Theoretical and Computational Analysis of the Two-Stage Capacitated Plant Location Problem

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Abstract

Mathematical models for plant location problems form an important class of integer and mixed-integer linear programs. The Two-Stage Capacitated Plant Location Problem (TSCPLP), the subject of this thesis, consists of a three level structure: in the first or upper-most level are the production plants, the second or central level contains the distribution depots, and the third level is the customers. The decisions to be made are: the subset of plants and depots to open; the assignment of customers to open depots, and therefore open plants; and the flow of product from the plants to the depots, to satisfy the customers' service or demand requirements at minimum cost.

The formulation proposed for the TSCPLP is unique from previous models in the literature because customers can be served from multiple open depots (and plants) and the capacity of both the set of plants and the set of depots is restricted. Surrogate constraints are added to strengthen the bounds from relaxations of the problem. The need for more understanding of the strength of the bounds generated by this procedure for the TSCPLP is evident in the literature. Lagrangian relaxations are chosen based more on ease of solution than the knowledge that a strong bound will result. Lagrangian relaxation has been applied in heuristics and also inserted into branch-and-bound algorithms, providing stronger bounds than traditional linear programming relaxations. The current investigation provides a theoretical and computational analysis of Lagrangian relaxation bounds for the TSCPLP directly.

Results are computed through a Lagrangian heuristic and CPLEX. The test problems for the computational analysis cover a range of problem size and strength of capacity constraints. This is achieved by scaling the ratio of total depot capacity to customer demand and the ratio of total plant capacity to total depot capacity on subsets of problem instances. The analysis shows that there are several constraints in the formulation that if dualized in a Lagrangian relaxation provide strong bounds on the optimal solution to the TSCPLP. This research has applications in solution techniques for the TSCPLP and can be extended to some transformations of the TSCPLP. These include the single-source TSCPLP, and the multi-commodity TSCPLP which accommodates for multiple products or services.

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Introduction

1.1 Preface

The modelling of plant location problems for locating facilities and satisfying customers' demand is an important area of Decision Science. Plant location models form an important class of integer and mixed-integer linear programming problems, with applications in many areas such as the distribution and transportation industries. Where to locate themselves and their respective distribution facilities and/or production plants to satisfy the needs of their customers, or to reach the destinations of their products, is a critical issue for any business. When facilities are located well the business can grow and stand firm in the increasingly competitive business world. This problem of locating plants and facilities has been researched under the names of the plant, facility, or warehouse location problem and these names all refer to the same principles and are interchangeable.

Plant location studies formally started in 1909 when Alfred Weber investigated the problem of how to position a single warehouse so as to minimize the total distance between the warehouse and several customers, but workable and realistic models and algorithms began to emerge only in the mid-1960s with the arrival of computers (ReVelle & Laporte, 1996). Since then, several models and algorithms have been developed to solve specific location problems. Many of these involve Lagrangian relaxation in either a heuristic or an exact algorithm and the aim of this thesis is to analyse a plant location problem that models a useful scenario. The aim of the analysis is to theoretically and computationally compare the bounds from several Lagrangian relaxations to show the strength of the resulting bounds on the optimal solution. As Lagrangian relaxation is a widely used technique for problems such as this, this information is extremely useful when designing a solution strategy.

The current research into plant location problems is varied and extensive. The model involves choosing the best locations for facilities. Given a set of potential locations for facilities and a set of customers, the plant location problem is to locate facilities in such a way that the total cost for assigning customers to facilities and satisfying the service (or demand) required by customers is minimized. The cost considered is the sum of the fixed costs of opening facilities and the costs for assigning customers to specific facilities which depend on, for example, the distance between them. The facility location problem can be classified into different categories depending on the restrictions assumed. In the *Uncapacitated* or *Simple Plant Location Problem* (SPLP) each facility is assumed to have no limits on its capacity. Here, each customer receives all the required service or demand from one facility. When each facility has a limited capacity the problem is called the *Capacitated Plant Location Problem* (CPLP). This problem is then broken down into other sub-categories such as the *Capacitated Plant Location Problem with Single-Source constraints* (CPLPSS) where each customer must receive their demand from one open facility, as opposed to receiving their total demand in smaller fractions from two or more open depots.

Agar & Salhi (1998) provide many applications of plant location problems across all sectors of society. The problem of choosing the location of facilities in order to serve a set of customers at minimum cost can be encountered in the public sector (libraries/health centres), in the private sector (factories/depots), or in managing the environment (waste disposal/chemical factories).

The plant location models mentioned above, such as the SPLP and CPLP, all involve two decision levels. In the first level the decision to be made is the choice of the subset of facilities or plants to open. In the second level the decision is which customers are assigned to the chosen subset of plants. These type of formulations have received significant attention. The next stage from here is to add a third decision level. The problem under consideration in this thesis is the *Two-Stage Capacitated Plant Location Problem* (TSCPLP) where there are in fact three stages, but potentially more than three decision levels. The first or upper-most stage is the production plants, where the decision to be made is the choice of which plants to open, the second or central stage is the distribution depots and the decision here is which subset of depots to open. The third stage is the customers and the decisions to be made here are to assign customers to open depots, and therefore open plants, to satisfy their service or demand requirements. Included in this is also the decision of the flow of product from the plants to the depots. Overall the solution to the

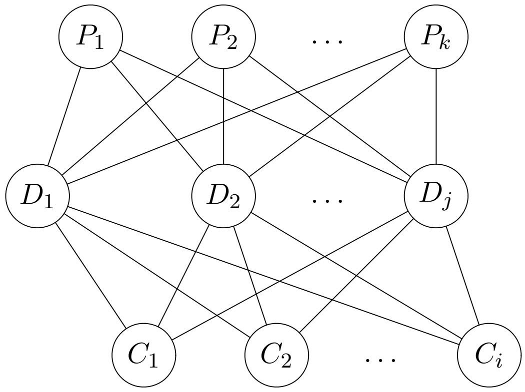

Figure 1.1: The structure of a two-stage capacitated plant location problem with P_k plants, D_j depots, and C_i customers.

TSCPLP defines which plants and depots are opened and the flow of demand through the system from plants–depots–customers. This is known as a two-level or three-echelon structure, whereas the CPLP has a one-level/two-echelon structure. The structure of the system being considered is best demonstrated in Figure 1.1.

The next section introduces the mathematical formulation of the CPLP and the TSCPLP.

1.2 Formulations

There are many location systems that can be modelled with a two-level structure. Manufacturing plants or factories produce a product that is then shipped to an intermediate facility (a regional storage depot or distribution centre) where the product is shipped in smaller quantities directly to the customers. The Two-Stage Capacitated Plant Location Problem is to decide which plants and depots to open from a given set of potential plant and depot sites, which customers to assign to the open depots and to determine the product flow from the plants to the depots. The objective is to minimize the total distribution cost under the constraints that the demand of all customers has to be met and that the capacities of the plants and depots must not be violated. The total cost of the system consists of the cost of shipping the product from the plants to the depots, the cost of satisfying the customers' demand from the depots, the cost of throughput at each depot and the fixed costs of maintaining and/or establishing the plants and depots.

The three levels in the system could take many forms. The upper-most level could be a manufacturing plant producing cars, timber, or appliances; the central stage could be a local car dealership, a hardware centre, or department stores; and the customers could be businesses with a fleet of cars, or physical people who take deliveries or travel to the stores to buy goods. The cost of supplying demand in the latter cases would include the costs incurred by the customers to reach the store. The location of such stores would need to consider the target market for their product and the travelling distance required to reach their store as opposed to a competitor's store. The applications of this problem appear in all aspects of business.

In this thesis the background of the TSCPLP will be given in terms of the CPLP since this is where most of the research in this area has been focused. The analysis and solution techniques of the CPLP provide a good background for work on two-stage problems. Formulations for both these problems are now given, starting with the standard CPLP with n potential plants and m customers, as a mixed-integer linear program (MILP). Firstly define:

$I = \{1, \ldots, m\}$ as the set of customers/demand nodes

$J = \{1, \ldots, n\}$ as the set of potential locations for plants

$c_{ij} =$ total transportation cost from plant j to serve customer i

$d_i =$ demand of cutomer i

$s_j =$ capacity of plant j

$f_j =$ fixed cost associated with operating plant j

$x_{ij} =$ the fraction of customer i's demand supplied by plant$j$

$$y_j = \begin{cases} 0, & \text{if plant } j \text{ is not opened} \\ 1, & \text{if plant } j \text{ is opened} \end{cases}$$

Then the objective is:

$$\text{Minimize} \quad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} f_j y_j \tag{1.1}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad (i = 1, \ldots, m) \tag{1.2}$$

$$\sum_{i=1}^{m} d_i x_{ij} \leqslant s_j y_j \qquad (j = 1, \ldots, n) \tag{1.3}$$

$$0 \leqslant x_{ij} \leqslant y_j \leqslant 1 \qquad (i = 1, \ldots, m; j = 1, \ldots, n) \tag{1.4}$$

$$y_j \in \{0, 1\} \qquad (j = 1, \ldots, n) \tag{1.5}$$

The objective function (1.1) minimizes the sum of the transportation costs of meeting the demand of the customers and the fixed costs for opening plants. Demand constraints (1.2) ensure that all customers demand is satisfied. Constraints (1.3) ensure that the total demand of the customers assigned to a plant does not exceed its maximum capacity. The last two sets of constraints are the non-negativity, upper bound and integrality constraints. Constraints (1.4) also say that the customer's demand can only be satisfied by an open plant.

The Two-Stage Capacitated Plant Location Problem (TSCPLP) can also be formulated as a mixed-integer linear program. The connection between the two formulations can be seen since the constraints of the CPLP remain intact in the TSCPLP formulation and there are common terms between the two objective functions.
Let:

$I = \{1, \ldots, m\}$ be the set of customers,
$J = \{1, \ldots, n\}$ be the set of potential depot locations,
$K = \{1, \ldots, p\}$ be the set of potential plant locations,
$c_{ij} =$ total cost of transportation from depot j to serve customer $i, \forall i \in I, \forall j \in J$,
$f_j =$ fixed cost associated with depot $j, \forall j \in J$,
$g_k =$ fixed cost associated with plant $k, \forall k \in K$,
$b_{kj} =$ unit cost of transportation from plant k to depot $j, \forall k \in K, \forall j \in J$,
$d_i =$ demand of cutomer $i, \forall i \in I$,
$s_j =$ capacity of depot $j, \forall j \in J$,
$a_k =$ capacity of plant $k, \forall k \in K$,

The decision variables are defined as:

$x_{ij} =$ fraction of the demand of customer i supplied from depot $j, \forall i \in I, \forall j \in J$,

$$y_j = \begin{cases} 0, & \text{if depot } j \text{ is closed} \\ 1, & \text{if depot } j \text{ is open } \forall i \in I, \forall j \in J \end{cases}$$

$w_{kj} =$ units of demand transported from plant k to depot $j, \forall k \in K, \forall j \in J$,

$$z_k = \begin{cases} 0, & \text{if plant } k \text{ is closed} \\ 1, & \text{if plant } k \text{ is open } \forall k \in K. \end{cases}$$

The problem can now be stated as:

(P1)

$$\text{Minimize} \qquad Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj}w_{kj} \tag{1.6}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \in I \tag{1.7}$$

$$\sum_{i=1}^{m} d_i x_{ij} \leqslant s_j y_j \qquad \forall j \in J \tag{1.8}$$

$$x_{ij} \leqslant y_j \qquad \forall i \in I, \forall j \in J \tag{1.9}$$

$$\sum_{j=1}^{n} w_{kj} \leqslant a_k z_k \qquad \forall k \in K \tag{1.10}$$

$$\sum_{k=1}^{p} w_{kj} = \sum_{i=1}^{m} d_i x_{ij} \qquad \forall j \in J \tag{1.11}$$

$$w_{kj} \geqslant 0 \qquad \forall k \in K, \forall j \in J \tag{1.12}$$

$$y_j \text{ integer} \qquad \forall j \in J \tag{1.13}$$

$$z_k \text{ integer} \qquad \forall k \in K \tag{1.14}$$

$$0 \leqslant x_{ij} \leqslant 1 \qquad \forall i \in I, \forall j \in J \tag{1.15}$$

$$0 \leqslant y_j \leqslant 1 \qquad \forall j \in J \tag{1.16}$$

$$0 \leqslant z_k \leqslant 1 \qquad \forall k \in K \tag{1.17}$$

The objective function (1.6) minimizes the fixed costs of opening both plants and depots, and the transportation costs of moving demand from plants to depots and from depots to customers.

Constraints (1.7) state that each customer's demand must be fully met by the depots. Constraints (1.8) guarantee that open depots do not supply more than their capacity, i.e., that for each depot the sum of the demand of the customers it is supplying is less than or equal to its capacity. Constraints (1.9) ensure that customers are only served from open depots. Constraints (1.10) guarantee that open plants do not supply more than their capacity, i.e., that for each plant the sum of the demand leaving it is less than or equal to the capacity it can hold.

Constraints (1.11) are conservation of flow constraints for the depots. That is, that for each depot the amount of demand entering the depot from the plants is equal to the demand leaving the depot to be transported to the customers. Constraints (1.12) are non-negativity constraints on the amount of demand transported from plants to depots. Constraints (1.13) and (1.14) are integrality constraints on both plants and depots. Constraints (1.15), (1.16), and (1.17) are non-negativity and simple upper bound constraints also restricting the fractional values of customer demand to be less than 1.

This formulation will be presented again in Chapter 3 for the theoretical analysis of possible Lagrangian relaxations.

1.3 Outline of thesis

In Chapter 2 the literature on the Two-Stage Capacitated Plant Location Problem is reviewed. Firstly some of the formulations that have been covered in the literature and the differences arising from including problem characteristics such as single-source constraints and multi-commodity capabilities are discussed. The review also covers solution methods applied to two-stage problems including heuristics, such as Lagrangian heuristics and primal heuristics, and exact methods including the branch-and-bound method. The multi-period and multi-product two-stage capacitated plant location problems are also presented. This review will place the current research in context and identify potential areas where the research of this thesis can be applied.

Chapters 3 and 4 provide a new theoretical formulation of the two-stage capacitated problem that has not been studied before where customers can be served from any open depot, and there are explicit capacities placed on both plants and depots. This is what has been lacking in previous formulations of the problem. The non-single-source problem is interesting to investigate since practical applications of this problem do exist in business. Consider, for example, a customer who requires an order of timber and one supplier can only fill 75% of their demand. Then they may receive the further 25% from another supplier, hence a non-single source problem is being modelled.

Several theorems will be presented that classify bounds on the problem showing that the bounds generated from the solution of particular Lagrangian relaxations are 'tighter' or closer to the optimal solution than others. Chapter 3 deals with *trivial* bounds — those that are equivalent to the linear programming bound, and Chapter 4 categorises bounds into *equivalent* and *dominant* classes. Equivalent bounds are defined as pairs of Lagrangian dual bounds that give the same lower bound value. Dominant bounds are defined as $A < B$ where for any given problem instance A is less than or equal to B, but for some problem instance A is strictly less than B. This theoretical analysis is then confirmed computationally in Chapter 5.

Chapter 5 will computationally analyse the relative strengths of some of the possible Lagrangian relaxations of the problem and corroborate the results found in Chapters 3 and 4. A solution method in the form of a Lagrangian heuristic is provided and a computer code written, to solve the problem which utilises the commercial software CPLEX. Surrogate constraints are included in the formulation that strengthen the lower bound values, ensuring that at each lower bound solution there is a set of plants and depots open that can satisfy the customer demand. It is shown that certain relaxations of the problem give stronger bounds that, when used in a Lagrangian relaxation scheme, converge to the optimal solution quicker than others and can be appropriate for solving both large and small scale problems. The solution method most suitable for solving a given set of problem data is recommended. An analysis of the effect on solution quality of the strength of the capacity constraints is included. This scales the ratio of total depot capacity to customer demand ($\frac{\sum_j s_j}{\sum_i d_i}$) and the ratio of total plant capacity to total depot capacity ($\frac{\sum_k a_k}{\sum_j s_j}$) to be 1.5, 3, 5, or 10.

In Chapter 6 the Two-Stage Capacitated Plant Location Problem with Single-Source constraints (TSCPLPSS) is analysed. The formulation, based on that of the TSCPLP, is presented as a mixed-integer linear program. A theoretical and computational analysis is undertaken, following the format of Chapter 5. A Lagrangian heuristic is used to compare the bounds from several Lagrangian relaxations of the TSCPLPSS and the investigation into adjusting depot capacity to customer demand and plant capacity to depot capacity ratios is again completed.

Finally in Chapter 7 some concluding remarks are made and consequences or areas for further extension of this research are presented.

Chapter 2

Literature survey

2.1 Introduction

In this chapter the literature on the Two-Stage Capacitated Plant Location Problem (TSCPLP) is discussed. A formulation for this problem is given in Chapters 1, 3, 4 and 5. The literature review aims to place the current research in context with previous work and to identify areas in the literature that are lacking in, or could benefit from, the analysis provided by this thesis. Part of this chapter is focused on the Capacitated Plant Location Problem (CPLP). As the area of two-stage models is relatively new, the solution methods most readily available for the TSCPLP are those methods developed for the CPLP. The TSCPLP is a transformation or extension of the CPLP, so the literature is reviewed in this context. The solution methods that have been applied to solving the CPLP that have been, or could be, adapted for solving the TSCPLP are presented in Sections 2.4 and 2.5. These methods include both heuristics and exact algorithms. The area of Lagrangian relaxation and subgradient optimization that is used in the heuristic presented in Chapter 5 is reviewed in Section 2.3. Several theoretical results are presented that will be used in the theoretical analysis of Chapters 3 and 4. This chapter also includes a survey of a small area of literature focused on multi-commodity and multi-period plant location problems, in Sections 2.7 and 2.8, that carry the two-stage structure also. These are also extensions of the CPLP, and the outcomes of the analysis of this thesis can be applied, in part, to solving these complex problems.

2.2 Plant location problems and their variations

Several variations on the CPLP have been studied in the literature. These are discussed here to provide a background and context for the TSCPLP. One variation that is often included in the literature is to add a surrogate constraint, which can be used to strengthen the original formulation or to replace an existing constraint. Thizy (1994) formulated a Facility Location Problem with Aggregate Capacity where a constraint is added to force enough facilities to be open so that together they satisfy the overall customer demand. It is formulated as (P1) in Chapter 1 but (1.8) is replaced by

$$\sum_{i=1} d_i \leqslant \sum_{j} s_j y_j \tag{2.1}$$

This is also presented in Cornejouls *et al.* (1991). This constraint is actually derived by summing (1.8) over all the plants and using the equalites in (1.7). It is redundant in the original formulation of (P1) but has been shown to strengthen some of the relaxations (Sridharan, 1995). This concept is also applied to the formulation of TSCPLP in this thesis starting in Chapter 3, with the aim of strengthening the bounds resulting from Lagrangian relaxations.

Several researchers including Sridharan (1993), Hindi & Pienkosz (1999), Klose (1999), Rönnqvist *et al.* (1999), Delmaire *et al.* (1999), Holmberg *et al.* (1999), and Scaparra (2001) studied the *Capacitated Plant Location Problem with Single-Source Constraints* (CPLPSS). This problem is characterized by the addition of the restriction that each customer is served by a single facility, rather than receiving fractional quantities of their demand from two or more facilities, which in practice may allow a closer scrutiny of supply and demand. The problem is given below:

$$\text{Minimize} \quad Z(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j \tag{2.2}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad (i = 1, \ldots, m) \tag{2.3}$$

$$\sum_{i=1}^{m} d_i x_{ij} \leqslant s_j y_j \qquad (j = 1, \ldots, n) \tag{2.4}$$

$$x_{ij} \leqslant y_j \qquad (i = 1, \ldots, m; j = 1, \ldots, n) \tag{2.5}$$

$$y_j \in \{0, 1\} \qquad (j = 1, \ldots, n) \tag{2.6}$$

$$x_{ij} \in \{0, 1\} \qquad (i = 1, \ldots, m; j = 1, \ldots, n) \tag{2.7}$$

All coefficients and variables have the same definition as in Chapter 1 except for

$$x_{ij} = \begin{cases} 1 \text{ if customer } i \text{ is assigned to plant } j \\ 0 \text{ otherwise, } \forall i \in I, \forall j \in J \end{cases} \tag{2.8}$$

This problem has received much attention in the literature and most of the research into two-stage models, which is the focus of this thesis, has been into single-source models. The formulation presented in Chapter 3 however is not a single-source model, making this formulation unique. The non-single-source case is an interesting problem that exists in real-life and deserves the same attention as single-source models. The solution technique applied to the TSCPLP will also be applied to the single-source version of the formulation in Chapter 6, to cover the analysis of both models.

The decisions associated with plant location are often very strategic and frequently involve large outlays of capital and long-term planning horizons. They impact on not only the owners or providers of the plant/facility but also their users and neighbours, considering that several types of plants or facilities to be located could have expected lifetimes of 20 to 50 years (Current *et al.*, 2002). Consequently plant location decisions often involve many stake holders and can consider multiple, often conflicting, objectives. These *Multi-Objective Plant Location Problems* have been studied by Jayaraman (1999) whose objectives were to minimize the fixed costs of opening facilities, minimize the total variable costs for satisfying demands, and to minimize the average response time or distance for serving demand nodes. Fernández & Puerto (2002) also discuss several multi-objective models. A bi-objective problem is presented by Myung *et al.* (1997) where one objective is to maximize the net profit and the other to maximize the profitability of the investment. These extensions of plant location models are difficult to solve, however well they may model a real-life accounting scenario. Jayaraman (1999) recognises that the objectives are often in conflict and no one solution exists that is optimal for all of the objectives. These types of models for plant location problems are an interesting extension but cannot necessarily be generalized to a model such as the TSCPLP.

Marin & Pelegrin (1997) present an interesting extension to the CPLP called the *Return Plant Location Problem* where there exists a return product from each customer in a system of suppliers and customers. This return product might be a transformed version of the product, the product returned after use or renting, containers such as reusable bottles, bundles of recyclable material, defective items of the product or dangerous residues, among many others. The formulation is very similar to a CPLP. The objective function to be minimized is the sum of the opening costs of the plants, the supply costs for the primary product and the return costs for the secondary product. Constraints on the problem guarantee that:

- the demand for the first product by each customer will be satisfied in any feasible solution of the problem
- the prescribed amount of the secondary product sent by the customers will be returned to the open plants
- the demand for the primary or secondary product will only be assigned to open plants
- all plants receive the same prescribed ratio of the total amount of the secondary product returned from the customers with respect to the primary product demand that is supplied.

The solutions were found using a Lagrangian decomposition method that is incorporated into a heuristic and a branch-and-bound method.

When a plant produces more than one product or commodity we have the *Multi-Commodity Plant Location Problem* which can be capacitated or uncapacitated. Customers in the system have a distinct demand for each of the multiple products that must be satisfied. A plant maybe able to handle only one type of product, or many. This concept can also be combined with the TSCPLP to give the *Multi-Commodity, Two-Stage Capacitated Plant Location Problem.* This extension will be discussed in more detail in sections 2.7 and 2.8.

All of the problems described so far have been concerned with a single time period but when a location decision needs to consider the effect time has on the problem parameters, the problem is a dynamic model. They are stated as *Dynamic location problems* or *Multi-period plant location problems* and have been addressed by Klincewicz *et al.* (1988), Hormozi & Khumawala (1996), Canel & Khumawala (1997), Pirkul & Jayaraman (1998), Canel & Das (1999), Hinjosa *et al.* (2000), and Canel *et al.* (2001). Multi-period modelling allows for variation in pricing, demand, transportation costs, and production costs, with time. It can also be used to incorporate trends, seasonality, and other time-related factors into the problem formulation. This allows for sensitivity analysis on these factors for further insight. The multi-period plant location problem involves decisions on how many facilities to open, where to locate them, and how to allocate the production of each facility, in each period, to a list of known demand centers. These decisions are made for a predetermined planning horizon in response to expected changes in demand from customers. These problems provide useful solutions to common business scenarios, but are difficult to solve due to the variability of the problem parameters across the planning horizon. Such problems can become very complex with only a minor change to a problem characteristic. They are usually solved using the technique of dynamic programming although many of the authors employ Lagrangian relaxation in finding bounds at various stages of their method. The analysis in this thesis of a static model for the TSCPLP could therefore be utilised in a multi-period model at any given time period.

A formulation for the multi-commodity uncapacitated plant location problem is given by Klincewicz & Luss (1987). It is a single-stage, uncapacitated model but the principle can be extended to a capacitated problem and a two-stage model. It is presented here to illustrate the concept of a multi-commodity model on a simple scale, and to show the effect on the formulation of allowing for demand of multiple products. In general, an extra index is placed on the decision variables; to separate out the demand for differing products or services.

Define the sets:

$$
\begin{aligned}
&I = \{1, \dots, m\} \text{ as the set of customers/demand nodes,}\\
&J = \{1, \dots, n\} \text{ as the set of potential locations of plants,}\\
&K = \{1, \dots, p\} \text{ as the set of possible products,}
\end{aligned}
$$

The cost coefficients are defined as:

$$
\begin{aligned}
c_{ijk} = \;& \text{the assignment cost for serving the requirements of customer } i\\
& \quad \text{for product } k \text{ from plant } j,\\
f_j = \;& \text{fixed cost associated with operating plant } j,\\
E_{jk} = \;& \text{the fixed setup cost for equipping location } j \text{ to handle product } k;
\end{aligned}
$$

The decision variables are defined as:

$$z_{jk} = \begin{cases} 1 \text{ if equipment for product } k \text{ is installed at plant} j \\ 0 \text{ if not} \end{cases}$$

$$x_{ij} = \text{ fraction of the demand for product } k \text{ by customer } i \text{ that is provided by plant} j$$

$$y_j = \begin{cases} 0 \text{ if plant } j \text{ is not opened} \\ 1, \text{ if plant } j \text{ is opened} \end{cases}$$

Then the problem is to:

$$\text{Minimize} \qquad \sum_j f_j y_j + \sum_j \sum_k E_{jk} z_{jk} + \sum_i \sum_j \sum_k c_{ijk} x_{ijk} \tag{2.9}$$

Subject to:

$$\sum_j x_{ijk} = 1 \qquad \forall i, j, k \tag{2.10}$$

$$z_{jk} \leqslant y_j \qquad \forall j, k \tag{2.11}$$

$$0 \leqslant x_{ijk} \leqslant z_{jk} \qquad \forall i, j, k \tag{2.12}$$

$$z_{jk}, yj \in \{0, 1\} \qquad \forall j, k \tag{2.13}$$

The first set of constraints (2.10) ensure that every customer's demand is satisfied. The second set of constraints (2.11) ensures that only open facilities are equipped to handle products. The third constraint set (2.12) ensures that a customer is assigned their demand for a given product only from a facility that handles that product.

One of the most complex variations of a plant location problem incorporates all of the above concepts to give the *Multi-Period, Multi-Commodity, Two-Stage Capacitated Plant Location Problem.* There is little literature on this problem but it will be discussed later in the chapter. The formulations are very problem specific rather than generalized, however they provide a context for the current work.

This concludes the review of the literature on the formulation of plant location models. The focus now moves to reviewing the solution methods that have been applied to plant location problems. Solution techniques for these problems, and for the CPLP and TSCPLP in particular, can be classified into two groups; *heuristic* and *exact.* A heuristic can be defined as a rule of thumb, simplification, or educated guess that reduces or limits the search for solutions in domains that are difficult and poorly understood. Unlike exact algorithms, heuristics do not guarantee optimal or even feasible solutions (in the case of dual heuristics) and are often used with no theoretical guarantee (On-Line Dictionary of Computing, 2007). Heuristics generally are quicker in solving the problem than exact algorithms, so there is a trade off in accuracy of results for solution time.

The most common solution method of either type computes both lower and upper bounds on the objective function solution. Lower bounds are usually found using a Linear Programming or Lagrangian relaxation technique. In this situation the upper bounding procedure defines whether the method is classified as heuristic or exact. If the problem is of polynomial complexity then an exact method can be defined, normally utilising a branch-and-bound technique. If the problem is classified as NP-hard or NP-complete then a heuristic must be used to find an upper bound. These upper bounds are iteratively adjusted until some criteria are reached: usually a suitably small gap between the upper and lower bounds, or an iteration limit. The details of these methods are given in later sections.

Another solution technique called Dynamic Programming is used to solve the multi-period problems outlined in the previous section. Dynamic programming involves working backward from the end of a problem toward the beginning; with the aim of breaking up a large problem into a series of smaller, more solvable problems. This method will be briefly discussed in Section 2.8.

The next sections of this chapter give details of solution techniques applied specifically to the CPLP and the TSCPLP, and are separated into the two categories mentioned: heuristics and exact methods.

2.3 Lagrangian relaxation and subgradient optimization

Lagrangian relaxation and subgradient optimization are important tools in generating solutions to combinatorial optimization problems. The method of solving the TSCPLP in this thesis incorporates Lagrangian relaxation into a heuristic. This section addresses the review of Lagrangian relaxation and subgradient optimization and their application to location problems. Several theoretical results on Lagrangian relaxation will be presented that will be utilised in the theoretical results of Chapters 3.

Fisher (1981) observed that one of the most computationally useful ideas of the 1970s is the observation that many hard combinatorial optimization problems can be viewed as easy problems complicated by a relatively small set of side constraints. Dualizing the side constraints produces a Lagrangian problem that is easy to solve and whose optimal value is a lower bound (for minimization problems) on the optimal value of the original problem. The "birth" of the Lagrangian approach as it is today occurred in 1970 when Held & Karp used a Lagrangian problem based on minimum spanning trees to devise a dramatically successful algorithm for the travelling salesman problem (Fisher, 1981). Motivated by Held and Karp's success, Lagrangian methods were applied in the early 1970s to scheduling problems and the general integer programming problem. By 1974 Lagrangian methods had gained considerable standing when Geoffrion gave the approach the name "Lagrangian relaxation". Fisher (1981) mentioned that since then the list of applications of Lagrangian relaxation had grown to include over a dozen of the most infamous combinatorial optimization problems. And:

> "for most of these problems, Lagrangian relaxation has provided the best existing algorithm for the problem and has enabled the solution of problems of practical size."

In order to illustrate the concept of Lagrangian relaxation, consider the following general integer program (P) in matrix form:

$$\text{Minimize} \quad cx \tag{2.14}$$

Subject to:

$$Ax = b \tag{2.15}$$

$$Dx \leqslant e \tag{2.16}$$

$$x \in \{0,1\} \tag{2.17}$$

A lower bound for the above program can be found by introducing a Lagrange multiplier vector $u = (u_1, \ldots, u_m)$ for the first constraint to get the Lagrangian lower bound program or Lagrangian relaxation (Nemhauser & Wolsey, 1988). This is often denoted by LR_u and is given by:

$$Z_D(u) = \min cx + u(Ax - b) \tag{2.18}$$

Subject to:

$$Dx \leqslant e \tag{2.19}$$

$$x \in \{0,1\} \tag{2.20}$$

Where the subscript D is referring to the constraint that was relaxed, or dualized. Now the Lagrangian dual Z_D is defined to be

$$Z_D = \max_u Z_D(u).$$

It is clear that LR_u can be easily solved to give a solution X with a corresponding lower bound given by $cx + u(Ax - b)$. The best choice of u gives the optimal solution to the Lagrangian dual problem Z_D.

Theorem 2.3.1 $Z_D(u) \leqslant Z$.

Proof Fisher (1981) provides a validation for this by assuming an optimal solution x^* to (P) and observing that

$$Z_D(u) \leqslant cx^* + u(Ax^* - b) = Z$$

The inequality in this relation follows from the definition of $Z_D(u)$ and the equality from $Z = cx^*$ and $Ax^* - b = 0$. If $Ax = b$ is replaced by $Ax \leqslant b$ in (P), then we require $u \geqslant 0$ and the argument becomes

$$Z_D(u) \leqslant cx^* + u(Ax^* - b) \leqslant Z$$

where the second inequality follows from $Z = cx^*, u \geqslant 0$ and $Ax^* - b \leqslant 0$. Similarly for $Ax \geqslant b$ we require $u \leqslant 0$ for $Z_D(u) \leqslant Z$ to hold. ∎

The fact that $Z_D(u) \leqslant Z$ allows LR_u to be used in place of a linear programming relaxation to provide lower bounds in a branch and bound algorithm for (P). Fisher (1981) presents other uses for LR_u — it can be used for selecting branching variables and choosing the next branch to explore in a branch and bound algorithm. Good feasible solutions to (P) can frequently be obtained by perturbing nearly feasible solutions to LR_u. Lagrangian relaxation has also been used as an analytic tool for establishing worst-case bounds on the performance of certain heuristics.

The aim of this procedure is to obtain a Lagrangian relaxation which is easier to solve than the original problem because some special structure in the remaining constraints can be exploited. The selection of a suitable relaxation is one of the important issues to be considered when forming a solution method based on Lagrangian relaxation. Two key factors in the evaluation of a relaxation are its ease of solution and the tightness of the bounds generated. The ease of solution depends on the methods available for solving the

Lagrangian subproblem. The possibility of generating such smaller and easier problems, as compared to the original problem, depends on the structure of the original problem and the degree of separability obtained by relaxing certain constraints. Generally, a relaxation which gives a tighter bound will use greater computation time, whereas an easily solved relaxation problem is likely to give poor bounds (Geoffrion & McBride, 1978).

The main property of the dual problem $Z_D(u)$ is that the dual function is always concave so any local optimal solution is also a global one (Bazaraa & Sherali, 1981). The constraints are just non-negativity constraints on the Lagrange multipliers (or dual variables) associated with the inequality constraints. In the case of an integer formulation, we also have that the dual function is non-differentiable so standard ascent methods based on gradients cannot be used for its solution. The function is nondifferentiable at any $\bar{u}$ where $LR_{\bar{u}}$ has multiple optima. Although it is differentiable almost everywhere, it generally is nondifferentiable at an optimal point. Methods that can take the non-differentiability into account need to be adopted. There are a number of such methods available such as subgradient optimization, steepest ascent, improved subgradient, and the Bundle method. The most commonly used is the subgradient optimization method due to its ease of programming and because it has worked well on many practical problems. Applying subgradient optimization generates a sequence of Lagrange multipliers in an attempt to maximize the lower bound obtained from LR_u. Since it is the most popular method of determining u, it is the method employed in the heuristic of Chapter 5 in this thesis.

The subgradient method for updating the value of u is now given. At the points were LR_u is non-differentiable, the subgradient method chooses arbitrarily from the set of alternative optimal Lagrangian solutions and uses the vector $Ax - b$ for this solution as though it were the gradient of LR_u. Given an initial value u^0, a sequence $\{u^k\}$ is obtained using the rule:

$$u^{k+1} = u^k + t_k(Ax^k - b)$$

where x^k is an optimal solution to LR_{u^k} and t_k is a positive step size. The fundamental theoretical result on the convergence of the subgradient method discussed in Held, Wolfe & Crowder (1974) is that $Z_D(u^k) \to Z_D$ if $t_k \to 0$ and $\sum_{i=0}^{k} t_i \to \infty$. The step size most commonly used is defined as:

$$t_k = \lambda_k(Z^* - Z_D(u^k)) \,/\, (Ax^k - b)^2$$

where $0 < \lambda_k \leqslant 2$ and Z^* is an upper bound on Z_D obtained by using a suitable heuristic or algorithm such as those discussed in Section 2.4.3.

General theoretical results for Lagrangian relaxation are now presented that will be useful for inclusion in the theoretical analysis of the TSCPLP. Firstly, the relationship between Lagrangian relaxations and the linear programming relaxation is defined.

Theorem 2.3.2 $Z_D \geqslant Z_{LP}$

Proof Fisher (1981). This can be established by the following series of relations between optimization problems.

$$Z_D = \max_u \ \{\min_{\ x} \ cx + u(Ax - b)\} \tag{2.21}$$

Subject to:

$$Dx \leqslant e \tag{2.22}$$

$$x \in \{0, 1\} \tag{2.23}$$

$$\geqslant \max_u \ \{\min \ cx + u(Ax - b)\} \tag{2.24}$$

Subject to:

$$Dx \leqslant e \tag{2.25}$$

$$x \geqslant 0 \tag{2.26}$$

$$\text{(By LP duality)} \ = \max_u \ \max_{\nu \geqslant 0} \ \nu e - ub \tag{2.27}$$

Subject to:

$$\nu D \leqslant c + uA \tag{2.28}$$

$$\text{(By LP duality)} \ = \min_x \ cx \tag{2.29}$$

Subject to:

$$Ax = b \tag{2.30}$$

$$Dx \leqslant e \tag{2.31}$$

$$x \geqslant 0 \tag{2.32}$$

$$= Z_{LP} \tag{2.33}$$

$$\tag{2.34}$$

∎

This also reveals a sufficient condition for $Z_D = Z_{LP}$: whenever $Z_D(u)$ is not increased by removing the integrality restriction on x from the constraints of the Lagrangian problem (Fisher, 1981 and Corollary 6.6: Nemhauser & Wolsey, 1988).

Geoffrion (1974) and Nemhauser & Wolsey (1988) present several theoretical results on the definition and solution of Lagrangian relaxations.

Theorem 2.3.3 *The primal linear programming problem of finding a convex combination of points in* $\{x \in \{0,1\} : Dx \leqslant e\} \neq \emptyset$ *that also satisfy the complicating constraint* $Ax = b$ *is dual to the Lagrangian dual.*

Proof See the proof of Theorem 6.2 in Nemhauser & Wolsey (1988).

The above result also links in with Theorem 2.3.2 above.

Theorem 2.3.4

1. $Z_{LP} \leqslant Z_D \leqslant Z$,
2. $Z_D(u) \leqslant Z,\ \forall u \geqslant 0$
3. *If for a given* u *a vector* x *satisfies the three conditions*
   (a) x *is optimal in* LR_u,
   (b) $Ax \leqslant b$,
   (c) $u(b - Ax) = 0$

   then x *is an optimal solution of* (P)*. If* x *satisfies (3a) and (3b) but not (3c), then* x *is an* ϵ*-optimal solution of* (P) *with* $\epsilon = u(Ax - b)$.

Proof See the proof of Theorem 1 in Geoffrion (1974).

The results above record the relationships between a problem (P) and its linear programming relaxation, Z_{LP}; and Lagrangian relaxations Z_D, for various constraints D. A multiplier u yields a Lagrangian relaxation that is at least as good as Z_{LP} and hopefully will be better. Part (3c) of 2.3.4 indicates the conditions under which a solution of a Lagrangian relaxation is also optimal or near-optimal in (P). The position of Z_D in the interval $[Z_{LP}, Z]$ is the question of central concern when analyzing the potential value of a Lagrangian relaxation applied to a particular problem (Geoffrion, 1974).

All of these results are utilised in the formation of the analysis of this thesis. A general result that is required for the theorems in the next chapter follows.

Theorem 2.3.5 $Z_A \geqslant Z_{A,B}$ *where* A *and* B *are sets of constraints dualized in a Lagrangian fashion.*

Proof This result is the obvious relation that a known bound cannot be improved by relaxing a further constraint. This follows from the definition of LR_u when a 'penalty' term is created for each constraint that is dualized. This further constraint could also be redundant, having no effect on the bound, in which case $Z_A = Z_{A,B}$ holds. ∎

This chapter now moves onto discussing one of the biggest areas of application for Lagrangian relaxation — heuristics.

2.4 Heuristics for the TSCPLP

Heuristics have been widely applied to plant location models in the literature due to the combinatorial nature of the problems and the computational effort required to enumerate solutions in an algorithm. This section will focus on the main category — Lagrangian heuristics, and also on the more recent but lesser used applications of Tabu Search and other metaheuristics. A metaheuristic can be defined as a top-level general strategy which guides other heuristics to search for feasible solutions in domains where the task is hard (On-Line Dictionary of Computing, 2007). This section will also outline the heuristics that could be used in the upper bounding procedure of a Lagrangian heuristic to find feasible solutions.

2.4.1 Lagrangian heuristics

Lagrangian heuristics form the main class of heuristics applied to location problems. These are also classified as dual heuristics, as they are concerned mainly with the dual problem formed through the relaxation of some constraint. The solutions are initially infeasible and work towards a feasible solution. Other heuristics such as construction heuristics are classified as primal heuristics as the solutions generated are always feasible and are iteratively improved to attempt optimality.

The basic motivation behind a Lagrangian heuristic is that the information contained in LR_u (from Section 2.3) at each subgradient iteration can be used in an attempt to construct a feasible solution to the original problem through the application of some heuristic. Essentially we generate a sequence of Lagrange multipliers (defining lower bounds) and a sequence of feasible solutions (defining upper bounds). This sequence is not monotonic, since worse (or better) upper and lower bounds can be generated at any stage of the process. Unless a u^k is found that results in $Z_D(u^k) = Z^*$, there is no way of proving optimality in the subgradient method. Therefore this process is repeated until the gap between the best objective function value of the feasible solutions, Z^*, and the largest lower bound, $Z_D(u^k)$, is acceptably small, or a specified iteration limit has been reached. At the end of the Lagrangian heuristic, the best feasible solution found is a heuristic solution to the original problem.

The combination of Lagrangian relaxation with subgradient optimization has been adopted by researchers in this field for over 20 years, beginning with A. Geoffrion in 1978, using the concept of Lagrangian multipliers first introduced by J. L. Lagrange in 1788. Since one of the most important issues to consider when using Lagrangian relaxation is the choice of a suitable relaxation, it makes sense to have some prior knowledge of the nature and strength of possible Lagrangian relaxations on the problem, or on a problem that has a similar structure. The analysis of this thesis aims to provide this knowledge for the TSCPLP, which could also be applied to related problems.

The rest of this section will outline some of the application of Lagrangian heuristics in the literature specifically to the CPLP and its variations. Cornejouls *et al.* (1991) compared Lagrangian relaxations that had been studied in the literature. These included relaxations of the demand constraints, capacity constraints, or both. Their comparison was based on new theoretical and computational results. Dominance relationships between the various relaxations found in the literature were identified and were tested using a known data set. Several of these relaxations could be used to generate heuristic feasible solutions. The theoretical analysis of Chapters 3 and 4 aims to achieve a similar result for the TSCPLP, identifying which Lagrangian relaxations are likely to give strong bounds.

Agar & Salhi (1998) applied Lagrangian heuristics to a variety of location problems including the single-source multi-capacitated plant location problem which was claimed to have not been addressed in the literature. This problem had the added possibility of a choice for the capacity of plants at a given site. Their heuristics were based on a relaxation of the demand and the capacity constraints. Sridharan (1991) developed a Lagrangian heuristic for the capacitated plant location problem with side constraints, where there is an upper and lower limit on the number of open plants. Lower bounds were obtained by solving a Lagrangian relaxation of the problem where the demand constraints are relaxed, which reduces to continuous knapsack problems. Sridharan (1993) also discusses a Lagrangian heuristic for the capacitated plant location problem with single source constraints. Two relaxations were presented based on the capacity and the demand constraints. It was noted that many combinatorial problems can be derived as special cases of the problem covered, and that the approach used could be used to solve those problems as well. It is anticipated that the results of the analysis of the TSCPLP could be applied to other related combinatorial problems in the same manner.

Hindi & Pienkosz (1999) address large scale, single-source capacitated plant location problems. Again the demand constraints were chosen to be relaxed. Klose (1995) applied Lagrangian relaxation, cross decomposition, Dantzig-Wolfe decomposition and a reassignment heuristic to the two-stage capacitated facility location problem. In a more recent paper (Klose, 2000) a Lagrangian-relax-and-cut approach is taken, based on a branch-and-bound method. The capacity constraints on both stages are relaxed and again reassignment procedures are applied.

Tragantalerngsak *et al.* (1997) applied Lagrangian heuristics to the two-echelon, single-source, capacitated facility location problem. They present six relaxations based on relaxing various combinations of the constraints. Feasible solutions are found by solving a generalized assignment problem. This forms somewhat of a computational analysis of the relaxations for the TSCPLPSS, however the formulation presented is lacking in some important features. Here the movement of demand through the two stages does not remain as two distinct movements, since the decision variable for satisfying customer demand is x_{ijk} which equals 1 if facility i served by depot k services customer j, and 0 otherwise. The depots in the upper-most level do not have an explicit capacity so there is no capacity constraint included in the formulation for this stage. The formulation is also purely an integer program and is a single-source model. This thesis aims to provide an alternate formulation that accommodates the capacity restriction that would almost certainly exist on the upper-most stage of the system. The analysis includes a theoretical component as well as a computational analysis of both the single-source case (the TSCPLPSS) and the case where fractional demand allocation is permitted (the TSCPLP).

2.4.2 Tabu Search and other metaheuristics

This section briefly covers another category of heuristics, metaheuristics, that have been applied to plant location problems. The most commonly used is Tabu Search, introduced by Glover (1989), which is a method for intelligent problem solving. Its power is based on the use of adaptive memory to record historical information for guiding the search process. The simple form of Tabu Search uses only the short-term memory component. It is a form of aggressive exploration that seeks to make smart moves through the feasible solution set, subject to the constraints implied by Tabu restrictions. The primary goal of such constraints is to go beyond points of local optimality. Long-term memory is also used for intensification and diversification purposes which are two highly important components of Tabu Search. Intensification strategies exploit features historically found to be useful, while diversification strategies encourage the search process to explore unvisited regions. The use of longer term memory for intensification and diversification purposes makes the Tabu Search methodology significantly stronger than regular heuristics.

Metaheuristics are difficult to implement in practice and little literature is available on metaheuristics for plant location problems. Delmaire *et al.* (1998) applied Reactive GRASP (Greedy Randomized Adaptive Search) and Tabu Search-based metaheuristics to the single-source capacitated plant location problem. GRASP is a metaheuristic proposed by Feo and Resende (1995) with an iterative method that, within each iteration, contains two phases. The first phase builds a solution from scratch, while the second phase is a local search that tries to improve the solution in the first stage. Reactive GRASP self-tunes the parameters of the problem in an attempt to achieve higher performance. The authors presented a Reactive GRASP heuristic, a Tabu Search heuristic, and two approaches that combined elements of the GRASP and Tabu Search methodologies. The Tabu Search metaheuristic uses the GRASP methodology as a diversification mechanism in the first phase and the second phase is an intensification phase. The results showed that the proposed methods were very efficient, however extending this concept to a two-stage model such as the TSCPLP would expand the complexity of the method significantly.

2.4.3 Primal heuristics

This section covers another category of heuristics, known as primal heuristics, which are most often used to calculate and update upper bounds during a Lagrangian heuristic. The details of these heuristics will not be given as most are problem specific. In a Lagrangian heuristic the information provided by the solution to the Lagrangian lower bound problem is used to construct a feasible solution through the application of one of these heuristics.

Authors in the literature have found upper bounds on Lagrangian subproblems via the following techniques or heuristics:

- Greedy heuristic
- Exchange heuristic
- Matching heuristic
- Assignment heuristic

- Interchange heuristic
- Solving a network flow problem
- Solving a transportation problem
- Solving a knapsack problem

Some researchers have chosen to use some of the above techniques outright instead of working through the Lagrangian heuristic scheme. Rönnqvist *et al.* (1999) uses a repeated matching heuristic for the single source capacitated plant location problem. The approach is based on a repeated matching algorithm which essentially solves a series of matching problems until certain convergence criteria are satisfied. The method generates feasible solutions in each iteration. This method would be potentially very difficult to implement on a two-stage problem due to the inter-connectedness of the two-stages.

Scaparra (2001) presents a multi-exchange heuristic for the single source case of the CPLP also. The neighbourhood structure proposed relies on the exchange of sets of customers among facilities along special paths or cycles detected in an improvement graph. Starting with an initial feasible solution, each multi-exchange allows the replacement of the current solution with an improved solution in its neighbourhood until some termination criteria is satisfied. Due to the large scale of the neighbourhood structure, an improved solution is computed heuristically by exploring the neighbourhood space through efficient network flow based improvement algorithms. Again, this method is likely to be nontransferable to a two-stage model as it relies on the separability of the stages.

The focus now moves to the literature presenting methods that find exact solutions to plant location problems.

2.5 Exact methods for the TSCPLP

The solution methods discussed in the previous section have all been approximation or heuristic methods that cannot guarantee that the optimal solution has been found. The next methods are classified as *exact* in that they find the optimal solution. They usually take much longer computational time to find their solutions. The exact methods that have been researched for plant location problems often involve some heuristic steps incorporated into a traditional exact method such as the branch-and-bound method. Branch-and-bound methods find the optimal solution to an integer program by efficiently enumerating the points in a subproblem's feasible region often using linear programming relaxations of the integer constraints (Winston, 1994).

Therefore the branch-and-bound method is classified as an enumeration algorithm. A relaxation of the problem P is solved at each node of the enumeration tree. When the relaxed problem meets the constraints of P, we have solved P; otherwise we obtain a lower bound for P. The relaxations are usually a Linear Programming (LP) or a Lagrangian relaxation. The linear programming relaxation is formed by relaxing the integer constraints on the variables to reduce the integer or mixed-integer program to a linear program.

Dual ascent and adjustment procedures can be inserted into a branch-and-bound framework, in order to produce the exact solution. The dual ascent procedure is used to solve the subproblem in each node of the branch-and-bound tree, yielding lower bounds on the optimal objective function value, but often also primal solutions. Branches are cut off when the lower bound exceeds the best upper bound known. Since, in the worst case, the branch-and-bound method will enumerate all z-solutions, the optimum will be found within a finite number of steps by the method.

Decomposition methods have been applied both as stand alone methods and as part of a branch-and-bound method. The idea of such methods is to decompose a mathematical program into two or more sets of variables and associated constraints. The purpose is to separate some portion of the problem with a special structure from the rest of the mathematical program. Decomposition methods include Cross decomposition, Dantzig-Wolfe decomposition, and Benders decomposition; the details of which will not be discussed here.

Holmberg *et al.* (1999) proposed an exact algorithm for the capacitated facility location problem with single sourcing. The solution procedure is based on a Lagrangian heuristic (using subgradient optimization), a strong primal heuristic (using repeated matching) and the branch-and-bound algorithm. A strong dual approach (the Lagrangian dual) is combined with a strong primal approach (the repeated matching heuristic). The dual scheme generates lower bounds and the primal scheme generates feasible solutions and upper bounds. The solutions from the Lagrangian subproblems act as initial solutions for the primal heuristic. The branch-and-bound procedure utilizes information easily obtainable from the Lagrangian relaxation to speed up the bounding process.

Klose & Drexl (2005) investigated decomposition methods for solving subproblems exactly. They used a column generation procedure to compute exact Lagrangian lower bounds for the CPLP, and different strategies for stabilizing the process were employed.

Strong lower and upper bounds for the CPLP were obtainable from a Lagrangian relaxation of the capacity constraints via a partitioning of the plant set. A branch-and-price algorithm based on the column generation method was claimed to give good results for problem instances with tight capacity constraints, and could be applied to a variety of other assignment type problems.

Branch-and-bound-based procedures based on Lagrangian relaxation and subgradient optimization have been found by many authors to be effective in solving facility location problems. One reason is that the lower bound from the Lagrangian relaxation will be at least comparable to, and most often much better than, the bound obtained from the LP relaxation. This has the effect of lowering the number of nodes required in the branch-and-bound tree. The analysis of this thesis aims to provide information on the strength of the bound likely from a given Lagrangian relaxation. This has the potential to speed up the branch-and-bound process even further.

The focus of the literature survey now turns to the problem under consideration for the rest of the thesis, the TSCPLP.

2.6 The TSCPLP and the TSCPLPSS

The TSCPLP is the main problem to be analysed by this thesis, and is formulated and described in Chapters 1, 3, 4, and 5. A small number of researchers have studied this problem, and their work is outlined below to provide a context for the current work, and to present the methods currently being applied to such models. The majority of the work into two-stage capacitated plant location problems has been focused on single-source formulations. This problem is referred to as the *Two-Stage Capacitated Plant Location Problem with Single Source constraints* (TSCPLPSS). The formulation and analysis of this problem is presented in Chapter 6, however as mentioned previously the main formulation of the TSCPLP is not a single-source model. This allows for customers to have their demand satisfied by more than one open depot, and for depots to receive demand from more than one open plant.

Klose (1999) proposed a heuristic solution procedure for a two-stage capacitated facility location problem with single-source constraints. The formulation is very similar to the one proposed for the TSCPLP in this thesis. However there is no decision variable for the opening of plants in the upper-most level, they are simply assumed to all be open. The solution approach is based on linear programming, and iteratively refines the LP formulation using valid inequalities for various relaxations of the problem. These inequalities have been presented in the literature and include cover inequalities for knapsack problems, odd hole inequalities for the set packing problem, and flow cover and submodular inequalities for the CPLP. After each re-optimization step of the LP formulation, a feasible solution (upper bound) is obtained from the current fractional solution using different heuristics to determine the set of open depots and simple reassignment procedures to find a feasible customer assignment. If the assignment is feasible, a transportation problem is solved to find the best current solution.

The computational results confirmed the use of the inequalities to improve the LP bound, however it was noted that the approach may become inefficient in problems with a large number of customers or in the case of weak capacity constraints where too many variable upper bounds are needed to solve the LP. This LP approach can be surpassed by the use of Lagrangian relaxation which provides stronger lower bounds than LP relaxations.

Klose (2000) presented a Lagrangian relax-and-cut procedure for a two-stage, single-source, capacitated facility location problem. The formulation of the problem is the same as in the 1999 paper discussed above. The solution approach in this paper is based on relaxing the plant and depot capacity constraints. The resulting Lagrangian subproblem is an aggregate capacitated plant location problem and can be solved efficiently by a branch-and-bound method based on dual ascent and subgradient optimization. Feasible solutions are found using reassignment heuristics. The lower bound is further improved by adding valid inequalities such as those mentioned above.

The method was able to find upper and lower bounds close to optimality but it was noted that the computational effort required to achieve such results is substantial. Klose mentioned that in the case of small or 'easy' problems it can take less time to solve the problem to optimality using an LP-based cutting plane approach and a 'state-of-the-art' mixed-integer linear program solver than to apply the Lagrangian relaxation procedure. Conversely, in the case of large or more difficult problems the approach has given much better upper bounds (and sometimes also lower bounds) in far less running times than the application of an LP-based approach and the use of a mixed integer solver.

Tragantalerngsak *et al.* (1997) propose six heuristics based on Lagrangian relaxation for the solution of the two-echelon, single-source, capacitated facility location problem. The six relaxations studied are based on various combinations of constraints; including relaxing the connection between the upper two levels, the demand constraints, and the capacity constraints, or some combination of the above. Some relaxations also have surrogate constraints added to strengthen the resulting bounds. These take the form of the surrogate constraints mentioned earlier that force enough depots or plants open to satisfy the customer demand. This concept is also applied to the TSCPLP in this thesis.

To solve the lower bound problems, a subgradient optimization procedure is used. Numerical results are presented for a large number of test problems. These indicate that the lower bounds obtained from some of the relaxations frequently have a duality gap one third of the one obtained from the traditional linear programming relaxation. Also the overall solution times for the heuristics are less than the time required to solve the problem using the LP relaxation. This indicates that Lagrangian relaxation can provide stronger bounds than the LP relaxation, and that a Lagrangian heuristic is both a suitable and useful technique for solving the TSCPLP.

The same authors also presented an exact method for a two-echelon, single-source, capacitated facility location problem (Tragantalerngsak *et al.*, 1999). A three-stage branch-and-bound algorithm was proposed. Surrogate constraints were again added to strengthen the lower bound subproblems. In the first stage the branch-and-bound tree is created. In general, not all candidate problems at terminal nodes — where all the upper level variables have been set — will be fathomed (discontinued due to the current solution value). The second stage of the process is continued from these unfathomed terminal nodes, involving the variables determining the flow of demand from plants to depots. Finally a third stage is carried out, starting from the unfathomed terminal nodes from the second stage, to assign customers to open facilities. The problem here is reduced to a generalized assignment problem, for which efficient solution algorithms exist.

Lagrangian relaxation incorporating subgradient optimization is used to compute lower bounds for the first two stages of the method. Comparison between the results of the Lagrangian relaxation-based branch-and-bound algorithm presented in the paper and those from an existing LP-based code shows that the Lagrangian relaxation approach produces significantly smaller branch-and-bound trees and consumes much less computing time than the LP-based method. It is noted that the efficiency of the branch-and-bound method is critically dependent on the quality of the bounds produced at each node of the tree. The Lagrangian relaxation provided much stronger lower bounds and used less computational time than the lower bounds obtained from the LP code. The lower bounds for some problem sets were about 95% of the optimal objective value, compared to 85% for the LP relaxation.

The literature that focused particularly on the TSCPLP and TSCPLPSS shows that the use of Lagrangian relaxation in either a heuristic or a branch-and-bound algorithm can improve the solution quality and computational time, when compared to a traditional LP relaxation approach. The choice of a Lagrangian relaxation is a critical decision to be made. Researchers in the literature have often chosen the constraints to relax based on the ease of solution of the lower bound problems, rather than the relative strength of the likely resulting bound. Some researchers of two-stage problems have referred to inequalities and relationships between bounds that have been reported for Lagrangian relaxations of the CPLP, and this thesis aims to provide the equivalent information for the TSCPLP. This will increase the information available for the possible bounds; which can be used when choosing a suitable Lagrangian relaxation for solving the TSCPLP and its extensions, some of which are discussed in the next section.

2.7 The Multi-Commodity TSCPLP

Recall that when a customer has a unique demand for more than one distinct product or service, the problem is known as a multi-commodity plant location problem. A few researchers in the literature have discussed this problem as a two-stage capacitated model, such as Pirkul and Jayaraman (1998). They researched a *Multi-Commodity, Multi-Plant, Capacitated Facility Location Problem* where a number of production plants supply warehouses with multiple products which in turn distribute these products to customer outlets based on their specified demand quantities of different products. The formulation includes decision variables for the location of open plants and warehouses; and the flow of each product from the plants to the warehouses, and from the warehouses to the customers. Both the plants and the warehouses carry capacity restrictions. The constraints are of a similar format to a TSCPLP, as the model is not single-source.

A Lagrangian heuristic is presented, based on relaxing the demand constraints, where feasible solutions are found using an assignment heuristic, attempting to assign customers to the warehouses for the least cost. Then the demand for products is allocated to the warehouses from the plants using a similar technique. The results showed that the feasible solution procedure consistently provides stable solutions to the problem, and the heuristic performs well in terms of both an approximation to optimality and solution times regardless of problem size and structure.

The findings of this thesis on the strength of the bounds from Lagrangian relaxation have the potential to also be applied to multi-commodity models.

2.8 The Multi-Period, Multi-Commodity TSCPLP

These problems are possibly the most complex of all of the plant location problems as they incorporate all of the individual problem characteristics into one model. Variables in the formulation of these problems may have up to five indices: one each for the plants, depots, customers, commodities, and time periods. This is a relatively new area of research but some of the work is outlined in this section. Formulations are not included as these are often problem specific.

The solution methods for these types of problems are most likely to include dynamic programming. Hinojosa *et al.* (2000) present a Lagrangian relaxation scheme incorporating a dual ascent method together with a heuristic construction phase which showed in computational tests to provide good feasible solutions for their two-echelon, multi-period, multi-commodity capacitated facility location problem. It was assumed that the capacities of the plants and warehouses, as well as the demands and transportation costs, change over time. It was also assumed that the sets of customers and products, and the feasible locations for the plants and the warehouses, are known and fixed. Therefore they would not change over the planning horizon. It was noted that the typical time period considered in multi-period location problems would be a season or a month.

A standard Lagrangian relaxation as described in previous sections is employed and an ascent procedure is used to generate a good solution for the relaxed problem. This solution is very often infeasible for the original problem. Therefore, an alternative procedure must be developed that starting from this solution constructs a good feasible solution for the original problem. The proposed scheme consists of two different steps. The first step looks for capacities in each time period t, both for plants and warehouses. Once these capacities have been established for meeting the demand, the second step looks for the best transportation plan between plants and warehouses, and between warehouses and customers.

Canel *et al.* (2001) developed an algorithm to solve the capacitated, multi-commodity, multi-period (dynamic), multi-stage facility location problem. The proposed algorithm consists of two parts: in the first part a branch-and-bound method is used to generate a list of candidate solutions for each period and then dynamic programming is used to find the optimal sequence of configurations over the multi-period planning horizon.

Sambasivan & Yahya (2005) utilise a Lagrangian heuristic in a solution procedure for a multi-plant, multi-item, multi-period capacitated lot-sizing problem with inter-plant transfers. The model is based on a real-world problem in a large steel corporation in United States of America where customers place demand for steel which may be satisfied by transporting steel from another plant at the company's expense to the plant where the customer placed the demand. The model considers capacities, production rates, setup times, and costs of production and carrying inventory of each product at each plant during a given time period. The problem is solved using a technique developed for the uncapacitated relaxation incorporating a lot shifting–splitting–merging routine to alleviate the capacity violations.

These types of problems, although complex, provide a context for the work in this thesis on the TSCPLP and TSCPLPSS. Multi-commodity and/or multi-period models are extensions of a single-product, static TSCPLP. The analysis of the TSCPLP can be applied to any number of transformations or extensions of the original problem.

2.9 Summary

This concludes the survey of the literature on plant location problems, and in particular the context of the TSCPLP and TSCPLPSS. A niche for the current work has been identified and the formulation to be considered for analysis in the next chapters has characteristics that have been lacking in the models presented in the literature. Lagrangian relaxation has been a common solution technique across many types of formulations including single-source and multi-commodity models. The analysis of this thesis is concerned with Lagrangian relaxation bounds and their relative strengths. This information can be applied to many scenarios that have been discussed in the literature.

The next two chapters focus on the theoretical analysis of the TSCPLP. This begins with the presentation of a formulation that has not been studied before; where customers can be served from any open depot, and where explicit capacities are placed on both the set of plants and the set of depots. Several theorems will be presented that classify bounds on the problem showing that the bounds generated from the solution of particular Lagrangian relaxations are 'tighter' or closer to the optimal solution than others. The relationships between the bounds from the Lagrangian relaxations are classified into three categories: *trivial*, *equivalent*, and *dominant*. Chapter 3 begins the theoretical analysis by giving the results for the Lagrangian bounds that are classified as *trivial*.

Chapter 3

Theorectical analysis of the TSCPLP - Part One

3.1 Introduction

In the next two chapters a theoretical analysis of the Two-Stage Capacitated Plant Location Problem (TSCPLP) is presented. This shows the relative strength of the lower bounds resulting from several Lagrangian relaxations to get an insight into the problem and the structure of the constraints for use in solution techniques. The next section contains two formulations of the problem — one with surrogate constraints added to strengthen some of the relaxations. These formulations are unique from any other two-stage problem previously presented in the literature that was discussed in Chapter 2. Customers in this system can be served from any open depot and therefore the demand is not required to be single-source. There are explicit capacities included for both the set of plants and the set of depots, and the set of open plants is not already known at the beginning of the problem as in some previous models. The solution to this problem defines which plants and depots are opened and the flow of demand from plants to depots to customers. This means that the mathematical model can be used to model a situation where the location for potential production plants is a decision to be made, and the fixed costs can represent building and setup costs and/or relocation costs.

Section 3.3 contains a theorem on the computational complexity of the problem and Section 3.4 presents and compares different relaxations (either linear programming or Lagrangian). This identifies and classifies the bounds on the dual problems in terms of strength or closeness to the optimal solution. The bounds are classified into three groups:

trivial bounds, *equivalent* bounds, and *dominant* bounds. Trivial bounds are those that give a resulting value that is only as good as the linear programming relaxation. Equivalent bounds are those that can be said to equal each other. Dominant bounds, for example $A < B$, are defined as: for any given instance A is less than or equal to B and for some instance A is strictly less than B. The trivial bounds will be presented in this chapter and the remaining bounds can be found in Chapter 4.

3.2 Formulation

The TSCPLP can be formulated as a mixed-integer linear program as below. This is the general formulation under consideration for theoretical and computational analysis in the next two chapters. Let:

$I = \{1, \dots, m\}$ be the set of customers,
$J = \{1, \dots, n\}$ be the set of potential depot locations,
$K = \{1, \dots, p\}$ be the set of potential plant locations,
$c_{ij} =$ total cost of transportation from depot j to serve customer $i, \forall i \in I, \forall j \in J$,
$f_j =$ fixed cost associated with depot $j, \forall j \in J$,
$g_k =$ fixed cost associated with plant $k, \forall k \in K$,
$b_{kj} =$ unit cost of transportation from plant k to depot $j, \forall k \in K, \forall j \in J$,
$d_i =$ demand of cutomer $i, \forall i \in I$,
$s_j =$ capacity of depot $j, \forall j \in J$,
$a_k =$ capacity of plant $k, \forall k \in K$,

The decision variables are:

$x_{ij} =$ fraction of the demand of customer i supplied from depot $j, \forall i \in I, \forall j \in J$,

$$y_j = \begin{cases} 0, & \text{if depot } j \text{ is closed} \\ 1, & \text{if depot } j \text{ is open } \forall i \in I, \forall j \in J \end{cases}$$

$w_{kj} =$ units of demand transported from plant k to depot $j, \forall k \in K, \forall j \in J$,

$$z_k = \begin{cases} 0, & \text{if plant } k \text{ is closed} \\ 1, & \text{if plant } k \text{ is open } \forall k \in K. \end{cases}$$

The problem can now be stated as:

(P3.1)

$$Z = \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj}w_{kj} \tag{3.1}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \in I \tag{3.2}$$

$$\sum_{i=1}^{m} d_i x_{ij} \leqslant s_j y_j \qquad \forall j \in J \tag{3.3}$$

$$x_{ij} \leqslant y_j \qquad \forall i \in I, \forall j \in J \tag{3.4}$$

$$\sum_{j=1}^{n} w_{kj} \leqslant a_k z_k \qquad \forall k \in K \tag{3.5}$$

$$\sum_{k=1}^{p} w_{kj} = \sum_{i=1}^{m} d_i x_{ij} \qquad \forall j \in J \tag{3.6}$$

$$w_{kj} \geq 0 \qquad \forall k \in K, \forall j \in J \tag{3.7}$$

$$y_j \text{ integer} \qquad \forall j \in J \tag{3.8}$$

$$z_k \text{ integer} \qquad \forall k \in K \tag{3.9}$$

$$0 \leqslant x_{ij} \leqslant 1 \qquad \forall i \in I, \forall j \in J \tag{3.10}$$

$$0 \leqslant y_j \leqslant 1 \qquad \forall j \in J \tag{3.11}$$

$$0 \leqslant z_k \leqslant 1 \qquad \forall k \in K \tag{3.12}$$

All constraints can be interpreted as described in Chapter 1.

Surrogate constraints (3.13) and (3.14) can be added as follows:

$$\sum_{k=1}^{p}\sum_{j=1}^{n} w_{kj} \leqslant \sum_{k=1}^{p} a_k z_k \tag{3.13}$$

$$\sum_{j=1}^{n} s_j y_j \geqslant \sum_{i=1}^{m} d_i \tag{3.14}$$

(3.13) is derived by summing (3.5) over all k plants and states that the total capacity of the plants is at least as large as the total demand being transported from them to the depots. (3.14) is derived by summing (3.3) over all j depots and using the equalities (3.2) and states that the total capacity of the depots is at least as large as the total demand being transported from them to the customers. These two constraints are redundant in the original formulation but strengthen some of the relaxations.

The second formulation of the TSCPLP with the surrogate constraints added is:

(P3.2)

$$z = \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj}w_{kj}$$

Subject to: (3.2) – (3.14)

3.3 Computational complexity

This section focuses on proving the computational complexity of the TSCPLP. It shows that the problem is complex and optimal solutions are not able to be found in polynomial time. Recall that a problem is said to be NP-hard if the existence of a polynomial-time algorithm for the problem would imply that all other NP-complete problems can also be solved in polynomial time.

Theorem 3.3.1 *The TSCPLP is NP-hard.*

Proof To show this we reduce the TSCPLP to the 3-dimensional matching problem (3-DM). The 3-DM is known to be NP-complete (Garey & Johnson, 1979). Let $X = \{1, \ldots .n\}$ represent the set of customers, $Y = \{1, \ldots, n\}$ be the set of depots, and $Z = \{1, \ldots, n\}$ represent the set of plants. Associate a 0 or 1 to each of the customers in X and the depots in Y which represents the transportation cost to the customer from a depot or from a plant to a depot. Let the fixed cost of opening a depot or plant be $\epsilon > 0$. The customers demand comes from the set $\{1, \ldots, k\}$, $k \geqslant 1$. We need to satisfy all of the customers demand and minimise the cost. Let $S = X \times Y \times Z$. The triplet (x, y, z) defines relationships of the sort: customer x receives demand from depot y which has received demand from plant z. The TSCPLP is then to decide whether there exists a perfect matching M, a subset of S, such that$|M| = n$ and every element of $X, Y,$ and Z occurs exactly once in a triple $(x, y, z) \in M$. The solution to this problem gives a solution to the 3-DM problem.
This proves that if a solution of the TSCPLP is found then a solution to the 3-DM can be found. The 3-DM is NP-complete and therefore the TSCPLP is NP-hard when the capacities are from the set $\{1, \ldots, k\}, k \geqslant 1$. ∎

3.4 Relaxations and bound relationships

Several relaxations will be considered for theoretical analysis and are of two types: Linear Programming (LP) relaxations and Lagrangian relaxations. A linear programming relaxation is the relaxation of the original problem formed by removing the integrality constraints on variables y and z. The linear programming relaxation of program (P3.1) is defined as:
(P3.1:LP)

$$Z = \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj}w_{kj}$$

$$\text{Subject to: } (3.2) - (3.7), (3.10) - (3.12)$$

The linear programming relaxation of (P3.2) is (P3.2:LP) defined as (P3.1:LP) above subject to the constraints $(3.2) - (3.7), (3.10) - (3.12)$ as defined previously, and the surrogate constraints (3.13) and (3.14). The bounds from these problems shall be referred to as Z_{LP} and z_{LP} respectively.

As mentioned in Chapter 2, Lagrangian relaxation exploits the observation that many difficult integer programming problems can be modelled as a relatively easy problem complicated by a set of side constraints. These complicating constraints are replaced with a penalty term in the objective function involving the amount of violation of the constraints and their dual variables. For an outline of the procedure see Beasley (1993) or section 2.4.1 of the previous chapter.

The selection of a suitable relaxation is one of the important issues to be considered when aiming to solve integer or mixed-integer linear programs such as the TSCPLP. Two key factors in the evaluation of a relaxation are its ease of solution and the tightness of the bounds generated. The ease of solution depends on the methods available for solving the Lagrangian subproblem, and generally a relaxation which gives a tighter bound will use greater computation time whereas an easily solved relaxation problem is likely to give poor bounds (Geoffrion & McBride, 1978).

The next sections aim to classify several of the Lagrangian duals to show the tightness of bounds resulting from various Lagrangian relaxations. In general, the sub indices of Z and z will indicate the sets of relaxed constraints in problems (P3.1) and (P3.2) respectively. The Lagrange multipliers that will be associated with each set of constraints will remain consistent throughout the sections and are given as follows:

$$\begin{array}{lll}
\text{Constraints (3.2):} & \mu = (\mu_1, \ldots, \mu_m) \Rightarrow & \displaystyle\sum_{i=1}^{m} \mu_i \left(1 - \sum_{j=1}^{n} x_{ij}\right) \\
\text{Constraints (3.3):} & \lambda = (\lambda_1, \ldots, \lambda_n) \Rightarrow & \displaystyle\sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} d_i x_{ij} - s_j y_j\right) \\
\text{Constraints (3.4):} & \alpha = (\alpha_{11}, \ldots, \alpha_{mn}) \Rightarrow & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij} \left(x_{ij} - y_j\right) \\
\text{Constraints (3.5):} & \beta = (\beta_1, \ldots, \beta_p) \Rightarrow & \displaystyle\sum_{k=1}^{p} \beta_k \left(\sum_{j=1}^{n} w_{kj} - a_k z_k\right) \\
\text{Constraints (3.6):} & \delta = (\delta_1, \ldots, \delta_j) \Rightarrow & \displaystyle\sum_{j=1}^{n} \delta_j \left(\sum_{i=1}^{m} d_i x_{ij} - \sum_{k=1}^{p} w_{kj}\right) \\
\text{Constraints (3.13):} & \theta \Rightarrow & \displaystyle\theta \left(\sum_{k=1}^{p}\sum_{j=1}^{n} w_{kj} - \sum_{k=1}^{p} a_k z_k\right) \\
\text{Constraints (3.14):} & \sigma \Rightarrow & \displaystyle\sigma \left(\sum_{i=1}^{m} d_i - \sum_{j=1}^{n} s_j y_j\right)
\end{array}$$

The constraints defining the sign restrictions on the Lagrange multipliers are as follows:

$$\mu_i \text{ unrestricted in sign} \qquad \forall i \in I \tag{3.15}$$

$$\lambda_j \geqslant 0 \qquad \forall j \in J \tag{3.16}$$

$$\alpha_{ij} \geqslant 0 \qquad \forall i \in I, \forall j \in J \tag{3.17}$$

$$\beta_k \geqslant 0 \qquad \forall k \in K \tag{3.18}$$

$$\delta_j \text{ urs} \qquad \forall j \in J \tag{3.19}$$

$$\theta \geqslant 0 \tag{3.20}$$

$$\sigma \geqslant 0 \tag{3.21}$$

3.4.1 Trivial bounds

A Lagrangian relaxation is said to produce a *trivial* bound when the Lagrangian dual problem gives a dual value that is the same as the bound resulting from a linear programming relaxation of the original problem. It is well known (see Fisher, 1985) that the worst, in this case lowest, bound that can be obtained using Lagrangian relaxation cannot be worse or lower than the bound resulting from the linear programming relaxation of the problem, i.e., $Z_{LP} \leqslant Z_D$ for any relaxation D. The inequality in this logic is between the Lagrangian problem and the Lagrangian problem with the integrality constraints relaxed. So $Z_{LP} < Z_D$ only if the inequality holds strictly and $Z_{LP} = Z_D$ if the Lagrangian problem is unaffected by removing the integrality requirements on the variables, i.e., the optimal values of the variables will be integer whether we require it or not, and the bound is therefore trivial. This also shows that we can improve the lower bound by using a Lagrangian relaxation in which the variables are not naturally integrals (Fisher, 1985).

Some notation used in the following theorems: if a is any real number then a^- denotes $\min\{0, a\}$.

Theorem 3.4.1

1. *$Z_{2,5}$ is trivial.*
2. *$z_{2,5,13,14}$ is trivial.*

Proof

1. The optimal solution of the Lagrangian problem obtained by relaxing constraint sets (3.2) and (3.5) in (P3.1) can be obtained by solving:

$$Z_{2,5} = \max_{\mu,\beta} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \mu_i)\, x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} (g_k - \beta_k a_k)\, z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} (b_{kj} + \beta_k)\, w_{kj} + \sum_{i=1}^{m} \mu_i \quad (3.22)$$

$$\text{Subject to: } (3.3), (3.4), (3.6) - (3.12), (3.15), (3.18)$$

Now take $z_k = 1$ if $g_k - \beta_k a_k < 0$ and $z_k = 0$ otherwise. This is possible because the only constraints on z_k are that z_k must be integer and $0 \leqslant z_k \leqslant 1$. The optimal values for z_k, regardless of the values for x_{ij}, y_j and z_k, will be those that contribute the negative values in the objective function. So if $g_k - \beta_k a_k < 0$, setting $z_k = 1$ will contribute a negative term in the objective function and setting all other $z_k = 0$ eliminates the z_k that would contribute positively.

Constants aside the following subproblem results:

$$\max_{\mu,\beta} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \mu_i)\, x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} \sum_{j=1}^{n} (b_{kj} + \beta_k)\, w_{kj} \quad (3.23)$$

$$\text{Subject to: } (3.3), (3.4), (3.6) - (3.8), (3.10), (3.11), (3.15), (3.18)$$

The optimal value of this sub problem does not change for any μ_i or β_k when the integrality constraint (3.8) is deleted since the new optimal value is reached by a solution (x, y, w) such that $y_j \in \{0, 1\}, \forall j$.

To see this, note that this problem decomposes into n subproblems, one for each value of j. For any possible values of j, x_{ij}, and y_j, the optimal values of w_{kj} are obtained by choosing an index $k(j)$ such that $b_{k(j)j} + \beta_{k(j)} = \min_k\{b_{kj} + \beta_k\}$ and setting $w_{k(j)j} = \sum_{i=1}^{m} d_i x_{ij}$ and $w_{kj} = 0, \forall k \neq k(j)$. This is possible because the remaining constraints can be separated for the index j creating a subproblem for each j. Constraints (3.6) and (3.7) are the only ones involving the variable w_{kj} so for any value of x_{ij} and y_j, w_{kj} needs to satisfy $\sum_{k=1}^{p} w_{kj} = \sum_{i=1}^{m} d_i x_{ij}$ and we need to minimise $b_{kj} + \beta_k$ in the objective function. So for each index j we choose the index k, denoted by $k(j)$, that minimises $b_{kj} + \beta_k$. Then for this value of k set $w_{kj} = \sum_{i=1}^{m} d_i x_{ij}$ to satisfy the constraint and set $w_{kj} = 0$ for all other values of k to minimise the objective function contribution.

Therefore the linear programming relaxation of the problem for any j becomes:

$$Z_{2,5}(\mu, \beta : j) = \min_{x,y} \sum_{i=1}^{m} \left[c_{ij} - \mu_i + d_i \left(b_{k(j)j} + \beta_{k(j)}\right)\right] x_{ij} + f_j y_j \tag{3.24}$$

$$\text{Subject to: } (3.3), (3.4), (3.10), (3.11)$$

The solution is characterised by a value v such that

$$x_{ij} = \quad y_j \quad \text{for } i \text{ with } c_{ij} - \mu_i + d_i(b_{k(j)j} + \beta_{k(j)}) < v, \tag{3.25}$$

$$x_{ij} = \quad 0 \quad \text{for } i \text{ with } c_{ij} - \mu_i + d_i(b_{k(j)j} + \beta_{k(j)}) > v, \tag{3.26}$$

and any remaining x_{ij} get values which make constraints (3.4) tight, as seen below.

The value of v is determined by the following argument. If y_j is fixed (say to its optimal value) the resulting problem is equivalent to a linear knapsack problem. To see this, transform to the variables u_{ij} which satisfy $x_{ij} = y_j u_{ij}$. To make the notation simpler, also drop the index j and substitute:

$$\gamma_i = c_{ij} - \mu_i + d_i \left(b_{k(j)j} + \beta_{k(j)}\right).$$

Linear program (3.24) with y fixed becomes:

$$\begin{aligned} &\min_u \sum_{i=1}^{m} \gamma_i y u_i + f y \\ &\text{Subject to:} \\ &\quad \sum_{i=1}^{m} d_i y u_i \leqslant s y \\ &\quad 0 \leqslant y u_i \leqslant y \ \forall i \in I \\ &\qquad\qquad\qquad (A) \end{aligned}$$

Since y is fixed the common factor can be removed everywhere and the problem becomes the well-known linear knapsack problem and independent of y. There exists an optimal solution to (A) with the following structure (see for instance Theorem 2 in Martello & Toth, 1990). Without loss of generality assume that the u_i are ordered such that $\frac{\gamma_1}{d_1} \leqslant \frac{\gamma_2}{d_2} \leqslant \ldots \leqslant \frac{\gamma_k}{d_k} < 0 \leqslant \frac{\gamma_{k+1}}{d_{k+1}} \leqslant \ldots \leqslant \frac{\gamma_m}{d_m}$, then $u_i = 0$ for $i = k+1, \ldots, m$ and there is an index $q \leqslant k+1$ (called the *critical item*) such that $u_i = 1$ for $i = 1, \ldots, q-1$. Also, $q \leqslant k$ means that $\sum_{i=1}^{q-1} d_i \leqslant s < \sum_{i=1}^{q} d_i$ and the solution has $u_q = \frac{s - \sum_{i=1}^{q-1} d_i}{d_q} (< 1)$.

The value of v given in (3.25) and (3.26) is therefore $v = \frac{\gamma_q}{d_q}$ if $q \leqslant k$ or $v = 0$ if $q = k+1$.

Substituting the optimal values of u back into (A) gives the following linear program.

$$Z_{2,5}(\mu, \beta : j) = \min_y \left(\sum_{i=1}^{m} u_i + f_j \right) y_j \tag{3.27}$$

Subject to: (3.11).

The optimal solution for y_j in the above LP is

$$y_j = \begin{cases} 1 \text{ if } \sum_{i=1}^{m} u_i + f_j < 0 \\ 0 \text{ if } \sum_{i=1}^{m} u_i + f_j > 0 \\ 0 \text{ or } 1 \text{ if } \sum_{i=1}^{m} u_i + f_j = 0 \end{cases} \tag{3.28}$$

It is known that when the optimal solution does not change after deleting the integrality constraints the associated lower bound is trivial, therefore the subproblems of (3.23) have integer solutions and therefore $Z_{2,5}$ is trivial. $\blacksquare$

2. Similarly for part 2, this result holds as a consequence of Theorem 2.3.5 for relaxing a redundant constraint. $\blacksquare$

Lemma 3.4.2 *A known bound cannot be improved by relaxing a further constraint.*

Proof This result is found in Theorem 2.3.5.

Theorem 3.4.3

1. $Z_{2,3,5}$ *is trivial.*

2. $z_{2,3,5,13,14}$ *is trivial.*

Proof

1. This result follows from Lemma 3.4.2 and Theorem 3.4.1, the known bound of $Z_{2,5}$. Constraint (3.3) is the further constraint in this case.

2. Similarly for part 2, this result holds as a consequence of Theorem 2.3.5 for relaxing a redundant constraint/s. $\blacksquare$

Corollary 3.4.4

1. $Z_{2,4,5}$ *is trivial.*

2. $z_{2,4,5,13,14}$ *is trivial.*

Corollary 3.4.5

1. $Z_{2,5,6}$ *is trivial.*

2. $z_{2,5,6,13,14}$ *is trivial.*

Corollary 3.4.6

1. $Z_{2,3,5,6}$ *is trivial.*

2. $z_{2,3,5,6,13,14}$ *is trivial.*

Corollary 3.4.7

1. $Z_{2,4,5,6}$ *is trivial.*

2. $z_{2,4,5,6,13,14}$ *is trivial.*

Corollaries 3.4.4, 3.4.5, 3.4.6, and 3.4.7 all follow as consequences of the previous Theorem 3.4.3 and therefore of Theorem 3.4.1 and Lemma 3.4.2.

Theorem 3.4.8 $Z_{3,4,5,6}$ *and* $z_{3,4,5,6,13,14}$ *are trivial.*

Proof If we can show that the LP relaxations of the above Lagrangian relaxations have integer solutions then we can say the bounds are trivial. When these constraints are relaxed in the respective problems the resulting constraint matrix is totally unimodular:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \in I \qquad (3.2)$$

$$w_{kj} \geqslant 0 \qquad \forall k \in K, \forall j \in J \qquad (3.7)$$

$$0 \leqslant x_{ij} \leqslant 1 \qquad \forall i \in I, \forall j \in J \qquad (3.10)$$

$$0 \leqslant y_j \leqslant 1 \qquad \forall j \in J \qquad (3.11)$$

$$0 \leqslant z_k \leqslant 1 \qquad \forall k \in K \qquad (3.12)$$

Hence we have integer solutions for the linear programming relaxation, proving the theorem. ∎

3.5 Summary

This concludes the analysis of the trivial bounds and the computational complexity of the TSCPLP. The formulation of the TSCPLP has been presented, along with a variation including the addition of surrogate constraints which strengthen some of the Lagrangian relaxations. It has been shown that the TSCPLP is NP-hard apart from one special case that can be solved in polynomial time, as a transshipment problem. The bounds from the Lagrangian relaxations that are the same as those found via a linear programming relaxation have been classified in this chapter as *trivial.* In general the relaxations that result in trivial bounds are those that dualize more than three sets of constraints.

The next chapter continues the theoretical analysis of the problem and presents results relating to *equivalent* and *dominant* bounds.

Chapter 4

Theoretical analysis of the TSCPLP - Part Two

4.1 Introduction

In this chapter the theoretical analysis of the Two-Stage Capacitated Plant Location Problem (TSCPLP) is continued. The next section contains the formulations as outlined in Chapter 3 for reference. Section 4.3 presents and compares different Lagrangian relaxations. This identifies and classifies the bounds on the dual problems in terms of their strength or closeness to the optimal solution. Recall from Chapter 3 that the bounds are classified into three groups: trivial bounds, equivalent bounds, and dominant bounds.

Trivial bounds are those that give a resulting value that is only as good as the linear programming relaxation and were discussed in Chapter 3. *Equivalent* bounds are those that can be said to equal each other. *Dominant* bounds, e.g. $A < B$, are defined as: for any given instance A is less than or equal to B and for some instance A is strictly less than B. This chapter is concerned with the equivalent and dominant bound relationships. These relationships are important when selecting a solution technique or a relaxation to use as part of a solution strategy. This information can be used to choose a relaxation that is going to result in strong bounds on the optimal solution to the problem. This can then be weighed up with the ease of solution and likely computational time required to compute such a bound.

4.2 Formulation

The TSCPLP can be formulated as a mixed-integer linear program as in Chapter 1 and is given below for reference.

(P4.1)

$$Z = \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj} w_{kj} \tag{4.1}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \in I \tag{4.2}$$

$$\sum_{i=1}^{m} d_i x_{ij} \leqslant s_j y_j \qquad \forall j \in J \tag{4.3}$$

$$x_{ij} \leqslant y_j \qquad \forall i \in I, \forall j \in J \tag{4.4}$$

$$\sum_{j=1}^{n} w_{kj} \leqslant a_k z_k \qquad \forall k \in K \tag{4.5}$$

$$\sum_{k=1}^{p} w_{kj} = \sum_{i=1}^{m} d_i x_{ij} \qquad \forall j \in J \tag{4.6}$$

$$w_{kj} \geqslant 0 \qquad \forall k \in K, \forall j \in J \tag{4.7}$$

$$y_j \text{ integer} \qquad \forall j \in J \tag{4.8}$$

$$z_k \text{ integer} \qquad \forall k \in K \tag{4.9}$$

$$0 \leqslant x_{ij} \leqslant 1 \qquad \forall i \in I, \forall j \in J \tag{4.10}$$

$$0 \leqslant y_j \leqslant 1 \qquad \forall j \in J \tag{4.11}$$

$$0 \leqslant z_k \leqslant 1 \qquad \forall k \in K \tag{4.12}$$

Surrogate constraints (4.13) and (4.14) can again be added as follows:

$$\sum_{k=1}^{p}\sum_{j=1}^{n} w_{kj} \leqslant \sum_{k=1}^{p} a_k z_k \tag{4.13}$$

$$\sum_{j=1}^{n} s_j y_j \geqslant \sum_{i=1}^{m} d_i \tag{4.14}$$

The second formulation of the TSCPLP with the surrogate constraints added is:

(P4.2)

$$z = \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj} w_{kj}$$

Subject to: (4.2) − (4.14)

All constraints can be interpreted as described in Chapter 1.

4.3 Relaxations and bound relationships

Several relaxations will be considered for further theoretical analysis and in this chapter are all Lagrangian relaxations. Recall that Lagrangian relaxation exploits the observation that many difficult integer programming problems can be modelled as a relatively easy problem complicated by a set of side constraints. These complicating constraints are replaced with a penalty term in the objective function involving the amount of violation of the constraints and their dual variables.

When aiming to solve integer or mixed-integer linear programs, such as the TSCPLP, one of the important issues to be considered is the selection of a suitable relaxation. The quality of a relaxation can be evaluated by the ease of its solution, and the tightness of the bounds that it generates. The ease of solution depends on the methods available for solving the Lagrangian subproblem, and generally a relaxation which gives a tighter bound will use greater computation time whereas an easily solved relaxation problem is likely to give poor bounds (Geoffrion & McBride, 1978).

The next sections aim to classify several of the Lagrangian dual problems to show the tightness of bounds resulting from various Lagrangian relaxations. In general, the sub indices of Z and z will indicate the sets of relaxed constraints in problems (P4.1) and (P4.2) respectively. The Lagrange multipliers that will be associated with each set of constraints are the same as were used in Chapter 3. These will remain consistent throughout the sections and are given as follows:

$$\text{Constraints (4.2):} \quad \mu = (\mu_1, \ldots, \mu_m) \Rightarrow \quad \sum_{i=1}^{m} \mu_i \left(1 - \sum_{j=1}^{n} x_{ij}\right)$$

$$\text{Constraints (4.3):} \quad \lambda = (\lambda_1, \ldots, \lambda_n) \Rightarrow \quad \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} d_i x_{ij} - s_j y_j\right)$$

$$\text{Constraints (4.4):} \quad \alpha = (\alpha_{11}, \ldots, \alpha_{mn}) \Rightarrow \quad \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij} \left(x_{ij} - y_j\right)$$

$$\text{Constraints (4.5):} \quad \beta = (\beta_1, \ldots, \beta_p) \Rightarrow \quad \sum_{k=1}^{p} \beta_k \left(\sum_{j=1}^{n} w_{kj} - a_k z_k\right)$$

$$\text{Constraints (4.6):} \quad \delta = (\delta_1, \ldots, \delta_j) \Rightarrow \quad \sum_{j=1}^{n} \delta_j \left(\sum_{i=1}^{m} d_i x_{ij} - \sum_{k=1}^{p} w_{kj}\right)$$

Constraints (4.13): $\theta \Rightarrow$ $\theta\left(\sum_{k=1}^{p}\sum_{j=1}^{n} w_{kj} - \sum_{k=1}^{p} a_k z_k\right)$

Constraints (4.14): $\sigma \Rightarrow$ $\sigma\left(\sum_{i=1}^{m} d_i - \sum_{j=1}^{n} s_j y_j\right)$

The constraints defining the sign restrictions on the Lagrange multipliers are as follows:

$$\mu_i \text{ unrestricted in sign} \qquad \forall i \in I \qquad (4.15)$$

$$\lambda_j \geqslant 0 \qquad \forall j \in J \qquad (4.16)$$

$$\alpha_{ij} \geqslant 0 \qquad \forall i \in I, \forall j \in J \qquad (4.17)$$

$$\beta_k \geqslant 0 \qquad \forall k \in K \qquad (4.18)$$

$$\delta_j \text{ urs} \qquad \forall j \in J \qquad (4.19)$$

$$\theta \geqslant 0 \qquad (4.20)$$

$$\sigma \geqslant 0 \qquad (4.21)$$

Some notation used in the following theorems: if a is any real number then a^- denotes $\min\{0, a\}$.

4.3.1 Equivalent bounds

The following theorems result from the properties of the surrogate constraints in problem (P4.2).

Theorem 4.3.1

1. $z_2 = z_{2,13} \geqslant Z_2$,
2. $z_3 = z_{3,13} \geqslant Z_3$,
3. $z_4 = z_{4,13} = Z_4$,
4. $z_6 = z_{6,13} = Z_6$.

Proof By definition constraint set (4.13) is simply both sides of constraint set (4.5) summed over k. In all of the above cases constraint set (4.5) is still satisfied so relaxing constraint (4.13) has no effect on the solution and constraint (4.13) is therefore redundant and the theorem follows.

1.

$$z_2 = \max_{\mu} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \mu_i) x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} + \sum_{i=1}^{m} \mu_i \quad (4.22)$$

Subject to: (4.3) – (4.15).

$$z_{2,13} = \max_{\mu,\theta} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \mu_i) x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} + \sum_{i=1}^{m} \mu_i + \theta \left(\sum_{k=1}^{p} \sum_{j=1}^{n} w_{kj} - \sum_{k=1}^{p} a_k z_k \right) \quad (4.23)$$

Subject to: (4.3) – (4.12), (4.14), (4.15), (4.20).

Since constraint set (4.5) is satisfied, the last term in (4.23) is 0 at optimality and these two bound values are equal.

$$\text{Now } Z_2 = \max_{\mu} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \mu_i) x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} + \sum_{i=1}^{m} \mu_i \quad (4.24)$$

Subject to: (4.3) – (4.12), (4.15).

The inequality exists between $z_{2,13}$ and Z_2 since constraint set (4.14) is derived from constraints (4.2), which are relaxed, and therefore the feasible set for $Z_{2,13}$: $\{(4.3) - (4.12), (4.14), (4.15), (4.20)\}$ is smaller than that of $\{(4.3) - (4.12), (4.15)\}$, the feasible set of Z_2. ∎

2. Similarly part 2 holds by the same logic, replacing constraint set (4.3) for (4.2):

$$z_3 = \max_{\lambda} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \lambda_j d_i) x_{ij} + \sum_{j=1}^{n} (f_j - \lambda_j s_j) y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} \quad (4.25)$$

Subject to: (4.2), (4.4) – (4.14), (4.16).

$$z_{3,13} = \max_{\lambda,\theta} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \lambda_j d_i)\, x_{ij} + \sum_{j=1}^{n} (f_j - \lambda_j s_j)\, y_j + \sum_{k=1}^{p} g_k z_k$$
$$+ \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} + \theta \left(\sum_{k=1}^{p} \sum_{j=1}^{n} w_{kj} - \sum_{k=1}^{p} a_k z_k \right) \tag{4.26}$$

Subject to: (4.2), (4.4) − (4.12), (4.14), (4.16), (4.20).

Since constraint set (4.5) is satisfied, the last term in (4.26) is 0 at optimality and these two bound values are equal.

$$\text{Now } Z_3 = \max_{\lambda} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \lambda_j d_i)\, x_{ij} + \sum_{j=1}^{n} (f_j - \lambda_j s_j)\, y_j$$
$$+ \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} \tag{4.27}$$

Subject to: (4.2), (4.4) − (4.12), (4.16).

The inequality exists between $z_{3,13}$ and Z_3 since constraint set (4.14) is derived from constraints (4.2), which are relaxed, and therefore the feasible set for $Z_{3,13}$: $\{(4.2), (4.4)-(4.12), (4.14), (4.16), (4.20)\}$ is smaller than that of $\{(4.2), (4.4)-(4.12), (4.16)\}$, the feasible set of Z_3. ∎

3.

$$z_4 = \max_{\alpha} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \alpha_{ij})\, x_{ij} + \sum_{j=1}^{n} \left(f_j - \sum_{i=1}^{m} \alpha_{ij} \right) y_j$$
$$+ \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} \tag{4.28}$$

Subject to: (4.2), (4.3), (4.5) − (4.14), (4.17).

$$z_{4,13} = \max_{\alpha,\theta} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \alpha_{ij})\, x_{ij} + \sum_{j=1}^{n} \left(f_j - \sum_{i=1}^{m} \alpha_{ij} \right) y_j + \sum_{k=1}^{p} g_k z_k$$
$$+ \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} + \theta \left(\sum_{k=1}^{p} \sum_{j=1}^{n} w_{kj} - \sum_{k=1}^{p} a_k z_k \right) \tag{4.29}$$

Subject to: (4.2), (4.3), (4.5) − (4.12), (4.14), (4.17), (4.20).

Since constraint set (4.5) is satisfied, the last term in (4.29) is 0 at optimality and these two bound values are equal.

$$\text{Now } Z_4 = \max_{\alpha} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n} (c_{ij} + \alpha_{ij})\, x_{ij} + \sum_{j=1}^{n} \left(f_j - \sum_{i=1}^{m} \alpha_{ij} \right) y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj} w_{kj} \tag{4.30}$$

$$\text{Subject to: } (4.2), (4.3), (4.5) - (4.12), (4.17).$$

Since constraints (4.2),(4.3), and (4.5) are all satisfied, constraints (4.13) and (4.14) are automatically satisfied and are therefore redundant, and the last equality also holds. ∎

4. Similarly part 4 holds by the same logic, replacing constraints (4.4) with constraints (4.6):

$$z_6 = \max_{\delta} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n} (c_{ij} + \delta_j d_i)\, x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} (b_{kj} - \delta_j)\, w_{kj} \tag{4.31}$$

$$\text{Subject to: } (4.2) - (4.5), (4.7) - (4.14), (4.19).$$

$$z_{6,13} = \max_{\delta,\theta} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n} (c_{ij} + \delta_j d_i)\, x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} (b_{kj} - \delta_j)\, w_{kj} + \theta \left(\sum_{k=1}^{p}\sum_{j=1}^{n} w_{kj} - \sum_{k=1}^{p} a_k z_k \right) \tag{4.32}$$

$$\text{Subject to: } (4.2) - (4.5), (4.7) - (4.12), (4.14), (4.19), (4.20).$$

Since constraint set (4.5) is satisfied, the last term in (4.32) is 0 at optimality and these two bound values are equal.

$$\text{Now } Z_6 = \max_{\delta} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n} (c_{ij} + \delta_j d_i)\, x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} (b_{kj} - \delta_j)\, w_{kj} \tag{4.33}$$

$$\text{Subject to: } (4.2) - (4.5), (4.7) - (4.12), (4.19).$$

Since constraints (4.2),(4.3), and (4.5) are all satisfied, constraints (4.13) and (4.14) are automatically satisfied and are therefore redundant, and the last equality also holds. ▮

Theorem 4.3.2

1. $z_4 = z_{4,14} = Z_4$,

2. $z_6 = z_{6,14} = Z_6$.

Proof

1.

$$z_4 = \max_{\alpha} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}+\alpha_{ij})\,x_{ij} + \sum_{j=1}^{n}\left(f_j - \sum_{i=1}^{m}\alpha_{ij}\right)y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj}w_{kj} \tag{4.34}$$

$$\text{Subject to: } (4.2),(4.3),(4.5)-(4.14),(4.17).$$

$$z_{4,14} = \max_{\alpha,\theta} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}+\alpha_{ij})\,x_{ij} + \sum_{j=1}^{n}\left(f_j - \sum_{i=1}^{m}\alpha_{ij}\right)y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj}w_{kj} + \sigma\left(\sum_{i=1}^{m} d_i - \sum_{j=1}^{n} s_j y_j\right) \tag{4.35}$$

$$\text{Subject to: } (4.2),(4.3),(4.5)-(4.13),(4.17),(4.21).$$

Since constraints sets (4.2) and (4.3) are satisfied the last term in (4.35) is 0 at optimality and these two bounds are equal.

$$\text{Now } Z_4 = \max_{\alpha} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}+\alpha_{ij})\,x_{ij} + \sum_{j=1}^{n}\left(f_j - \sum_{i=1}^{m}\alpha_{ij}\right)y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj}w_{kj} \tag{4.36}$$

$$\text{Subject to: } (4.2),(4.3),(4.5)-(4.12),(4.17).$$

Since constraints (4.2),(4.3), and (4.5) are all satisfied, constraints (4.13) and (4.14) are automatically satisfied and are therefore redundant, and the last equality also holds. ▮

2.

$$z_6 = \max_{\delta} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \delta_j d_i) x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} (b_{kj} - \delta_j) w_{kj} \quad (4.37)$$

Subject to: $(4.2) - (4.5), (4.7) - (4.14), (4.19)$.

$$z_{6,14} = \max_{\delta,\theta} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \delta_j d_i) x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} (b_{kj} - \delta_j) w_{kj} + \sigma \left(\sum_{i=1}^{m} d_i - \sum_{j=1}^{n} s_j y_j \right) \quad (4.38)$$

Subject to: $(4.2) - (4.5), (4.7) - (4.13), (4.19), (4.21)$.

Since constraints sets (4.2) and (4.3) are satisfied the last term in (4.38) is 0 at optimality and these two bounds are equal.

$$\text{Now } Z_6 = \max_{\delta} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \delta_j d_i) x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} (b_{kj} - \delta_j) w_{kj} \quad (4.39)$$

Subject to: $(4.2) - (4.5), (4.7) - (4.12), (4.19)$.

Since constraints (4.2),(4.3), and (4.5) are all satisfied, constraints (4.13) and (4.14) are automatically satisfied and are therefore redundant, and the last equality also holds. ∎

Theorem 4.3.3 $Z_5 = z_{5,13}$

Proof

$$Z_5 = \max_{\beta} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} (g_k - \beta_k a_k) z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} (b_{kj} + \beta_k) w_{kj} \quad (4.40)$$

Subject to: $(4.2) - (4.4), (4.6) - (4.12), (4.18)$.

$$z_{5,13} = \max_{\beta,\theta} \min x, y, z, w \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} (g_k - (\beta_k + \theta) a_k) z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} (b_{kj} + (\beta_k + \theta)) w_{kj} \tag{4.41}$$

Subject to: $(4.2) - (4.4), (4.6) - (4.12), (4.14), (4.18), (4.20).$

Now constraint (4.14) is redundant since constraints sets (4.2) and (4.3) are not relaxed so this constraint has no effect on the proposed relationship.

Substituting $\beta'_k = \beta_k + \theta$ for every j, we get:

$$z_{5,13} = \max_{\beta'>0} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} \left(g_k - \beta'_k a_k\right) z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} \left(b_{kj} + \beta'_k\right) w_{kj} \tag{4.42}$$

Subject to: $(4.2) - (4.4), (4.6) - (4.12),$

which is exactly Z_5 where we have β' for β. Hence $Z_5 = z_{5,13}$. ∎

Theorem 4.3.4 $Z_3 = z_{3,14}$

Proof

$$Z_3 = \max_{\lambda} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n} (c_{ij} + \lambda_j d_i) x_{ij} + \sum_{j=1}^{n} (f_j - \lambda_j s_j) y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj} w_{kj} \tag{4.43}$$

Subject to: $(4.2), (4.4) - (4.12), (4.16).$

$$z_{3,14} = \max_{\lambda,\sigma} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n} (c_{ij} + \lambda_j d_i) x_{ij} + \sum_{j=1}^{n} (f_j - (\lambda_j + \sigma) s_j) y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj} w_{kj} + \sigma \sum_{i=1}^{m} d_i \tag{4.44}$$

Subject to: $(4.2), (4.4) - (4.13), (4.16), (4.21).$

Now constraint (4.13) is redundant since constraint (4.5) is not relaxed so this constraint has no effect on the proposed relationship.

Also we know that $\sum_{j=1}^{n} x_{ij} = 1$ so we can write $\sigma \sum_{i=1}^{m} d_i$ as

$$\left(\sigma \sum_{i=1}^{m} d_i\right)\left(\sum_{j=1}^{n} x_{ij}\right) = \sum_{j=1}^{n} \sigma \sum_{i=1}^{m} d_i x_{ij} \tag{4.45}$$

Using (4.45) we can write $z_{3,14}$ as

$$z_{3,14} = \max_{\lambda,\sigma} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n} \left(c_{ij} + (\lambda_j + \sigma)\, d_i\right) x_{ij} + \sum_{j=1}^{n} \left(f_j - (\lambda_j + \sigma)\, s_j\right) y_j$$
$$+ \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj} w_{kj} \tag{4.46}$$

Substituting $\lambda'_j = \lambda_j + \sigma$ for every j, we get:

$$z_{3,14} = \max_{\lambda'>0} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n} \left(c_{ij} + \lambda'_j d_i\right) x_{ij} + \sum_{j=1}^{n} \left(f_j - \lambda'_j s_j\right) y_j$$
$$+ \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj} w_{kj} \tag{4.47}$$

$$\text{Subject to: } (4.2), (4.4) - (4.12),$$

which is exactly Z_3 where we have λ' for λ. Hence $Z_3 = z_{3,14}$. ∎

Theorem 4.3.5 $z_{13,14} = z$, *the optimal value of problem (P4.2).*

Proof

$$z_{13,14} = \max_{\theta,\sigma} \min_{x,y,z,w} \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj} w_{kj}$$
$$+\theta \left(\sum_{k=1}^{p}\sum_{j=1}^{n} w_{kj} - \sum_{k=1}^{p} a_k z_k\right) + \sigma \left(\sum_{i=1}^{m} d_i - \sum_{j=1}^{n} s_j y_j\right) \tag{4.48}$$

$$\text{Subject to: } (4.2) - (4.12), (4.20), (4.21).$$

Since constraints sets (4.2), (4.3), and (4.5) are satisfied the last two terms of (4.48) will be 0 at optimality. For example the term: $\sigma \left(\sum_{i=1}^{m} d_i - \sum_{j=1}^{n} s_j y_j\right)$ in the above objective function is minimizing over $x, y, z,$ and w a negative quantity: $\sum_{i=1}^{m} d_i - \sum_{j=1}^{n} s_j y_j$ but maximizing this over σ will push the whole term to 0. Therefore relaxing constraints (4.13) and (4.14) has no effect on the bound value and since by definition they are surrogate to the formulation this bound is equal to the optimal value of the problem, z. ▮

Theorem 4.3.6 $z_4 = z_{4,13,14} = Z_4$.

Proof

$$z_4 = \max_{\alpha} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \alpha_{ij}) x_{ij} + \sum_{j=1}^{n} \left(f_j - \sum_{i=1}^{m} \alpha_{ij} \right) y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} \tag{4.49}$$

Subject to: (4.2), (4.3), (4.5) – (4.14), (4.17).

$$z_{4,13,14} = \max_{\alpha,\theta,\sigma} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \alpha_{ij}) x_{ij} + \sum_{j=1}^{n} \left(f_j - \sum_{i=1}^{m} \alpha_{ij} \right) y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} + \theta \left(\sum_{k=1}^{p} \sum_{j=1}^{n} w_{kj} - \sum_{k=1}^{p} a_k z_k \right) + \sigma \left(\sum_{i=1}^{m} d_i - \sum_{j=1}^{n} s_j y_j \right) \tag{4.50}$$

Subject to: (4.2), (4.3), (4.5) – (4.12), (4.17), (4.20), (4.21).

Since constraints sets (4.2), (4.3), and (4.5) are satisfied the last two terms of (4.50) will be 0 at optimality and these two bounds are equal.

$$Z_4 = \max_{\alpha} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \alpha_{ij}) x_{ij} + \sum_{j=1}^{n} \left(f_j - \sum_{i=1}^{m} \alpha_{ij} \right) y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} b_{kj} w_{kj} \tag{4.51}$$

Subject to: (4.2), (4.3), (4.5) – (4.12), (4.17).

Since constraints (4.2),(4.3), and (4.5) are all satisfied, constraints (4.13) and (4.14) are automatically satisfied and are therefore redundant, and the last equality also holds. ▮

Theorem 4.3.7 $z_6 = z_{6,13,14} = Z_6$.

Proof

$$z_6 = \max_{\delta} \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + \delta_j d_i) x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k$$

$$+\sum_{k=1}^{p}\sum_{j=1}^{n}\left(b_{kj}-\delta_j\right)w_{kj} \tag{4.52}$$

Subject to: $(4.2)-(4.5),(4.7)-(4.14),(4.19)$.

$$z_{6,13,14}=\max_{\delta,\theta,\sigma}\min_{x,y,z,w}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}+\delta_j d_i\right)x_{ij}+\sum_{j=1}^{n}f_j y_j+\sum_{k=1}^{p}g_k z_k$$

$$+\sum_{k=1}^{p}\sum_{j=1}^{n}\left(b_{kj}-\delta_j\right)w_{kj}+\theta\left(\sum_{k=1}^{p}\sum_{j=1}^{n}w_{kj}-\sum_{k=1}^{p}a_k z_k\right)+\sigma\left(\sum_{i=1}^{m}d_i-\sum_{j=1}^{n}s_j y_j\right) \tag{4.53}$$

Subject to: $(4.2)-(4.5),(4.7)-(4.12),(4.19)-(4.21)$.

Since constraints sets (4.2), (4.3), and (4.5) are satisfied the last two terms of (4.50) will be 0 at optimality and these two bounds are equal.

$$\text{Now } Z_6=\max_{\delta}\min_{x,y,z,w}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}+\delta_j d_i\right)x_{ij}+\sum_{j=1}^{n}f_j y_j+\sum_{k=1}^{p}g_k z_k$$

$$+\sum_{k=1}^{p}\sum_{j=1}^{n}\left(b_{kj}-\delta_j\right)w_{kj} \tag{4.54}$$

Subject to: $(4.2)-(4.5),(4.7)-(4.12),(4.19)$.

Since constraints (4.2),(4.3), and (4.5) are all satisfied, constraints (4.13) and (4.14) are automatically satisfied and are therefore redundant, and the last equality also holds. ∎

Theorem 4.3.8 $Z_{2,3,6}=Z_{2,3,4,6}$

Proof This proof is split into the two problems in question and then it is shown that the two parts are equal.

1.

$$Z_{2,3,6}\left(\mu,\lambda,\delta\right)=\min_{x,y,z,w}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}-\mu_i+\lambda_j d_i+\delta_j d_i\right)x_{ij}+\sum_{j=1}^{n}\left(f_j-\lambda_j s_j\right)y_j$$

$$+\sum_{k=1}^{p}g_k z_k+\sum_{k=1}^{p}\sum_{j=1}^{n}\left(b_{kj}-\delta_j\right)w_{kj}+\sum_{i=1}^{m}\mu_i \tag{4.55}$$

Subject to: $(4.4),(4.5),(4.7)-(4.12)$.

This problem decomposes into:

$$\sum_{i=1}^{m} \mu_i \tag{4.56}$$

$$+\min_{x,y} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \mu_i + \lambda_j d_i + \delta_j d_i)\, x_{ij} + \sum_{j=1}^{n} (f_j - \lambda_j s_j)\, y_j \tag{4.57}$$

Subject to: $(4.4), (4.8), (4.10), (4.11)$.

$$+\min_{z,w} \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} (b_{kj} - \delta_j)\, w_{kj} \tag{4.58}$$

Subject to: $(4.5), (4.7), (4.9), (4.12)$.

The optimal value of the first subproblem (4.57) is:

$$\sum_{j=1}^{n} \left(f_j - \lambda_j s_j + \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \mu_i + d_i (\lambda_j + \delta_j))^- \right)^- \tag{4.59}$$

2.

$$Z_{2,3,4,6}(\mu, \lambda, \alpha, \delta) = \min_{x,y,z,w} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \mu_i + \lambda_j d_i + \alpha_{ij} + \delta_j d_i)\, x_{ij}$$

$$+ \sum_{j=1}^{n} \left(f_j - \lambda_j s_j - \sum_{i=1}^{m} \alpha_{ij} \right) y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} (b_{kj} - \delta_j)\, w_{kj} + \sum_{i=1}^{m} \mu_i \tag{4.60}$$

Subject to: $(4.5), (4.7) - (4.12)$.

This problem decomposes into:

$$\sum_{i=1}^{m} \mu_i \tag{4.61}$$

$$+\min_{x} \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \mu_i + \alpha_{ij} + \lambda_j d_i + \delta_j d_i)\, x_{ij} \tag{4.62}$$

Subject to: (4.10).

$$+\min_{y} \sum_{j=1}^{n} \left(f_j - \lambda_j s_j - \sum_{i=1}^{m} \alpha_{ij} \right) y_j \tag{4.63}$$

Subject to: $(4.8), (4.11)$.

$$+\min_{z,w} \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p} \sum_{j=1}^{n} (b_{kj} - \delta_j)\, w_{kj} \tag{4.64}$$

Subject to: $(4.5), (4.7), (4.9), (4.12)$.

The optimal value of the first subproblem (4.62) is:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}-\mu_i+\alpha_{ij}+d_i\left(\lambda_j+\delta_j\right)\right)^- \tag{4.65}$$

The optimal value of the second subproblem (4.63) is:

$$\sum_{j=1}^{n}\left(f_j-\lambda_j s_j-\sum_{i=1}^{m}\alpha_{ij}\right)^- \tag{4.66}$$

The third subproblem (4.64) is the same as (4.58) as in part 1.

Now we know that $Z_{2,3,4,6} \leqslant Z_{2,3,6}$. To prove $Z_{2,3,4,6} \geqslant Z_{2,3,6}$ consider $\left(\hat{\mu},\hat{\lambda},\hat{\delta}\right)$, an optimal solution of dual problem (4.55) and take

$$\hat{\alpha_{ij}} = -\left(c_{ij}-\hat{\mu_i}+d_i\left(\hat{\lambda_j}+\hat{\delta_j}\right)\right)^- \geqslant 0\ \forall i,j \tag{4.67}$$

Using part 2 and then part 1 it follows:

$$\begin{aligned}
&Z_{2,3,4,6}\left(\hat{\mu},\hat{\lambda},\hat{\alpha},\hat{\delta}\right)-\sum_{i=1}^{m}\mu_i-\min_{z,w}\sum_{k=1}^{p}g_k z_k+\sum_{k=1}^{p}\sum_{j=1}^{n}\left(b_{kj}-\delta_j\right)w_{kj}\\
&\qquad\text{Subject to: } (4.5),(4.7),(4.9),(4.12)\\
&=\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}-\hat{\mu_i}+\hat{\alpha_{ij}}+d_i\left(\hat{\lambda_j}+\hat{\delta_j}\right)\right)^-+\sum_{j=1}^{n}\left(f_j-\hat{\lambda_j}s_j-\sum_{i=1}^{m}\hat{\alpha_{ij}}\right)^-
\end{aligned} \tag{4.68}$$

Using (4.67):

$$\begin{aligned}
(4.68) &= \sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}-\hat{\mu_i}+d_i\left(\hat{\lambda_j}+\hat{\delta_j}\right)-\left(c_{ij}-\hat{\mu_i}+d_i\left(\hat{\lambda_j}+\hat{\delta_j}\right)\right)^-\right)^-\\
&\quad+\sum_{j=1}^{n}\left(f_j-\hat{\lambda_j}s_j+\sum_{i=1}^{m}\left(c_{ij}-\hat{\mu_i}+d_i\left(\hat{\lambda_j}+\hat{\delta_j}\right)\right)^-\right)^- && (4.69)\\
&=\sum_{j=1}^{n}\left(f_j-\hat{\lambda_j}s_j+\sum_{i=1}^{m}\left(c_{ij}-\hat{\mu_i}+d_i\left(\hat{\lambda_j}+\hat{\delta_j}\right)\right)^-\right)^- && (4.70)\\
&=Z_{2,3,6}\left(\hat{\mu},\hat{\lambda},\hat{\delta}\right)-\sum_{i=1}^{m}\mu_i-\min_{z,w}\sum_{k=1}^{p}g_k z_k+\sum_{k=1}^{p}\sum_{j=1}^{n}\left(b_{kj}-\delta_j\right)w_{kj} && (4.71)\\
&\qquad\text{Subject to: } (4.5),(4.7),(4.9),(4.12)
\end{aligned}$$

Therefore

$$Z_{2,3,4,6}=\max_{\mu,\lambda,\alpha,\delta} Z_{2,3,4,6}\left(\mu,\lambda,\alpha,\delta\right) \geqslant Z_{2,3,4,6}\left(\hat{\mu},\hat{\lambda},\hat{\alpha},\hat{\delta}\right) = Z_{2,3,6}\left(\hat{\mu},\hat{\lambda},\hat{\delta}\right) = Z_{2,3,6} \quad \blacksquare$$

4.3.2 Dominant bounds

Dominant bounds, e.g. $A < B$, are defined as: for any given instance A is less than or equal to B and for some instance A is strictly less than B. These bounds will be proved by providing problem instances that show the strict inequalities. In general it will be shown that there is at least one feasible solution of B which is greater than the optimal solution of A. The optimal solutions of A are guaranteed by the solutions providing subgradients that are 0, or a convex combination of them exists that results in a 0 vector. This ensures that the solution is optimal over all multipliers in an equivalent fashion as a first-derivative being equal to 0 in differential calculus.

Theorem 4.3.9 $z_{5,6} < z_6$

Proof We know that z_{LP} is less than or equal to all the Lagrangian bounds and these are less than or equal to z. Obviously the bound from relaxing two constraints is further from the optimal than the bound obtained by relaxing a single constraint. So to prove the theorem it is sufficient to provide a numerical example showing $z_{5,6} < z_6$.

Consider an instance of TSCPLP where:

$$c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad d_i = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \quad g_k = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_k = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix},$$

$$f_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad s_j = \begin{bmatrix} 9 & 9 & 9 \end{bmatrix}.$$

A feasible solution for z_6 is $z_6 = 6$ with:

$$y_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

For $z_{5,6}$, take $\beta = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and $\delta = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$. Then $z_{5,6} = 4$ with optimal solutions:

1. $y_j = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; z_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. $y_j = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $x_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $z_k = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $w_{kj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. $y_j = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $x_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $z_k = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $w_{kj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4. $y_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $w_{kj} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

The subgradients resulting from the above solutions are:

	$\sum_j w_{kj} - a_k z_k$	$\sum_i d_i x_{ij} - \sum_k w_{kj}$
1.	$\begin{bmatrix} -3 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 6 & 0 & 0 \end{bmatrix}$
2.	$\begin{bmatrix} 0 & -3 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 6 & 0 \end{bmatrix}$
3.	$\begin{bmatrix} 0 & 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 6 \end{bmatrix}$
4.	$\begin{bmatrix} -1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} -2 & -2 & -2 \end{bmatrix}$

A convex combination of the subgradient vectors can easily be found that results in a zero vector. Therefore $z_{5,6}$ has an optimal solution with $\beta = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, $\delta = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ and $z_{5,6} = 4$. This shows that $z_6 > z_{5,6}$. ∎

Theorem 4.3.10 $z_{2,6} < z_6$

Proof We know that z_{LP} is less than or equal to all the Lagrangian bounds and these are less than or equal to z. Obviously the bound from relaxing two constraints is further from the optimal than the bound obtained by relaxing a single constraint. So to prove the theorem it is sufficient to provide a numerical example showing $z_{2,6} < z_6$.

Consider an instance of TSCPLP where:

$$c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad d_i = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \quad g_k = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_k = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix},$$

$$f_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad s_j = \begin{bmatrix} 9 & 9 & 9 \end{bmatrix}.$$

A feasible solution for z_6 is $z_6 = 6$ with:

$$y_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

For $z_{2,6}$, take $\mu = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and $\delta = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$. Then $z_{2,6} = 4$ with optimal solutions:

1. $y_j = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; z_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. $y_j = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; z_k = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. $y_j = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; z_k = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4. $y_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

The subgradients resulting from the above solutions are:

	$\sum_j x_{ij} - 1$	$\sum_i d_i x_{ij} - \sum_k w_{kj}$
1.	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 6 & 0 & 0 \end{bmatrix}$
2.	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 6 & 0 \end{bmatrix}$
3.	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 6 \end{bmatrix}$
4.	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -2 & -2 & -2 \end{bmatrix}$

A convex combination of the subgradient vectors can easily be found that results in a zero vector. Therefore $z_{2,6}$ has an optimal solution with $\mu = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, $\delta = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ and $z_{2,6} = 4$. This shows that $z_6 > z_{2,6}$. ∎

Theorem 4.3.11 $z_{2,3,6} < z_4$

Proof We know that z_{LP} is less than or equal to all the Lagrangian bounds and these are less than or equal to z. Obviously the bound from relaxing two constraints is further from the optimal than the bound obtained by relaxing a single constraint.

From Theorems 4.3.8 and 4.3.1(3): $Z_{2,3,6} = Z_{2,3,4,6} \leqslant Z_4 = z_4$ so to prove the theorem it is sufficient to provide a numerical example showing $z_{2,3,6} < z_4$.

Consider an instance of TSCPLP where:

$$c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad d_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad g_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad a_k = \begin{bmatrix} 4 \\ 4 \end{bmatrix},$$

$$f_j = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad s_j = \begin{bmatrix} 3 & 3 \end{bmatrix}.$$

A feasible solution for z_4 is $z_4 = 4$ with:

$$y_j = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; z_k = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For $z_{2,3,6}$, take $\mu = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $\lambda = \begin{bmatrix} 0 & 0 \end{bmatrix}$, and $\delta = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Then $z_{2,3,6} = 3$ with optimal solutions:

1. $y_j = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; z_k = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
2. $y_j = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \; z_k = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
3. $y_j = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; z_k = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

The subgradients resulting from the above solutions are:

	$\sum_j x_{ij} - 1$	$\sum_i d_i x_{ij} - s_j y_j$	$\sum_i d_i x_{ij} - \sum_k w_{kj}$
1.	$\begin{bmatrix} 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 \end{bmatrix}$
2.	$\begin{bmatrix} 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 \end{bmatrix}$
3.	$\begin{bmatrix} 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -2 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \end{bmatrix}$

A convex combination of the subgradient vectors can easily be found that results in a zero vector. Therefore $z_{2,6}$ has an optimal solution with $\mu = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $\lambda = \begin{bmatrix} 0 & 0 \end{bmatrix}$, and $\delta = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $z_{2,3,6} = 3$. This shows that $z_4 > z_{2,3,6}$. ∎

Theorem 4.3.12 $Z_{2,4} < z_4$

Proof We know that z_{LP} is less than or equal to all the Lagrangian bounds and these are less than or equal to z. By Theorem 4.3.1(3) $Z_4 = z_4$ and the bound from relaxing two constraints is further from the optimal than the bound obtained by relaxing a single constraint. So then $Z_{2,4} \leqslant Z_4 = z_4$ and to prove the theorem it is sufficient to provide a numerical example showing $Z_{2,4} < z_4$.

Consider an instance of TSCPLP where:

$$c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \quad d_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad g_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad a_k = \begin{bmatrix} 4 \\ 4 \end{bmatrix},$$

$$f_j = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad s_j = \begin{bmatrix} 3 & 3 \end{bmatrix}.$$

A feasible solution for z_4 is $z_4 = 5$ with:

$$y_j = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; z_k = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For $Z_{2,4}$, take $\mu = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $\alpha = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Then $Z_{2,4} = 3$ with optimal solutions:

1. $y_j = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $x_{ij} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $z_k = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $w_{kj} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

2. $y_j = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $x_{ij} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $z_k = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $w_{kj} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

The subgradients resulting from the above solutions are:

	$\sum_j x_{ij} - 1$	$x_{ij} - y_j$
1.	$\begin{bmatrix} 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
2.	$\begin{bmatrix} 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

A convex combination of the subgradient vectors can easily be found that results in a zero vector. Therefore $Z_{2,4}$ has an optimal solution with $\mu = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $\alpha = \begin{bmatrix} 0 & 0 \end{bmatrix}$, and $Z_{2,4} = 3$. This shows that $z_4 > Z_{2,4}$. ▮

Theorem 4.3.13 $z_{2,6} < z_2$

Proof We know that z_{LP} is less than or equal to all the Lagrangian bounds and these are less than or equal to z. Obviously the bound from relaxing two constraints is further from the optimal than the bound obtained by relaxing a single constraint. So to prove the theorem it is sufficient to provide a numerical example showing $z_{2,6} < z_2$.

Consider an instance of TSCPLP where:

$$c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad d_i = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \quad g_k = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_k = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix},$$

$$f_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad s_j = \begin{bmatrix} 9 & 9 & 9 \end{bmatrix}.$$

A feasible solution for z_2 is $z_2 = 6$ with:

$$y_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

For $z_{2,6}$, take $\mu = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and $\delta = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$. Then $z_{2,6} = 4$ with optimal solutions:

1. $y_j = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; z_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. $y_j = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; z_k = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. $y_j = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; z_k = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4. $y_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $w_{kj} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

The subgradients resulting from the above solutions are:

	$\sum_j x_{ij} - 1$	$\sum_i d_i x_{ij} - \sum_k w_{kj}$
1.	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 6 & 0 & 0 \end{bmatrix}$
2.	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 6 & 0 \end{bmatrix}$
3.	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 6 \end{bmatrix}$
4.	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -2 & -2 & -2 \end{bmatrix}$

A convex combination of the subgradient vectors can easily be found that results in a zero vector. Therefore $z_{2,6}$ has an optimal solution with $\mu = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, $\delta = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ and $z_{2,6} = 4$. This shows that $z_2 > z_{2,6}$. ▮

4.3.3 Other relationships

Theorem 4.3.14 *There is at least one instance where $z_2 > Z_3$ and at least one instance where the converse is true i.e. $z_2 < Z_3$.*

Proof The theorem will be proved by providing an actual example for each case. To show $z_2 > Z_3$ consider an instance of TSCPLP where:

$$c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad d_i = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \quad g_k = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_k = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix},$$

$$f_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad s_j = \begin{bmatrix} 4 & 4 & 4 \end{bmatrix}.$$

A feasible solution for z_2 is $z_2 = 6$ with:

$$y_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \; x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \; w_{kj} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

For Z_3, take $\lambda = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$. This gives $Z_3 = 5.25$ with optimal solutions:

1. $y_j = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $w_{kj} = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

2. $y_j = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $x_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $w_{kj} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

3. $y_j = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $x_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $w_{kj} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

4. $y_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $w_{kj} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

The subgradients defined by $\sum_i d_i x_{ij} - s_j y_j$ resulting from the above solutions are:

1. $\begin{bmatrix} 5 & 0 & 0 \end{bmatrix}$
2. $\begin{bmatrix} 0 & 5 & 0 \end{bmatrix}$
3. $\begin{bmatrix} 0 & 0 & 5 \end{bmatrix}$
4. $\begin{bmatrix} -1 & -1 & -1 \end{bmatrix}$

A convex combination of the subgradient vectors can easily be found that results in a zero vector. Therefore Z_3 has an optimal solution with $\lambda = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$ and $Z_3 = 5.25$. This shows that $z_2 > Z_3$.

To show $z_2 < Z_3$ consider an instance of TSCPLP where:

$$c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad d_i = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad g_k = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_k = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix},$$

$$f_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad s_j = \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}.$$

A feasible solution for Z_3 is $Z_3 = 6$ with:

$$y_j = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},\ x_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ z_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},\ w_{kj} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For z_2, take $\mu = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. This gives $z_2 = 4.5$ with optimal solutions:

1. $y_j = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},\ x_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ z_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},\ w_{kj} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. $y_j = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},\ x_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ z_k = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},\ w_{kj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. $y_j = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},\ x_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ z_k = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},\ w_{kj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

4. $y_j = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},\ x_{ij} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},\ z_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},\ w_{kj} = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

The subgradients defined by $\sum_i x_{ij} - 1$ resulting from the above solutions are:

1. $\begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$
2. $\begin{bmatrix} 0 & -1 & 0 \end{bmatrix}$
3. $\begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

A convex combination of the subgradient vectors can easily be found that results in a zero vector. Therefore z_2 has an optimal solution with $\mu = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $z_2 = 4.5$. This shows that $z_2 < Z_3$. ∎

Strength	Theoretical result	Proposed dominance
Strongest	2, 4, 24	4
	5, 45	2, 24
	3, 6, 46	5, 45
		3
		23
		34
		234
		25
		6, 46
	23, 34, 35	245
	26, 36, 56	26, 246
	234, 246	56
		36
		35
		235
		345
		356
		236
	25, 235, 245	456
	236, 345, 346	346
Weakest	256, 356, 456	256

Table 4.1: Summary of theoretical results.

4.4 Summary of relationships

Before moving onto the computational analysis a summary of the theoretical results is presented, including a list of relationships that need to be investigated further. Table 4.1 groups the relaxations into three main categories from the theoretical results. The groups are ordered from the strongest to weakest bounds (closest to farthest) from the optimal solution. The proposed dominance list is a combination of the theoretical results and the results from the computational work later in the thesis. Since the computational work is based on a heuristic the exact structure cannot be guaranteed to follow this list. The theoretical relationships that need to be investigated further to prove their equivalence or dominance are:

- $z_2 > z_3$, $z_2 > z_5$, and $z_2 > z_6$.
- $z_5 > z_3$ and $z_5 > z_6$.
- The equivalence or dominance between z_3 and z_6.
- The dominance or equivalence between relaxations in the inner group:

- compared to each other
- and compared to the linear programming bound.

The discussion now turns to the nature of the Lagrangian relaxation bounds in relation to the linear programming bound and the optimal solution. These results are formed from the theoretical work in Chapters 3 and 4, and the results of the computational study in Chapter 5.

- The relaxations that can be said to be equal to the linear programming bound are:
  - $z_{2,5}$, $z_{2,3,5}$, $z_{2,3,6}$,
  - $z_{2,4,5}$, $z_{2,5,6}$, $z_{3,4,5}$,
  - $z_{3,4,6}$, $z_{3,5,6}$, $z_{4,5,6}$.
- The relaxations that have instances which are strictly better than the linear programming bound are:
  - z_2, z_3, z_4, z_5, z_6,
  - $z_{2,3}$, $z_{2,4}$, $z_{3,4}$,
  - $z_{4,5}$, $z_{4,6}$, $z_{2,3,4}$.
- The relaxations whose strength relative to the linear programming bound is not known are:
  - $z_{2,6}$, $z_{3,5}$, $z_{3,6}$,
  - $z_{5,6}$, $z_{2,4,6}$.
- The relaxation that is equal to the optimal solution is:
  - z_4.
- The relaxations that have instances for which the optimal solution can be found are:
  - z_2, z_3, z_5, z_6,
  - $z_{2,3}$, $z_{2,4}$, $z_{2,5}$, $z_{2,6}$,
  - $z_{3,4}$, $z_{3,5}$, $z_{4,5}$, $z_{4,6}$, $z_{5,6}$,
  - $z_{2,3,4}$, $z_{2,3,5}$, $z_{2,4,5}$, $z_{2,4,6}$, $z_{2,5,6}$,
  - $z_{3,4,5}$, $z_{3,4,6}$, $z_{4,5,6}$.

- The relaxations for which it is not known if there exist instances for which the optimal solution can be found are:
  - $z_{3,6}$, $z_{2,3,6}$, $z_{3,5,6}$.

4.5 Conclusion

This concludes the theoretical analysis of the TSCPLP. Over the last two chapters it has been shown that there are relaxations of the problem that lead to bounds on the optimal solution that are closer than the bounds from other relaxations. These are important qualities to know when considering a solution method that involves Lagrangian relaxation. A relaxation that yields a poor bound will result in poor solutions for the overall problem. When strong bounds are incorporated into a solution method such as a branch-and-bound algorithm, the solution method can work more intelligently towards quality solutions, such as fathoming nodes sooner in a branch-and-bound algorithm.

The results found theoretically over the last two chapters will be used to form a solution method in the next chapter, which is concerned with the computational analysis of the relaxations discussed. A Lagrangian heuristic is presented for solving and analysing various relaxations. The performance of the relaxations is tested with data sets derived from the literature, and an analysis of the effect of several problem characteristics is undertaken.

Chapter 5

Computational analysis of the TSCPLP

5.1 Introduction

In this chapter the computational analysis of the Two-Stage Capacitated Plant Location Problem (TSCPLP) is presented. The theoretical results behind this analysis can be found in Chapters 3 and 4. The relationships found in those chapters will be corroborrated here with the computational work. This chapter provides a Lagrangian heuristic to solve the TSCPLP, using the commercial software CPLEX to find upper and lower bounds. The Lagrangian relaxations are tested with data sets dervied from those outlined in the literature and the effect of several problem characteristics on solution quality is examined. These include the ratio of depot capacity to customer demand and plant capacity to depot capacity.

The aim of this chapter is to investigate the relationships between constraints and solution quality. As Lagrangian relaxations are used in many solution techniques such as heuristics, and in branch-and-bound type enumeration methods, it is important to understand further the effect the constraints have on solution quality. Since a good relaxation has the potential to cut solution time in an enumeration method considerably, it is important to choose a set or sets of constraints to dualize that will result in upper and lower bounds that are close together. This guarantees that the optimal solution is between these values and the feasible upper bound can be taken as the optimal or near-optimal solution. The analysis performed in this chapter can accurately compare the upper and lower bound solutions to the actual optimal solutions and linear programming bounds for some of the data sets through the use of CPLEX. When computation time is restricted, several relaxations provide bounds on the optimal solution before CPLEX can find an optimal solution. This shows that even when the optimal solution is difficult to find, Lagrangian relaxation can be used to find bounds on the optimal solution.

Firstly, the formulation for the TSCPLP is presented again for reference in this chapter.

(P5.1)

$$Z = \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj}w_{kj} \tag{5.1}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \in I \tag{5.2}$$

$$\sum_{i=1}^{m} d_i x_{ij} \leqslant s_j y_j \qquad \forall j \in J \tag{5.3}$$

$$x_{ij} \leqslant y_j \qquad \forall i \in I, \forall j \in J \tag{5.4}$$

$$\sum_{j=1}^{n} w_{kj} \leqslant a_k z_k \qquad \forall k \in K \tag{5.5}$$

$$\sum_{k=1}^{p} w_{kj} = \sum_{i=1}^{m} d_i x_{ij} \qquad \forall j \in J \tag{5.6}$$

$$w_{kj} \geqslant 0 \qquad \forall k \in K, \forall j \in J \tag{5.7}$$

$$y_j \text{ integer} \qquad \forall j \in J \tag{5.8}$$

$$z_k \text{ integer} \qquad \forall k \in K \tag{5.9}$$

$$0 \leqslant x_{ij} \leqslant 1 \qquad \forall i \in I, \forall j \in J \tag{5.10}$$

$$0 \leqslant y_j \leqslant 1 \qquad \forall j \in J \tag{5.11}$$

$$0 \leqslant z_k \leqslant 1 \qquad \forall k \in K \tag{5.12}$$

All constraints can be interpreted as described in Chapter 1.

Surrogate constraints (5.13) and (5.14) can be added as follows:

$$\sum_{k=1}^{p} a_k z_k \geqslant \sum_{i=1}^{m} d_i \tag{5.13}$$

$$\sum_{j=1}^{n} s_j y_j \geqslant \sum_{i=1}^{m} d_i \tag{5.14}$$

This is a revision of the surrogate constraint (Chapter #.13) used in previous chapters. It is derived by using the original version of (5.13) e.g.

$$\sum_{k=1}^{p}\sum_{j=1}^{n} w_{kj} \leqslant \sum_{k=1}^{p} a_k z_k,$$

and the equalities defined by (5.6) and then (5.2). This ensures that the total capacity of the open plants is at least as large as the total customer demand moving through the system. This is therefore of a similar structure as surrogate constraint (5.14) which is derived by summing (5.3) over all j depots and using the equalities (5.2) and states that the total capacity of the open depots is at least as large as the total demand being transported from them to the customers. These two constraints are redundant in the original formulation but strengthen some of the relaxations.

The second formulation of the TSCPLP with the surrogate constraints added is:

(P5.2)

$$z = \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj} w_{kj}$$

Subject to: (5.2) – (5.14)

This revision results in an equivalent formulation as those presented in previous chapters and will be used for the computational purposes due to the more workable structure of the constraints. The two surrogate constraints are now both 'knapsack' constraints that are directly linked to the demand in the system. This results in a tidier set of surrogate constraints than those in Chapters 3 and 4. If the relaxations are to be used for generating feasible upper bounds, the knapsack constraints allow the use of primal heuristics that cope well with knapsack constraints in polynomial time.

The reason behind the change in surrogate constraints is that when the formulation was being programmed for the computational analysis the original surrogate constraints made finding solutions for the lower bound problems more difficult than if both the surrogate constraints were of a knapsack form. The new surrogate constraint was considered to be a tighter constraint since it retains more information from the main constraint set than the previous version. Therefore the revision results in a computationally more efficient and 'tighter' solution process.

The original surrogate constraint was based on constraint set (5.5), and the new version is based on (5.2), (5.5), and (5.6) so there could be some slight differences in the computational results from the theoretical results. Any relaxations that involve dualizing constraint set (5.5) but not (5.6) may give different results than the theoretical work. The difference that could arise from the change in the formulation would most likely be that the bounds found computationally may be tighter than those from the original formulation, due to more information being retained by the updated surrogate constraints.

The tested relaxations that correspond directly to the theoretical results are:

- z_2, z_3, z_4, z_6
- $z_{2,3}$, $z_{2,4}$, $z_{2,6}$, $z_{3,4}$, $z_{3,6}$, $z_{4,6}$, $z_{5,6}$
- $z_{2,3,4}$, $z_{2,3,6}$, $z_{2,4,6}$, $z_{2,5,6}$, $z_{3,4,6}$, $z_{3,5,6}$, $z_{4,5,6}$

The tested relaxations that may differ to the theoretical results due to the slightly different surrogate constraints are:

- z_5
- $z_{2,5}$, $z_{3,5}$, $z_{4,5}$
- $z_{2,3,5}$, $z_{2,4,5}$, $z_{3,4,5}$

The aim of the computational analysis is to corroborate the theoretical results, not to prove them. If a rogue result is found it can come from two sources: incorrect programming or incorrect theory. There are no unusual results in the computational analysis that follows so there is no evidence that the theory is wrong, despite the difference in formulations. Even if the theoretical results equal the computational results you cannot prove the theory with the computational work. The solution method is a heuristic approach that is not exact so can only provide a large compilation of evidence from the studied problem instances.

The next section gives the details of the Lagrangian heuristic method to solve several relaxations from Chapters 3 and 4. The performance of these heuristics will be tested with data sets described in Section 5.3 and the results presented in Section 5.4. The quality of the solutions from the heuristics provided in Section 5.2 will be compared with the solutions from the direct use of the commercial solver package CPLEX. A discussion of the results in terms of solution quality and computational times can be found in Section 5.5. This section also compares the computational results to the theoretical results as shown in Chapters 3 and 4. Although the formulations are not identical they are equivalent and the relevant conclusions can still be drawn. Section 5.6 provides some practical applications of this problem and draws conclusions based on the results of the analysis.

5.2 Solving the relaxations

To get a useful analysis of the Lagrangian relaxations available in the problem it is necessary to be able to solve both the lower bound problem created by relaxing a certain set or sets of constraints, and to find a feasible solution, or an upper bound. The nature of this formulation is such that only very weak relaxations can be solved optimally to find lower bounds. These procedures would follow a similar format as in the trivial theorems of Chapter 3. What complicates the relaxations used in this chapter further is that the surrogate constraints are not relaxed which is strengthening the lower bounds but in turn creating difficult problems to solve.

Therefore the analysis of the relaxations is undertaken using CPLEX to perform both the lower bound and the upper bound calculations in a Lagrangian heuristic, coded in C. The lower bound problems result when one or more sets of constraints are relaxed in a Lagrangian fashion as illustrated in section 2.4.1 and in Chapters 3 and 4.

There are five constraints, (5.2) – (5.6), that can be relaxed; giving 30 combinations of relaxed constraints. When four constraints are relaxed only one original constraint set along with the surrogates is left in the subproblem and this is likely to yield very poor lower bounds. Therefore these are not included in the main analysis. The 25 relaxations considered are of the form z_a, $z_{a,b}$ or $z_{a,b,c}$ where a, b and c are the numbers of the constraint set or sets being relaxed from the set {2,3,4,5,6}.

The symbols representing the Lagrange multipliers that will be associated with each set of constraints will remain consistent throughout the heuristics and are given below. These are the same definitions as those used in Chapters 3 and 4 for continuity:

$$\text{Constraints (5.2):} \quad \mu = (\mu_1, \ldots, \mu_m) \Rightarrow \quad \sum_{i=1}^{m} \mu_i \left(1 - \sum_{j=1}^{n} x_{ij} \right)$$

$$\text{Constraints (5.3):} \quad \lambda = (\lambda_1, \ldots, \lambda_n) \Rightarrow \quad \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} d_i x_{ij} - s_j y_j \right)$$

$$\text{Constraints (5.4):} \quad \alpha = (\alpha_{11}, \ldots, \alpha_{mn}) \Rightarrow \quad \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \left(x_{ij} - y_{ij} \right)$$

$$\text{Constraints (5.5):} \quad \beta = (\beta_1, \ldots, \beta_p) \Rightarrow \quad \sum_{k=1}^{p} \beta_k \left(\sum_{j=1}^{n} w_{kj} - a_k z_k \right)$$

$$\text{Constraints (5.6):} \quad \delta = (\delta_1, \ldots, \delta_j) \Rightarrow \quad \sum_{j=1}^{n} \delta_j \left(\sum_{i=1}^{m} d_i x_{ij} - \sum_{k=1}^{p} w_{kj} \right)$$

The constraints defining the sign restrictions on the Lagrange multipliers are as follows:

$$\mu_i \text{ unrestricted in sign} \qquad \forall i \in I \tag{5.15}$$

$$\lambda_j \geqslant 0 \qquad \forall j \in J \tag{5.16}$$

$$\alpha_{ij} \geqslant 0 \qquad \forall i \in I, \forall j \in J \tag{5.17}$$

$$\beta_k \geqslant 0 \qquad \forall k \in K \tag{5.18}$$

$$\delta_j \text{ urs} \qquad \forall j \in J \tag{5.19}$$

The exact resultant formulations for the 25 relaxations will not be presented here in their entirety but many can be found in Chapters 3 and 4 in a similar form. The effect however is that the constraint being relaxed is removed from the constraints set, and the terms as described above are added to the objective function.

The heuristics described in the following section do not have the two surrogate constraints (5.13) and (5.14) relaxed. The effect of this is that for each lower bound found there will be a feasible set of open plants and depots to satisfy the customer demand. This will strengthen the lower bound values and in turn is an attempt to strengthen the upper bounds found.

5.2.1 Lagrangian heuristic

The general solution method applied to the relaxations under consideration is a Lagrangian heuristic coded in C. This method has been used by many researchers in the literature as it has been found to generate quality solutions in reasonable computational time. This heuristic is being analysed with many different combinations of dualized constraints to further understand the nature of the constraints on the problem, and to discover which constraints are the best to dualize for a Lagrangian relaxation used in heuristics or for enhancing an enumeration method such as the branch-and-bound method.

The general outline of the heuristic remains the same regardless of which relaxation is being used to find lower bounds. The details of the general method are given below. It was coded in C language, calling the CPLEX routines as necessary.

1. Create, define and read into the program the:

    - Initial value for the best lower bound, z_{LB}
    - Initial value for the best upper bound, z_{UB}
    - Maximum iteration count

- Problem parameters:
  - Number of plants, depots, and customers
  - Which constraints are to be relaxed/dualized
  - Objective function coefficients — fixed and transportation costs, customer demands, depot and plant capacities
  - Initial step length
- Initial values for the Lagrange multipliers required for the given relaxation
- Convergence criteria for the duality gaps, ϵ.

2. Create the Lagrangian lower bound problem. This is the original problem formulation with the appropriate terms for the Lagrange multipliers corresponding to the relaxed constraints included in the objective function, subject to the remaining constraints; and defining the variables z and y to be binary. The terms to be used are those given above.

3. Solve the lower bound problem as a mixed-integer linear program using CPLEX's *mipopt*() function.

4. Read in the lower bound solution, storing the solution and the values for the objective function and the variables.

5. If the lower bound is greater than the current best lower bound, update:

$$z_{LB} = \max\{z_{LB},\ \text{solution to Lagrangian sub-problem}\}$$

6. Check if the current set of open plants and depots had been found in the previous iteration. If it had, there is no need to calculate the new upper bound, read in the stored solution for the upper bound from the previous iteration and move to checking the convergence criteria (11). Otherwise continue to creating the upper bound problem (7).

7. Create the upper bound problem. With the set of open depots and plants found in the solution to the lower bound problem create the upper bound problem, which is the original formulation subject to all the constraints, fixing the 0–1 variables to the appropriate values as defined by the lower bound solution.

8. Solve the upper bound problem as a linear program using CPLEX's *optimize*() function.

9. Read in the upper bound solution, storing the solution and the values for the objective function and the variables.

10. If the upper bound is less than the current best upper bound, update:

$$z_{UB} = \min\{z_{UB}, \text{ feasible solution created from lower bound}\}$$

11. Check the convergence criteria:

    - If $z_{UB} = z_{LB}$, stop — an optimal solution has been found.
    - Or if $(z_{UB} - z_{LB})/z_{LB} < \epsilon$
    - Or if iteration count = maximum iteration count,

    Stop — The current Lagrangian dual is chosen as the optimal and the solution corresponding to the best upper bound is chosen as the optimal primal solution.

12. Otherwise, compute the subgradients and update the Lagrange multipliers.

13. Update the iteration count (iteration count = iteration count + 1) and continue back to the lower bound problem creation step (2).

14. Continue until one of the convergence criteria is met.

The formulae for the subgradients in step (12) specific to each set of constraints are now given:

- Let x^*_{ij}, y^*_j, z^*_k and w^*_{kj} be the optimal solutions to the Lagrangian problem, and j be the indices of the open depots. The subgradient calculations for each set of constraints are now given:

$$\text{For constraint (5.2)} \quad N\mu(i) = 1 - \sum_j x^*_{ij} \tag{5.20}$$

$$\text{For constraint (5.3)} \quad N\lambda(j) = \sum_i d_i x^*_{ij} - s_j \tag{5.21}$$

$$\text{For constraint (5.4)} \quad N\alpha(ij) = x^*_{ij} - y^*_j \tag{5.22}$$

$$\text{For constraint (5.5)} \quad N\beta(k) = \sum_j w^*_{kj} - a_k \tag{5.23}$$

$$\text{For constraint (5.6)} \quad N\delta(j) = \sum_i d_i x^*_{ij} - \sum_k w^*_{kj} \tag{5.24}$$

- Then the multipliers are updated as:

$$\mu_i^{k+1} = \mu_i^k + t_k N\mu(i), \tag{5.25}$$

$$\lambda_j^{k+1} = \max\{0, \lambda_j^k + t_k N\lambda(j)\}, \tag{5.26}$$

$$\alpha_{ij}^{k+1} = \max\{0, \alpha_{ij}^k + t_k N\alpha(ij)\}, \tag{5.27}$$

$$\beta_k^{k+1} = \max\{0, \beta_k^k + t_k N\beta(k)\}, \tag{5.28}$$

$$\delta_j^{k+1} = \delta_j^k + t_k N\delta(j), \tag{5.29}$$

  k is the iteration count and

$$t_k = \lambda(z_{UB} - z_{LB})/Norm$$

  where the norm is taken as the Euclidean norm of the subgradient. Typically the steplength (λ) starts at a value between 0 and 2 and is halved when the lower bound does not improve over a set number of iterations. For the heuristic used in this thesis λ starts at 0.6, and is halved after 80 iterations.

The program source code and sample input and output files from the heuristic can be found in the Appendices and on the accompanying CD-ROM.

5.3 Test problems and experimental design

The Lagrangian heuristic described above can be used to solve any of the relaxations described in Section 5.2. The data that the performance of the Lagrangian heuristic is tested with has been generated in a similar fashion as previous researchers, to give a useful interpretation of the effect of the constraints.

Rardin & Uzsoy (2001) discuss the experimental evaluation of heuristic optimization algorithms. The key points to be considered in a computational analysis are the experimental design, sources and quality of test instances, and the relevant measures of performance. Items relating to experimental design that are covered by the method used in the analysis for this chapter are a combination of a sequential design, a full factorial design, and blocking. The sequential design aspect is covered by running the computational analysis in two parts with the second part using a subset of the relaxations from the first. The first stage employs a full factorial design by testing all the relaxations with the problem instances in the given sets. The analysis incorporates blocking by having all the combinations of relaxations run against all the same problem instances.

The ideal qualities of test instances to be used in a computational analysis as covered by Rardin & Uzsoy (2001) that are taken into consideration by the analysis of the following sections are the use of published and online libraries of data sets, and random generation. The data as described and generated below is based on published data sets that have been used, with or without modification, by the majority of researchers into plant location problems of a similar nature as the TSCPLP. The approach also incorporates random generation of the problem factors in its method of creating problem instances, as outlined below.

Klose (1999) and (2000), and Sridharan (1991) and (1993), have based their computational study around data sets generated using a method described in Cornejouls *et al.* (1991). It is a proposal for test data for the Capacitated Plant Location Problem however Klose adapted and applied it to his Two-Stage, Single-Source formulation. The key elements of this data generation approach are that the locations of customers, depots, and plants are generated as points and the transportation costs are set as a function of the distance between the points. Capacities and demands are randomly generated from uniform distributions, where $U[a,b]$ denotes values from the uniform distribution in the interval $[a,b]$, and fixed costs are calculated to reflect economies of scale.

The actual test data used is generated as follows (all values are rounded to the nearest integer):

- c_{ij}: depot to customer transport costs $= 0.01 \times \sum_i d_i \times$Euclidean distance from customer i to depot j
- f_j: fixed depot costs $= U[0,90] + U[100,110] \times \sqrt{s_j}$
- g_k: fixed plant costs $= U[0,90] + U[100,110] \times \sqrt{a_k}$
- b_{kj}: plant to depot transport costs $= 7.5\times$ the Euclidean distance from plant k to depot j
- d_i: customer demands $= U[5,35]$
- s_j: depot capacities $= U[10,160]$
- a_k: plant capacities $= U[20,730]$, and scaled so that the total plant capacity is at least $1.25\times$ total depot capacity, i.e., $\sum_k a_k \geqslant 1.25 \times \sum_j s_j$
- Customer sites are randomly selected from the square 100×100

- Depot sites are randomly selected from the square 100 × 100
- Plant sites are randomly selected from the square 100 × 100
- If two points are $P(x, y)$ and $Q(x, y)$, the distance between them is calculated as $\sqrt{(P_x - Q_x)^2 + (P_y - Q_y)^2}$

The convergence criteria is set as:

- Maximum iteration count = 300
- $\epsilon = 0.005$ or 0.5%
- Maximum computation time allowable = 30 minutes
- Initial steplength = 0.6, halved after 80 iterations
- All problem sizes described in this chapter are quoted as: number of plants × number of depots × number of customers.

In the initial testing stage it was found that the initial lower and upper bounds were significantly distanced from each other and so the steplength calculation resulted in the terms for the multipliers going straight to 0. Fisher (1985) illustrated this scenario and concluded that "if the stepsize converges to 0 too quickly, then the subgradient method will converge to a point other than the optimal solution". It was found that halving the steplength every 80 iterations and multiplying it (and therefore the stepsize) by a value ranging from 0.1% to 10% in most cases resulted in a better performing heuristic. The exact multipliers used in this step for each data set and relaxation can be found on the accompanying CD-ROM.

Overall, the computational study has two parts. The first is a broad analysis of all 25 relaxations as mentioned in Section 5.2. These were tested with three problems of each of the following sizes:

- 3×5×10
- 5×10×25
- 10×33×50
- 15×40×60
- 20×50×75

- 30×60×120
- 40×80×200
- 50×100×250

Four heuristics with interesting properties were then chosen for an in-depth analysis for part two. The most interesting factor affecting each problem is the number and capacity of the central stage — the depots. As the number of depots increases, relative to the number of plants and customers, the problem becomes denser and the number of feasible solutions increases as the number of potential customer–depot and depot–plant combinations increases. Therefore the four relaxations chosen for further analysis were tested with three classes of problem size that were generated using the same distributions as before to give problem sizes of 5×8×25, 5×16×25, and 5×25×25.

An analysis that was carried out by Cornejouls *et al.* (1991) and Klose (2000) is of the ratio of total depot capacity to total customer demand. This concept was used in the analysis in part two. After the problems were generated, groups of three of each class size were then adjusted so that the ratio of depot capacity to customer demand, $\frac{\sum_j s_j}{\sum_i d_i}$, was 1.5, 3, 5, or 10. This gives 36 test problems, 3 of each ratio for each of the 3 sizes. This concept was also adapted to analyse the ratio of total plant capacity to total depot capacity. The problem instances were generated using the problems of size 5×8×25 where the ratio of depot capacity to customer demand was equal to 3. This gives three problems which were then adjusted so that the ratio of plant capacity to depot capacity, $\frac{\sum_k a_k}{\sum_j s_j}$, was again 1.5, 3, 5, or 10. These two sets of ratios will be referred to in tables and figures as the 'depot ratios' and the 'plant ratios' respectively.

The data files can be found on the accompanying CD-ROM. The numerical results from both sections of the analysis are presented in the next section.

5.4 Computational results

This section gives details of the results from the relaxations studied in Section 5.2 and tested with the data as outlined in Section 5.3. Rardin & Uzsoy (2001) discuss measures of performance and solution quality when evaluating heuristic methods. They mention that instances where heuristics are useful are likely to give weak bounds on optimal solution values — for if the bounds were sharp an algorithm would surely exist that would render the heuristic unnecessary. So problem instances must be considered that truly evaluate the heuristic solution quality. Concepts that are covered by the analysis presented here are firstly the exact solution of small instances with which to compare the results from the heuristics. The downfall of this approach is that the results can be misleading. Therefore bounds on optimal values to the problems which are generated by the Lagrangian heuristic are compared with one another. This means we can give an upper bound on the deviation from the optimal solution of the results from the heuristic. In the context of this analysis these are the duality gaps.

The results for the initial stage where all 25 relaxations were tested begin with Tables 5.1 – 5.4. Tables 5.1 and 5.2 contain the minimum, maximum, and average duality gaps from the best upper bound to the best lower bound for each relaxation on three different problems for each size. The average solution time for each relaxation on that problem size is also given.

Tables 5.3 and 5.4 present the same information with respect to the gaps from the best upper bound to the optimal solution, and the best lower bound to the linear programming bound respectively. The gaps from the lower bound to the linear programming bounds are reported in order of magnitude not simply the positivity or negativity of the gap.

Tables 5.5 – 5.10 contain the minimum, maximum, and average duality gaps for each relaxation that found solutions for a range of larger sized problems within the maximum allowed time of 30 minutes. Entries of "not found" in the tables imply that a solution was not found given the level of computer resources and time.

The four heuristics chosen for the in-depth study in part two were z_2, z_5, $z_{3,5}$ and $z_{2,3,4}$. These were chosen as they had some of the better duality gaps between upper and lower bounds, and in some cases were able to find the optimal solution as the upper bound. z_2 is useful to investigate as it results when the customer demand constraint is dualized. In practical terms this is often the constraint which might have the least effect on overall costs of supplying demand. z_5 dualizes the capacity constraint on the plants, and $z_{3,5}$ is dualizing the capacity constraints on the plants and the depots. This is an interesting investigation into the effect of capacity on the problem. Finally $z_{2,3,4}$ has just two constraints remaining apart from the surrogates; the capacity constraint for the plants, and the conservation of flow constraints for the depots. These constraints are the only two involving the w_{kj} variables. This gives good solutions to the bound problems since the conservation of flow equalities strengthen the solutions, linking the w_{kj} variables to the x_{ij} and y_j variables.

The results for the ratios of depot capacity to customer demand are presented in

Constraints	Duality gaps from UB to LB			Avg time
Dualized	Min gap	Max gap	Avg gap	(secs)
2	0.45	4.20	2.02	6.77
3	3.87	45.32	20.05	11.46
4	0.00	0.00	0.00	0.58
5	2.80	6.37	4.81	8.13
6	3.94	99.97	40.14	10.69
2,3	7.08	13.92	9.99	12.92
2,4	0.45	4.20	2.02	6.98
2,5	3.89	38.89	16.19	9.84
2,6	11.61	131.99	57.89	9.01
3,4	2.39	48.15	20.98	12.09
3,5	7.35	59.87	26.88	9.79
3,6	36.85	256.71	129.18	11.65
4,5	2.80	6.37	4.81	9.15
4,6	3.94	99.97	40.14	11.57
5,6	32.22	119.73	87.83	11.43
2,3,4	1.50	12.30	6.65	12.37
2,3,5	8.24	89.43	50.20	8.66
2,3,6	51.51	297.16	149.10	9.72
2,4,5	3.89	38.89	16.19	8.49
2,4,6	1.61	117.68	53.12	9.28
2,5,6	61.85	82.51	70.40	8.65
3,4,5	11.87	48.16	31.92	10.27
3,4,6	35.88	291.71	121.91	13.25
3,5,6	63.11	301.57	185.95	10.53
4,5,6	40.96	148.75	101.72	10.60

Table 5.1: Percentage duality gaps from upper to lower bounds on problems sized $3\times5\times10$.

Constraints Dualized	Duality gaps from UB to LB Min gap	Max gap	Avg gap	Avg time (secs)
2	2.26	11.93	5.66	14.40
3	23.36	41.27	29.47	17.68
4	0.00	0.00	0.00	1.95
5	5.21	29.45	14.79	12.16
6	23.03	62.08	38.89	15.95
2,3	17.73	50.17	38.02	16.51
2,4	2.26	36.08	16.08	11.40
2,5	54.85	108.71	73.24	11.59
2,6	65.51	104.37	88.83	14.24
3,4	20.78	60.41	40.98	19.79
3,5	26.36	149.84	93.23	12.98
3,6	59.17	99.73	82.02	13.26
4,5	5.21	29.45	14.79	10.51
4,6	25.71	62.08	39.79	13.38
5,6	60.86	191.97	106.76	16.82
2,3,4	16.90	47.22	28.92	17.64
2,3,5	31.60	97.27	60.26	12.82
2,3,6	64.39	257.94	146.09	14.71
2,4,5	53.87	108.71	72.91	13.25
2,4,6	80.70	96.61	87.39	14.54
2,5,6	161.59	234.47	208.33	14.20
3,4,5	77.64	149.84	101.71	14.13
3,4,6	44.41	307.55	134.94	15.52
3,5,6	32.65	184.89	107.13	14.07
4,5,6	24.17	192.36	92.46	20.25

Table 5.2: Percentage duality gaps from upper to lower bounds on problems sized $5\times10\times25$.

Constraints	Gaps from UB to Optimal			Gaps from LB to LP		
Dualized	Min gap	Max gap	Avg gap	Min gap	Max gap	Avg gap
2	0.00	0.86	0.29	2.34	7.37	5.62
3	0.00	33.85	11.28	1.97	-2.81	-0.31
4	0.00	0.00	0.00	3.77	10.71	7.45
5	0.00	3.84	1.28	2.39	4.92	3.80
6	0.00	2.69	0.90	3.77	-44.63	-16.47
2,3	0.00	0.86	0.29	2.48	-5.32	-1.98
2,4	0.00	0.86	0.29	2.34	7.37	5.62
2,5	0.00	0.00	0.00	3.82	-20.29	-6.13
2,6	0.00	21.73	7.65	-3.37	-41.91	-21.50
3,4	0.00	33.85	11.28	1.35	-4.04	-0.89
3,5	0.00	0.30	0.10	-1.57	-32.54	-12.48
3,6	3.01	24.13	16.38	-18.82	-61.47	-38.84
4,5	0.00	3.84	1.28	2.39	4.92	3.80
4,6	0.00	2.69	0.90	3.77	-44.63	-16.47
5,6	0.00	81.52	27.27	-7.46	-49.46	-26.14
2,3,4	0.00	0.86	0.29	-0.56	2.24	1.10
2,3,5	0.00	24.13	8.04	-0.36	-32.15	-19.99
2,3,6	11.46	22.59	15.77	-19.37	-65.83	-42.32
2,4,5	0.00	0.00	0.00	3.82	-20.29	-6.13
2,4,6	0.00	21.73	7.65	-3.37	-38.09	-20.23
2,5,6	0.00	21.73	7.24	-26.16	-35.88	-32.46
3,4,5	0.00	24.13	8.04	-7.24	-20.54	-11.68
3,4,6	0.00	22.73	11.40	-14.88	-65.31	-34.04
3,5,6	3.84	81.52	36.50	-33.22	-65.78	-44.31
4,5,6	0.00	81.52	34.42	-9.13	-45.82	-27.11
Average Optimal solution time:			0.55 secs	Average LP bound time:		0.64 secs

Table 5.3: Percentage gaps from upper bounds to optimal solutions and lower bounds to linear programming bounds on problems sized 3×5×10.

Constraints	Gaps from UB to Optimal			Gaps from LB to LP		
Dualized	Min gap	Max gap	Avg gap	Min gap	Max gap	Avg gap
2	0.00	0.15	0.05	7.55	-7.62	1.42
3	7.62	32.68	17.72	3.82	-7.37	-2.99
4	0.00	0.00	0.00	3.40	10.54	6.82
5	0.00	14.49	4.88	1.42	-8.55	-2.12
6	0.00	19.62	7.37	-11.25	-23.69	-16.98
2,3	0.00	37.01	13.07	3.61	-31.15	-11.68
2,4	0.00	2.72	0.96	4.34	-24.02	-5.46
2,5	1.78	19.66	8.08	-30.56	-36.62	-32.85
2,6	0.00	13.05	5.08	-34.22	-45.91	-40.23
3,4	12.85	31.33	24.79	2.41	-13.54	-4.84
3,5	0.00	35.11	14.49	-8.65	-55.76	-31.92
3,6	0.30	14.49	5.66	-30.35	-45.49	-37.53
4,5	0.00	14.49	4.88	1.42	-8.55	-2.12
4,6	0.00	19.62	8.05	-11.41	-23.69	-17.03
5,6	0.00	65.43	22.44	-31.28	-41.42	-35.96
2,3,4	0.99	5.54	2.91	-10.68	-20.75	-14.22
2,3,5	1.50	32.68	15.14	-3.44	-41.69	-20.99
2,3,6	0.30	17.04	8.32	-32.56	-67.97	-48.16
2,4,5	1.78	19.66	8.08	-30.56	-36.62	-32.70
2,4,6	0.00	13.05	7.32	-37.24	-40.55	-38.87
2,5,6	3.12	32.16	13.79	-56.98	-65.35	-60.44
3,4,5	0.00	28.97	14.49	-22.65	-55.76	-37.26
3,4,6	0.00	70.50	25.01	-22.88	-56.75	-37.26
3,5,6	4.54	35.11	16.40	-16.05	-50.96	-35.87
4,5,6	0.00	65.43	22.54	-12.31	-41.49	-28.36
Average Optimal solution time:			2.04 secs	Average LP bound time:		2.90 secs

Table 5.4: Percentage gaps from upper bounds to optimal solutions and lower bounds to linear programming bounds on problems sized 5×10×25.

Constraints	Duality gaps from UB to LB		
Dualized	Min gap	Max gap	Avg gap
2	3.23	14.77	9.01
3	16.16	73.83	45.00
4	0.00	0.00	0.00
5	10.56	45.94	28.16
6	25.17	114.85	69.94
2,3	16.33	73.83	45.12
2,4	4.67	21.33	12.96
2,5	24.79	113.21	69.08
2,6	33.96	150.80	92.38
3,4	20.12	91.68	55.87
3,5	55.62	255.96	155.69
3,6	50.21	229.71	139.83
4,5	10.06	45.94	28.05
4,6	25.50	116.50	71.07
5,6	45.81	201.82	123.62
2,3,4	14.73	67.27	40.94
2,3,5	57.57	267.45	162.47
2,3,6	65.61	295.34	181.22
2,4,5	26.22	118.96	72.75
2,4,6	26.01	122.81	74.44
2,5,6	91.96	420.04	255.70
3,4,5	62.50	285.49	174.16
3,4,6	73.27	329.80	201.36
3,5,6	64.66	295.34	179.83
4,5,6	70.41	321.59	196.15

Table 5.5: Percentage duality gaps from upper to lower bounds on problems sized $10\times33\times50$.

Constraints	Duality gaps from UB to LB		
Dualized	Min gap	Max gap	Avg gap
2	4.31	19.69	12.05
3	22.25	98.45	60.42
4	0.00	0.00	0.00
5	15.74	73.83	45.02
6	not found	not found	not found
2,3	22.85	101.73	62.40
2,4	6.47	29.53	18.05
2,5	29.30	139.47	84.88
2,6	34.72	159.16	97.32
3,4	27.40	127.98	78.00
3,5	67.89	310.11	189.57
3,6	58.19	265.81	162.57
4,5	18.85	73.83	46.06
4,6	not found	not found	not found
5,6	52.45	239.55	146.41
2,3,4	21.95	96.81	59.42
2,3,5	71.13	324.87	198.63
2,3,6	80.47	367.53	224.72
2,4,5	34.15	146.03	90.00
2,4,6	36.93	162.44	99.76
2,5,6	135.79	620.21	379.18
3,4,5	76.11	357.69	218.04
3,4,6	93.88	423.32	259.22
3,5,6	83.34	380.66	232.84
4,5,6	93.04	424.96	259.98

Table 5.6: Percentage duality gaps from upper to lower bounds on problems sized $15\times40\times60$.

Constraints	Duality gaps from UB to LB		
Dualized	Min gap	Max gap	Avg gap
2	6.13	24.61	15.24
3	28.04	129.62	78.88
4	not found	not found	not found
5	22.73	103.37	63.02
6	not found	not found	not found
2,3	29.97	141.11	85.68
2,4	7.47	34.46	20.98
2,5	38.22	162.44	99.90
2,6	42.26	193.61	117.93
3,4	32.03	146.03	89.01
3,5	78.48	357.69	218.08
3,6	not found	not found	not found
4,5	21.65	103.37	62.72
4,6	not found	not found	not found
5,6	70.61	323.23	196.93
2,3,4	29.60	136.18	82.93
2,3,5	81.03	370.82	225.96
2,3,6	96.44	439.73	268.04
2,4,5	38.22	170.64	104.28
2,4,6	46.12	210.02	128.04
2,5,6	152.41	697.33	424.94
3,4,5	96.10	438.09	267.08
3,4,6	107.92	493.87	300.95
3,5,6	103.48	474.18	288.87
4,5,6	108.50	493.87	301.14

Table 5.7: Percentage duality gaps from upper to lower bounds on problems sized $20 \times 50 \times 75$.

Constraints	Duality gaps from UB to LB		
Dualized	Min gap	Max gap	Avg gap
2	7.18	32.82	20.05
3	32.69	149.31	90.90
4	not found	not found	not found
5	not found	not found	not found
6	not found	not found	not found
2,3	39.87	182.13	110.77
2,4	10.78	49.22	30.10
2,5	44.18	201.82	122.87
2,6	52.09	237.91	145.12
3,4	42.03	191.97	116.89
3,5	88.37	403.63	246.18
3,6	not found	not found	not found
4,5	not found	not found	not found
4,6	not found	not found	not found
5,6	not found	not found	not found
2,3,4	39.16	178.84	109.10
2,3,5	96.27	439.73	267.82
2,3,6	105.25	480.75	292.81
2,4,5	48.14	219.86	133.87
2,4,6	55.68	254.32	155.13
2,5,6	184.28	841.72	513.34
3,4,5	105.97	484.03	294.79
3,4,6	133.99	612.01	373.50
3,5,6	114.23	521.77	317.63
4,5,6	not found	not found	not found

Table 5.8: Percentage duality gaps from upper to lower bounds on problems sized $30\times60\times120$.

Constraints	Duality gaps from UB to LB		
Dualized	Min gap	Max gap	Avg gap
2	not found	not found	not found
3	37.72	172.28	103.79
4	not found	not found	not found
5	not found	not found	not found
6	not found	not found	not found
2,3	47.42	216.58	131.93
2,4	not found	not found	not found
2,5	not found	not found	not found
2,6	not found	not found	not found
3,4	52.09	237.91	145.08
3,5	106.69	487.31	297.12
3,6	not found	not found	not found
4,5	not found	not found	not found
4,6	not found	not found	not found
5,6	not found	not found	not found
2,3,4	45.62	208.38	126.93
2,3,5	109.20	498.80	304.11
2,3,6	120.34	549.66	335.12
2,4,5	not found	not found	not found
2,4,6	not found	not found	not found
2,5,6	not found	not found	not found
3,4,5	128.96	589.04	358.86
3,4,6	160.21	731.79	445.83
3,5,6	132.91	607.09	370.25
4,5,6	not found	not found	not found

Table 5.9: Percentage duality gaps from upper to lower bounds on problems sized $40 \times 80 \times 200$.

Constraints	Duality gaps from UB to LB		
Dualized	Min gap	Max gap	Avg gap
2	not found	not found	not found
3	not found	not found	not found
4	not found	not found	not found
5	not found	not found	not found
6	not found	not found	not found
2,3	56.04	255.96	155.87
2,4	not found	not found	not found
2,5	not found	not found	not found
2,6	not found	not found	not found
3,4	not found	not found	not found
3,5	110.64	505.36	308.10
3,6	not found	not found	not found
4,5	not found	not found	not found
4,6	not found	not found	not found
5,6	not found	not found	not found
2,3,4	52.09	237.91	144.89
2,3,5	117.47	536.53	327.19
2,3,6	not found	not found	not found
2,4,5	not found	not found	not found
2,4,6	not found	not found	not found
2,5,6	not found	not found	not found
3,4,5	153.75	702.25	427.75
3,4,6	179.97	822.03	500.71
3,5,6	156.98	717.02	436.78
4,5,6	not found	not found	not found

Table 5.10: Percentage duality gaps from upper to lower bounds on problems sized $20 \times 50 \times 250$.

Size	Constraints Dualized	Ratio of sum s_j to sum d_i 1.5	3	5	10	Avg time (secs)
A	2	7.53	2.78	2.43	0.00	8.78
(5x8x25)	5	13.53	11.76	11.32	3.84	9.42
	3,5	47.96	59.44	36.40	4.60	9.98
	2,3,4	10.17	19.13	26.16	5.49	14.32
B	2	13.33	11.97	5.61	5.11	13.69
(5x16x25)	5	14.82	12.78	22.80	16.55	11.69
	3,5	82.74	95.65	46.10	26.02	14.70
	2,3,4	51.23	57.81	44.56	18.32	19.43
C	2	17.65	5.97	4.02	1.84	21.04
(5x25x25)	5	11.80	10.54	31.86	18.94	16.74
	3,5	105.43	127.49	208.85	26.28	19.41
	2,3,4	92.83	63.59	21.57	12.96	27.61
Average time for ratio		15.33	15.37	15.98	15.58	

Table 5.11: Average duality gaps from upper to lower bounds for varying depot ratios.

Tables 5.11 – 5.13. These tables present the average duality gaps from upper to lower bounds, upper bounds to optimal solutions, and lower bounds to linear programming bounds across three problems of each size. Additionally in Table 5.11 the average time for the heuristic solutions is given, along with the overall average solution time for each ratio of depot capacity to customer demand. The ratios considered were 1.5, 3, 5, and 10.

The results for the ratios of plant capacity to depot capacity are presented in Tables 5.14 – 5.16. These tables present the average duality gaps from upper to lower bounds, upper bounds to optimal solutions, and lower bounds to linear programming bounds across three problems. The data was generated from Class A where the ratio of depot capacity to customer demand is 3, by adjusting the values for the plant capacity. Additionally in Table 5.14 the average time for the heuristic solutions is given, along with the overall average solution time for each ratio of plant capacity to depot capacity. The ratios considered were 1.5, 3, 5, and 10.

The next section discusses the results presented here in terms of the theoretical results found in Chapters 3 and 4, and the effect of various problem characteristics on duality gaps and computational time.

5.5 Discussion of results

This section discusses the results presented in the previous section. The theoretical results found in Chapters 3 and 4 will be checked against the computational results, for those

Size	Constraints Dualized	Ratio of sum s_j to sum d_i 1.5	3	5	10
A	2	0.30	0.00	0.00	0.00
(5x8x25)	5	5.96	0.00	1.63	0.00
	3,5	16.38	31.60	14.99	0.00
	2,3,4	0.28	0.00	5.07	1.02
B	2	6.07	8.30	1.37	2.51
(5x16x25)	5	7.90	2.20	2.86	8.67
	3,5	29.31	45.43	4.54	5.11
	2,3,4	18.21	11.30	19.13	5.95
C	2	2.17	0.03	3.13	0.00
(5x25x25)	5	2.59	1.54	14.12	8.57
	3,5	24.59	32.60	4.85	5.69
	2,3,4	22.98	10.32	1.31	0.00

Table 5.12: Average gaps for upper bounds to optimal solutions for varying depot ratios.

Size	Constraints Dualized	Ratio of sum s_j to sum d_i 1.5	3	5	10
A	2	-0.95	3.20	12.30	13.14
(5x8x25)	5	-1.28	-4.72	4.93	8.98
	3,5	-16.35	-11.94	-2.13	8.17
	2,3,4	-3.54	-9.39	-4.22	8.37
B	2	-2.86	6.68	18.01	13.76
(5x16x25)	5	-2.58	-0.37	3.18	9.00
	3,5	-26.66	-18.03	-11.41	-2.02
	2,3,4	-18.98	-21.20	1.12	4.44
C	2	-6.64	3.64	19.11	16.67
(5x25x25)	5	-1.60	0.85	3.90	8.12
	3,5	-35.03	-33.34	-47.11	-0.48
	2,3,4	-31.52	-25.40	0.49	5.30

Table 5.13: Average gaps for lower bounds to linear programming bounds for varying depot ratios.

Constraints Dualized	Ratio of sum a_k to sum s_j 1.5	3	5	10	Avg time (secs)
2	0.85	2.59	4.24	4.65	9.70
5	11.98	11.87	22.55	22.06	8.55
3,5	31.61	35.47	40.01	41.82	8.85
2,3,4	8.75	21.46	17.82	33.37	14.47
Average time for ratio	10.12	10.53	10.27	10.66	

Table 5.14: Average duality gaps from upper to lower bounds for varying plant ratios.

Constraints	Ratio of sum a_k to sum s_j			
Dualized	1.5	3	5	10
2	0.00	0.00	0.00	0.07
5	0.00	0.27	8.49	2.35
3,5	9.46	13.49	14.95	11.44
2,3,4	0.00	8.47	4.47	8.66

Table 5.15: Average gaps from upper bounds to optimal solutions for varying plant ratios.

Constraints	Ratio of sum a_k to sum s_j			
Dualized	1.5	3	5	10
2	3.71	12.61	15.44	26.49
5	-6.14	3.53	6.29	10.80
3,5	-12.06	-2.93	-1.35	4.22
2,3,4	-3.60	3.20	6.69	8.30

Table 5.16: Average gaps from lower bounds to linear programming bounds for varying plant ratios.

that are appropriate. Firstly, an identification of the surrogate constraint redundancy is presented:

- z_4 has both surrogate constraints redundant,
- z_3 and $z_{3,4}$ have constraint (5.13) redundant,
- z_5, z_6, $z_{4,5}$, $z_{4,6}$, $z_{5,6}$, and $z_{4,5,6}$ have constraint (5.14) redundant,
- $z_{2,5,6}$ utilises surrogate constraint (5.13) fully,
- $z_{2,3}$, $z_{2,3,4}$, $z_{2,3,5}$, and $z_{2,3,6}$ utilise surrogate constraint (5.14) fully.

Next, the results from the broad analysis of many relaxations are discussed.

Broad analysis

The results shown in Section 5.4 show that in general as the problem instance size increases the solutions use more computational time and the upper and lower bounds are farther apart at the iteration limit. Relaxing more sets of constraints also increases the duality gaps as can be seen from following down Tables 5.1–5.10. Figure 5.1 shows the ratio of the upper and lower bounds to the optimal solution for problem 2 of size 3×5×10. The increasing duality gaps can be seen as the two lines move further apart from each other as the number of constraints dualized increases. In this figure, and the following figures, the labels refer to the set of constraints dualized, i.e., "35" refers to relaxation $z_{3,5}$. For

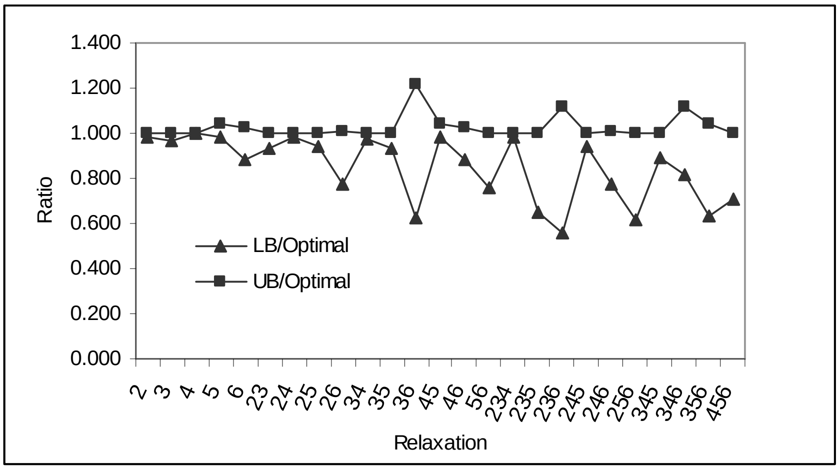

Figure 5.1: Ratio of upper and lower bounds to optimal solution, problem size 3×5×10.

the purposes of the discussion, the constraints will be referred to as the number of the constraint, i.e., "constraint 2" refers to equation (5.2).

Constraint 4 is redundant in the formulation when constraint 3 remains therefore dualizing constraint 4 has no effect on the relaxed problem. This results in z_4 finding the optimal solutions and therefore has duality gaps of 0%. This also means that $z_2 \equiv z_{2,4}$, $z_5 \equiv z_{4,5}$, $z_6 \equiv z_{4,6}$, $z_{2,5} \equiv z_{2,4,5}$, $z_{2,6} \equiv z_{2,4,6}$, and $z_{5,6} \equiv z_{4,5,6}$.

From the initial analysis of 25 relaxations there are a few combinations of dualized constraints that give very poor bounds on the problem, and some relaxations that perform very well. Firstly, dualizing constraint 6 results in poor bounds and the combination of dualizing constraint 6 with any other also compounds the poor results. Constraint 6 is the conservation of flow constraint through the depots and since this is an equality relationship involving all the x_{ij} and w_{kj} variables, relaxing this constraint leads to poor initial lower bounds which generate poor upper bounds. This is the constraint that links the customer–depot relationship with the depot–plant relationship. Removing the link means the heuristic finds many different, and often infeasible, solutions to the dualized problem and is unable to bring the bounds very close together within the iteration limit. Dualizing constraint 2 (z_2) or 5 (z_5) results in good solutions from the heuristic and the combinations 2,5 ($z_{2,5}$); 3,4 ($z_{3,4}$); 3,5($z_{3,5}$); and 2,3,4 ($z_{2,3,4}$) are satisfactory also. Most other combinations of three constraints dualized give poor duality gaps up to an average of 200%.

Relaxation $z_{2,3,4}$ is a surprise because it performs better than all other relaxations that have three sets of constraints dualized. It uses slightly more computational time to achieve its results, but the gain in solution quality is worth considering when choosing a relaxation to use in a solution method. The results for $z_{2,3,4}$ in tables 5.1 – 5.4 show that $z_{2,3,4}$ also performs better than many of the relaxations of two constraints. In fact, the combination of relaxing constraints 2, 3, and 4 ($z_{2,3,4}$) gives smaller duality gaps than those from just relaxing constraint 3 (z_3) or dualizing constraints 2 and 3 ($z_{2,3}$). Obviously the remaining constraints have enough of the overall problem structure to give useable lower bounds that produce good feasible solutions as upper bounds.

Tables 5.3 and 5.4 show the relative strength of the upper and lower bounds to the optimal solution and linear programming bounds respectively. It can be seen from the earlier tables that as the number of constraints relaxed increases, the computational time increases also. This is due to the extra time needed to calculate the upper bounds by the heuristic on problems with many constraints relaxed which have more solution combinations of open plants and depots. Almost all of the relaxations are able to find the optimal solution as the best upper bound in at least one problem instance (resulting in a 0% gap). The few that do not for the small sized problems are the relaxations involving dualizing both constraint sets 3 and 6. This results from the capacity constraint for the depots and the conservation of flow constraints being relaxed. This leads to many infeasible solutions being found as lower bounds resulting in poor upper bounds. For most of the larger of the two sizes of problems the upper bounds are further from the optimal solutions than for the upper bounds on the smaller sized problems. There are a number of relaxations that give better upper bounds for the larger sized problems than the smaller problems. These are z_2, $z_{2,6}$, $z_{3,6}$, $z_{3,5,6}$, and $z_{4,5,6}$. This could be due to specific qualities of the individual larger problems that were restricting the possible feasible sets of open depots and plants, giving upper bounds that are closer to the optimal solution. Also the ratios of the number of plants to the number of depots, and the number of depots to the number of customers, are not equally scaled between the small and large sized problems. This could be influencing the set of feasible solutions significantly.

Looking at the gaps from the lower bounds to the linear programming bounds shows that most relaxations have a positive gap as the minimum but average gaps are still negative. A negative value in the table indicates that the lower bound is less than the linear programming bound at the iteration limit. However as the heuristic terminates after 300 iterations, most lower bound problems are not solved to optimality so the table only gives a general indication of comparison. Only a few are close to the LP bound on larger sized problems. This shows that it would take more than 300 iterations for this heuristic to solve the lower bound problems to optimality.

Tables 5.5 – 5.10 investigate the need for relaxations when the problem instance size increases. These problems were generated in the same way as the earlier sizes and the ratio of plant capacity to depot capacity, and depot capacity to customer demand was set at approximately 3. These tables show that the relaxations exhibit the same patterns of overall strength of bounds as in the first two tables discussed above. However they also show that some relaxations are more useful than others. The total solution time for the heuristic was restricted to 30 minutes and for many of these largest sized problems the majority of relaxations were unable to generate upper or lower bounds. These instances were recorded in the tables as "not found". The optimal solution was not able to be found in the allowed time for all problems above size 30×60×120.

Relaxations involving dualizing constraint set 6 are the first to be restricted by time and result in the inability to find solutions. This is in line with the behaviour seen in the smaller sizes, where dualizing constraint 6 results in poor bounds. This shows that relaxing constraint 6 has a negative effect on solution quality but also on solution time. As the problem size increases, more relaxations are unable to find solutions in the allotted time. As the optimal solution is not known for most of these larger sized problems the quality of the upper bounds compared to the optimal solution cannot be known. All conclusions must be drawn from the duality gaps. Also, since the problem instances are not solved optimally in a restricted time, the relaxations become useful for finding solutions. Several relaxations find bounds on even the largest problems when optimal solutions are not computed. If the allotted time were to be increased the optimal solution may be found and more relaxations may be able to generate bounds, but the current restriction illustrates the trend.

The trend of increasing duality gaps with increasing problem size continues for the larger problem sizes, however as the size increases the bounds generated become far apart, by as much as 800%. These are not particularly useful for making conclusions on the optimal solution, but some relaxations have average duality gaps of less than 150% on even the larger problems, which is comparable with results for weaker relaxations seen in Tables 5.1 and 5.2. The performance of individual relaxations exhibit similar patterns as observed in Tables 5.1 and 5.2. Relaxation z_2 copes with larger sizes well to a point, then is unable to generate bounds in the allowed time. Relaxations z_3, $z_{2,3}$, and $z_{3,4}$ perform well on almost all the instance sizes considered. Relaxations z_5, $z_{2,5}$, and $z_{3,5}$ perform well on the first few of the larger sized instances, but z_5 and $z_{2,5}$ are unable to find bounds on problems larger than size 30×60×120; and even though $z_{3,5}$ finds bound for all problems, the gaps are not as tight as those from other relaxations. Ultimately the best performing heuristic overall is $z_{2,3,4}$ which finds reasonable bounds on all problem sizes considered.

Corroborating the theoretical results

Although the formulation used for the computational study is slightly different than the one used for the theoretical analysis, the results can still be effectively confirmed. Comparing the lower bounds to the linear programming bounds confirms the results for the trivial bound theorems found in Chapter 3. Tables 5.3 and 5.4 show that apart from relaxation $z_{2,3,4}$, all the relaxations involving three dualized constraints give gaps that are no better than the linear programming bound. This is in line with the theoretical results, however the theorems for many of the trivial bounds required the surrogate constraints to be relaxed also and here they are not. The effect of the surrogate constraints is only to *strengthen* the bound, not to worsen it, so the results shown here eventuate from there being only two of the original constraints left in the constraint set. This means that many lower bound solutions will be infeasible and the lower bound problems are not able to be solved to optimality by the heuristic within the required iteration limit. Many of the theorems of Chapter 3 were also related to four sets of constraints being dualized and although the computational study only looked at sets up to size 3, the results for four sets of constraints relaxed would only be as good as, or worse than, those for the sets of three.

The equivalent bounds of Chapter 4 are not easily seen from the computational study due to only one formulation with surrogate constraints being considered for the computational analysis.

The dominant bounds of Chapter 4 are the most accessible for comparison with the computational results. From Tables 5.1 and 5.2 it can be seen that the duality gaps for z_6 are significantly less than those of $z_{5,6}$ and $z_{2,6}$ — an average gap of 40.14% for z_6 compared to average gaps of 87.83% and 57.89% for $z_{5,6}$ and $z_{2,6}$ respectively on smaller sized problems, and comparisons of 38.89% versus 106.76% and 88.83% on larger sized problems. This illustrates the results of Theorems 4.3.9 and 4.3.10. Also the result that z_4 gives the optimal solutions for each problem instance illustrates Theorems 4.3.11 and 4.3.12 irrespective of the differing formulation. Theorem 4.3.13 can be corroborated by looking at the average gap for z_2 of 2.02% for small problems and 5.66% for larger problems and comparing these values to those of $z_{2,6}$ which are 57.89% and 88.83% respectively.

The result of Theorem 4.3.14 cannot be directly derived from the results of the computational analysis since the heuristic required that the surrogate constraints be satisfied for all relaxations. The attention now turns to the in-depth analysis of problem characteristics for the TSCPLP.

In-depth analysis

From Tables 5.11 – 5.13 the effect of the capacity to demand ratio on the problem can be seen. In general there is not much overall difference in the computational time for the varying ratios although ratio 5 uses the most and 1.5 the least. Relaxations z_2 and $z_{2,3,4}$ follow the same pattern where the gaps decrease with increasing ratio. Relaxations z_5 and $z_{3,5}$ also follow a pattern where ratio 5 is the most difficult to solve, however the gaps for relaxation z_5 are far superior to those from relaxation $z_{3,5}$. For all relaxations the lower bound to LP bound gap improves and becomes the most positive with increasing ratio.

Regarding the gaps from the upper bounds to the optimal solutions, relaxation z_2 gives good upper bounds across all problem sizes and ratios. Relaxation $z_{3,5}$ performs poorly across all ratios and problem sizes with only small problems with a ratio of 10 giving upper bounds comparable to the optimal solutions. In general the Class B problems, size (5×16×25), give upper bounds that are the farthest from the optimal solutions across all relaxations. This could be explained by there being enough depots in the middle echelon to give numerous combinations of open plants and depots that are feasible with respect to the required depot–capacity ratios. With too few depots to choose from, the feasible set reduces; and conversely with a large number of depots with small depot–capacity ratios the capacities on the respective depots are minimal and the feasible set will have most to all of the depots open to satisfy the demand.

Figure 5.2 shows the average duality gap for the four heuristics chosen across varying depot capacity to customer demand ratios. The relaxation of constraint 2 dominates all others, relaxation $z_{3,5}$ is dominated by all others, and relaxation z_5 and $z_{2,3,4}$ are close only when the ratio is 10, otherwise relaxation z_5 performs better. Relaxations z_2 and z_5 maintain duality gaps under 22% for all ratios considered. Relaxation $z_{3,5}$ gives very poor duality gaps (well over 70%) for all but the ratio of 10, which is also the ratio that gives the minimum gap across all relaxations.

Figure 5.3 shows the average gaps from the upper bounds found in the heuristic to the actual optimal solution to the problem, across several ratios of depot capacity to customer demand. There is no dominating relaxation, however all four considered achieved gaps under 10% for ratios 5 and 10. Relaxation $z_{3,5}$ has the largest change across the ratios,

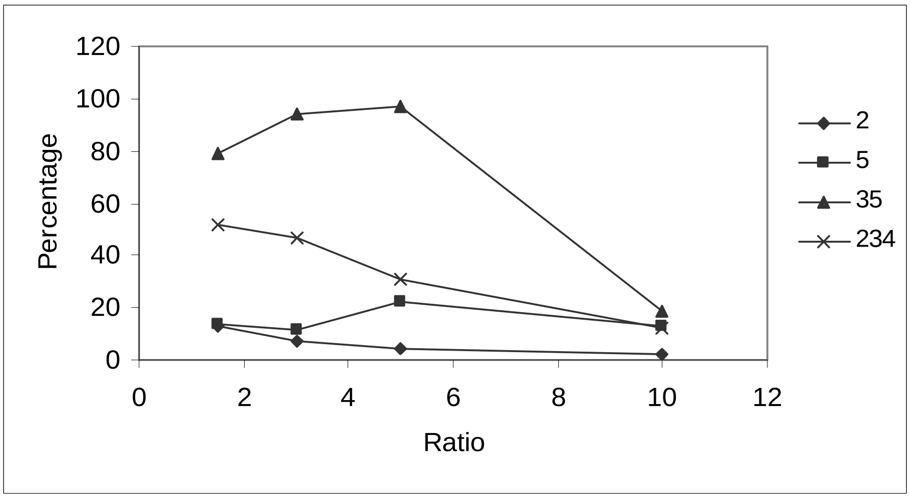

Figure 5.2: Duality gaps from upper to lower bounds for varying depot ratios on four relaxations.

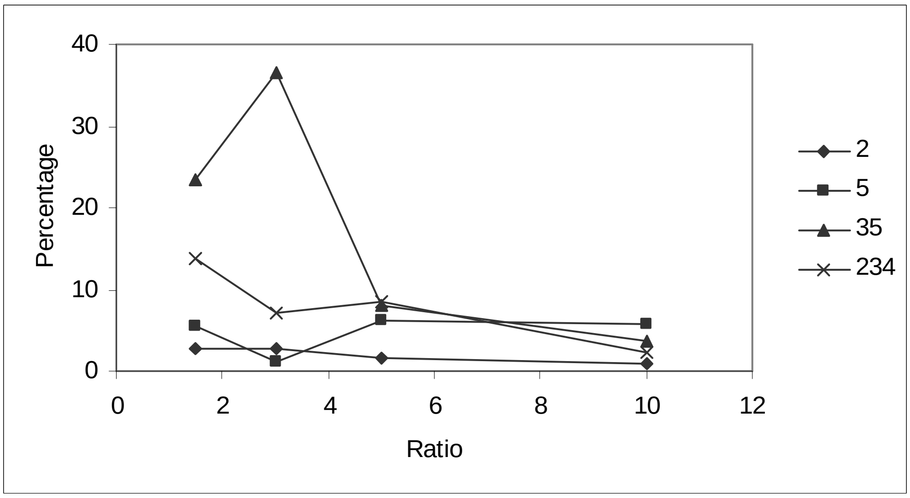

Figure 5.3: Percentage gaps from upper bounds to optimal solutions for varying depot ratios on four relaxations.

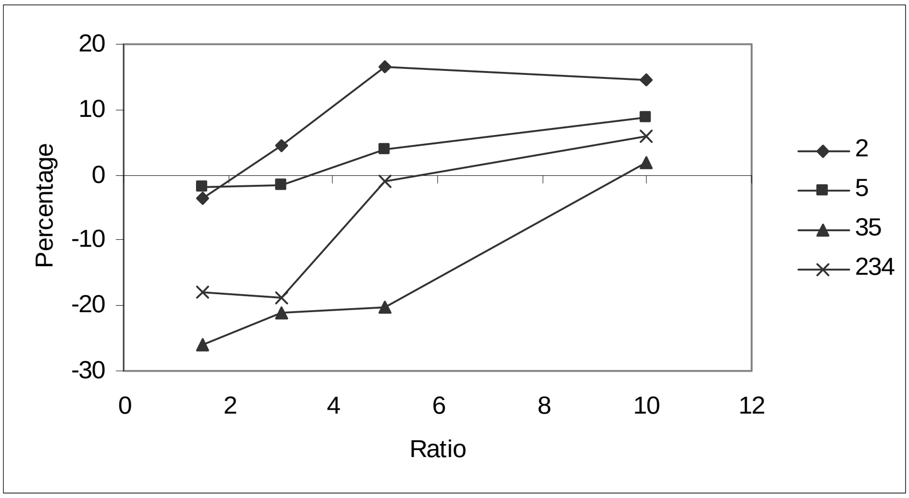

Figure 5.4: Percentage gaps from lower bounds to linear programming bounds for varying depot ratios on four relaxations.

showing also that the ratio of 3 has the most effect on the quality of the upper bounds found by the heuristic. This could be explained by both the capacity constraints being relaxed in relaxation $z_{3,5}$ and the capacity to demand ratio of 3 giving many infeasible solutions. The relaxation of constraint 2 results in upper bounds that are no more than 3% above the optimal solution across all ratios.

In Figure 5.4 the average percentage gaps from the lower bounds found by the heuristic to the linear programming bounds are shown for the four relaxations across the various ratios. Relaxation 2 dominates for all but the lowest ratio of 1.5. All four relaxations give lower bounds greater than the linear programming bound for ratio 10, and relaxations z_2, z_5, and $z_{3,5}$ are equivalent to, or better than, the LP bound from a ratio of 5. This can be explained by the linear programming solution allowing fractional values for the y_j and z_k variables that will be much less than 1 due to the large amount of capacity of the depots relative to the demand.

Similarly now discussing the results for the ratios of plant capacity to depot capacity. Figure 5.5 shows the average duality gaps from upper to lower bounds of the four relaxations across all ratios. Relaxation z_2 dominates all others and relaxation $z_{3,5}$ is the worst performer, with gaps over 30% throughout. The overall trend is the opposite to that observed in Figure 5.2 where the gaps were decreasing as the ratio increased. Here the duality gaps increase approximately 10% from when the ratio is 1.5 to the ratio of 10 for each relaxation. This shows that the lower and upper bounds are closer together when the plant capacity to depot capacity ratios are lower. If the plants have a significantly

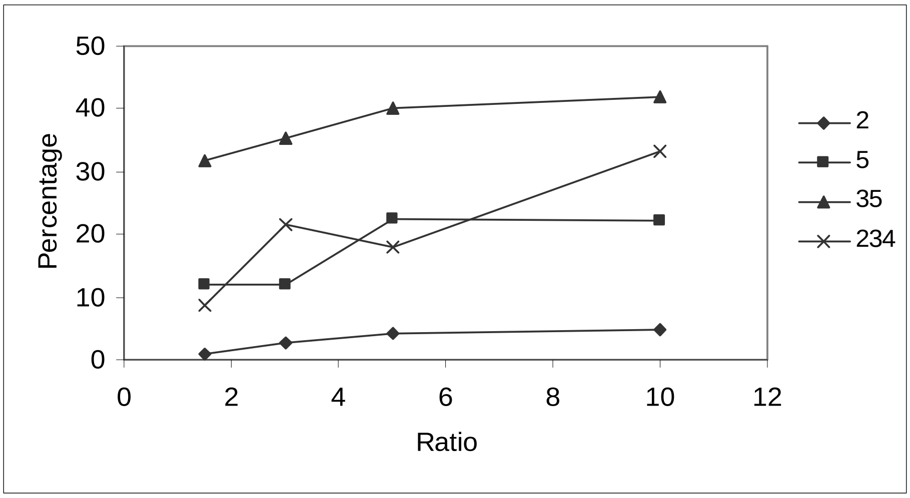

Figure 5.5: Duality gaps from upper to lower bounds for varying plant ratios on four relaxations.

larger capacity than is needed by the depots for satisfying the customer demand, the number of combinations of possible open plants and depots to satisfy the demand increases significantly, leading to more feasible solutions for the upper bounds.

Figure 5.6 presents the average gaps from the best heuristic upper bound to the optimal solution for several ratios of plant to depot capacity. Compared to Figure 5.3, the gaps from the upper bounds to the optimal solutions are smaller for the capacity to capacity ratios than for the capacity to demand ratios. The weakest upper bounds are found when the ratio is 3 or 5. Again the relaxation of constraints 3 and 5 gives the weakest upper bounds. Relaxation z_2 finds the optimal solution as the upper bound for all but the largest ratio, with the gap being negligible at 0.07%.

In Figure 5.7 the average gaps from the best heuristic lower bound to the linear programming bound across several ratios are shown. This figure shows the same trend as for the depot capacity to demand ratio where the gaps improve as the ratio increases. Overall the lower bounds here perform better than for the capacity to demand ratios, with all but relaxation $z_{3,5}$ giving positive bounds by a ratio of 3.

Figures 5.2 – 5.7 show that the four relaxations studied cannot be placed strictly in a hierarchy or dominance structure, since some of the lines on the graphs cross over each other as the ratios are increasing. This is not unexpected when the results for a single problem instance, as illustrated in Figure 5.1, are considered. The results in Figures 5.2 – 5.7 are for varying ratios of depot capacity to customer demand, and plant capacity to depot capacity. Each individual problem instance may have characteristics that favour a

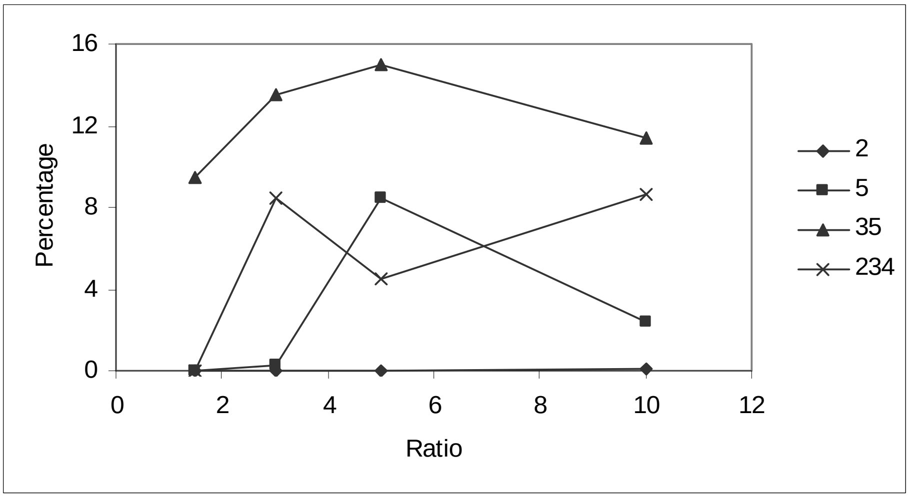

Figure 5.6: Percentage gaps from upper bounds to optimal solutions for varying plant ratios on four relaxations.

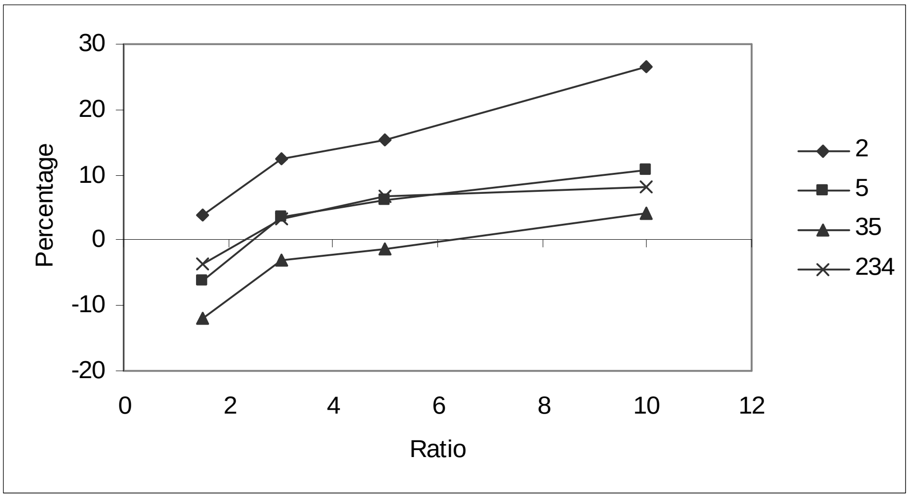

Figure 5.7: Percentage gaps from lower bounds to linear programming bounds for varying plant ratios on four relaxations.

certain relaxation over another for finding quality upper bounds, or lower bounds. Figure 5.1 shows that many relaxations find upper bounds that are close, or identical, to those found from other relaxations. And the equivalent statement can be made for the lower bounds. So the crossing of the lines in the figures result when one relaxation performs better on a certain ratio than another, and the reverse occurs for another ratio.

Overall there is no single ratio of depot capacity to customer demand or plant capacity to depot capacity that is the 'hardest' to solve. However ratios of 3 and 5 of each type have proved to be difficult to solve for several of the relaxations considered. This is a result that was seen in Sridharan(1987) from a similar analysis of capacity–demand ratios for the CPLP. Sridharan found that the most difficult CPLP problems were those where the ratio of depot capacity to customer demand was between 3 and 6. An intuitive observation was presented:

> "the combinatorial nature of the problem is exhibited clearly for these levels of capacities. When the capacity restriction is tight, then many plants should be kept open to service the demands, and when the capacity restriction is very loose, only a few plants need to be opened to satisfy all the demand. The number of combinations of open plants that can totally satisfy all the demand is highest when the ratio of $\frac{\sum s_j}{\sum d_i}$ is between 3 and 6 and this leads to the difficulty in solving such problems."

This concept also applies to the TSCPLP but is further complicated by the strength of the capacity constraints on the plants.

5.6 Applications

The analysis that has been presented over Chapters 3, 4, and 5 provides information that can be used in many ways. Firstly several applications of this information in terms of solution techniques are discussed.

The knowledge of the nature of the various Lagrangian relaxations provided by this analysis can be used to form a heuristic solution method that is efficient in finding bounds on the optimal solution. This could take the form of a Lagrangian heuristic as proposed by many authors in Chapter 2, or in the place of a traditional linear programming relaxation in a heuristic setting, as it was shown that many of the Lagrangian relaxations provide stronger bounds than the linear programming relaxation.

Lagrangian relaxations have also been successfully incorporated into exact solution methods, such as in a Lagrangian relaxation based branch-and-bound algorithm. They have been shown (Trangantalerngsak *et al.*, 2000) to provide smaller branch-and-bound trees and require less computational time than a traditional linear programming based branch-and-bound method. The information regarding the strength of the bounds resulting from the Lagrangian relaxations can be used in this sense to provide a strong bound on the branches of the tree, to enable nodes to be fathomed sooner than in a traditional method. This reduces the size of the tree required for the search.

A solution strategy can also be recommended for various problem characteristics. As the problem size increases, the Lagrangian dual lower bounds and resulting feasible upper bounds are farther apart from each other and therefore from the optimal solution. It is necessary in this instance to choose a relaxation for a solution method that results in strong bounds. The choices for medium sized problems would be to relax the customer demand constraints (set 2), or the capacity constraints on the plants (set 5). If these subproblems are too computationally intensive to solve, relaxation $z_{2,3,4}$ provides reasonable bounds on the optimal solution. For the largest problem instances considered in Section 5.4, relaxation $z_{2,3,4}$ performs the best in terms of solution quality and the ability to generate solutions for the problem when most other relaxations are unable to in the restricted time.

If the system allows a large amount of capacity for the depots compared to the demand of the customers then any of the four relaxations studied provides strong bounds on the problem, for the sizes considered. When the capacity of the depots relative to the demand is low, relaxing constraints 2 or 5 results in the strongest bounds. The most difficult problems to solve are those where the ratio is between 3 and 5, and in this situation using the relaxations z_2, z_5, or $z_{2,3,4}$ is recommended.

When the system has a large capacity for the upper stage (the plants) compared to the capacity of the central stage (the depots), the relaxation of constraint 2 stands out as the best option. When the total capacity of the plants is closer to the total capacity of the depots, relaxations z_2, z_5, and $z_{2,3,4}$ all provide strong bounds on the solution. For the most difficult problems where the ratio is between 3 and 5 the relaxation z_2 gives the strongest bounds, but z_5 and $z_{2,3,4}$ are also reasonable to use as relaxations.

The ultimate choice of which relaxation to use depends on the solution techniques available to solve the Lagrangian lower bound problems, and the computational time required to obtain the solutions — this will be very formulation specific.

5.7 Summary

The aim of this chapter was to identify, computationally, the influence the constraints have on the solution quality of the Two-Stage Capacitated Plant Location Problem. This was achieved through analysing the upper and lower bounds resulting from various combinations of constraints dualized in a Lagrangian heuristic setting.

It was found that the tightest bounds result from dualizing constraint set 2 (z_2) for smaller problems, and for larger problems $z_{2,3,4}$ provides the tightest bounds even when the optimal solution is difficult to find. The worst bounds on average are when any three or more constraints are relaxed, or when constraint set 6 is dualized. The theoretical results from Chapters 3 and 4 were also corroborated from the computational results.

The performance of a selected group of relaxations was tested with varying ratios of depot capacity to customer demand, and of plant capacity to depot capacity. This showed varied results for the different relaxations across problem size and ratios. In general the ratios of 3 and 5 proved to be the hardest to solve in terms of yielding poor bounds with larger duality gaps at the termination of the heuristic.

This information could be used to create a heuristic solution method separate from the use of CPLEX to find useful bounds on the problem. The main hurdle to this would always be the computational complexity of the Lagrangian relaxation subproblems.

This concludes the analysis of the TSCPLP. The next chapter presents a formulation based on the TSCPLP for the single-source case, as this is a commonly studied version of two-stage problems. This problem is analysed in a similar manner to the TSCPLP to see the effect of the single-source constraints on the problem.

Chapter 6

The TSCPLP with single-source constraints

6.1 Introduction

When customers require that their demand is satisfied from only one source the problem becomes known as *single-source*. The formulation used for the TSCPLP in the previous work is adapted so that in this chapter the Single-Source version of the Two-Stage Capacitated Plant Location Problem (TSCPLPSS) is presented. The TSCPLPSS is also NP-hard as shown in Section 6.1.1. The formulation for the TSCPLPSS is given in Section 6.2 and then a computational analysis is undertaken in Section 6.3. This takes a similar format to the one applied to the TSCPLP for direct comparisons. The effect on the formulation is that x_{ij} becomes a binary variable and the problem is a mixed-integer linear program. Unlike other single-source models it is not an integer program since the definition of the w_{kj} variables does not restrict them to be integers.

As mentioned in Section 2.6 of Chapter 2, most of the previous research into two-stage plant location models have been single-source problems. The single-source constraint perhaps better models a real-life scenario of a distribution/transportation model where customers require their demand to be satisfied from one source only, as often what they are demanding does not exist in fractional components. This chapter aims to extend the computational analysis on the TSCPLP to a single-source case of the formulation in Chapter 5, to investigate the effect the single-source constraint has on the solutions. In general, the extra 0–1 variable makes the problem harder to solve as the upper bound problems on the Lagrangian relaxation are mixed-integer linear programs, compared with the linear programs resulting from the TSCPLP relaxations.

6.1.1 Computational complexity

Theorem 6.1.1 *The TSCPLPSS is NP-hard.*

Proof This result follows as a consequence of Theorem 3.3.1 of Chapter 3. ∎

The next section reiterates the formulation, and Section 6.3 provides a computational analysis of the problem. The approach is the same as was used for the TSCPLP in Chapter 5, and the reader is referred to this for comparisons and exact definitions.

6.2 Formulation

As mentioned in Section 2.6 of Chapter 2, the Single-Source Two-Stage Capacitated Plant Location Problem can be formulated as an integer program. The formulation of the TSCPLP used in this thesis has the variable w_{kj} as non-negative. This means that the TSCPLPSS is not an integer program but is still a mixed-integer linear program. The formulation of the TSCPLPSS being presented here is that of the TSCPLP with the added restriction that the variables x_{ij} must be integer.

Let the sets:

$I = \{1, \ldots, m\}$ be the set of customers,
$J = \{1, \ldots, n\}$ be the set of potential depot locations,
$K = \{1, \ldots, p\}$ be the set of potential plant locations,

The objective function coefficients are defined as:

$c_{ij} =$ total cost of transportation from depot j to serve customer $i, \forall i \in I, \forall j \in J$,
$f_j =$ fixed cost associated with depot $j, \forall j \in J$,
$g_k =$ fixed cost associated with plant $k, \forall k \in K$,
$b_{kj} =$ unit cost of transportation from plant k to depot $j, \forall k \in K, \forall j \in J$,
$d_i =$ demand of cutomer $i, \forall i \in I$,
$s_j =$ capacity of depot $j, \forall j \in J$,
$a_k =$ capacity of plant $k, \forall k \in K$,

The decision variables are:

$$x_{ij} = \begin{cases} 1, \text{ if the demand of customer } i \text{ is satisfied by depot } j \\ 0, \text{ otherwise } \forall i \in I, \forall j \in J \end{cases}$$

$$y_j = \begin{cases} 1, \text{ if depot } j \text{ is open } \forall i \in I, \forall j \in J \\ 0, \text{ if depot } j \text{ is closed} \end{cases}$$

$$w_{kj} = \text{ units of demand transported from plant } k \text{ to depot } j, \forall k \in K, \forall j \in J,$$

$$z_k = \begin{cases} 1, \text{ if plant } k \text{ is open } \forall k \in K \\ 0, \text{ if plant } k \text{ is closed.} \end{cases}$$

The problem can now be stated as:

(P6.1)

$$Z^{ss} = \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj}w_{kj} \tag{6.1}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \in I \tag{6.2}$$

$$\sum_{i=1}^{m} d_i x_{ij} \leqslant s_j y_j \qquad \forall j \in J \tag{6.3}$$

$$x_{ij} \leqslant y_j \qquad \forall i \in I, \forall j \in J \tag{6.4}$$

$$\sum_{j=1}^{n} w_{kj} \leqslant a_k z_k \qquad \forall k \in K \tag{6.5}$$

$$\sum_{k=1}^{p} w_{kj} = \sum_{i=1}^{m} d_i x_{ij} \qquad \forall j \in J \tag{6.6}$$

$$w_{kj} \geqslant 0 \qquad \forall k \in K, \forall j \in J \tag{6.7}$$

$$x_{ij} \text{ integer} \qquad \forall i in I, \forall j \in J \tag{6.8}$$

$$y_j \text{ integer} \qquad \forall j \in J \tag{6.9}$$

$$z_k \text{ integer} \qquad \forall k \in K \tag{6.10}$$

$$0 \leqslant x_{ij} \leqslant 1 \qquad \forall i \in I, \forall j \in J \tag{6.11}$$

$$0 \leqslant y_j \leqslant 1 \qquad \forall j \in J \tag{6.12}$$

$$0 \leqslant z_k \leqslant 1 \qquad \forall k \in K \tag{6.13}$$

The constraints have the same definitions as previously mentioned but are given here again for refreshing. The objective function (6.1) minimizes the fixed costs of opening both plants (z_k) and depots (y_j), and the transportation costs of moving demand from plants to depots (w_{kj}) and from depots to customers (x_{ij}). Constraints (6.2) state that each customer's demand must be fully met by the depots. Constraints (6.3) guarantee that open depots do not supply more than their capacity. Constraints (6.4) ensure that customers are only served from open depots. Constraints (6.5) guarantee that open plants do not supply more than their capacity.

Constraints (6.6) are conservation of flow constraints for the depots. Constraints (6.7) are non-negativity constraints on the amount of demand transported from plants to depots. Constraints (6.8), (6.9), and (6.10) are integrality constraints on satisfying customer demand, and the plants and depots. Constraints (6.11), (6.12), and (6.13) are non-negativity and simple upper bound constraints.

Surrogate constraints (6.14) and (6.15) can be added as follows:

$$\sum_{k=1}^{p} a_k z_k \geqslant \sum_{i=1}^{m} d_i \tag{6.14}$$

$$\sum_{j=1}^{n} s_j y_j \geqslant \sum_{i=1}^{m} d_i \tag{6.15}$$

This ensures that the total capacity of the open plants and depots is at least as large as the total customer demand moving through the system. These two constraints are redundant in the original formulation but strengthen some of the relaxations. These are the same surrogate constraints that were included in the formulation for the computational study of the TSCPLP in Chapter 5.

The second formulation of the TSCPLPSS with the surrogate constraints added is:

(P6.2)

$$z^{ss} = \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{k=1}^{p} g_k z_k + \sum_{k=1}^{p}\sum_{j=1}^{n} b_{kj}w_{kj}$$

Subject to: (6.2) – (6.15)

6.3 Computational analysis

In this section the computational analysis of the TSCPLPSS is presented. This is an extension of the analysis in Chapter 5 that focused on the TSCPLP. Again, an analysis is undertaken of the effect of relaxing constraints in a Lagrangian fashion on the bounds of the problem solutions. The Lagrangian heuristic used here is adapted from the one given in Section 5.2.1. The method is tested with a number of data sets which are the same as, or adapted from, those used for the TSCPLP for a direct comparison of solutions.

6.3.1 Solving the relaxations

The analysis of the relaxations from the TSCPLPSS is again undertaken using CPLEX to perform both the lower bound and the upper bound calculations in a Lagrangian heuristic as in Chapter 5. The overall method of the heuristic is the same as for the TSCPLP.

Again, there are five constraints, (6.2) – (6.6), that can be relaxed; giving 30 combinations of relaxed constraints. As for the TSCPLP, the subproblems resulting when four sets of constraints are relaxed will not be included in the main analysis since they are likely to yield very poor lower bounds. The 25 relaxations considered are of the form z_a^{ss}, $z_{a,b}^{ss}$ or $z_{a,b,c}^{ss}$ where a, b and c are the numbers of the constraint set or sets being relaxed from the set {2,3,4,5,6}.

The symbols representing the Lagrange multipliers that will be associated with each set of constraints will remain consistent throughout the heuristics and are given below. These are the same definitions as those used in Chapters 3, 4, and 5 for continuity:

$$\text{Constraints (6.2):} \quad \mu = (\mu_1, \ldots, \mu_m) \Rightarrow \quad \sum_{i=1}^{m} \mu_i \left(1 - \sum_{j=1}^{n} x_{ij}\right)$$

$$\text{Constraints (6.3):} \quad \lambda = (\lambda_1, \ldots, \lambda_n) \Rightarrow \quad \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} d_i x_{ij} - s_j y_j\right)$$

$$\text{Constraints (6.4):} \quad \alpha = (\alpha_{11}, \ldots, \alpha_{mn}) \Rightarrow \quad \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \left(x_{ij} - y_{ij}\right)$$

$$\text{Constraints (6.5):} \quad \beta = (\beta_1, \ldots, \beta_p) \Rightarrow \quad \sum_{k=1}^{p} \beta_k \left(\sum_{j=1}^{n} w_{kj} - a_k z_k\right)$$

$$\text{Constraints (6.6):} \quad \delta = (\delta_1, \ldots, \delta_j) \Rightarrow \quad \sum_{j=1}^{n} \delta_j \left(\sum_{i=1}^{m} d_i x_{ij} - \sum_{k=1}^{p} w_{kj}\right)$$

The constraints defining the sign restrictions on the Lagrange multipliers are as follows:

$$\mu_i \text{ unrestricted in sign} \qquad \forall i \in I \tag{6.16}$$

$$\lambda_j \geqslant 0 \qquad \forall j \in J \tag{6.17}$$

$$\alpha_{ij} \geqslant 0 \qquad \forall i \in I, \forall j \in J \tag{6.18}$$

$$\beta_k \geqslant 0 \qquad \forall k \in K \tag{6.19}$$

$$\delta_j \text{ urs} \qquad \forall j \in J \tag{6.20}$$

The heuristics described in the following section do not have the two surrogate constraints (6.14) and (6.15) relaxed. The effect of this is that at each lower bound solution there will be a feasible set of open plants and depots to satisfy the customer demand. This will strengthen the lower bound values.

As for the TSCPLP, the general solution method applied to these relaxations is a Lagrangian heuristic coded in C, calling the routines from CPLEX as necessary. The general outline remains the same regardless of which relaxation is being used to find lower bounds. The details of the general method are given below.

1. Create, define and read into the program the:
   - Initial value for the best lower bound, z_{LB}^{ss}
   - Initial value for the best upper bound, z_{UB}^{ss}
   - Maximum iteration count
   - Problem parameters:
     - Number of plants, depots, and customers
     - Which constraints are to be relaxed/dualized
     - Objective function coefficients — fixed and transportation costs, customer demands, depot and plant capacities
     - Initial step length
   - Initial values for the Lagrange multipliers required for the current relaxation
   - Convergence criteria for the duality gap, ϵ.
2. Create the Lagrangian lower bound problem. This is the formulation of the original problem with the appropriate terms for the Lagrange multipliers corresponding to the relaxed constraint included in the objective function, subject to the remaining constraints; and defining the variables x, y, and z to be binary. The Lagrange terms to be included are those given above.
3. Solve the Lagrangian lower bound problem as a mixed-integer linear program using CPLEX's *mipopt*() function.
4. Read in the lower bound solution, storing the solution and the values for the objective function and the variables.

5. If the lower bound is greater than the current best lower bound, update:

$$z_{LB}^{ss} = \max\{z_{LB}^{ss}, \text{ solution to the Lagrangian sub-problem}\}$$

6. Check if the current set of open plants and depots had been found in the previous iteration. If it had, there is no need to calculate the new upper bound, read in the stored solution for the upper bound from the previous iteration and move to checking the convergence criteria (11). Otherwise continue to creating the upper bound problem (7).

7. Create the upper bound problem. With the set of open depots and plants found in the solution to the lower bound problem create the upper bound problem, which is the original formulation subject to all the constraints, fixing the 0–1 y_j and z_k variables to the appropriate values as found in the lower bound solution.

8. Solve the upper bound problem as a mixed-integer linear program using CPLEX's *mipopt*() function.

9. Read in the upper bound solution, storing the solution and the values for the objective function and the variables.

10. If the upper bound is less than the current best upper bound, update:

$$z_{UB}^{ss} = \min\{z_{UB}^{ss}, \text{ feasible solution created from lower bound}\}$$

11. Check the convergence criteria:

    - If $z_{UB}^{ss} = z_{LB}^{ss}$, stop — an optimal solution has been found.
    - Or if $(z_{UB}^{ss} - z_{LB}^{ss})/z_{LB}^{ss} < \epsilon$
    - Or if iteration count = maximum iteration count,

    Stop. The current Lagrangian dual is chosen as the optimal and the solution corresponding to the best upper bound is chosen as the optimal primal solution.

12. Otherwise, compute the subgradients and update the Lagrange multipliers.

13. Update the iteration count (iteration count = iteration count + 1) and continue back to the lower bound problem creation step (2.)

14. Continue until one of the convergence criteria is met.

The formulae for the subgradients in step (12) specific to each set of constraints relaxed are now given:

- Let x^*_{ij}, y^*_j, z^*_k and w^*_{kj} be the optimal solutions to the Lagrangian problem, and j be the indices of the open depots. The subgradient calculations for each set of constraints are now given:

$$\begin{aligned}
&\text{For constraint (6.2)} \quad & N\mu(i) &= 1 - \sum_j x^*_{ij} && (6.21)\\
&\text{For constraint (6.3)} \quad & N\lambda(j) &= \sum_i d_i x^*_{ij} - s_j && (6.22)\\
&\text{For constraint (6.4)} \quad & N\alpha(ij) &= x^*_{ij} - y^*_j && (6.23)\\
&\text{For constraint (6.5)} \quad & N\beta(k) &= \sum_j w^*_{kj} - a_k && (6.24)\\
&\text{For constraint (6.6)} \quad & N\delta(j) &= \sum_i d_i x^*_{ij} - \sum_k w^*_{kj} && (6.25)
\end{aligned}$$

- Then the multipliers are updated as:

$$\mu_i^{k+1} = \mu_i^k + t_k N\mu(i), \tag{6.26}$$

$$\lambda_j^{k+1} = \max\{0, \lambda_j^k + t_k N\lambda(j)\}, \tag{6.27}$$

$$\alpha_{ij}^{k+1} = \max\{0, \alpha_{ij}^k + t_k N\alpha(ij)\}, \tag{6.28}$$

$$\beta_k^{k+1} = \max\{0, \beta_k^k + t_k N\beta(k)\}, \tag{6.29}$$

$$\delta_j^{k+1} = \delta_j^k + t_k N\delta(j), \tag{6.30}$$

  k is the iteration count and

$$t_k = \lambda(z^{ss}_{UB} - z^{ss}_{LB})/Norm$$

  where the norm is taken as the Euclidean norm of the subgradient. Typically the steplength (λ) starts at a value between 0 and 2 and is halved when the lower bound does not improve over a set number of iterations. For this heuristic λ starts at 0.6, and is halved every 80 iterations.

The program source code and sample input and output files from the heuristic can be found in the Appendices and on the accompanying CD-ROM.

6.3.2 Test problems

The Lagrangian heuristic described above can be used to solve any of the relaxations described in Section 6.3.1. The data that the performance of the Lagrangian heuristic is tested with is from the data sets used to test the TSCPLP relaxations.

Since the data generation and evaluation approach is the same as for the TSCPLP in Chapter 5, the key elements from Rardin & Uzsoy (2001) for the evaluation of heuristic methods that are covered in Sections 5.3 and 5.4 are all addressed here for the TSCPLPSS. These include the experimental design of the computational analysis, the generation approach of test data instances, and the appropriate measures of solution evaluation.

The key elements of the data generation approach are that the locations of customers, depots, and plants are generated as points and the transportation costs set as a function of the distance between the points. Capacities and demands are randomly generated from uniform distributions, where $U[a, b]$ denotes values from the uniform distribution in the interval $[a, b]$, and fixed costs are calculated to reflect economies of scale.

The actual test data used is generated as follows (all values are rounded to the nearest integer):

- c_{ij}: depot to customer transport costs $= 0.01 \times \sum_i d_i \times$ Euclidean distance from customer i to depot j
- f_j: fixed depot costs $= U[0, 90] + U[100, 110] \times \sqrt{s_j}$
- g_k: fixed plant costs $= U[0, 90] + U[100, 110] \times \sqrt{a_k}$
- b_{kj}: plant to depot transport costs $= 7.5\times$ the Euclidean distance from plant k to depot j
- d_i: customer demands $= U[5, 35]$
- s_j: depot capacities $= U[10, 160]$
- a_k: plant capacities $= U[20, 730]$ and scaled so that the total plant capacity is at least $1.25\times$ total depot capacity, i.e. $\sum_k a_k \geqslant 1.25 \times \sum_j s_j$
- Customer sites are randomly selected from square 100×100
- Depot sites are randomly selected from square 100×100
- Plant sites are randomly selected from square 100×100

- If two points are $P(x, y)$ and $Q(x, y)$, the distance between them is calculated as $\sqrt{(P_x - Q_x)^2 + (P_y - Q_y)^2}$

The convergence criteria is set as:

- Maximum iteration count = 300
- $\epsilon = 0.005$ or 0.5%
- Maximum computation time allowable = 30 minutes
- Initial steplength = 0.6, halved after 80 iterations
- All problem sizes described in this chapter are quoted as: number of plants × number of depots × number of customers.

The issue of adjusting steplengths for the TSCPLP was again encountered for the TSCPLPSS, and the exact multipliers used in the steplength calculations can be found on the accompanying CD-ROM.

The computational study for the TSCPLPSS has two parts, in the same style as for the TSCPLP. The first is a broad analysis of all 25 relaxations. These were tested with three problems of each of the following sizes:

- 3×5×10
- 10×33×50
- 15×40×60
- 20×50×75
- 30×60×120

The same four heuristics as investigated for the TSCPLP were then chosen for an in-depth analysis for part two, which takes a similar structure as the analysis of the TSCPLP to get a comparison across problem instances where the only change is the inclusion of the single-source constraint. The four relaxations chosen for further analysis were tested with the three problems from part one of size 3×5×10, and each one generated eight new problem instances where the ratio of various quantities was adjusted accordingly.

Firstly for the analysis into the ratio of depot capacity to customer demand, each of the three problems were adjusted so the ratio of depot capacity to customer demand, $\frac{\sum_j s_j}{\sum_i d_i}$, was 1.5, 3, 5, or 10. This gives 12 test problems, 3 of each ratio. The total plant capacity was adjusted to remain at least 1.25 times the total depot capacity. This concept was also applied to the ratio of total plant capacity to total depot capacity. The problem instances were generated from the same three problems as before which were then adjusted so that the ratio of plant capacity to depot capacity, $\frac{\sum_k a_k}{\sum_j s_j}$, was again 1.5, 3, 5, or 10.

The quantities that where changed to create these problem instances were the values of a_k, and therefore g_k, for the depot ratios; and both s_j and a_k, and therefore f_j and g_k, for the plant ratios. The data files for these problems can be found on the accompanying CD-ROM. The numerical results from both sections of the analysis are presented in the next section.

6.3.3 Computational results

This section gives details of the results from the relaxations studied in Section 6.3.1 and tested with the data as outlined in Section 6.3.2. The results for the initial stage where all 25 relaxations were tested are shown in Tables 6.1 and 6.2. Table 6.1 contains the minimum, maximum, and average duality gaps from the best upper bound to the best lower bound for each relaxation on three different problems. The average solution time for each relaxation is also given. Table 6.2 presents the same information regarding to the gaps from the best upper bound to the optimal solution, and the best lower bound to the linear programming bound respectively. The gaps from the lower bound to the linear programming bounds are reported in order of magnitude not simply the positivity or negativity of the gap.

Tables 6.3 – 6.6 contain the minimum, maximum, and average duality gaps for each relaxation that found solutions for a range of larger sized problems within the maximum allowed time of 30 minutes. Entries of "not found" in the tables imply that a solution was not found given the level of computer resources and time.

The four heuristics chosen for the in-depth study in part two were z_2^{ss}, z_5^{ss}, $z_{3,5}^{ss}$ and $z_{2,3,4}^{ss}$. These were chosen as they were the four used in the analysis of the TSCPLP, where they had performed well in terms of low duality gaps. This is again seen for the TSCPLPSS case and so are a good choice here also, but choosing the same relaxations as the previous investigation gives an insight into the impact the single-source constraint has on the problem.

The results for the ratios of depot capacity to customer demand are presented in Tables 6.7 – 6.9. These tables present the average duality gaps from upper to lower bounds, upper

Constraints	Duality gaps from UB to LB			Avg time
Dualized	Min gap	Max gap	Avg gap	(secs)
2	0.49	2.71	1.23	4.49
3	2.24	44.70	19.38	9.64
4	0.00	0.00	0.00	0.46
5	1.26	6.34	3.46	5.97
6	3.95	43.76	22.47	8.34
2,3	1.60	8.02	5.38	9.84
2,4	0.49	2.71	1.24	4.52
2,5	8.98	15.18	11.17	8.52
2,6	18.44	127.60	55.47	8.42
3,4	1.10	48.92	21.60	9.81
3,5	10.12	55.97	28.96	7.26
3,6	30.39	222.49	98.38	7.51
4,5	1.26	6.34	3.46	6.17
4,6	3.95	43.76	22.47	8.21
5,6	34.98	112.92	77.16	8.37
2,3,4	5.25	13.99	8.80	10.14
2,3,5	8.50	63.62	41.89	6.56
2,3,6	51.79	300.89	150.43	7.72
2,4,5	8.98	15.18	11.17	8.62
2,4,6	18.44	117.69	52.16	8.60
2,5,6	28.39	120.87	60.06	7.63
3,4,5	19.27	41.67	27.49	7.02
3,4,6	47.86	330.59	144.46	9.32
3,5,6	92.38	283.63	156.62	8.68
4,5,6	34.98	112.92	77.16	9.76

Table 6.1: Percentage duality gaps from upper to lower bounds on problems sized $3\times5\times10$.

Constraints	Gaps from UB to Optimal			Gaps from LB to LP		
Dualized	Min gap	Max gap	Avg gap	Min gap	Max gap	Avg gap
2	0.00	0.00	0.00	3.78	8.30	6.56
3	0.00	33.92	11.31	-2.77	2.95	0.73
4	0.00	0.00	0.00	4.30	11.24	7.89
5	0.00	0.00	0.00	3.00	5.20	4.27
6	0.00	21.62	8.00	4.03	-10.80	-4.22
2,3	0.00	0.00	0.00	1.52	2.98	2.38
2,4	0.00	0.00	0.00	3.78	8.30	6.56
2,5	0.00	1.04	0.35	-0.77	-4.62	-2.60
2,6	0.00	23.85	8.74	-8.70	-39.47	-19.82
3,4	0.00	33.92	11.31	3.17	-5.80	-0.87
3,5	0.00	5.23	1.74	-3.08	-30.67	-13.01
3,6	3.51	24.99	11.31	-17.20	-56.89	-31.32
4,5	0.00	0.00	0.00	3.00	5.20	4.27
4,6	0.00	21.62	8.00	4.03	-10.80	-4.22
5,6	1.04	81.15	28.19	-8.00	-38.78	-22.56
2,3,4	0.00	2.23	0.74	-0.91	0.93	-0.07
2,3,5	0.00	14.44	4.81	-0.33	-32.08	-18.20
2,3,6	11.24	23.85	16.11	-19.32	-65.63	-42.18
2,4,5	0.00	1.04	0.35	-0.77	-4.63	-2.60
2,4,6	0.00	23.85	8.74	-9.70	-36.71	-18.90
2,5,6	0.00	23.05	8.47	-16.84	-38.03	-24.09
3,4,5	0.00	17.84	6.73	-7.48	-11.01	-9.66
3,4,6	3.00	23.60	10.44	-26.12	-68.07	-40.77
3,5,6	0.00	24.99	9.50	-43.88	-63.76	-50.62
4,5,6	1.04	81.15	28.19	-8.00	-38.78	-22.56
Average Optimal solution time:			1.04 secs	Average LP bound time:		0.64 secs

Table 6.2: Percentage gaps from upper bounds to optimal solutions and lower bounds to linear programming bounds on problems sized 3×5×10.

Constraints	Duality gaps from UB to LB		
Dualized	Min gap	Max gap	Avg gap
2	3.62	16.52	10.10
3	16.53.75	51.46	18.
4	not found	not found	not found
5	not found	not found	not found
6	not found	not found	not found
2,3	17.09	81.40	49.53
2,4	5.67	25.90	15.84
2,5	24.99	114.16	69.81
2,6	34.29	156.57	96.24
3,4	20.09	92.79	56.66
3,5	58.85	261.02	160.19
3,6	not found	not found	not found
4,5	not found	not found	not found
4,6	not found	not found	not found
5,6	not found	not found	not found
2,3,4	16.27	71.92	44.16
2,3,5	60.45	274.59	168.03
2,3,6	65.73	296.39	181.71
2,4,5	26.68	124.22	75.87
2,4,6	27.53	131.10	79.44
2,5,6	92.89	424.28	259.45
3,4,5	63.95	292.11	178.63
3,4,6	72.41	333.50	203.74
3,5,6	66.04	301.66	184.47
4,5,6	not found	not found	not found

Table 6.3: Percentage duality gaps from upper to lower bounds on problems sized $10{\times}33{\times}50$.

Constraints	Duality gaps from UB to LB		
Dualized	Min gap	Max gap	Avg gap
2	5.92	24.09	14.81
3	24.47	111.77	68.48
4	not found	not found	not found
5	not found	not found	not found
6	not found	not found	not found
2,3	24.29	110.93	67.99
2,4	9.63	45.44	27.39
2,5	not found	not found	not found
2,6	38.80	174.90	107.19
3,4	not found	not found	not found
3,5	not found	not found	not found
3,6	not found	not found	not found
4,5	not found	not found	not found
4,6	not found	not found	not found
5,6	not found	not found	not found
2,3,4	26.76	110.83	68.20
2,3,5	73.04	329.47	200.75
2,3,6	82.70	374.05	228.57
2,4,5	not found	not found	not found
2,4,6	40.08	176.98	108.55
2,5,6	136.97	625.64	380.46
3,4,5	79.37	362.54	221.38
3,4,6	95.05	434.15	265.09
3,5,6	86.33	395.92	241.50
4,5,6	not found	not found	not found

Table 6.4: Percentage duality gaps from upper to lower bounds on problems sized $15 \times 40 \times 60$.

Constraints	Duality gaps from UB to LB		
Dualized	Min gap	Max gap	Avg gap
2	not found	not found	not found
3	not found	not found	not found
4	not found	not found	not found
5	not found	not found	not found
6	not found	not found	not found
2,3	32.41	141.70	86.96
2,4	not found	not found	not found
2,5	not found	not found	not found
2,6	not found	not found	not found
3,4	not found	not found	not found
3,5	not found	not found	not found
3,6	not found	not found	not found
4,5	not found	not found	not found
4,6	not found	not found	not found
5,6	not found	not found	not found
2,3,4	30.18	141.57	86.42
2,3,5	84.58	383.14	233.00
2,3,6	not found	not found	not found
2,4,5	not found	not found	not found
2,4,6	not found	not found	not found
2,5,6	not found	not found	not found
3,4,5	97.60	445.13	272.00
3,4,6	109.73	501.52	306.40
3,5,6	106.81	489.07	297.70
4,5,6	not found	not found	not found

Table 6.5: Percentage duality gaps from upper to lower bounds on problems sized $20\times50\times75$.

Constraints	Duality gaps from UB to LB		
Dualized	Min gap	Max gap	Avg gap
2	not found	not found	not found
3	not found	not found	not found
4	not found	not found	not found
5	not found	not found	not found
6	not found	not found	not found
2,3	55.71	254.45	154.98
2,4	not found	not found	not found
2,5	not found	not found	not found
2,6	not found	not found	not found
3,4	not found	not found	not found
3,5	not found	not found	not found
3,6	not found	not found	not found
4,5	not found	not found	not found
4,6	not found	not found	not found
5,6	not found	not found	not found
2,3,4	54.41	250.62	152.46
2,3,5	131.84	615.22	373.98
2,3,6	not found	not found	not found
2,4,5	not found	not found	not found
2,4,6	not found	not found	not found
2,5,6	not found	not found	not found
3,4,5	147.89	677.17	413.06
3,4,6	not found	not found	not found
3,5,6	159.58	729.61	444.16
4,5,6	not found	not found	not found

Table 6.6: Percentage duality gaps from upper to lower bounds on problems sized $30\times60\times120$.

Constraints	Ratio of sum s_j to sum d_i				Avg time
Dualized	1.5	3	5	10	(secs)
2	1.91	2.53	0.32	0.32	3.88
5	3.50	5.87	2.83	0.94	5.12
3,5	26.46	21.52	3.31	4.32	7.18
2,3,4	18.93	35.82	8.57	0.04	6.37
Average time for ratio	6.73	7.38	4.99	3.09	

Table 6.7: Average duality gaps from upper to lower bounds for varying depot ratios.

Constraints	Ratio of sum s_j to sum d_i			
Dualized	1.5	3	5	10
2	0.00	0.00	0.00	0.00
5	0.00	0.07	0.00	0.03
3,5	4.84	1.66	0.00	0.00
2,3,4	0.48	2.96	1.15	0.00

Table 6.8: Average duality gaps from upper bounds to optimal solutions for varying depot ratios.

bounds to optimal solutions, and lower bounds to linear programming bounds across three problems. Additionally in Table 6.7 the average time for the heuristic solutions is given, along with the overall average solution time for each ratio of depot capacity to customer demand. The ratios considered were 1.5, 3, 5, and 10.

The results for the ratios of plant capacity to depot capacity are presented in Tables 6.10 – 6.12. These tables present the average duality gaps from upper to lower bounds, upper bounds to optimal solutions, and lower bounds to linear programming bounds across three problems. Additionally in Table 6.10 the average time for the heuristic solutions is given, along with the overall average solution time for each ratio of plant capacity to depot capacity. The ratios considered were 1.5, 3, 5, and 10.

Constraints	Ratio of sum s_j to sum d_i			
Dualized	1.5	3	5	10
2	7.49	10.39	8.29	13.00
5	5.90	6.97	5.59	12.18
3,5	-8.34	-4.41	5.16	8.51
2,3,4	-6.47	-11.87	1.86	13.30

Table 6.9: Average duality gaps from lower bounds to linear programming bounds for varying depot ratios.

Constraints Dualized	Ratio of sum a_k to sum s_j 1.5	3	5	10	Avg time (secs)
2	1.66	1.91	1.93	1.83	4.54
5	6.60	3.32	4.71	5.99	5.63
3,5	14.74	17.02	21.19	22.85	7.41
2,3,4	6.34	8.21	11.27	10.82	9.69
Average time for ratio	7.29	6.85	6.23	6.91	

Table 6.10: Average duality gaps from upper to lower bounds for varying plant ratios.

Constraints Dualized	Ratio of sum a_k to sum s_j 1.5	3	5	10
2	0.00	0.32	0.38	0.37
5	3.24	0.00	0.11	0.36
3,5	3.99	2.75	4.37	0.83
2,3,4	0.75	0.39	0.49	0.37

Table 6.11: Average gaps from upper bound to optimal solutions for varying plant ratios.

Constraints Dualized	Ratio of sum a_k to sum s_j 1.5	3	5	10
2	4.62	6.60	9.67	14.07
5	3.05	4.82	6.53	9.67
3,5	-3.63	-4.85	-4.13	-4.59
2,3,4	0.78	0.52	0.61	4.92

Table 6.12: Average gaps from lower bounds to linear programming bounds for varying plant ratios.

6.3.4 Discussion of results

This section discusses the results presented in the previous section. The results from the TSCPLPSS will be compared with the results from the TSCPLP. Firstly, the results from the broad analysis of 25 relaxations are discussed.

Broad analysis

The results in Section 6.3.3 show that the TSCPLPSS follows similar trends to the TSC-PLP. In general as the problem instance size increases the solutions use more computational time and the upper and lower bounds are farther apart at the iteration limit. Relaxing more sets of constraints also increases the duality gaps. However, comparing Table 6.1 to the equivalent table for the TSCPLP (Table 5.1), the average solution time is lower by approximately 2–3 seconds for each relaxation. This shows that the TSCPLPSS heuristic finds solutions slightly faster than the heuristic for the TSCPLP but the quality of those solutions is no superior, at least for smaller sized problems.

Figure 6.1 shows the ratio of the upper and lower bounds to the optimal solution for problem 2 of size 3×5×10. Comparing this to Figure 5.1, which is the same problem instance, the upper bounds found by the TSCPLPSS heuristic are closer to the optimal solution than the bounds for the TSCPLP were. This can be attributed to the nature of the single-source problem restricting more variables to be integer, so the lower bounds result in solutions that are more likely to yield upper bound solutions that are the same or very close to the optimal solution. In this figure and the following figures the labels refer to the set of constraints dualized, i.e., “56” refers to relaxation $z^{ss}_{5,6}$. For the purposes of the discussion, the constraints will be referred to as the number of the constraint, i.e., “constraint 3” refers to equation (6.3).

Again for the TSCPLPSS, constraint 4 is redundant in the formulation when constraint 3 remains, therefore dualizing constraint 4 has no effect on the relaxed problem. This results in z^{ss}_4 finding the optimal solutions and therefore has duality gaps of 0%. This also means that $z^{ss}_2 \equiv z^{ss}_{2,4}$, $z^{ss}_5 \equiv z^{ss}_{4,5}$, $z^{ss}_6 \equiv z^{ss}_{4,6}$, $z^{ss}_{2,5} \equiv z^{ss}_{2,4,5}$, $z^{ss}_{2,6} \equiv z^{ss}_{2,4,6}$, and $z^{ss}_{5,6} \equiv z^{ss}_{4,5,6}$.

The relaxation of constraint 2 (z^{ss}_2) results in the best quality solutions for the TSC-PLPSS on small and medium sized problems. Others of note from Table 6.1 are dualizing 5 (z^{ss}_5), and the combinations of dualizing 2,5 ($z^{ss}_{2,5}$); 3,4 ($z^{ss}_{3,4}$); 3,5 ($z^{ss}_{3,5}$); and 2,3,4 ($z^{ss}_{2,3,4}$) also give reasonable results as they did for the TSCPLP. Relaxation $z^{ss}_{3,4,5}$ gives closer solutions on the TSCPLPSS than it did for the TSCPLP. Over most of the relaxations the average duality gap has decreased for the TSCPLPSS, except for relaxations $z^{ss}_{3,4}$,

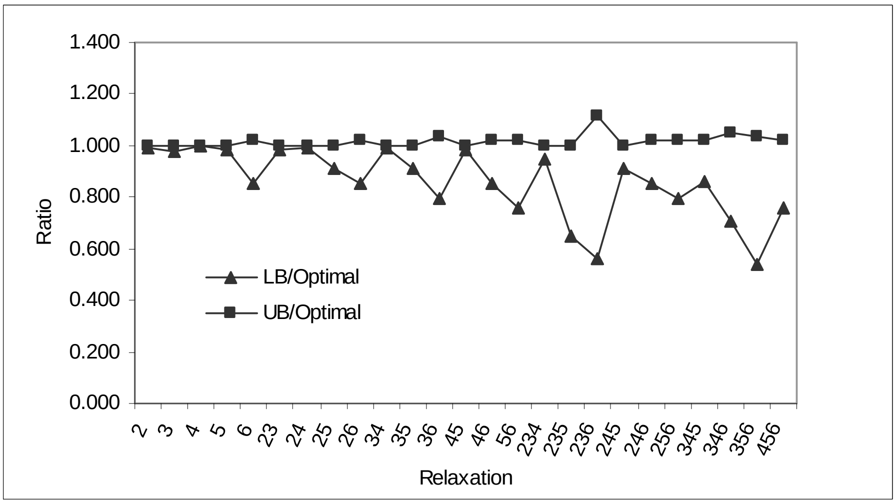

Figure 6.1: Ratio of upper and lower bounds to optimal solution, problem size 3×5×10.

$z^{ss}_{3,5}$, $z^{ss}_{2,3,4}$, $z^{ss}_{2,3,6}$, and $z^{ss}_{3,4,6}$. Overall, the effect of the single-source constraints on solution quality is minimal, but will be significant in the actual solution values. For example the best lower bound for z_2 on small problem number 2 is 66135.24 and the best upper bound found is 67061.32; but for z^{ss}_2 the best lower bound is 67064.99 and the best upper bound is 67400.00.

Table 6.2 shows the relative strength of the upper and lower bounds to the optimal solution and linear programming bounds respectively. Almost all of the relaxations are able to find the optimal solution as the best upper bound in at least one problem instance (resulting in a 0% gap). The few that do not, involve dualizing constraint 6 and either constraint 3 or 5 also. These are the capacity, demand, and conservation of flow constraints. When these are relaxed the lower bounds may never have enough constraints on them to produce a feasible set of open plants and depots that is optimal within the iteration limit of the heuristic. Several of the relaxations were able to find the optimal solution as the upper bound across all three problems analysed. These were relaxations z^{ss}_2, z^{ss}_4, z^{ss}_5, $z^{ss}_{2,3}$, $z^{ss}_{2,4}$, and $z^{ss}_{4,5}$.

Comparing the lower bounds to the linear programming bounds confirms that the results found in Chapter 3 for the TSCPLP also could apply to the TSCPLSS. That is, that the bounds from relaxing sets of three or more constraints result in lower bounds only equivalent to the linear programming relaxation. As the iteration limit here was 300, the heuristic terminates before many of the lower bound problems are solved to optimality, so are still showing that the lower bounds are less than the linear programming bound. Most have a positive gap as the best result but average gaps are still negative.

Larger sized problems were considered in Tables 6.3 – 6.6. These tables show that the relaxations exhibit similar trends to the results shown by the TSCPLP on larger problems. Entries in the tables of "not found" refer to the heuristic not being able to generate upper and lower bounds in 30 minutes. The optimal solutions were not able to be found in the allotted time for all problems above size 10×33×50. This also shows that the single-source problems are computationally more difficult to solve with less problems being solved optimally in the restricted time.

Relaxations involving dualizing constraint sets 5 or 6 are the first to be restricted by time and become unable to generate bounds in the allowable time. As the problem size increases, more relaxations are unable to find solutions. Compared to the results for the TSCPLP, less of the relaxations of the TSCPLPSS are able to generate bounds on the optimal solution. On the largest sized problem considered (30×60×120) only 5 out of the 25 relaxations were able to generate bounds on the optimal solution, compared to 17 out of 25 on the same sized problem for the TSCPLP. Since the problem instances are not solved optimally in a restricted time frame, Lagrangian relaxations become very useful for finding solutions.

As the problem size increases, the trend of increasing duality gaps continues,; however as the size increases the gaps between the bounds generated become as much as 700%. These are not particularly useful for making conclusions on the optimal solution, but some relaxations have average duality gaps of approximately 150% on even the larger problems, which is comparable with results for some of the weaker relaxations seen in Table 6.1. The performance of individual relaxations exhibit similar patterns as observed in Table 6.1. Relaxation z_2 copes with larger sizes well to a point, then is unable to generate bounds in the allowed time. Overall however, relaxations $z_{2,3}$ and $z_{2,3,4}$ are the only relaxations that find useful bounds on all sized problems, and would be the relaxations to recommend.

In-depth analysis

Tables 6.7 – 6.9 show the effect varying the depot capacity to customer demand ratio has on the problem. The larger ratios were solved in shorter times than those for ratios of 1.5 or 3, which had the largest average solution time. The relaxation of constraint 2 (z_2^{ss}) is solved faster than the others and relaxation $z_{3,5}^{ss}$ takes the most time. Relaxations z_2^{ss}, z_5^{ss}, and $z_{2,3,4}^{ss}$ follow the same pattern where the duality gaps peak at a ratio of 3 and are at a minimum for larger ratios. Relaxation $z_{3,5}^{ss}$ is hardest to solve when the ratio is smaller.

Comparing the upper bounds to the optimal solutions found by CPLEX shows that

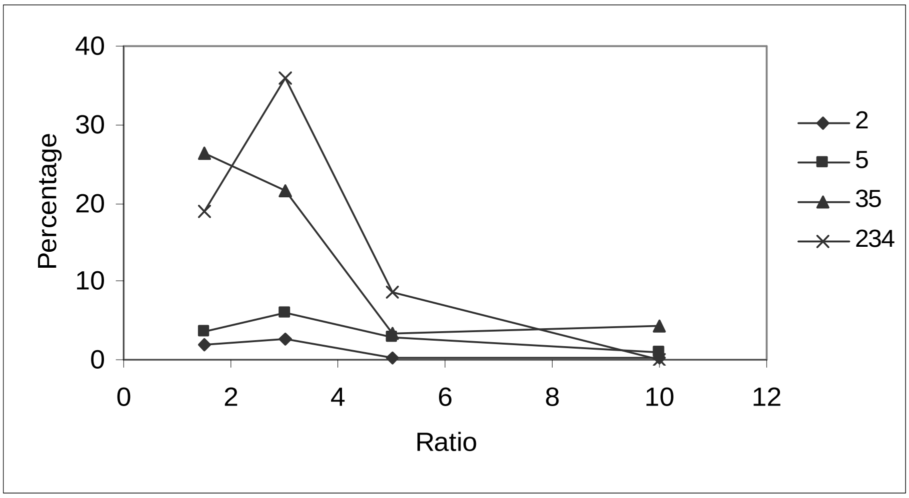

Figure 6.2: Duality gaps from upper to lower bounds for varying depot ratios on four relaxations.

the TSCPLPSS is capable of finding upper bounds that are equal to the optimal solution across all problems and ratios for relaxation z_2^{ss}. Relaxation z_5^{ss} produced optimal solutions as upper bounds for all but one problem of ratio 3 and one of ratio 10. Relaxation $z_{3,5}^{ss}$ did not find the optimal solution for any of the problems with ratio 1.5 and for one of ratio 3, but for the higher ratios the optimal solution was found as the upper bound. Relaxing constraints 2, 3, and 4 ($z_{2,3,4}^{ss}$) gave optimal solutions as the upper bounds for at least one problem of each ratio, but the ratio of 3 gave the weakest upper bounds — still acceptable however, at 2.96%.

Figure 6.2 shows the average duality gap for the four heuristics chosen across varying depot capacity to customer demand ratios. The relaxation of constraint 2 (z_2^{ss}) or 5 (z_5^{ss}) perform best across all the ratios, with gaps less than 3% or 6% respectively. Relaxations $z_{3,5}^{ss}$ and $z_{2,3,4}^{ss}$ are much harder to solve when the ratio is 1.5 and 3, but all four are similar when the ratio is 5 or 10, maintaining duality gaps less than 10%. Compared to the TSCPLP, the duality gaps for the TSCPLPSS are less — a maximum duality gap of 36%, compared to a maximum gap of 97% for the TSCPLP.

Figure 6.3 shows the average gaps from the upper bounds found in the heuristic to the actual optimal solution to the problem, across several ratios of depot capacity to customer demand. The relaxation of constraint 2 (z_2^{ss}) finds the optimal solution across all ratios and constraint 5 (z_5^{ss}) does for ratios of 1.5 and 5. However all four relaxations achieve gaps of less than 5% from the upper bound to the actual optimal solution, which is a very acceptable result. This is a better result than for the TSCPLP, where fewer relaxations

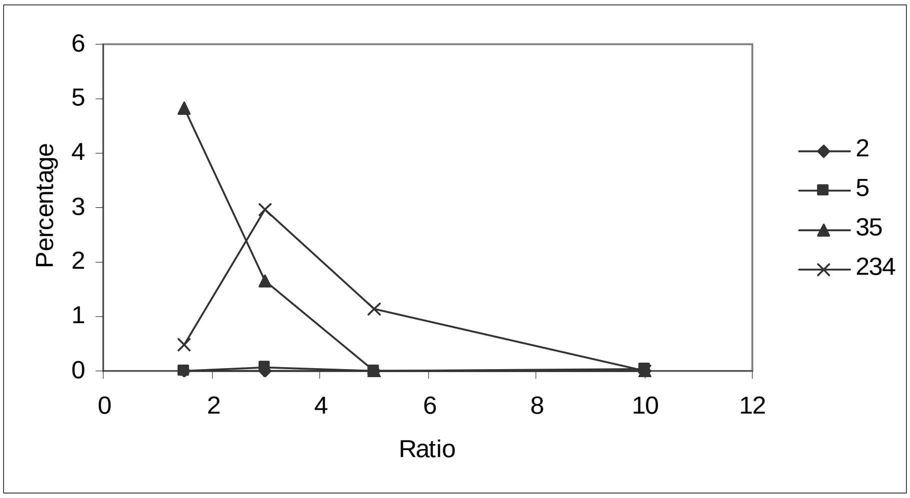

Figure 6.3: Percentage gaps from upper bounds to optimal solutions for varying depot ratios on four relaxations.

consistently found the optimal solution as the heuristic upper bound.

In Figure 6.4 the average percentage gaps from the lower bounds found by the heuristic to the linear programming bounds are shown for the four relaxations across the various ratios. Relaxation z_2^{ss} dominates for all but the highest ratio of 10. All four relaxations give lower bounds greater than the linear programming bound for ratios of 5 and 10. This can be explained by the linear programming solution allowing fractional values for the x_{ij}, y_j, and z_k variables that will be much less than 1 due to the large amount of capacity of the depots relative to the demand of the customers. The results of the TSCPLPSS give lower bounds that are stronger than the linear programming bound for lower ratios than in the TSCPLP. Relaxations z_2^{ss} and z_5^{ss} give bounds stronger than the LP bound for all ratios, and relaxations $z_{3,5}^{ss}$ and $z_{2,3,4}^{ss}$ are stronger than the LP bound for ratios 5 and 10. Overall the gaps from the lower bounds from TSCPLPSS are closer to the LP bounds than those from the TSCPLP in both directions (not as positive, but also not as negative).

Similarly now discussing the results for the ratios of plant capacity to depot capacity. Figure 6.5 shows the average duality gaps from upper to lower bounds of the four relaxations across all ratios. Relaxation z_2^{ss} dominates all others and is relatively constant across the ratios with duality gaps of approximately 2%. Relaxation $z_{3,5}^{ss}$ is the worst performer with gaps over 14% throughout. The overall trend is for the solution quality to remain the same or slightly decrease as the ratio increases.

Figure 6.6 presents the average gaps from the best heuristic upper bound to the optimal solution for several ratios of plant to depot capacity. Compared to Figure 6.3, the gaps

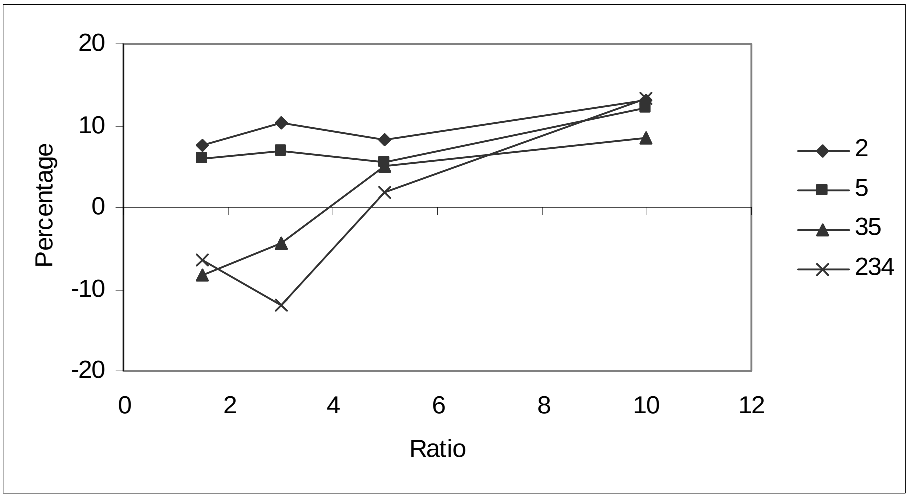

Figure 6.4: Percentage gaps from lower bounds to linear programming bounds for varying depot ratios on four relaxations.

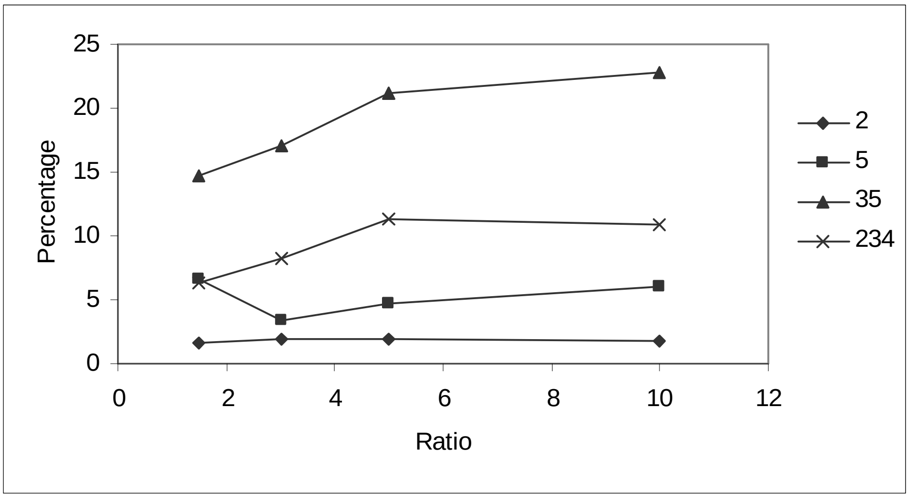

Figure 6.5: Duality gaps from upper to lower bounds for varying plant ratios on four relaxations.

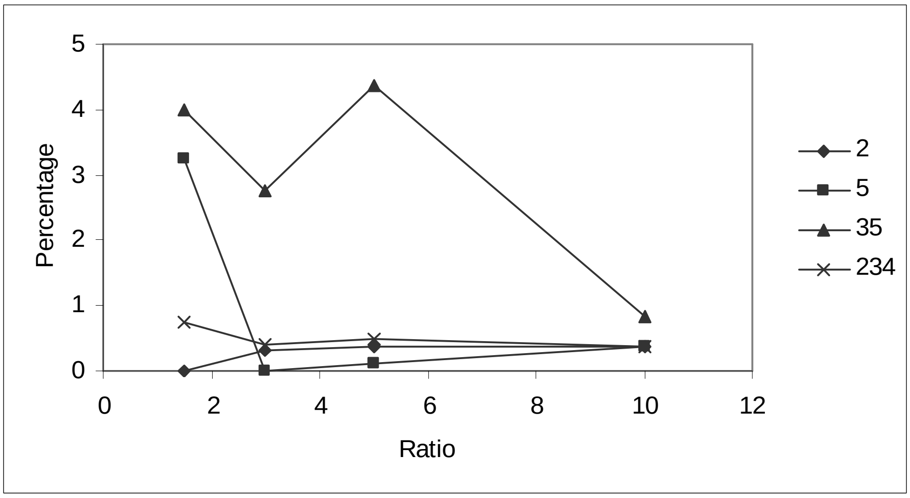

Figure 6.6: Percentage gaps from upper bounds to optimal solutions for varying plant ratios on four relaxations.

from the upper bounds to the optimal solutions are slightly larger for the capacity to capacity ratios, than for the capacity to demand ratio. Relaxation $z^{ss}_{2,3,4}$ performs better here, with gaps less than 1% throughout, compared to up to 3% in Figure 6.3.

In Figure 6.7 the average gaps from the best heuristic lower bound to the linear programming bound across several ratios are shown. This figure shows the same trend as for the depot capacity to customer demand ratios, where the gaps improve as the ratio increases. However the four relaxations' results do not cross each other, meaning that relaxation z^{ss}_2 dominates. Unlike for the depot ratios, the solutions to the lower bound problems from relaxations z^{ss}_2, z^{ss}_5, and $z^{ss}_{2,3,4}$ are stronger than the linear programming bound for all four ratios, however the bounds from relaxation $z^{ss}_{3,5}$ remain relatively constant, at 4% below the linear programming bound. Relaxing the capacity constraints on the plants and the depots has resulted in very weak lower bounds on the problem.

Figures 6.2 – 6.6 exhibit the same crossing of lines for different relaxations across the varying ratios as was seen for the TSCPLP. Figure 6.1 also follows the same pattern as Figure 5.1, so the same result is not unexpected. The crossing of results for varying ratios means that there is no clear dominating relaxation across all problem characteristics.

As was the case for the depot capacity to customer demand ratios, there is no clear ratio that is the most difficult to solve across all four relaxations. However ratios of 3 and 5 have been the most difficult for relaxations $z^{ss}_{3,5}$ and $z^{ss}_{2,3,4}$ to solve. This is a similar result to the TSCPLP, however the ratio of 1.5 for both the depot capacity–customer demand and the plant capacity–depot capacity ratios gave weaker bounds for $z^{ss}_{3,5}$ and $z^{ss}_{2,3,4}$, which

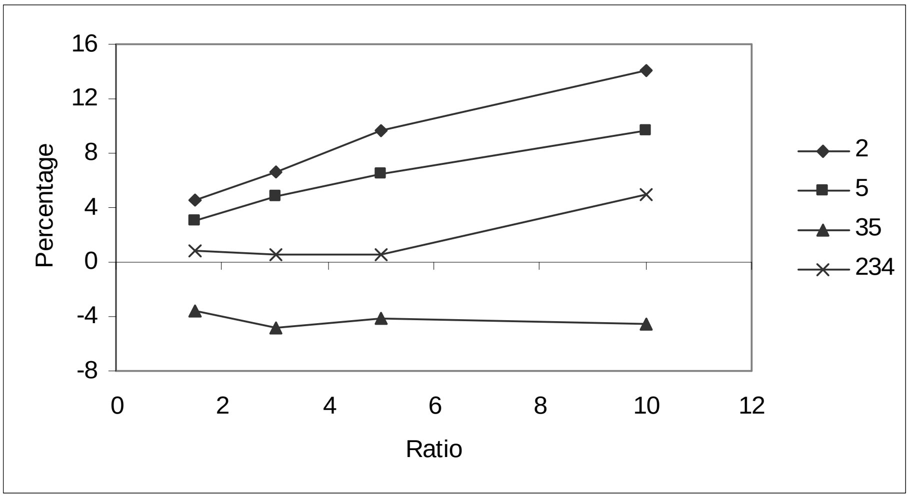

Figure 6.7: Percentage gaps from lower bounds to linear programming bounds for varying plant ratios on four relaxations.

was not seen for the TSCPLP.

6.4 Applications

The analysis of the TSCPLPSS from this chapter provides information about the problem that can be used in many ways. Firstly several applications of this information in terms of solution techniques are discussed.

Knowledge of the effect on solution quality of relaxing certain sets of constraints was provided by the analysis. This can be used in a heuristic solution method, to improve the efficiency of finding bounds on the optimal solution. This could take the form of a Lagrangian heuristic for the TSCPLPSS as proposed by many authors in Chapter 2, or in the place of a linear programming relaxation used in a heuristic, as it has been shown that many of the Lagrangian relaxations provide stronger bounds than the linear programming relaxation.

As mentioned in Section 5.6, Lagrangian relaxations have been successfully incorporated into exact solution methods, such as Lagrangian relaxation based branch-and-bound algorithms. The information regarding the strength of the bounds resulting from the Lagrangian relaxations compared to those from the linear programming bound can be used in this sense to provide a strong bound on the branches of the tree, to enable nodes to be fathomed sooner than in a traditional method. This reduces the size of the tree required for the search.

Solution strategies can also be recommended for various problem characteristics. As the number of plants, depots, and customers increase, the lower bounds from the Lagrangian dual problems and the feasible solutions defining the upper bounds are further apart from each other and therefore from the optimal solution. In this case it is necessary to choose a relaxation to use in a solution method that results in strong bounds. The choices for small and medium sized problems would be to relax the customer demand constraints (set 2), and/or the capacity constraints on the plants (set 5). If these subproblems are too computationally intensive to solve, relaxations $z^{ss}_{2,3}$, $z^{ss}_{3,5}$, or $z^{ss}_{2,3,4}$ also provide reasonable bounds on the optimal solution. For the largest problem instances considered in Section 6.3.3, relaxations $z^{ss}_{2,3}$ and $z^{ss}_{2,3,4}$ perform the best in terms of solution quality and the ability to generate solutions for the problem in restricted time when most other relaxations are unable to.

If the nature of the system has a large amount of capacity for the depots compared to the demand of the customers (a ratio of $\frac{\sum_j s_j}{\sum_i d_i} \approx 10$), then any of the four relaxations studied provides strong bounds on the problem. When the capacity of the depots relative to the demand is low (a ratio of $\frac{\sum_j s_j}{\sum_i d_i} \approx 1.5$), relaxing constraints 2 or 5 results in the strongest bounds. The most difficult problems to solve are those where the ratio is between 3 and 5, and in this situation using the relaxations z^{ss}_2, z^{ss}_5, or $z^{ss}_{3,5}$ is recommended.

When the system has a large capacity for the upper stage (the plants) compared to the capacity of the central stage (the depots), i.e., $\frac{\sum_k a_k}{\sum_j s_j} \approx 10$, the relaxation of constraint 2 (z^{ss}_2) or 5 (z^{ss}_5) stand out as the best options. When the total capacity of the plants is closer to the total capacity of the depots ($\frac{\sum_k a_k}{\sum_j s_j} \approx 1.5$), relaxations z^{ss}_2, z^{ss}_5, and $z^{ss}_{2,3,4}$ all provide acceptable bounds on the solution. For the most difficult problems where the ratio $\frac{\sum_k a_k}{\sum_j s_j}$ is between 3 and 5, the relaxation z^{ss}_2 gives the strongest bounds, but z^{ss}_5 and $z^{ss}_{2,3,4}$ are also reasonable to use as relaxations.

As for the TSCPLP, the ultimate choice of which relaxation to use in a solution method depends on the techniques available to solve the Lagrangian lower bound problems, and the computational time required to obtain the solution. These will be specific to the nature of each individual formulation.

6.5 Summary

This chapter presented a formulation and an analysis of the single-source version of the Two-Stage Capacitated Plant Location Problem under consideration in previous chapters of this thesis. It was shown that the TSCPLPSS is computationally NP-hard. The influence the single-source constraints have on the solution quality of the problem was shown computationally through the use of a Lagrangian heuristic. A subset of relaxations was chosen for a more intensive analysis of the effect of varying the ratio of depot capacity to customer demand, and plant capacity to depot capacity.

It was found that the TSCPLPSS is similar to the TSCPLP with regards to the nature of the constraints, and the results from the in-depth analysis of depot capacity to customer demand ratios and plant capacity to depot capacity ratios also reflect this. The strongest bounds resulted when constraint set 2 was dualized on smaller problems, and for larger problems $z^{ss}_{2,3}$ and $z^{ss}_{2,3,4}$ provide the tightest bounds even when the optimal solution is difficult to find. The weakest bounds result when at least three sets of constraints are relaxed or the relaxation involves constraint set 6, the constraint on the conversation of flow through the depots. This is a crucial constraint that needs to remain intact to give quality bounds on the formulation. It links the depot–customer relationship with the depot–plant relationship.

The performance of the same four relaxations as studied with the TSCPLP were tested with varying ratios of depot capacity to customer demand, and of plant capacity to depot capacity. This showed varied results for the different relaxations across problem size and ratio, although the problems studied for the TSCPLPSS were smaller in instance size. As for the TSCPLP, the ratios of 3 and 5 proved to be the hardest to solve, in terms of yielding poor bounds with larger duality gaps at the termination of the heuristic; however, the ratio of 1.5 gave weaker results than those from the TSCPLP. This information on the strength of the bounds could be incorporated into a heuristic solution method to provide useful bounds on the solution to the original problem, or used in an exact method to reduce the computational time.

This concludes the analysis of the TSCPLPSS. The next chapter summarizes the research and presents some items for further investigation.

Chapter 7

Conclusions

7.1 Introduction

The purpose of this chapter is to collate and summarize all of the findings of this thesis. The investigation into the Two-Stage Capacitated Plant Location Problem (TSCPLP) was formed into a theoretical analysis, a computational analysis, and an extension into a related problem — the Two-Stage Capacitated Plant Location Problem with Single-Source constraints (TSCPLPSS).

The aim of the analysis was to provide knowledge of the bounds on the solution resulting from Lagrangian relaxations of the constraints, to be able to improve solution techniques that utilise Lagrangian relaxation. The strength of the upper and lower bounds were compared to the optimal solutions and the linear programming relaxation bounds found using CPLEX. The analysis included the effect of varying problem characteristics such as problem size, and the ratios of plant capacity to depot capacity, and depot capacity to customer demand.

The next section will summarize the research conclusions and the final part of the thesis provides some future research directions and applications of this work.

7.2 Research conclusions

This research problem was encountered after reviewing the literature on two-stage location problems. The major issue that arose from the formulations studied by previous researchers was that the location and status of the facilities in the upper level were assumed to be known. This may be the case in some practical situations, but for a system that is being designed from scratch the actual location of the production plants is unknown. The problem in this case requires the solution to include the decision of which potential locations to be used, dependent on the location of the distribution depots and customers.

Chapter 1 presented a new formulation for the TSCPLP. This was not a single-source model as presented by the majority of researchers in the literature. The location of the production plants is the same type of decision to be made as the location of the distribution depots. This significantly distinguishes the formulation from previous models for two-stage location problems. Each possible production plant and distribution depot has a unique capacity attributed to it, and each customer has a demand to be satisfied by the system. The solution to the TSCPLP provides the locations of plants and depot to be opened and defines the flow of demand from the plants to the depots and on to the customers.

Chapter 2 reviewed the literature into location problems and specifically the techniques applied to two-stage problems. Section 2.2 discussed the TSCPLP in terms of the CPLP and other one-stage models. Various model characteristics were introduced, such as single-source, multi-commodity, multi-period, and combinations of these characteristics. Formulations were presented in whole or part for simple problems, highlighting the effect of the characteristics on the formulation.

The concept of Lagrangian relaxation and subgradient optimisation was presented in Section 2.3. The background and some crucial properties and results of Lagrangian relaxation were discussed for use in the theoretical analysis of the TSCPLP in the thesis. Section 2.4 introduced heuristics that had been applied to solving plant location problems in general, and also specificly to capacitated and two-stage problems. These included Lagrangian heuristics which were used to solve the TSCPLP in this thesis. It was noted that the choice of a suitable relaxation is one of the most important issues to consider. The researchers who presented Lagrangian heuristics to solve the CPLP and the TSCPLP (in various different forms) did not analyse the nature of the bounds resulting from different relaxations. Other heuristics were briefly covered, including Tabu Search and heuristics that have been used to find feasible upper bounds such as exchange and matching heuristics.

Section 2.5 covered methods applied to location problems that find the optimal solutions. Some of these methods, such as branch-and-bound algorithms, involve the use of Lagrangian relaxation and the work in this thesis can be applied to these algorithms to improve their performance in terms of computational complexity and time.

Section 2.6 reviewed the literature specific to the TSCPLP and the TSCPLPSS. A need was identified for a formulation that allowed the upper most level to be associated with a decision variable as opposed to a given capacity quantity that already exists. Several researchers proposed Lagrangian heuristics or Lagrangian relaxation based branch-and-bound methods that could have been improved with a greater knowledge of the strength of the bounds. Tragantalerngsak *et al.* (1997) studied six Lagrangian heuristics based on different combinations of dualized constraints and the duality gaps reported for their formulation are equivalent or surpassed by the results found in the computational study of Chapter 5. Their formulation was a single-source model and their test data generated in a similar manner as that used in this thesis, however the heuristics they used were individual to each corresponding Lagrangian relaxation. They reported gaps for many of the relaxations of between 10 and 20%, and upwards to a maximum of 144%. This indicates that the large duality gaps found for some of the relaxations in Chapter 5 are not unexpected. Their best performing relaxation achieved duality gaps of 5.3%.

Sections 2.7 and 2.8 covered two of the extensions of the TSCPLP — the Multi-Commodity TSCPLP and the Multi-Period, Multi-Commodity TSCPLP. The solution strategies applied to these problems included Lagrangian relaxation and dynamic programming. The work in this thesis could be extended to these problem types also.

The review of the literature in Chapter 2 emphasised the need for more understanding of the performance of Lagrangian relaxations that are used in solution methods. Most researchers mention that Lagrangian relaxations provide stronger bounds than linear programming bounds, but an in-depth analysis of the comparative strengths of the bounds has not been undertaken. This thesis aimed to provide this analysis. This information can be used to choose a Lagrangian relaxation that will provide strong bounds on the optimal solution to the original problem. This could be in a heuristic or an algorithm setting.

Chapter 3 provided the first part of the theoretical analysis into the TSCPLP. The formulation to be used for the theoretical work was presented and surrogate constraints included that would further strengthen the bounds from some of the relaxations. Section 3.3 proved that the TSCPLP is an NP-Hard problem and Section 3.4.1 began to classify the bounds from various Lagrangian relaxations, starting with the trivial bounds — those that give a lower bound no better than the bound from the linear programming relaxation. It was shown that the relaxations involving dualizing three sets of constraints tended to produce trivial bounds.

Chapter 4 continued the theoretical analysis with Section 4.3.1 classifying bounds as equivalent and Section 4.3.2 as dominant. Equivalent bounds were defined as pairs of

Lagrangian dual bounds that give the same lower bound value. Dominant bounds were defined as $A < B$, where for any given problem instance A is less than or equal to B but for some problem instance A is strictly less than B. This creates a hierarchy structure within all the possible Lagrangian relaxations. Section 4.4 summarized all of the relaxations and their relationship to each other, the linear programming bound, and the optimal solution. Identifications included which relaxations strictly dominate the linear programming bound and which are strictly dominated by the optimal solution.

Chapter 5 moved onto analysing the TSCPLP through compuation of results with the use of a Lagrangian heuristic and the commercial solver CPLEX. The formulation was slightly adjusted to re-work one of the surrogate constraints to make it of the same form as the other. The surrogate constraints, if left in the formulation, ensured that at each lower bound solution there was a set of open plants and depots that could satisfy the customer demand. This would therefore strengthen the lower bounds in the heuristic.

Section 5.2 presented the Lagrangian heuristic to be used to analyse the strength of the Lagrangian relaxations. This included the steps taken by the overall heuristic coded in C and the CPLEX routines used to find solutions to the subproblems. The formulae for the subgradient and stepsize calculations relevant to the dualized constraints were included.

Section 5.3 outlined the test problems used to test the TSCPLP bounds. This was adapted from a procedure used by researchers for the CPLP and the TSCPLP, and covered some of the recommendations from Rardin & Uzsoy (2001) for the evaluaton of heuristic optimization algorithms. The test problems covered a range of problem size and strength of capacity constraints. This was achieved by scaling some of the problems, adjusting the ratio of total depot capacity to customer demand ($\frac{\sum_j s_j}{\sum_i d_i}$) and the ratio of total plant capacity to total depot capacity ($\frac{\sum_k a_k}{\sum_j s_j}$), to be 1.5, 3, 5, or 10. The ratios of 3 and 5 proved to be the most difficult to solve, a result seen by Sridharan (1987) for the CPLP.

Section 5.4 and 5.5 corroborated the theoretical results that were found in Chapters 3 and 4. They also showed that z_2 provided the strongest bounds on the optimal solution for all but the two largest sized problems considered, and that $z_{2,3,4}$ performed much better than anticipated. This relaxation provided bounds with a duality gap of under 20% on many problems, and was able to give upper bounds within 3% of the optimal value. The upper bounds were fairly robust against changes in capacity ratios. This was an interesting finding considering that $z_{2,3,4}$ dualized three out of five of the main group of constraints. Even when CPLEX was unable to provide optimal solutions for large problem instances in a set time frame, $z_{2,3,4}$ gave upper and lower bounds of an acceptable standard. Relaxations involving constraint set 6 proved difficult to solve. This, along with the performance of $z_{2,3,4}$ shows that the conservation of flow constraints need to remain in the lower bound formulations to give reasonable bounds. If the capacity constraint on the plants is relaxed, the resulting bounds are stronger than those from relaxing the capacity constraint on the depots.

Section 5.6 provided some practical applications of this work in terms of recommending solution strategies based on problem characteristics, and improving both heuristic solution methods and exact algorithms.

Chapter 6 was concerned with an extension of the TSCPLP — the TSCPLPSS, where customers require that their demand is satisfied from only one open depot. Section 6.2 provided the formulation of the TSCPLPSS to be considered for analysis. This was kept the same as the TSCPLP for direct comparison. Section 6.1.1 showed that like the TSCPLP, the TSCPLP is NP-Hard.

Section 6.3 provided the main analysis of the TSCPLP. This took a similar format to the computational analysis of the TSCPLP. Again the heuristic was presented and the test data discussed. The same analysis into the ratio of total depot capacity to customer demand ($\frac{\sum_j s_j}{\sum_i d_i}$), and the ratio of total plant capacity to total depot capacity ($\frac{\sum_k a_k}{\sum_j s_j}$), was completed. The results showed that the TSCPLPSS is harder to solve than the TSCPLP, as problem sizes where the relaxations were unable to generate bounds were of a smaller size than for the TSCPLP. For many of the smaller problem instances however, the TSCPLPSS provided stronger bounds on the solution for many of the relaxations, due to the extra integer constraint on the $x_{i,j}$ variables.

Overall, the analysis showed that there are several constraints in the formulation that if dualized in a Lagrangian relaxation provide strong bounds on the optimal solution to the TSCPLP and the TSCPLPSS. These problems are often easier to solve than the original and provide information that can be used in a wide range of solution methods to reduce computational time and improve solution quality.

It could be argued that in the current environment of powerful computers and readily available software packages like CPLEX, there is no need to use a Lagrangian heuristic, or any other heuristic, to solve problems such as the TSCPLP and the TSCPLPSS. Yes, CPLEX is able to perform complex calculations and solve problems such as the TSCPLP and the TSCPLPSS in short computational time and in many ways is superior in solution quality, however the research of this thesis is a useful understanding of the processes that even CPLEX potentially uses. CPLEX uses modified branch-and-bound algorithms to solve mixed-integer linear programs, and this thesis has shown how the different Lagrangian relaxations can speed up this and other similar algorithms and/or heuristics.

The solutions for the TSCPLP and the TSCPLPSS could be found by direct use of CPLEX instead of through a Lagrangian heuristic. However many researchers may not have access to CPLEX, and still choose to use heuristics as a smaller part of a bigger research problem. The analysis of the Lagrangian relaxations in this thesis has provided insight into how the constraints affect the solution quality. This information can be used in many ways apart from an obvious solution strategy, as discussed in Chapters 5 and 6. It could provide knowledge into how a system reacts to changes in demand and/or capacity. For example if demand increases significantly, this analysis could be used to determine if a new plant or depot is required to maintain minimum costs or if the current system can cope with the tighter capacity constraints.

The analysis into larger problem sizes in Chapters 5 and 6 shows that certain Lagrangian relaxations can find upper and lower bounds on the optimal solution in shorter time than CPLEX can find an optimal solution. This isn't to say that CPLEX cannot find the optimal solution, just that in a given time frame some relaxations find bounds in reasonable time when the optimal solution is obviously difficult to find. This shows that studying Lagrangian relaxations is necessary. These difficult problem instances are characterized by the system having ratios of capacity to demand of approximately 3, and where there are a large number of depots compared to the number of plants. This means that the number of customers is approximately 5 times the number of plants, and the number of depots is approximately twice the number of plants. These problem characteristics result in a large number of possible combinations of open plants and depots, due to the combinatorial nature of plant location problems.

Referring back now to the practical examples presented in Chapter 1, where the plants could represent a manufacturer of appliances, depots represent department chain stores and customers are physical businesses or people. The information provided by the analysis of this thesis could be used in the following scenario. If a new subdivision was being planned in a major city, the information could be used to aid in the analysis of the decision to serve the new customers with a new department store (new depot) or with the current set of stores which could handle the extra capacity needed. This would also need to take into consideration whether the distance to a competitors equivalent store is sufficiently large that new customers would travel to the current store, or if the business needs a presence closer to the competition or to the source of the new custom.

7.3 Future research directions

There are several areas for further investigation that result from the work in this thesis. Firstly the nature of problems with ratios of total depot capacity to customer demand ($\frac{\sum_j s_j}{\sum_i d_i}$), and ratios of total plant capacity to total depot capacity ($\frac{\sum_k a_k}{\sum_j s_j}$), between 3 and 6 warrants further research. These problem instances proved difficult to solve and this area and the underlying reasons behind this could be investigated further.

There are several theoretical relationships that require further analysis to prove their equivalence or dominance. These include investigating:

- $z_2 > z_3$, $z_2 > z_5$, and $z_2 > z_6$.
- $z_5 > z_3$ and $z_5 > z_6$.
- The equivalence or dominance between z_3 and z_6.
- The dominance or equivalence between relaxations $z_{2,3}$, $z_{3,4}$, $z_{3,5}$, $z_{2,6}$, $z_{3,6}$, $z_{5,6}$, $z_{2,3,4}$, and $z_{2,4,6}$:
  - compared to each other
  - and compared to the linear programming bound.

From the combination of the theoretical and computational work there are a few relaxations whose strength relative to the linear programming bound is not yet known. These are:

- $z_{2,6}$, $z_{3,5}$, $z_{3,6}$, $z_{5,6}$, and $z_{2,4,6}$.

The relaxations for which it is not yet known if there exist instances where the optimal solution can be found are:

- $z_{3,6}$, $z_{2,3,6}$, and $z_{3,5,6}$.

These are areas identified from the thesis that could be further investigated.

The knowledge of the bounds resulting from Lagrangian relaxations of the TSCPLP and TSCPLPSS formulations in this thesis can be applied to many more problems presented in the literature. New research into two-stage problems can involve separating out and distinguishing between different products that are demanded by the customers, as opposed to one bulk delivery of a quantity that is demanded. This is the multi-commodity version of the TSCPLP or the TSCPLPSS. The effect on the formulation is that an extra index is needed to separate out the differing products. This has been solved using a heuristic based on a Lagrangian relaxation by Jayaraman and Pirkul (1998).

The formulation for the TSCPLP and the TSCPLPSS could also be extended into a multi-period or stochastic model where each customer's demand is changing over time. These are usually solved using dynamic programming, however the knowledge of the bounds from Lagrangian relaxations could be used in a technique similar to that of Canel *et al.* (2001). Their solution technique involved finding candidate solutions for each time period through a branch-and-bound method (which could incorporate Lagrangian relaxation), then using dynamic programming to find the optimal solution over the entire planning horizon.

A direct consequence of this work is to find an exact method, such as the branch-and-bound method, for obtaining optimal or near optimal solutions using the knowledge of the bounds from the Lagrangian dual problems to reduce the size of the branch-and-bound trees significantly. Lagrangian relaxations have been shown to provide stronger bounds than the linear programming relaxations at the equivalent branches of the tree. Traganterlangsak *et al.* (2000) recognised that the Lagrangian relaxation based branch-and-bound algorithm is superior to the equivalent linear programming based algorithm. This reduces the size of the branch-and-bound tree and the computational time required for the algorithm.

A different kind of extension to this problem is to add a third stage. A practical application of this would be an agricultural model. The upper-most stage could be the growers or farms, the inner two stages the processing plants and storage or distribution depots, and the final stage the set of customers. The demand units in this model could be any number of different types of items including fruits, vegetables, crops, or animals. The formulation presented in this thesis is easily extendable in this way, more so than formulations containing variables with three indices as discussed in Chapter 2 which would then require up to four indices for each variable. The extension would involve changing the current plant stage to be a throughput level like the current depots, and then adding the fourth level (say farms) as the source and variables describing the movement of demand from farms to the plants. The solution would then define the open set of farms, plants, and depots; and the movement of product from the farms through the plants and depots to the customers.

The main application of this research is in heuristics, using the knowledge of the bounds to reduce solution time and to give strong bounds on the problem. This knowledge can be used to ensure the heuristics work efficiently and smartly, to reach near optimal or even optimal solutions, in short amounts of computational time. Whether the heuristic involves Lagrangian relaxation or not, the analysis provided by this thesis shows which constraints tighten the solutions and cannot be ‘ignored’, which is a useful concept to understand across many applications.

Appendix A

Program source code

The full source code for the programs used in the computational analysis can be found on the accompanying CD-ROM. There are five different programs used.

- "heuristicmain.cpp" runs the heuristic as outlined in the thesis on the TSCPLP.
- "lpboundmain.cpp" finds the linear programming bound on either the TSCPLP or TSCPLPSS.
- "optimalmain.cpp" finds the optimal solution from CPLEX for the TSCPLP.
- "ssheuristicmain.cpp" runs the heuristic for the TSCPLPSS.
- "ssoptimalmain.cpp" finds the optimal solution for the TSCPLPSS from CPLEX.

Appendix B

Example files

The following sections contain an example of an input and an output file used by the heuristics. The input file (B.1) is for the size 3×5×10, problem 3. The output file (B.2) is the solution to this problem when constraint 2 is dualized. This file has been abridged, but shows how the solutions are updated at each iteration and the timing information that is available. This particular problem was solved to within an acceptably small duality gap before reaching the iteration limit of 300.

Full input and output files used in analysing both the TSCPLP and the TSCPLPSS can be found on the accompanying CD-ROM.

B.1 Input file

```
[Problem initialisation]
Customers=10
Depots=5
Plants=3
InitialLowerBound=-1000000000
InitialUpperBound=1000000000
Epsilon=0.005
Steplength=0.6
[CostArray]
Customer Transport Costs1,1=078
Customer TransportCosts1,2=0119
Customer Transport Costs1,3=021
Customer TransportCosts1,4=0111
Customer Transport Costs1,5=059
Customer Transport Costs2,1=0136
Customer Transport Costs2,2=0165
```

Customer Transport Costs2,3=075
Customer Transport Costs2,4=0166
Customer Transport Costs2,5=0108
Customer Transport Costs3,1=070
Customer Transport Costs3,2=0133
Customer Transport Costs3,3=0102
Customer Transport Costs3,4=07
Customer Transport Costs3,5=074
Customer Transport Costs4,1=076
Customer Transport Costs4,2=085
Customer Transport Costs4,3=038
Customer Transport Costs4,4=0139
Customer Transport Costs4,5=0115
Customer Transport Costs5,1=0120
Customer Transport Costs5,2=0157
Customer Transport Costs5,3=062
Customer Transport Costs5,4=0144
Customer Transport Costs5,5=077
Customer Transport Costs6,1=0110
Customer Transport Costs6,2=0171
Customer Transport Costs6,3=086
Customer Transport Costs6,4=083
Customer Transport Costs6,5=042
Customer Transport Costs7,1=0130
Customer Transport Costs7,2=0193
Customer Transport Costs7,3=0113
Customer Transport Costs7,4=088
Customer Transport Costs7,5=076
Customer Transport Costs8,1=0125
Customer Transport Costs8,2=0168
Customer Transport Costs8,3=070
Customer Transport Costs8,4=0138
Customer Transport Costs8,5=055
Customer Transport Costs9,1=096
Customer Transport Costs9,2=0160
Customer Transport Costs9,3=087
Customer Transport Costs9,4=059
Customer Transport Costs9,5=056
Customer Transport Costs10,1=051
Customer Transport Costs10,2=088
Customer Transport Costs10,3=011
Customer Transport Costs10,4=0104
Customer Transport Costs10,5=065

Depot Transport Costs1,1=0285
Depot Transport Costs1,2=0547
Depot Transport Costs1,3=0272
Depot Transport Costs1,4=0207
Depot Transport Costs1,5=0145
Depot Transport Costs2,1=0501
Depot Transport Costs2,2=0760
Depot Transport Costs2,3=0435
Depot Transport Costs2,4=0333
Depot Transport Costs2,5=0344
Depot Transport Costs3,1=0559
Depot Transport Costs3,2=0813
Depot Transport Costs3,3=0470
Depot Transport Costs3,4=0399
Depot Transport Costs3,5=0393
Plant Fixed Costs1=02910
Plant Fixed Costs2=01915
Plant Fixed Costs3=01050
Depot Fixed Costs1=0890
Depot Fixed Costs2=01100
Depot Fixed Costs3=01233
Depot Fixed Costs4=01241
Depot Fixed Costs5=01178
[CapacityDemandArray]
Customer Demand1=022
Customer Demand2=08
Customer Demand3=012
Customer Demand4=022
Customer Demand5=022
Customer Demand6=013
Customer Demand7=021
Customer Demand8=030
Customer Demand9=010
Customer Demand10=022
Plant capacities1=0711
Plant capacities2=0302
Plant capacities3=086
Depot Capacities1=068
Depot Capacities2=0108
Depot Capacities3=0136
Depot Capacities4=0126
Depot Capacities5=0115

B.2 Output file

Constraint 2 is relaxed
Constraint 3 is not relaxed
Constraint 4 is not relaxed
Constraint 5 is not relaxed
Constraint 6 is not relaxed
The time to read the input and initialise the problem = 562 ms

Iteration 1
The time to create lower bound problem = 16 ms
The time to compute the lower bound = 141 ms
The lower bound solution is: z = 3983.000000 + 443.000000 = 4426.000000
The time to read the lower bound solution = 0 ms
The time to compute the upper bound = 15 ms
The upper bound solution is: z = 77789.500000
The optimal solution has not been found yet.

Iteration 2
The time to create lower bound problem = 0 ms
The time to compute the lower bound = 16 ms
The lower bound solution is: z = 2278.570000 + 22452.050000 = 24730.620000
The time to read the lower bound solution = 0 ms
The time to check if computing the upper bound is needed = 0 ms
The time to create the upper bound problem = 15 ms
The time to compute the upper bound = 16 ms
The upper bound solution is: z = 36575.545455
The time to read the upper bound solution = 0 ms
The optimal solution has not been found yet.

Iteration 3
The time to create lower bound problem = 0 ms
The time to compute the lower bound = 94 ms
The lower bound solution is: z = 3220.075528 + 23873.441055 = 27093.516583
The time to read the lower bound solution = 15 ms
The time to check if computing the upper bound is needed = 0 ms
The time to create the upper bound problem = 0 ms
The time to compute the upper bound = 16 ms
The upper bound solution is: z = 41427.500000
The time to read the upper bound solution = 0 ms
The optimal solution has not been found yet.

Iteration 4
The time to create lower bound problem = 15 ms
The time to compute the lower bound = 0 ms
The lower bound solution is: z = 3220.075528 + 25580.206252 = 28800.281780
The time to read the lower bound solution = 16 ms
This set of open plants and depots has been found before.
The time for checking, creating and reading the upper bound problem = 0 ms
The upper bound solution is: z = 41427.500000
The optimal solution has not been found yet.

Iteration 5
The time to create lower bound problem = 0 ms
The time to compute the lower bound = 16 ms
The lower bound solution is: z = 3191.103988 + 26979.753713 = 30170.857701
The time to read the lower bound solution = 0 ms
This set of open plants and depots has been found before.
The time for checking, creating and reading the upper bound problem = 0 ms
The upper bound solution is: z = 41427.500000
The optimal solution has not been found yet.

⋮

Iteration 19
The time to create lower bound problem = 15 ms
The time to compute the lower bound = 16 ms
The lower bound solution is: z = -1786.755508 + 35727.294381 = 33940.538873
The time to read the lower bound solution = 0 ms
This set of open plants and depots has been found before.
The time for checking, creating and reading the upper bound problem = 0 ms
The upper bound solution is: z = 36575.545455
The optimal solution has not been found yet.

Iteration 20
The time to create lower bound problem = 15 ms
The time to compute the lower bound = 0 ms
The lower bound solution is: z = -1607.786254 + 35756.938164 = 34149.151910
The time to read the lower bound solution = 16 ms
The time to check if computing the upper bound is needed = 0 ms
The time to create the upper bound problem = 0 ms
The time to compute the upper bound = 31 ms
The upper bound solution is: z = 41427.500000
The time to read the upper bound solution = 0 ms
The optimal solution has not been found yet.

⋮

Iteration 40
The time to create lower bound problem = 16 ms
The time to compute the lower bound = 15 ms
The lower bound solution is: z = -2577.473002 + 38019.150673 = 35441.677671
The time to read the lower bound solution = 0 ms
The time to check if computing the upper bound is needed = 0 ms
The time to create the upper bound problem = 0 ms
The time to compute the upper bound = 16 ms
The upper bound solution is: z = 41427.500000
The time to read the upper bound solution = 0 ms
The optimal solution has not been found yet.

Iteration 41
The time to create lower bound problem = 15 ms
The time to compute the lower bound = 16 ms
The lower bound solution is: z = -2959.767803 + 38359.311246 = 35399.543443
The time to read the lower bound solution = 0 ms
The time to check if computing the upper bound is needed = 0 ms
The time to create the upper bound problem = 0 ms
The time to compute the upper bound = 16 ms
The upper bound solution is: z = 36575.545455
The time to read the upper bound solution = 0 ms
The optimal solution has not been found yet.

Iteration 42
The time to create lower bound problem = 31 ms
The time to compute the lower bound = 31 ms
The lower bound solution is: z = -2979.004237 + 38438.682139 = 35459.677902
The time to read the lower bound solution = 0 ms
This set of open plants and depots has been found before.
The time for checking, creating and reading the upper bound problem = 0 ms
The upper bound solution is: z = 36575.545455
The optimal solution has not been found yet.

⋮

Iteration 60
The time to create lower bound problem = 16 ms
The time to compute the lower bound = 15 ms
The lower bound solution is: z = -2709.185621 + 38853.416929 = 36144.231308

The time to read the lower bound solution = 0 ms
The time to check if computing the upper bound is needed = 0 ms
The time to create the upper bound problem = 15 ms
The time to compute the upper bound = 0 ms
The upper bound solution is: z = 36575.545455
The time to read the upper bound solution = 0 ms
The optimal solution has not been found yet.

Iteration 61
The time to create lower bound problem = 16 ms
The time to compute the lower bound = 16 ms
The lower bound solution is: z = -2766.483103 + 38909.487515 = 36143.004412
The time to read the lower bound solution = 0 ms
This set of open plants and depots has been found before.
The time for checking, creating and reading the upper bound problem = 0 ms
The upper bound solution is: z = 36575.545455
The optimal solution has not been found yet.

Iteration 62
The time to create lower bound problem = 15 ms
The time to compute the lower bound = 16 ms
The lower bound solution is: z = -2622.114689 + 38855.686744 = 36233.572056
The time to read the lower bound solution = 0 ms
The time to check if computing the upper bound is needed = 0 ms
The time to create the upper bound problem = 0 ms
The time to compute the upper bound = 16 ms
The upper bound solution is: z = 41427.500000
The time to read the upper bound solution = 0 ms
The optimal solution has not been found yet.

⋮

Iteration 70
The time to create lower bound problem = 0 ms
The time to compute the lower bound = 15 ms
The lower bound solution is: z = -2596.403713 + 38969.560965 = 36373.157252
The time to read the lower bound solution = 0 ms
The time to check if computing the upper bound is needed = 0 ms
The time to create the upper bound problem = 16 ms
The time to compute the upper bound = 16 ms
The upper bound solution is: z = 41427.500000
The time to read the upper bound solution = 15 ms
The optimal solution has not been found yet.

Iteration 71
The time to create lower bound problem = 0 ms
The time to compute the lower bound = 16 ms
The lower bound solution is: z = -2599.556360 + 38989.293815 = 36389.737455
The time to read the lower bound solution = 15 ms
This set of open plants and depots has been found before.
The time for checking, creating and reading the upper bound problem = 0 ms
The upper bound solution is: z = 41427.500000
The optimal solution has not been found yet.

Iteration 72
The time to create lower bound problem = 0 ms
The time to compute the lower bound = 16 ms
The lower bound solution is: z = -2609.488411 + 39019.952135 = 36410.463725
The time to read the lower bound solution = 0 ms
This set of open plants and depots has been found before.
The time for checking, creating and reading the upper bound problem = 0 ms
The upper bound solution is: z = 41427.500000

The duality gap is acceptably small.
The current upper (and lower) bound are taken as optimal.
The best upper bound = 36575.545455, the best lower bound = 36410.463725

TIMINGS:
Total run time for the heuristic = 104453 ms
The total time to calculate the lower bounds with CPLEX = 1218 ms
The total time to calculate the upper bounds with CPLEX = 379 ms
The total time to create the lower bound problems = 642 ms
The total time to read the lower bound solutions = 247 ms
The total time to check if computing the upper bounds are needed = 0 ms
The total time to create the upper bound problems = 249 ms
The total time to read the upper bound solutions = 77 ms

The best upper bound found = 36575.545455,
the best lower bound found = 36410.463725
Therefore the duality gap is 0.453391

Appendix C

Data from computational results

C.1 TSCPLP

The results from the TSCPLP heuristic are given here. The first six tables (C.1 - C.6) are the runs where all combinations of dualized constraints were considered. Then the results from the varying ratios of depot capacity to customer demand (C.13 - C.16), and plant capacity to depot capacity (C.17 - C.20), are given. There is one table for each relaxation studied (dualizing constraints 2; 5; 3 and 5; 2, 3, and 4). Each table shows the best lower and upper bound found by the heuristic, and the optimal solution and linear programming bound for the problem. All times are given in milliseconds, and some tables quote total run times for the heuristic as well as the time taken calculating the upper and lower bounds individually.

Full data, including comprehensive timing information, can be found on the accompanying CD-ROM.

Dualized	Best LB	Best UB	LBs time	UBs time	Total time
2	53385.83	55627.61	6938	2095	9471
3	50796.62	73189.86	6936	2267	9641
4	55152.86	55152.86	141	0	625
5	51848.86	55152.86	7813	77	8343
6	27580.22	55152.86	7306	206	7887
2,3	51051.35	55627.61	7099	5046	12567
2,4	53385.83	55627.61	7095	2169	9686
2,5	39708.70	55152.86	7202	531	8139
2,6	28939.22	67134.86	6777	738	7922
3,4	49826.71	73819.86	6298	6184	12904
3,5	49035.65	55316.86	6631	2214	9235
3,6	19191.89	68458.86	7482	1799	9687
4,5	51848.86	55152.86	6988	77	7472
4,6	27580.22	55152.86	7327	235	8031
5,6	25175.30	55316.86	7912	218	8630
2,3,4	49534.29	55627.61	7178	3667	11267
2,3,5	36139.48	68458.86	8205	250	8892
2,3,6	17023.46	67609.61	7044	1345	8827
2,4,5	39708.70	55152.86	7451	484	8373
2,4,6	30841.51	67134.86	7567	637	8641
2,5,6	36783.29	67134.86	7283	656	8345
3,4,5	46205.88	68458.86	7133	1930	9500
3,4,6	17208.14	67688.86	7846	5366	13650
3,5,6	17047.80	68458.86	7204	1856	9513
4,5,6	26989.17	67134.86	7031	640	8093
LP bound	49815.26				562
Optimal		55152.86			547

Table C.1: Size 3×5×10, problem 1 results.

Dualized	Best LB	Best UB	LBs time	UBs time	Total time
2	66135.24	67061.32	6934	79	7451
3	64563.22	67061.32	7576	5297	13279
4	67061.32	67061.32	125	15	547
5	66165.79	69638.73	6766	32	7173
6	59104.13	68867.73	6813	656	7844
2,3	62630.04	67061.32	6593	4891	11859
2,4	66135.24	67061.32	7547	63	8032
2,5	63384.57	67061.32	10267	110	10799
2,6	52191.81	67885.32	7544	1567	9533
3,4	65497.65	67061.32	7418	4676	12531
3,5	62469.34	67061.32	7078	1157	8798
3,6	42178.99	81817.73	7484	5030	12951
4,5	66165.79	69638.73	10760	63	11260
4,6	59104.13	68867.73	7549	874	8845
5,6	50720.85	67061.32	7470	639	8515
2,3,4	66070.58	67061.32	6687	5373	12435
2,3,5	43848.07	67061.32	6917	519	7826
2,3,6	37631.15	74748.83	6983	1408	8735
2,4,5	63384.57	67061.32	8866	92	9365
2,4,6	52191.81	67885.32	7652	1628	9718
2,5,6	41433.60	67061.32	7930	1990	10358
3,4,5	59945.38	67061.32	8372	1455	10265
3,4,6	55009.70	74748.83	8243	3038	11797
3,5,6	42695.31	69638.73	8263	1390	10137
4,5,6	47574.91	67061.32	9251	3596	13300
LP bound	64622.98				563
Optimal		67061.32			547

Table C.2: Size 3×5×10, problem 2 results.

Dualized	Best LB	Best UB	LBs time	UBs time	Total time
2	36410.46	36575.55	2107	705	3374
3	32959.98	36575.55	9141	1893	11456
4	36575.55	36575.55	125	16	563
5	35580.55	36575.55	8157	47	8860
6	35189.86	36575.55	8097	7702	16330
2,3	32106.69	36575.55	8326	5582	14330
2,4	36410.46	36575.55	2011	785	3218
2,5	35206.52	36575.55	8303	1868	10593
2,6	32770.74	36575.55	8472	655	9565
3,4	32541.30	36575.55	8403	1875	10840
3,5	22878.38	36575.55	8591	2315	11328
3,6	27350.59	37675.55	8523	3257	12311
4,5	35580.55	36575.55	8247	47	8731
4,6	35189.86	36575.55	8516	8870	17839
5,6	31382.67	66393.03	8810	7896	17159
2,3,4	34460.63	36575.55	8599	4335	13418
2,3,5	33790.02	36575.55	8562	234	9249
2,3,6	27342.72	41427.50	8581	2559	11593
2,4,5	35206.52	36575.55	6064	1312	7735
2,4,6	32770.74	36575.55	8487	577	9486
2,5,6	21924.45	36575.55	6610	280	7249
3,4,5	26946.26	36575.55	8420	2173	11030
3,4,6	26479.38	36575.55	8434	5424	14312
3,5,6	22646.29	66393.03	8203	3263	11935
4,5,6	30815.56	66393.03	8859	1077	10421
LP bound	33912.30				782
Optimal		36575.55			547

Table C.3: Size 3×5×10, problem 3 results.

Dualized	Best LB	Best UB	LBs time	UBs time	Total time
2	60850.96	62224.54	8753	155	10533
3	54018.67	66861.01	8865	4997	15409
4	62129.54	62129.54	110	16	1751
5	59144.54	62224.54	8530	64	10313
6	51755.40	63675.54	11645	1339	14531
2,3	53936.24	63498.01	9275	5665	16752
2,4	60850.96	62224.54	9420	233	11419
2,5	40495.59	63233.54	9643	345	11831
2,6	38363.94	63498.01	9998	860	12811
3,4	50420.42	80880.01	9473	8854	20171
3,5	53276.50	67321.01	9736	3515	15219
3,6	31792.26	63498.01	8503	3231	13375
4,5	59144.54	62224.54	8266	47	9859
4,6	51664.36	64949.01	8377	623	10532
5,6	37808.52	63308.20	8609	439	10688
2,3,4	51773.09	63498.01	8407	7608	17609
2,3,5	47917.95	63062.14	9984	1656	13562
2,3,6	18679.36	66861.01	11031	643	13502
2,4,5	40495.59	63233.54	10439	406	12908
2,4,6	36602.99	67660.14	10744	613	13264
2,5,6	25090.77	65636.01	10391	468	12844
3,4,5	45108.76	80129.01	10370	2943	15282
3,4,6	44975.25	64949.01	10377	3672	16080
3,5,6	48961.12	64949.01	9534	1702	12924
4,5,6	51139.15	63498.01	9920	8611	20187
LP bound	58319.22				4625
Optimal		62129.54			1984

Table C.4: Size 5×10×25, problem 1 results.

Dualized	Best LB	Best UB	LBs time	UBs time	Total time
2	42065.52	43235.00	11456	2436	15673
3	40605.93	57365.66	10654	3517	16187
4	43235.00	43235.00	125	15	2031
5	39410.00	43235.00	10999	94	12953
6	32862.67	43235.00	11204	1700	14670
2,3	40524.58	59237.28	10781	8987	21752
2,4	40406.71	44412.16	11454	95	13283
2,5	24787.40	51734.53	10843	686	13514
2,6	21155.38	43235.00	10994	4018	16887
3,4	40053.91	56781.00	11149	9351	22578
3,5	17305.09	43235.00	10705	1533	14035
3,6	27242.86	43363.14	10862	3353	16121
4,5	39410.00	43235.00	10737	78	12580
4,6	32862.67	43235.00	11069	1619	14516
5,6	26876.94	43235.00	10288	10413	22685
2,3,4	30996.45	45632.16	11315	7982	21063
2,3,5	37766.54	57365.66	10608	2893	15236
2,3,6	26377.74	43363.14	11547	5123	18983
2,4,5	24787.40	51734.53	11359	689	14111
2,4,6	23926.15	43235.00	11238	3062	16175
2,5,6	13553.66	44583.14	11574	3343	16807
3,4,5	17305.09	43235.00	11427	2837	16232
3,4,6	28284.00	43235.00	11401	7038	20392
3,5,6	23231.16	47358.92	11196	3739	16732
4,5,6	26876.94	43235.00	10865	10198	23110
LP bound	39112.08				2078
Optimal		43235.00			1937

Table C.5: Size 5×10×25, problem 2 results.

Dualized	Best LB	Best UB	LBs time	UBs time	Total time
2	97965.61	109651.05	14460	782	17007
3	100311.50	123743.62	10338	9346	21450
4	109651.05	109651.05	156	16	2078
5	96979.30	125539.94	11064	93	13219
6	80921.57	131161.17	9993	7163	18640
2,3	73018.95	109651.05	7983	1673	11031
2,4	80576.77	109651.05	7875	218	9484
2,5	72788.48	112715.05	7892	171	9438
2,6	63048.51	123958.05	8611	3031	13017
3,4	102454.51	123743.62	8718	6487	16627
3,5	72807.29	148150.94	7765	501	9688
3,6	67075.59	125539.94	8123	721	10281
4,5	96979.30	125539.94	7626	62	9078
4,6	80921.57	131161.17	7998	5706	15094
5,6	62127.67	181393.90	8266	7514	17092
2,3,4	94725.24	110732.05	7992	4758	14235
2,3,5	61833.08	121977.94	7466	831	9672
2,3,6	59429.59	128332.13	7779	2503	11642
2,4,5	73251.65	112715.05	10971	343	12720
2,4,6	63048.51	123958.05	9254	3480	14187
2,5,6	43474.18	145408.05	8605	2927	12954
3,4,5	70670.54	125539.94	8409	1028	10875
3,4,6	45873.24	186957.45	8251	421	10079
3,5,6	52003.46	148150.94	8638	2502	12562
4,5,6	62044.80	181393.90	8814	7202	17438
LP bound	106048.14				2000
Optimal		109651.05			2203

Table C.6: Size 5×10×25, problem 3 results.

	Size 10×33×50		Size 15×40×60		Size 20×50×75	
Dualized	Best LB	Best UB	Best LB	Best UB	Best LB	Best UB
2	96550.43	105261.30	42559.95	47729.94	142851.28	151602.51
3	84568.56	122613.45	31571.67	50688.88	123113.50	157639.32
4	100746.01	100746.01	46142.29	46142.29	not found	not found
5	90948.48	116388.45	36918.48	53714.15	124776.67	153137.67
6	80537.94	136742.90	not found	not found	not found	not found
2,3	75811.33	110071.43	29561.10	48073.95	115732.49	150412.24
2,4	96670.36	109124.10	42577.26	50310.00	144908.02	155731.92
2,5	67745.47	114658.05	27843.37	51752.32	113748.57	157224.12
2,6	73679.90	141751.68	32550.95	64482.02	111108.24	158062.92
3,4	81942.12	127673.71	30872.84	55141.76	119355.40	157586.85
3,5	50334.14	128599.04	30254.75	87953.45	102616.42	183154.55
3,6	55512.25	132988.80	35402.01	93358.91	not found	not found
4,5	90828.48	116388.45	36918.67	53713.45	125491.48	152665.31
4,6	80337.48	137547.27	not found	not found	not found	not found
5,6	64469.29	143928.87	32599.28	80600.05	110997.86	189377.13
2,3,4	77168.10	108666.02	32088.58	51179.23	116212.44	150606.46
2,3,5	43669.58	114586.47	28579.26	85707.00	113546.93	205548.45
2,3,6	54350.57	153660.00	33460.18	109134.85	113815.25	223583.84
2,4,5	67167.49	116235.91	27855.10	52877.37	113748.57	157224.12
2,4,6	83457.28	145624.75	29367.06	58714.00	103437.42	151144.65
2,5,6	40245.19	142916.20	19155.94	92244.09	98019.90	247414.49
3,4,5	55578.58	152557.72	25600.02	82003.74	103324.60	202622.10
3,4,6	45245.37	136187.45	25255.05	91035.00	103182.90	214542.82
3,5,6	55595.28	155386.00	35475.18	118675.20	120665.81	245528.35
4,5,6	63356.85	187829.76	33426.12	120982.30	112253.69	234045.91

Table C.7: Larger sizes, problem 1 results.

	Size 30×60×120		Size 40×80×200		Size 50×100×250	
Dualized	Best LB	Best UB	Best LB	Best UB	Best LB	Best UB
2	189883.68	252195.00	not found	not found	not found	not found
3	141521.82	352828.91	158504.44	218290.01	not found	not found
4	not found	not found	not found	not found	not found	not found
5	not found	not found	not found	not found	not found	not found
6	not found	not found	not found	not found	not found	not found
2,3	131514.89	371037.83	147296.67	217141.21	134239.97	343142.33
2,4	191146.59	285235.20	not found	not found	not found	not found
2,5	123880.11	373889.29	not found	not found	not found	not found
2,6	146530.88	495146.14	not found	not found	not found	not found
3,4	137617.13	401801.80	154131.19	234414.29	not found	not found
3,5	135585.67	682851.28	151855.96	313870.46	138088.92	563810.79
3,6	not found	not found	not found	not found	not found	not found
4,5	not found	not found	not found	not found	not found	not found
4,6	not found	not found	not found	not found	not found	not found
5,6	not found	not found	not found	not found	not found	not found
2,3,4	142972.42	398670.83	160129.11	233182.38	145917.49	357007.86
2,3,5	149749.95	808242.25	167719.94	350877.09	152425.14	651709.37
2,3,6	150368.13	873258.87	168412.31	371079.87	not found	not found
2,4,5	123915.00	396359.43	not found	not found	not found	not found
2,4,6	136669.39	484247.33	not found	not found	not found	not found
2,5,6	129254.26	1217210.64	not found	not found	not found	not found
3,4,5	136520.47	797318.95	152902.93	350088.83	139341.67	734667.99
3,4,6	136069.43	968827.07	152397.76	396560.31	138881.96	833478.59
3,5,6	159014.87	988701.48	178096.65	414810.18	162267.95	870304.90
4,5,6	not found	not found	not found	not found	not found	not found

Table C.8: Larger sizes, problem 1 results continued.

	Size 10×33×50		Size 15×40×60		Size 20×50×75	
Dualized	Best LB	Best UB	Best LB	Best UB	Best LB	Best UB
2	85645.57	88413.92	42671.17	44510.59	115091.02	132343.17
3	79568.45	92430.24	39643.30	48463.52	85780.80	153529.74
4	87463.25	87463.25	43235.79	43235.79	not found	not found
5	81648.28	90268.54	38039.85	44027.38	99921.34	162839.18
6	77246.57	96693.35	not found	not found	not found	not found
2,3	78912.67	91799.75	37651.40	46255.33	79715.99	148253.14
2,4	85645.67	89644.55	42663.28	45421.90	115840.42	140174.17
2,5	83257.39	103893.47	35933.18	46462.32	75058.86	149406.92
2,6	79897.67	107031.16	35669.17	48052.66	88836.59	193598.37
3,4	85945.47	103234.17	38929.17	49595.37	83424.32	157634.17
3,5	67567.76	105145.79	30340.73	50940.07	82153.14	261310.57
3,6	72367.09	108703.09	35534.63	56213.77	not found	not found
4,5	94653.57	104173.46	37045.25	44027.00	99503.19	162336.90
4,6	93246.93	117028.32	not found	not found	not found	not found
5,6	71678.57	104511.71	32661.29	49791.08	88738.29	263493.31
2,3,4	87640.92	100547.79	40162.52	48979.12	86639.95	158569.42
2,3,5	62567.84	98590.24	33680.64	57636.46	90747.54	295869.59
2,3,6	63159.71	104599.73	33560.41	60565.21	91142.20	335366.50
2,4,5	73257.05	92467.45	34941.30	46874.41	75110.00	153204.00
2,4,6	78897.04	99417.83	37467.68	51302.96	82839.94	188852.25
2,5,6	67548.27	129666.09	29253.71	68976.37	78325.92	411263.57
3,4,5	65890.93	107074.55	31728.69	55875.82	82729.68	303654.63
3,4,6	62428.47	108171.31	30337.29	58819.29	82456.32	330689.94
3,5,6	66496.92	109492.55	35596.68	65262.97	96382.65	374889.59
4,5,6	71678.46	122145.01	33566.61	64796.67	89952.95	360751.43

Table C.9: Larger sizes, problem 2 results.

	Size 30×60×120		Size 40×80×200		Size 50×100×250	
Dualized	Best LB	Best UB	Best LB	Best UB	Best LB	Best UB
2	88512.39	106334.86	not found	not found	not found	not found
3	66143.52	126143.12	112953.11	227453.88	not found	not found
4	not found	not found	not found	not found	not found	not found
5	not found	not found	not found	not found	not found	not found
6	not found	not found	not found	not found	not found	not found
2,3	61573.61	129498.32	103207.67	239209.80	201059.96	313731.90
2,4	89001.74	115962.26	not found	not found	not found	not found
2,5	57910.72	128917.90	not found	not found	not found	not found
2,6	68281.08	167533.64	not found	not found	not found	not found
3,4	64321.33	139360.28	107791.83	264334.99	not found	not found
3,5	63173.31	218925.67	106199.17	422007.70	207283.38	436623.91
3,6	not found	not found	not found	not found	not found	not found
4,5	not found	not found	not found	not found	not found	not found
4,6	not found	not found	not found	not found	not found	not found
5,6	not found	not found	not found	not found	not found	not found
2,3,4	66620.46	139445.77	112190.38	254445.16	218576.24	332427.17
2,3,5	69983.31	257170.57	117303.96	474311.99	228937.72	497862.43
2,3,6	70271.79	275775.15	117788.62	512815.48	not found	not found
2,4,5	57927.00	135315.18	not found	not found	not found	not found
2,4,6	63679.05	162636.58	not found	not found	not found	not found
2,5,6	60218.66	369753.36	not found	not found	not found	not found
3,4,5	63809.55	251652.74	107132.05	491277.11	208712.50	529603.72
3,4,6	63299.07	300350.58	106778.43	582464.24	208022.94	582404.60
3,5,6	74406.94	310185.00	124467.66	585937.98	243101.93	624725.72
4,5,6	not found	not found	not found	not found	not found	not found

Table C.10: Larger sizes, problem 2 results continued.

	Size 10×33×50		Size 15×40×60		Size 20×50×75	
Dualized	Best LB	Best UB	Best LB	Best UB	Best LB	Best UB
2	99567.34	114270.04	84618.40	101279.18	126589.12	157744.78
3	89378.45	155370.17	63066.77	125153.82	94347.88	216642.82
4	101432.45	101432.45	91877.31	91877.31	not found	not found
5	92693.71	135278.80	73456.87	127693.68	109891.47	223485.05
6	82303.99	176833.66	not found	not found	not found	not found
2,3	77565.58	134836.06	58607.35	118227.49	87676.59	211394.15
2,4	97124.94	117841.77	85181.19	110338.57	127431.06	171339.08
2,5	68237.73	145492.07	55205.04	132197.27	82586.74	216738.01
2,6	51821.69	129971.14	65298.97	169225.70	97687.25	286821.05
3,4	81613.22	156439.94	61326.71	139812.94	91744.75	225718.74
3,5	52174.34	185720.31	60421.42	247792.26	90390.45	413707.25
3,6	57134.22	188376.44	70993.08	259696.71	not found	not found
4,5	92568.80	135096.51	73230.29	127299.80	109552.51	222795.70
4,6	81778.56	177046.56	not found	not found	not found	not found
5,6	44506.84	134328.52	65234.03	221504.29	97590.11	413033.40
2,3,4	78645.65	131552.00	63713.20	125391.26	95314.95	225119.03
2,3,5	45397.17	166810.29	66733.49	283532.97	99833.30	470030.46
2,3,6	34589.97	136747.86	67008.97	313289.57	100245.42	541052.52
2,4,5	67403.58	147588.70	55220.59	135858.69	82610.00	223576.27
2,4,6	60123.20	133959.25	60904.36	159835.47	91112.93	282467.66
2,5,6	41540.41	216026.17	57599.94	414842.39	86169.51	687055.10
3,4,5	56415.96	217480.71	60838.46	278448.91	91013.65	489732.73
3,4,6	46160.20	198394.66	60637.02	317325.64	90712.95	538720.19
3,5,6	56577.63	223673.81	70862.24	340606.43	106009.91	608692.16
4,5,6	64513.88	271985.38	66149.23	347257.60	98959.25	587692.74

Table C.11: Larger sizes, problem 3 results.

	Size 30×60×120		Size 40×80×200		Size 50×100×250	
Dualized	Best LB	Best UB	Best LB	Best UB	Best LB	Best UB
2	141779.82	151965.97	not found	not found	not found	not found
3	105669.63	140212.40	144239.04	392736.17	not found	not found
4	not found	not found	not found	not found	not found	not found
5	not found	not found	not found	not found	not found	not found
6	not found	not found	not found	not found	not found	not found
2,3	98197.78	137353.05	134039.97	424346.96	160847.97	572556.06
2,4	142722.79	158103.63	not found	not found	not found	not found
2,5	92497.15	133366.63	not found	not found	not found	not found
2,6	109409.72	166398.52	not found	not found	not found	not found
3,4	102754.13	145940.90	140259.38	473953.97	not found	not found
3,5	101237.30	190699.86	138188.92	811597.91	165826.70	1003846.78
3,6	not found	not found	not found	not found	not found	not found
4,5	not found	not found	not found	not found	not found	not found
4,6	not found	not found	not found	not found	not found	not found
5,6	not found	not found	not found	not found	not found	not found
2,3,4	106752.74	148552.25	145717.49	449361.46	174860.99	590877.13
2,3,5	111813.29	219458.29	152625.14	913913.01	183150.17	1165812.54
2,3,6	112274.87	230447.20	153255.20	995637.56	not found	not found
2,4,5	92523.20	137060.21	not found	not found	not found	not found
2,4,6	102046.48	158865.71	not found	not found	not found	not found
2,5,6	96509.85	274360.16	not found	not found	not found	not found
3,4,5	101935.29	209957.27	139141.67	958739.65	166970.00	1339520.21
3,4,6	101598.51	237730.99	138681.96	1153537.14	166418.35	1534424.92
3,5,6	118731.10	254361.56	162067.95	1145961.48	194481.54	1588951.18
4,5,6	not found	not found	not found	not found	not found	not found

Table C.12: Larger sizes, problem 3 results continued.

Size	Ratio	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
A:1	1.5	62165.81	63927.84	63392.84	59633.97	8092	189
A:2	1.5	57398.04	66790.71	66790.71	62994.61	8154	205
A:3	1.5	65591.23	67822.85	67784.32	64429.31	8452	173
A:1	3	54892.31	55293.94	55293.94	52934.34	8280	298
A:2	3	26954.44	28464.56	28464.56	26707.90	8859	1436
A:3	3	47849.31	48803.15	48803.15	45577.01	8188	327
A:1	5	24488.29	24903.20	24903.20	20820.27	9062	5658
A:2	5	26892.72	28264.00	28264.00	24501.96	8288	4792
A:3	5	31011.27	31166.15	31166.15	28317.02	7688	2062
A:1	10	14349.00	14349.00	14349.00	12550.84	126	46
A:2	10	11793.00	11793.00	11793.00	10420.44	156	31
A:3	10	16052.00	16052.00	16052.00	14341.60	265	94
B:1	1.5	124064.00	135269.83	135244.86	132656.33	9515	891
B:2	1.5	109843.84	110387.58	110387.58	104021.99	6705	249
B:3	1.5	96689.50	126155.44	106730.43	104756.83	10364	2182
B:1	3	75649.45	77216.71	77216.71	70387.46	9295	547
B:2	3	62692.88	83467.52	66826.29	61788.58	9435	4081
B:3	3	46353.89	46674.70	46674.70	41725.38	10570	32
B:1	5	27379.58	28860.39	28860.39	23486.24	9402	1645
B:2	5	28290.31	31097.19	29869.19	23537.40	9426	2338
B:3	5	36674.98	37229.97	37229.97	31277.74	8690	310
B:1	10	19237.84	19376.00	19376.00	17230.67	8442	5339
B:2	10	22395.88	23718.20	23718.20	19876.25	8370	5396
B:3	10	21638.25	23523.50	21877.33	18502.39	11428	809
C:1	1.5	106701.85	126116.54	118420.59	111491.75	13139	407
C:2	1.5	82799.86	89749.59	89749.59	82799.62	13578	610
C:3	1.5	76136.16	96202.83	96202.83	90240.90	12904	828
C:1	3	53281.41	58484.95	58484.95	52427.13	12953	891
C:2	3	74928.77	77560.05	77495.63	70021.02	13400	3237
C:3	3	54341.62	56865.65	56865.65	53133.92	13673	172
C:1	5	46015.34	46281.72	46149.23	39162.62	13311	2708
C:2	5	30448.23	30871.13	30871.13	24579.58	12925	78
C:3	5	47667.00	52471.44	48087.91	41103.91	12735	3641
C:1	10	22130.77	22245.42	22245.42	16809.39	13519	10156
C:2	10	23038.00	23770.97	23770.97	22373.82	12622	8019
C:3	10	27489.55	27990.20	27990.20	23823.01	13923	8046

Table C.13: Depot capacity to customer demand ratio results, dualizing constraint 2. Problem sizes A (5×8×25), B (5×16×25) and C (5×25×25).

Size	Ratio	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
A:1	1.5	58564.90	63927.84	63392.84	59633.97	7564	31
A:2	1.5	61896.41	66790.71	66790.71	62994.61	7718	219
A:3	1.5	64225.78	79339.30	67784.32	64429.31	8126	123
A:1	3	45873.44	55293.94	55293.94	52934.34	8125	94
A:2	3	27432.81	28464.56	28464.56	26707.90	7797	16
A:3	3	43971.45	48803.15	48803.15	45577.01	8218	79
A:1	5	21843.20	24903.20	24903.20	20820.27	8001	47
A:2	5	25487.00	28264.00	28264.00	24501.96	7955	62
A:3	5	29975.00	32692.00	31166.15	28317.02	8858	48
A:1	10	13502.88	14349.00	14349.00	12550.84	7548	47
A:2	10	11608.38	11793.00	11793.00	10420.44	7922	47
A:3	10	15483.00	16052.00	16052.00	14341.60	9218	63
B:1	1.5	122750.87	165598.42	135244.86	132656.33	10719	265
B:2	1.5	104972.62	111634.62	110387.58	104021.99	9095	78
B:3	1.5	103528.14	106854.35	106730.43	104756.83	9358	79
B:1	3	71670.85	08536.71	77216.71	70387.46	9063	125
B:2	3	59105.36	68358.29	66826.29	61788.58	9266	171
B:3	3	42309.70	46674.70	46674.70	41725.38	9061	109
B:1	5	25156.95	28860.39	28860.39	23486.24	8921	78
B:2	5	23646.36	31843.75	29869.19	23537.40	9187	157
B:3	5	31893.47	37960.97	37229.97	31277.74	8046	63
B:1	10	18587.06	19376.20	19376.20	17230.67	8061	32
B:2	10	21142.20	27307.20	23718.20	19876.25	8124	16
B:3	10	20865.67	24258.08	21877.33	18502.39	10533	92
C:1	1.5	103351.81	121819.12	118420.59	111491.75	12251	219
C:2	1.5	84270.30	90958.20	89749.59	82799.62	9314	141
C:3	1.5	90899.83	99633.77	96202.83	90240.90	12357	204
C:1	3	52769.13	60517.07	58484.95	52427.13	12392	218
C:2	3	70263.63	78378.05	77495.63	70021.02	12846	264
C:3	3	53965.57	56865.65	56865.65	53133.92	12282	94
C:1	5	40266.23	61835.93	46149.23	39162.62	11857	94
C:2	5	25969.92	32538.83	30871.13	24579.58	12107	142
C:3	5	42429.91	49519.07	48087.91	41103.91	12034	94
C:1	10	18718.88	24142.88	22245.42	16809.39	12421	124
C:2	10	23484.93	26437.97	23770.97	22373.82	12158	93
C:3	10	25734.00	29660.00	27990.20	23823.01	13145	78

Table C.14: Depot capacity to customer demand ratio results, dualizing constraint 5. Problem sizes A (5×8×25), B (5×16×25) and C (5×25×25).

Size	Ratio	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
A:1	1.5	52861.94	65008.84	63392.84	59633.97	7842	611
A:2	1.5	51569.62	84124.33	66790.71	62994.61	8082	1012
A:3	1.5	51819.87	81764.79	67784.32	64429.31	8077	564
A:1	3	42455.55	68221.74	55293.94	52934.34	8267	234
A:2	3	26509.79	38015.56	28464.56	26707.90	8157	905
A:3	3	38617.02	67282.57	48803.15	45577.01	8230	629
A:1	5	20230.96	30908.91	24903.20	20820.27	8286	1619
A:2	5	22173.66	32372.00	28264.00	24501.96	8440	1347
A:3	5	30001.87	33131.00	31166.15	28317.02	7641	46
A:1	10	13502.88	14349.00	14349.00	12550.84	8095	45
A:2	10	11242.68	11793.00	11793.00	10420.44	8172	78
A:3	10	15637.44	16052.00	16052.00	14341.60	8000	141
B:1	1.5	97337.47	192560.17	135244.86	132656.33	9438	4953
B:2	1.5	84418.26	140572.18	110387.58	104021.99	9175	2589
B:3	1.5	68605.20	126155.44	106730.43	104756.83	9338	2287
B:1	3	60527.49	109991.63	77216.71	70387.46	9001	421
B:2	3	53041.21	102835.83	66826.29	61788.58	9381	1480
B:3	3	30910.40	65327.69	46674.70	41725.38	10270	1310
B:1	5	23434.82	30372.60	28860.39	23486.24	9161	2026
B:2	5	19433.07	30906.75	29869.19	23537.40	10632	2464
B:3	5	26097.17	39054.47	37229.97	31277.74	10765	2734
B:1	10	18091.56	21128.00	19376.20	17230.67	8182	2912
B:2	10	20724.57	25209.00	23718.20	19876.25	11154	2564
B:3	10	15666.68	21877.33	21877.33	18502.39	10751	1907
C:1	1.5	71524.44	149527.28	118420.59	111491.75	12249	3295
C:2	1.5	55586.95	110120.34	89749.59	82799.62	9255	5105
C:3	1.5	57410.49	120052.70	96202.83	90240.90	10856	1564
C:1	3	34361.00	82384.40	58484.95	52427.13	12359	967
C:2	3	32359.06	86982.37	77495.63	70021.02	12581	2998
C:3	3	46654.42	82289.35	56865.65	53133.92	12592	3849
C:1	5	34362.47	46149.23	46149.23	39162.62	12393	3653
C:2	5	6767.20	35363.83	30871.13	24579.58	12985	3470
C:3	5	17831.70	48087.91	48087.91	41103.91	10427	3306
C:1	10	16589.64	24142.88	22245.42	16809.39	12297	2312
C:2	10	21603.16	24984.97	23770.97	22373.82	12686	3749
C:3	10	24610.26	28954.20	27990.20	23823.01	12673	3594

Table C.15: Depot capacity to customer demand ratio results, dualizing constraints 3 and 5. Problem sizes A (5×8×25), B (5×16×25) and C (5×25×25).

Size	Ratio	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
A:1	1.5	59213.07	63927.84	63392.84	59633.97	8188	5140
A:2	1.5	57717.61	66790.71	66790.71	62994.61	8918	6379
A:3	1.5	63445.72	67784.32	67784.32	64429.31	9273	6353
A:1	3	54511.62	55293.94	55293.94	52934.34	8581	4873
A:2	3	20124.19	28464.56	28464.56	26707.90	8642	3795
A:3	3	42614.06	48803.15	48803.15	45577.01	8877	6217
A:1	5	20070.79	26335.20	24903.20	20820.27	8443	5087
A:2	5	23925.97	30938.22	28264.00	24501.96	8517	1467
A:3	5	26418.72	31166.15	31166.15	28317.02	8209	1683
A:1	10	13890.12	14349.00	14349.00	10420.44	8550	7840
A:2	10	11260.76	11793.00	11793.00	10420.44	8748	425
A:3	10	15256.79	16544.00	16052.00	14341.60	8456	3872
B:1	1.5	101408.80	170586.62	135244.86	132656.33	10093	7640
B:2	1.5	87827.16	140806.55	110387.58	104021.99	11091	8502
B:3	1.5	86101.97	107746.98	106730.43	104756.83	11110	8045
B:1	3	44956.13	83261.69	77216.71	70387.46	9983	8628
B:2	3	57655.01	81814.92	66826.29	61788.58	10139	5077
B:3	3	33059.92	48375.87	46674.70	41725.38	10019	6466
B:1	5	21241.34	30372.60	28860.39	23486.24	11455	8076
B:2	5	25602.70	31616.27	29869.19	23537.40	9270	7527
B:3	5	32577.26	54466.96	37229.97	31277.74	9142	7188
B:1	10	17353.60	19901.00	19376.20	17230.67	8965	1363
B:2	10	21383.01	27307.20	23718.20	19876.25	11508	3351
B:3	10	19433.48	21877.33	21877.33	18502.39	12169	6490
C:1	1.5	82576.34	147450.76	118420.59	82799.62	13047	10998
C:2	1.5	55267.29	114319.60	89749.59	82799.62	9903	7596
C:3	1.5	58320.08	112606.82	96202.83	90240.90	13244	11148
C:1	3	37021.26	68908.51	58484.95	52427.13	13630	8809
C:2	3	50579.19	84276.16	77495.63	70021.02	13300	11199
C:3	3	43014.83	59367.43	56865.65	53133.92	10308	7239
C:1	5	37367.70	46281.72	46149.23	39162.62	14549	12060
C:2	5	27675.53	31802.48	30871.13	24579.58	13834	10978
C:3	5	38418.31	48385.27	48087.91	41103.91	13921	10503
C:1	10	19797.70	22245.42	22245.42	16809.39	13883	8463
C:2	10	20281.41	23770.97	23770.97	22373.82	14067	10075
C:3	10	25604.06	27990.20	27990.20	23823.01	14714	11738

Table C.16: Depot capacity to customer demand ratio results, dualizing constraints 2, 3 and 4. Problem sizes A (5×8×25), B (5×16×25) and C (5×25×25).

Ratio	Prob	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	54892.31	55293.94	55293.94	52934.34	7408	219
1.5	2	28811.83	28955.00	28955.00	27893.72	5523	2211
1.5	3	47602.23	48238.15	48238.15	45715.29	7515	2126
3	1	48965.68	49768.77	49768.77	43352.34	7395	1509
3	2	28486.64	30009.56	30009.56	26028.65	7727	2368
3	3	45778.42	46141.65	46141.65	39651.02	7425	2874
5	1	48627.63	50792.77	50792.77	43018.51	7219	187
5	2	29043.25	31109.56	31109.56	25864.26	7206	1450
5	3	47638.27	48189.65	48189.65	39374.58	7110	2047
10	1	51174.95	53707.77	53707.77	42681.42	7595	374
10	2	32361.49	35114.56	35043.49	25373.35	7345	1838
10	3	51507.99	51764.23	51764.23	39014.27	5735	79

Table C.17: Plant capacity to depot capacity ratio results, dualizing constraint 2. Problem size A (5×8×25), with depot capacity to customer demand ratio set to 3.

Ratio	Prob	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	45873.44	55293.94	55293.94	52934.34	7077	47
1.5	2	28537.42	28955.00	28995.00	27893.72	7187	63
1.5	3	42331.41	48238.15	48238.15	45715.29	7158	46
3	1	44680.77	50175.77	49768.77	43352.34	7123	80
3	2	27416.56	30009.56	30009.56	26028.65	7282	62
3	3	40525.65	46141.65	46141.65	39651.02	6890	64
5	1	45194.77	50792.77	50792.77	43018.51	7907	124
5	2	28153.56	39033.56	31109.56	28564.26	6953	48
5	3	41326.65	48189.65	48189.65	39374.58	7249	111
10	1	46466.77	56545.77	53707.77	42681.42	7453	78
10	2	29218.56	35114.56	35043.49	25373.35	7373	47
10	3	42284.65	52569.65	51764.23	39014.27	8220	31

Table C.18: Plant capacity to depot capacity ratio results, dualizing constraint 5. Problem size A (5×8×25), with depot capacity to customer demand ratio set to 3.

Ratio	Prob	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	43342.62	68221.74	55293.94	52934.34	6953	157
1.5	2	27038.74	28955.00	28955.00	27893.72	7338	940
1.5	3	38860.30	50652.15	48238.15	45715.29	7251	265
3	1	42308.24	60168.77	49768.77	43352.34	7329	125
3	2	26190.07	30009.56	30009.56	26028.65	7465	332
3	3	36870.71	55167.15	46141.65	39651.02	6858	111
5	1	44652.58	60632.77	50792.77	43018.51	7095	218
5	2	27259.52	39033.56	31109.56	25864.26	7049	497
5	3	34164.92	48189.65	48189.65	39374.58	7595	186
10	1	40041.31	61904.77	53707.77	42681.42	7484	140
10	2	28890.62	41172.56	35043.49	25373.35	7275	1193
10	3	40955.77	52569.65	51764.23	39014.27	7578	859

Table C.19: Plant capacity to depot capacity ratio results, dualizing constraints 3 and 5. Problem size A (5×8×25), with depot capacity to customer demand ratio set to 3.

Ratio	Prob	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	54511.62	55293.94	55293.94	52934.34	8155	3938
1.5	2	26023.79	28955.00	28955.00	27893.72	7624	5094
1.5	3	42483.70	48238.15	48238.15	45715.29	7440	5841
3	1	43913.81	57511.74	49768.77	43352.34	7873	5986
3	2	26691.84	31277.56	30009.56	26028.65	7761	4254
3	3	41927.92	48732.65	46141.65	39651.02	7581	5967
5	1	44591.29	57447.77	50792.77	43018.51	7726	5495
5	2	28134.44	31109.56	31109.56	25864.26	7714	3474
5	3	42382.25	48342.23	48189.65	39374.58	8195	6493
10	1	41506.18	61678.74	53707.77	42681.42	8175	5138
10	2	29926.40	35114.56	35043.49	25373.35	7519	7012
10	3	42801.38	57422.65	51764.23	39014.27	8379	6921

Table C.20: Plant capacity to depot capacity ratio results, dualizing constraints 2, 3 and 4. Problem size A (5×8×25), with depot capacity to customer demand ratio set to 3.

Dualized	Best LB	Best UB	LBs time	UBs time	Total time
2	53950.90	55413.00	6077	1625	8046
3	51284.51	74208.00	6056	5116	11516
4	55413.00	55413.00	110	15	468
5	52109.00	55413.00	5486	63	5861
6	46879.52	67395.00	5686	141	6140
2,3	51297.42	55413.00	5947	4070	10345
2,4	53948.65	55413.00	5921	1921	8186
2,5	48612.05	55989.00	6018	2654	9016
2,6	30153.78	68629.00	5910	1325	7563
3,4	49831.98	74208.00	6468	5406	12203
3,5	48279.74	58313.00	5597	1278	7218
3,6	21476.86	69261.00	5688	1437	7453
4,5	52109.00	55413.00	5875	79	6282
4,6	46879.52	67395.00	5595	171	6079
5,6	30498.48	55989.00	5674	171	6157
2,3,4	49693.69	56647.00	5985	3608	9922
2,3,5	38758.52	63417.00	5955	107	6406
2,3,6	17119.23	68629.00	6112	1090	7530
2,4,5	48612.05	55989.00	5959	2649	8952
2,4,6	31526.23	68629.00	5829	764	6937
2,5,6	30870.66	68184.00	5894	843	7080
3,4,5	46089.42	65296.00	5703	453	6500
3,4,6	15906.22	68491.00	5865	4730	10938
3,5,6	18053.99	69261.00	5813	1421	7562
4,5,6	30498.48	55989.00	5610	187	6125
LP bound	49815.26				562
Optimal		55413.00			1422

Table C.21: Size 3×5×10, single-source problem 1 results.

C.2 TSCPLPSS

The results from the TSCPLPSS heuristic are given here in the same manner as the previous tables. The first three (C.21 - C.23) are the runs where all combinations of dualized constraints were considered. Then the results from the varying ratios of depot capacity to customer demand (C.30 - C.33), and plant capacity to depot capacity (C.34 - C.37), are given. There is one table for each relaxation studied (dualizing constraints 2; 5; 3 and 5; 2, 3, and 4). Each table shows the best lower and upper bound found by the heuristic and the optimal solution and linear programming bound for the problem. All times are given in milliseconds, and some tables quote total run times for the heuristic as well as the time taken calculating the upper and lower bounds individually.

Full data, including comprehensive timing information, can be found on the accompanying CD-ROM.

Dualized	Best LB	Best UB	LBs time	UBs time	Total time
2	67065.00	67400.00	2127	169	2703
3	65924.62	67400.00	5502	3248	9093
4	67400.00	67400.00	110	0	438
5	66560.53	67400.00	5626	47	6017
6	57645.63	68995.00	6077	532	6952
2,3	66335.94	67400.00	6037	4400	10765
2,4	67065.00	67400.00	2141	156	2641
2,5	61635.86	67400.00	6015	250	6609
2,6	57322.99	68995.00	6042	1349	7735
3,4	66668.50	67400.00	5988	3215	9547
3,5	61203.56	67400.00	6030	502	6876
3,6	53506.64	69766.00	5721	748	6782
4,5	66560.53	67400.00	5576	46	5934
4,6	57645.63	68995.00	5748	534	6610
5,6	51113.28	68995.00	6124	516	6984
2,3,4	64037.31	67400.00	6041	4726	11111
2,3,5	43893.80	67400.00	6048	265	6657
2,3,6	37749.61	74977.00	5669	1284	7297
2,4,5	61632.21	67400.00	5937	234	6515
2,4,6	57322.99	68995.00	6666	1474	8562
2,5,6	53739.55	68995.00	6230	1129	7703
3,4,5	57845.86	68995.00	6063	437	6843
3,4,6	47740.58	70590.00	6108	940	7376
3,5,6	36265.12	69766.00	5874	611	6813
4,5,6	51113.28	68995.00	6065	498	6907
LP bound	64622.98				563
Optimal		67400.00			656

Table C.22: Size 3×5×10, single-source problem 2 results.

Dualized	Best LB	Best UB	LBs time	UBs time	Total time
2	36492.00	36672.00	2047	344	2735
3	32973.87	36672.00	6101	1868	8297
4	36672.00	36672.00	110	15	468
5	35677.00	36672.00	5626	47	6017
6	35278.44	36672.00	5841	5754	11923
2,3	34429.30	36672.00	5890	2204	8422
2,4	36492.00	36672.00	1997	409	2734
2,5	33651.52	36672.00	5867	3727	9938
2,6	30962.39	36672.00	6161	3463	9953
3,4	31944.01	36672.00	5742	1602	7672
3,5	23512.36	36672.00	5920	1410	7674
3,6	27177.37	38662.00	5990	1947	8281
4,5	35677.00	36672.00	5953	15	6297
4,6	35278.51	36672.00	5866	5728	11954
5,6	31200.36	66431.00	5907	5718	11969
2,3,4	34226.17	36672.00	6121	2924	9389
2,3,5	33799.03	36672.00	6124	141	6609
2,3,6	27360.75	41530.00	6098	1887	8344
2,4,5	33651.52	36672.00	6133	3929	10406
2,4,6	30962.39	36672.00	6431	3507	10297
2,5,6	28009.94	36672.00	6154	1595	8093
3,4,5	30177.88	36672.00	6137	1223	7703
3,4,6	24379.18	37772.00	6155	3126	9656
3,5,6	18918.84	36672.00	8464	2583	11656
4,5,6	31200.36	66431.00	8446	7342	16241
LP bound	33912.30				782
Optimal		36672.00			1047

Table C.23: Size 3×5×10, single-source problem 3 results.

	Size 10×33×50		Size 15×40×60	
Dualized	Best LB	Best UB	Best LB	Best UB
2	96985.05	106850.00	42215.44	48300.00
3	84677.14	124042.00	31316.35	52985.00
4	not found	not found	not found	not found
5	not found	not found	not found	not found
6	not found	not found	not found	not found
2,3	76045.92	114155.00	29171.33	49228.00
2,4	96830.28	112269.00	42306.10	53766.00
2,5	67933.78	115677.00	not found	not found
2,6	58141.76	115041.00	32079.89	66689.00
3,4	82369.49	129420.00	not found	not found
3,5	50761.38	132330.00	not found	not found
3,6	not found	not found	not found	not found
4,5	not found	not found	not found	not found
4,6	not found	not found	not found	not found
5,6	not found	not found	not found	not found
2,3,4	77623.22	111993.00	31983.62	53420.00
2,3,5	43900.08	118111.00	28378.79	85064.00
2,3,6	40527.80	114703.00	33094.44	108868.00
2,4,5	72572.38	128246.00	not found	not found
2,4,6	60477.15	108663.00	28872.62	60224.00
2,5,6	40372.75	145824.00	18983.45	90887.00
3,4,5	55778.64	156085.00	25322.98	81597.00
3,4,6	45590.50	139193.00	24755.31	90623.00
3,5,6	55896.65	159703.00	35037.91	119925.00
4,5,6	not found	not found	not found	not found

Table C.24: Larger sizes, single-source problem 1 results.

	Size 20×50×75		Size 30×60×120	
Dualized	Best LB	Best UB	Best LB	Best UB
2	not found	not found	not found	not found
3	not found	not found	not found	not found
4	not found	not found	not found	not found
5	not found	not found	not found	not found
6	not found	not found	not found	not found
2,3	149149.10	197492.00	86303.06	219886.00
2,4	not found	not found	not found	not found
2,5	not found	not found	not found	not found
2,6	not found	not found	not found	not found
3,4	not found	not found	not found	not found
3,5	not found	not found	not found	not found
3,6	not found	not found	not found	not found
4,5	not found	not found	not found	not found
4,6	not found	not found	not found	not found
5,6	not found	not found	not found	not found
2,3,4	147161.69	191575.00	87468.65	220733.00
2,3,5	132528.24	244618.00	77776.63	369344.00
2,3,6	not found	not found	not found	not found
2,4,5	not found	not found	not found	not found
2,4,6	not found	not found	not found	not found
2,5,6	not found	not found	not found	not found
3,4,5	121404.40	239897.00	59133.38	304022.00
3,4,6	121207.76	254212.00	not found	not found
3,5,6	101619.18	210159.00	54369.71	295387.00
4,5,6	not found	not found	not found	not found

Table C.25: Larger sizes, single-source problem 1 results continued.

	Size 10×33×50		Size 15×40×60	
Dualized	Best LB	Best UB	Best LB	Best UB
2	85706.82	88807.00	32938.46	34888.00
3	79658.76	92828.00	31447.92	39143.00
4	not found	not found	not found	not found
5	not found	not found	not found	not found
6	not found	not found	not found	not found
2,3	78398.85	91796.00	29604.01	36794.00
2,4	85912.05	90783.00	33202.90	36401.00
2,5	83387.48	104229.00	not found	not found
2,6	73342.84	98494.00	29515.05	40966.00
3,4	82279.01	98806.00	not found	not found
3,5	67003.83	106436.00	not found	not found
3,6	not found	not found	not found	not found
4,5	not found	not found	not found	not found
4,6	not found	not found	not found	not found
5,6	not found	not found	not found	not found
2,3,4	78990.90	91846.00	29152.62	36954.00
2,3,5	58008.25	93072.00	23496.69	40659.00
2,3,6	63505.19	105246.00	23382.21	42720.00
2,4,5	81312.93	103010.00	not found	not found
2,4,6	81029.09	103338.00	28971.02	40584.00
2,5,6	67593.21	130380.00	28904.95	68497.00
3,4,5	66323.49	108739.00	25621.50	45958.00
3,4,6	62894.33	108436.00	24902.99	48574.00
3,5,6	66842.24	110988.00	29939.41	55785.00
4,5,6	not found	not found	not found	not found

Table C.26: Larger sizes, single-source problem 2 results.

	Size 20×50×75		Size 30×60×120	
Dualized	Best LB	Best UB	Best LB	Best UB
2	not found	not found	not found	not found
3	not found	not found	not found	not found
4	not found	not found	not found	not found
5	not found	not found	not found	not found
6	not found	not found	not found	not found
2,3	140578.92	339779.00	112063.98	174493.00
2,4	not found	not found	not found	not found
2,5	not found	not found	not found	not found
2,6	not found	not found	not found	not found
3,4	not found	not found	not found	not found
3,5	not found	not found	not found	not found
3,6	not found	not found	not found	not found
4,5	not found	not found	not found	not found
4,6	not found	not found	not found	not found
5,6	not found	not found	not found	not found
2,3,4	152594.03	368620.00	111249.24	171778.00
2,3,5	129033.88	623409.00	107369.62	248927.00
2,3,6	not found	not found	not found	not found
2,4,5	not found	not found	not found	not found
2,4,6	not found	not found	not found	not found
2,5,6	not found	not found	not found	not found
3,4,5	125685.27	685154.00	96133.39	238307.00
3,4,6	115449.32	694456.00	not found	not found
3,5,6	119943.01	706547.00	85420.63	221739.00
4,5,6	not found	not found	not found	not found

Table C.27: Larger sizes, single-source problem 2 results continued.

	Size 10×33×50		Size 15×40×60	
Dualized	Best LB	Best UB	Best LB	Best UB
2	99820.31	116314.00	84267.28	104570.00
3	89742.43	157511.00	62919.24	133243.00
4	not found	not found	not found	not found
5	not found	not found	not found	not found
6	not found	not found	not found	not found
2,3	77915.18	141340.00	58574.55	123550.00
2,4	97153.12	122315.00	85119.27	123795.00
2,5	68381.15	146445.00	not found	not found
2,6	52254.53	134071.00	64907.64	178431.00
3,4	84073.03	162081.00	not found	not found
3,5	52571.02	189792.00	not found	not found
3,6	not found	not found	not found	not found
4,5	not found	not found	not found	not found
4,6	not found	not found	not found	not found
5,6	not found	not found	not found	not found
2,3,4	78882.14	135617.00	63580.85	134046.00
2,3,5	45622.29	170898.00	66264.12	284587.00
2,3,6	34671.57	137433.00	66555.29	315505.00
2,4,5	66865.87	149925.00	not found	not found
2,4,6	59464.60	137424.00	60583.86	167808.00
2,5,6	41633.57	218275.00	57542.49	417550.00
3,4,5	56613.74	221988.00	60702.20	280771.00
3,4,6	46609.50	202053.00	60603.88	323717.00
3,5,6	57017.37	229018.00	70809.59	351156.00
4,5,6	not found	not found	not found	not found

Table C.28: Larger sizes, single-source problem 3 results.

	Size 20×50×75		Size 30×60×120	
Dualized	Best LB	Best UB	Best LB	Best UB
2	not found	not found	not found	not found
3	not found	not found	not found	not found
4	not found	not found	not found	not found
5	not found	not found	not found	not found
6	not found	not found	not found	not found
2,3	93519.28	174656.00	78444.78	278048.00
2,4	not found	not found	not found	not found
2,5	not found	not found	not found	not found
2,6	not found	not found	not found	not found
3,4	not found	not found	not found	not found
3,5	not found	not found	not found	not found
3,6	not found	not found	not found	not found
4,5	not found	not found	not found	not found
4,6	not found	not found	not found	not found
5,6	not found	not found	not found	not found
2,3,4	92529.36	173503.00	84874.47	297589.00
2,3,5	86222.59	285650.00	89158.73	637677.00
2,3,6	not found	not found	not found	not found
2,4,5	not found	not found	not found	not found
2,4,6	not found	not found	not found	not found
2,5,6	not found	not found	not found	not found
3,4,5	83323.52	311010.00	81293.37	631784.00
3,4,6	76766.21	313170.00	not found	not found
3,5,6	79495.34	315779.00	94794.44	786427.00
4,5,6	not found	not found	not found	not found

Table C.29: Larger sizes, single-source problem 3 results continued.

Size	Ratio	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	53670.88	55041.00	55041.00	49817.33	5979	1129
1.5	2	67232.51	67564.00	67564.00	64995.72	1719	156
1.5	3	58311.97	59869.00	59869.00	52397.64	5938	31
3	1	19706.12	20862.00	20862.00	16622.18	6603	3739
3	2	56202.40	56896.00	56896.00	53567.41	5856	4346
3	3	36739.16	36919.00	36919.00	34115.47	1484	94
5	1	10340.00	10340.00	10340.00	9022.92	109	31
5	2	50387.02	50634.00	50634.00	49551.91	921	32
5	3	36770.75	36946.00	36946.00	33866.47	1392	46
10	1	11828.00	11828.00	11828.00	9303.03	125	48
10	2	52187.76	52436.00	52436.00	49702.71	1001	46
10	3	31373.28	31528.00	31528.00	29362.89	1594	16

Table C.30: Depot capacity to customer demand ratio results, dualizing constraint 2. Single-source, problem size 3×5×10.

Size	Ratio	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	51737.00	55041.00	55041.00	49817.33	5594	31
1.5	2	65970.00	67564.00	67564.00	64995.72	5702	63
1.5	3	58874.00	59869.00	59869.00	52397.64	5595	31
3	1	18893.00	20862.00	20862.00	16622.18	5673	15
3	2	54604.00	57017.00	56896.00	53567.41	5798	47
3	3	35924.00	36919.00	36919.00	34115.47	5672	31
5	1	9935.00	10340.00	10340.00	9022.92	5639	32
5	2	50366.00	50634.00	50634.00	49551.91	5611	61
5	3	35567.00	36946.00	36946.00	33866.47	5563	47
10	1	11521.00	11828.00	11828.00	9303.03	5641	30
10	2	52356.00	52436.00	52436.00	49702.71	204	31
10	3	31528.00	31528.00	31503.00	29362.89	172	62

Table C.31: Depot capacity to customer demand ratio results, dualizing constraint 5. Single-source, problem size 3×5×10.

Size	Ratio	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	48117.40	55830.00	55041.00	49817.33	6003	685
1.5	2	61380.96	69291.00	67564.00	64995.72	6017	810
1.5	3	43984.40	66181.00	59869.00	52397.64	5997	157
3	1	15027.82	21902.00	20862.00	16622.18	6003	248
3	2	54659.64	56896.00	56896.00	53567.41	5876	2937
3	3	32180.38	36919.00	36919.00	34115.47	5877	1968
5	1	10099.14	10340.00	10340.00	9022.92	6016	78
5	2	49803.79	50634.00	50634.00	49551.91	6158	63
5	3	34894.08	36946.00	36946.00	33866.47	5833	3136
10	1	11001.91	11828.00	11828.00	9303.03	5956	60
10	2	50738.21	52436.00	52436.00	49702.71	6187	109
10	3	30881.09	31528.00	31528.00	29362.89	5810	96

Table C.32: Depot capacity to customer demand ratio results, dualizing constraints 3 and 5. Single-source, problem size 3×5×10.

Size	Ratio	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	48117.40	55830.00	55041.00	49817.33	5998	737
1.5	2	65839.27	67564.00	67564.00	64995.72	6262	4738
1.5	3	43335.67	59869.00	59869.00	52397.64	6032	1359
3	1	12731.07	22243.00	20862.00	16622.18	6057	1160
3	2	45896.11	58183.00	56896.00	53567.41	6263	3564
3	3	34838.99	36919.00	36919.00	34115.47	6007	3477
5	1	10340.00	10340.00	10340.00	9022.92	828	110
5	2	44172.16	52386.00	50634.00	49551.91	5966	1392
5	3	34494.17	36946.00	36946.00	33866.47	5902	880
10	1	11828.00	11828.00	11828.00	9303.03	1863	464
10	2	52414.35	52436.00	52436.00	49702.71	971	325
10	3	31504.01	31528.00	31528.00	29362.89	1515	548

Table C.33: Depot capacity to customer demand ratio results, dualizing constraints 2, 3 and 4. Single-source, problem size 3×5×10.

Ratio	Prob	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	53123.95	54581.00	54581.00	50042.17	6112	543
1.5	2	68665.23	69848.00	69848.00	66749.17	5667	3426
1.5	3	35802.05	35981.00	35981.00	34150.03	1999	470
3	1	53962.75	56523.00	55989.00	49777.39	5909	966
3	2	67067.06	67400.00	67400.00	64622.98	2204	124
3	3	36492.00	36672.00	36672.00	33912.30	1941	403
5	1	55337.37	58002.00	57344.00	49476.83	5917	1159
5	2	69284.39	69626.00	69626.00	64020.18	1002	61
5	3	36852.00	37032.00	37032.00	33828.67	2001	405
10	1	57478.40	60069.00	59411.00	49264.23	6030	954
10	2	70831.39	71173.00	71173.00	63711.10	1062	79
10	3	38432.07	38622.00	38622.00	33609.67	1719	266

Table C.34: Plant capacity to depot capacity ratio results, dualizing constraint 2. Single-source, problem size 3×5×10.

Ratio	Prob	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	52223.00	54581.00	54581.00	50042.17	5904	95
1.5	2	66969.45	76629.00	69848.00	66749.17	5657	31
1.5	3	35677.00	35981.00	35981.00	34150.03	5672	31
3	1	52109.00	55989.00	55989.00	49777.39	5685	64
3	2	66759.92	67400.00	67400.00	64622.98	5672	16
3	3	36111.00	36672.00	36672.00	33912.30	5608	48
5	1	52893.00	57344.00	57344.00	49476.83	5660	124
5	2	66170.00	69854.00	69626.00	64020.18	5764	48
5	3	37019.00	37032.00	37032.00	33828.67	204	16
10	1	53969.00	59411.00	59411.00	49264.23	5701	80
10	2	67135.15	71944.00	71173.00	63711.10	5656	48
10	3	38339.00	38622.00	38622.00	33609.67	5705	15

Table C.35: Plant capacity to depot capacity ratio results, dualizing constraint 5. Single-source, problem size 3×5×10.

Ratio	Prob	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	49640.42	55815.00	54581.00	50042.17	6127	1092
1.5	2	64073.58	76629.00	69848.00	66749.17	5670	1158
1.5	3	32070.14	35981.00	35981.00	34150.03	5812	766
3	1	48279.74	59281.00	55989.00	49777.39	6072	475
3	2	62633.68	68995.00	67400.00	64622.98	5981	424
3	3	31046.68	36672.00	36672.00	33912.30	6089	1910
5	1	48050.30	59933.00	57344.00	49476.83	6107	2129
5	2	62232.52	75608.00	69626.00	64020.18	6014	784
5	3	31555.36	37032.00	37032.00	33828.67	5777	897
10	1	43937.23	60069.00	59411.00	49264.23	6012	2410
10	2	60499.42	72151.00	71173.00	63711.10	6000	500
10	3	34311.30	38622.00	38622.00	33609.67	6091	534

Table C.36: Plant capacity to depot capacity ratio results, dualizing constraints 3 and 5. Single-source, problem size 3×5×10.

Ratio	Prob	Best LB	Best UB	Optimal	LP bound	LBs time	UBs time
1.5	1	50840.50	55185.00	54581.00	50042.17	6617	4680
1.5	2	67419.14	69848.00	69848.00	66749.17	5754	2418
1.5	3	34059.03	35981.00	35981.00	34150.03	5494	2224
3	1	49693.69	56647.00	55989.00	49777.39	6401	3521
3	2	63900.18	67400.00	67400.00	64622.98	6116	3698
3	3	34876.76	36672.00	36672.00	33912.30	5895	2839
5	1	51653.58	58002.00	57344.00	49476.83	6081	1326
5	2	64593.03	69854.00	69626.00	64020.18	6181	4977
5	3	32660.42	37032.00	37032.00	33828.67	6148	1961
10	1	51578.36	60069.00	59411.00	49264.23	6342	3924
10	2	64277.75	71173.00	71173.00	63711.10	6224	4965
10	3	36690.88	38622.00	38622.00	33609.67	5945	2462

Table C.37: Plant capacity to depot capacity ratio results, dualizing constraints 2, 3 and 4. Single-source, problem size 3×5×10.

Appendix D

Accompanying CD-ROM

Described below are the contents of the files on the accompanying CD-ROM. The descriptions are arranged in alphabetical order of folder names.

- input files

  Contains subfolders containing the input and data files used by the Lagrangian heuristic.

  - Class A

    Contains Excel spreadsheets and *.ini* files for problem size 5×8×25 and ratios for depot capacity to customer demand of 1.5, 3, 5 and 10. Each spreadsheet has three problem instances. The *.ini* files are the actual input files used by the C program.

    * Plant ratio data

      Contains Excel spreadsheets and *.ini* files used for analysing the ratios of plant capacity to depot capacity for the TSCPLP.

  - Class B

    As for Class A but for problems sized 5×16×25.

  - Class C

    As for Class A but for problems sized 5×25×25.

  - Larger

    Contains an Excel spreadsheet and folders with the *.ini* files used for the larger sized problems of sizes:

    * 10×33×50

      * 15×40×60
      * 20×50×75
      * 30×60×120
      * 40×80×200
      * 50×100×250

  - medium

    Contains an Excel spreadsheet and *.ini* files for the three problems used to analyse all the relaxations, sized 5×10×25.

  - small

    Contains an Excel spreadsheet and *.ini* files for the three problems used to analyse all the relaxations sized 3×5×10.

      * singlesource ratio data
          · Depot ratios

            Excel and *.ini* files for analysing depot ratios

          · Plant ratios

            Excel and *.ini* files for analysing plant ratios

  - steplength multipliers
      * TSCPLP

        Excel spreadsheets containing the multiplier values used in the subgradient and stepsize calculations in the C program.

      * TSCPLPSS

        Excel spreadsheets containing multiplier values used in the subgradient and stepsize calculations in the C program.

- original results

  Contains Excel spreadsheets of numerical results for analysing all the relaxations (two tabs on "depot ratio originalresults.xls") and the four ("2", "5", "35", "234") where ratios are changing, for both plants and depots.

  - 4 Heuristics depot ratios

    .txt files of the CPLEX solutions to each problem instance for the linear programming bounds and the optimal solutions. Example of file name descriptions:

"lpA1.5-1.txt" refers to the linear programming relaxation, Class A, ratio 1.5, problem 1.

  * Heuristic 1 - 2
  * Heuristic 2 - 5
  * Heuristic 3 - 35
  * Heuristic 4 - 234

  All four folders contain the corresponding output *.txt* files from the C program. Files are named as "Class Ratio-Relaxation-problem number.txt" e.g. "B10-35-2.txt" is Class B, ratio 10, relaxing constraints 3 and 5, problem 2.

- 4 Heuristics plant ratios

  .txt files of the CPLEX solutions to each problem instance for the linear programming bounds and the optimal solutions. Example of file name descriptions: "optimal5-3.txt" refers to the optimal solution, ratio 5, problem 3.

  * Heuristic 1 - 2
  * Heuristic 2 - 5
  * Heuristic 3 - 35
  * Heuristic 4 - 234

  All four folders contain the corresponding output *.txt* files from the C program. Files are named as "Ratio-Relaxation-problem number.txt" e.g. "3-5-1.txt" is ratio 3, relaxing constraint 5, and problem 1.

- All heuristics
  * Larger
    · 10×33×50
    · 15×40×60
    · 20×50×75
    · 30×60×120
    · 40×80×200
    · 50×100×250

  These six folders contain the output *.txt* files from the C program for the analysis of all the relaxations on larger sized problems. Files are named as "relaxation-problem number.txt" e.g. "234-1.txt" is the relaxation of constraints 2, 3, and 4, on problem 1 of the size given by the folder name.

    * Medium (refers to size 5×10×25)
    * Small (refers to size 3×5×10)

      These two folders contain the output *.txt* files from the C program for the analysis of all the relaxations. Files are named as "relaxation size problem number.txt" e.g. "345medium2.txt" is the relaxation of constraints 3, 4, and 5, on problem 2 of medium size. The linear programming bounds and optimal solutions are also included.

- program files
  - heuristicmain.cpp

    The heuristic for the TSCPLP.
  - lpboundmain.cpp

    Finds the linear programming bound for both the TSCPLP and the TSCPLPSS from CPLEX.
  - optimalmain.cpp

    Finds the optimal solution for the TSCPLP from CPLEX.
  - ssheuristicmain.cpp

    The heuristic for the TSCPLPSS.
  - ssoptimalmain.cpp

    Finds the optimal solution for the TSCPLPSS from CPLEX.

- singlesource results

  Contains Excel spreadsheets of numerical results for analysing all the relaxations (a tab on "depot ratio singlesourceresults.xls") and the four ("2", "5", "35", "234") where ratios are changing, for both plants and depots.

  - 4 Heuristics depot ratios

    .txt files of the CPLEX solutions to each problem instance for the linear programming bounds and the optimal solutions. Example of file name descriptions: "lp3-2.txt" refers to the linear programming relaxation, ratio 3, problem 2.

    * Heuristic 1 - 2
    * Heuristic 2 - 5
    * Heuristic 3 - 35

    * Heuristic 4 - 234

      All four folders contain the corresponding output *.txt* files from the C program. Files are named as "d Ratio-Relaxation-problem number.txt" e.g. "d5-234-1.txt" is depot ratio 5, relaxing constraints 2, 3 and 4, problem 1.

- 4 Heuristics plant ratios

  .txt files of the CPLEX solutions to each problem instance for the linear programming bounds and the optimal solutions. Example of file name descriptions: "optimal10-2.txt" refers to the optimal solution, ratio 10, problem 2.

    * Heuristic 1 - 2
    * Heuristic 2 - 5
    * Heuristic 3 - 35
    * Heuristic 4 - 234

      All four folders contain the corresponding output *.txt* files from the C program. Files are named as "p Ratio-Relaxation-problem number.txt" e.g. "p1.5-2-2.txt" is plant ratio 1.5, relaxing constraint 2, and problem 2.

- All heuristics

    * Larger

        · 10×33×50
        · 15×40×60
        · 20×50×75
        · 30×60×120

      All four folders contain the corresponding output *.txt* files from the C program for the analysis of all the single-sourced relaxations on larger sized problems. Files are named as "relaxation-problem number.txt" e.g. "345-3.txt" is the relaxation of constraints 3, 4, and 5, on problem 3 of the size given by the folder name.

    * Small

      Contains the output *.txt* files from the C program for the analysis of all the relaxations. Files are named as "ss relaxation size problem number.txt" e.g. "ss56small1.txt" is the single-source relaxation of constraints 5 and 6, on problem 1 of small size. The linear programming bounds and optimal solutions are also included.

Appendix E

References

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