Conformal Energy of Quasisymmetric and Quasimöbius Mappings

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Abstract

Conformal energy was introduced to solve minimisation problems concerning the mean distortion of homeomorphisms of the disk. Interesting new connections were observed between conformal energy and extremal quasiconformal mappings associated with quasisymmetric homeomorphisms of the circle.

This thesis outlines the mathematical foundations of quasiconformality and explores its connection with quasisymmetry further, giving explicit formulae for lowest energy quasisymmetric mappings, asymptotics for bilipschitz mappings and other related estimates.

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Chapter 1

Mathematical Motivation of Quasiconformality

1.1 Introduction

Conservation laws and equations of motion or state in physics and mathematics are naturally described by divergence-type second-order partial differential equations.

Hamilton's principle of least action – that a system acts so as to minimise some action functional – applies quite generally to classical fields and even quantum fields. We are therefore led to study minima of energy functionals (which typically satisfy an associate Euler-Lagrange equation that appears in divergence form as a result of integration by parts in its derivation), regularity of minimisers and other aspects of the calculus of variations. Similar equations apply to conservation laws, due to the basic assumption of continuum physics that the gain of a physical quantity in a domain corresponds to the loss of this quantity across the domain's boundary.

It is a remarkable fact that the existence, regularity and singular set structure of this most important class of PDEs in applications are determined by the mathematics underpinning the geometry, structure and dimension of fractal sets, moduli spaces of Riemann surfaces and conformal dynamical systems. All of these topics are inextricably linked in two dimensions by the theory of quasiconformal mappings, which is introduced and discussed herewith. In the first two chapters we introduce definitions, ideas and useful results from the literature. It is not intended to give a thorough introduction to the area and as such Sobolev theory is not discussed in any depth, as there are various books which do that - for example *Sobolev*

Spaces with Applications to Elliptic Partial Differential Equations by V.G Maz'ya [8]. On the other hand, most of the results of Chapter 3 are new to the literature.

1.2 Conformal Invariance and the Dirichlet Problem for the Unit Disk

We will first discuss the classical approach of complex analysis, while reviewing some key results in conformal geometry. Starting with homeomorphisms of the closed unit disk, we will establish the motivation for the minimisation problems we consider, as well as the notion of quasiconformality - that can be roughly described as a homeomorphism between plane domains which takes small circles to small ellipses of bounded eccentricity.

Problem 1. *Given a homeomorphism of the closed unit disk $f_o : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, find the homeomorphism of minimal energy $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ subject to prescribed boundary values*

$$f|\partial\mathbb{D} = f_o \tag{1.1}$$

We will have to make a priori Sobolev regularity assumptions about f_o and f of course. The first example is when the energy of a mapping is defined as the Dirichlet integral, so we are looking to minimise

$$\int_{\mathbb{D}} \|Df(z)\|^2 dz \tag{1.2}$$

subject to boundary condition (1.1). Necessarily $f_o, f \in W^{1,2}(\mathbb{D})$.

This is the Dirichlet problem in complex analysis, and it is known that the least energy map from the unit disk to itself satisfying (1.1) is the harmonic map, which is obtained through the Poisson integral of $f_o|\partial\mathbb{D}$:

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{1-|z|^2}{|\xi - z|^2}\frac{f_o(\xi)}{\xi}\, d\xi$$

Proposition 1.2.1. *Energy is conformally invariant.*

Proof. Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is a Möbius transformation of the form $\varphi(z) = \zeta\frac{z-a}{1-\bar{a}z}$ with $\zeta = e^{i\theta}$ and $|a| < 1$. By fundamental properties of Möbius transformations, φ is injective, analytic and hence continuous. Furthermore, φ is conformal since

$\varphi' = \frac{1-|a|^2}{(1-\bar{a}z)^2} \neq 0$, and has a continuous inverse function φ^{-1}, so it is a homeomorphism. Given our problem, $f \circ \varphi$ describes the boundary values of $f_o \circ \varphi$ which thus becomes a strong candidate for the minimiser.

By the change of variable formula, we have

$$\int_{\mathbb{D}} \|Df(z)\|^2 dz = \int_{\mathbb{D}} \|Df(\varphi)\|^2 |J(z,\varphi)| dz \tag{1.3}$$

For any map, the determinant of the Jacobian is given by $J(z,f) = |f_z|^2 - |f_{\bar{z}}|^2$. In particular, for a homeomorphism f, $J(z,f) \geq 0$ so $|f_z| \geq |f_{\bar{z}}|$. Additionally, if f is conformal, $f_{\bar{z}} = 0$ and $J(z,f) = |f_z|^2$. Subsituting into (1.3) yields:

$$\begin{aligned}\int_{\mathbb{D}} \|Df(z)\|^2 dz &= \int_{\mathbb{D}} \|Df(\varphi)\|^2 |\varphi'(z)|^2 dz \\ &= \int_{\mathbb{D}} (|f_z(\varphi)|^2 + |f_{\bar{z}}(\varphi)|^2)|\varphi'(z)|^2 dz \\ &= \int_{\mathbb{D}} (|f_z(\varphi)\varphi'|^2 + |f_{\bar{z}}(\varphi)\bar{\varphi}'|^2) dz \\ &= \int_{\mathbb{D}} \|D(f \circ \varphi)\|^2 dz\end{aligned}$$

This shows energy is indeed conformally invariant - it does not change via Möbius transformations, so we conclude that the map f has the same energy as the map $f \circ \varphi$. □

Now let's assume that a function g exists, with the same boundary values but smaller energy. Then, since φ^{-1} is conformal, the map $g \circ \varphi^{-1}$ has the same boundary values and energy as f_o. As f is assumed to be a minimiser, this contradiction shows no such g exists.

For the Dirichlet problem, this means that a Möbius transformation which changes the boundary values of the initial problem will also change the minimiser.

Remark. We have not shown uniqueness - that is if $\int_{\mathbb{D}} \|Df\|^2 = \int_{\mathbb{D}} \|Dg\|^2$ and $g = f$ on $\partial\mathbb{D}$, then $f \equiv g$. In the current problem it will follow because both f and g will be harmonic and thus satisfy the maximum principle. More generally, uniqueness is quite a challenging question.

1.3 Least Energy Mappings for the Extended Dirichlet Problem

Consider deforming the unit disk $\mathbb{D}$ to another simply connected domain Ω, minimising energy. There are no boundary value restrictions.

Problem 2. *Given a simply connected domain Ω of finite area, find the homeomorphism of minimal energy mapping $\mathbb{D}$ to Ω, and find the minimiser subject to prescribed boundary values [3]. In other words, find over all homeomorphisms*

$$\min_{f:\mathbb{D}\to\Omega}\left\{\int_{\mathbb{D}}\|Df(z)\|^2 dz\right\},$$

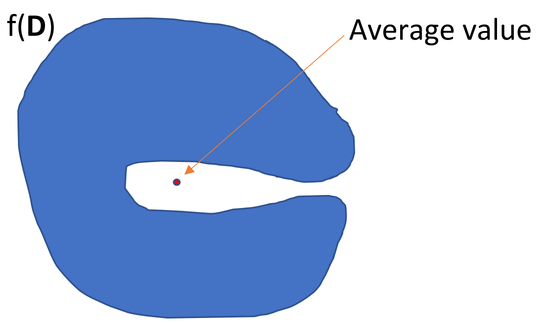
$\|Df\|^2 = |f_z|^2 + |f_{\bar{z}}|^2$, *with the possible restriction* $f|\partial\mathbb{D} = f_o$.

Using the average value property of harmonic mappings [1] - that the value of a harmonic function f at the centre of a disk contained in its domain is given by the average of all values of f on the disk, i.e.

$$f(0) = \frac{1}{\pi}\int_{\mathbb{D}} f(z)\,dz,$$

there are two cases to consider:
Case 1: average value is mapped inside the domain Ω
Case 2: average value is mapped outside the domain Ω

In case 2, if the "centre of mass" weighted with respect to our chosen boundary values lies outside of the domain Ω, the harmonic map is not valued in Ω and so we do not have a candidate for the energy minimiser.



We now consider the minimisation problem in case 1, i.e. minimising

$$\int_{\mathbb{D}} \|Df\|^2 dz$$

for a homeomorphism $f : \mathbb{D} \to \Omega$.

In this case a minimiser always exists - this was proved by Riemann and used in the approach to his mapping theorem. Using Hadamard's inequality for 2×2 matrices, $\|A\|^2 = \operatorname{tr}(A^T A) \geq 2 \det A$, for an arbitrary mapping we have

$$\|Df\|^2 \geq 2J(z, f), \tag{1.4}$$

which gives a lower bound on the minimum:

$$\int_{\mathbb{D}} \|Df\|^2 \geq 2 \int_{\mathbb{D}} J(z, f) = 2|f(\mathbb{D})| = 2|\Omega|$$

Consequently, if there is an absolute minimiser f achieving this lower bound, it must satisfy (1.4) with equality:

$$\|Df\|^2 = 2J(z, f) \tag{1.5}$$

Because $f_z = \frac{1}{2}(f_x - i f_y)$ and $f_{\bar{z}} = \frac{1}{2}(f_x + i f_y)$, the factor of 2 in (1.5) can be dropped to obtain $\|Df\|^2 = J(z, f)$, which can be written

$$|f_z|^2 + |f_{\bar{z}}|^2 = |f_z|^2 - |f_{\bar{z}}|^2$$

and only holds if $f_{\bar{z}} = 0$, i.e. if f is conformal. Such a conformal mapping exists by the Riemann mapping theorem.

Another approach is to vary a supposed minimiser f by a parametrised family of homeomorphisms of $\mathbb{D}$ that are the identity near the boundary, say φ_t, normalised such that $\varphi_0(z) = z$. Since f is a minimiser we must have

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathbb{D}} \|D(f \circ \varphi_t)\|^2 = 0,$$

leading to the second order Euler-Lagrange equation for f, $\operatorname{div} Df = \Delta f = 0$.

In the case of the boundary value problem (BVP) the minimum should be a harmonic mapping, and the problem reduces to whether our prescribed boundary values f_o have a harmonic homeomorphic extension to $\mathbb{D}$.

Riemann showed there is always a least energy mapping (which is particularly obvious in physics). Weierstrass proved Riemann's argument was not generally valid, but Hilbert later showed that Riemann's proof can be made to work, ultimately requiring some regularity of $\partial\Omega$, and finally gave a complete proof for the general Riemann mapping problem.

Chapter 2

Theory of Quasiconformal Mappings

2.1 The Linear Distortion

Given two domains Ω and Ω' and a homeomorphism $f : \Omega \to \Omega'$, we can introduce a quantity that measures the deviation from f to a conformal mapping.

Consider the disk $\mathbb{D}(z_0, r) \subset \Omega$, with $r < d(z, \partial\Omega)$. By Taylor's theorem, an analytic function $f(z)$ can be expanded locally around z_0:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f"(z_0)}{2!}(z - z_0)^2 + \dots$$

If $z - z_0 = re^{i\theta}$,

$$f(z) = f(z_0) + f'(z_0)re^{i\theta} + r^2 e^{2i\theta} \frac{f"(z_0)}{2!} + \dots$$

Since f is conformal at z_0, $f'(z_0) \neq 0$ and then

$$\frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)| + \frac{r^2}{r}\left(\frac{|f"(z_0)|}{2!} + \dots\right) \tag{2.1}$$

As z gets arbitrarily close to z_0, i.e. as $r \to 0$

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|$$

Hence

$$\lim_{r \to 0} \frac{\max_{|z-z_0|=r} |f(z) - f(z_0)|}{r} = \lim_{r \to 0} \frac{\min_{|z-z_0|=r} |f(z) - f(z_0)|}{r} = |f'(z_0)| \neq 0$$

which implies that

$$\lim_{r\to 0} \frac{\max_{|z-z_0|=r} |f(z)-f(z_0)|}{\min_{|z-z_0|=r} |f(z)-f(z_0)|} = 1 \tag{2.2}$$

In the image $\Omega' = f(\Omega)$, we can define the largest and smallest distance from $f(z_0)$ on the circle $|z-z_0| = r < d(z_0, \partial\Omega)$:

$$L = L(z, f, r) = \max_{|z-z_0|=r} |f(z)-f(z_0)| \text{ (the maximal stretching)}$$

and

$$l = l(z, f, r) = \min_{|z-z_0|=r} |f(z)-f(z_0)| \text{ (the minimal stretching)}$$

With this notation in place, (2.2) shows that if f is conformal in Ω, then

$$\lim_{r\to 0} \frac{L(z,f,r)}{l(z,f,r)} = 1, \text{ for all } z \in \Omega$$

The converse is also true [3], which leads to the next theorem.

Theorem 2.1.1. *A homeomorphic map $f : \Omega \to \mathbb{C}$ is conformal if and only if*

$$\lim_{r\to 0} \frac{L(z,f,r)}{l(z,f,r)} = 1$$

Note that not all homeomorphisms satisfy this property.

For example, consider the linear transformation (over $\mathbb{R}$) that maps a circle to an ellipse, $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$.

Clearly in this case, the eccentricity of the ellipse $\frac{L}{l} \neq 1$ (where L and l represent the semi-major and semi-minor axes respectively).

Definition 2.1.1. *The linear distortion of a homeomorphism f is the measurable function*

$$H(z,f) = \limsup_{r\to 0} = \frac{\max_{|\zeta=r|} |f(z+\zeta)-f(z)|}{\min_{|\zeta=r|} |f(z+\zeta)-f(z)|} = \limsup_{r\to 0} \frac{L(z,f,r)}{l(z,f,r)}$$

More generally, if $f(z)$ is differentiable, we can use the complex differential operators

$$\frac{\partial f}{\partial z} = f_z = \frac{1}{2}(f_x - i f_y),\ \frac{\partial f}{\partial \bar{z}} = f_{\bar{z}} = \frac{1}{2}(f_x + i f_y)$$

to obtain $J(z,f) = |f_z|^2 - |f_{\bar{z}}|^2$ and $H(z,f) = \frac{|f_z(z)|+|f_{\bar{z}}(z)|}{|f_z(z)|-|f_{\bar{z}}(z)|}$.

Let us now consider the linear (homeomorphic) mapping of $\mathbb{C}$, $f:\Omega \to \Omega'$ given by $f: z \mapsto az + b\bar{z}$, where $a, b \in \mathbb{C}$, i.e.

$$a = a_1 + i a_2$$
$$b = b_1 + i b_2$$

Then

$$\begin{aligned} f(z) &= (a_1 + i a_2)(x + i y) + (b_1 + i b_2)(x + i y) \\ &= (a_1 + b_1)x + (b_2 - a_2)y + i[(a_2 + b_2)x + (a_1 - b_1)y] \end{aligned}$$

This can be written in matrix form

$$\begin{pmatrix} a_1 + b_1 & b_2 - a_2 \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The trace of the operator matrix is $2a_1$ and its determinant is $|a|^2 - |b|^2$, so the following cases arise:
Case 1: if $|a| > |b|$, the map is injective and orientation preserving
Case 2: if $|a| = |b|$, the image of z under f is a line
Case 3: if $|a| < |b|$, the map is injective and is orientation reversing

Assuming $|a| > |b|$ and writing $L = |a| + |b|$ and $l = |a| - |b|$, we have $H(z,f) = \frac{|a|+|b|}{|a|-|b|} = \frac{L}{l}$, $J(z,f) = |a|^2 - |b|^2 = lL$ and $\|Df\|^2 = |a|^2 + |b|^2 = l^2 + L^2$.

Definition 2.1.2. *A homeomorphism $f:\Omega \to \Omega'$ is called a quasiconformal mapping (of bounded distortion) if it is orientation preserving and its linear distortion is uniformly bounded,* $\sup_{z\in\Omega} H(z,f) < \infty$.

In our previous example, the geometry of the mapping is bounded since $H(z,f) = \frac{L}{l}$ clearly satisfies $1 \le H(z,f) < \infty$.

2.2 Quasiconformal and Quasiregular Maps

Definition 2.2.1. *A homeomorphic map $f:\Omega\to\Omega'$ is called K-quasiconformal if $H(z,f)\le K<\infty$ in Ω. The constant $K=K(f)$ is called the distortion of the mapping.*

If f is differentiable, then

$$K=\sup_{z\in\Omega}\frac{|f_z|+|f_{\bar z}|}{|f_z|-|f_{\bar z}|}=\sup_{z\in\Omega}\frac{1+\frac{|f_{\bar z}|}{|f_z|}}{1-\frac{|f_{\bar z}|}{|f_z|}}=\frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}$$

where

$$\mu(z)=\mu_f(z)=\frac{f_{\bar z}(z)}{f_z(z)} \tag{2.3}$$

is called the Beltrami coefficient of f, and $\|\cdot\|_\infty$ is the $L^\infty(\Omega)$ norm.

Therefore a quasiconformal mapping tautologically satisfies the Beltrami equation $f_{\bar z}(z)=\mu(z)f_z(z)$, which we will discuss in the next section.

Writing $k=\|\mu\|_\infty$ we note the relations $K=\frac{1+k}{1-k}$ and $k=\frac{K-1}{K+1}$.

The definition of a quasiconformal map can also be formulated in terms of Sobolev spaces.

Definition 2.2.2. *A homeomorphism $f:\Omega\to\Omega'$ is called K-quasiconformal if it is orientation preserving, if*

$$f\in W^{1,2}_{loc}(\Omega) \tag{2.4}$$

and if the directional derivatives satisfy

$$\max_\alpha|\partial_\alpha f(z)|\le K\min_\alpha|\partial_\alpha f(z)| \tag{2.5}$$

for almost every $z\in\Omega$.

The equivalence of these two definitions of a quasiconformal mapping is a non-trivial theorem [3]. In particular, a mapping f is 1-quasiconformal if and only if it is conformal. The requirement $f\in W^{1,2}_{loc}(\Omega)$ implies that f has partial derivatives f_x and f_y almost everywhere, but being a Sobolev function is not enough for f to be differentiable almost everywhere. For (2.5) to be meaningful, we only need to set

$$\partial_\alpha f(z)=\cos(\alpha)f_x(z)+\sin(\alpha)f_y(z),\ 0\le\alpha\le2\pi \tag{2.6}$$

Then, for almost all $z \in \Omega$, condition (2.5) is well defined. The condition ties together the partial derivatives of f and hence provides geometric information on the mapping. As all homeomorphic Sobolev functions are differentiable almost everywhere [3], the directional derivatives thus retain their usual meaning,

$$\partial_\alpha f(z) = \lim_{r \to 0} \frac{f(z + re^{i\alpha}) - f(z)}{r}$$

Now we want to remove the condition that f is injective, in order to be able to define a wider class of mappings particularly useful in the study of planar elliptic PDEs.

We will only require that $f \in W^{1,2}_{loc}(\Omega)$, that f is orientation preserving, so $J(z, f) \geq 0$ almost everywhere, and that it satisfies conditions (2.5) and (2.6) on partial derivatives. These maps are called K-quasiregular.

One may wonder about the choice of the Sobolev regularity $W^{1,2}_{loc}$ in the definitions of quasiconformal and quasiregular mappings. For the homeomorphic $W^{1,1}_{loc}(\Omega)$-maps, this regularity follows from the distortion inequality (2.5):

Since $1 \leq \frac{|f_z|+|f_{\bar{z}}|}{|f_z|-|f_{\bar{z}}|} \leq K$, we can write $1 \leq \frac{(|f_z|+|f_{\bar{z}}|)^2}{|f_z|^2-|f_{\bar{z}}|^2} \leq K$, which implies

$$(|f_z| + |f_{\bar{z}}|)^2 \leq K(|f_z|^2 - |f_{\bar{z}}|^2) = KJ(z, f) \tag{2.7}$$

It is a general fact for Sobolev homeomorphisms that $\int_\Omega J(z, f) \leq |f(\Omega)|$.

Integrating both sides of (2.7), for any compact subset $A \subset \Omega$,

$$\int_A |Df|^2 \leq K(f) \int_A J(z, f) \leq K(f)|f(A)| < \infty$$

since the image of a compact subset is a compact subset (and consequently has finite area).

However, the last estimate fails for general non-homeomorphic Sobolev mappings, and hence the precise regularity $f \in W^{1,2}_{loc}(\Omega)$ is necessary. This condition guarantees the local integrability of the Jacobian, the key property in the geometric study of mappings.

2.3 The Beltrami Equation and the Existence Theorem for Quasiconformal Mappings

Suppose we are just given a measurable bounded function $\mu : \Omega \to \mathbb{D}$ with

$$\|\mu\|_{L^\infty(\Omega)} = k < 1$$

where $k = \frac{K-1}{K+1}$ (which is equavalent to $|\mu(z)| \leq k < 1$ almost everywhere in Ω).

Can we find a homeomorphic quasiconformal map $f : \Omega \to \mathbb{C}$ in $W^{1,2}_{loc}(\Omega)$ satisfying $\mu_f(z) = \mu(z)$?

In other words, we are looking for solutions to the linear partial differential equation $f_{\bar{z}}(z) = \mu(z) f_z(z)$, called the Beltrami equation.

Linearity over $\mathbb{C}$ is particularly useful and can be easily checked.

Indeed, given $\alpha, \beta \in \mathbb{C}$ and two solutions $f(z), g(z)$:

$$(\alpha f + \beta g)_{\bar{z}} = \alpha f_{\bar{z}} + \beta g_{\bar{z}} = \alpha\mu(z) f z + \beta\mu(z) g_z = \mu(z)(\alpha f_z + \beta g_z) = \mu(z)(\alpha f + \beta g)_z$$

Note that not all equations are linear over $\mathbb{C}$. For example, the PDE

$$f_{\bar{z}} = \nu_1(z) f_z + \nu_2(z) \bar{f}_z \tag{2.8}$$

is only linear over $\mathbb{R}$.

If $\nu_1(z) + \nu_2(z)| \leq k < 1$, then a homeomorphic solution is quasiconformal as

$$f_{\bar{z}} = \nu_1(z) f_z + \nu_2(z) \bar{f}_z = \left(\nu_1(z) + \frac{\nu_2(z)\bar{f}_z}{f_z}\right) f_z$$

and

$$|\mu(z)| = \left|\frac{f_{\bar{z}}}{f_z}\right| = \left|\nu_1(z) + \nu_2(z)\frac{\bar{f}_z}{f_z}\right| \leq |\nu_1(z)| + |\nu_2(z)| \leq k < 1$$

So if given $f_{\bar{z}}$ by (2.8), we can solve the Beltrami equation by defining $\mu(z) = \frac{f_{\bar{z}}(z)}{f_z(z)}$ when $f_z(z) \neq 0$, and $\mu(z) = 0$ otherwise. We have thus shown the following theorem:

Theorem 2.3.1. *Suppose $f : \Omega \to \Omega'$ is a homeomorphic $W^{1,2}_{loc}$-mapping. Then f is K-quasiconformal if and only if $f_{\bar{z}}(z) = \mu(z) f_z(z)$ for almost every $z \in \Omega$ where μ (called the Beltrami coefficient of f) is a bounded measurable function satisfying*

$$\|\mu\|_\infty \leq \frac{K-1}{K+1} < 1$$

Linearity of the Beltrami equation over $\mathbb{C}$ implies that if there is one solution then we can build others by taking linear combinations. But even if two solutions $f(z), g(z)$ are homeomorphic, a linear combination $\alpha f(z) + \beta g(z)$ will likely not be (for example we could choose α and β such that $\alpha f + \beta g$ vanishes twice).

Theorem 2.3.2. *Suppose $f_{\bar{z}} = \mu(z) f_z$ almost everywhere in Ω and that f is quasiconformal. Let $\Phi : f(\Omega) \to \mathbb{C}$ be a holomorphic map. Then $\Phi \circ f$ is a solution to the Beltrami equation and is quasiconformal if Φ is injective, i.e. conformal.*

Proof. Since Φ is holomorphic, it is infinitely differentiable, so $\Phi \circ f : \Omega \to \mathbb{C}$ is differentiable almost everywhere and $(\Phi \circ f)_z, (\Phi \circ f)_{\bar{z}}$ are locally L^2 functions.

In fact, for $w = f(z)$ we can simply compute

$$(\Phi \circ f)_z = \Phi_w(f) f_z + \Phi_{\bar{w}}(f) \bar{f}_z$$

and

$$(\Phi \circ f)_{\bar{z}} = \Phi_w(f) f_{\bar{z}} + \Phi_{\bar{w}}(f) \bar{f}_{\bar{z}}$$

Since Φ is holomorphic, $\Phi_{\bar{w}} \equiv 0$ and then

$$(\Phi \circ f)_{\bar{z}} = \Phi_w(f) f_{\bar{z}} = \Phi_w(f) \mu_f(z) f_z = \mu_f(z) \Phi_w(f) f_z = \mu_f(z) (\Phi \circ f)_z$$

So $\Phi \circ f$ solves the Beltrami equation. □

We will now briefly explore a fundamental result in the theory of quasiconformal mappings, namely the measurable Riemann mapping theorem, which is also known as the existence theorem for quasiconformal maps.

Theorem 2.3.3. *Given a domain $\Omega \subset \mathbb{C}$ and a map $\mu : \Omega \to \mathbb{D}$ with $|\mu| \le k < 1$ almost everywhere in Ω, there exists a quasiconformal homeomorphism $f : \mathbb{D} \to \mathbb{D}$ solving the Beltrami equation $f_{\bar{z}} = \mu(z) f_z(z)$. Additionally, f_z and $f_{\bar{z}}$ are in a higher degree of integrability, i.e. there exists $\epsilon = \epsilon(k) > 0$ such that*

$$f_z, f_{\bar{z}} \in L^{2+\epsilon}_{loc}(\Omega) \text{ and } f \in W^{1,2+\epsilon}_{loc}(\Omega) \tag{2.9}$$

Kari Astala showed that the best choice for ϵ is any number $\epsilon(k) < 1 + \frac{1}{k}$. This result is also known as the higher integrability theorem.

As such, condition (2.9) can be written $f_z \in L^{1+\frac{1}{k}}(\mathbb{D})$ and $f \in W^{1,1+\frac{1}{k}}(\mathbb{D})$, where $1 + \frac{1}{k} > 2$.

Recalling that a function f is Hölder continuous (typically with exponent $\alpha < 1$) if $|f(z) - f(w)| \leq C|z-w|^\alpha$, the Sobolev embedding theorems [8] imply that K-quasiconformal mappings are uniformly Hölder continuous. In fact, all functions $f \in W^{1,p}_{loc}$ are Hölder continuous for $p > 2$.

2.4 Homeomorphic Extension to the Riemann Sphere

Proposition 2.4.1. *Suppose $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is quasiconformal in $\mathbb{D}$. The reflection*

$$F(z) = \begin{cases} f(z), & \text{if } |z| \leq 1 \\ \frac{1}{\overline{f(\frac{1}{\bar{z}})}}, & \text{if } |z| > 1 \end{cases} \tag{2.10}$$

is a quasiconformal homeomorphism of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Proof. If $|z| = 1$, $|z\bar{z}| = 1$ and $z = \frac{1}{\bar{z}}$.

Since $|f(z)| = 1$, $\frac{1}{\overline{f(\frac{1}{\bar{z}})}} = \frac{1}{\bar{f}(z)} = f(z)$ so the extension agrees.

Now define $g(z) = \frac{1}{\overline{f(\frac{1}{\bar{z}})}}$ and $\phi(z) = \frac{1}{\bar{z}}$.

It is easy to see that $\phi_z = 0$ and $\phi_{\bar{z}} = -\frac{1}{\bar{z}^2}$.

On the outside of $\mathbb{D}$ (i.e. on $\mathbb{C} \setminus \mathbb{D}$), $F(z) = \phi \circ f \circ \phi$. Since $\phi(\phi(z)) = z$, $\phi = \phi^{-1}$ and then $\phi \circ F = f \circ \phi$.

Differentiating with respect to z gives:

$$\phi_w(F)F_z + \phi_{\bar{w}}(F)\bar{F}_{\bar{z}} = f_w(\phi)\phi_z + f_{\bar{w}}(\phi)\bar{\phi}_{\bar{z}}$$

Differentiating with respect to $\bar{z}$ gives:

$$\phi_w(F)F_{\bar{z}} + \phi_{\bar{w}}(F)\bar{F}_z = f_w(\phi)\phi_{\bar{z}} + f_{\bar{w}}(\phi)\bar{\phi}_z$$

Since $\phi_w(F) = \phi_z = 0$,

$$-\frac{1}{\bar{F}^2}\bar{F}_{\bar{z}} = -f_{\bar{w}}\left(\frac{1}{\bar{z}}\right)\left(\overline{\frac{1}{\bar{z}^2}}\right) \tag{2.11}$$

and

$$-\frac{1}{\bar{F}^2}\bar{F}_z = -f_w\left(\frac{1}{\bar{z}}\right)\left(\frac{1}{\bar{z}^2}\right) \tag{2.12}$$

Dividing (2.11) by (2.12), for $|z| > 1$,

$$\left(\overline{\frac{F_{\bar{z}}}{F_z}}\right) = \frac{f_{\bar{w}}\left(\frac{1}{\bar{z}}\right)}{f_w\left(\frac{1}{\bar{z}}\right)}\left(\frac{\bar{z}}{z}\right)^2 = \mu_f\left(\frac{1}{\bar{z}}\right)\left(\frac{\bar{z}}{z}\right)^2$$

Thus

$$\frac{F_{\bar{z}}}{F_z} = \bar{\mu}_f\left(\frac{1}{\bar{z}}\right)\left(\frac{z}{\bar{z}}\right)^2 = \mu_F$$

and the map F is quasiconformal, since $F_{\bar{z}} = \mu_F F_z$ and $|\mu_F| = |\mu_f| \le k < 1$. □

Furthermore, we can conclude that F is quasiconformal if and only if f is quasiconformal, so the distortion does not change:

$$K = \frac{1+|\mu_F|}{1-|\mu_F|} = \frac{1+|\mu_f|}{1-|\mu_f|}$$

Remark. f need only be a mapping $f : \mathbb{D} \to \mathbb{D}$ by the quasiconformal boundary extension theorem [3], which implies that f extends homeomorphically to $\overline{\mathbb{D}} \to \overline{\mathbb{D}}$. To show this requires significant geometric reasoning using the modulus of curve families and the modulus of continuity of f and f^{-1}.

2.5 Inverse Quasiconformal Maps

Theorem 2.5.1. *The inverse of a surjective homeomorphic quasiconformal map* $f:\mathbb{D}\to\mathbb{D}$ *is quasiconformal.*

Proof. We omit the long technical details showing $f_z \neq 0$ almost everywhere. Then

$$J(z,f) = (1-|\mu_f|^2)|f_z|^2 \neq 0$$

almost everywhere and hence, by the inverse function theorem $(f^{-1})_w$ and $(f^{-1})_{\bar{w}}$ exist almost everywhere. Differentiating $f\circ f^{-1}(z) = z$ with respect to z and $\bar{z}$ we obtain the system

$$\begin{cases} f_z(f^{-1})(f^{-1})_z + f_{\bar{z}}(f^{-1})\overline{(f^{-1})_{\bar{z}}} = 1 \\ f_z(f^{-1})(f^{-1})_{\bar{z}} + f_{\bar{z}}(f^{-1})\overline{(f^{-1})_z} = 0 \end{cases}$$

with solutions $f_{\bar{z}}(f^{-1}) = \frac{-(f^{-1})_{\bar{z}}}{J(z,f^{-1})}$ and $f_z(f^{-1}) = \frac{\overline{(f^{-1})_z}}{J(z,f^{-1})}$.

If $g = f^{-1}$ then $\frac{-g_{\bar{z}}}{\overline{g_z}} = \frac{f_{\bar{z}}(g)}{f_z(g)} = \mu_f(g)$, so g solves the equation $g_{\bar{z}} = -\mu_f(g)\overline{g_z}$.

Note that this is not a Beltrami equation, but we can write

$$g_{\bar{z}} = -\mu_f(g)\frac{\overline{g_z}}{g_z}g_z \tag{2.13}$$

and then

$$\frac{g_{\bar{z}}}{g_z} = \mu_g = -\mu_f(g)\frac{\overline{g_z}}{g_z} \tag{2.14}$$

which makes (2.13) a Beltrami equation. The relation (2.14) between Beltrami coefficients of f and g implies $|\mu_g| = |-\mu_f(g)\frac{\overline{g_z}}{g_z}| = |\mu_f(g)| \leq k < 1$, which shows that $g = f^{-1}$ is also quasiconformal. □

We have thus shown that the distortion of g at a point is the same as the distortion of f at the image point, and have found a formula for the local change of distortion:

$$K_g(z,g) = \frac{1+|\mu_g(z)|}{1-|\mu_g(z)|} = \frac{1+|\mu_f(g)|}{1-|\mu_f(g)|} = K_f(g,f)$$

2.6 Quasiconformal Extensions for the Dirichlet Problem

Let us now denote the boundary values of the quasiconformal mapping $f:\overline{\mathbb{D}}\to\overline{\mathbb{D}}$ as $f_o:\mathbb{S}\to\mathbb{S}$ where $\mathbb{S}=\partial\mathbb{D}$ is the unit circle.

Given a homeomorphism $f_o:\mathbb{S}\to\mathbb{S}$, one should first ask when is there a quasiconformal mapping $f:\overline{\mathbb{D}}\to\overline{\mathbb{D}}$ with $f|\mathbb{S}=f_o$? In other words, we are looking for a necessary and sufficient condition that f_o should satisfy for a quasiconformal extension to exist (i.e. that f_o should represent the boundary values for a quasiconformal map).

The above question is difficult to answer because: for example, there exists a homeomorphism $h:\mathbb{S}\to\mathbb{S}$ that is nowhere differentiable but has a quasiconformal extension. In fact, these homeomorphisms are most common in Teichmüller theory.

Let us now observe that a quasiconformal mapping of the unit disk $\mathbb{D}$ corresponds to a quasiconformal mapping of the upper half plane $\mathbb{H}$.

Indeed, consider the conformal map $\Phi:\mathbb{H}\to\mathbb{D}$ and its inverse $\Phi^{-1}:\mathbb{D}\to\mathbb{H}$, with $\Phi(1,-1,0)\mapsto(0,\infty,i)$ and $\Phi^{-1}(z)=-i\frac{z-1}{z+1}$.

If f is a quasiconformal mapping of $\mathbb{D}$, then $F=\Phi\circ f\circ\phi^{-1}$ is a quasiconformal mapping of the upper half plane, $F:\mathbb{H}\to\mathbb{H}$.

Because Φ and f extend homeomorphically to the closed unit disk $\overline{\mathbb{D}}$, F extends homeomorphically to $\hat{\mathbb{H}}=\mathbb{H}\cup\mathbb{R}\cup\{\infty\}$ and $F_0=F|\mathbb{R}$ extends to $\hat{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$.

Consequently $F_0:\mathbb{R}\to\mathbb{R}$ is a real valued function of a real variable, and therefore the concept of the derivative $F'(t)$ is well defined ($t\in\mathbb{R}$). In fact, there exists a homeomorphism $F_0:\mathbb{R}\to\mathbb{R}$ representing the boundary values of a quasiconformal mapping with $F_0'(t)=0$ almost everywhere, which cannot be a Sobolev function of course.

Since F_0 is a homeomorphism of $\mathbb{R}$, it is either strictly increasing or strictly decreasing, so $F_0(b)-F_0(a)\neq 0$ when $a\neq b$, and then $\int_b^a F_0'(t)dt\neq 0$ which is a contradiction since $\int_b^a F_0'(t)dt=0$.

This arises from the fact that $F_0(b)-F_0(a)=\int_b^a F_0'(t)dt$ does not hold unless F_0 is absolutely continuous.

2.7 Quasisymmetry

The definition of a quasiconformal mapping assumes the function to be defined on an open set. However, as we have seen in the previous sections, often we are dealing with restrictions of quasiconformal mappings to smaller sets or mappings perhaps defined on sets that might have a quasiconformal extension. This suggests that we seek a notion of (quasi)conformality applicable in general subsets of the plane [3].

Definition 2.7.1. *If $z_0 \in \hat{\mathbb{C}}$, the cross ratio $[z_0, z_1, z_2, z_3]$ is the image of z_0 under the unique Möbius transformation which takes z_1 to 1, z_2 to 0, and z_3 to ∞.*

Proposition 2.7.2. *If $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ are distinct and T is any Möbius transformation of $\hat{\mathbb{C}}$, then*

$$[z_0, z_1, z_2, z_3] = [T(z_0), T(z_1), T(z_2), T(z_3)]$$

In other words, the cross ratio is conformally invariant.

Proof. Let $S(z) = [z, z_1, z_2, z_3]$ be a Möbius transformation. Given any Möbius transformation T we can define $M = S \circ T^{-1}$ (since T is invertible), and then

$$M(T(z_1)) = S \circ T^{-1}(T(z_1)) = S(z_1) = 1$$
$$M(T(z_2)) = S \circ T^{-1}(T(z_2)) = S(z_2) = 0$$
$$M(T(z_3)) = S \circ T^{-1}(T(z_3)) = S(z_3) = \infty$$

So

$$S \circ T^{-1}(z) = S(T^{-1}(z)) = M(z) = [z, T(z_1), T(z_2), T(z_3)]$$

and

$$S(T^{-1}(T(z))) = S(z) = [T(z), T(z_1), T(z_2), T(z_3)]$$

□

This result for conformal mappings suggests that we "could" expect the cross ratio to be quasi-invariant for quasiconformal mappings.

Definition 2.7.3. *Let $\eta : [0,\infty) \to [0,\infty)$ be an increasing homeomorphism and $\Omega \subset \mathbb{C}$ (not necessarily open). A map $f : \Omega \to \mathbb{C}$ is called η-quasisymmetric if for each triple $z_0, z_1, z_2 \in \Omega$ we have*

$$\frac{|f(z_0) - f(z_1)|}{|f(z_0) - f(z_2)|} \le \eta\left(\frac{|z_0 - z_1|}{|z_0 - z_2|}\right) \tag{2.15}$$

Should f be defined on an open set, we will assume that it is orientation preserving and further, we say f is quasisymmetric if there is some η as above for which f is η-quasisymmetric.

Notice that by definition $\eta(0) = 0$ and $\eta(t) \to \infty$.

As we have seen, it is not at all obvious that the inverse of a quasiconformal homeomorphism is quasiconformal. However, for quasisymmetric mappings this is an elementary observation.

Lemma 2.7.1. *Let $f : \Omega \to \Omega'$ be a surjective η-quasisymmetric mapping. Then $f^{-1} : \Omega' \to \Omega$ is σ-quasisymmetric with*

$$\sigma(t) = \frac{1}{\eta^{-1}(1/t)}$$

Proof. As f is injective and onto, given a triple of points a_i, we may write

$$z_i = f^{-1}(a_i),\ i = 0,1,2$$

Since f is η-quasisymmetric,

$$\frac{|a_0 - a_2|}{|a_0 - a_1|} \le \eta\left(\frac{|z_0 - z_2|}{|z_0 - z_1|}\right)$$

and hence

$$\frac{|f^{-1}(a_0) - f^{-1}(a_1)|}{|f^{-1}(a_0) - f^{-1}(a_2)|} \le 1/\eta^{-1}\left(\frac{|a_0 - a_2|}{|a_0 - a_1|}\right)$$

□

Lemma 2.7.2. *Every entire quasisymmetric mapping f (that is defined in the whole complex plane $\mathbb{C}$) is a surjection, $f(\mathbb{C}) = \mathbb{C}$.*

Proof. The condition (2.15) implies $|f(z)| \to \infty$ when $|z| \to \infty$ since we may fix z_0, z_1 and let $z_2 \to \infty$. Consequently f extends continuously to $\hat{\mathbb{C}}$ with $f(\infty) = \infty$ and thus $f(\hat{\mathbb{C}})$ is open and closed in $\hat{\mathbb{C}}$, that is $f(\hat{\mathbb{C}}) = \hat{\mathbb{C}}$. □

It is an immediate consequence of the definition that a quasisymmetric mapping is continuous and injective, and by Lemma 2.7.1 above we see that it is a homeomorphism onto its image. Furthermore, when Ω is an open set, $z_0 \in \Omega$ so with a suitable choice of z_1 and z_2, on the circle $\{|z-z_0| = r < \operatorname{dist}(z_0, \partial\Omega)\}$ we have

$$\frac{\max_{|z-z_0|=r} |f(z)-f(z_0)|}{\min_{|z-z_0|=r} |f(z)-f(z_0)|} = \frac{|f(z_1)-f(z_0)|}{|f(z_2)-f(z_0)|} \leq \eta(1) \tag{2.16}$$

The right hand side of (2.16) is independent of r so we may take the limit as $r \to 0$ on the left to see that $H(z, f) \leq \eta(1)$, so quasisymmetric mappings defined on open sets are, in particular, mappings of bounded distortion, i.e. quasiconformal [3].

This suggests a strong relationship between quasiconformality and quasisymmetry which we explore next.

2.8 Relationship between Quasiconformality and Quasisymmetry

We previously saw that quasisymmetric mappings defined on open sets are quasiconformal. However, the converse - that quasiconformality implies quasisymmetry - is more difficult to show.

Theorem 2.8.1. *If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, then it is η-quasisymmetric with $\eta(t) = Ct^k$ for constants $C = \eta(1) \geq 1$.*

The proof usually makes use of Mori's lemma and the modulus of curve families.

We already showed that a quasiconformal map $f : \mathbb{D} \to \mathbb{D}$ extends homeomorphically (by reflection) to the map $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, with $F|\mathbb{D} = f$. Since quasiconformality implies quasisymmetry, F is quasisymmetric so f is quasisymmetric on $\mathbb{D}$, and $f_o = f|\partial\mathbb{D}$ is quasisymmetric.

Thus, on the unit circle

$$\frac{|f_0(z_0) - f_0(z_1)|}{|f_0(z_0) - f_0(z_2)|} \leq \eta\left(\frac{|z_0 - z_1|}{|z_0 - z_2|}\right) \tag{2.17}$$

which leads to the Ahlfors-Beurling theorem below.

Theorem 2.8.2. *If $f_o : \mathbb{S} \to \mathbb{S}$ is a homeomorphism which is η-quasisymmetric, then there is a quasiconformal mapping $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ with $f|\partial\mathbb{D} = f_o$.*

Below we present two theorems that outline the most fundamental properties of quasiconformal mappings, which are established using the concept of quasisymmetry.

Theorem 2.8.3. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping from the domain $\Omega \in \mathbb{C}$ onto $\Omega' \in \mathbb{C}$ and let $g : \Omega' \to \mathbb{C}$ be a K'-quasiconformal mapping. Then*

- *$f^{-1} : \Omega' \to \Omega$ is K-quasiconformal*
- *$g \circ f : \Omega \to \mathbb{C}$ is KK'-quasiconformal*
- *For all measurable sets $E \in \Omega$, $|E| = 0$ if and only if $|f(E)| = 0$*
- *The Jacobian determinant $J(z, f) > 0$ almost everywhere in Ω.*

Theorem 2.8.4. *Let $f_\nu : \Omega \to \mathbb{C}$, $\nu = 1, 2, \ldots$, be a bounded sequence of K-quasiconformal mappings defined on the domain $\Omega \in \mathbb{C}$. Then there is a sequence converging locally uniformly on Ω to a mapping f,*

$$f_{\nu_k} \to f,$$

and f is either a K-quasiconformal mapping or a constant.

2.9 Extremal Mappings and the Teichmüller Special Case

In 1936 a special case arising from quasiconformal maps of a surface in a given homotopy class led Teichmüller to look for the "best map" in terms of the distortion K. Such a map should satisfy

$$1 \leq \inf\{K : f : \mathbb{D} \to \mathbb{D} \text{ is quasiconformal with } f|\partial\mathbb{D} = f_o\} < \infty$$

By Theorem 2.8.4, the set of all quasiconformal mappings $f : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$, $f|\partial\mathbb{D} = f_o$ is compact, and the limit of K-quasiconformal mappings is K-quasiconformal.

Thus, if $\{f_j\}$ is a minimising sequence, $f_j \to f$ uniformly on $\bar{\mathbb{D}}$ and $f_j|\partial\mathbb{D} = f_o$ implies $f|\partial\mathbb{D} = f_o$, and an extremal mapping therefore exists.

The complete proof was given by Ahlfors in 1953, showing that in this case the Beltrami coefficient has a special form, which we will look at next.

Consider the L^p-norm ($p \geq 1$) of functions f satisfying

$$1 \leq \inf\left\{\int_{\mathbb{D}} K(z, f)^p dz : f : \bar{\mathbb{D}} \to \bar{\mathbb{D}}, f|\partial\mathbb{D} = f_o\right\} < \infty \tag{2.18}$$

Note that the class of functions satisfying (2.18) can be expanded, since such maps may not be quasiconformal but could have finite distortion. Also, any extension of f_o which is quasiconformal acts as a barrier (which is required for the infimum to exist).

Next, using the modulus of continuity and normal families of holomorphic mappings, Ahlfors proved that the infimum is actually the minimum, which means there exists a (not necessarily quasiconformal) extremal map $\hat{f} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ that achieves the minimum. No assertion is made that $\hat{f}$ is a homeomorphism.

We will now consider the case $p = 1$.

For small $|t|$, $t \in \mathbb{R}$ and any test function $\varphi \in C_0^\infty(\mathbb{D})$ whose support is compactly contained in $\mathbb{D}$, define $f^t = f + t\varphi$. Since $\varphi = 0$ on $\partial\mathbb{D}$,

$$f^t|_{\partial\mathbb{D}} = f|_{\partial\mathbb{D}} = f_o$$

and $\int_{\mathbb{D}} \|Df^t\|^2 < \infty$, so f^t is a smooth function of t and therefore each value of t yields a corresponding energy $E(t) = \int_{\mathbb{D}} \|Df^t\|^2$.

When $t = 0$, $f^0 = f$ and $E(0)$ is a minimum, so $E'(0) = 0$.

Then

$$0 = E'(0) = \frac{d}{dt}\bigg|_{t=0} \int_{\mathbb{D}} \|Df^t\|^2 = \int_{\mathbb{D}} 2\mathrm{Re}(\bar{f}_z \varphi_z + \bar{f}_{\bar{z}} \varphi_{\bar{z}}) \tag{2.19}$$

Replacing φ with $i\varphi$ in the definition of f^t gives

$$0 = E'(0) = \frac{d}{dt}\bigg|_{t=0} \int_{\mathbb{D}} \|Df^t\|^2 = \int_{\mathbb{D}} 2\mathrm{Im}(\bar{f}_z \varphi_z + \bar{f}_{\bar{z}} \varphi_{\bar{z}}) \tag{2.20}$$

Adding (2.19) and (2.20)

$$\begin{aligned} 0 &= \int_{\mathbb{D}} \left(\bar{f}_z \varphi_z + \bar{f}_{\bar{z}} \varphi_{\bar{z}}\right) \\ &= \int_{\mathbb{D}} \left[(\bar{f})_{\bar{z}} \varphi_z + (\bar{f})_z \varphi_{\bar{z}}\right] \\ &= -\int_{\mathbb{D}} \left(\bar{f} \varphi_{z\bar{z}} + \bar{f} \varphi_{z\bar{z}}\right) \\ &= -2 \int_{\mathbb{D}} \bar{f} \varphi_{z\bar{z}} \end{aligned}$$

Since this holds for any test function φ, $\int_{\mathbb{D}} f \Delta\varphi = 0$ and thus by Weyl's lemma f is harmonic, so $\Delta f = 4 f_{z\bar{z}} = 0$.

Now consider the function $\Phi = f_z \bar{f}_{\bar{z}}$. As f is harmonic, its weak second derivatives are guaranteed and then

$$\Phi_{\bar{z}} = f_{z\bar{z}} \bar{f}_{\bar{z}} + f_z \overline{f_{z\bar{z}}} = 0,$$

so Φ is holomorphic.

Furthermore,

$$\mu_f = \frac{f_{\bar{z}}}{f_z} \frac{\bar{f}_z}{\bar{f}_z} = \frac{\overline{\Phi}}{|f_z|^2} \frac{|f_{\bar{z}}|}{|f_{\bar{z}}|} = \frac{|f_{\bar{z}}|}{|f_z|} \frac{\overline{\Phi}}{|\Phi|} = |\mu_f| \frac{\overline{\Phi}}{|\Phi|}$$

A similiar approach is taken when $p > 1$.

For any test function $\varphi \in C_0^\infty(\mathbb{D})$, define $g^t(z) = z + t\varphi$, $t \in (-\frac{1}{2}, \frac{1}{2})$. When $\xi \in \partial\mathbb{D}$, $g^t(\xi) = \xi$ so g^t is a homeomorphism on the unit circle.

Then $g_z = 1 + t\varphi_z$ and $g_{\bar{z}} = t\varphi_{\bar{z}}$, so $J(z, g) = |1 + t\varphi_z|^2 - |t\varphi_z|^2 > 0$.

Thus g^t is locally injective and furthermore, it is quasiconformal for small t.

If $\hat{f}$ is an extremal map, $\hat{f} \circ g^t : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ and $\hat{f} \circ g^t\big|_{\partial\mathbb{D}} = f_o$ (since g^t is the identity on the unit circle), so we can define

$$E(t) = \int_{\mathbb{D}} \mathbb{K}(z, \hat{f} \circ g^t)^p dt$$

where $\mathbb{K} = K + \frac{1}{K} = \frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2}$

By Fatou's lemma and writing $\mathbb{K}_t = \mathbb{K}(z, g^t)$,

$$E(t) \leq \int_{\mathbb{D}} \mathbb{K}(z, \hat{f})^p \mathbb{K}_t^p \leq \mathbb{K}_t^p \int_{\mathbb{D}} \mathbb{K}(z, \hat{f}) < \infty$$

which shows that $E(t)$ is smooth in t.

In particular, $E(0)$ is a minimum and $\frac{d}{dt}\Big|_{t=0} E(t) = 0$.

Differentiating with respect to t under the integral sign,

$$p \int_{\mathbb{D}} \left[\frac{d}{dt}\bigg|_{t=0} \mathbb{K}(z, \hat{f} \circ g^t) \right] \mathbb{K}(z, \hat{f})^{p-1} dz = 0$$

Since this holds for every test function $\varphi \in C_0^\infty(\mathbb{D})$, we obtain a partial differential equation for $\hat{f}$ that leads to

$$\mu_{\hat{f}} = |\mu_{\hat{f}}| \frac{\overline{\Phi}}{|\Phi|}$$

where $\Phi : \mathbb{D} \to \mathbb{C}$ is holomorphic and $\Phi = \mathbb{K}(z, f)^{p-1} f_z \bar{f}_{\bar{z}}$.

The full details of this computation are in [3]. In fact, one can show that $|\mu_f| = k < 1$ is constant and that $|\mu_{\hat{f}}| \to \mu_f$. The Beltrami coefficient in this special case is thus unique, i.e. constant with holomorphic argument, $\mu_f = k\frac{\overline{\Phi}}{|\Phi|}$, $k < 1$.

Chapter 3

The Conformal Energy of Quasisymmetric Maps

3.1 Introduction

The notion of conformal energy was introduced in [4] to solve a minimisation problem concerning the mean distortion of homeomorphisms of the disk. There (and in a sequel [2]), interesting new phenomena were observed in these minimisation problems. There are also connections between conformal energy, which is in a sense dual to the energy introduced by Douglas [5], and extremal quasiconformal mappings associated with quasisymmetric homeomorphisms of the circle. Here we explore this connection further, giving explicit formulae for lowest energy quasisymmetric mappings, asymptotics for bilipschitz mappings and other related estimates.

Let $\mathbb{S}$ denote the unit circle and

$$g_o : \mathbb{S} \to \mathbb{S}$$

a homeomorphism, which we view as the boundary values of a homeomorphism of the unit disk, $g : \mathbb{D} \to \mathbb{D}$. We say g_o has finite *conformal energy* if the real number

$$\mathscr{E}(g_o) = -\frac{1}{2\pi^2} \iint_{\mathbb{S}\times\mathbb{S}} \log|g_o(\zeta) - g_o(\eta)|\, d\zeta\, d\bar{\eta} < \infty$$

When finite, this integral coverges absolutely for homeomorphisms.

For linear fractional transformations

$$\varphi(z) = \zeta \frac{z-a}{1-\bar{a}z}, |\zeta| = 1$$

The identity

$$\frac{|\varphi(z)-\varphi(w)|}{|z-w|} = \frac{1-|a|^2}{|z-a||w-a|}$$

shows the conformal energy of $\varphi \circ g_o$ to be

$$\begin{aligned}
&-2\pi^2 \mathscr{E}(\varphi \circ g_o) \\
&= \iint_{\mathbb{S}\times\mathbb{S}} \log|\varphi(g_o(\zeta)) - \varphi(g_o(\eta))|\, d\zeta d\bar{\eta} \\
&= \iint_{\mathbb{S}\times\mathbb{S}} \log \frac{|g_o(\zeta)-g_o(\eta)|(1-|a|^2)}{|g_o(\zeta)-a||g_o(\eta)-a|}\, d\zeta d\bar{\eta} \\
&= -2\pi^2 \mathscr{E}(g_o) - \iint_{\mathbb{S}\times\mathbb{S}} \left[\log|g_o(\zeta)-a| + \log|g_o(\eta)-a|\right]\, d\zeta d\bar{\eta} \\
&= -2\pi^2 \mathscr{E}(g_o)
\end{aligned}$$

Conformal energy is *conformally invariant.*

The energy of the identity is

$$\begin{aligned}
\mathscr{E}(id) &= -\frac{1}{2\pi^2} \iint_{S\times S} \log|\zeta - \eta|\, d\zeta d\bar{\eta} \\
&= \frac{1}{2\pi^2} \int_0^{2\pi}\int_0^{2\pi} \log \frac{1}{|e^{it}-e^{is}|} \cos(t-s)\, dt ds \\
&= \frac{1}{\pi} \int_0^{2\pi} \log \frac{1}{2sin(t/2)} \cos(t)\, dt = 1
\end{aligned}$$

$\mathscr{E}(\varphi) = 1$ for every Möbius transformation

The main reason for studying the conformal energy of a homeomorphism is the following: as earlier, define the pointwise distortion of a homeomorphism $g : \mathbb{D} \to \mathbb{D}$ at a point $z \in \mathbb{D}$ as

$$\mathbb{K}(z) = \frac{1}{2}\left(K(z) + \frac{1}{K(z)}\right) = \frac{1+|\mu(z)|^2}{1-|\mu(z)|^2}$$

where
$$K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$$
and $\mu(z) = g_{\bar{z}}/g_z$ is the Beltrami coefficient as introduced earlier. Here we must assume some regularity of the mapping g and the correct notion is that of *mappings of finite distortion* [6]. Namely g is assumed to lie in the Sobolev space $W^{1,1}_{loc}(\mathbb{D},\mathbb{D})$ of mappings that have a weak first derivative which is locally integrable, the Jacobian determinant $J(z,g) \in L^1_{loc}(\mathbb{D})$ and the distortion function $\mathbb{K}(z)$ is usually defined via the distortion inequality $||D_g(z)||^2 \le \mathbb{K}(z)J(z,g)$ and is assumed finite almost everywhere.

The function $\mathbb{K}$ has far better convexity properties than K and is better suited to minimisation problems. Of course $K = \mathbb{K} + \sqrt{\mathbb{K}^2-1}$ is increasing so that $||K||_\infty$ and $||\mathbb{K}||_\infty$ have the same minimisers. We have the following theorem of [4]:

Theorem 3.1.1. *Let $g_o : \mathbb{S} \to \mathbb{S}$ be a homeomorphism. Consider the minimisation problem*
$$\inf_g \left\{ \frac{1}{\pi} \iint_{\mathbb{D}} \mathbb{K}_g(z)|dz|^2 \right\}$$
where the infimum is taken over all homeomorphisms $g : \mathbb{D} \to \mathbb{D}$ of finite distortion for which $g|\mathbb{S} = g_o$. Then there is a unique minimiser if and only if g_o has finite energy. This minimum value is $\mathscr{E}(g_o)$.

Therefore if g_o has a K-quasiconformal extension to the disk,
$$1 \le \mathscr{E}(g_o) \le \mathbb{K}$$
Let us say that g_o is a *highest energy map* if $\mathscr{E}(g_o) = \mathbb{K}$. Of course the lowest energies possible are described below.

Corollary 3.1.2. $\mathscr{E}(g_o) = 1$ *if and only if g_o represents the boundary values of a Möbius transformation.*

We will give a direct proof of this fact using the calculus of variations. Next, if g_o is bilipschitz, so there is a constant $L \ge 1$ such that
$$\frac{1}{L}|\zeta - \eta| \le |g_o(\zeta) - g_o(\eta)| \le L|\zeta - \eta|$$

we may compute

$$2\pi^2(\mathscr{E}(g_o)-1)=\int_0^{2\pi}\int_0^{2\pi}\log\left|\frac{e^{i(t+s)-e^{is}}}{g_o(e^{i(t+s)})-g_o(e^{is})}\right|\cos(t)\,dt\,ds$$

Where $\cos(t)$ is positive and negative we have

$$\begin{aligned}&\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\log\left|\frac{e^{it}-1}{g_o(e^{i(t+s)})-g_o(e^{is})}\right|\cos(t)\,dt\,ds\\ +&\int_0^{2\pi}\int_{\pi/2}^{3\pi/2}\log\left|\frac{e^{it}-1}{g_o(e^{i(t+s)})-g_o(e^{is})}\right|\cos(t)\,dt\,ds\\ &\leq 4\pi\int_{-\pi/2}^{\pi/2}\log L\cos(t)\,dt=8\pi\log L\end{aligned}$$

Theorem 3.1.3. *If g_o is L-bilipschitz, then*

$$\boxed{1\leq\mathscr{E}(g_o)\leq 1+\frac{4}{\pi}\log L}$$

If $g_o:\mathbb{S}\to\mathbb{S}$ is L-bilipschitz, then in general the best we might expect is that an extension to the disk $g:\mathbb{D}\to\mathbb{D}$ is $K=L^2$ quasiconformal. Hence for L-bilipschitz maps we might expect the estimate $\mathscr{E}(g_o)\leq(L^2+L^{-2})/2$ but we have achieved the estimate

$$\mathscr{E}(g_o)\leq 1+\frac{2}{\pi}\log L^2$$

which is generally much better, and definitely better as soon as $L\geq 1.771\ldots$.

Later we will give examples to show this logarithmic estimate is generally as good as possible. Of course $L=1$ is an absolute minimum and $\mathscr{E}(g_o)=1$ for isometries. Thus for L close to 1 we must (at least) expect

$$\mathscr{E}(g_o)=1+O((L-1)^2)$$

Again we will give an explicit estimate here.

What appears to be happening (as seen in our later examples) with the minimisers of mean distortion [the problem introduced in Theorem 3.1.1] is that the distortion is concentrating at the boundary. Actually the following surprising result is known [4] which strongly links conformal energy, quasiconformal mappings and bilipschitz homeomorphisms of the circle.

Theorem 3.1.4. *(AIMO, Martio, Pavlović) If g_o has finite energy, then the unique minimiser of the mean distortion is quasiconformal if and only if g_o is bilipschitz.*

In fact what happens is that the distortion "accumulates" at points where there is no bilipschitz estimate - i.e. those points where there is no finite non-zero derivative.

It is an interesting problem to quantify this. A sample result might be

Theorem 3.1.5. *Suppose that g_o is L-bilipschitz at $1 \in \mathbb{S}$ and of finite energy. Thus*

$$\frac{1}{L} \leq \liminf_{z\to 1, z\in\mathbb{S}} \frac{|g_o(z)-g_o(1)|}{|z-1|} \leq \limsup_{z\to 1, z\in\mathbb{S}} \frac{|g_o(z)-g_o(1)|}{|z-1|} \leq L$$

and $\mathscr{E}(g_o) < \infty$. Then there is a constant K_L such that

$$\lim_{z\to 1, z\in\mathbb{D}} K(g,z) \leq K_L$$

where g is the unique minimiser of mean distortion with boundary values g_o.

This result is more or less contained in the proof of Martio's result. Perhaps a direct approach via the Poisson kernel would be OK, however if we have an α-Hölder estimate ($\alpha \neq 1$) at 1 we expect that $K(z) \to \infty$ and we should be able to describe this rate.

The relationship between quasisymmetric mappings (of the circle) and energy seems unnatural. We can already see this for Möbius transformations

$$\frac{|\varphi(\zeta)-\varphi(\xi)|}{|\varphi(\zeta)-\varphi(\nu)|} = \frac{|1-\bar{a}\nu||\zeta-\xi|}{|1-\bar{a}\xi||\zeta-\nu|}$$

so the best general estimate we might find is

$$\frac{|\varphi(\zeta)-\varphi(\xi)|}{|\varphi(\zeta)-\varphi(\nu)|} \leq \left(\frac{1+|a|}{1-|a|}\right)\frac{|\zeta-\xi|}{|\zeta-\nu|}$$

which depends on the particular transformation when we might hope to get a uniform bound if we are to use it in an energy estimate.

Therefore the correct notion to use here is that of *quasimöbius* maps introduced by Väisälä.

3.2 Quasimöbius Maps and Energy

A homeomorphism of the circle is said to be η-quasimöbius ($\eta : [0,\infty] \to [0,\infty]$ is increasing with $\eta(0) = 0$ and $\eta(\infty) = \infty$) if for every quadruple of points

$$\frac{|g_o(a) - g_o(b)||g_o(c) - g_o(d)|}{|g_o(a) - g_o(c)||g_o(b) - g_o(d)|} \leq \eta\left(\frac{|a-b||c-d|}{|a-c||b-d|}\right)$$

Writing the cross ratio as

$$[a,b,c,d] = \frac{|a-b||c-d|}{|a-c||b-d|}$$

and recalling the invariance property

$$g[a,b,c,d] = [g(a), g(b), g(c), g(d)]$$

immediately implies the two sided inequality

$$\frac{1}{\eta(1/[a,b,c,d])} \leq g[a,b,c,d] \leq \eta([a,b,c,d])$$

So $1 \leq \eta(t)\eta(1/t)$ and $\eta(1) \geq 1$. We now observe

$$\begin{aligned}
\mathscr{E}(g_o) &= -\frac{1}{2\pi^2}\iint_{\mathbb{S}\times\mathbb{S}} \log|g_o(\zeta) - g_o(\xi)|\, d\zeta d\bar{\xi} \\
&= \frac{1}{2\pi^2}\iint_{\mathbb{S}\times\mathbb{S}} \log|g_o(\zeta) - g_o(-\xi)|\, d\zeta d\bar{\xi} \\
&= -\frac{1}{2\pi^2}\iint_{\mathbb{S}\times\mathbb{S}} \log\frac{1}{|g_o(\zeta) - g_o(-\xi)|}\, d\zeta d\bar{\xi}
\end{aligned}$$

Therefore

$$\begin{aligned}
&4\mathscr{E}(g_0) = \\
&= \frac{1}{2\pi^2}\iint_{\mathbb{S}\times\mathbb{S}} \log\frac{|g_o(\zeta) - g_o(-\xi)||g_o(-\zeta) - g_o(\xi)|}{|g_o(\zeta) - g_o(\xi)||g_o(-\zeta) - g_o(-\xi)|}\, d\zeta d\bar{\xi} \\
&= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log g_o[e^{it}, e^{is}, -e^{it}, -e^{is}]\cos(t-s)\, dt ds \\
&= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log g_o[e^{i(t+s)}, e^{is}, -e^{i(t+s)}, -e^{is}]\cos(t)\, dt ds
\end{aligned}$$

Again we observe where $\cos(t)$ is positive and negative

$$[e^{i(t+s)}, e^{is}, -e^{i(t+s)}, -e^{is}] = [e^{it}, 1, -e^{it}, -1] = \cot^2(t/2)$$

Theorem 3.2.1. *Let $g_o : \mathbb{S} \to \mathbb{S}$ be η-quasimöbius. Then*

$$\boxed{\mathcal{E}(g_o) \leq \frac{1}{\pi}\int_0^{\pi/2} \log\eta(\cot^2(t/2)\cos(t)dt)}$$

This estimate is sharp.

$$\boxed{\eta(t) = Kt, t \geq 1}$$

Since any extension of g_o is K-quasiconformal,

$$\mathcal{E}(g_o) \leq 1 + \frac{1}{\pi}\log K, ||\mathbb{K}||_\infty \geq \frac{1}{2}(K + 1/K)$$

so for K large the mean distortion minimisers are again far from extremal quasiconformal mappings (of "expected" constant distortion).

3.3 Examples

Define a homeomorphism $\theta : [0, 2\pi] \to [0, 2\pi]$ by

$$\theta(t) = \begin{cases} t/\lambda, & 0 \leq t\lambda \\ 1 + (2\pi - 1)/(2\pi - \lambda)t, & \lambda \leq t \leq 2\pi \end{cases} \tag{3.1}$$

Now define a homeomorphism of the circle as

$$f_o(e^{it}) = e^{i\theta(t)} \tag{3.2}$$

and compute the energy of the map f_o.

$\theta(t)$: Low energy quasimöbius $f = e^{i\theta(t)}$ with $\mathcal{E}(f_o) << \mathcal{K}(f_o)$

Set $\alpha = (2\pi - 1)/(2\pi - \lambda)$ and $F(t, s) = \log|e^{i\theta(t)} - e^{i\theta(s)}|\cos(t - s)$

With $\zeta = e^{it}, \eta = e^{is}$ we see (using symmetry),

$$\begin{aligned}
\mathscr{E}(f) &= \iint_{\mathbb{S}\times\mathbb{S}} \log|f(\zeta)-f(\eta)|\,d\zeta\, d\bar{\eta} \\
&= \int_0^{2\pi}\int_0^{2\pi} \log|e^{i\theta(t)} - e^{i\theta(s)}|\cos(t-s)\,dt\,ds \\
&= \int_0^{\lambda}\int_0^{\lambda} F(s,t)\,dt\,ds + \int_{\lambda}^{2\pi}\int_{\lambda}^{2\pi} F(s,t)\,dt\,ds + 2\int_0^{\lambda}\int_{\lambda}^{2\pi} F(t,s)\,dt\,ds
\end{aligned}$$

We will now deal with each of these three integrals in turn.

$$\begin{aligned}
&\int_0^{\lambda}\int_0^{\lambda} F(s,t)\,dt\,ds \\
&= \int_0^{\lambda}\int_0^{\lambda} \log|1-e^{i(t-s)/\lambda}|\cos(t-s)\,dt\,ds \\
&= \lambda\int_0^{\lambda}\int_{-s/\lambda}^{1-s/\lambda} \log|1-e^{iu}|\cos(\lambda u)\,du\,ds \\
&\le \lambda^2 \int_0^{2\pi} \Big|\log|1-e^{iu}|\Big|\,du
\end{aligned}$$

where we have used the periodicity of the integrand. As $\log|1-e^{iu}| \in L^1[0,2\pi]$ this integral tends to 0 with λ:

$$\int_0^{\lambda}\int_0^{\lambda} F(s,t)\,dt\,ds \to 0 \text{ as } \lambda \to 0 \tag{3.3}$$

Next,

$$\begin{aligned}
&\int_{\lambda}^{2\pi}\int_{\lambda}^{2\pi} F(s,t)\,dt\,ds \\
&= \int_{\lambda}^{2\pi}\int_{\lambda}^{2\pi} \log|1-e^{i\alpha(t-s)}|\cos(t-s)\,dt\,ds \\
&= \int_{\lambda}^{2\pi}\int_{\alpha(\lambda-s)}^{\alpha(2\pi-s)} \log|1-e^{iu}|\cos(u/\alpha)\,dt\,ds
\end{aligned}$$

Here we can apply the dominated convergence theorem to see

$$\int_{\lambda}^{2\pi}\int_{\lambda}^{2\pi} F(s,t)\,dt\,ds \to \int_0^{2\pi}\int_{-as}^{a(2\pi-s)} \log|1-e^{i(t-s)}|\cos(a(t-s))\,dt\,ds \tag{3.4}$$

as $\lambda \to 0$ and $a = 2\pi/(2\pi - 1)$. This last number can be computed numerically. Finally we have to deal with the cross terms, namely

$$\begin{aligned}
&\left|\int_0^\lambda \int_\lambda^{2\pi} F(t,s)\,dt\,ds\right| \\
&= \left|\int_0^\lambda \int_\lambda^{2\pi} \log|1 - e^{i(1+\alpha t - s/\lambda)}|\cos(t-s)\,dt\,ds\right| \\
&\leq \frac{\lambda}{\alpha}\int_0^1 \int_{1+\alpha\lambda}^{1+2\pi\alpha} \left|\log|1 - e^{i(u-v)}|\right|\,du\,dv
\end{aligned}$$

And so as above we deduce

$$\int_0^\lambda \int_\lambda^{2\pi} F(t,s)\,dt\,ds \to 0 \,\text{as}\, \lambda \to 0 \tag{3.5}$$

When we deal with the inverse function the analysis is similar. However is it clear that the behaviour of the integral is dominated by the term

$$\begin{aligned}
&\int_0^1 \int_0^1 \log|e^{i\lambda t} - e^{i\lambda s}|\cos(t-s)\,dt\,ds \\
&= \int_0^1 \int_0^1 \log|1 - e^{i\lambda(t-s)}|\cos(t-s)\,dt\,ds \\
&= \int_0^1 \int_0^1 \log|\lambda(t-s)[1 + h(\lambda(t-s))]|\cos(t-s)\,dt\,ds
\end{aligned}$$

Here h is the remainder term in the Taylor series expansion and can be assumed uniformly small if we choose λ small enough. Thus

$$\begin{aligned}
&\int_0^1 \int_0^1 \log|e^{i\lambda t} - e^{i\lambda s}|\cos(t-s)\,dt\,ds \\
&= \log\lambda \int_0^1 \int_0^1 \cos(t-s)dt\,ds + \int_0^1 \int_0^1 \log|(t-s)|\cos(t-s)\,dt\,ds + O(1) \\
&\approx (2 - 2\cos(1))\log\lambda + O(1)
\end{aligned}$$

which divergees with λ. The constants associated with the definition of quasimöbius are estimated by considering the four points

$$[1, e^{i\lambda}, e^{i\pi}, e^{3i\pi/2}] = \frac{|1 - e^{i\lambda}||e^{i\pi} - e^{3i\pi/2}|}{|1 - e^{i\pi}||e^{i\lambda} - e^{3i\pi/2}|} \approx \frac{\lambda}{2}$$

The image cross ratio is

$$[1, e^i, e^{i\alpha\pi}, e^{3i\alpha\pi/2}] = \frac{|1-e^i||e^{i\alpha\pi} - e^{3i\alpha\pi/2}|}{|1-e^{i\alpha\pi}||e^i - e^{3i\alpha\pi/2}|} \approx 0.541\ldots$$

So

$$f_o[1, e^{i\lambda}, e^{i\pi}, e^{3i\pi/2}] = C\lambda[1, e^{i\lambda}, e^{i\pi}, e^{3i\pi/2}]$$

with $C \approx 1.08\ldots$.

In particular, for any function η for which f_o is η-quasimöbius we have

$$\eta(2) \geq \frac{1}{\lambda}$$

Clearly the bilipshitz constant is $1/\lambda$. The map f_o has finite energy independent of λ while its inverse has energy $\approx \log\lambda$. This is aligned with our more general calculations earlier and shows those estimates to be of the correct orders.

3.4 A Map and Inverse of Finite Energy but not Quasimöbius

A quasimöbius map always has finite energy and since the inverse of a quasimöbius map is again quasimöbius, the inverse will also have finite energy. We next give an example to show this does not characterise quasimöbius maps. Consider the map $\theta : [0, 2\pi] \to [0, 2\pi]$ by

$$\theta(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ t, & 1 \leq t \leq 2\pi \end{cases} \tag{3.6}$$

and define a homeomorphism of the circle as

$$g_o(e^{it}) = e^{i\theta(t)} \tag{3.7}$$

As in the calculations above, to see that g_o and g_o^{-1} have finite energy, we first examine the integrals

$$\int_0^1 \int_0^1 \left|\log|e^{it^2} - e^{is^2}|\right| dt\, ds \text{ and } \int_0^1 \int_0^1 \left|\log|e^{i\sqrt{t}} - e^{i\sqrt{s}}|\right| dt\, ds$$

As above we consider the power series expansion of the exponential function to see

$$\begin{aligned}
&\int_0^1\int_0^1 \left|\log|e^{it^2}-e^{is^2}|\right| dt ds \\
&\quad = \int_0^1\int_0^1 \left|\log|t^2-s^2|\right| dt ds + O(1) \\
&\quad \le \int_0^1\int_0^1 \left|\log|t-s|\right| dt ds + \int_0^1\int_0^1 \left|\log|t+s|\right| dt ds + O(1) \\
&\quad \le 2\int_0^1\int_0^1 \left|\log|t-s|\right| dt ds + O(1)
\end{aligned}$$

and this last integral is finite. Further

$$\begin{aligned}
&\int_0^1\int_0^1 \left|\log|e^{i\sqrt{t}}-e^{i\sqrt{s}}|\right| dt ds \\
&\quad \le \int_0^1\int_0^1 \left|\log|\sqrt{t}-\sqrt{s}|\right| dt ds + O(1) \\
&\quad \le \int_0^1\int_0^1 \left|\log|t-s|\right| dt ds + O(1)
\end{aligned}$$

is again finite. A typical cross term

$$\int_0^1\int_1^{2\pi} \left|\log|e^{it^2}-e^{is}|\right| dt ds = \int_0^1\int_1^{2\pi} \left|\log|t^2-s|\right| dt ds + O(1)$$

and $|t-s| \le |t^2-s| \le 2\pi$. So this integral is dominated by those above.

3.5 Highest Energy Quasisymmetric and Quasimöbius Mappings

Given an η-quasimöbius map $g_o : \mathbb{S} \to \mathbb{S}$ we can define the number

$$\mathscr{K}(g_o) = \min\left\{\mathbb{K} = \frac{1}{2}\left(K + \frac{1}{K}\right) : g : \mathbb{D} \to \mathbb{D} \text{ is } K\text{-quasiconformal and } g|\mathbb{S} = g_o\right\}$$

We have already observed

$$\mathscr{E}(g_o) \le \mathscr{K}(g_o)$$

and defined maps where equality holds to be of highest energy. We can now completely describe all the highest energy maps.

Theorem 3.5.1. *Let $g_o : \mathbb{S} \to \mathbb{S}$ be a highest energy quasisymmetric map. Then g_o is the sewing map of an ellipse. Thus, up to a Möbius transformation, there is $\alpha \in \mathbb{C}$, $|\alpha| < 1$ and real constants k and K determined by α such that*

$$g_o(z) = \sin\left(\frac{\pi}{2K} F\left(\frac{\sqrt{k}}{1-|\alpha|^2}\left(z - \frac{\alpha}{z}\right)\right)\right)$$

where

$$F(z) = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \tag{3.8}$$

Proof. For a highest energy quasisymmetric map g_o we consider the quasiconformal extension minimising the mean distortion and obtain the equality

$$\|\mathbb{K}(g)\|_\infty = \frac{1}{\pi}\int_{\mathbb{D}} \mathbb{K}(g) \tag{3.9}$$

which immediately implies that the distortion function $\mathbb{K}$ is constant. At this point we are also aware by Theorem 3.1.4 that g_o is bilipshitz. Now (3.9) quickly implies the modulus of the complex dilatation μ_g is constant. Let $f = g^{-1} : \mathbb{D} \to \mathbb{D}$. Then f is harmonic [4] and also has constant distortion since $|\mu_f(g)| = |\mu_g|$, [7]. We can write the harmonic function f in the form

$$f(z) = \phi(z) + \overline{\psi(z)}$$

for analytic functions φ and ψ defined on the disk. We compute the complex distortion of the map f to be

$$\mu_f(z) = \frac{\overline{\psi'(z)}}{\phi'(z)}$$

and since this has constant modulus $|\psi'(z)| = k|\phi'(z)|$ we find the analytic functions ψ' and ϕ' to be proportional. Thus for constants $\alpha \in \mathbb{C}$, $|\alpha| < 1$ and $\beta \in \mathbb{C}$ we have

$$f(z) = \phi(z) + \alpha\overline{\phi(z)} + \beta \tag{3.10}$$

We can absorb the constant β into ϕ (as $|\alpha| < 1$) and so ignore it. Notice that $f(\mathbb{D}) = \mathbb{D}$ and f is a homeomorphism. We can solve for ϕ using (3.10):

$$\phi(z) = \frac{1}{1-|\alpha|^2}(f(z) - \alpha\overline{f(z)}) \tag{3.11}$$

which we can recognise as

$$\phi = L_\alpha \circ f$$

where

$$L_\alpha(z) = \frac{1}{1-|\alpha|^2}(z - \alpha\bar{z})$$

is a linear endomorphism of $\mathbb{C}$. It follows that ϕ is a conformal homeomorphism from the disk to the ellipse $L_\alpha(\mathbb{D})$. The map ϕ is uniquely determined up to a Möbius transformation of the disk. The conformal homeomorphism

$$\varphi(z) = \frac{1}{1-|\alpha|^2}(z - \frac{\alpha}{z}) : \hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}} \setminus L_\alpha(\mathbb{D})$$

agrees with L_α on the circle (all maps here extend homeomorphically to the boundary). Thus

$$g_o = f^{-1}|\mathbb{S} = (\phi^{-1} \circ L_\alpha)|\mathbb{S} = (\phi^{-1} \circ \varphi)|\mathbb{S}$$

If $\Omega = L_\alpha(\mathbb{D})$ is the ellipse, then $\phi : \mathbb{D} \to \Omega$ and $\varphi : \hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}} \setminus \Omega$ are Riemann mappings and so g_o is a sewing map as claimed. We have an explicit formula for ϕ which can be found in Nehari's book [9] pp 296 & 300. For suitable choice of (elliptic) constants,

$$\phi(z) = \sqrt{k}sn\left(\frac{2K}{\pi}\arcsin(z)\right)$$

Therefore, for $\zeta \in \mathbb{S}$ there is a constant $\alpha \in \mathbb{C}$ such that

$$g_o(\zeta) = \sin\left(\frac{\pi}{2K}F\left(\frac{\sqrt{k}}{1-|\alpha|^2}\left(z - \frac{\alpha}{2}\right)\right)\right)$$

where $F(z)$ given by (3.8) is the inverse of the elliptic sn function. The constants k and K are determined by α. □

The following problems immediately suggest themselves:

- Describe a more general relationship between conformal energy and properties of sewing maps (or more reasonably the associated Jordan domains). A sample result might be that if g_o has finite energy and is a sewing map, then the associated domain is a John domain.
- Give an example of a map of finite energy which admits no conformal sewing (presumably the standard examples of sewing maps which are not quasisymmetric have finite energy?)

- Show that if $\mathscr{E}(g_o) << \mathscr{K}(g_o)$, (a low energy mapping) then the Hausdorff dimension of the boundary if the sewing domain associated with g_o is close to 1.

3.6 Critical Points

Here we consider the critical points of the energy functional and associated variational problems. A homeomorphism $g_o : \mathbb{S} \to \mathbb{S}$ can be written in the form

$$g_o(e^{it}) = e^{i\theta(x)},\ x \in [0, 2\pi]$$

After some normalisations, $\theta : [0, 2\pi] \to [0, 2\pi]$ may be assumed to be an increasing continuous function $\theta(0) = 0$, $\theta(2\pi) = 2\pi$. A variation of g_o can be obtained by considering a smooth function $\phi : [0, 2\pi] \to \mathbb{R}$ with $\phi(0) = \phi(2\pi) = 0$. Notice that

$$e^{i(\theta(x)+i\phi(x))},\ x \in [0, 2\pi]$$

need not be a homeomorphism of the circle, though it is a map $\mathbb{S} \to \mathbb{S}$ and one can still speak of its conformal energy. For simplicity we will write $\mathscr{E}(\theta + t\phi)$ for $\mathscr{E}(e^{\theta + t\phi})$. Further, we extend θ and ϕ periodically to the real line and consider

$$\begin{aligned}
&\frac{d}{dt}\mathscr{E}(\theta + t\phi) \\
&= \frac{d}{dt}\left\{\int_0^{2\pi}\int_0^{2\pi} \log\left|e^{i(\theta(x)+t\phi(x))} - e^{i(\theta(y)+t\phi(y))}\right| \cos(x-y)\, dxdy\right\} \\
&= \frac{1}{2}\frac{d}{dt}\left\{\int_0^{2\pi}\int_0^{2\pi} \log[2 - 2\cos(\theta(x) + t\phi(x) - (\theta(y) + t\phi(y)))] \cos(x-y)\, dxdy\right\} \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \cot\left(\frac{\theta(x) - \theta(y) + t\phi(x) - t\phi(y)}{2}\right)(\phi(x) - \phi(y)) \cos(x-y)\, dxdy
\end{aligned}$$

At $t = 0$ this gives

$$\begin{aligned}
&\frac{d}{dt}\mathscr{E}(\theta + t\phi) \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \cot\left(\frac{\theta(x) - \theta(y)}{2}\right)(\phi(x) - \phi(y)) \cos(x-y) dxdy
\end{aligned}$$

Here we must discuss the convergence of this integral. The function ϕ is smooth and so near the diagonal $\{x = y\}$ where the singularity occurs we have

$$\phi(x) - \phi(y) = \phi'(s)(x - y)$$

for some $s \in [x, y]$. As we may assume that ϕ' is bounded, we must consider if

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{x-y}{\theta(x)-\theta(y)}\right| dxdy < \infty \tag{3.12}$$

Notice that we are already assuming that $\int_0^{2\pi}\int_0^{2\pi}|\log[1-\cos(\theta(x)-\theta(y))]|dxdy$ converges in our assumption that g_o has finite energy. Equivalently

$$\int_0^{2\pi}\int_0^{2\pi}\left|\log[\theta(x)-\theta(y)]\right| dxdy < \infty \tag{3.13}$$

Of course (3.13) will not in general imply (3.12). To move forward, we therefore make the assumption that there are constants α and $p < 2$ such that

$$|\theta(x) - \theta(y)| \geq \alpha|x - y|^p \tag{3.14}$$

Now at least we can discuss convergent integrals. Using 2π-periodicity and a linear change of variables $(x \to x + y)$:

$$\begin{aligned}&\frac{d}{dt}\mathscr{E}(\theta+t\phi)\bigg|_{t=0}\\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\frac{\sin(\theta(x+y)-\theta(y))}{1-\cos(\theta(x+y)-\theta(y))}(\phi(x+y)-\phi(y))\cos(x)\,dxdy\end{aligned}$$

Now for each $x \in [0, 2\pi]$ the integral

$$F(x) = \int_0^{2\pi}\frac{\sin(\theta(x+y)-\theta(y))}{1-\cos(\theta(x+y)-\theta(y))}(\phi(x+y)-\phi(y))dy$$

converges and with our assumptions defines a continuous bounded function of x. Similarly

$$G(y) = \int_0^{2\pi}\frac{\sin(\theta(x+y)-\theta(y))}{1-\cos(\theta(x+y)-\theta(y))}(\phi(x+y)-\phi(y))\cos(x)dx$$

defines a continuous and bounded function of y. Set

$$A(x) = \int_0^{2\pi} \frac{\sin(\theta(x+y)-\theta(y))}{1-\cos(\theta(x+y)-\theta(y))}\phi(x+y)dy$$

$$B(x) = -\int_0^{2\pi} \frac{\sin(\theta(x+y)-\theta(y))}{1-\cos(\theta(x+y)-\theta(y))}\phi(y)dy$$

So that for $x \in (0,2\pi)$ we have $F(x) = A(x)+B(x)$. Of course as $x \to 0$, both $A(x)$ and $B(x)$ tend to ∞ (mod 2π). Using periodicity again we have

$$B(x) = -\int_0^{2\pi} \frac{\sin(\theta(2x+y)-\theta(x+y))}{1-\cos(\theta(2x+y)-\theta(x+y))}\phi(x+y)dy$$

Putting these back together gives

$$\begin{aligned} &F(x) \\ &= \int_0^{2\pi} \left(\frac{\sin(\theta(x+y)-\theta(y))}{-\cos(\theta(x+y)-\theta(y))} - \frac{\sin(\theta(2x+y)-\theta(x+y))}{1-\cos(\theta(2x+y)-\theta(x+y))} \right) \phi(x+y)dy \\ &= \int_0^{2\pi} \left(\frac{\sin(\theta(u)-\theta(u-x))}{-\cos(\theta(u)-\theta(u-x))} - \frac{\sin(\theta(u+x)-\theta(u)}{1-\cos(\theta(u+x)-\theta(u))} \right) \phi(u)du \end{aligned}$$

Then

$$\begin{aligned} &\int_0^{2\pi} F(x)\cos(x)dx \\ &= \int_0^{2\pi} \left[\int_0^{2\pi} \left(\cot\frac{\theta(u)-\theta(u-x)}{2} - \cot\frac{\theta(u+x)-\theta(u)}{2} \right) \phi(u)du \right] \cos(x)dx \end{aligned}$$

Fubini's theorem allows us to change the order of integration here provided

$$\cot\frac{\theta(u)-\theta(u-x)}{2} - \cot\frac{\theta(u+x)-\theta(u)}{2} \tag{3.15}$$

is integrable. Let us assume this for now and discuss the conditions later. This leads us to the formula

$$\begin{aligned} &\frac{d}{dt}\mathscr{E}(\theta+t\phi)\bigg|_{t=0} \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \left(\cot\frac{\theta(y)-\theta(y-x)}{2} - \cot\frac{\theta(y+x)-\theta(y)}{2} \right) \cos(x)dx\phi(y)dy \end{aligned}$$

It further follows from Fubini's theorem together with the fact that ϕ is an essentially arbitrary function and so we see that at a critical point of the conformal energy (homeomorphism $e^{i\theta(x)}$) we must have

$$\frac{d}{dt}\mathscr{E}(\theta + t\phi)\bigg|_{t=0} = 0$$

and thus for every $y \in [0,2\pi]$ the integral

$$\int_0^{2\pi} \left(\cot\frac{\theta(y)-\theta(y-x)}{2} - \cot\frac{\theta(y+x)-\theta(y)}{2}\right)\cos(x)dx = 0 \qquad (3.16)$$

The convergence of this integral will negate our postponed discussion of the convergence of the integral at (3.15). If θ is $C^{1,\alpha}(\mathbb{S})$ for any $\alpha > 0$ then this integral is finite.

Conjecture 1. *If θ is a sufficiently regular homeomorphism $[0,2\pi] \to [0,2\pi]$, with $\theta(\pi) = \pi$, and if for every $y \in [0,2\pi]$ the integral*

$$\int_0^{2\pi} \left(\cot\frac{\theta(y)-\theta(y-x)}{2} - \cot\frac{\theta(y+x)-\theta(y)}{2}\right)\cos(x)dx = 0 \qquad (3.17)$$

then there is $a > 0$ such that

$$\theta(t) = \arctan\left(\frac{(1-a^2)\sin(t)}{(1+a^2)\cos(t)-2a}\right)$$

We note that if $f(z) = (z-a)/(1-az) = e^{i\theta(t)}$, $a \in \mathbb{R}$ is a Möbius transformation fixing ± 1, the formula for θ is

$$\begin{aligned}\theta(t) &= \arg((e^{it}-a)/(1-ae^{it}))\\ &= \arg(e^{it}-2a+a^2e^{-it})\\ &= \arg((1+a^2)\cos(t)-2a+i(1-a^2)\sin(t))\\ &= \arctan\left(\frac{(1-a^2)\sin(t)}{(1+a^2)\cos(t)-2a}\right)\end{aligned}$$

Conjecture 2. *Möbius transformations are the only homeomorphic critical points of the conformal energy functional $\mathscr{E} : Hom(\mathbb{S}) \to [1,\infty)$.*

Next, going back to the calculation of the derivative, we would like to calculate the second derivative at the critical point $\theta(x) = x$ (which we now know is a critical point):

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}\int_0^{2\pi}\frac{\sin(x+t\phi(x)-y-t\phi(y))(\phi(x)-\phi(y))}{1-\cos(x+t\phi(x)-y-t\phi(y))}\cos(x-y)dxdy$$
$$=-\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{1-\cos(x+t\phi(x)-y-t\phi(y))}\cos(x-y)(\phi(x)-\phi(y))^2dxdy$$

which for $t=0$ gives

$$-\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos(x-y)}{1-\cos(x-y)}(\phi(x)-\phi(y))^2dxdy$$

At this point we need

Lemma 3.6.1. *Let $g_o:\mathbb{S}\to\mathbb{S}$ be a normalised L-bilipschitz homeomorphism with*

$$g_o(e^{it})=e^{i\theta(t)},\ \theta(0)=0,\ \theta(\pi/2)=\pi/2,\ \theta(\pi)=\pi$$

with θ monotone increasing. Then

$$-\frac{\pi}{2L}\le\frac{\theta(t)-1}{L-1}\le\frac{\pi}{2}$$

and

$$\left|\frac{\theta(x)-\theta(y)-(x-y)}{L-1}\right|\le\pi$$

Proof. The map g_o fixes ±1 and i. Each of these three sectors are mapped onto themselves. As g_o is L-bilipschitz θ is absolutely continuous, $\frac{1}{L}<\theta'<L$ and so (in one of the largest sectors $t\in[-\pi/2,0]$)

$$\left(\frac{1}{L}-1\right)t\le\theta(t)-t=\int_0^t\left[\theta'(t)-1\right]dt\le(L-1)t$$

We can make similar estimates in other sectors changing the intervals of integration to be based at the closest fixed point. This shows that all estimates are worst when $t=-\frac{\pi}{2}$. Similarly,

$$\left|\frac{\theta(x)-\theta(y)-(x-y)}{L-1}\right|\le\frac{1}{L-1}\int_z^y|\theta'(t)-1|dt\le|x-y|\le\pi$$

where of course all distances are mod 2π. □

Notice that in fact we could improve this estimate because $(\theta(t)-t)/(L-1)$ is controlled by the angular distance between e^{it} and ± 1, i.

Now we consider a family of normalised L-bilipschitz mappings $g_L : \mathbb{S} \to \mathbb{S}$ with $L \to 1$. We write

$$g_L(e^{ix}) = e^{i\theta_L(x)} = e^{i[x+\theta_L(x)-x]} = e^{i[x+(L-1)(\theta_L(x)-x)/(L-1)]}$$

and set

$$\phi_L(x) = \frac{\theta_L(x)-x}{L-1}$$

We have the uniform estimates of Lemma 3.6.1,

$$-\frac{\pi}{2L} \le \phi_L(x) \le \frac{\pi}{2}$$

Because of this uniformity we may use the functions ϕ_L in our Taylor expansion with $t = (L-1)$ to get the expansion

$$\begin{aligned} \mathscr{E}(g_L) &= \mathscr{E}(e^{i\theta}) = \mathscr{E}(e^{i[x+(L-1)(\theta_L(x)-x)/(L-1)]} = \mathscr{E}\left(e^{i[x+(L-1)\phi_L(x)]}\right) \\ &= 1 + \frac{(L-1)^2}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos(x-y)}{1-\cos(x-y)}(\phi_L(x)-\phi_L(y))^2\,dx\,dy + O((L-1)^3) \end{aligned}$$

Now we have to account for the change of sign in cosine in this integral when making an estimate. Using 2π-periodicity,

$$\begin{aligned} &\int_0^{2\pi}\left[\int_0^{2\pi}\frac{\cos(u)}{1-\cos(u)}\left(\phi_L(u+y)-\phi_L(y)\right)^2 du\right] dy \\ &\le \int_0^{2\pi}\left[\int_{-\pi/2}^{\pi/2}\frac{\cos(u)}{1-\cos(u)}\left(\phi_L(u+y)-\phi_L(y)\right)^2 du\right] dy \\ &\le \int_0^{2\pi}\left[\int_{-\pi/2}^{\pi/2}\frac{\cos(u)}{1-\cos(u)}u^2 du\right] dy \\ &= 2\pi\int_{-\pi/2}^{\pi/2}\frac{\cos(u)}{1-\cos(u)}u^2 du \\ &= 2\pi\left(8c - \frac{\pi}{12}(6\pi+\pi^2-12\log 4)\right) \approx 26.1647\ldots \end{aligned}$$

where c is the Catalan number. Hence

$$\mathscr{E}(g_L) \le 1 + \alpha(L-1)^2 + O(L-1)^3 \tag{3.18}$$

where

$$\alpha = \frac{2c}{\pi} - \frac{\pi}{8} - \frac{\pi^2}{48} + \log\sqrt{2} < \frac{1}{3}$$

converges uniformly as $L \to 1$. Possibly a more rigorous approach (we could use Taylor's formula with error bound) would give an explicit estimate with no $O(L-1)^3$ term.

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