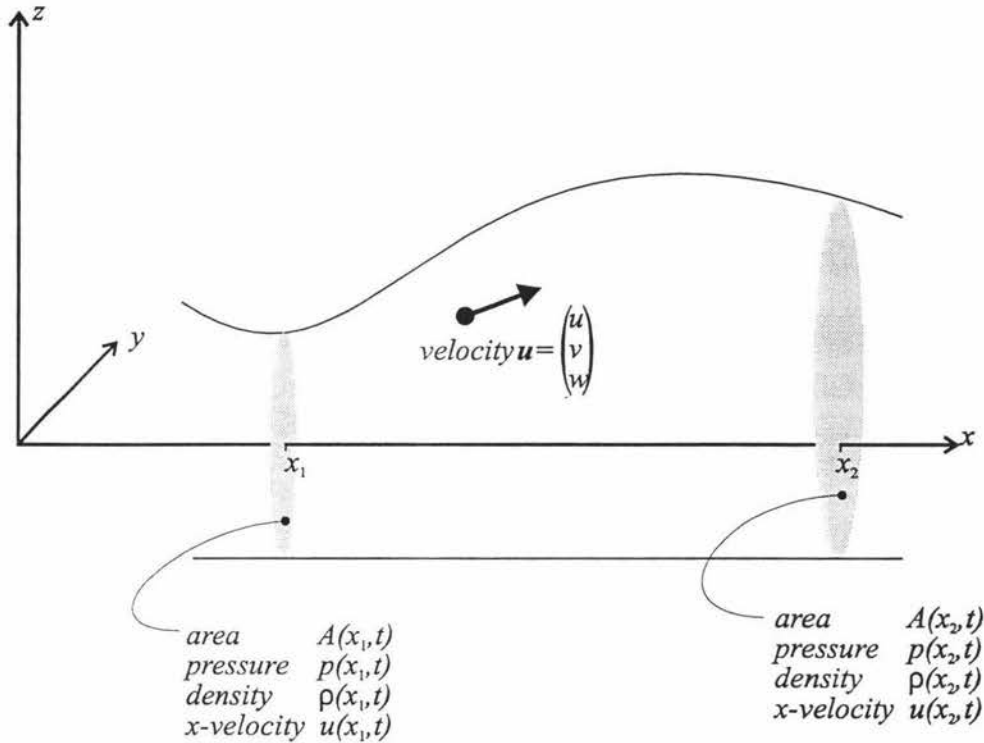


Figure 6.32: Quasi-1D Fluid Motion in the  $x$ -Direction

Quasi-1D Continuity Equation:

The assumption of quasi-1D fluid motion allows the general Equation of Continuity (Equation 2.3) to be simplified as follows on account of the uniformity of the fluid density in any plane perpendicular to the  $x$ -axis.

$$\begin{aligned} & \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \Rightarrow & \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \nabla \cdot \mathbf{u} = 0 \\ \Rightarrow & \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \nabla \cdot \mathbf{u} = 0 \end{aligned} \tag{6.59}$$

### Expression for the Velocity Divergence:

The motion of the fluid within the duct can be related to the motion of the walls of the duct using a control volume analysis as follows. A deformable control volume is employed which is, at every instant of time, coincident with the interior of the duct wall between two specified points along the  $x$ -axis.

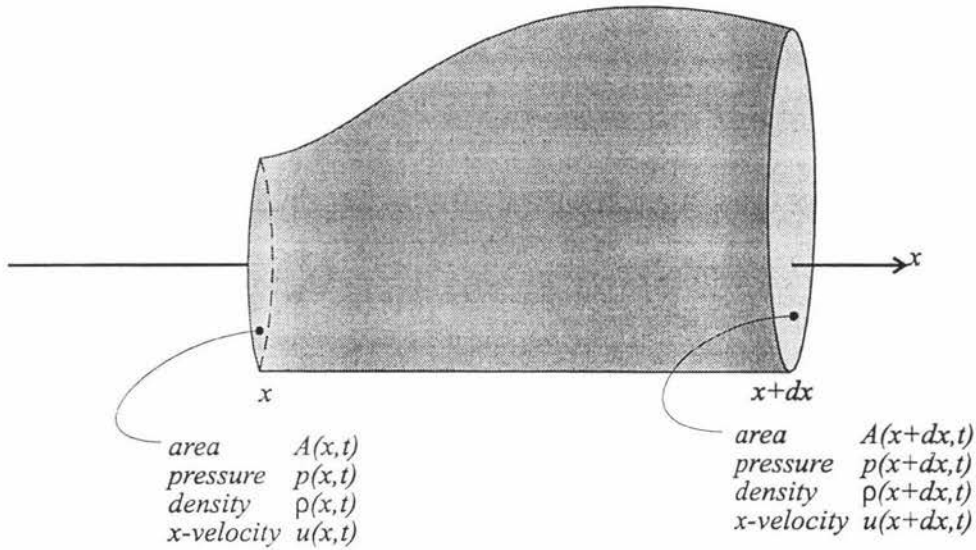


Figure 6.33: Deformable Control Volume of Length  $dx$

Relations shall now be obtained which express mathematically the principle of conservation of mass within the deformable control volume. The cross-sectional area at each place from position  $x$  to position  $x + \Delta x$  varies with time. Mass transport will be considered between an initial time instant  $t$  and a later time instant  $t + \Delta t$ .

During the elapsed time  $\Delta t$  (between initial time  $t$  and later time  $t + \Delta t$ ) the total mass flux into the control volume at the upstream end of the control volume (i.e. at position  $x$ ) is described by the following integral taken over the time elapsed:

$$\text{mass influx} = \int_t^{t+\Delta t} \rho(x,t) u(x,t) A(x,t) dt$$

During that same time interval, the total mass flux out of the downstream end of the control volume (i.e. at position  $x + \Delta x$ ) is:

$$\text{mass outflux} = \int_t^{t+\Delta t} \rho(x + \Delta x,t) u(x + \Delta x,t) A(x + \Delta x,t) dt$$

The amount of fluid mass contained within the deformable control volume at the initial time  $t$  is described by the following integration taken along the length of the control volume :

$$\text{initial mass} = \int_x^{x+\Delta x} \rho(x,t) A(x,t) dx$$

The amount of fluid mass contained within the deformable control volume at the later time  $t + \Delta t$  is described by the following integration taken along the length of the control volume :

$$\text{final mass} = \int_x^{x+\Delta x} \rho(x,t + \Delta t) A(x,t + \Delta t) dx$$

The principal of conservation of mass states that the difference in fluid mass contained by the control volume during time interval  $\Delta t$  is balanced by the net transfer of mass into the control volume during that time interval. Mathematically, this statement may be expressed using the previous four integrals as follows :

$$\left[ \int_x^{x+\Delta x} \rho(x,t) A(x,t) dx \right]_t^{t+\Delta t} = - \int_t^{t+\Delta t} \left[ \rho(x,t) u(x,t) A(x,t) \right]_x^{x+\Delta x} dt$$

Both sides may be converted to double integrals as follows:

$$\begin{aligned} \int_t^{t+\Delta t} \frac{\partial}{\partial t} \left[ \int_x^{x+\Delta x} \rho A dx \right] dt &= - \int_t^{t+\Delta t} \int_x^{x+\Delta x} \frac{\partial}{\partial x} (\rho u A) dx dt \\ \Rightarrow \int_t^{t+\Delta t} \int_x^{x+\Delta x} \left[ \frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) \right] dx dt &= 0 \end{aligned}$$

Because the boundary positions  $x$  and  $x + \Delta x$ , and the time instants  $t$  and  $t + \Delta t$  were chosen arbitrarily, the integrand itself of the above equation can be equated to zero:

$$\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0$$

This differential equation is valid anywhere along the  $x$ -axis of a quasi-1D flow. It can be expanded as follows:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \left[ \frac{\partial u}{\partial x} + \frac{1}{A} \left( \frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} \right) \right] = 0 \quad (6.60)$$

This equation can be compared with Equation (6.59), which is repeated below:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \nabla \cdot \mathbf{u} = 0$$

comparison yields

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{1}{A} \left( \frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} \right) \quad (6.61)$$

Note in particular that the velocity divergence is independent of the  $y$  and  $z$  co-ordinates.

#### Quasi-1D Momentum Equations:

The assumption of quasi-1D motion allows for simplification of the usual differential equations that describe the conservation of fluid momentum for a general fluid motion.

The Navier-stokes equations, for general motion of a Newtonian fluid which has constant coefficients of viscosity and which is negligibly affected by gravitational forces (and other body forces), are:

$$\begin{aligned} \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) &= -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + (\lambda + \mu) \frac{\partial}{\partial x} (\nabla \cdot \mathbf{u}) \\ \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) &= -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + (\lambda + \mu) \frac{\partial}{\partial y} (\nabla \cdot \mathbf{u}) \\ \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) &= -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + (\lambda + \mu) \frac{\partial}{\partial z} (\nabla \cdot \mathbf{u}) \end{aligned}$$

From the definition of quasi-1D motion, the fluid pressure is considered to be uniform across any cross-section of the flow taken perpendicular to the  $x$ -axis. This means that there are no pressure gradients in the  $y$  and  $z$  directions.

$$\frac{\partial p}{\partial y} = 0 \quad \frac{\partial p}{\partial z} = 0$$

Also from the definition of quasi-1D motion, recall that the  $x$ -component of the fluid velocity is also considered to be uniform across any flow cross-section taken perpendicular to the  $x$ -axis. This means that there are no velocity gradients in the  $y$  and  $z$  directions, i.e. that

$$\begin{aligned} \frac{\partial u}{\partial y} = 0 \quad \frac{\partial u}{\partial z} = 0 \\ \frac{\partial^2 u}{\partial y^2} = 0 \quad \frac{\partial^2 u}{\partial z^2} = 0 \end{aligned}$$

The equation obtained already for the velocity divergence (Equation 6.61) assumes only that the motion is quasi-1D. The velocity divergence was found to be always independent

of the  $y$  and  $z$  co-ordinates (i.e. it also is uniform across any cross-section perpendicular to the  $x$ -axis):

$$\frac{\partial}{\partial y}(\nabla \cdot \mathbf{u}) = 0 \quad \frac{\partial}{\partial z}(\nabla \cdot \mathbf{u}) = 0$$

With these simplifications the Navier-Stokes equations simplify to:

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + (\lambda + \mu) \frac{\partial}{\partial x}(\nabla \cdot \mathbf{u}) \quad (6.62)$$

$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad (6.63)$$

$$\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

The last term in Equation (6.62) can be expanded using Equation (6.61) derived earlier:

$$\begin{aligned} \frac{\partial}{\partial x}(\nabla \cdot \mathbf{u}) &= \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial x} + \frac{1}{A} \left( \frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} \right) \right] \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{1}{A} \left( \frac{\partial^2 A}{\partial x \partial t} + \frac{\partial u}{\partial x} \frac{\partial A}{\partial x} + u \frac{\partial^2 A}{\partial x^2} \right) - \frac{1}{A^2} \frac{\partial A}{\partial x} \left( \frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} \right) \end{aligned}$$

This substitution leads to the following equation involving the unknown quantities  $p$ ,  $\rho$  and  $u$ :

$$\begin{aligned} \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) &= -\frac{\partial p}{\partial x} + (2\mu + \lambda) \frac{\partial^2 u}{\partial x^2} \\ &+ \frac{\lambda + \mu}{A} \left( \frac{\partial^2 A}{\partial x \partial t} + \frac{\partial u}{\partial x} \frac{\partial A}{\partial x} + u \frac{\partial^2 A}{\partial x^2} \right) - \frac{\lambda + \mu}{A^2} \frac{\partial A}{\partial x} \left( \frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} \right) \end{aligned} \quad (6.64)$$

Equation (6.60), repeated below, adds no new unknowns

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \left[ \frac{\partial u}{\partial x} + \frac{1}{A} \left( \frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} \right) \right] = 0$$

If it can be assured that the motion of the fluid is barotropic, then a system of three equations in three unknowns is formed by combining with Equations (6.60) and (6.64) a barotropic equation of state, of the form

$$p = p(\rho)$$

Once this system is solved, the values obtained for  $p$ ,  $\rho$  and  $u$  can be used to solve Equations (6.63) for the remaining unknowns  $v$  and  $w$  (although this extension to

complete the entire three-dimensional solution is not required for the purposes described in this thesis).

### 6.3.2 Analytical Solution for Incompressible Fluid Motion:

An analytical solution is available for the solution of quasi-1D flow for situations where the fluid may be treated as having constant density. This is the case for the study of flow through a constriction whose characteristic dimensions are very much smaller than the wavelength of any acoustic motions present. On the other hand, the assumptions of quasi-1D flow required that the characteristic dimensions of the flow problem are very much larger than the boundary layer thickness for the flow. It will be assumed here that there is a common ground where both approximations are valid.

When partial derivatives of density are identically zero, Equation (6.60) simplifies to

$$\frac{\partial u}{\partial x} + \frac{1}{A} \left( \frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} \right) = 0 \quad (6.65)$$

Equation (6.59) becomes  $\nabla \cdot \mathbf{u} = 0$ , so that the first of the Navier-Stokes equation, expressing conservation of  $x$ -momentum, becomes

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} \quad (6.66)$$

Equation (6.65) may also be written as

$$\frac{\partial A}{\partial t} = - \frac{\partial}{\partial x} (uA) \quad (6.67)$$

This equation shows that the product  $uA$  remains spatially continuous everywhere if the cross-sectional area variation is always continuous in time. The cross-sectional area  $A$  in some quasi-1D flow geometries may have a spatial jump discontinuity, and Equation (6.67) shows that in such cases the velocity  $u$  will have a discontinuity also. For this reason, quasi-1D fluid motion is usually described in terms of the *volume velocity*, defined by  $q = uA$ , rather than the particle velocity. In terms of the volume velocity, Equation (6.67) can be written as

$$\frac{\partial q}{\partial x} = - \frac{\partial A}{\partial t}$$

This equation can be integrated with respect to  $x$  to yield an integral equation for the volume velocity. Assume that the value of  $q$  is known for all time at some point  $x = x_Q$ , then

$$q(x, t) = - \int_{x_Q}^x \frac{\partial A}{\partial t} dx + q(x_Q, t) \quad (6.68)$$

In terms of the fluid velocity, an equivalent equation is found as follows:

$$\begin{aligned} u(x, t)A(x, t) &= - \int_{x_Q}^x \frac{\partial A}{\partial t} dx + u(x_Q, t)A(x_Q, t) \\ \Rightarrow u(x, t) &= - \frac{1}{A(x, t)} \int_{x_Q}^x \frac{\partial A}{\partial t} dx + \frac{u(x_Q, t)A(x_Q, t)}{A(x, t)} \end{aligned} \quad (6.69)$$

It is possible to rearrange Equation (6.66) to give an integral expression for the pressure  $p$  in terms of  $u$  and its first derivatives:

$$\begin{aligned} \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) &= - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} \\ \Rightarrow \frac{\partial p}{\partial x} &= \mu \frac{\partial^2 u}{\partial x^2} - \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \left[ \mu \frac{\partial u}{\partial x} - \rho \int \frac{\partial u}{\partial t} dx - \frac{1}{2} \rho u^2 \right] \\ \Rightarrow [p]_{x_R}^x &= \mu \left[ \frac{\partial u}{\partial x} \right]_{x_R}^x - \rho \int_{x_R}^x \frac{\partial u}{\partial t} dx - \frac{1}{2} \rho [u^2]_{x_R}^x \\ \Rightarrow p(x, t) &= p(x_R, t) + \mu \left[ \frac{\partial u}{\partial x}(x, t) - \frac{\partial u}{\partial x}(x_R, t) \right] - \rho \int_{x_R}^x \frac{\partial u}{\partial t} dx \\ &\quad - \frac{1}{2} \rho [u^2(x, t) - u^2(x_R, t)] \end{aligned} \quad (6.70)$$

Equation (6.70) requires knowledge of the variation of fluid pressure  $p$  at a position  $x_R$  in the duct, in addition to that information already needed to fix the fluid velocity using Equation (6.69). It is possible to choose  $x_R = x_Q$ .

Expressions for the partial derivatives of the velocity are obtained by differentiating Equation (6.69). The partial derivative with respect to time is

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= \frac{\partial}{\partial t} \left[ -\frac{1}{A(x,t)} \int_{x_Q}^x \frac{\partial A}{\partial t} dx + \frac{u(x_Q,t)A(x_Q,t)}{A(x,t)} \right] \\ &= \frac{1}{A^2(x,t)} \frac{\partial A}{\partial t} \left[ \int_{x_Q}^x \frac{\partial A}{\partial t} dx - u(x_Q,t)A(x_Q,t) \right] - \frac{1}{A(x,t)} \int_{x_Q}^x \frac{\partial^2 A}{\partial t^2} dx\end{aligned}\quad (6.71)$$

For the case of the partial derivative with respect to  $x$ , the following result can alternatively be obtained by substituting Equation (6.69) into Equation (6.65):

$$\begin{aligned}\frac{\partial u}{\partial x}(x,t) &= \frac{\partial}{\partial x} \left[ -\frac{1}{A(x,t)} \int_{x_Q}^x \frac{\partial A}{\partial t} dx + \frac{u(x_Q,t)A(x_Q,t)}{A(x,t)} \right] \\ &= \frac{1}{A^2(x,t)} \frac{\partial A}{\partial x} \left[ \int_{x_Q}^x \frac{\partial A}{\partial t} dx - u(x_Q,t)A(x_Q,t) \right] - \frac{1}{A(x,t)} \int_{x_Q}^x \frac{\partial^2 A}{\partial x \partial t} dx \\ &= \frac{1}{A^2(x,t)} \frac{\partial A}{\partial x} \left[ \int_{x_Q}^x \frac{\partial A}{\partial t} dx - u(x_Q,t)A(x_Q,t) \right] - \frac{1}{A(x,t)} \left[ \frac{\partial A}{\partial t} \right]_{x_Q}\end{aligned}\quad (6.72)$$

By substituting these expressions into Equation (6.70), it is possible to write the solution for the pressure distribution without any reference to the fluid velocity, except at the two positions  $x_Q$  and  $x_R$ . The extent of the benefits afforded by such a substitution is subject to the choice of geometry profile for the constriction. Except for the simplest constriction geometries, in situations where the constriction motion is unaffected by the nature of the fluid motion, numerical solution appears to be a more viable option.

#### Solution Strategy:

Equation (6.69) repeated below can be solved explicitly for the fluid velocity at every position along the constriction if knowledge of the constriction's cross-sectional area variations is available throughout space and time, and if the fluid velocity at one position  $x_Q$  is known always.

$$u(x,t) = -\frac{1}{A(x,t)} \int_{x_Q}^x \frac{\partial A}{\partial t} dx + \frac{u(x_Q,t)A(x_Q,t)}{A(x,t)}$$

Once Equation (6.69) has been used to determine the velocity solution, then Equation (6.70) solves explicitly for the fluid pressure at every position along the constriction if additional knowledge of the pressure variations at an arbitrary position  $x_R$  is available throughout time.

### 6.3.3 Determination of a Suitable Test Geometry:

Consider a rigid duct, aligned with the  $x$ -axis, in which is inserted a time-varying constriction centred at the position  $x = 0$ . The constriction is always at its narrowest at this central position, and the cross-sectional area there at time  $t$  is denoted  $A_0(t)$ . Except for the constriction region, the duct itself is fixed in time and its constant cross-sectional area is denoted  $A_x$ . Consider that the constriction within the duct is symmetrical about the position  $x = 0$ ; the constriction extends a length  $X$  either side of the central position and joins smoothly with the duct wall at the positions  $x = \pm X$ .

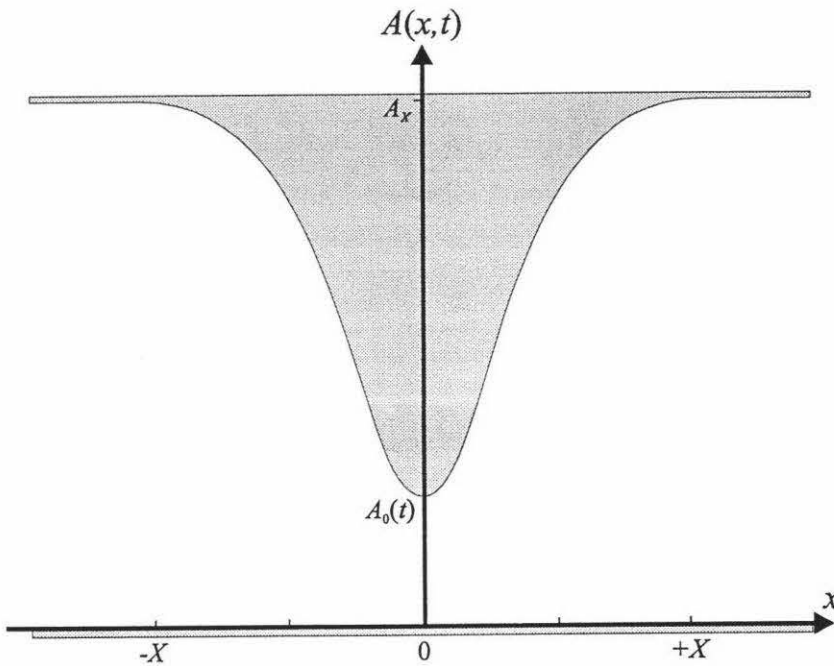


Figure 6.34: Test Geometry for a Time-Varying Constriction at time  $t$

Consider the piecewise function defined by:

$$A(x,t) = \begin{cases} A_x, & \forall t & x \leq -X \\ A_x - [A_x - A_0(t)]H'(x) & -X \leq x \leq X \\ A_x, & \forall t & x \geq X \end{cases} \quad (6.73)$$

Any function of this form will satisfy the above conditions upon the constriction geometry if  $H'(x)$  has a local maximum at the position  $(0,1)$  and further stationary points at  $(-X,0)$  and  $(X,0)$ . Two possible geometries will be studied.

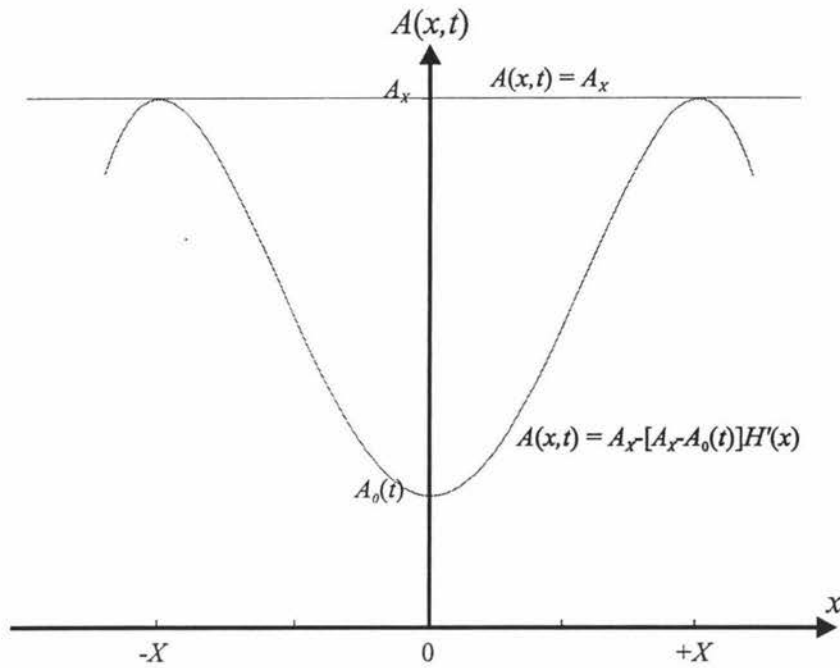


Figure 6.35: Piecewise Composition of Test Geometry

#### Polynomial Test Geometry:

Because the function  $H'(x)$  (in Equation 6.73) is required to have stationary points for  $x \in \{-X, 0, X\}$ , the function  $H''(x)$ , (where a prime is used to denote differentiation of a function of  $x$ ) must have roots for these values of  $x$ . When  $H''(x)$  is chosen to be a polynomial, a cubic will suffice :

$$\begin{aligned} H'' &= k_1 x(x - X)(x + X) \\ &= k_1 (x^3 - X^2 x) \end{aligned}$$

(Note that because it has been specified that  $H'(x)$  is symmetrical, and it has a maximum at  $x = 0$ , then the distribution of roots of  $H''(x)$  must be symmetrical about  $x = 0$ , and there must be an odd number of roots at  $x = 0$  itself.) The form of the function  $H'(x)$  is found by integration:

$$H' = k_1 \left[ \frac{x^4}{4} - \frac{X^2 x^2}{2} \right] + k_2$$

The condition that the function passes through the point  $(0,1)$  dictates that  $k_2 = 1$ , and further function points at  $(\pm X, 0)$  then leads to the conclusion that  $k_1 = \frac{4}{X^4}$ , thence to

$$\begin{aligned}
0 &= k_1 \left( \frac{X^4}{4} - \frac{X^4}{2} \right) + k_2 \\
k_1 &= \frac{4}{X^4} \\
\Rightarrow H' &= \frac{4}{X^4} \left( \frac{X^4}{4} - \frac{X^2 x^2}{2} \right) + 1 \\
&= \left( \frac{x}{X} \right)^4 - 2 \left( \frac{x}{X} \right)^2 + 1 \\
&= \left[ 1 - \left( \frac{x}{X} \right)^2 \right]^2
\end{aligned} \tag{6.74}$$

#### Alternative Trigonometrical Test Geometry:

Recall that the function  $H'(x)$  (in Equation 6.73) is required to have stationary points for  $x \in \{-X, 0, X\}$ . It is a simple matter to deduce that another solution is available of the form

$$H' = k_3 + k_4 \cos\left(\frac{\pi x}{X}\right)$$

The condition of a local maximum at  $(0, 1)$  leads to the conclusion that  $k_3 + k_4 = 1$  (and also that  $k_4$  is positive, since the cosine function reaches a maximum as  $x \rightarrow 0$ ). The condition that the solution passes through the points  $(\pm X, 0)$  dictates that  $k_3 - k_4 = 0$ . Consequently there is a unique solution for  $k_3$  and  $k_4$ :

$$H' = \frac{1}{2} \left[ 1 + \cos\left(\frac{\pi x}{X}\right) \right] \tag{6.75}$$

#### **6.3.4 Description of the Constriction as an Acoustical Element:**

The current investigation of the oscillating constriction in the duct is based upon the following question: "Given knowledge of two of the flow variables  $p(X, t)$ ,  $p(-X, t)$ ,  $u(X, t)$  and  $u(-X, t)$ , what is the effect of the constriction oscillation  $A_0(t)$  upon the other two flow variables?"

Solution for an Unknown Velocity:

Equation (6.69) is repeated below. This equation gives an expression for the fluid velocity at any position  $x$  when the velocity is already known at one particular position  $x_Q$ .

$$u(x,t) = -\frac{1}{A(x,t)} \int_{x_Q}^x \frac{\partial A}{\partial t} dx + \frac{u(x_Q,t)A(x_Q,t)}{A(x,t)}$$

Equation (6.69) is now written, in particular, for the case of  $x_Q = -X$  and a solution sought for the velocity at  $x = X$  :

$$u(X,t) = -\frac{1}{A(X,t)} \int_{-X}^X \frac{\partial A}{\partial t} dx + \frac{u(-X,t)A(-X,t)}{A(X,t)} \quad (6.76)$$

Recall Equation (6.73), repeated below, for the specific type of geometry under consideration:

$$A(x,t) = \begin{cases} A_x, & \forall t & x \leq -X \\ A_x - [A_x - A_0(t)]H'(x) & -X \leq x \leq X \\ A_x, & \forall t & x \geq X \end{cases}$$

Equation (6.76) can be simplified by noting that  $A(X) = A(-X) = A_x$ , and by using Equation (6.73) to expand the integration term as follows:

$$\begin{aligned} \int_{-X}^X \frac{\partial A}{\partial t} dx &= \int_{-X}^X \dot{A}_0(t)H'(x) dx \\ &= \dot{A}_0(t) \int_{-X}^X H'(x) dx \\ &= \dot{A}_0(t)[H(X) - H(-X)] \end{aligned}$$

Substitution into Equation (6.76) gives

$$u(X,t) = -\frac{\dot{A}_0}{A_x} [H(X) - H(-X)] + u(-X,t) \quad (6.77)$$

Polynomial Test Geometry:

The function  $H'(x)$ , for a quartic test geometry, was given in Equation (6.74):

$$\begin{aligned}
 H(X) - H(-X) &= \int_{-X}^X \left[ \left( \frac{x}{X} \right)^4 - 2 \left( \frac{x}{X} \right)^2 + 1 \right] dx \\
 &= X \left[ \frac{1}{5} \left( \frac{x}{X} \right)^5 - \frac{2}{3} \left( \frac{x}{X} \right)^3 + \left( \frac{x}{X} \right) \right]_{-X}^X \\
 &= 2X \left( \frac{1}{5} - \frac{2}{3} + 1 \right) \\
 &= \frac{16X}{15}
 \end{aligned}$$

This information, used in Equation (6.77) gives

$$u(X, t) = -\frac{16X}{15A_x} \dot{A}_0 + u(-X, t) \quad (6.78)$$

### Trigonometrical Test Geometry:

The  $H'(x)$  function obtained for the alternative trigonometrical geometry was given in Equation (6.75).

$$\begin{aligned}
 H(X) - H(-X) &= \int_{-X}^X \frac{1}{2} \left[ 1 + \cos \left( \frac{\pi x}{X} \right) \right] dx \\
 &= \frac{1}{2} \left[ x + \frac{X}{\pi} \sin \left( \frac{\pi x}{X} \right) \right]_{-X}^X \\
 &= X
 \end{aligned}$$

In Equation (6.77), this gives:

$$u(X, t) = -\frac{X}{A_x} \dot{A}_0 + u(-X, t) \quad (6.79)$$

### Solution for an Unknown Pressure:

Equation (6.70) is repeated below. This equation solves for the pressure at any position  $x$  when a solution is known already for the velocity, and the pressure variation at any point  $x_R$ :

$$p(x, t) = p(x_R, t) + \mu \left[ \frac{\partial u}{\partial x}(x, t) - \frac{\partial u}{\partial x}(x_R, t) \right] - \rho \int_{x_R}^x \frac{\partial u}{\partial t} dx - \frac{1}{2} \rho [u^2(x, t) - u^2(x_R, t)]$$

Equation (6.70), in particular, for the case of  $x_R = -X$  and  $x = X$ , is:

$$p(X,t) = p(-X,t) + \mu \left[ \frac{\partial u}{\partial x}(X,t) - \frac{\partial u}{\partial x}(-X,t) \right] - \rho \int_{-X}^X \frac{\partial u}{\partial t} dx - \frac{1}{2} \rho [u^2(X,t) - u^2(-X,t)] \quad (6.80)$$

Equation (6.72), repeated below, may be used to simplify the viscous term above

$$\frac{\partial u}{\partial x}(x,t) = \frac{1}{A^2(x,t)} \frac{\partial A}{\partial x}(x,t) \left[ \int_{x_0}^x \frac{\partial A}{\partial t} dx - u(x_0,t)A(x_0,t) \right] - \frac{1}{A(x,t)} \left[ \frac{\partial A}{\partial t} \right]_{x_0}^x$$

Recall that  $x_0 = -X$ , and that  $A(x_0,t) = A_x$ , a constant:

$$\frac{\partial u}{\partial x}(x,t) = \frac{1}{A^2(x,t)} \frac{\partial A}{\partial x}(x,t) \left[ \int_{-X}^x \frac{\partial A}{\partial t} dx - u(-X,t)A_x \right] - \frac{1}{A(x,t)} \frac{\partial A}{\partial t}(x,t)$$

Consider now that  $x = \pm X$ .  $A(x,t)$  is again constant, and the above equation shows that

$$\frac{\partial u}{\partial x}(\pm X,t) = 0$$

Substitution into Equation (6.80) gives

$$p(X,t) = p(-X,t) - \rho \int_{-X}^X \frac{\partial u}{\partial t} dx - \frac{1}{2} \rho [u^2(X,t) - u^2(-X,t)] \quad (6.81)$$

This is an unfortunate result, as it signifies that there can be no account of momentum loss due to fluid viscosity between any two places in the duct when the rate of change of cross-sectional area is locally zero.

The integral term and the final term of Equation (6.81) can both be evaluated from the solution of the continuity equation (Equation 6.69) alone. (The assumption of incompressible flow effectively de-couples the continuity and momentum equations.) Because viscosity does not affect the solution of Equation (6.69), this also indicates that viscosity has no effect upon the solution of Equation (6.81) at any point from  $-X$  to  $X$ .

Equation (6.71) is repeated below:

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{A^2(x,t)} \frac{\partial A}{\partial t} \int_{x_0}^x \frac{\partial A}{\partial t} dx - \frac{1}{A^2(x,t)} \frac{\partial A}{\partial t} u(x_0,t)A(x_0,t) - \frac{1}{A(x,t)} \int_{x_0}^x \frac{\partial^2 A}{\partial t^2} dx$$

Substituting  $x_0 = -X$  converts this to

$$\frac{\partial u}{\partial t} = \frac{1}{A^2} \frac{\partial A}{\partial t} \int_{-X}^x \frac{\partial A}{\partial t} dx - \frac{1}{A^2} \frac{\partial A}{\partial t} u(-X,t)A_x - \frac{1}{A} \int_{-X}^x \frac{\partial^2 A}{\partial t^2} dx$$

For the specific types of geometries under consideration, the cross-sectional area in the constriction region obeys  $A(x,t) = A_x - [A_x - A_0(t)]H'(x)$ . Consequently, the above equation leads to

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\dot{A}_0^2 H'(x)[H(x) - H(-X)] - A_x \dot{A}_0 H'(x)u(-X)}{\{A_x - [A_x - A_0]H'(x)\}^2} - \frac{\ddot{A}_0 H'(x)}{A_x - [A_x - A_0]H'(x)} \\ &= \frac{\dot{A}_0^2 H'(x)H(x) - [\dot{A}_0^2 H(-X) + A_x \dot{A}_0 u(-X)]H'(x)}{\{A_x - [A_x - A_0]H'(x)\}^2} - \frac{\ddot{A}_0 H'(x)}{A_x - [A_x - A_0]H'(x)} \end{aligned}$$

This equation shows that the value of the integral  $\int_{-X}^x \frac{\partial u}{\partial t} dx$  depends intricately upon the detail of the geometry throughout the constriction and the variation of the fluid velocity at the position  $-X$ .

Consequently, further development of a description of the oscillating constriction as an acoustical element - i.e. description in terms of flow variables  $p(X,t)$ ,  $p(-X,t)$ ,  $u(X,t)$  and  $u(-X,t)$  - is thwarted.

Numerical solution of Equation (6.81) is capable of yielding results for incompressible, inviscid fluid through the time-varying constriction. To yield a viscous solution, however, it is necessary to use the compressible system equations instead, viz. Equations 6.60, 6.64 and an equation of state (refer to Section 6.3.1), or to abandon the quasi-1D description and adopt a multi-dimensional incompressible viscous-fluid approach. It appears that there may be no common ground for which a simultaneously quasi-1D and viscous description is possible (c.f. introductory comments of Section 6.3.2).

### 6.3.5 Numerical Solution for Case of Zero Viscosity:

Numerical solution will now be described for quasi-1D flow of an incompressible inviscid fluid through the time-varying constriction. Constriction geometries will be considered which have the form of Equation (6.73), repeated here:

$$A(x,t) = \begin{cases} A_x, & \forall t & x \leq -X \\ A_x - [A_x - A_0(t)]H'(x) & -X \leq x \leq X \\ A_x, & \forall t & x \geq X \end{cases}$$

Solution for an Unknown Velocity:

Equation (6.69) is repeated below. This equation solves for the fluid velocity at any position  $x$  when the velocity is already known at one particular position  $x_Q$ .

$$u(x, t) = -\frac{1}{A(x, t)} \int_{x_Q}^x \frac{\partial A}{\partial t} dx + \frac{u(x_Q, t)A(x_Q, t)}{A(x, t)}$$

Equation (6.69) is now written, in particular, for the case of  $x_Q = \pm X$ . Note that  $A(x_Q, t) = A_x$  throughout time in such cases:

$$u(x, t) = -\frac{1}{A(x, t)} \int_{\pm X}^x \frac{\partial A}{\partial t} dx + \frac{u(\pm X, t)A_x}{A(x, t)} \quad (6.82)$$

Equation (6.73) may be used to expand the integration term as follows:

$$\begin{aligned} \int_{\pm X}^x \frac{\partial A}{\partial t} dx &= \int_{\pm X}^x \dot{A}_0(t) H'(x) dx \\ &= \dot{A}_0(t) \int_{\pm X}^x H'(x) dx \\ &= \dot{A}_0(t) [H(x) - H(\pm X)] \end{aligned}$$

Substitution into Equation (6.82) gives

$$u(x, t) = -\frac{1}{A(x, t)} \left\{ \dot{A}_0 [H(x) - H(\pm X)] - u(\pm X, t)A_x \right\} \quad (6.83)$$

The function  $H'(x)$ , for a quartic test geometry, was given in Equation (6.74), repeated here:

$$\begin{aligned} H' &= \left[ 1 - \left( \frac{x}{X} \right)^2 \right]^2 \\ \Rightarrow H(x) - H(\pm X) &= \int_{\pm X}^x \left[ \left( \frac{x}{X} \right)^4 - 2 \left( \frac{x}{X} \right)^2 + 1 \right] dx \\ &= X \left[ \frac{1}{5} \left( \frac{x}{X} \right)^5 - \frac{2}{3} \left( \frac{x}{X} \right)^3 + \left( \frac{x}{X} \right) \right]_{\pm X}^x \\ &= \frac{X}{15} \left\{ \left[ 3 \left( \frac{x}{X} \right)^5 - 10 \left( \frac{x}{X} \right)^3 + 15 \left( \frac{x}{X} \right) \right] \mp (3 - 10 + 15) \right\} \\ &= \frac{X}{15} \left[ 3 \left( \frac{x}{X} \right)^5 - 10 \left( \frac{x}{X} \right)^3 + 15 \left( \frac{x}{X} \right) \mp 8 \right] \quad (6.84) \end{aligned}$$

Equation (6.84) is illustrated in Figure (6.36) for the case of  $x_0 = -X$  :

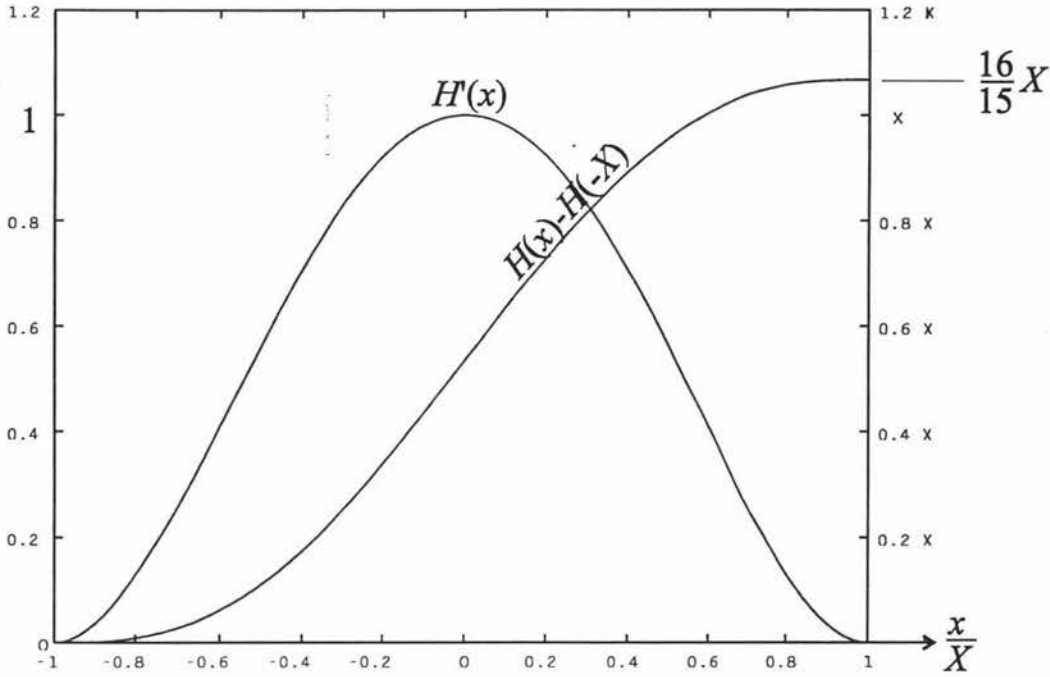


Figure 6.36: Quartic Geometry

The  $H'(x)$  function obtained for the alternative trigonometrical geometry was given in Equation (6.75), and this is repeated here:

$$\begin{aligned}
 H' &= \frac{1}{2} \left[ 1 + \cos\left(\frac{\pi x}{X}\right) \right] \\
 \Rightarrow H(x) - H(\pm X) &= \int_{\pm X}^x \frac{1}{2} \left[ 1 + \cos\left(\frac{\pi x}{X}\right) \right] dx \\
 &= \frac{1}{2} \left[ x + \frac{X}{\pi} \sin\left(\frac{\pi x}{X}\right) \right]_{\pm X}^x \\
 &= \frac{1}{2} \left[ (x \mp X) + \frac{X}{\pi} \sin\left(\frac{\pi x}{X}\right) \right] \tag{6.85}
 \end{aligned}$$

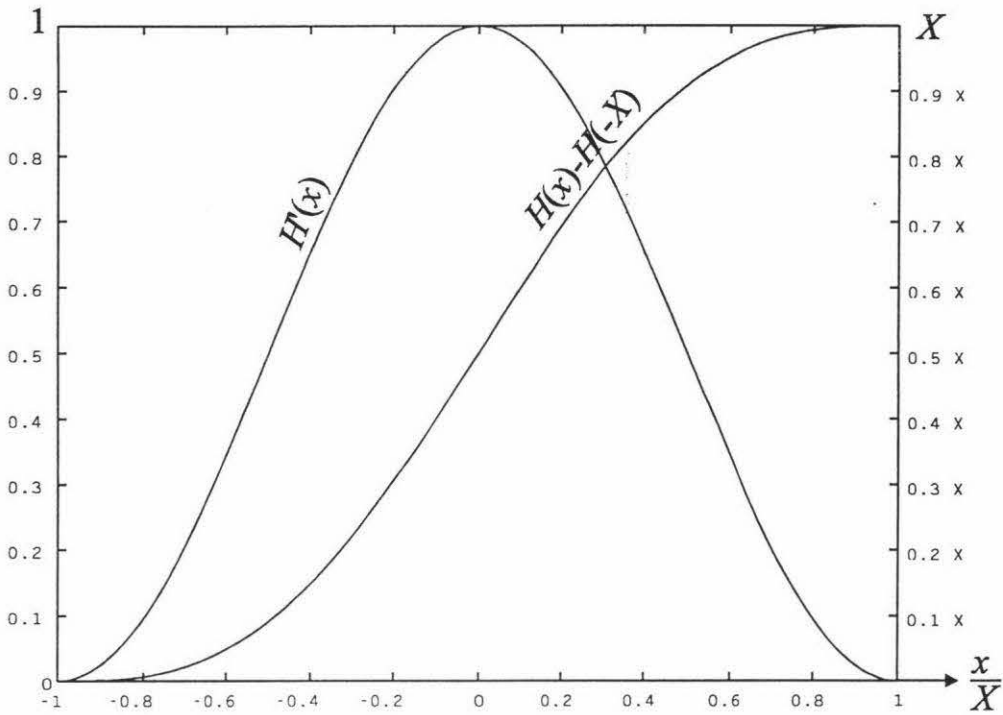


Figure 6.37: Trigonometrical Geometry

#### Solution for an Unknown Pressure:

Equation (6.70) is repeated below. This equation solves for the pressure at any position  $x$  when a solution is known already for the fluid velocity, and the pressure variation at any point  $x_R$ :

$$p(x, t) = p(x_R, t) + \mu \left[ \frac{\partial u}{\partial x}(x, t) - \frac{\partial u}{\partial x}(x_R, t) \right] - \rho \int_{x_R}^x \frac{\partial u}{\partial t} dx - \frac{1}{2} \rho [u^2(x, t) - u^2(x_R, t)]$$

Equation (6.70) is now written, in particular, for the case of zero viscosity and  $x_R = \pm X$ .

$$p(x, t) = p(\pm X, t) - \rho \int_{\pm X}^x \frac{\partial u}{\partial t} dx - \frac{1}{2} \rho [u^2(x, t) - u^2(\pm X, t)] \quad (6.86)$$

#### Fluid Force upon the Constriction:

The fluid force exerted upon the constriction by the inviscid flow is determined by the fluid pressure and the interior surface area of the constriction at each position along it. Knowledge is first required of the relationship between the cross-sectional area variations and the corresponding variations of the surface area of the constriction. For simplicity, it will be assumed that the constriction fits snugly in a duct of rectangular cross-section, so

that the perimeter of the boundary is, at any place, equal to  $2w + 2\frac{A(x,t)}{w}$  where  $w$  denotes the width of the duct. This situation is illustrated below in Figure 6.38.

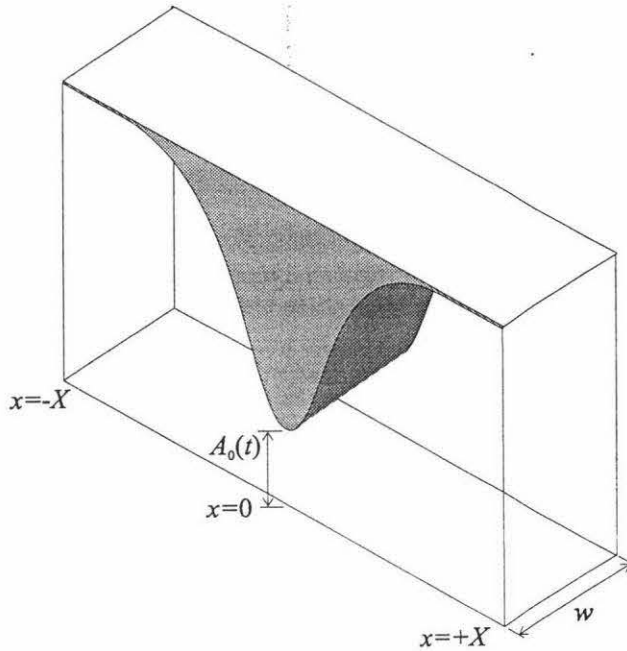


Figure 6.38: Perimeter at  $x, t$  is a Linear Function of Cross-Sectional Area

The upwards pressure force upon the constriction (i.e. between  $-X$  and  $X$ ) is then given by

$$F = \int_{-X}^X p(x,t)w dx = w \int_{-X}^X p(x,t) dx \quad (6.87)$$

#### Summary of System Equations:

The geometry under consideration varies according to Equation (6.73):

$$A(x,t) = \begin{cases} A_x, & \forall t & x \leq -X \\ A_x - [A_x - A_0(t)]H'(x) & & -X \leq x \leq X \\ A_x, & \forall t & x \geq X \end{cases}$$

Within the constriction region, i.e.  $-X < x < X$ , the cross-sectional area is a separable function of  $x$  and  $t$ . The functions  $A_0(t)$  and  $H'(x)$  are specified by the user in the current investigation.

The fluid velocity at any position along the duct is given by Equation (6.83):

$$u(x,t) = -\frac{1}{A(x,t)} \left\{ \dot{A}_0 [H(x) - H(\pm X)] - u(\pm X, t) A_x \right\}$$

The fluid velocity along the duct is determined by the choices made for the functions  $A_0(t)$  and  $H'(x)$ , and also by the time-variation of the fluid velocity immediately upstream or downstream of the constriction region  $u(\pm X, t)$ .

The fluid pressure at any position along the duct is given by Equation (6.86):

$$p(x, t) = p(\pm X, t) - \rho \int_{\pm X}^x \frac{\partial u}{\partial t} dx - \frac{1}{2} \rho [u^2(x, t) - u^2(\pm X, t)]$$

Solution of this equation requires prior solution of Equation (6.83) for the fluid velocity. A finite difference approximation can then be used to estimate the partial derivative of velocity with respect to time, but initial values of the derivative must be provided by the user to enable approximation by finite differences at later time-steps. The integration in Equation (6.86) can be performed numerically using, for example, the Trapezoidal Rule or Simpson's Rule. Additional knowledge of the pressure variation immediately upstream or downstream of the constriction, i.e. of  $p(\pm X, t)$ , is also required before the value of the fluid pressure can be known everywhere.

The upwards pressure force upon the constriction (i.e. between  $-X$  and  $X$ ), for the arrangement depicted in Figure 6.36, is given by Equation (6.87):

$$F = w \int_{-X}^X p(x, t) dx$$

This equation provides another opportunity for numerical integration.

The complete system can be programmed with ease using MATLAB. The user is required to specify  $A_0(t)$  and  $H'(x)$ , one of  $u(\pm X, t)$  and one of  $p(\pm X, t)$  before results can be obtained, but then Equations (6.83), (6.86) and (6.87) can be evaluated individually via the substitution of known quantities.

#### Sample Solution:

The results which follow are obtained for a hypothetical duct, 20 mm square (i.e.  $A_x = 4 \times 10^{-4}$  m and  $w = 2 \times 10^{-2}$  m).

The oscillating constriction within the duct has a trigonometrical profile, and a half-length of  $X = 2 \times 10^{-3}$  m. The aperture cross-section at the midpoint of the constriction varies sinusoidally throughout time according to

$$A_0(t) = \bar{A}_0 - \frac{1}{2} A_{PP} \cos(\omega_A t) \tag{6.88}$$

In this equation,  $\bar{A}_0$  represents the time-averaged aperture area at  $x = 0$ ; a value of  $10^{-4}$  is used. The same value is used for the constant  $A_{PP}$  representing the peak-to-peak amplitude of the time-variations of the cross-sectional area at  $x = 0$ . The frequency of the oscillation is taken as 450 Hz, so that  $\omega_A = \frac{450}{2\pi} \text{ s}^{-1}$ .

The variation of cross-sectional area, at various positions along the duct, is illustrated in Figure 6.39 to follow.

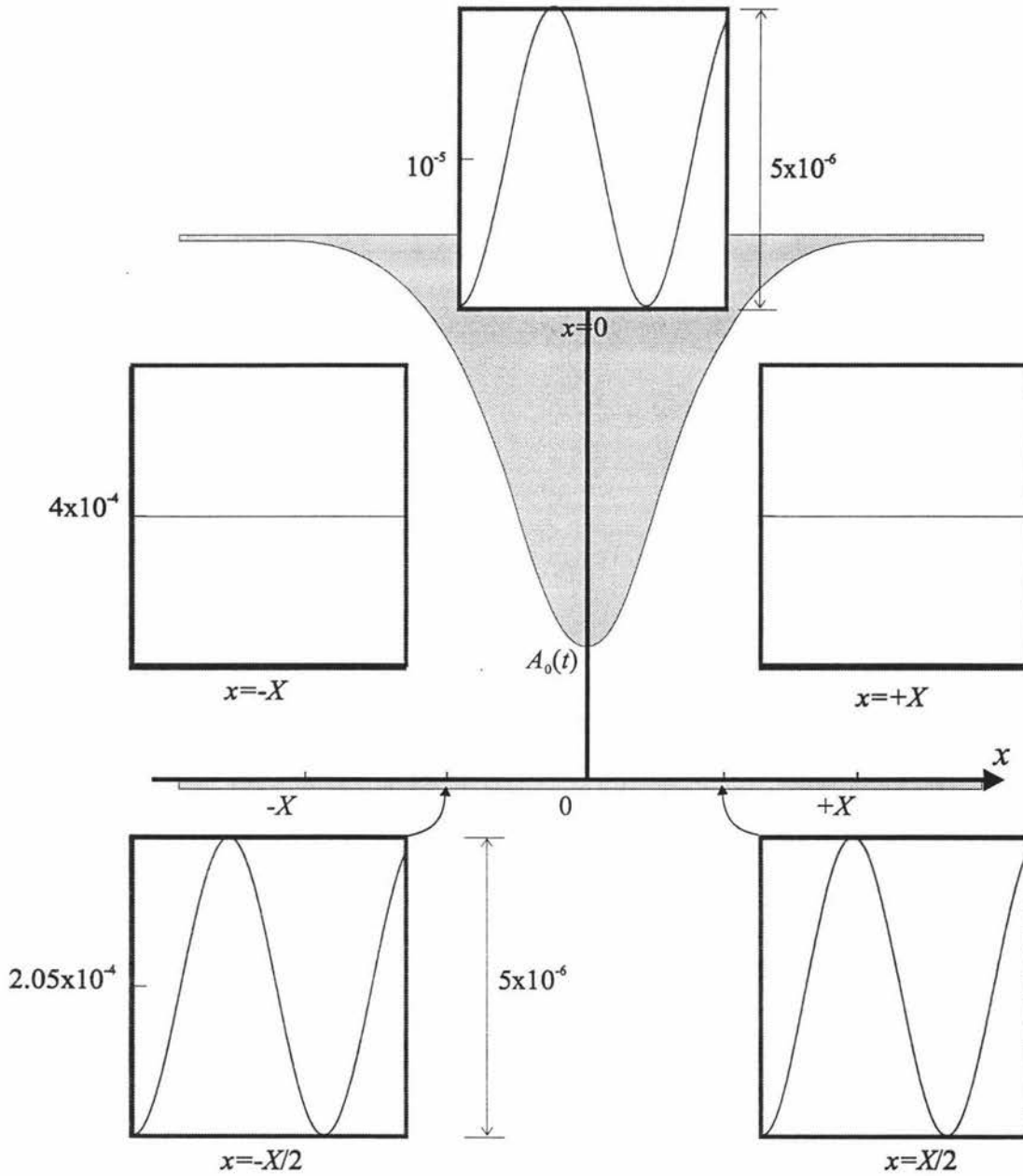


Figure 6.39: Cross-Sectional Area ( $\text{m}^2$ ) at Various Positions

The upstream velocity  $u(-X, t)$  is assumed constant, and is given the value of  $15 \text{ ms}^{-1}$ . Fluid velocities at various other locations within the constriction region are depicted in Figure 6.40. Note the unrealistically high velocities due to the neglect of viscosity.

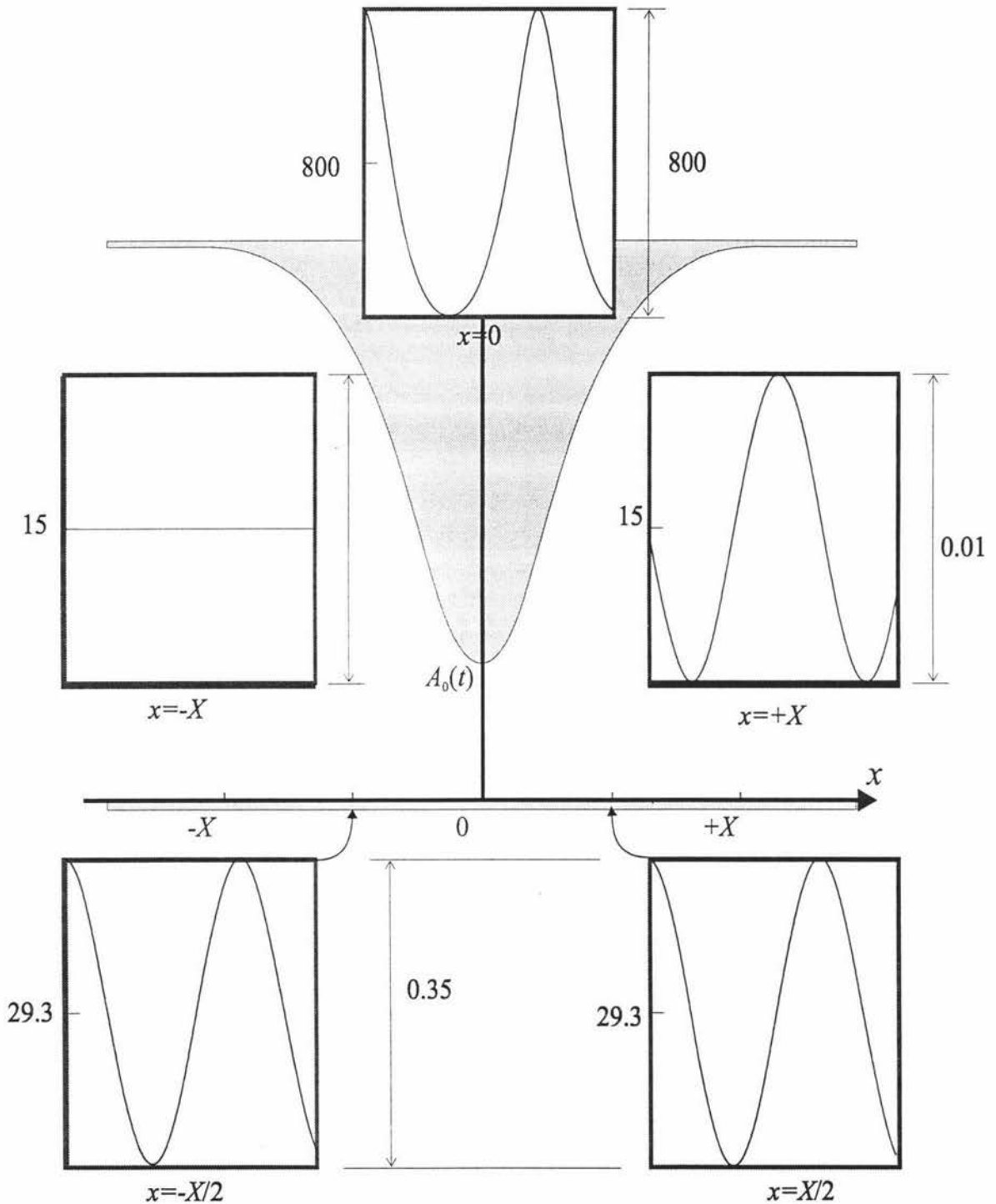


Figure 6.40: Time-Variation of Velocity ( $\text{m.s}^{-1}$ ) at Various Positions

The upstream pressure  $p(-X, t)$  is also assumed constant, having the value 103.82 kPa. (Atmospheric air corresponds to 101.325 kPa.) Pressure fluctuations within the constriction are illustrated in Figure 6.41.

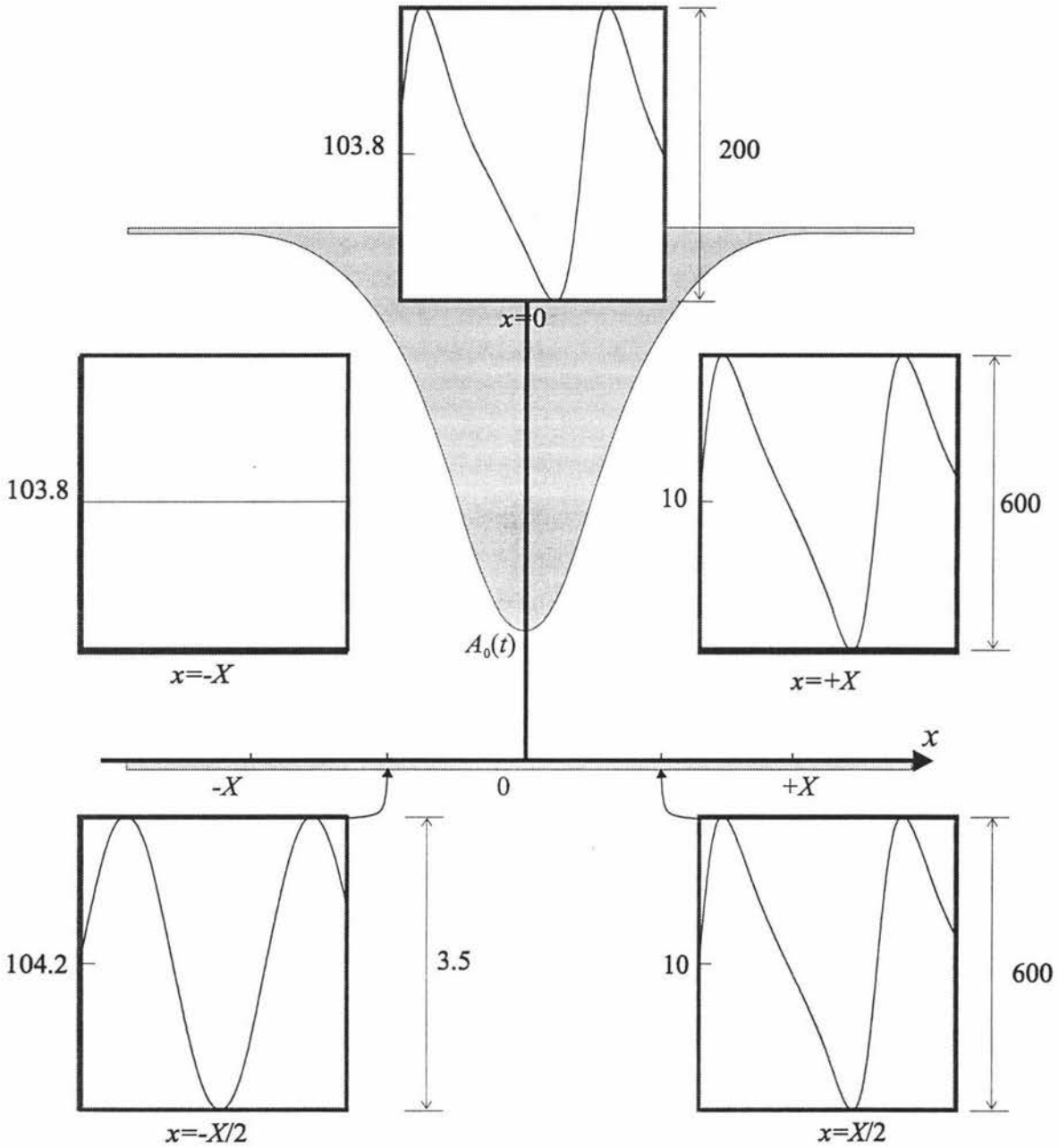


Figure 6.41: Time-Variation of Pressure (kPa) at Various Positions

The fluid force upon the moving constriction, arising from the resulting pressure distribution throughout the constriction region, is shown in Figure 6.42 to follow:

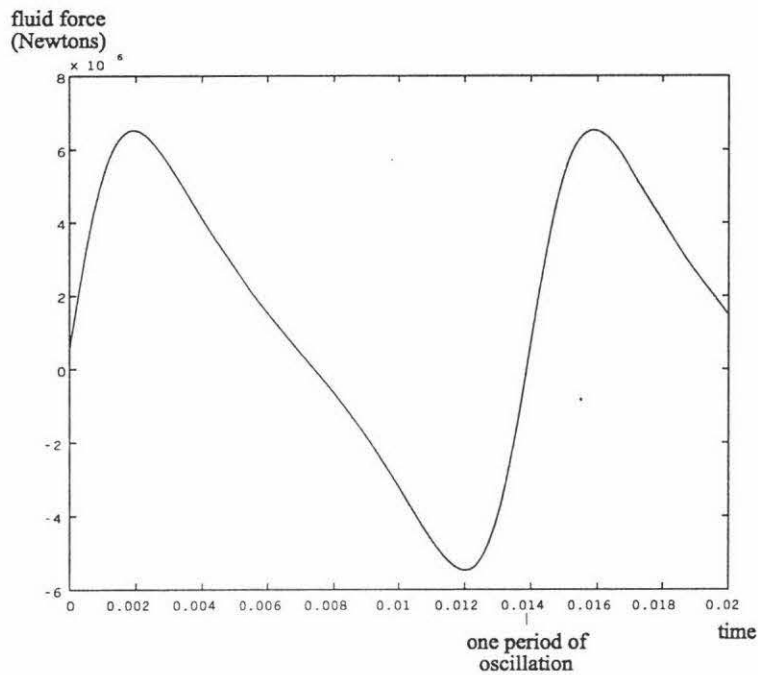


Figure 6.42: Time-variation of Fluid Force Upon Constriction Walls

Further sets of results are readily obtained for any different set of model parameter values. The situation studied above is an especially simple case, since the upstream velocity and pressure have both been assumed constant, and the time-variation of the aperture cross-section was specified as being monochromatic. Some attention is now given to the question of whether the model equations may be simplified a little, which would facilitate the introduction of more complicated (more realistic) boundary conditions for the model.

For the Flow-Rate to be Approximately Independent of  $x$ :

Recall Equation (6.83), which is repeated here:

$$u(x,t) = -\frac{1}{A(x,t)} \left\{ \dot{A}_0 [H(x) - H(\pm X)] - u(\pm X, t) A_x \right\}$$

The two terms within the curly brackets are now compared in magnitude. The form chosen for the function  $\dot{A}_0$  (see Equation 6.88) indicates that the magnitude of this function never exceeds  $\frac{1}{2} A_{pp} \omega_A$ , while Equation (6.79) indicates that the difference

$H(x) - H(\pm X)$  is never larger than  $X$ . Defining  $f_A = \frac{\omega_A}{2\pi}$  leads to the following

sufficient condition for Equation (6.83) to be simplified by removal of the first term in the curly brackets.

$$\begin{aligned} \dot{A}_0[H(x) - H(\pm X)] &\ll u(\pm X, t)A_x \\ \Rightarrow f_A &\ll \frac{u_{\min}A_x}{\pi A_{pp}X} \end{aligned} \quad (6.89)$$

Substitution of the values chosen for the parameters  $A_x$ ,  $A_{pp}$ ,  $X$  and the boundary condition  $u(-X, t) = 15 \text{ ms}^{-1}$  leads to a condition that  $f_A \ll 10 \text{ kHz}$ . The flow rate is approximately constant throughout the constriction region when this condition is fulfilled, since Equation (6.83) then becomes

$$u(x, t)A(x, t) \cong u(\pm X, t)A_x \quad (6.90)$$

$$\Rightarrow u(-X, t) \cong u(+X, t) \quad (6.91)$$

Notice from Figure 6.40 that the error of such an assumption, for the example simulation illustrated, is only of the order of  $\frac{1}{15}\%$ .

#### Question of Suitability of an Unsteady Bernoulli Equation:

Equation (6.88), repeated below, is an expression of Bernoulli's Equation for an inviscid fluid in the absence of body forces.

$$p(x, t) = p(\pm X, t) - \rho \int_{\pm X}^x \frac{\partial u}{\partial t} dx - \frac{1}{2} \rho [u^2(x, t) - u^2(\pm X, t)]$$

For a fluid flow situation which is time-independent, the integral term in this equation is identically zero. The "unsteady Bernoulli equation" is useful for describing such phenomena:

$$p(x, t) = p(\pm X, t) - \frac{1}{2} \rho [u^2(x, t) - u^2(\pm X, t)] \quad (6.92)$$

In some situations which are not strictly time-independent, Equation (6.90) may still give an acceptable approximation. Such a flow situation is referred to as being *quasi-steady*. Suitability of a quasi-steady description of the given oscillating constriction model is now tested by examining the contribution of the integral term in Equation (6.88) to the overall pressure value. The results for an oscillation of 70 Hz are shown in Figure 6.43 to follow:

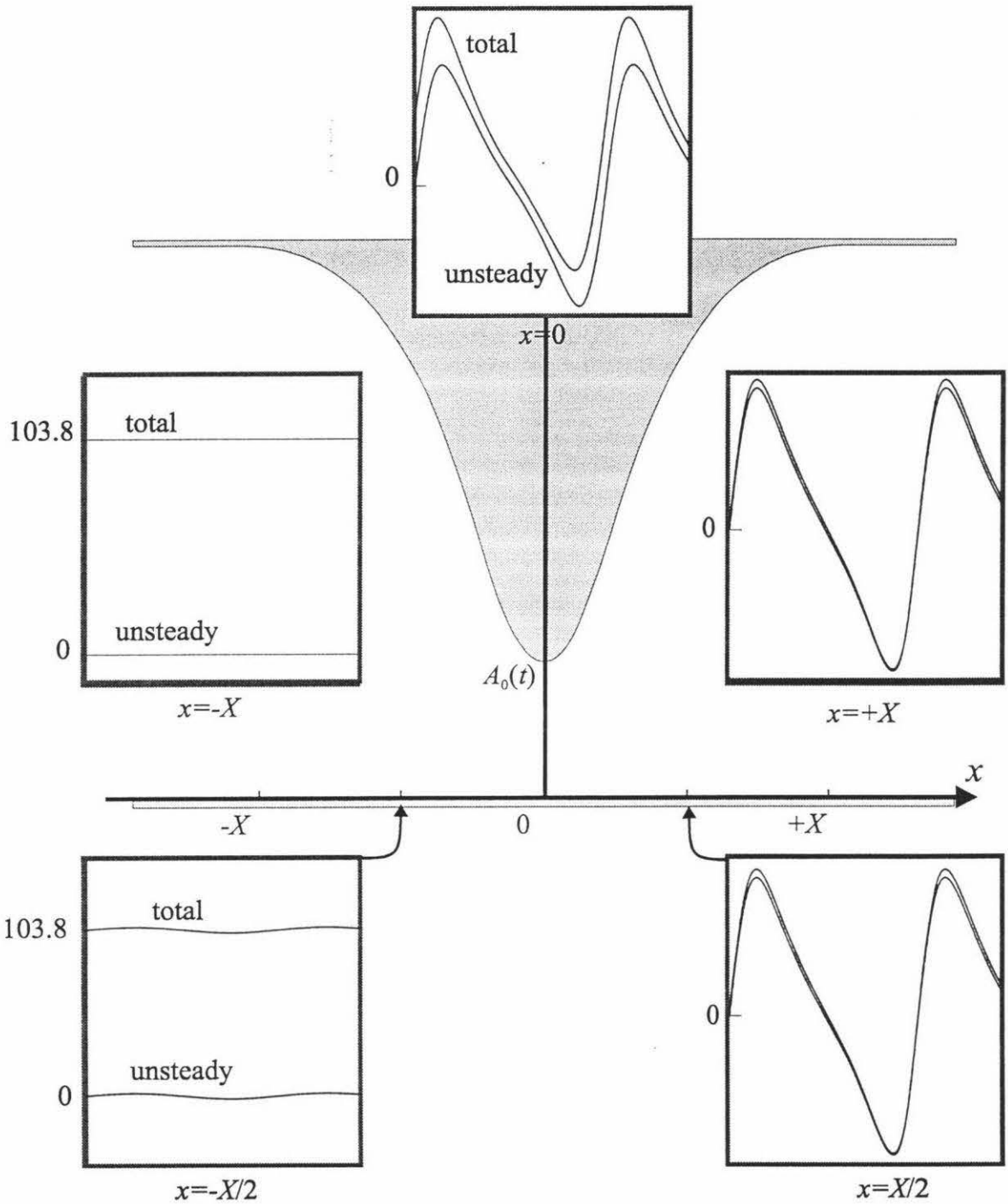


Figure 6.43: “Unsteady Term” Contribution to Total Pressure Value

A quasi-static description does not appear to be appropriate in the study of the proposed model of the flow through the constriction. At the downstream side of the constriction, almost all pressure variation is due to the unsteady term.

### 6.3.6 Derivation of a New Model Flow Equation

The condition given in Section 6.3.5 (Equation 6.89) for the flow-rate to be approximately independent of  $x$ , will be assumed satisfied, viz.:

$$f_A \ll \frac{u_{\min} A_x}{\pi A_{pp} X}$$

As stated already, this allows simplification of Equation (6.83) leading to Equation (6.91)

$$u(-X, t) \cong u(+X, t)$$

This can now be used to simplify Equation (6.81) of Section 6.3.4, repeated below

$$\begin{aligned} p(X, t) &= p(-X, t) - \rho \int_{-X}^X \frac{\partial u}{\partial t} dx - \frac{1}{2} \rho [u^2(X, t) - u^2(-X, t)] \\ \Rightarrow p(X, t) &\cong p(-X, t) - \rho \int_{-X}^X \frac{\partial u}{\partial t} dx \end{aligned}$$

It is useful to rewrite this equation in terms of the volume velocity  $q = uA$ , since Equation (6.90) shows that  $q$  is independent of  $x$  within the confines of the current approximation.

$$p(X, t) \cong p(-X, t) - \rho \int_{-X}^X \left( \frac{1}{A} \frac{\partial q}{\partial t} - \frac{q}{A^2} \frac{\partial A}{\partial t} \right) dx$$

The pressure drop across the constriction can be written as

$$\boxed{p(-X, t) - p(X, t) \cong \rho \dot{q} \int_{-X}^X \frac{dx}{A} - \rho q \int_{-X}^X \frac{1}{A^2} \frac{\partial A}{\partial t} dx} \quad (6.93)$$

This is the proposed Flow Modulation Equation for the case of modulation by a time-varying area constriction in a duct. It is suitable when the flow is quasi-1D, inviscid and incompressible, and when the constriction moves "as a whole" - meaning that  $A(x, t)$  is known everywhere if it is known at one place, as in Equation (6.73) - and when the condition given in Equation (6.89) holds.

Case of Oscillating Square-Edged Piston in a Duct:

For the special case scenario considered in Sections 6.1 and 6.2 (and writing  $L = 2X$  for consistency of notation), Equation (6.93) simplifies to

$$\begin{aligned} p(-X, t) - p(X, t) &\cong \rho \dot{q} \frac{L}{A} - \rho q \frac{L \dot{A}}{A^2} \\ &= \rho L \left( \frac{\dot{q}A - q\dot{A}}{A^2} \right) \\ &= \rho L \frac{d}{dt} \left( \frac{q}{A} \right) \end{aligned}$$

Alternatively

$$p(-X, t) - p(X, t) \cong \rho L \dot{u} \tag{6.94}$$

This might be compared with Equation (1.5) employed in the traditional trumpet model:

$$p_1 - p_2 = \frac{\rho}{2} u^2$$

Recall the following Bernoulli equation which appears in the Abstract.

$$\int_{x_1}^{x_2} \frac{\partial u}{\partial t} dx + \frac{1}{2} u^2 \Big|_{x_1}^{x_2} + \frac{p}{\rho} \Big|_{x_1}^{x_2} = 0$$

Equation (6.94), which describes the flow in the aperture beneath the oscillating piston, retains the same two terms of the Bernoulli equation that are retained in the derivation of the equations of linear acoustics (Section 4.1: compare Equations 4.25).



## Application Of Results

### To Modelling Of Sound-Production

#### In A Trumpet

The work described in this thesis does not solve the question of sound-production in the trumpet. However, it does provide a framework upon which further investigation into the fluid-dynamical aspects of trumpet operation can be built.

The mathematics presented in Chapter Four showed that pseudosound cannot be described by the equations of one-dimensional inviscid fluid flow. A study of the characteristics of the describing equations revealed that all oscillatory flow energy of a one-dimensional inviscid flow is propagated.

Chapter Five used experimental means to confirm the existence of pseudosound in a real modulated flow. However, for the flow situation examined, the pseudosound pressures made a relatively small contribution to the overall pressure signal. They were of measurable size only during the first few cycles of the 'low' notes played. The results suggest that pseudosound does not represent a significant proportion of the overall pressure measured in a lip-excited cylindrical tube. A logical next step would seem to be a similar experimental procedure carried out with an actual trumpet, rather than a length of pipe.

It is worth noting at this point that the fluid velocities associated with sound and pseudosound pressure fluctuations are not of the same order of magnitude (Ffowcs Williams, 1969). In linear acoustics (see Section 4.1) the order of magnitude of the pressure fluctuation is  $\bar{p}\bar{c}\bar{u}$ , whereas for quasi-steady convective motion, the pressure

variations are of the order of  $\bar{\rho}\bar{u}\tilde{u}$ . This can be demonstrated using a steady Bernoulli equation as follows:

$$\begin{aligned} (\bar{p} + \tilde{p}) + \frac{1}{2}\bar{\rho}(\bar{u} + \tilde{u})^2 &= \bar{p} + \frac{1}{2}\bar{\rho}\bar{u}^2 \\ \Rightarrow \tilde{p} + \frac{1}{2}\bar{\rho}(2\bar{u}\tilde{u} + \tilde{u}^2) &= 0 \\ \Rightarrow \tilde{p} + \bar{\rho}(\bar{u} + \frac{1}{2}\tilde{u})\tilde{u} &= 0 \end{aligned}$$

This signifies that, for a given amplitude of pressure variation, the corresponding velocity fluctuation is different by a factor of  $M$  (where  $M$  is a Mach number based upon the mean flow velocity  $M = \bar{u}/\bar{c}$ ) depending upon whether the measured pressure corresponds to sound or to pseudosound. This suggests that perhaps future work in the experimental investigation of the modulated flow into the trumpet should include flow-velocity sensors (hot-wires) as well.

The importance of this realisation is that the above conclusion of the insignificance of the pseudosound pressures in comparison to the acoustic pressures does not rule out the significance of convective velocities in comparison to propagated velocities. Pseudosound has not yet been shown to be unimportant to the mechanism of sound-production in the trumpet.

Chapter Six introduced a fluid-dynamical approach to the study of flow modulation by an oscillating constriction in a flow duct. The work is based upon the premise that a quasi-1D description of the flow is adequate. There is considerable scope for extension and refinement of this work (e.g. incorporation of fluid viscosity, using a two- or three-dimensional flow description). Other analytical and experimental studies of related systems might also be compared (Allen & Watters, 1959; Meyer, 1969; Glendinning, Nelson & Elliott, 1990; Chapman & Glendinning, 1990). From the investigation described thus far, the accepted equation for the flow between the trumpeter's lips (Equation 1.5) appears to be unsuitable, even in the event that a quasi-1D flow description is adequate. An alternative quasi-1D flow equation is proposed (Equation 6.93).

Suitability of the quasi-1D flow description could be investigated by calculating the flow field computationally using a two-dimensional or three-dimensional description (see Section 3.5 for some simplifications that are available). Calculations of flow fields for related situations are to be found in the literature (Pedley & Stephanoff, 1985; Ralph &

Pedley, 1988, 1989). From the results of such a study it may be possible to determine enough information regarding the flow behaviour to devise a low-order model, which yet includes the effects of higher-dimensional flow phenomena without requiring complete calculation of the flow field. Simple models of other multi-dimensional flow phenomena have been devised already (Katz, Chan & Moreno, 1969; Hartlen & Currie, 1970; Skop & Griffin, 1973; Iwan & Blevins, 1974; Flandro & Jacobs, 1975; Blake, Gerschfeld & Maga, 1985).

Also, from multi-dimensional calculation of an incompressible flow in a compact region, it is possible to apply basic principles of aeroacoustics (Lighthill, 1952, 1954; Powell, 1964, 1995; Crow, 1970; Howe, 1975) to determine any far field sound caused by the flow. This is the basic premise of the field of study known as *computational aeroacoustics* (Hardin & Lamkin, 1984, 1985).

## 7.1 Proposal for a New Trumpet Model:

An alternative to the traditional trumpet model is here proposed. A quasi-1D flow description is again assumed adequate. The lip oscillations are modelled as depicted in Figure 7.1 below:

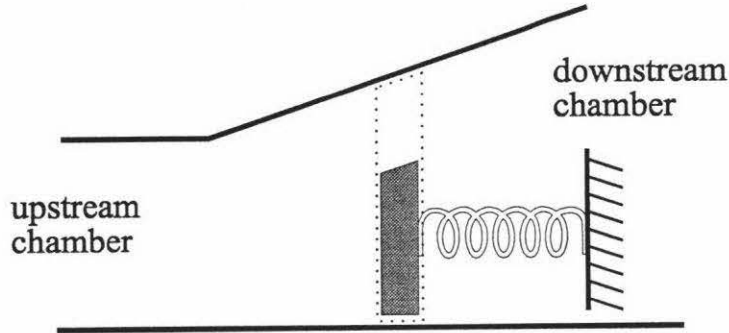


Figure 7.1: Plunger-in-Diverging-Channel Model

The longitudinal motion of the plunger in the diverging channel has a three-fold influence:

- it affects the value of the “volume above the plunger”,
- it affects the value of the “volume upstream of the plunger”, and
- it affects the value of the “volume downstream of the plunger”.

A control volume may be defined which moves longitudinally with the plunger, and stretches vertically to the top and bottom of the diverging channel. The dotted outline in Figure 7.1 above illustrates this control volume at one instant of time.

The results of Chapter Six are applicable (through a suitable change of variables) to this “plunger control volume” as it moves.

That volume which is, at any instant, upstream of the plunger control volume is defined as the “upstream control volume”. The right-hand boundary of this control volume moves so that it is always coincident with the left-hand boundary of the plunger control volume. Similarly, a “downstream plunger volume” is defined which has its left-hand boundary coincident with the right-hand boundary of the plunger control volume. In this manner, all of the fluid is included in one, and only one, of the three control volumes as the plunger moves.

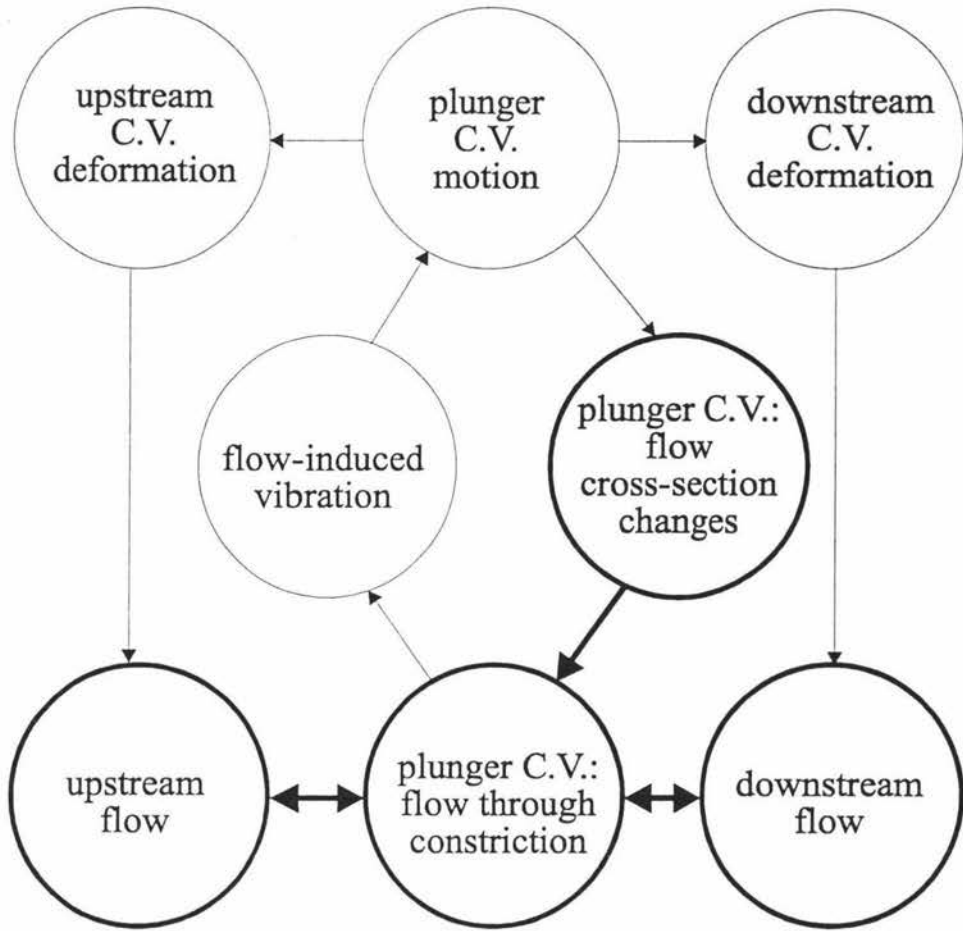


Figure 7.2: Cause-and-Effect for Plunger-in-Diverging-Channel Analysis

In Figure 7.2, the bold portion of the diagram indicates the subsystem which was considered (in isolation from the rest of the system) in Chapter Six.

The model of Figures 7.1 is given as a suggestion for future investigation. It is intended that the mathematical description of the fluid behaviour in the three defined regions be approached using control volume analysis; similar analyses have been presented in Chapter Six. The resulting system of six flow equations, yielded by application of the principles of conservation of fluid mass and momentum in the three control volumes, would describe flow properties at the four positions shown in Figure 7.3 to follow, (where positions 1 and 2 move with the plunger).

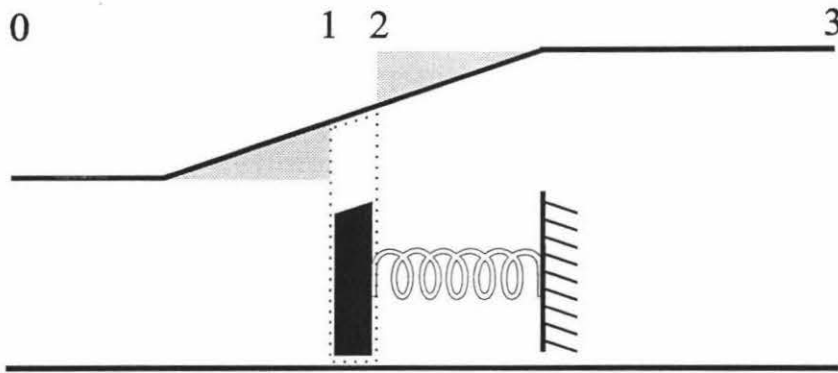


Figure 7.3: Locations for Pressure and Flow Variables

Six flow equations will be coupled to describe the passage of fluid through the model. This is considerably more complex than the description of the flow between the lips in the traditional trumpet model (Equation 1.5).

It is expected, however, that a number of simplifying approximations will be possible. For example, the volumes of the shaded areas of Figure 7.3 might be shown to be unimportant to the model behaviour for some channel geometries.

It is the author's opinion that a pre-tension force should be included in description of the plunger dynamics (Section 1.3.5). Also, it is not considerably more difficult to consider the effects of possible flexing (with damping) of the muscle walls of the oral cavity, in response to the fluid pressures within the mouth; *locally-reacting* duct walls have been incorporated into models of speech production for many years (Section 1.3.6).

A first attempt at providing suitable boundary conditions for the flow might ascribe the values  $p_0 = 103.82$  kPa (a steady mouth pressure measured by Vivona, 1968) and  $p_3 = 101.325$  kPa (atmospheric conditions).

## Appendix:

### Series Solution of the 1-D Modified Wave Equation

#### - Alternative Dimensionless Parameter

In Section 4.2.2, a series solution was sought for the following 1-D Modified Wave Equation (4.37):

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x}\right)^2 \tilde{p} - \bar{c}^2 \frac{\partial^2 \tilde{p}}{\partial x^2} \equiv 0$$

The equation was non-dimensionalised, and in terms of a Mach number defined as  $M = \frac{\bar{u}}{\bar{c}}$ , became:

$$\frac{\partial^2 P}{\partial T^2} + 2M \frac{\partial^2 P}{\partial X \partial T} + (M^2 - 1) \frac{\partial^2 P}{\partial X^2} = 0$$

A series solution was found, using  $M$  as an expansion parameter, to be (Equation 4.65):

$$P(X, T) = f_0(T - X) + \sum_{j=1}^{\infty} \left\{ \frac{1}{j!} \left[ \frac{MX}{1+M} \right]^j f_0^{(j)}(T - X) \right\}$$

The form of the solution suggests that perhaps a different dimensionless parameter could have been more efficient in reaching the final solution. Such a parameter might be defined by

$$N = \frac{M}{M+1} \tag{A1}$$

Equation (A1) leads to

$$M = \frac{N}{1-N} \quad \text{and} \quad M^2 - 1 = \frac{2N-1}{(1-N)^2}$$

Substitution into Equation (4.39) gives a new dimensionless equation:

$$(N-1)^2 \frac{\partial^2 P}{\partial T^2} - 2N(N-1) \frac{\partial^2 P}{\partial X \partial T} + (2N-1) \frac{\partial^2 P}{\partial X^2} = 0 \tag{A2}$$

A series solution for Equation (A2) is now proposed, in which  $P$  has the form

$$P(X, T) = P_0(X, T) + NP_1(X, T) + N^2 P_2(X, T) + \dots \tag{A3}$$

Convergence of the series requires that  $|N| < 1$ . Equation (A1) indicates that this is automatically satisfied for any non-negative choice of  $M$ .

Substitution of Equation (A3) into Equation (A2) leads to the following:

$$\left( [N-1]^2 \frac{\partial^2}{\partial T^2} - 2N[N-1] \frac{\partial^2}{\partial X \partial T} + [2N-1] \frac{\partial^2}{\partial X^2} \right) [P_0 + NP_1 + N^2 P_2 + \dots] = 0$$

This expression may be regrouped as follows, in terms of ascending powers of  $N$ :

$$\begin{aligned} & \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_0 + N \left[ \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_1 + 2 \left( -\frac{\partial^2}{\partial T^2} + \frac{\partial^2}{\partial X \partial T} + \frac{\partial^2}{\partial X^2} \right) P_0 \right] \\ & + \sum_{i=2}^{\infty} \left\{ N^i \left[ \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_i + 2 \left( -\frac{\partial^2}{\partial T^2} + \frac{\partial^2}{\partial X \partial T} + \frac{\partial^2}{\partial X^2} \right) P_{i-1} + \left( \frac{\partial^2}{\partial T^2} - 2 \frac{\partial^2}{\partial X \partial T} \right) P_{i-2} \right] \right\} = 0 \end{aligned} \quad (\text{A4})$$

#### Order Zero Terms:

The order zero terms in the above equation necessarily lead to the same solution as the order zero terms in the equivalent equation expanded in terms of the alternative parameter  $M$  (Equation 4.40). Again it is seen that the function  $P_0(X, T)$  is a solution to a linear wave equation:

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_0 = 0$$

Any function of the form  $P_0 = P_0(X - T)$  is known to be a solution to the above equation. Imposing the boundary condition that  $P(0, T) = f_0(T)$  requires again that (c.f. Equation 6.42)

$$P_0(X - T) = f_0(T - X) \quad (\text{A5})$$

#### Order $N$ Terms:

By retaining in Equation (A4) only those terms which are of order  $N$ , the following non-homogeneous linear wave equation results:

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_1 = 2 \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X \partial T} - \frac{\partial^2}{\partial X^2} \right) P_0$$

Utilising from above that  $P_0 = f_0(T - X)$  leads to:

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_1 = 2f_0''(T - X)$$

This expression is identical to Equation (4.43), and so its solution is the same as Equation (4.47) repeated below as Equation (A6):

$$P_1(X, T) = X f_0'(T - X) \quad (\text{A6})$$

### Squared $N$ Terms:

When retaining only those terms of Equation (A4) which are of second order in  $N$ , the following equation results:

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_2 = 2 \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X \partial T} - \frac{\partial^2}{\partial X^2} \right) P_1 + \left( 2 \frac{\partial^2}{\partial X \partial T} - \frac{\partial^2}{\partial T^2} \right) P_0$$

Substitution of the expressions obtained already for  $P_0$  and  $P_1$  (Equations A5 and A6) follows:

$$\begin{aligned} f_0 &= f_0(T - X) \\ \Rightarrow \left( 2 \frac{\partial^2}{\partial X \partial T} - \frac{\partial^2}{\partial T^2} \right) P_0 &= 2 \left[ -f_0''(T - X) \right] - \left[ f_0''(T - X) \right] \\ &= -3f_0''(T - X) \\ P_1(X, T) &= X f_0'(T - X) \\ \Rightarrow \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X \partial T} - \frac{\partial^2}{\partial X^2} \right) P_1 &= \left[ X f_0'''(T - X) \right] \\ &\quad - \left[ f_0''(T - X) - X f_0'''(T - X) \right] \\ &\quad - \left[ -2f_0''(T - X) + X f_0'''(T - X) \right] \\ &= f_0''(T - X) + X f_0'''(T - X) \\ \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_2 &= 2 \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X \partial T} - \frac{\partial^2}{\partial X^2} \right) P_1 + \left( 2 \frac{\partial^2}{\partial X \partial T} - \frac{\partial^2}{\partial T^2} \right) P_0 \\ &= -f_0'' + 2X f_0''' \end{aligned} \quad (\text{A7})$$

Consider a solution of Equation (A7) of the form

$$P_2(X, T) = g_{21} X f_0'(T - X) + g_{22} X^2 f_0''(T - X)$$

If such a solution is available, then its partial derivatives must satisfy Equation (A7):

$$\frac{\partial P_2}{\partial X} = g_{21} f_0'(T - X) + (2g_{22} - g_{21}) X f_0''(T - X) - g_{22} X^2 f_0'''(T - X)$$

$$\begin{aligned} \frac{\partial^2 P_2}{\partial X^2} &= (2g_{22} - 2g_{21})f_0''(T - X) \\ &\quad + (g_{21} - 2(2g_{22}))X f_0'''(T - X) + g_{22}X^2 f_0^{(iv)}(T - X) \\ \frac{\partial P_2}{\partial T} &= g_{21}X f_0''(T - X) + g_{22}X^2 f_0'''(T - X) \\ \frac{\partial^2 P_2}{\partial T^2} &= g_{21}X f_0'''(T - X) + g_{22}X^2 f_0^{(iv)}(T - X) \\ \therefore \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_2 &= 2(g_{21} - g_{22})f_0'' + 4g_{22}X f_0''' \end{aligned}$$

This can now be compared to Equation (A7) to give:

$$g_{22} = \frac{1}{2} \qquad \Rightarrow g_{21} = 0$$

Thus the appropriate solution for the order  $N^2$  terms in Equation (A4) is

$$P_2(X, T) = \frac{1}{2} X^2 f_0''(T - X) \tag{A8}$$

General Solution for Order  $i$  Terms in  $N$ :

When the index  $i$  in Equation (A4) is two or greater, the equation to be solved after collecting terms in  $N^i$  has the form

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_i + 2 \left( -\frac{\partial^2}{\partial T^2} + \frac{\partial^2}{\partial X \partial T} + \frac{\partial^2}{\partial X^2} \right) P_{i-1} + \left( \frac{\partial^2}{\partial T^2} - 2\frac{\partial^2}{\partial X \partial T} \right) P_{i-2} = 0 \tag{A9}$$

As previously for the solution in the power series of  $M$ , a sequence of solutions for the power series in  $N$  is sought having the following form:

$$P_i(X, T) = \sum_{j=1}^i g_{ij} X^j f_0^{(j)}(T - X) \tag{A10}$$

This leads to, for  $i - 1$  and  $i - 2$ ,

$$P_{i-1}(X, T) = \sum_{j=1}^{i-1} g_{(i-1)j} X^j f_0^{(j)}(T - X) \text{ and } P_{i-2}(X, T) = \sum_{j=1}^{i-2} g_{(i-2)j} X^j f_0^{(j)}(T - X).$$

These expressions may be substituted into Equation (A9) to determine the elements of a  $g$ -matrix (see Section 4.2.2).

$$\left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} \right) P_i + 2 \left( -\frac{\partial^2}{\partial T^2} + \frac{\partial^2}{\partial X \partial T} + \frac{\partial^2}{\partial X^2} \right) P_{i-1} + \left( \frac{\partial^2}{\partial T^2} - 2\frac{\partial^2}{\partial X \partial T} \right) P_{i-2} = 0$$

$$\begin{aligned}
&\Rightarrow \sum_{j=1}^i g_{ij} X^j P_0^{(j+2)} - \sum_{j=1}^i g_{ij} \left[ j(j-1) X^{j-2} P_0^{(j)} - 2j X^{j-1} P_0^{(j+1)} + X^j P_0^{(j+2)} \right] \\
&\quad - 2 \sum_{j=1}^{i-1} g_{(i-1)j} X^j P_0^{(j+2)} + 2 \sum_{j=1}^{i-1} g_{(i-1)j} \left[ j X^{j-1} P_0^{(j+1)} - X^j P_0^{(j+2)} \right] \\
&\quad + 2 \sum_{j=1}^{i-1} g_{(i-1)j} \left[ j(j-1) X^{j-2} P_0^{(j)} - 2j X^{j-1} P_0^{(j+1)} + X^j P_0^{(j+2)} \right] \\
&\quad + \sum_{j=1}^{i-2} g_{(i-2)j} X^j P_0^{(j+2)} - 2 \sum_{j=1}^{i-2} g_{(i-2)j} \left[ j X^{j-1} P_0^{(j+1)} - X^j P_0^{(j+2)} \right] = 0 \\
&\Rightarrow - \sum_{j=1}^i g_{ij} j(j-1) X^{j-2} P_0^{(j)} + 2 \sum_{j=1}^i g_{ij} j X^{j-1} P_0^{(j+1)} \\
&\quad - 2 \sum_{j=1}^{i-1} g_{(i-1)j} X^j P_0^{(j+2)} - 2 \sum_{j=1}^{i-1} g_{(i-1)j} j X^{j-1} P_0^{(j+1)} + 2 \sum_{j=1}^{i-1} g_{(i-1)j} j(j-1) X^{j-2} P_0^{(j)} \\
&\quad + 3 \sum_{j=1}^{i-2} g_{(i-2)j} X^j P_0^{(j+2)} - 2 \sum_{j=1}^{i-2} g_{(i-2)j} j X^{j-1} P_0^{(j+1)} = 0
\end{aligned}$$

It is helpful to change the indices of some summations to aid the collection of like terms:

$$\begin{aligned}
&\Rightarrow - \sum_{j=1}^i g_{ij} j(j-1) X^{j-2} P_0^{(j)} + 2 \sum_{j=2}^{i+1} g_{i(j-1)} (j-1) X^{j-2} P_0^{(j)} \\
&\quad - 2 \sum_{j=3}^{i+1} g_{(i-1)(j-2)} X^{j-2} P_0^{(j)} - 2 \sum_{j=2}^i g_{(i-1)(j-1)} (j-1) X^{j-2} P_0^{(j)} + 2 \sum_{j=1}^{i-1} g_{(i-1)j} j(j-1) X^{j-2} P_0^{(j)} \\
&\quad + 3 \sum_{j=3}^i g_{(i-2)(j-2)} X^{j-2} P_0^{(j)} - 2 \sum_{j=2}^{i-1} g_{(i-2)(j-1)} (j-1) X^{j-2} P_0^{(j)} = 0
\end{aligned}$$

Both terms for which  $j = 1$  are zero, since they each have a factor of  $j - 1$  contained within them. Terms for  $j = 2$  give:

$$-2g_{i2} + 2g_{i1} - 2g_{(i-1)1} + 4g_{(i-1)2} - 2g_{(i-2)1} = 0 \quad (\text{A11})$$

Terms for values of  $j$  ranging from 3 to  $i - 1$  give:

$$\begin{aligned}
&\quad - j(j-1)g_{ij} + 2(j-1)g_{i(j-1)} \\
&\quad - 2g_{(i-1)(j-2)} - 2(j-1)g_{(i-1)(j-1)} + 2j(j-1)g_{(i-1)j} \\
&\quad + 3g_{(i-2)(j-2)} - 2(j-1)g_{(i-2)(j-1)} = 0
\end{aligned} \quad (\text{A12})$$

Terms for  $j = i$  are:

$$-i(i-1)g_{ii} + 2(i-1)g_{i(i-1)} - 2g_{(i-1)(i-2)} - 2(i-1)g_{(i-1)(i-1)} + 3g_{(i-2)(i-2)} = 0 \quad (\text{A13})$$

Terms for  $j = i + 1$  are:

$$2ig_{ii} - 2g_{(i-1)(i-1)} = 0 \quad (\text{A14})$$

In a similar fashion to the process described in Section 4.2.2, all of the  $g_{ij}$  coefficients are determined uniquely by Equations (A11) through (A14) if the  $g_{(i-1)j}$  and  $g_{(i-2)j}$  coefficients are known already.

It can be shown as follows that the resulting  $g$ -matrix is actually a diagonal matrix. First notice that Equation (A14) dictates that each diagonal element is dependent only upon its predecessor. For example

$$ig_{ii} = g_{(i-1)(i-1)}$$

and  $(i-1)g_{(i-1)(i-1)} = g_{(i-2)(i-2)}$  (A15)

These results may be usefully employed as follows:

$$\begin{aligned} 3g_{(i-2)(i-2)} &= 3(i-1)g_{(i-1)(i-1)} \\ &= 2(i-1)g_{(i-1)(i-1)} + (i-1)g_{(i-1)(i-1)} \\ &= 2(i-1)g_{(i-1)(i-1)} + i(i-1)g_{ii} \end{aligned}$$

This expression may be used to simplify Equation (A13):

$$\begin{aligned} -i(i-1)g_{ii} + 2(i-1)g_{i(i-1)} - 2g_{(i-1)(i-2)} - 2(i-1)g_{(i-1)(i-1)} + 3g_{(i-2)(i-2)} &= 0 \\ \Rightarrow 2(i-1)g_{i(i-1)} - 2g_{(i-1)(i-2)} &= 0 \\ \Rightarrow (i-1)g_{i(i-1)} &= g_{(i-1)(i-2)} \end{aligned}$$

This expression shows that the value of  $g_{i(i-1)}$  is independent of the values of any diagonal elements. Equation (A15) may be used to cancel any diagonal elements that occur within Equation (A12) repeated below:

$$\begin{aligned} -j(j-1)g_{jj} + 2(j-1)g_{i(j-1)} - 2g_{(i-1)(j-2)} - 2(j-1)g_{(i-1)(j-1)} + 2j(j-1)g_{(i-1)j} \\ + 3g_{(i-2)(j-2)} - 2(j-1)g_{(i-2)(j-1)} = 0 \end{aligned}$$

This equation is valid for  $j = 3$  to  $j = i - 1$ . However diagonal elements feature only for the case of  $j = i - 1$ :

$$\begin{aligned} -(i-1)(i-2)g_{i(i-1)} + 2(i-2)g_{i(i-1)} - 2g_{(i-1)(i-3)} - 2(i-2)g_{(i-1)(i-2)} + 2(i-1)(i-2)g_{(i-1)(i-1)} \\ + 3g_{(i-2)(i-3)} - 2(i-2)g_{(i-2)(i-2)} = 0 \\ \Rightarrow -(i-1)(i-2)g_{i(i-1)} + 2(i-2)g_{i(i-1)} - 2g_{(i-1)(i-3)} - 2(i-2)g_{(i-1)(i-2)} + 3g_{(i-2)(i-3)} = 0 \end{aligned}$$

Similarly, all diagonal elements can be cancelled from Equation (A11), and it follows that every off-diagonal element of the entire matrix is independent of the values of each of the diagonal elements. The solution for the  $N^2$  terms of the power series (Equation A8)

shows that  $g_{21} = 0$ . Consequently every other off-diagonal element is zero also, and the coefficient matrix is a diagonal matrix.

It is also apparent from Equation (A14), together with the initial values  $g_{11} = 1$  and  $g_{22} = \frac{1}{2}$ , that each diagonal element obeys the simple rule  $g_{ii} = \frac{1}{i!}$ .

Consequently the general solution for  $P_i(X, T)$  is simply (c.f. Equation A10):

$$P_i(X, T) = \frac{X^i}{i!} f_0^{(i)}(T - X)$$

This formula, though only proved for  $i > 2$ , can be shown to be valid for all whole numbers. Substitution into Equation (A3) gives an equation exactly equivalent to Equation (4.65):

$$\begin{aligned} P(X, T) &= P_0(X, T) + NP_1(X, T) + N^2P_2(X, T) + \dots \\ &= \sum_0^{\infty} \frac{(NX)^i}{i!} f_0^{(i)}(T - X) \end{aligned} \tag{A16}$$

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