FUNDAMENTALS OF RIEMANNIAN GEOMETRY AND ITS EVOLUTION

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Abstract

In this thesis we study the theory of Riemannian manifolds: these are smooth manifolds equipped with Riemannian metrics, which allow one to measure geometric quantities such as distances and angles.

The main objectives are:

(i) to introduce some of the main ideas of Riemannian geometry, i.e. geometry of curved spaces.

(ii) to present the basic concepts of Riemannian geometry such as Riemannian connections, geodesics, curvature (which describes the most important geometric features of universes) and Jacobi fields (which provide the relationship between geodesics and curvature).

(iii) to show how we can generalize the notion of Gaussian curvature for surfaces to the notion of sectional curvature for Riemannian manifolds using the second fundamental form associated with an isometric immersion. Finally we compute the sectional curvatures of our model Riemannian manifolds - Euclidean spaces, spheres and hyperbolic spaces.

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Chapter 0

Introduction

0.1 The Evolution of Riemannian Geometry

Geometry is the branch of mathematics that deals with the relationships, properties and measurements of solids, surfaces, lines and angles. It also considers spatial relationships, the theory of space and figures in space. The name comes from Greek words meaning, "land" and "to measure". Geometry was first used by the Egyptians to measure lands. Later it was highly developed by the great Greek mathematicians.

About 300 B.C, Euclid was a Greek mathematician. Elements of Euclid is a scientific work containing the foundations of ancient mathematics: elementary geometry, number theory, algebra, the general theory of proportion and a method for the determination of areas and volumes. The geometry based on the assumptions of Euclid and dealing with the study of plane and solid or space geometry is called Euclidean geometry. In the 19^{th} century, new kinds of geometry, called Non-Euclidean geometry, were created. Any kind of geometry not based upon Euclid's assumptions is called Non-Euclidean geometry. E.g:- Differential geometry (Surface geometry), Hyperbolic geometry, Riemannian geometry, etc. Classical differential geometry consisted of the study of curves and surfaces (embedded in three-dimensional Euclidean space) by means of the differential and integral calculus.

The founders of Non-Euclidean geometry were Gauss, Riemann, Bolyai and Lobachevski. All of them investigated the possibilities of changing Euclid's parallel postulate, which said that one and only one line parallel to a given line could be drawn through a point outside that line. Until the 19^{th} century, this was accepted as a " self -evident truth ". The replacement of this postulate led to new geometries. In the early part of the 19^{th} century, Carl Friedrich Gauss (1777-1855) was considered to be one of the most original mathematicians living in Germany. He was a pioneer in Non-Euclidean geometry, statistics and probability. He developed the theory of functions and the geometry of curved surfaces. Gauss defined a notion of curvature (Gaussian curvature) for surfaces, which measures the amount that the surface deviates from its tangent plane at each point on the surface.

Towards the end of his life (1855) Gauss was fortunate to have an excellent student, Gerg Friedrich Riemann (1826-1866), who was the founder of Riemannian geometry. Riemann's life was short but marvelously creative. He took up the ideas of Gauss. On June 10^{th} in 1854, he delivered his inaugural lecture, entitled " On the Hypotheses that lie at the foundations of geometry ". Several vital concepts of modern mathematics appeared for the first time from his lecture. In particular, he

1. Introduced the concept of a manifold.
2. Explained how different metric relations could be defined on a manifold.
3. Extended Gauss's notion of curvature of a surface to higher dimensional manifolds.

The concepts of Riemannian geometry played an important role in the formulation of the general theory of relativity. Riemannian geometry is a special geometry, the geometry of curved spaces, associated with differentiable manifolds and has many applications to Physics. During the closing decades of the 19^{th} century, Levi-civita (1873-1941), took up the ideas of Riemann and contributed the concept of parallel displacement or parallel transport, which plays an important role in Riemannian geometry.

0.2 Generalization of Surface Theory to Riemannian Geometry

Surface is one of the basic concepts in geometry. The definitions of a surface in various fields of geometry differ substantially. In elementary geometry, one considers planes, multifaceted surfaces, as well as certain curved surfaces (for example, spheres). Each curved surface is defined in a special way, very often as a set of points or lines. The general concept of surface is only explained, not defined, in elementary geometry: one says that a surface is the boundary of a body, or the trace of a moving line, etc. In analytic and algebraic geometry, a surface is considered as a set of points the coordinates of which satisfy equations of a particular form. In three-dimensional Euclidean space, $\Re^3$, a surface is obtained by deforming pieces of the plane and arranging them in such a way that the resulting figure has no sharp points, edges, or self-intersection. We must require that a surface be smooth and two-dimensional, so that the usual notions of calculus can be extended to it. A surface is defined by means of the concept of a surface patch, which is a homeomorphic image of a square in $\Re^3$. A surface is understood to be a connected set, which is the union of surface patches (for example, a sphere is the union of two hemispheres, which are surface patches). Usually, a surface is specified in $\Re^3$ by a vector function

$$r = r(x(u,v), y(u,v), z(u,v)), \text{ where } 0 \le u,\ v \le 1.$$

The first example of a manifold, is a regular surface in $\Re^3$.

0.2.1 Definition

A subset $S \subset \Re^3$ is a *regular surface,* if, for every point $p \in S$, there exists a neighborhood V of p in $\Re^3$ and a mapping x: $U \subset \Re^2 \to V \cap S$ of an open set $U \subset \Re^2$ onto $V \cap S$, such that:

(a) x is a differentiable homeomorphism;

(b) The differential $(dx)_q : \Re^2 \to \Re^3$ is injective for all $q \in U$

The mapping x is called a *parametrization* of S at p. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood.*

A major defect of the definition of regular surface is its dependence on $\Re^3$. This situation gradually became clear to the mathematicians of 19th century. Riemann drew the correct conclusion, which says that there must exist a geometrical theory of surfaces completely independent of $\Re^3$. His idea was to replace the dot product by a arbitrary inner product on each tangent plane of S. He observed that all the notions of the intrinsic geometry (for example, Gaussian curvature) only depended on the choice of an inner product on each tangent plane of S. Next we will introduce the notion of abstract surface which is an outgrowth of the definition of the regular surface.

Historically, it took a long time to appear due to the fact that the fundamental role of the change of parameters in the definition of a surface in $\Re^3$ was not clearly understood.

0.2.2 Definition

An *abstract surface* (differentiable manifold of dimension 2) is a set S together with a family of one-to-one mappings $x_\alpha : U_\alpha \to S$ of open sets $U_\alpha \subset \Re^2$ into S such that

(i) $\bigcup_\alpha x_\alpha(U_\alpha) = S$.

(ii) For each pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \phi$, we have that $x_\alpha^{-1}(W), x_\beta^{-1}(W)$ are open sets in $\Re^2$, and $x_\alpha^{-1} \circ x_\beta, x_\beta^{-1} \circ x_\alpha$ are differentiable mappings.

The pair (U_α, x_α) with $p \in x_\alpha(U_\alpha)$ is called a *parametrization* of S around p. $x_\alpha(U_\alpha)$ is called a *coordinate neighborhood* at p. The family $\{U_\alpha, x_\alpha\}$ is called a *differentiable structure* for S.

Shifting then from surfaces in $\Re^3$ to abstract surfaces and, from the dot product to arbitrary inner products, we get the following definition.

0.2.3 Definition

A *geometric surface* is an abstract surface furnished with an inner product $\langle\ ,\ \rangle$, on each of its tangent planes. This inner product is required to be differentiable in the sense that if V and W are differentiable vector fields on S then $\langle V, W\rangle$ is a differentiable real-valued function on S.

We emphasize that each tangent plane T_pS of S has its own inner product. An assignment of inner products to tangent planes as in the above definition is called a *geometric structure* (or metric tensor or "ds^2") on S. We emphasize that the same surface furnished with two different geometric structures gives rise to two different geometric surfaces.

If we look back to the definition of abstract surface, we see that the number 2 has played no essential role. Thus, we can extend that definition to an arbitrary n and this may be useful in future.

0.2.4 Definition

A *differentiable manifold* of dimension n is a set M and a family of injective mappings $x_\alpha : U_\alpha \subset \Re^n \to M$ of open sets U_α of $\Re^n$ into M such that

(I) $\bigcup_\alpha x_\alpha(U_\alpha) = M$

(II) For any pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \phi$, the sets $x_\alpha^{-1}(W), x_\beta^{-1}(W)$ are open sets in $\Re^n$ and the mappings $x_\alpha^{-1} \circ x_\beta, x_\beta^{-1} \circ x_\alpha$ are differentiable.

(III) The family $\{ (U_\alpha, x_\alpha) \}$ is maximal relative to the conditions (I) and (II).

The pair (U_α, x_α) with $p \in x_\alpha(U_\alpha)$ is called a *parametrization* of M around p. $x_\alpha(U_\alpha)$ is called a *coordinate neighborhood* at p. A family $\{ (U_\alpha, x_\alpha) \}$ satisfying (I) and (II) is called a *differentiable structure* on M

For example, curves are one-dimensional manifolds because every point of a curve can be located by a single parameter. Also surfaces are two-dimensional manifolds since for each piece of a surface, every point can be located by surface coordinates. Generalizing, we say that an n-dimensional manifold is a set, such that on every piece, of it, we can locate points by using n coordinates.

0.2.5 The metric coefficients of the surface

Gauss presented the most important formula in surface geometry in 1827. This appeared in his paper " General investigation of curved surface ".

$$ds^2 = a_{11}du_1^2 + 2a_{12}du_1du_2 + a_{22}du_2^2 \qquad (1)$$

It expresses the distance between two infinitesimally close points on the surface in terms of surface coordinates u_1, u_2. He considered that the geometry of a surface is Euclidean in infinitesimal neighborhoods. Thus, a surface can be regarded as an infinite collection of Euclidean spaces that are smoothly joined together. Another way of thinking about this is to regard the surface as the envelope of its tangent planes.

The proof of this formula for a surface in $\Re^3$ is briefly as follows.

Let p be any point on the surface and (u_1 , u_2) be the surface coordinates of p. Let s be the value of arc length.

Then the rectangular Cartesian coordinates of p are $(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2))$.

So $$dx = \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2 \qquad \text{(a)}$$

$$dy = \frac{\partial y}{\partial u_1} du_1 + \frac{\partial y}{\partial u_2} du_2 \qquad \text{(b)}$$

$$dz = \frac{\partial z}{\partial u_1} du_1 + \frac{\partial z}{\partial u_2} du_2 \qquad \text{(c)}$$

We know that $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ (Pythagorean formula)

Substituting from (a), (b) and (c),

$$(ds)^2 = (\frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2)^2 + (\frac{\partial y}{\partial u_1} du_1 + \frac{\partial y}{\partial u_2} du_2)^2 + (\frac{\partial z}{\partial u_1} du_1 + \frac{\partial z}{\partial u_2} du_2)^2$$

Simplifying the terms in brackets and taking,

$$a_{11} = (\frac{\partial x}{\partial u_1})^2 + (\frac{\partial y}{\partial u_1})^2 + (\frac{\partial z}{\partial u_1})^2$$

$$a_{12} = \frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2} + \frac{\partial y}{\partial u_1}\frac{\partial y}{\partial u_2} + \frac{\partial z}{\partial u_1}\frac{\partial z}{\partial u_2} = a_{21}$$

$$a_{22} = (\frac{\partial x}{\partial u_2})^2 + (\frac{\partial y}{\partial u_2})^2 + (\frac{\partial z}{\partial u_2})^2$$

So, $ds^2 = a_{11}du_1^2 + 2a_{12}du_1du_2 + a_{22}du_2^2$. Hence the result.

The expression (1) appearing on the right hand side of the equation is called the *first fundamental form* and a_{11}, a_{12}, a_{22} are called the *metric coefficients*. They vary from point to point as one moves across the surface. But in the Euclidean plane we can choose coordinates so that the metric coefficients are constants.

Consider a horizontal plane lying in three-dimensional Euclidean space.

The equation of this plane is z = constant. We can choose the coordinates $u_1 = x, u_2 = y$ on the plane.

Then $\frac{\partial x}{\partial u_1} = 1, \frac{\partial x}{\partial u_2} = 0, \frac{\partial y}{\partial u_1} = 0, \frac{\partial y}{\partial u_2} = 1, \frac{\partial z}{\partial u_1} = 0, \frac{\partial z}{\partial u_2} = 0$. Therefore, we can show that $a_{11} = 1, a_{12} = 0, a_{22} = 1$. That is, the metric coefficients are constant for the plane. Therefore $ds^2 = du_1^2 + du_2^2$. (from (1))

Consider the sphere with radius *r*, centered at the origin. Let θ and ϕ be surface coordinates (except at the poles) of any point *p*, where $u_1 = \theta, u_2 = \phi$. The Cartesian coordinates of *p* can be expressed in terms of θ and ϕ as

$$x = r\cos\phi\cos\theta$$

$$y = r\cos\phi\sin\theta$$

$$z = r\sin\phi, \qquad \text{where } 0 \le \theta < 2\pi, \ -\pi/2 \langle \phi \langle \pi/2$$

Taking partial derivatives of the functions in these expressions,

$$\frac{\partial x}{\partial \theta} = -r\cos\phi\sin\vartheta, \qquad \frac{\partial x}{\partial \phi} = -r\sin\phi\cos\theta$$

$$\frac{\partial y}{\partial \theta} = r\cos\phi\cos\theta, \qquad \frac{\partial y}{\partial \phi} = -r\sin\phi\sin\theta$$

$$\frac{\partial z}{\partial \theta} = 0, \qquad \frac{\partial z}{\partial \phi} = r\cos\phi$$

Substituting these expressions into equations (a), (b), (c) and using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, then $a_{11} = r^2\cos^2\phi$, $a_{12} = 0$, $a_{22} = r^2$.

Hence, equation (1) becomes

$$ds^2 = r^2\cos^2\phi d\theta^2 + r^2 d\phi^2.$$

This is the expression for the square of the length of an infinitesimal line element on the sphere. It is clear that the metric coefficient a_{11} varies with ϕ.

0.2.6 Generalization of metric coefficients of surfaces to Riemannian space

Generalizing the formula which Gauss obtained and extending it to n-dimensional manifolds, Riemann explained some basic concepts of a n-dimensional manifold. Consider a point p in an n-dimensional manifold and let $u_1, u_2, \ldots, u_n$ be its coordinates. Take a second point q whose coordinates $u_1 + du_1, u_2 + du_2, \ldots, u_n + du_n$ differ only infinitesimally from those of p. Riemann suggested that the square of the length ds of the line element joining p to q is given by

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} du_i du_j. \tag{2}$$

where g_{ij} are functions of $u_1, u_2, \ldots, u_n$. This directly generalizes the formula (1) Gauss obtained for the line element of a surface. The expression on the right hand side of the equation (2) is a quadratic form in the variables $du_1, du_2, \ldots, du_n$, where ds^2 is positive unless q and p coincide. Therefore the quadratic form is said to be positive definite.

Using the expression (2) for determining length, he defined a *Riemannian metric* (see the definition in chapter 1) on the differentiable manifold. It provides the ability to calculate the length of paths in the manifold, and angles between tangent vectors in the same tangent space of the manifold. A manifold furnished with a Riemannian metric is called a *Riemannian manifold* or a *Riemannian space.*

For an example, in an n-dimensional Euclidean space, the square of the length of a line segment is given by the Pythagorean formula.

$$ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2 \ , \qquad (3)$$

where $x_1, x_2, \ldots, x_n$ are rectangular Cartesian coordinates. It is clear that (3) is a special case of (2) with $g_{11} = 1, g_{22} = 1, \ldots, g_{nn} = 1$. Thus, Euclidean space is a special case of Riemannian space. Riemann called Euclidean spaces flat. A Riemannian space is locally Euclidean which means that an infinitesimal neighborhood of a point appears to be Euclidean. Just as the surface can be regarded as the envelope of its tangent planes, we may think of a Riemannian space as a collection of Euclidean spaces. We may say that a Riemannian space is infinitesimally flat or locally Euclidean.

0.2.7 Generalizing Gaussian curvature into Riemannian Geometry

In 1760, L. Euler described the curvature of a surface in space by two numbers at each point, called the *principal curvatures*. He defined the principal curvatures k_1 and k_2 of a surface by considering the curvature of curves, k_n, obtained by intersecting the surface with planes normal to the surface at an arbitrary point and taking $k_1 = \max k_n$ and $k_2 = \min k_n$. But at the time of Gauss, it was not clear that the principal curvatures would be an adequate definition of curvature. Gauss was the first to realize that surfaces have an intrinsic metric geometry that is independent of the surrounding space. More precisely, a property of surfaces in $\Re^3$ is called intrinsic if it is preserved by isometries. Even though the principal curvatures are not intrinsic, Gauss made the surprising discovery in 1827, that the product of the principal curvatures, now called the Gaussian curvature, is intrinsic. Gauss was amazed by his wonderful results and then named the theorem as Theorema Egregium, which is in colloquial American English can be translated roughly as "Totally Awesome Theorem". To get an idea of what Gaussian curvature tells us about surfaces, let's look at few examples. Simplest of all is the plane, which has both principal curvatures equal to zero and therefore has constant Gaussian curvature equal to zero. Another simple example is a sphere of radius r. Any normal planes intersect the sphere in great circles, which have radius r and therefore curvatures are $\pm 1/r$ (sign depends on whether we choose the outward pointing or inward pointing normal). Thus the principal curvatures are both equal to $\pm 1/r$, and the Gaussian curvature is equal to $1/r^2$ and always positive on the sphere.

The model spaces of surface theory are the surfaces with constant Gaussian curvature. We have discussed two of them: the Euclidean plane $\Re^2$ and the sphere of radius r. The third model is a surface of constant negative curvature, which is not so easy to visualize. Let's just mention that the upper half plane $\{(x, y): y > 0\}$ with the Riemannian metric $g = R^2(dx^2 + dy^2)/y^2$ has constant negative curvature $-1/R^2$, where R is a constant. In the special case $R = 1$ the curvature is -1. This is called the *hyperbolic plane.*

Here again generalizing the ideas of Gauss, Riemann defined the intrinsic geometry of a Riemannian space. Just as the notion of Gaussian curvature he thought that Riemannin curvature is a measure of the degree to which a Riemannian space differs from Euclidean space. In Euclidean space, he considered that the Riemannian curvature is zero everywhere. As with the surfaces, the basic geometric invariant is curvature. But the curvature becomes much more complicated quantity in higher dimensions because a manifold may curve in so many directions. The curvature can vary from point to point, but there are important special cases in which Riemann's measure is constant across the entire space. As with the surfaces, the model spaces of Riemannian geometry are the manifolds with constant sectional curvature (see chapter 3). In the end of the chapter 5, we introduce three classes of highly symmetric model Riemannian manifolds:- Euclidean spaces, spheres, and hyperbolic spaces. All most all of the properties of Riemannian geometry are related to the curvature. Therefore as in surface geometry, we can say that the curvature was the main source to develop Riemannian geometry.

The main objective of this thesis is to discuss more details about the curvature of the Riemannian manifold.

Chapter 1

Preliminaries and Notations

As stated in chapter 0, this thesis is mainly concerned with the curvature of a Riemannian manifold. Therefore we need to know the basic definitions, results (without proofs) and notations in Riemannian geometry. The purpose of this chapter is to familiarize the reader with the basic language of Riemannian geometry as a review and to provide a quick reference. Further details can be found in the following sources: [DC 1], [DC 2] and [JML].

1.1 Tangent space (T_pM)

Let M be a differentiable manifold with dimension n. A differentiable function $\alpha : (-\varepsilon,\varepsilon) \subset \Re \to M$ is called a (differentiable) curve in M, where $\Re$ is the set of all real numbers. Suppose that $\alpha(0) = p \in M$ and let D be the set of functions on M that are differentiable at p.

The tangent vector to the curve α at $t = 0$ is a function $\alpha'(0) : D \to \Re$ given by

$$\alpha'(0)f = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0}, \qquad f \in D.$$

A *tangent vector* at p is the tangent vector at $t = 0$ of some curve $\alpha: (-\varepsilon, \varepsilon) \rightarrow M$ with $\alpha(0) = p$. The set of all tangent vectors to M at p will be denoted by T_pM.

If we choose a parametrization $x: U \subset \Re^n \rightarrow M^n$ around $p = x(0)$ with $x(x_1,\dots,x_n) = q \in x(U)$, we can express the function f and the curve α in this paramerization by

$$f \circ x(q) = f(x_1,\dots,x_n)\text{, where } (x_1,\dots,x_n) \in U \text{ and}$$

$$x^{-1} \circ \alpha(t) = (x_1(t),\dots,x_n(t))\text{, where } t \in (-\varepsilon,\varepsilon)\,.$$

Then
$$\alpha(t) = x(x_1(t),\dots,x_n(t))$$

$$f \circ \alpha(t) = f(x_1(t),\dots,x_n(t))$$

Therefore
$$\begin{aligned}\alpha'(0)f &= \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} \\ &= \frac{d}{dt}(f(x_1(t),\dots,x_n(t))\Big|_{t=0} \\ &= \sum_{i=1}^{n} x_i'(0)\left(\frac{\partial f}{\partial x_i}\right)_0 \\ &= \sum_{i=1}^{n} x_i'(0)\left(\frac{\partial}{\partial x_i}\right)_0 f\end{aligned}$$

In other words, the vector $\alpha'(0)$ can be expressed in the parametrization x by

$$\alpha'(0) = \sum_{i=1}^{n} x_i'(0)\left(\frac{\partial}{\partial x_i}\right)_0 \tag{1}$$

It is clear that $\left(\frac{\partial}{\partial x_i}\right)_0$ is the tangent vector at p of the coordinate curve $x_i \rightarrow x(0,\dots,x_i,\dots,0)$. It follows from (1) that the set T_pM forms a vector space of dimension n with an associated basis $\left\{\left(\frac{\partial}{\partial x_1}\right)_0,\dots,\left(\frac{\partial}{\partial x_n}\right)_0\right\}$. Then the vector space T_pM is called the *tangent space* of M at p.

Using the idea of the tangent space, we can define the differential of a differentiable mapping.

1.2 Proposition

Let M_1^n and M_2^m be differentiable manifolds and let $\varphi : M_1 \to M_2$ be a differentiable mapping. For every $p \in M_1$ and for each $v \in T_pM_1$, choose a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to M_1$ with $\alpha(0) = p, \alpha'(0) = v$. Take $\beta = \varphi \circ \alpha$. The mapping $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ given by $d\varphi_p(v) = \beta'(0)$ is a linear mapping that does not depend on the choice of α. (Proof, see [DC 2]). The mapping $d\varphi_p$ is called the *differential* of φ at *p*.

1.3 Definition

Let M_1 and M_2 be differentiable manifolds. A mapping $\varphi : M_1 \to M_2$ is a *diffeomorphism,* if it is differentiable, bijective, and its inverse, φ^{-1}, is differentiable.

1.4 Definition

Let M^m and N^n be differentiable manifolds. A differentiable mapping $\varphi : M \to N$ is said to be an *immersion* if $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is injective for all $p \in M$. If, in addition, φ is a homeomorphism onto $\varphi(M) \subset N$, where $\varphi(M)$ has the subspace topology induced from *N*, we say that φ is an *embedding.* If $M \subset N$ and the inclusion $i : M \subset N$ is an embedding, we say that *M* is a *submanifold* of *N.*

1.5 The tangent bundle

Let M^n be a differentiable manifold and let $TM = \{(p,v);\, p \in M,\, v \in T_pM\}$. The set *TM* with a differentiable structure of dimension $2n$ is called the *tangent bundle* of *M.*

1.6 Definition

A *vector field X* on a differentiable manifold *M* is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. In terms of mappings, *X* is a mapping of *M* into the tangent bundle *TM.* The vector field is *differentiable* if the mapping $X : M \to TM$ is a differentiable mapping.

Considering a parametrization x: $U \subset \Re^n \rightarrow M$, we can write

$$X(p) = \sum_{i=1}^{n} \alpha_i(p) \frac{\partial}{\partial x_i}, \tag{2}$$

where each α_i: $U \rightarrow \Re$ is a function on U and $\frac{\partial}{\partial x_i}$, $i = 1,\ldots,n$ is the basis associated with the parameterization x. Therefore from (2) we can say that $X(p)$ is differentiable if and only if the α_i's are differentiable functions for all parametrization. It is convenient to think of a vector field as a mapping, X: $D \rightarrow F$ from the set D of differentiable functions on M to the set of functions on M, defined in the following way

$$(Xf)(p) = \sum_{i=1}^{n} \alpha_i(p) \frac{\partial f}{\partial x_i}(p), \text{ for all } f \in D.$$

1.7 Lemma

Let X and Y be differentiable vector fields on a differentiable manifold M. Then there exists a unique vector field Z such that, for all $f \in D$, $Zf = (XY - YX)f$.

The vector field Z is called the *bracket* $[X,Y] = XY - YX$ of X and Y. It is clear that Z is differentiable.

There are well-known properties of the bracket.

1.8 Proposition

If X, Y and Z are differentiable vector fields on M, and a, b are real numbers, and f, g are differentiable functions, then:

(a) $[X,Y] = -[Y,X]$

(b) $[aX + bY, Z] = a[X,Z] + b[Y,Z]$

(c) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (*Jacobi identity*)

(d) $[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$.

(Proof, see [DC 2])

1.9 Definition

A *Riemannian metric* on a differentiable manifold M is a correspondence which associates to each p of M an inner product $\langle\ ,\ \rangle_p$, which is a symmetric, bilinear, positive definite form, on the tangent space T_pM. If x: $U \subset \Re^n \to M$ is a system of coordinates around p, with $x(x_1,...,x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = dx_q(0,...,1,...,0)$, then

$\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \rangle_q = g_{ij}(x_1,...,x_n)$ is a differentiable function on U.

It is possible to delete the index p in the function $\langle\ ,\ \rangle_p$ whenever there is no chance of confusion. The function $g_{ij}(=g_{ji})$ is called the *local representation of the Riemannian metric* in the coordinate system x: $U \subset \Re^n \to M$.

1.10 Example

The almost trivial example is $M = \Re^n$ with $\frac{\partial}{\partial x_i} = e_i = (0,...,1,...,0)$. The metric is given by $\langle e_i, e_j \rangle = \delta_{ij}$. Therefore $\Re^n$ is called *Euclidean space of dimension n* and the Riemannian geometry of this space is metric Euclidean geometry.

1.11 Definition

Let M and N be Riemannian manifolds. A diffeomorphism f: $M \to N$ is called an *isometry* if: $\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}$ for all $p \in M$, and $u, v \in T_pM$.

1.12 Definition

A differentiable mapping c: $I \to M$ of an open interval $I \subset \Re$ into a differentiable manifold M is called a (parametrized) *curve*. A *vector field* V along a curve c: $I \to M$ is a differentiable mapping that associates to every $t \in I$ a tangent vector $V(t) \in T_{c(t)}M$. To say that V is *differentiable* means that for any differentiable function f on M, the function $t \to V(t)f$ is a differentiable function on I. The vector field, $dc\left(\frac{d}{dt}\right)$, denoted by $\frac{dc}{dt}$, is called *velocity field* (or tangent vector field) of the curve.

Chapter 2

Connections

2.1 Introduction

Before defining curvature on Riemannian manifolds, we need to study geodesics, the Riemannian generalizations of straight lines. A curve in Euclidean space is a straight line if and only if its acceleration is identically zero. This is the property that we choose to take as a defining property of geodesics on a Riemannian manifold. To make sense of this idea, we are going to introduce a new object on manifolds, called a connection. We give a rather general definition of a connection, called affine connection, in terms of directional derivatives of sections of the tangent bundles. After deriving some basic properties of connections, we show how to use them to differentiate vector fields along curves, to define geodesics and parallel transport of vector fields.

2.2 Affine Connections

2.2.1 Definition

Let us indicate by $\aleph(M)$ the set of vector fields of class C^∞ on M and by $D(M)$ the ring of real valued functions of class C^∞ defined on M. An *affine connection* ∇ on a differentiable manifold M is a mapping

$$\nabla:\ \aleph(M)\times\aleph(M)\to\aleph(M),$$

which is denoted by $(X, Y) \to \nabla_X Y$ and which satisfies the following properties:

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

(ii) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

(iii) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$, where $X, Y, Z \in \aleph(M)$ and $f, g \in D(M)$.

2.2.2 Proposition

Let M be a differentiable manifold with an affine connection ∇. There exists a unique correspondence, which associates to a vector field V along the differentiable curve $c: I \to M$ another vector field $\frac{DV}{dt}$ along c, called the *covariant derivative* of V along c, such that:

(a) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$, where W is a vector field along c.

(b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$, where f is a differentiable function on I.

(c) If V is induced by a vector field $Y \in \aleph(M)$, that is, $V(t) = Y(c(t))$,

then $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$, where $Y(c(t)) \in T_{c(t)}M$.

The notion of covariant derivative has many important consequences.

2.2.3 Remark

Let M^n be a differentiable manifold and $p \in M$. Choose a system of coordinates $(x_1, \ldots, x_n)$ about p and write $X = \sum_{i=1}^{n} x_i X_i$, where $X_i = \frac{\partial}{\partial x_i}$ and $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$ is a basis in T_pM. Let $(x_1(t), \ldots, x_n(t))$ be the local expression of $c(t)$, $t \in I$. Then we can express the field, $V = \sum_{j=1}^{n} v^j X_j$, where $v^j = v^j(t)$ and $X_j = \frac{\partial}{\partial x_j} = X_j(c(t))$.

Taking the covariant derivative of V along c,

$$\frac{DV}{dt} = \sum_{j=1}^{n}\left(\frac{dv^j}{dt}\right)X_j + \sum_{j=1}^{n} v^j \frac{DX_j}{dt}, \quad \text{(using (b) of proposition 2.2.2)} \qquad (1)$$

By using (c) of proposition 2.2.2 and (i) of definition 2.2.1,

$$\frac{DX_j}{dt} = \nabla_{\frac{dc}{dt}} X_j(c(t))$$

$$= \nabla_{\sum_{i=1}^{n}\left(\frac{dx_i}{dt}\right)X_i} X_j(c(t))$$

$$= \sum_{i=1}^{n}\left(\frac{dx_i}{dt}\right)\nabla_{X_i} X_j$$

Then from (1), $$\frac{DV}{dt} = \sum_{j=1}^{n}\frac{dv^j}{dt}X_j + \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{dx_i}{dt}v^j\nabla_{X_i}X_j \qquad (2)$$

This differential equation provides the following results.

2.2.4 Example

Let M be Riemannian manifold and let p be a fixed point of M. Consider a constant curve c: $I \to M$ given by $c(t) = p$, for all $t \in I$. Let V be a vector field along c (that is, V is a differentiable mapping of I into T_pM). Then we can show that $\frac{DV}{dt} = \frac{dV}{dt}$.

Proof:

Let M be Riemannian manifold with dimension n. Take $c(t) = p = (x_1,...,x_n)$, where $x_1,...,x_n$ are constants, since $c(t)$ is a constant curve.

Therefore $\frac{dx_i}{dt} = 0$, where $i = 1,\ldots,n$.

Substituting into equation (2),

$$\frac{DV}{dt} = \sum_{j=1}^{n}\frac{dv^j}{dt}X_j = \frac{dV}{dt}.$$

Hence the result. That is, the covariant derivative coincides with the usual derivative of V if V is a vector field along a constant curve.

2.2.5 Definition

Let M be a differentiable manifold with an affine connection ∇. A vector field V along curve c: $I \to M$ with $\frac{DV}{dt} = 0$, for all $t \in I$ is called a *parallel* vector field.

2.2.6 Proposition

Let M be a differentiable manifold with an affine connection ∇. Let c: $I \to M$ be a differentiable curve in M and let V_0 be a vector tangent to M at $c(t_0), t_0 \in I$ (i.e. $V_0 \in T_{c(t_0)}M$). Then there exists a unique parallel vector field V along c, such that $V(t_0) = V_0$. The vector field $V(t)$ is called the *parallel transport* of the vector $V(t_0)$ along c.

2.2.7 Remark

If $V(t)$ is a parallel vector field then $\frac{DV}{dt} = 0$.

From equation (2), we have

$$\sum_{j=1}^{n} \frac{dv^j}{dt} X_j + \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{dx_i}{dt} v^j \nabla_{X_i} X_j = 0. \tag{3}$$

We put $\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma_{ij}^k X_k$, where Γ_{ij}^k are differentiable functions on M and are called the coefficients of the connection ∇ or *Christoffel symbols* of the connection.

Then from equation (3),

$$\sum_{j=1}^{n}\frac{dv^j}{dt}X_j+\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{dx_i}{dt}v^j(\sum_{k=1}^{n}\Gamma_{ij}^kX_k)=0$$

$$\sum_{j=1}^{n}\frac{dv^j}{dt}X_j+\sum_{i,j,k}\frac{dx_i}{dt}v^j\Gamma_{ij}^kX_k=0$$

Replacing j with k in the first term, we can obtain

$$\sum_{k=1}^{n}\left\{\frac{dv^k}{dt}+\sum_{i,j}v^j\frac{dx_i}{dt}\Gamma_{ij}^k\right\}X_k=0$$

Since the X_k's are linearly independent in T_pM, we have

$$\frac{dv^k}{dt}+\sum_{i,j}v^j\Gamma_{ij}^k\frac{dx_i}{dt}=0,\ k=1,\dots,n \tag{4}$$

This is the system of n differential equations in $v^k(t)$, which gives a unique solution satisfying the initial conditions $v^k(t_0)=v_0^k$. It then follows that, if V exists, it is unique.

2.3 Riemannian Connection

Among all possible metric connections, the most important is the Riemannian connection (sometimes called the Levi-Civita connection) which is given by the fundamental theorem of Riemannian geometry. Before that we need to know the following definitions.

2.3.1 Definition

Let M be a differentiable manifold with an affine connection ∇ and a Riemannian metric $\langle\ ,\ \rangle$. A connection is said to be *compatible* with the metric $\langle\ ,\ \rangle$, when for any smooth curve c and any pair of parallel vector fields P and Q along c, we have $\langle P,Q\rangle$ = constant.

This definition shows that if ∇ is compatible with $\langle\ ,\ \rangle$ then we are able to differentiate the inner product by the usual "product rule".

2.3.2 Proposition

Let M be a Riemannian manifold. A connection ∇ on M is compatible with a metric if and only if for any vector fields V and W along the differentiable curve $c: I \to M$ we have

$$\frac{d}{dt}\langle V,W\rangle = \langle \frac{DV}{dt},W\rangle + \langle V,\frac{DW}{dt}\rangle, \quad t \in I \tag{5}$$

2.3.3 Corollary

A connection ∇ on a Riemannian manifold M is compatible with the metric if and only if $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$, for all $X, Y, Z \in \aleph(M)$. (6)

2.3.4 Definition

An affine connection ∇ on a smooth manifold is said to be *symmetric* when

$$\nabla_X Y - \nabla_Y X = [X,Y], \quad \text{for all } X, Y \in \aleph(M).$$

2.3.5 Remark

If ∇ is symmetric then $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Since, for an arbitrary function f,

$$[X_i, X_j]f = (X_i X_j - X_j X_i)f, \quad \text{(from 1.7 lemma)}$$

$$[X_i, X_j]f = \left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} \right) f$$

$$= \left(\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial f}{\partial x_i} \right) = 0$$

$$\Rightarrow \ [X_i, X_j] = 0, \text{ for arbitrary function } f$$

Therefore $\nabla_{X_i} X_j - \nabla_{X_j} X_i = 0.$

That is $\sum_{k=1}^{n} (\Gamma_{ij}^k - \Gamma_{ji}^k) X_k = 0$, since X_k's are linearly independent.

So $\Gamma_{ij}^k = \Gamma_{ji}^k$

We are now in a position to state the fundamental theorem of Riemannian geometry. If we are going to use geodesics and covariant derivatives as tools for studying Riemannian geometry, it is evident that we need a way to single out a particular connection on a Riemannian manifold that is determined by the metric. We describe two properties that determine a unique connection on any Riemannian manifold.

2.3.6 Theorem. (Levi-Civita)

Given a Riemannian manifold M, there exists a unique affine connection ∇ on M satisfying the following conditions:

(a) ∇ is symmetric.

(b) ∇ is compatible with the Riemannian metric. (Proof, see [DC 2])

Then ∇ is called the *Levi- Civita (or Riemannian)* connection on M.

2.3.7 Remark: Calculating the Christoffel symbols of the Riemannian connection in terms of the metric coefficients.

Take $X, Y, Z \in \aleph(M)$.

If ∇ is compatible then

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \qquad (7)$$

Similarly, $$Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \qquad (8)$$

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \qquad (9)$$

Adding (7) and (8) and subtracting (9), then using the symmetry of ∇, we have

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = \langle Y,[X,Z]\rangle + \langle X,[Y,Z]\rangle + \langle \nabla_X Y, Z\rangle + \langle Z, \nabla_Y X\rangle$$

Subtracting and adding the term $\langle \nabla_Y X, Z\rangle$ in the right hand side,

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = \langle Y,[X,Z]\rangle + \langle X,[Y,Z]\rangle + \langle Z,[X,Y]\rangle + 2\langle Z, \nabla_Y X\rangle$$

Taking $X = X_i = \dfrac{\partial}{\partial x_i}$, $Y = X \; = \dfrac{\partial}{\partial x_j}$ and $Z = X_k = \dfrac{\partial}{\partial x_k}$

$$X_i\langle X_j, X_k\rangle + X_j\langle X_k, X_i\rangle - X_k\langle X_i, X_j\rangle = 2\langle X_k, \nabla_{X_j} X_i\rangle \qquad (10)$$

Since ∇ is symmetric, $[X,Z] = [X_i, X_k] = 0$, similarly $[Y,Z] = 0$ and $[X,Y] = 0$.

Taking $\langle X_i, X_j\rangle = g_{ij}$, $\langle X_j, X_k\rangle = g_{jk}$ and $\langle X_k, X_i\rangle = g_{ki}$

From equation (10),

$$\langle X_k, \nabla_{X_j} X_i\rangle = \frac{1}{2}\left\{\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right\}$$

$$\langle X_k, \sum_{l=1}^{n}\Gamma_{ij}^{l} X_l\rangle = \frac{1}{2}\left\{\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right\}, \text{ since } \nabla_{X_j} X_i = \sum_{l=1}^{n}\Gamma_{ij}^{l} X_l$$

$$\sum_{l=1}^{n}\Gamma_{ij}^{l}g_{kl} = \frac{1}{2}\left\{\frac{\partial g_{jk}}{\partial x_i}+\frac{\partial g_{ki}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_k}\right\} \tag{11}$$

Multiplying equation (11) by the inverse matrix g^{km} and noting that $\sum_{k=1}^{n} g_{kl}g^{km} = \delta_l^m$

$$\sum_{l=1}^{n}\Gamma_{ij}^{l}g_{kl}g^{km} = \frac{1}{2}\left\{\frac{\partial g_{jk}}{\partial x_i}+\frac{\partial g_{ki}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_k}\right\} g^{km}$$

Summing over *k*,

$$\sum_{l=1}^{n}\Gamma_{ij}^{l}(\sum_{k=1}^{n} g_{kl}g^{km}) = \frac{1}{2}\sum_{k=1}^{n}\left\{\frac{\partial g_{jk}}{\partial x_i}+\frac{\partial g_{ki}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_k}\right\}g^{km}$$

$$\sum_{l=1}^{n}\Gamma_{ij}^{l}\delta_l^m = \frac{1}{2}\sum_{k=1}^{n}\left\{\frac{\partial g_{jk}}{\partial x_i}+\frac{\partial g_{ki}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_k}\right\}g^{km}$$

$$\Gamma_{ij}^{m} = \frac{1}{2}\sum_{k=1}^{n}\left\{\frac{\partial g_{jk}}{\partial x_i}+\frac{\partial g_{ki}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_k}\right\}g^{km} \quad \text{(since } \delta_l^m = 1\text{, if } m = l\text{)} \tag{12}$$

This formula provides the ability to compute the Christoffel symbols of the Riemannian connection in any coordinate.

For example, in Euclidean space all g_{ij}'s are constant. Then we have $\Gamma_{ij}^{m} = 0$.

Hence from equation (2),

$\frac{DV}{dt} = \sum_{k=1}^{n}\frac{dv^k}{dt}X_k = \frac{dV}{dt}$. Therefore the covariant derivative coincides with the usual derivative in Euclidean space. In Riemannian space the covariant derivative differs from the usual derivative by terms which involve the Christoffel symbols.

2.3.8 Example

Consider the upper half plane $\Re_+^2 = \{(x,y) \in \Re^2; y > 0\}$ with the metric $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$, (the metric of Lobatchevski's non Euclidean geometry). We can show that the Christoffel symbols of the Riemannian connection are $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0$, $\Gamma_{11}^2 = \frac{1}{y}$, $\Gamma_{12}^1 = \Gamma_{22}^2 = -\frac{1}{y}$.

Proof: Considering equation (11),

$$\sum_{l=1}^{2} \Gamma_{ij}^l g_{kl} = \frac{1}{2}\left\{\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right\}, \text{ where } i, j, k = 1, 2.$$

$$\Gamma_{ij}^1 g_{k1} + \Gamma_{ij}^2 g_{k2} = \frac{1}{2}\left\{\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right\}$$

Taking $i = j = k = 1$ and $x_1 = x,\ x_2 = y$,

$$\Gamma_{11}^1 g_{11} + \Gamma_{11}^2 g_{12} = \frac{1}{2}\left\{\frac{\partial g_{11}}{\partial x} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial x}\right\}$$

$\Gamma_{11}^1 = 0$, since $g_{21} = 0$ and $\frac{\partial g_{11}}{\partial x} = 0$, where $g_{11} = \frac{1}{y^2}$

Similarly, when $i = 1,\ j = 2,\ k = 2 \quad \Rightarrow \Gamma_{12}^2 = 0$

$i = 1,\ j = 2,\ k = 1 \quad \Rightarrow \Gamma_{12}^1 = -\frac{1}{y}$

$i = 1,\ j = 1,\ k = 2 \quad \Rightarrow \Gamma_{11}^2 = \frac{1}{y}$, similarly others.

On any Riemannian manifold, we will always use the Riemannian connection from now on without further comment.

2.4 Geodesics

Having defined the covariant differentiation along curves, we can now introduce the notion of a geodesic as a curve with zero acceleration.

2.4.1 Definition

Let M be a manifold with a Riemannian connection ∇, and γ be a curve in M. The acceleration of γ is the vector field $\frac{D}{dt}\frac{d\gamma}{dt}$ along γ. A curve γ is called a *geodesic* with respect to ∇ if its acceleration is zero.

2.4.2 Remark

If $\gamma : [a,b] \to M$ is a geodesic, then $\frac{d}{dt}\langle\frac{d\gamma}{dt},\frac{d\gamma}{dt}\rangle = 2\langle\frac{D}{dt}\frac{d\gamma}{dt},\frac{d\gamma}{dt}\rangle = 0.$

This implies that $\langle\frac{d\gamma}{dt},\frac{d\gamma}{dt}\rangle$ = Constant.$\Rightarrow \left|\frac{d\gamma}{dt}\right| = \left|\gamma'(t)\right|$ = Constant (c)

That is, the length of the tangent vector is constant. We assume that $\left|\gamma'(t)\right| = c \neq 0$, that is, we exclude the geodesics which reduce to points. The arc length s of γ, starting from a fixed origin, say $t = t_0$, is given by

$$s(t) = \int_{t_0}^{t}\left|\frac{d\gamma}{d\tau}\right|d\tau = c(t-t_0).$$

When $c = 1$, the parameter is actually arc length and $\left|\gamma'(t)\right| = 1$. In this case we say that the geodesic γ is *normalized.* Now we are going to determine the local equations satisfied by a geodesic γ in a system of coordinates (U,x) about $\gamma(t_0)$ on M. In U, $\gamma(t) = (x_1(t),\ldots,x_n(t))$ will be a geodesic if and only if $\frac{D}{dt}\frac{d\gamma}{dt} = 0,$

$$\text{where} \quad \frac{d\gamma}{dt} = \left(\frac{dx_1(t)}{dt}, \ldots, \frac{dx_n(t)}{dt} \right).$$

Using equation (4) of remark 2.2.7 and taking $v^k = \dfrac{dx_k}{dt}$ and $v^j = \dfrac{dx_j}{dt}$,

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{dx_j}{dt} \frac{dx_i}{dt} = 0, \quad \text{where } k = 1, \ldots, n. \tag{13}$$

Next we will discuss a geodesic frame which will also be useful in future situations

2.4.3 Example: Geodesic frame

Let M be a Riemannian manifold of dimension n and let $p \in M$. It can be shown that there exists a neighborhood $U \subset M$ of p and n vector fields $E_1, \ldots, E_n \in \aleph(U)$, orthonormal at each point of U, such that, at p, $\nabla_{E_i} E_j(p) = 0$. Such a family E_i of vector fields is called a (local) *geodesic frame* at p, where $i, j = 1, \ldots, n$.

Proof: We prove one special case.

Consider the special case in which $E_i = \dfrac{\partial}{\partial x_i}$, $E_j = \dfrac{\partial}{\partial x_j}$ and $E_k = \dfrac{\partial}{\partial x_k}$. $i, j, k = 1, \ldots, n$.

For any choices of the indices i and j, we can expand $\nabla_{E_i} E_j$ in terms of the same frame.

$$\nabla_{E_i} E_j = \sum_{m=1}^{n} \Gamma_{ij}^m E_m,$$

where $\Gamma_{ij}^m = \dfrac{1}{2} \sum_{k=1}^{n} \{ E_i \langle E_j, E_k \rangle + E_j \langle E_k, E_i \rangle - E_k \langle E_i, E_j \rangle \} g^{km}$ and (from equation (12))

$\langle E_j, E_k \rangle = g_{jk}$, $\langle E_k, E_i \rangle = g_{ki}$ and $\langle E_i, E_j \rangle = g_{ij}$.

But each term of the right hand side vanishes at each point of U, since E_i's are orthonormal at p.

Therefore $\quad \Gamma_{ij}^m = 0$.

Hence $\quad \nabla_{E_i} E_j(p) = 0$.

Using a geodesic frame next we are going to obtain the expressions of the gradient of a function on M as a vector field and divergence of a vector field on M as a function.

2.4.4 Example

Let $X \in \aleph(M)$ and $f \in D(M)$. Define the *divergence* of X as a function div X: $M \to \Re$ given by div $X(p)$ = trace of the linear mapping $Y(p) \to \nabla_Y X(p)$, $p \in M$, and the *gradient* of f as a vector field *grad f* on M defined by $\langle grad\, f(p), v \rangle = df_p(v)$, where $p \in M$ and $v \in T_p M$. Let E_i, $i = 1,\ldots,n$ be a geodesic frame at $p \in M$. We can show that, $grad\, f(p) = \sum_{i=1}^{n}(E_i(f)E_i)(p)$ and div $X(p) = \sum_{i=1}^{n} E_i(f_i)(p)$, where $X = \sum_{i=1}^{n} f_i E_i$.

Proof:

We know that *grad f(p)* is vector field on M at p, therefore *grad f(p)* can be expressed in terms of basis $E_1,\ldots,E_n$ at p.

$$grad\, f(p) = \sum_{i=1}^{n}\langle grad\, f(p), E_i\rangle E_i(p)$$

But $\langle grad\, f(p), E_i\rangle = df_p(E_i) = E_i(f)(p)$

Therefore $grad\, f(p) = \sum_{i=1}^{n}(E_i(f)E_i)(p)$. Hence the result.

For the second part, taking the covariant derivative of X in the direction E_i at p.

$$\nabla_{E_j} X(p) = \nabla_{E_j}\left(\sum_{i=1}^{n} f_i E_i\right)(p)$$

$$= \sum_{i=1}^{n}\nabla_{E_j}(f_i E_i)(p) \qquad \text{(using (ii) of definition 2.2.1)}$$

$$= \sum_{i=1}^{n}(f_i\nabla_{E_j}E_i + E_j(f_i)E_i)(p) \quad \text{(using (iii) of definition 2.2.1)}$$

$$\nabla_{E_j} X(p) = \sum_{i=1}^{n}(E_j(f_i)E_i)(p), \ \text{(since } \nabla_{E_j}E_i(p) = 0 \text{ in a geodesic frame).}$$

Considering $\nabla_{E_j} X(p)$ with $j = 1,\ldots,n$.

$$\nabla_{E_1} X(p) = E_1(f_1)(p)E_1(p) + \ldots\ldots\ldots\ldots\ldots\ldots + E_1(f_n)(p)E_n(p)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\nabla_{E_n} X(p) = E_n(f_1)(p)E_1(p) + \ldots\ldots\ldots\ldots\ldots\ldots + E_n(f_n)(p)E_n(p).$$

Then from the definition of the divergence, div $X(p)$ = trace of this linear mapping

$$\operatorname{div} X(p) = E_1(f_1)(p) + \dots\dots\dots\dots\dots\dots\dots + E_n(f_n)(p)$$

$$\operatorname{div} X(p) = \sum_{i=1}^{n} E_i(f_i)(p)$$. Hence the result.

If $M = \Re^n$, with coordinates $(x_1,\dots,x_n)$ and $\frac{\partial}{\partial x_i} = (0,\dots,1,\dots,0) = e_i$ then we can conclude that $grad\, f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i$ and $\operatorname{div} X(p) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$, where $X = \sum_{i=1}^{n} f_i e_i$.

Since, taking $E_i = e_i = \frac{\partial}{\partial x_i}$ and substituting into above expressions we can get the answer. Again we have the same argument that the *grad f* and the divergence of X on M generalize to the Euclidean space, which are familiar in applied mathematics.

2.4.5 Example

Let M be a Riemannian manifold. Define an operator Δ: $D(M) \rightarrow D(M)$ (the *Laplacian* of M) by

$$\Delta f = \operatorname{div} grad\, f, \qquad f \in D(M).$$

Let E_i be a geodesic frame at $p \in M$, $i = 1,\dots,n = \dim M$.

Then using example 2.4.4, we have $\Delta f(p) = \sum_{i=1}^{n} E_i(E_i(f))(p)$.

Since $$grad\, f(p) = \sum_{i=1}^{n} (E_i(f)E_i)(p) = \sum_{i=1}^{n} E_i(f)(p)E_i(p)$$

So $$\operatorname{div} grad\, f(p) = \sum_{i=1}^{n} E_i(E_i(f))(p) = \Delta f(p).$$

As above example if $M = \Re^n$. Δ coincides with the usual laplacian, namely,

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}. \qquad (\text{since } E_i = \frac{\partial}{\partial x_i})$$

Also we can prove that the following property.

$$\Delta(f\circ g) = f\circ\Delta g + g\circ \Delta f + 2\,\langle grad\, f,\ grad\ g\rangle, \text{ where } g\in D(M)$$

and $f\circ g$ denotes the product of the functions.

Proof:

Using the expression above $\Delta f(p) = \sum_{i=1}^{n} E_i(E_i(f))(p)$ and replacing f by $f\circ g$,

$$\Delta(f\circ g)(p) = \sum_{i=1}^{n} E_i(E_i(f\circ g))(p)\,. \tag{14}$$

Considering the right hand side and using the property of the directional derivative,

$$E_i\{E_i(f\circ g)\}(p) = E_i\{E_i(f(p))\circ g(p) + f(p)\circ E_i(g(p))\}$$

$$= E_i(E_i(f(p)))\circ g(p) + E_i(E_i(g(p)))\circ f(p) + 2E_i(f(p))\circ E_i(g(p))$$

Then substituting into equation (14),

$$\Delta(f\circ g)(p) = \sum_{i=1}^{n}\{E_i(E_i(f(p)))\circ g(p) + E_i(E_i(g(p)))\circ f(p) + 2E_i(f(p))\circ E_i(g(p))\}$$

$$= g(p)\circ \sum_{i=1}^{n} E_i(E_i(f))(p) + f(p)\circ \sum_{i=1}^{n} E_i(E_i(g))(p) +$$

$$2\sum_{i=1}^{n} E_i(f(p))\circ E_i(g(p))$$

Then using $\langle grad\, f(p),\ grad\ g(p)\rangle = \sum_{i=1}^{n} E_i(f(p))\circ E_i(g(p))$,

$$\Delta f(p) = \sum_{i=1}^{n} E_i(E_i(f))(p) \text{ and } \Delta g(p) = \sum_{i=1}^{n} E_i(E_i(g))(p)$$

So $\quad \Delta(f\circ g)(p) = f(p)\circ\Delta g(p) + g(p)\circ\Delta f(p) + 2\langle grad\, f(p),\ grad\ g(p)\rangle.$

$\Rightarrow \quad \Delta(f\circ g) = f\circ\Delta g + g\circ\Delta f + 2\langle grad\, f,\ grad\ g\rangle.$

Chapter 3

Curvature

3.1 Introduction

The notion of curvature in a Riemannin manifold generalizes the notion of Gaussian curvature of a surface, which was introduced by Riemann. He assumed that the curvature of Euclidean space is zero and the curvature of Riemannian manifold is the amount that a Riemannian manifold deviates from Euclidean space and introduced the idea of the curvature in a rather geometric manner as follows.

Using the way of defining the principal curvatures of a surface in $\Re^3$, he introduced the idea of cutting out curves by intersecting our manifold with planes. He thought that the geodesics (curves that are the shortest paths between nearby points) are the best tools for this purpose. The brief method of calculating curvature is given here. Let p be a point of a Riemannian manifold M. The most fundamental fact about geodesics is that given any point $p \in M$ and any vector v tangent to M at p, there is a unique geodesic starting at p with initial tangent vector v. Let σ be a two-dimensional subspace of the tangent space to M at p. Consider all the geodesics through p whose initial vectors lie in the selected plane σ. It turns out that near p these determine a two-dimensional submanifold S_σ of M, which has a Riemannian metric induced from M. Calculate the Gaussian curvature of S_σ at p. This gives a number, denoted by

$K(p,\sigma)$, called the sectional curvature of M at p associated with the plane σ. Again it is a natural generalization of the Gaussian curvature of surfaces. It is clear that if $M = \Re^n$, $K(p,\sigma) = 0$, for all p and σ. But Riemann did not explain a way to calculate the curvature starting with the metric of M, which was done a few years later by Christoffel.

3.2 Curvature

This section presents a definition of the Riemannian curvature tensor as a geometrical object characterizing the deviation of the Riemannian space from Euclidean space. It allows us to compute the sectional curvatures. The behavior of geodesics of Riemannian space is largely determined by its curvature tensor. Similarly to the Gaussian curvature of a surface, the curvature tensor of a Riemannian space M at p, which is a generalization of the Gaussian curvature, determines the properties of the space M in a neighborhood of the point p. Moreover, the curvature tensor gives rich information about the global properties of the Riemannian space and its topology.

3.2.1 Definition

The *curvature tensor* R of a Riemannian manifold M is a correspondence it associates a mapping $R(X,Y): \aleph(M) \rightarrow \aleph(M)$ with each pair of vectors $X, Y \in \aleph(M)$ given by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \qquad Z \in \aleph(M) \qquad (1)$$

where ∇ is the Riemannian connecton of M.

We can show that if M is Euclidean space, $\Re^n$, then $R(X,Y)Z = 0,\ X, Y, Z \in \aleph(M)$

Let $Z = (z_1,\ldots,z_n)$ be the natural coordinates of $\Re^n$, we obtain,

$$\nabla_X Z = (X(z_1),\ldots,X(z_n))$$

Differentiating in the direction of Y,

$$\nabla_Y \nabla_X Z = (YX(z_1),\dots,YX(z_n))$$

Similarly $$\nabla_X \nabla_Y Z = (XY(z_1),\dots,XY(z_n))$$

We know $$\nabla_{[X,Y]} Z = ([X,Y](z_1),\dots,[X,Y](z_n))$$

$$= ((XY - YX)(z_1),\dots,(XY - YX)(z_n)).$$

Substituting into equation (1), $R(X,Y)Z = 0$. Therefore we are now able to think of R as a way of measuring how much M deviates from being Euclidean space.

Now we are going to state the properties of the curvature tensor without giving proofs, (for the proofs see [DC 2]).

3.2.2 Proposition

The curvature tensor R of a Riemannian manifold has the following properties:

(a) R is bilinear in $\aleph(M) \times \aleph(M)$, that is,

$$R(fX_1 + gX_2, Y_1) = fR(X_1, Y_1) + gR(X_2, Y_1),$$

$$R(X_1, fY_1 + gY_2) = fR(X_1, Y_1) + gR(X_1, Y_2),$$

$$f, g \in D(M),\ X_1, X_2, Y_1, Y_2 \in \aleph(M)$$

(b) For any X, $Y \in \aleph(M)$, the curvature operator $R(X,Y) : \aleph(M) \to \aleph(M)$ is linear, that is,

$$R(X,Y)(Z + W) = R(X,Y)Z + R(X,Y)W,$$

$$R(X,Y)fZ = fR(X,Y)Z, \quad f \in D(M), \quad Z, W \in \aleph(M)$$

From now on, we shall write $\langle R(X,Y)Z,T\rangle = R(X,Y,Z,T) = (X,Y,Z,T)$. The curvature tensor on a Riemannian manifold has a number of symmetries.

3.2.3 Proposition (Symmetries of the curvature tensor)

The curvature tensor has the following symmetries for all $X, Y, Z, T \in \aleph(M)$.

(a) $(X,Y,Z,T) = -(Y,X,Z,T)$

(b) $(X,Y,Z,T) = -(X,Y,T,Z)$

(c) $(X,Y,Z,T) = (Z,T,X,Y)$

(d) $(X,Y,Z,T) + (Y,Z,X,T) + (Z,X,Y,T) = 0.$

The symmetry expressed in (d) is called the algebraic *Bianchi identity.*

3.2.4 Remark

It is convenient to express above identities using a coordinate system (U,x) based at the point $p \in M$ in terms of components with respect to any basis.

Let us indicate, as usual, $\dfrac{\partial}{\partial x_i} = X_i$, $i = 1,\ldots,n$.

We put, $R(X_i, X_j)X_k = \sum_{l=1}^{n} R_{ijk}^{l} X_l$. Thus R_{ijk}^{l} are the components of the curvature R.

To express R_{ijk}^{l} in terms of the coefficients Γ_{ij}^{k} of the Riemannian connection,

we write $R(X_i, X_j)X_k = \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{X_i}\nabla_{X_j}X_k + \nabla_{[X_i,X_j]}X_k$

Then $R(X_i, X_j)X_k = \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{X_i}\nabla_{X_j}X_k$ (since $[X_i, X_j] = 0$).

Taking $\nabla_{X_i}X_k = \sum_{l=1}^{n}\Gamma_{ik}^{l}X_l$ and $\nabla_{X_j}X_k = \sum_{l=1}^{n}\Gamma_{jk}^{l}X_l$

$$R(X_i, X_j)X_k = \nabla_{X_j}(\sum_{l=1}^{n}\Gamma_{ik}^{l}X_l) - \nabla_{X_i}(\sum_{l=1}^{n}\Gamma_{jk}^{l}X_l),$$

Then using the linear property of the Riemannian connection,

$$R(X_i, X_j)X_k = \sum_{l=1}^{n}\{\nabla_{X_j}(\Gamma_{ik}^{l} X_l) - \nabla_{X_i}(\Gamma_{jk}^{l} X_l)\}$$

Using (iii) property of definition 2.2.1,

$$= \sum_{l=1}^{n}\{(X_j(\Gamma_{ik}^{l})X_l + \Gamma_{ik}^{l}\nabla_{X_j}X_l) - (X_i(\Gamma_{jk}^{l})X_l + \Gamma_{jk}^{l}\nabla_{X_i}X_l)\}$$

Taking $X_j = \dfrac{\partial}{\partial x_j}$, $X_i = \dfrac{\partial}{\partial x_i}$, $\nabla_{X_j}X_l = \sum_{s=1}^{n}\Gamma_{jl}^{s}X_s$ and $\nabla_{X_i}X_l = \sum_{s=1}^{n}\Gamma_{il}^{s}X_s$

$$= \sum_{l=1}^{n}\{(\frac{\partial}{\partial x_j}(\Gamma_{ik}^{l})X_l + \Gamma_{ik}^{l}\sum_{s=1}^{n}\Gamma_{jl}^{s}X_s) - (\frac{\partial}{\partial x_i}(\Gamma_{jk}^{l})X_l + \Gamma_{jk}^{l}\sum_{s=1}^{n}\Gamma_{il}^{s}X_s)\}$$

Replacing l by s in the first and third terms,

$$R(X_i, X_j)X_k = \sum_{s=1}^{n}\{\frac{\partial}{\partial x_j}(\Gamma_{ik}^{s}) + \sum_{l=1}^{n}\Gamma_{ik}^{l}\Gamma_{jl}^{s} - \frac{\partial}{\partial x_i}(\Gamma_{jk}^{s}) - \sum_{l=1}^{n}\Gamma_{jk}^{l}\Gamma_{il}^{s}\}X_s$$

Taking $R(X_i, X_j)X_k = \sum_{s=1}^{n} R_{ijk}^{s} X_s$,

$$\sum_{s=1}^{n} R_{ijk}^{s} X_s = \sum_{s=1}^{n}\{\frac{\partial}{\partial x_j}(\Gamma_{ik}^{s}) + \sum_{l=1}^{n}\Gamma_{ik}^{l}\Gamma_{jl}^{s} - \frac{\partial}{\partial x_i}(\Gamma_{jk}^{s}) - \sum_{l=1}^{n}\Gamma_{jk}^{l}\Gamma_{il}^{s}\}X_s$$

Since X_s's are linearly independent, then

$$R_{ijk}^{s} = \frac{\partial}{\partial x_j}(\Gamma_{ik}^{s}) + \sum_{l=1}^{n}\Gamma_{ik}^{l}\Gamma_{jl}^{s} - \frac{\partial}{\partial x_i}(\Gamma_{jk}^{s}) - \sum_{l=1}^{n}\Gamma_{jk}^{l}\Gamma_{il}^{s}$$

This gives the components of the curvature tensor in terms of the Christoffel symbols of the connection.

Put $\langle R(X_i, X_j)X_k, X_s\rangle = \langle\sum_{l=1}^{n} R_{ijk}^{l} X_l, X_s\rangle = \sum_{l=1}^{n} R_{ijk}^{l} g_{ls} = R_{ijks}$, where $g_{ls} = \langle X_l, X_s\rangle$.

Therefore, $(X_i, X_j, X_k, X_s) = R_{ijks}$. Then we can write the identities of proposition 3.2.3 as:

(i) $R_{ijks} = -R_{jiks}$

(ii) $R_{ijks} = -R_{ijsk}$

(iii) $R_{ijks} = R_{ksij}$

(iv) $R_{ijks} + R_{jkis} + R_{kijs} = 0$.

3.3 Sectional Curvature

The Riemannian curvature tensor, R, is fairly complicated. Therefore we now define a simple real valued function which completely determines R. Before that we consider the following proposition.

3.3.1 Proposition

Let $\sigma \subset T_pM$ be a two-dimensional subspace of the tangent space T_pM and $x, y \in \sigma$ be two linearly independent vectors. Then $K(x,y) = \dfrac{(x,y,x,y)}{|x \wedge y|^2}$ does not depend on the choice of the vectors $x, y \in \sigma$, where $|x \wedge y|^2 = |x|^2|y|^2 - \langle x, y\rangle^2$.

Proof:

Using the symmetry and linearity properties of the curvature tensor in the right hand side of the above definition, we can replace the basis $\{x, y\}$ of σ from any other basis by iterating the following elementary transformations:

(a) $\{x, y\} \rightarrow \{y, x\}$

(b) $\{x, y\} \rightarrow \{\lambda x, y\}$

(c) $\{x, y\} \to \{x + \lambda y, y\}$

It is easy to see that $K(x, y)$ is invariant by such transformations.

(a)
$$K(y, x) = \frac{(y, x, y, x)}{|y \wedge x|^2}$$

$$= \frac{(x, y, x, y)}{|x \wedge y|^2} \quad \text{(from (a) and (b) proposition 3.2.3)}$$

$$= K(x, y)$$

(b) Consider
$$K(\lambda x, y) = \frac{(\lambda x, y, \lambda x, y)}{|\lambda x \wedge y|^2}$$

$$= \frac{\lambda^2 (x, y, x, y)}{\lambda^2 |x \wedge y|^2} \quad \text{(from (a) and (b) proposition 3.2.2)}$$

$$= K(x, y)$$

Similarly we can prove the other result.

3.3.2 Definition

Given a point $p \in M$ and a two-dimensional subspace $\sigma \subset T_p M$, the real number $K(x, y) = K(p, \sigma)$, where $\{x, y\}$ is any basis of σ, is called the *sectional curvature* of σ at p.

The sectional curvature has interesting geometrical interpretations. The following lemma shows that the sectional curvatures completely determine the curvature tensor.

3.3.3 Lemma

Let V be a vector space of dimension ≥ 2, provided with an inner product $\langle\ ,\ \rangle$. Let $R: V \times V \times V \to V$ and $R': V \times V \times V \to V$ be tri-linear mappings such that conditions (a), (b), (c) and (d) of proposition 3.2.3 are satisfied by

$$(x, y, z, t) = R(x, y, z, t) = \langle R(x, y)z, t \rangle,$$

$$(x, y, z, t)' = R'(x, y, z, t) = \langle R'(x, y)z, t \rangle.$$

If x, y are two linearly independent vectors, we may write,

$$K(\sigma) = \frac{(x, y, x, y)}{|x \wedge y|^2},\ K'(\sigma) = \frac{(x, y, x, y)'}{|x \wedge y|^2},$$ where σ is the two-dimensional subspace generated by x and y.

If for all $\sigma \subset V$, $K(\sigma) = K'(\sigma)$, then $R(x, y, z, t) = R'(x, y, z, t)$. (Proof, see [DC 2])

The Riemannian manifolds that have constant sectional curvature, which means that the sectional curvatures are the same for all planes at all points, played a fundamental role in the development of Riemannian geometry. The following lemma shows how the constant sectional curvature is related to the curvature tensors R and R'.

3.3.4 Lemma

Let M be a Riemannian Manifold and p a point of M. Define a tri-linear mapping $R': T_pM \times T_pM \times T_pM \to T_pM$ by

$$\langle R'(X,Y)W, Z\rangle = \langle X,W\rangle\langle Y,Z\rangle - \langle Y,W\rangle\langle X,Z\rangle, \text{ for all } X, Y, Z, W \in T_pM.$$

Then M has constant sectional curvature equal to K_0 if and only if $R(X,Y,W,Z) = K_0 R'(X,Y,W,Z)$, where R is the curvature of M.

Proof:

Let $K(p,\sigma)$ be the sectional curvature of M at p associated with the two-dimensional subspace $\sigma \subset T_pM$.

Take $K(p,\sigma) = K_0 =$ constant and set $\langle R'(X,Y)W,Z\rangle = (X,Y,W,Z)'$

We can show that R' satisfies the proposition 3.2.3.

We are given that $(X,Y,W,Z)' = \langle X,W\rangle\langle Y,Z\rangle - \langle Y,W\rangle\langle X,Z\rangle$ (2)

Similarly, $(Y,W,X,Z)' = \langle Y,X\rangle\langle W,Z\rangle - \langle W,X\rangle\langle Y,Z\rangle$

$$(W,X,Y,Z)' = \langle W,Y\rangle\langle X,Z\rangle - \langle X,Y\rangle\langle W,Z\rangle$$

Adding all these three equations,

$(X,Y,W,Z)' + (Y,W,X,Z)' + (W,X,Y,Z)' = 0$, this proves the property (d).

$-(Y,X,W,Z)' = -\{\langle Y,W\rangle\langle X,Z\rangle - \langle X,W\rangle\langle Y,Z\rangle\} = (X,Y,W,Z)'$, property (b).

Similarly we can prove the other identities.

Using the equation (2) and replacing W by X and Z by Y,

$$(X,Y,X,Y)' = \langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2$$

$$= |X \wedge Y|^2 \tag{3}$$

We know that the definition of the sectional curvature,

$$K(p,\sigma) = K_0 = \frac{(X,Y,X,Y)}{|X \wedge Y|^2}$$

Therefore $(X,Y,X,Y) = K_0|X \wedge Y|^2$

Then from equation (3), $(X,Y,X,Y) = K_0(X,Y,X,Y)'$

Using lemma 3.3.3, we can say that $(X,Y,W,Z) = K_0(X,Y,W,Z)'$, for all X, Y, W, Z.

That is, $R(X,Y,W,Z) = K_0 R'(X,Y,W,Z)$. Hence the result.

To prove the converse, assume that $R(X,Y,X,Y) = K_0 R'(X,Y,X,Y)$

$$(X,Y,X,Y) = K_0(X,Y,X,Y)'$$

$$= K_0|X \wedge Y|^2 \quad \text{(using (3))}$$

Therefore $K_0 = \dfrac{(X,Y,X,Y)}{|X \wedge Y|^2} = K(p,\sigma)$, which implies that M has constant sectional curvature equal to K_0.

3.3.5 Corollary

Let M be Riemannian manifold of dimension n and p be a point of M. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of T_pM. Define $R_{ijkl} = \langle R(e_i,e_j)e_k, e_l\rangle$, where $i, j, k, l = 1,\dots,n$. Then $K(p,\sigma) = K_0$ for all $\sigma \subset T_pM$, if and only if, $R_{ijkl} = K_0(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. In other words, $K(p,\sigma) = K_0$ for all $\sigma \subset T_pM$ if and only if $R_{ijij} = -R_{ijji} = K_0$, for all $i \neq j$, and $R_{ijkl} = 0$ in other cases.

Proof:

From lemma 3.3.4, $R(X,Y,W,Z) = K_0 R'(X,Y,W,Z)$

Replacing X, Y, W, Z by e_i, e_j, e_k, e_l respectively,

$$R(e_i,e_j,e_k,e_l) = K_0 R'(e_i,e_j,e_k,e_l),$$

where $\quad R'(e_i,e_j,e_k,e_l) = \langle e_i,e_k\rangle\langle e_j,e_l\rangle - \langle e_j,e_k\rangle\langle e_i,e_l\rangle = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.

Therefore $R_{ijkl} = R(e_i, e_j, e_k, e_l) = K_0(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$.

Then $\quad R_{ijij} = K_0 = -R_{ijji}$, if $k = i, l = j$ and for all $i \neq j$

We are now taking a Lie group G as the manifold, and obtaining expression for the curvature and sectional curvature in terms of left invariant vector fields. Before that, we need some preliminary facts about Lie groups.

A *Lie group* is a group G with a differentiable structure such that the mapping $G \times G \to G$ given by $(x, y) \to xy^{-1}, x, y \in G$ is differentiable. It follows that translations from the left L_x and translations from right R_x given by

$$L_x : G \to G, L_x(y) = xy;$$

$$R_x : G \to G, R_x(y) = yx; \text{ are diffeomorphims.}$$

We say that a Riemannian metric on G is left invariant if

$$\langle u, v\rangle_y = \langle d(L_x)_y u, d(L_x)_y v\rangle_{L_x(y)}, \text{ for all } x, y \in G, \quad u, v \in T_y G.$$

That is, a Riemannian metric on G is left invariant if L_x is an isometry. Also we say that a differentiable vector field X on a Lie group G is left invariant if $dL_x X = X$ for all $x \in G$. In other words, if $\langle \ , \ \rangle$ is *left invariant metric tensor* on G, then $\langle X,Y\rangle$ is constant for $X,Y \in \aleph(G)$. (see [BO 2])

Similarly, we can define a right invariant Riemannian metric. A Riemannian metric on G, which is both right and left invariant is, said to be *bi-invarant*. If G has a bi-invariant metric, the inner product that determines on $\aleph(G)$ satisfies the following relation:

For any $U, V, X \in \aleph(G)$, $\quad \langle [U, X], V\rangle = -\langle U, [X, V]\rangle$. (proof see [DC 2])

Using all properties mentioned above, we will prove an important formula in terms of Riemannian connection.

3.3.6 Example

If X, Y, $Z \in \aleph(G)$, left invariant vector fields on G with a bi-invariant metric $\langle\ ,\ \rangle$ then $\nabla_X X = 0$, for all $X \in \aleph(G)$, where ∇ is the Riemannian connection on G.

Proof:

Using remark 2.3.7,

$$2\langle Z, \nabla_Y X\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [X,Z], Y\rangle - \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle$$

Replacing Y by X and using the fact that the metric is left invariant,

$\langle Z, \nabla_X X\rangle = \langle [Z,X], X\rangle$ (inner product is constant so $X\langle Y, Z\rangle = 0$, similarly others)

Using the fact that the bi-invariance of $\langle\ ,\ \rangle$ implies that

$\langle Z, \nabla_X X\rangle = -\langle Z, [X,X]\rangle = 0$ (the property of the bracket, $[X, X] = 0$)

It follows that $\nabla_X X = 0$, for all $X \in \aleph(G)$, since Z is arbitrary vector field on G.

Similarly, $\nabla_Y Y = 0, \nabla_Z Z = 0$, for all $Y, Z \in \aleph(G)$.

3.3.7 Example

Let G be a Lie group with a bi-invariant metric $\langle\ ,\ \rangle$. Let X, Y, $Z \in \aleph(G)$ be unit left invariant vector fields on G. Then we can prove that the following expressions.

(a) $\nabla_X Y = \dfrac{1}{2}[X,Y]$

(b) $R(X,Y)Z = \dfrac{1}{4}[[X,Y],Z]$

(c) If X and Y are orthonormal, the sectional curvature $K(\sigma)$ of G with respect to the plane σ generated by X and Y is given by $K(\sigma) = \dfrac{1}{4}\|[X,Y]\|^2$.

Proof:

(a) $\nabla_Z Z = 0$, for all $Z \in \aleph(G)$.

Let $Z = fX + gY$, where $f, g \in D(G)$.

Then, $\nabla_{(fX+gY)}(fX + gY) = 0$

$\Rightarrow f^2\nabla_X X + fg\nabla_X Y + fg\nabla_Y X + g^2\nabla_Y Y = 0$

$\Rightarrow \nabla_X Y + \nabla_Y X = 0$, since $\nabla_X X = 0, \nabla_Y Y = 0$ and $fg \neq 0$.

$$\Rightarrow \frac{\nabla_X Y + \nabla_Y X}{2} = 0$$

$$\Rightarrow \nabla_X Y - \frac{1}{2}\nabla_X Y + \frac{1}{2}\nabla_Y X = 0$$

$\Rightarrow \nabla_X Y - \frac{1}{2}[X,Y] = 0.$ Hence the result.

(b) $R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$

$= \nabla_Y(\frac{1}{2}[X,Z]) - \nabla_X(\frac{1}{2}[Y,Z]) + \frac{1}{2}[[X,Y],Z]$ (using the result of (a))

$= \frac{1}{4}[Y,[X,Z]] - \frac{1}{4}[X,[Y,Z]] + \frac{1}{2}[[X,Y],Z]$ (using (a))

$= -\frac{1}{4}[[X,Z],Y] + \frac{1}{4}[[Y,Z],X] + \frac{1}{2}[[X,Y],Z]$ (using the property of the bracket, $[Y,[X,Z]] = -[[X,Z],Y]$ and $[X,[Y,Z]] = -[[Y,Z],X]$)

$R(X,Y)Z = \frac{1}{4}[[Z,X],Y] + \frac{1}{4}[[Y,Z],X] + \frac{1}{2}[[X,Y],Z]$ (since $[X,Z] = -[Z,X]$)

Using the Jacobi identity, $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

Then $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$. Hence the result.

(c) We know, $K(\sigma) = \dfrac{\langle R(X,Y)X,Y\rangle}{|X \wedge Y|^2}$

$K(\sigma) = \langle R(X,Y)X,Y\rangle$ (since X and Y are orthonormal, $|X \wedge Y|^2 = 1$.)

$K(\sigma) = \langle \frac{1}{4}[[X,Y],X],Y\rangle$ (using part (b))

But $\langle \nabla_X \nabla_X Y, Y\rangle = \langle \nabla_X (\frac{1}{2}[X,Y]), Y\rangle$ (using part (a))

$= \langle \frac{1}{4}[X,[X,Y]], Y\rangle$ (using part (a))

Using the property of bracket $[X,[X,Y]] = -[[X,Y],X]$,

$$= -\frac{1}{4}\langle [[X,Y],X],Y\rangle \qquad (4)$$

Also we know that $\langle Y,Y\rangle = 1$ (since Y is a unit vector on G)

Taking the covariant derivative in the direction of X,

$X\langle Y,Y\rangle = 0 \Rightarrow \langle \nabla_X Y, Y\rangle = 0.$

Again taking the covariant derivative in the direction of X,

$X\langle \nabla_X Y, Y\rangle = 0, \Rightarrow \langle \nabla_X \nabla_X Y, Y\rangle + \langle \nabla_X Y, \nabla_X Y\rangle = 0$

But $\langle \nabla_X Y, \nabla_X Y\rangle = \frac{1}{4}\langle [X,Y],[X,Y]\rangle = \frac{1}{4}\|[X,Y]\|^2$ (using part (a))

Therefore $\langle \nabla_X \nabla_X Y, Y\rangle = -\frac{1}{4}\|[X,Y]\|^2$

Then, from (4) $\langle [[X,Y],X],Y\rangle = \|[X,Y]\|^2$

$\Rightarrow K(\sigma) = \frac{1}{4}\|[X,Y]\|^2$. Hence the result.

Therefore, the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant metric is non-negative and is zero if and only if σ is generated by vectors X, Y which commute, that is, such that $[X,Y]=0$.

3.4 Tensors on Riemannian Manifolds

An extensive technical theory of Riemannian geometry is built up using tensors; indeed, Riemannian metrics themselves are tensors. The notion of curvature can be expressed in terms of a tensor. The idea of a tensor is a natural generalization of the idea of a vector field. Tensors can be differentiated covariantly as vector fields. Thus we begin with the basic definitions and properties of tensors on a Riemannian manifold.

3.4.1 Definition

A tensor T of order r on a Riemannian manifold, M, is a multilinear mapping

$$T\colon \aleph(M)\times\ldots\ldots\times\aleph(M)\rightarrow D(M).$$

This means that given $Y_1,\ldots,Y_r\in\aleph(M)$, $T(Y_1,\ldots,Y_r)$, is a differentiable function on M, and that T is linear in each argument, that is,

$$T(Y_1,\ldots,fX+gY,\ldots,Y_r)=fT(Y_1,\ldots,X,\ldots,Y_r)+gT(Y_1,\ldots,Y,\ldots,Y_r),$$

for all X, $Y\in\aleph(M)$ and f, $g\in D(M)$.

3.4.2 Example

The *curvature tensor*, R: $\aleph(M)\times\aleph(M)\times\aleph(M)\times\aleph(M)\rightarrow D(M)$, is defined by

$$R(X,Y,Z,W)=\langle R(X,Y)Z,W\rangle,\quad \text{for all } X,Y,Z,W\in\aleph(M).$$

Here R is a tensor of order 4, whose components in the frame $\{X_i = \frac{\partial}{\partial x_i}\}$ associated with the system of coordinates (x_i) are

$$R(X_i, X_j, X_k, X_l) = R_{ijkl}.$$

3.4.3 Example

The metric tensor G: $\aleph(M) \times \aleph(M) \to D(M)$ is defined by $G(X,Y) = \langle X,Y \rangle$, where $X, Y \in \aleph(M)$ and G is a tensor of order 2.

Taking $X = \sum_{i=1}^{n} x_i X_i$ and $Y = \sum_{j=1}^{n} y_j X_j$,

$$G(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j G(X_i, X_j)$$

Therefore using the given definition, it is clear that the components of G in the frame $\{X_i\}$ are the coefficients g_{ij} of the Riemannian metric in the given system of coordinates.

3.4.4 Definition

Let T be a tensor of order r. The *covariant differential* ∇T of T is a tensor of order $(r+1)$ given by

$$\nabla T(Y_1,...,Y_r,Z) = Z(T(Y_1,...,Y_r)) - T(\nabla_Z Y_1,...,Y_r) - ... - T(Y_1,...,Y_{r-1},\nabla_Z Y_r),$$

where $Y_1,...,Y_r,Z \in \aleph(M)$. For each $Z \in \aleph(M)$, the *covariant derivative* of the tensor T relative to Z, $\nabla_Z T$, is a tensor of order r given by $\nabla_Z T(Y_1,...,Y_r) = \nabla T(Y_1,...,Y_r,Z)$.

Using this definition, we will show that the covariant differential of the metric tensor of a Riemannian manifold is the zero tensor.

3.4.5 Example

Let M be a Riemannian manifold and G be the metric tensor defined by $G(X,Y) = \langle X,Y \rangle$. Then the covariant differential of the metric tensor is the zero tensor, for all $X, Y, Z \in \aleph(M)$.

Proof:

G is a tensor of order 2. Therefore by definition ∇G is a tensor of order 3.

Then using the definition of the covariant differential of the metric tensor G, we can write

$$\nabla G(X,Y,Z) = Z(G(X,Y)) - G(\nabla_Z X,Y) - G(X,\nabla_Z Y) \qquad (5)$$

Using the definition of the metric tensor, we have

$$G(X,Y) = \langle X,Y \rangle$$

$$G(\nabla_Z X,Y) = \langle \nabla_Z X,Y \rangle$$

$$G(X,\nabla_Z Y) = \langle X,\nabla_Z Y \rangle, \text{ for all } X, Y, Z, \nabla_Z X, \nabla_Z Y \in \aleph(M).$$

Then substituting into equation (5),

$$\nabla G(X,Y,Z) = Z\langle X,Y \rangle - \langle \nabla_Z X,Y \rangle - \langle X,\nabla_Z Y \rangle$$

But $Z\langle X,Y \rangle = \langle \nabla_Z X,Y \rangle + \langle X,\nabla_Z Y \rangle$, since ∇ is the Riemannian connection.

$$\nabla G(X,Y,Z) = \langle \nabla_Z X,Y \rangle + \langle X,\nabla_Z Y \rangle - \langle \nabla_Z X,Y \rangle - \langle X,\nabla_Z Y \rangle = 0.$$

Hence the result.

3.4.6 Remark

Now we will consider a vector field as a special case of a tensor and show that the covariant derivative of the tensor is a generalization of the covariant derivative of the vector field. For various reasons, it is convenient to identify the vector field $X \in \aleph(M)$ with the tensor $X : \aleph(M) \to D(M)$ given by $X(Y) = \langle X, Y \rangle$, for all $Y \in \aleph(M)$, where M is the Riemannian manifold.

The covariant derivative of the tensor X relative to the vector field $Z \in \aleph(M)$ is

$$\nabla_Z X(Y) = \nabla X(Y, Z) \qquad \text{(from definition 3.4.4)}$$

$$= Z(X(Y)) - X(\nabla_Z Y) \quad \text{(from definition 3.4.4)}$$

$$\nabla_Z X(Y) = Z\langle X, Y \rangle - \langle X, \nabla_Z Y \rangle \qquad (\text{since } X(Y) = \langle X, Y \rangle)$$

Since ∇ is the Riemannian connection

Then $\nabla_Z X(Y) = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle - \langle X, \nabla_Z Y \rangle = \langle \nabla_Z X, Y \rangle$.

Hence we can conclude that the tensor $\nabla_Z X$ can be identified with the vector field $\nabla_Z X$. This shows that the covariant derivative of a tensor is a generalization of the covariant derivative of the vector field.

3.4.7 Examples related to tensors on Riemannian manifolds

3.4.7.1 Example

Let $\gamma : [0, l] \to M$ be a geodesic and let $X \in \aleph(M)$ be such that $X(\gamma(0)) = 0$. Then we can show that $\nabla_{\gamma'} (R(\gamma', X)\gamma')(0) = (R(\gamma', X')\gamma')(0)$, where $X' = \dfrac{DX}{dt}$.

***Proof*:**

As defined in example 3.4.2, R is a tensor of order 4. The covariant differential ∇R of R is a tensor of order 5 given by

$$\nabla R(X,Y,Z,W,U) = U(R(X,Y,Z,W)) - R(\nabla_U X,Y,Z,W) - R(X,\nabla_U Y,Z,W) -$$
$$R(X,Y,\nabla_U Z,W) - R(X,Y,Z,\nabla_U W)$$

For each $U \in \aleph(M)$, the covariant derivative $\nabla_U R$ of R relative to U is a tensor of order 4 given by

$\nabla_U R(X,Y,Z,W) = \nabla R(X, Y, Z, W, U)$ (using definition 3.4.4)

Taking $X = \gamma'$, $Y = X$, $Z = \gamma'$, $W = Z$, and $U = \gamma'$,

So $\nabla_{\gamma'} R(\gamma', X, \gamma', Z) = \nabla R(\gamma', X, \gamma', Z, \gamma')$

$$= \gamma'(R(\gamma', X, \gamma', Z)) - R(\nabla_{\gamma'}\gamma', X, \gamma', Z) - R(\gamma', \nabla_{\gamma'} X, \gamma', Z) -$$

$$R(\gamma', X, \nabla_{\gamma'}\gamma', Z) - R(\gamma', X, \gamma', \nabla_{\gamma'} Z)$$

But we know that γ is a geodesic, therefore $\nabla_{\gamma'}\gamma' = 0$ so then

$$\nabla_{\gamma'} R(\gamma', X, \gamma', Z) = \gamma'(R(\gamma', X, \gamma', Z) - R(\gamma', \nabla_{\gamma'} X, \gamma', Z) - R(\gamma', X, \gamma', \nabla_{\gamma'} Z) \quad (6)$$

But we have, $\gamma'(R(\gamma', X, \gamma', Z)) = \gamma'\langle R(\gamma', X)\gamma', Z\rangle$ (using tensor notation).

Taking the covariant derivative in the direction of γ',

$$\gamma'(R(\gamma', X, \gamma', Z)) = \langle \nabla_{\gamma'}(R(\gamma', X)\gamma'), Z\rangle + \langle R(\gamma', X)\gamma', \nabla_{\gamma'} Z\rangle.$$

Then substituting into equation (6),

$$\nabla_{\gamma'} R(\gamma', X, \gamma', Z) = \langle \nabla_{\gamma'}(R(\gamma', X)\gamma'), Z\rangle - R(\gamma', \nabla_{\gamma'} X, \gamma', Z) \quad (7)$$

Also we know that $\nabla_{\gamma'} R(\gamma', X, \gamma', Z)$ is a tensor of order 4. This can be written as

$\nabla_{\gamma'} R(\gamma', X, \gamma', Z) = \nabla_{\gamma'} R(\gamma', Z, \gamma', X)$ (using (c) of proposition 3.2.3).

$$= \langle \nabla_{\gamma'} R(\gamma', Z)\gamma', X\rangle \text{ (using tensor notation).}$$

Evaluating at $t = 0$, $\nabla_{\gamma'} R(\gamma', X, \gamma', Z)(0) = 0$. Since, we are given that $X(\gamma(0)) = 0$. Therefore the left-hand side of the equation is zero at $t = 0$.

Substituting into equation (7) and evaluating at $t = 0$,

$$\langle \nabla_{\gamma'} (R(\gamma', X)\gamma'), Z\rangle(0) - R(\gamma', \nabla_{\gamma'} X, \gamma', Z)(0) = 0$$

$$\langle \nabla_{\gamma'} (R(\gamma', X)\gamma'), Z\rangle(0) - \langle R(\gamma', \nabla_{\gamma'} X)\gamma', Z\rangle(0) = 0$$

$$\langle \nabla_{\gamma'} (R(\gamma', X)\gamma'), Z\rangle(0) = \langle R(\gamma', \nabla_{\gamma'} X)\gamma', Z\rangle(0)$$

$$\nabla_{\gamma'} (R(\gamma', X)\gamma')(0) = (R(\gamma', \nabla_{\gamma'} X)\gamma')(0), \qquad \text{for all } Z \in \aleph(M)$$

$\nabla_{\gamma'} (R(\gamma', X)\gamma')(0) = (R(\gamma', X')\gamma')(0)$, since $X' = \dfrac{DX}{dt} = \nabla_{\gamma'} X$. Hence the result.

3.4.7.2 Example (Locally symmetric spaces)

Let M be a Riemannian manifold. M is locally symmetric space if $\nabla R = 0$, where R is the curvature tensor of M. (The geometric significance of this condition is given in exercise 14 of chapter (8) of [DC 2].) We can prove that the following properties in locally symmetric space using tensors.

(a) Let $\gamma\colon [0, l) \to M$ be a geodesic of M. Let X, Y, Z be parallel vector fields along γ. Then $R(X, Y)Z$ is parallel field along γ.

(b) If M is locally symmetric connected and has dimension two, then M has constant sectional curvature.

(c) If M has constant sectional curvature then M is locally symmetric space.

***Proof*:**

(a) Take $X = X(\gamma(t))$, $Y = Y(\gamma(t))$, and $Z = Z(\gamma(t))$

Then $\nabla_{\gamma'} X = 0, \nabla_{\gamma'} Y = 0, \nabla_{\gamma'} Z = 0$, since X, Y, Z are parallel vector fields along γ.

We need to prove that $\dfrac{D}{dt}(R(X,Y)Z) = 0$ or $\nabla_{\gamma'} (R(X,Y)Z) = 0$.

Let $U(\gamma(t))$ be any vector field along γ.

$$\nabla R(X, Y, Z, U, \gamma'(t)) = \gamma'(R(X, Y, Z, U)) - R(\nabla_{\gamma'} X, Y, Z, U) - R(X, \nabla_{\gamma'} Y, Z, U) -$$

$$R(X, Y, \nabla_{\gamma'} Z, U) - R(X, Y, Z, \nabla_{\gamma'} U) \qquad (8)$$

Using the fact that X, Y and Z are parallel vector fields, $\nabla_{\gamma'} X = 0, \nabla_{\gamma'} Y = 0, \nabla_{\gamma'} Z = 0$.

Consider $R(\nabla_{\gamma'} X, Y, Z, U) = -R(Y, \nabla_{\gamma'} X, Z, U)$ (using (a) of proposition 3.2.3)

$= -R(Z, U, Y, \nabla_{\gamma'} X)$ (using (c) of proposition 3.2.3)

$= -\langle R(Z,U)Y, \nabla_{\gamma'} X\rangle = 0$ (since $\nabla_{\gamma'} X = 0$)

Similarly we can show that $R(X, \nabla_{\gamma'} Y, Z, U) = 0$ and $R(X, Y, \nabla_{\gamma'} Z, U) = 0$.

Then from equation (8)

$$\nabla R(X, Y, Z, U, \gamma'(t)) = \gamma'(R(X, Y, Z, U)) - R(X, Y, Z, \nabla_{\gamma'} U)$$

$= \gamma'\langle R(X,Y)Z, U\rangle - \langle R(X,Y)Z, \nabla_{\gamma'} U\rangle$ (using tensor notation)

$= \langle \nabla_{\gamma'} R(X,Y)Z, U\rangle + \langle R(X,Y)Z, \nabla_{\gamma'} U\rangle - \langle R(X,Y)Z, \nabla_{\gamma'} U\rangle$

$= \langle \nabla_{\gamma'} R(X,Y)Z, U\rangle \qquad (9)$

We know that if M is locally symmetric space then $\nabla R = 0$.

Then from (9), $\langle \nabla_{\gamma'} R(X,Y)Z, U\rangle = 0$

Which implies that $\nabla_{\gamma'} R(X,Y)Z = 0$, for all $U \in \aleph(M)$.

Hence $R(X,Y)Z$ is a parallel field along γ.

(b) Let $p \in M$ and $\gamma(0) = p$. Assume that M has dimension two. Then T_pM has dimension 2.

Let $\{e_1, e_2\}$ be a orthonormal basis of T_pM and $e_i(t)$ be the parallel transport of e_i along γ, for $i = 1,2$. Therefore $\nabla_{\gamma'} e_1(t) = 0, \nabla_{\gamma'} e_2(t) = 0$

Using definition 3.4.4 and the fact that $e_1(t), e_2(t)$ are parallel vector fields,

$$\nabla R(e_1(t), e_2(t), e_1(t), e_2(t), \gamma'(t)) = \gamma'(t)(R(e_1(t), e_2(t), e_1(t), e_2(t)))$$

$$= \gamma'(t)\langle R(e_1(t), e_2(t))e_1(t), e_2(t)\rangle \quad \text{(tensor notation)}$$

$$\nabla R(e_1(t), e_2(t), e_1(t), e_2(t), \gamma'(t)) = \langle \nabla_{\gamma'}(R(e_1(t), e_2(t))e_1(t), e_2(t)\rangle + \langle R(e_1(t), e_2(t))e_1(t), \nabla_{\gamma'} e_2(t)\rangle$$

$$= \langle \nabla_{\gamma'}(R(e_1(t), e_2(t))e_1(t), e_2(t)\rangle \quad (\text{since } \nabla_{\gamma'} e_2(t) = 0)$$

But M is locally symmetric space, then $\nabla R(e_1(t), e_2(t), e_1(t), e_2(t), \gamma'(t)) = 0$

Therefore $\langle \nabla_{\gamma'}(R(e_1(t), e_2(t))e_1(t), e_2(t)\rangle = 0$ (10)

Let $K(t)$ be the sectional curvature of M at $\gamma(t)$.

$$K(t) = \frac{R(e_1(t), e_2(t), e_1(t), e_2(t))}{|e_1(t) \wedge e_2(t)|^2}$$

Then $$K(t) = R(e_1(t), e_2(t), e_1(t), e_2(t)) \qquad (\text{since } |e_1(t)| = 1, |e_2(t)| = 1)$$

$$= \langle R(e_1(t), e_2(t))e_1(t), e_2(t)\rangle \quad \text{(using tensor notation)}$$

Differentiating both sides with respect to t,

$$K'(t) = \langle \nabla_{\gamma'}(R(e_1(t), e_2(t))e_1(t), e_2(t)\rangle + \langle R(e_1(t), e_2(t))e_1(t), \nabla_{\gamma'} e_2(t)\rangle$$

Using equation (10) and $\nabla_{\gamma'} e_2(t) = 0$

So $K'(t) = 0$. Thus $K(t)$ is constant. Hence the result.

(c) If M has constant sectional curvature K_0 then using lemma 3.3.4,

$$R(X, Y, W, Z) = \langle R(X, Y)W, Z\rangle = K_0(\langle X, W\rangle\langle Y, Z\rangle - \langle Y, W\rangle\langle X, Z\rangle) \qquad (11)$$

Take $\nabla R(X, Y, W, Z, T) = T(R(X, Y, W, Z)) - R(\nabla_T X, Y, W, Z) - R(X, \nabla_T Y, W, Z) -$

$$R(X, Y, \nabla_T W, Z) - R(X, Y, W, \nabla_T Z) \text{ (from definition 3.4.4) (12)}$$

Consider $T(R(X, Y, W, Z)) = T(K_0(\langle X, W\rangle\langle Y, Z\rangle - \langle Y, W\rangle\langle X, Z\rangle))$ (from (11))

$$T(R(X, Y, W, Z)) = K_0\{T(\langle X, W\rangle\langle Y, Z\rangle) - T(\langle Y, W\rangle\langle X, Z\rangle)\}$$

$$T(R(X,Y,W,Z)) = K_0\{(\langle \nabla_T X, W\rangle + \langle X, \nabla_T W\rangle)\langle Y, Z\rangle + \langle X, W\rangle(\langle \nabla_T Y, Z\rangle\langle Y, \nabla_T Z\rangle)$$

$$-(\langle \nabla_T Y, W\rangle + \langle Y, \nabla_T W\rangle)\langle X, Z\rangle - \langle Y, W\rangle(\langle \nabla_T X, Z\rangle + \langle X, \nabla_T Z\rangle)\}$$

Using (11), we can write

$$R(\nabla_T X, Y, W, Z) = \langle R(\nabla_T X, Y)W, Z\rangle = K_0(\langle \nabla_T X, W\rangle\langle Y, Z\rangle - \langle Y, W\rangle\langle \nabla_T X, Z\rangle)$$

$$R(X, \nabla_T Y, W, Z) = \langle R(X, \nabla_T Y)W, Z\rangle = K_0(\langle X, W\rangle\langle \nabla_T Y, Z\rangle - \langle \nabla_T Y, W\rangle\langle X, Z\rangle)$$

$$R(X, Y, \nabla_T W, Z) = \langle R(X,Y)\nabla_T W, Z\rangle = K_0(\langle X, \nabla_T W\rangle\langle Y, Z\rangle - \langle Y, \nabla_T W\rangle\langle X, Z\rangle)$$

$$R(X, Y, W, \nabla_T Z) = \langle R(X,Y)W, \nabla_T Z\rangle = K_0(\langle X, W\rangle\langle Y, \nabla_T Z\rangle - \langle Y, W\rangle\langle X, \nabla_T Z\rangle)$$

Substituting all these relations into (12), $\nabla R(X, Y, W, Z, T) = 0$.

Hence if M has constant sectional curvature then M is locally symmetric space.

Chapter 4

Jacobi Fields

4.1 Introduction

A good part of the study of Riemannian geometry consists of understanding the relationship between geodesics and curvature. A basic tool for this is Jacobi fields which are vector fields along geodesics on manifolds. Before defining Jacobi fields, we need to study the collective behavior of geodesics. For this, we introduce the exponential map of an open set in the tangent bundle to the manifold, which is a way of collecting all of the geodesics of the manifold into a unique differentiable mapping. The exponential map provides a map from the tangent space of any given point of the manifold to the manifold itself, in which lines spreading from the origin of the tangent space are mapped to geodesics in the manifold. The properties of the exponential map are useful to the further study of Riemannian geometry. Using the exponential map, we next introduce Jacobi fields, which are vector fields along geodesics, defined by means of a differential equation. We then introduce the notion of conjugate points, which are pairs of points along a geodesic where some Jacobi field vanishes.

4.2 The Exponential Map

4.2.1 Definition

Let M be a smooth Riemannian manifold. The exponential map, $\exp_p$, at a point p in M maps the tangent space T_pM into M by sending a vector v in T_pM to the point in M a distance $|v|$ along the geodesic from p in the direction v.

That is, $exp_p : T_pM \to M$ is defined by $\exp_p(v) = \gamma_v(1)$, where γ_v denotes the unique geodesic of M with initial velocity v.

We also write $\exp_p(v)$ as $\exp_p(v) = \exp(p,v) = \gamma(1,p,v)$, for fixed p, where (p,v) is a point of TM(tangent bundle). Then as (p,v) is a point of TM, the change of notation to $\exp(p,v)$ shows $\exp$ as a mapping from a region of TM to M.

Using the fact that the homogeneity of a geodesic, that is, if the geodesic $\gamma(t, p, u)$ is defined on the interval $(-\delta, \delta)$, $\delta > 0$, then the geodesic $\gamma(t, p, au)$, $a \in \Re$, $a > 0$, is defined on the interval $(\frac{-\delta}{a}, \frac{\delta}{a})$ and $\gamma(at, p, u) = \gamma(t, p, au)$, and taking $t = 1$, $a = |v|$ and $u = \frac{v}{|v|}$, we can write, $\gamma(|v|, p, \frac{v}{|v|}) = \gamma(1, p, v)$. Roughly, it says that the point, one unit along the geodesic through p with initial velocity v is also the point, $|v|$ units along the geodesic through p with initial velocity $\frac{v}{|v|}$. Therefore the definition of the exponential map depends on the existence and uniqueness of a geodesic through p with initial velocity v. One consequence of the homogeneity condition is that $\gamma(1,p,v)$ is only guaranteed to be well defined for v near zero in T_pM, so $\exp$ is only defined on an open subset of T_pM.

Let $U \subset TM$ be an open set. Then the map $\exp : U \to M$ given by $\exp(p,v) = \exp_p(v) = \gamma(1,p,v) = \gamma(|v|, p, \frac{v}{|v|})$, where $(p,v) \in U$, is called the *exponential map* on U.

Geometrically, $\exp_p(v)$ is a point of M obtained by going out the length equal to $|v|$, starting from p, along a geodesic which passes through p with velocity equal to $\frac{v}{|v|}$. If $v = 0$ then $\exp_p(0) = p$. In most of the applications, we shall use the restriction of exp to an open subset of the tangent space T_pM.

$$\exp_p : B_\varepsilon(0) \subset T_pM \to M,$$

where $B_\varepsilon(0)$ is an open ball with center at the origin 0 of T_pM and of radius ε.

4.2.2 Remark

Suppose that there exists a unique geodesic γ: $[0,1] \to M$ such that $\gamma(0) = p$, $\gamma'(0) = v$.

Then the point $\gamma(1) \in M$ will be denoted by $\exp_p(v)$. The geodesic can be described by the following formula.

$$\gamma(1) = \exp_p(v)$$

$$= \gamma(1, p, v),$$

and $\gamma(t) = \gamma(t, p, v)$

$$= \gamma(1, p, tv) \quad \text{(using the homogeneity of a geodesic)}$$

So $\gamma(t) = \exp_p(tv)$

This is the parametric equation for the unique geodesic with $\gamma(0) = p$, $\gamma'(0) = v$.

This implies that $\gamma(0) = \exp_p(0) = p$ and $\gamma(1) = \exp_p(v)$.

4.2.3 Proposition

Given $p \in M$, there exists an $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0) \subset T_pM \rightarrow M$, is a diffeomorphism of $B_\varepsilon(0)$ onto an open subset of M.

Proof:

Let $(d\exp_p)_0$ be the differential of the function $\exp_p$ evaluated at $t = 0$. Now we are going to show that $(d\exp_p)_0$ is the identity map.

$$
\begin{aligned}
(d\exp_p)_0(v) &= \frac{d}{dt}(\exp_p(tv))\Big|_{t=0} \\
&= \frac{d}{dt}(\gamma(t, p, v))\Big|_{t=0} \\
&= \gamma'(t, p, v)\Big|_{t=0} \\
&= \gamma'(0) \\
&= v.
\end{aligned}
$$

Hence $(d\exp_p)_0$ is the identity map of T_pM, and it follows from the inverse function theorem [DC 1, page 131] that $\exp_p$ is a local diffeomorphism on a neighborhood of 0.

We shall use the following notation.

$$\exp_p : T_pM \rightarrow M$$

$$(d\exp_p)_v : T_v(T_pM) \rightarrow T_{\exp_p v}(M).$$

Where $v \in T_pM$ and $T_v(T_pM)$ is a tangent space of T_pM at (p,v). If $w \in T_v(T_pM)$ then $(d\exp_p)_v w \in T_{\exp_p v}(M)$.

4.3 The Jacobi Equation

Let M be a Riemannian manifold and let $p \in M$. Let $s \to v(s)$, $s \in (-\varepsilon, \varepsilon)$, be a parameterized curve in T_pM such that $v(0) = v$ and $v'(0) = w$.

Consider the parameterized surface,

$$f: A \to M, \quad A = \{ (t,s) ; 0 \le t \le 1, \ -\varepsilon < s < \varepsilon \} \text{ given by}$$

$$f(t, s) = \exp_p tv(s) .$$

The mapping f is differentiable, and the curves $t \to f_s(t) = f(t,s) = \exp_p tv(s)$ are geodesics, where $f_s(t)$ (= $\exp_p tv$) are functions of t (each fixed s).

Consider the curve $s \to tv(s)$ of T_pM and the tangent vector to this curve at $s = 0$ is

$$\frac{\partial}{\partial s}(tv(s))\Big|_{s=0} = tv'(0) = tw \qquad (\text{since } v'(0) = w)$$

Then we can say that $tw \in T_{tv(0)}(T_pM)$. That is, $tw \in T_{tv}(T_pM)$. (since $v(0) = v$)

Therefore $(d\exp_p)_{tv}(tw) \in T_{\exp_p(tv)}M$

That is, $\quad (d\exp_p)_{tv}(tw) \in T_{\gamma(t)}M \qquad$ (since $\gamma(t) = \exp_p(tv)$)

Hence $(d\exp_p)_{tv}(tw)$ is a vector field along the geodesic $\gamma(t) = \exp_p(tv)$, $0 \le t \le 1$.

Consider $\qquad f(t,s) = \exp_p(tv(s))$

Differentiating both sides with respect to s,

$$\frac{\partial f}{\partial s}(t,s) = (d\exp_p)_{tv(s)}\left(\frac{\partial}{\partial s}(tv(s))\right)$$

$$= (d\exp_p)_{tv(s)}(tv'(s))$$

Evaluating at $s = 0$, $\qquad \frac{\partial f}{\partial s}(t,0) = (d\exp_p)_{tv(0)}(tv'(0))$

$$\frac{\partial f}{\partial s}(t,0) = (d\exp_p)_{tv}(tw) \text{ (since } v'(0) = w \text{ and } v(0) = v)$$

Therefore $\frac{\partial f}{\partial s}(t,0)$ is a vector field along $\gamma(t)$ and which is the tangent vector to the curve $f_t(s)$ at $s = 0$, where $f_t(s)$ $(= \exp_p tv(s))$ are functions of s (each fixed t).

Let
$$\frac{\partial f}{\partial s}(t,0) = (d\exp_p)_{tv}(tw) = J(t). \tag{1}$$

It can be shown that $J(t)$ satisfies the Jacobi equation $\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0.$

Since γ is a geodesic, $\frac{D}{\partial t}\frac{\partial f}{\partial t} = 0$, for all (t,s), where $\gamma(t) = f_s(t) = \exp_p(tv)$.

Differentiating with respect to s,

$$\frac{D}{\partial s}\left(\frac{D}{\partial t}\frac{\partial f}{\partial t}\right) = 0, \tag{2}$$

Using the following lemma, we can show that this result can be linked with the curvature R.

Lemma ([DC 2], page 98)

Let f: $A \subset \Re^2 \to M$ be a parameterized surface and let (s,t) be the usual coordinates of $\Re^2$. Let $V = V(s,t)$ be a vector field along f. For each (s,t), it is possible to define $R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V$ as follows.

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V$$

Now we replace V by $\frac{\partial f}{\partial t}$,

$$\frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial t} - \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial f}{\partial t} = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}$$

$$\frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial t} = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} \text{ (using equation (2))} \tag{3}$$

Again considering the following lemma ([DC 2] page 68), which says that if M is a differentiable manifold with a symmetric connection and f: $A \subset \Re^2 \to M$ is a parameterized surface then:

$$\frac{D}{\partial s}\frac{\partial f}{\partial t} = \frac{D}{\partial t}\frac{\partial f}{\partial s}.$$

Substituting into (3),

$$\frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial f}{\partial s} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} = 0$$

$$\frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t} = 0, \text{ (using (a) of proposition 3.2.3)}$$

Evaluating at $(t,0)$, $\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0.$ (4)

Since $\gamma(t) = \exp_p(tv)$

$= \exp_p(tv(0))$ (since $v(0) = v$)

$= f(t,0)$

Then $\gamma'(t) = \frac{\partial f}{\partial t}(t,0)$

Equation (4) is called the *Jacobi equation* and $J(t)$ is an example of a Jacobi field.

4.3.1 Definition

Let γ: $[0,a] \to M$ be a geodesic in M. A vector field J along γ is said to be a *Jacobi field* if it satisfies the Jacobi equation (4), for all $t \in [0,a]$.

4.3.2 Remark

We can show that there are two trivial Jacobi fields along any geodesic which are $\gamma'(t)$ and $t\gamma'(t)$.

Proof:

Let $J(t) = \gamma'(t)$. Then $\frac{DJ}{dt} = \frac{D}{dt}(\gamma'(t)) = 0$. (since γ is a geodesic)

Also $R(\gamma'(t), J(t))\gamma'(t) = R(\gamma'(t), \gamma'(t))\gamma'(t) = 0$

Hence $\gamma'(t)$ satisfies (4). Therefore $\gamma'(t)$ is a Jacobi field. Similarly taking $J(t) = t\gamma'(t)$, we can show that $t\gamma'(t)$ is also a Jacobi field. The first field $(\gamma'(t))$ has zero derivative and is not equal to zero for all t. The second field $(t\gamma'(t))$ is zero if and only if $t = 0$ and $t\gamma'(t) \neq 0$ for all $t \neq 0$. Therefore for the second field we cannot consider the case where Jacobi fields vanish for $t \neq 0$. In order to avoid these facts, we shall consider Jacobi fields along γ that are normal to $\gamma'(t)$.

That is, $\langle J(t), \gamma'(t)\rangle = 0$.

We can also show that $J(0) = 0$.

From equation (1), $J(t) = (d\exp_p)_{tv} tw$

So $J(0) = (d\exp_p)_0 0 = 0$. (from, proposition 4.2.3, $(d\exp_p)_0$ is the identity map)

4.3.3 Example

Let γ: $[0,a] \to M$ be a geodesic in M and $J(t)$ be a Jacobi field along γ with $J(0) = 0$, $\langle J'(0), \gamma'(t)\rangle = 0$. Then $\langle J(t), \gamma'(t)\rangle = 0$ for all $t \in [0,a]$.

Proof:

Since $\gamma(t)$ is a geodesic, $\gamma''(t) = 0$, where $\gamma''(t) = \dfrac{D}{dt}(\gamma'(t))$.

Consider $\dfrac{d}{dt}\langle J(t), \gamma'(t)\rangle = \langle J(t), \gamma''(t)\rangle + \langle J'(t), \gamma'(t)\rangle$, where $J'(t) = \dfrac{DJ}{dt}$.

Then $\dfrac{d}{dt}\langle J(t), \gamma'(t)\rangle = \langle J'(t), \gamma'(t)\rangle$, (since $\gamma''(t) = 0$) (5)

Similarly, $\dfrac{d}{dt}\langle J'(t), \gamma'(t)\rangle = \langle J''(t), \gamma'(t)\rangle$, where $J''(t) = \dfrac{D^2 J}{dt^2}$. (6)

Since J(t) is a Jacobi field and using the Jacobi equation,

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0.$$

Taking the inner product with $\gamma'(t)$,

$$\langle J''(t), \gamma'(t)\rangle + \langle R(\gamma'(t), J(t))\gamma'(t), \gamma'(t)\rangle = 0$$

Using part (c) of proposition 3.2.3 and definition 3.2.1,

$$\langle R(\gamma'(t), J(t))\gamma'(t), \gamma'(t)\rangle = \langle R(\gamma'(t), \gamma'(t))\gamma'(t), J(t)\rangle = 0$$

Then, $\langle J''(t), \gamma'(t)\rangle = 0$

Using (6), $\dfrac{d}{dt}\langle J'(t), \gamma'(t)\rangle = 0 \Rightarrow \langle J'(t), \gamma'(t)\rangle =$ constant.

But we are given that $\langle J'(0), \gamma'(0)\rangle = 0$. Therefore $\langle J'(t), \gamma'(t)\rangle = 0$.

From (5), $\dfrac{d}{dt}\langle J(t), \gamma'(t)\rangle = 0, \Rightarrow \langle J(t), \gamma'(t)\rangle =$ constant.

But $J(0) = 0$, so $\langle J(0), \gamma'(0)\rangle = 0$,

Therefore, $\langle J(t), \gamma'(t)\rangle = 0$, for all $t \in [0,a]$. Hence the result.

4.3.4 Remark

We can also show that a Jacobi field J along the geodesic, γ, with $J(0) = 0$ can be written as $J(t) = (d\exp_p)_{t\gamma'(0)}(tJ'(0))$, for all $t \in [0, a]$.

Proof:

We have $J(t) = (d\exp_p)_{tv}(tw)$ (from (1))

Then $\dfrac{DJ(t)}{dt} = \dfrac{D}{dt}((d\exp_p)_{tv}(tw))$

$= \dfrac{D}{dt}(t(d\exp_p)_{tv}(w))$ (from linearity of the operator $d\exp_p$)

$= (d\exp_p)_{tv}(w) + t\dfrac{D}{dt}((d\exp_p)_{tv}(w))$

Evaluating at $t = 0$, $\dfrac{DJ(0)}{dt} = (d\exp_p)_0(w) = w$ (since $(d\exp_p)_0$ is the identity map)

$\Rightarrow J'(0) = w$.

Also we know $\gamma(t) = \exp_p(tv)$, where $\gamma'(0) = v$. (from remark 4.2.2)

Substituting into (1), $J(t) = (d\exp_p)_{t\gamma'(0)}(tJ'(0))$. Hence the result.

4.4 Jacobi Fields on Manifolds of Constant Curvature

4.4.1 Example

Let M be a Riemannian manifold of constant sectional curvature K, and let $\gamma : [0, l] \to M$ be a normalized geodesic on M and J be a Jacobi field along γ, normal to $\gamma'(t)$. We can show that the Jacobi equation can be written as $\dfrac{D^2J}{dt^2} + KJ = 0$ and

$$\begin{aligned} J(t) &= \sin(t\sqrt{K})w(t)/\sqrt{K}, && \text{if } K > 0, \\ &= tw(t), && \text{if } K = 0, \\ &= \sinh(t\sqrt{-K})w(t)/\sqrt{-K}, && \text{if } K < 0, \end{aligned}$$

is a solution of the Jacobi equation with initial conditions $J(0) = 0$, $J'(0) = w(0)$, where $w(t)$ is a parallel field along γ with $\langle \gamma'(t), w(t) \rangle = 0$ and $|w(t)| = 1$.

Proof:

Using lemma 3.3.4 and replacing X by $\gamma'(t)$, Y by $J(t)$, W by $\gamma'(t)$ and Z by T, where T is a arbitrary vector field along γ.

$\langle R(\gamma'(t), J(t))\gamma'(t), T\rangle = K\{\langle\gamma'(t),\gamma'(t)\rangle\langle J(t),T\rangle - \langle J(t),\gamma'(t)\rangle\langle\gamma'(t),T\rangle\}$, where R is the curvature tensor of M.

$\langle R(\gamma'(t), J(t))\gamma'(t), T\rangle = K\langle J(t), T\rangle$, since $\langle\gamma'(t),\gamma'(t)\rangle = 1$ and $\langle J(t),\gamma'(t)\rangle = 0$

Therefore $R(\gamma'(t), J(t))\gamma'(t) = KJ(t)$, for all vector fields T along γ.

Then from the Jacobi equation, $\dfrac{D^2 J(t)}{dt^2} + KJ(t) = 0$. Then it is easy to verify that above given solutions do satisfy this differential equation.

4.5 Conjugate Points

4.5.1 Definition

If γ is a geodesic segment joining p, $q \in M$, q is said to be *conjugate* to p along γ if there exists a Jacobi field J along γ vanishing at p and q but not identically zero. The maximum number of such linearly independent fields is called the *multiplicity* of the conjugate point q. We can observe that if q is conjugate to p then p is conjugate to q.

4.5.2 Example

Let $S^n = \{x \in \Re^n ; |x| = 1\}$.

In this example we assume the fact that the sectional curvatures of S^n are equal to 1. The Jacobi field on S^n given example 4.4.1 is $J(t) = \sin(t)w(t)$. Then $J(0) = J(\pi) = 0$. Therefore, along any geodesic γ which is a great circle of S^n, the antipodal point $\gamma(\pi)$ of $\gamma(0)$ is conjugate to $\gamma(0)$.

4.5.3 Example

Let $b < 0$ and let M be a manifold with constant negative sectional curvature equal to b. Let $\gamma : [0,a] \to M$ be a normalized geodesic, and let $v \in T_{\gamma(0)}(M)$ such that $\langle v, \gamma'(a)\rangle = 0$ and $|v| = 1$. Since M has negative curvature, $\gamma(a)$ is not conjugate to $\gamma(0)$

(see, example 4.5.5). We can show that the Jacobi field J along γ determined by $J(0)=0,\ J(a)=v$ is given by

$J(t)=\dfrac{\sinh(t\sqrt{-b})w(t)}{\sinh(a\sqrt{-b})}$, where $w(t)$ is the parallel transport of the vector

$w(0)=\dfrac{u_0}{|u_0|}$ along γ. $u_0=(d\exp_p)^{-1}_{a\gamma'(0)}(v)$, where u_0 is considered as a vector in

$T_{\gamma(0)}M$ by the identification $T_{\gamma(0)}M \approx T_{a\gamma'(0)}(T_{\gamma(0)}M)$.

Proof:

From example 4.4.1, the Jacobi field J_1 along γ satisfying $J_1(0)=0$,

$$J_1'(0)=w(0)=\frac{u_0}{|u_0|},\ \text{is given by } J_1(t)=\frac{\sinh(t\sqrt{-b})w(t)}{\sqrt{-b}}. \tag{7}$$

Using remark 4.3.4 and replacing t by a,

$$\begin{aligned} J_1(a) &= (d\exp_p)_{a\gamma'(0)}(aw(0)) \\ &= (d\exp_p)_{a\gamma'(0)}\left(\frac{au_0}{|u_0|}\right) \quad (\text{since } w(0)=\frac{u_0}{|u_0|}) \\ &= \frac{a}{|u_0|}(d\exp_p)_{a\gamma'(0)}(u_0) \end{aligned} \tag{8}$$

But we are given that $u_0=(d\exp_p)^{-1}_{a\gamma'(0)}(v) \Rightarrow (d\exp_p)_{a\gamma'(0)}(u_0)=v=J(a)$

From (8), $$J_1(a)=\frac{aJ(a)}{|u_0|}$$

Using the theorem in [PMM], which says that if two points t_0 and t_1 are not conjugate, then for arbitrary vectors $v\in T_{\gamma(t_0)}M$ and $u\in T_{\gamma(t_1)}M$, there exists one and only one Jacobi field, such that $J(t_0)=v, J(t_1)=u$, we can conclude that

$$J_1(t)=\frac{aJ(t)}{|u_0|},\ \text{for all } 0\le t\le a.$$

Substituting into (7),

$$\frac{\sinh(t\sqrt{-b})w(t)}{\sqrt{-b}}=\frac{aJ(t)}{|u_0|}$$

$$\Rightarrow \qquad J(t)=\frac{|u_0|\sinh(t\sqrt{-b})w(t)}{a\sqrt{-b}} \tag{9}$$

But we are given that $J(a) = v \Rightarrow |J(a)| = |v| = 1$

Therefore from (9),

$$|J(a)| = \frac{|u_0| \sinh(a\sqrt{-b}) |w(a)|}{a\sqrt{-b}} = 1$$

$$\Rightarrow \frac{|u_0|}{a} = \frac{\sqrt{-b}}{\sinh(a\sqrt{-b})} \qquad \text{(since } |w(t)| = 1\text{)}$$

Substituting into (9), $J(t) = \dfrac{\sinh(t\sqrt{-b})w(t)}{\sinh(a\sqrt{-b})}$. Hence the result.

4.5.4 Definition

The set of (first) conjugate points to the point $p \in M$, for all the geodesics that start at p, is called the *conjugate locus* of p and denoted by $C(p)$.

For example, on S^n, $C(p) = \{-p\}$, for all p.

4.5.5 Example

Let M be a Riemannian manifold with non positive sectional curvature. Then the conjugate locus $C(p)$ is empty, for all p.

Proof:

Assume that there exists a non trivial Jacobi field, $J(t)$, along the geodesic $\gamma : [0, a] \to M$ with $\gamma(0) = p$, $J(0) = J(a) = 0$. Let $K(p,\sigma)$ be the sectional curvature of M at p with respect to the plane, σ, generated by $\gamma'(t)$ and $J(t)$,

$$\text{where} \qquad K(p,\sigma) = \frac{\langle R(\gamma'(t), J(t))\gamma'(t), J(t)\rangle}{|\gamma'(t) \wedge J(t)|^2} \text{ and } \langle \gamma'(t), J(t)\rangle = 0.$$

But $K(p,\sigma) \le 0$, so $\langle R(\gamma'(t), J(t))\gamma'(t), J(t)\rangle \le 0$ (10)

Using the Jacobi equation and taking the inner product with J,

$$\langle \frac{D^2 J(t)}{dt^2}, J(t)\rangle + \langle R(\gamma'(t), J(t))\gamma'(t), J(t)\rangle = 0$$

Therefore $\langle \dfrac{D^2 J(t)}{dt^2}, J(t)\rangle \ge 0$ (using (10)) (11)

Consider $\dfrac{d}{dt}\langle \dfrac{DJ(t)}{dt}, J(t)\rangle = \langle \dfrac{D^2 J(t)}{dt^2}, J(t)\rangle + \langle \dfrac{DJ(t)}{dt}, \dfrac{DJ(t)}{dt}\rangle$

$$\frac{d}{dt}\langle\frac{DJ(t)}{dt},J(t)\rangle = \langle\frac{D^2J(t)}{dt^2},J(t)\rangle+\left|\frac{DJ(t)}{dt}\right|^2 \geq 0 \qquad \text{(using (11))}$$

$$\Rightarrow \qquad \frac{d}{dt}\langle\frac{DJ(t)}{dt},J(t)\rangle \geq 0$$

Take $T(t)=\langle\frac{DJ(t)}{dt},J(t)\rangle \Rightarrow T(0) = 0,\ T(a) = 0.$ (since $J(0) = J(a) = 0$)

Hence $\frac{dT(t)}{dt}$ cannot be positive for all $t \in [0,a]$.

It means that $T(t)$ = constant for all $t \in [0,a]$.

That is, $T(t) = T(0) = T(a) = 0 \qquad \Rightarrow T(t)=\langle\frac{DJ(t)}{dt},J(t)\rangle = 0.$ (12)

Consider $\frac{d}{dt}\langle J(t),J(t)\rangle = 2\langle\frac{DJ(t)}{dt},J(t)\rangle = 0$ (using (12))

Therefore $\langle J(t), J(t)\rangle$ = constant, $\Rightarrow |J(t)|^2$ = constant.

$\Rightarrow |J(t)|^2 = 0 \qquad$ (since $|J(0)|^2 = |J(a)|^2 = 0$)

$\Rightarrow J(t) = 0, \qquad$ for all $t \in [0,a]$

This contradicts the fact that J is a non zero Jacobi field along the geodesic γ with $J(0) = J(a) = 0$. No non-trivial Jacobi fields. That is, the conjugate locus $C(p)$ is empty.

4.5.6 Example. Jacobi fields and conjugate points on locally symmetric spaces

Let γ: $[0,\infty) \to M$ be a geodesic in a locally symmetric space M and let $v = \gamma'(0)$ be its velocity at $p = \gamma(0)$. Define a linear transformation $K_v : T_pM \to T_pM$ by

$$K_v(x) = R(v,x)v, \quad \text{where } x \in T_pM$$

Then we can prove that the following properties are true in a locally symmetric space.

(a) K_v is self adjoint.

(b) Choosing an orthonormal basis $\{e_1,...,e_n\}$ of T_pM that diagonalizes K_v, that is,

$$K_v(e_i) = \lambda_i e_i, \quad i = 1,\dots,n.$$

and extending the e_i to fields along γ by parallel transport, then

$K_{\gamma'(t)}(e_i(t)) = \lambda_i e_i(t)$, for all t, where λ_i does not depend on t.

(c) Let $J(t) = \sum_{i=1}^{n} x_i(t)e_i(t)$ be a Jacobi field along γ. Then the Jacobi equation is equivalent to the system, $\dfrac{d^2x_i}{dt^2} + \lambda_i x_i = 0$, $\quad i = 1,\ldots,n$.

(d) The conjugate points of p along γ are given by $\gamma(\pi k / \sqrt{\lambda_i})$, where k is a positive integer and λ_i is a positive eigenvalue of K_v.

Proof:

(a) $\quad \langle K_v(x), y\rangle = \langle R(v,x)v, y\rangle$, for all $x, y \in T_pM$.

$= (v, x, v, y)$ (from tensor notation) (13)

and $\langle x, K_v(y)\rangle = \langle x, R(v,y)v\rangle$

$= \langle R(v,y)v, x\rangle$

$= (v, y, v, x)$

$= (v, x, v, y)$ (using part(c) of proposition 3.2.3)

$\langle K_v(x), y\rangle = \langle x, K_v(y)\rangle$ (using (13))

That is, K_v is self adjoint.

(b) Let $\gamma'(t) = \alpha_1(t)e_1(t) + \ldots\ldots\ldots + \alpha_n(t)e_n(t)$, where $\{e_1(t),\ldots,e_n(t)\}$ is an orthonormal basis of $T_{\gamma(t)}M$.

Taking covariant derivative of $\gamma'(t)$ in the direction of $\gamma'(t)$,

$$\frac{D\gamma'(t)}{dt} = \alpha_1'(t)e_1(t) + \ldots\ldots\ldots + \alpha_n'(t)e_n(t)$$

Since $e_i(t)$'s are parallel vector fields so $\dfrac{De_i(t)}{dt} = 0, i = 1,\ldots,n$

$\Rightarrow \alpha_1'(t)e_1(t) + \ldots\ldots\ldots\ldots + \alpha_n'(t)e_n(t) = 0$, (since γ is a geodesic)

$\Rightarrow \alpha_1'(t) = 0,\ldots,\alpha_n'(t) = 0.$ (since $e_i(t)$'s are linearly independent)

$\Rightarrow \alpha_1(t) =$ constant,..., $\alpha_n(t)$ = constant

Take $\alpha_1(t) = \alpha_1,\ldots,\alpha_n(t) = \alpha_n$, where $\alpha_1,\ldots,\alpha_n$ are constants.

Therefore, $\gamma'(t) = \alpha_1 e_1(t) + \ldots\ldots\ldots\ldots + \alpha_n e_n(t)$

Using the definition of $K_v(x) = R(v,x)v$, we can write

$$K_{\gamma'(t)}(e_i(t)) = R(\gamma'(t), e_i(t))\gamma'(t).$$

$$K_{\gamma'(t)}(e_i(t)) = R(\sum_{k=1}^{n}\alpha_k e_k(t), e_i(t))\sum_{j=1}^{n}\alpha_j e_j(t)$$

$$= \sum_{k,j}^{n}\alpha_k\alpha_j R(e_k(t), e_i(t))e_j(t) \quad \text{(using the linearity of } R\text{)}$$

Taking the covariant derivative in the direction of $\gamma'(t)$,

$$\frac{DK_{\gamma'(t)}(e_i(t))}{dt} = \sum_{k,j}^{n}\alpha_k\alpha_j \frac{DR(e_k(t), e_i(t))e_j(t)}{dt}, \text{ (since } \alpha_k, \alpha_j\text{'s are constants)}$$

$= 0$ (since, in a locally symmetric space, from example 3 4.7.2 (a))

Therefore $K_{\gamma'(t)}(e_i(t))$ is a parallel vector field along γ. That is, $K_{\gamma'(t)}(e_i(t))$ is the parallel transport of $K_{\gamma'(0)}(e_i(0))$ along γ.

$\Rightarrow K_{\gamma'(0)}(e_i(0)) = K_v(e_i) = \lambda_i e_i$, where $e_i = e_i(0)$.

Therefore $K_{\gamma'(0)}(e_i(0)) = \lambda_i(0)e_i(0)$

That is, $K_{\gamma'(t)}(e_i(t)) = \lambda_i(t)e_i(t)$, for all t. (since $K_{\gamma'(t)}(e_i(t))$ is a parallel vector field)

Taking the inner product with $e_i(t)$,

$$\langle K_{\gamma'(t)}(e_i(t)), e_i(t)\rangle = \langle \lambda_i(t)e_i(t), e_i(t)\rangle$$

$$= \lambda_i(t)\langle e_i(t), e_i(t)\rangle$$

$$= \lambda_i(t) \qquad (14)$$

Taking the directional derivative of (14) in the direction of $\gamma'(t)$,

$$\gamma'(t)\langle K_{\gamma'(t)}(e_i(t)), e_i(t)\rangle = \gamma'(t)(\lambda_i(t))$$

$\Rightarrow \lambda_i'(t) = 0$ (since $K_{\gamma'(t)}(e_i(t))$ and $e_i(t)$ are parallel fields)

$\Rightarrow \lambda_i(t) =$ constant. That is, $\lambda_i(t)$ does not depend on t.

Therefore $K_{\gamma'(t)}(e_i(t)) = \lambda_i e_i(t)$, for all t. Hence the result.

(c) Take $J(t) = \sum_{i=1}^{n} x_i(t)e_i(t)$ (15)

Taking the covariant derivative in the direction of $\gamma'(t)$,

$$\frac{DJ(t)}{dt} = \sum_{i=1}^{n}\frac{dx_i(t)}{dt}e_i(t) \quad \text{(since } e_i(t)\text{'s are parallel vector fields)}$$

Again taking the covariant derivative in the direction of $\gamma'(t)$,

$$\frac{D^2J(t)}{dt^2}=\sum_{i=1}^{n}\frac{d^2x_i(t)}{dt^2}e_i(t)$$

Consider $R(\gamma'(t),J(t))\gamma'(t)=R(\gamma'(t),\sum_{i=1}^{n}x_i(t)e_i(t))\gamma'(t)$ (using (15))

$$=\sum_{i=1}^{n}x_i(t)R(\gamma'(t),e_i(t))\gamma'(t) \quad \text{(from linearity of } R\text{)}$$

$$=\sum_{i=1}^{n}x_i(t)K_{\gamma'(t)}(e_i(t)) \quad \text{(from the definition of } K_v\text{)}$$

$$=\sum_{i=1}^{n}x_i(t)\lambda_i e_i(t) \quad \text{(from part (b))}$$

Using the Jacobi equation,

$$\sum_{i=1}^{n}\frac{d^2x_i}{dt^2}e_i(t)+\sum_{i=1}^{n}x_i(t)\lambda_i e_i(t)=0$$

$$\sum_{i=1}^{n}\left(\frac{d^2x_i}{dt^2}+\lambda_i x_i(t)\right)e_i(t)=0$$

$\Rightarrow \quad \dfrac{d^2x_i}{dt^2}+\lambda_i x_i(t)=0$, $i=1,\ldots,n$ (since $e_i(t)$'s are linearly independent)

(d) Using $\dfrac{d^2x_i}{dt^2}+\lambda_i x_i(t)=0$,

The auxiliary equation is $r^2+\lambda_i=0$. w

The general solution is $x_i(t)=A\cos(\sqrt{\lambda_i}t)+B\sin(\sqrt{\lambda_i}t)$, where A and B are constants, where $\lambda_i>0$.

When $t=0$, $J(0)=0$. Therefore $x_i(0)=0 \Rightarrow A=0$.

$$x_i(t)=B\sin(\sqrt{\lambda_i}t)$$

Let $\gamma(t_0)$ be the conjugate point of $\gamma(0)$. That is, $J(t_0)=0$, where $t_0\neq 0$.

Therefore $x_i(t_0)=0$ (using (15))

$\Rightarrow \sin(\sqrt{\lambda_i}t_0)=0$ (from the general solution of the auxiliary equation)

$\sqrt{\lambda_i}t_0=k\pi$, $k=1,\ldots,n$, where $k\neq 0$, since if $k=0$ then $t_0=0$.

Thus, $t_0=\dfrac{k\pi}{\sqrt{\lambda_i}}$. So $\gamma(t_0)=\gamma\left(\dfrac{k\pi}{\sqrt{\lambda_i}}\right)$. Hence the result.

Chapter 5

Riemannian Submanifolds

5.1 Introduction

In this chapter, we shall consider the immersion of a manifold M into a Riemannian manifold ($\overline{M}$, $\overline{g}$), and examine the structures on M induced from the given structure on $\overline{M}$. This is a natural generalization of the study of surfaces in Euclidean three-dimensional space with properties induced from the Euclidean metric, which was the origin of the classical theory of surfaces. We first develop the basic concepts of the theory of Riemannian submanifolds and then define a tensor field called the second fundamental form, which measures the way a submanifold curves within the ambient manifold. We next prove the fundamental relationships between the intrinsic and extrinsic geometries of a submanifold: The Gauss formula relates the Riemannian connection on the submanifold to that of the ambient manifold, and the Gauss equation involving the second fundamental form relates their curvatures. Using these facts, we focus on the special case of hypersurfaces in $\Re^{n+1}$, and show how the second fundamental form is related to the principal curvatures and Gaussian curvature. Finally we compute the sectional curvatures of our model Riemannian manifolds- Euclidean spaces, spheres, and hyperbolic spaces.

5.2 The Second Fundamental Form

5.2.1 Definitions

Suppose $(\overline{M}, \overline{g})$ is a Riemannian manifold of dimension $m = n + k$, M is a manifold of dimension n, let $f: M \to \overline{M}$ be an immersion. If M is given the induced Riemannian metric g such that $g(u,v)_p = \overline{g}(df_p(u), df_p(v))_{f(p)}$, for all $p \in M$, and $u, v \in T_pM$, then f is said to be an *isometric immersion* of M into $\overline{M}$.

If in addition f is injective, so that M is an immersed submanifold of $\overline{M}$, then M is said to be a *Riemannian submanifold* of $\overline{M}$. In all of these situations, $\overline{M}$ is called the *ambient manifold.*

At each $p \in M$, the ambient tangent space $T_p\overline{M}$ splits as an orthogonal direct sum

$$T_p\overline{M} = T_pM \oplus (T_pM)^{\perp},$$

where $(T_pM)^{\perp}$ is the normal space at p with respect to the inner product $\overline{g}$ on $T_p\overline{M}$.

If $v \in T_p\overline{M}$, $p \in M$, we can write, $v = v^T + v^N$, where $v^T \in T_pM$ and $v^N \in (T_pM)^{\perp}$.

v^T and v^N are called the *tangential* and *normal* components of v respectively.

Consider the following example.

5.2.2 Example

As defined before, assume that M has the metric induced by f. Let $p \in M$ and $U \subset M$ be a neighborhood of p such that $f(U) \subset \overline{M}$ is a submanifold of $\overline{M}$. Further suppose that X, Y and Z are differentiable vector fields on $f(U)$ which can be extended to differentiable vector fields on an open set of $\overline{M}$.

Define $(\nabla_X Y)(p) =$ tangential component of $(\overline{\nabla}_X Y)(p)$,

where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Then we can prove that ∇ is the Riemannian connection of M.

***Proof*:**

We know that $\overline{\nabla}$ is the Riemannian connection on $\overline{M}$. Therefore from the Levi-Civita Theorem it suffices to show that ∇ is symmetric and compatible with the Riemannian metric induced by f.

To see that ∇ is symmetric, we use the symmetry of $\overline{\nabla}$ and the fact that $[X,Y]$ is tangent to M.

$$\nabla_X Y - \nabla_Y X = (\overline{\nabla}_X Y)^T - (\overline{\nabla}_Y X)^T, \text{ at } p. \text{ (using the given definition)}$$

$$= (\overline{\nabla}_X Y - \overline{\nabla}_Y X)^T, \text{ at } p.$$

$$= [X,Y]^T \quad \text{(using the symmetry of } \overline{\nabla}\text{)}$$

$$= [X,Y] \Rightarrow \nabla \text{ is symmetric.}$$

To prove compatibility, we use the compatibility of $\overline{\nabla}$ and evaluate at points of M

$$X\langle Y,Z\rangle = \langle \overline{\nabla}_X Y, Z\rangle + \langle Y, \overline{\nabla}_X Z\rangle$$

$$= \langle (\overline{\nabla}_X Y)^T, Z\rangle + \langle Y, (\overline{\nabla}_X Z)^T\rangle, \text{ at } p, \text{ where } Y, Z \in T_pM.$$

$$= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \quad \text{(using the given definition)}$$

Therefore ∇ is compatible. Hence ∇ is the Riemannian connection on M.

5.2.3 Remark

Our first main task is to compare the Riemannian connection of M with that of $\overline{M}$. The starting point for doing so is the orthogonal decomposition of sections of the *ambient tangent bundle* over M, $\coprod_{p\in M} T_p\overline{M}$, into tangential and orthogonal components as above.

If X, Y are vector fields in $\aleph(M)$, we can extend them to vector fields on $\overline{M}$, apply the ambient covariant derivative operator $\overline{\nabla}$, and then decompose at points of M to get

$$\overline{\nabla}_X Y = (\overline{\nabla}_X Y)^T + (\overline{\nabla}_X Y)^N. \qquad (1)$$

Next we consider the following definition.

5.2.4 Definition

The *second fundamental tensor form* B is a bilinear symmetric mapping $B: T_pM \times T_pM \to (T_pM)^\perp$ defined by $B(X,Y) = \overline{\nabla}_X Y - \nabla_X Y$, (2)

where $(T_pM)^\perp$ is the normal bundle of M in $\overline{M}$ and $\overline{\nabla}$ and ∇ are the Riemannian connections of M and $\overline{M}$ respectively.

Using equations (1), (2) and the fact that $(\overline{\nabla}_X Y)^T = \nabla_X Y$,

we can also conclude that $B(X,Y) = (\overline{\nabla}_X Y)^N \in \aleph(\overline{M})$

That is, $B(X,Y)$ is a vector field on $\overline{M}$ normal to M. It is easy to prove that $B(X,Y)$ does not depend on the extensions X, Y. Therefore we can use the same letter to denote both a vector field or function on M and its extension to M. It is also easy to show that $B(X,Y)$ is bilinear and symmetric. This formula, which relates the Riemannian connection on the submanifold to that of the ambient manifold is called the *Gauss formula.* Because Gauss first obtained this formula for surfaces embedded in Euclidean space $\Re^3$. Now we are in a position to define the second quadratic form.

5.2.5 Definition

Let $p \in M$ and $\eta \in (T_pM)^{\perp}$. The mapping $H_\eta : T_pM \times T_pM \to \Re$ given by

$H_\eta(x,y) = \langle B(x,y), \eta \rangle$, $x, y \in T_pM$, is a symmetric bilinear form. (since $B(x,y)$ is symmetric and bilinear.) The quadratic form $\mathrm{II}_\eta(x) = H_\eta(x,x)$ is called the *second fundamental form* of f at p along the normal vector η, where f is an isometric immersion of M into $\overline{M}$. We can show that the bilinear mapping H_η is associated to a linear self-adjoint operator $S_\eta : T_pM \to T_pM$ by

$$\langle S_\eta(x), y \rangle = H_\eta(x,y) = \langle B(x.y), \eta \rangle.$$

Since $\langle S_\eta(x), y \rangle = \langle B(x,y), \eta \rangle = \langle B(y,x), \eta \rangle = \langle S_\eta(y), x \rangle$.

Considering the following example, we can show that if S_η is a tensor of order 2 and symmetric then the tensor $\nabla_v S_\eta$ is symmetric, for all $v \in \aleph(M)$.

5.2.6 Example

Let f: $M^n \to \overline{M}^{m=n+k}$ be an isometric immersion and let S_η: $TM \to TM$ be the operator associated with the second fundamental form of f along the normal field η. Consider S_η as a tensor of order 2 given by $S_\eta(X,Y) = \langle S_\eta(X), Y \rangle$, where $X, Y \in \aleph(M)$. We can observe that if the operator S_η is self-adjoint then the tensor S_η is symmetric, that is, $S_\eta(X,Y) = S_\eta(Y,X)$, and the tensor $\nabla_v S_\eta$ is symmetric, for all $v \in \aleph(M)$.

***Proof*:**

Since S_η is self-adjoint, $\langle S_\eta(X), Y\rangle = \langle S_\eta(Y), X\rangle$ (3)

$$S_\eta(X,Y) = S_\eta(Y,X). \text{ That is, } S_\eta \text{ is symmetric.}$$

Differentiating (3) in the direction of v,

$$\langle \nabla_v(S_\eta(X)), Y\rangle + \langle S_\eta(X), \nabla_v Y\rangle = \langle \nabla_v X, S_\eta(Y)\rangle + \langle X, \nabla_v(S_\eta(y))\rangle \quad (4)$$

S_η is a tensor of order 2. Therefore the covariant differential ∇S_η of S_η is a tensor of order 3.

$$\nabla S_\eta(X,Y,v) = v(S_\eta(X, Y)) - S_\eta(\nabla_v X, Y) - S_\eta(X, \nabla_v Y) \quad \text{(from definition 3.4.4)}$$

$$= v\langle S_\eta(X), Y\rangle - \langle S_\eta(\nabla_v X), Y\rangle - \langle S_\eta(X), \nabla_v Y\rangle. \text{ (property of } S_\eta\text{)}$$

Since ∇ is the Riemannian connection on M,

$$\nabla S_\eta(X,Y,v) = \langle \nabla_v(S_\eta(X)), Y\rangle + \langle S_\eta(X), \nabla_v Y\rangle - \langle S_\eta(\nabla_v X), Y\rangle - \langle S_\eta(X), \nabla_v Y\rangle$$

$$= \langle \nabla_v(S_\eta(X)), Y\rangle - \langle S_\eta(\nabla_v X), Y\rangle$$

$$= \langle \nabla_v(S_\eta(X)), Y\rangle - \langle \nabla_v X, S_\eta(Y)\rangle \quad \text{(since } S_\eta \text{ is self-adjoint)}$$

But $\nabla S_\eta(X,Y,v) = (\nabla_v S_\eta)(X,Y)$ (from definition 3.4.4)

$$(\nabla_v S_\eta)(X,Y) = \langle \nabla_v(S_\eta(X)), Y\rangle - \langle \nabla_v X, S_\eta(Y)\rangle$$

Similarly $(\nabla_v S_\eta)(Y,X) = \langle \nabla_v(S_\eta(Y)), X\rangle - \langle \nabla_v Y, S_\eta(X)\rangle$.

Subtracting these two equations and substituting into (4),

$$(\nabla_v S_\eta)(X,Y) = (\nabla_v S_\eta)(Y,X)$$

$$\Rightarrow \quad \nabla_v S_\eta \text{ is symmetric.}$$

The following proposition shows that the linear operator associated with the second fundamental form can be used to evaluate covariant derivatives of normal vector fields.

5.2.7 Proposition

Let $p \in M$, $x \in T_pM$ and $\eta \in (T_pM)^\perp$. Let N be a local extension of η normal to M.

Then $S_\eta(x) = -(\overline{\nabla}_x \eta)^T$

Proof: Let $y \in T_pM$ and X, Y be local extensions of x, y respectively, which are tangent to M.

We have $\langle S_\eta(x), y\rangle = \langle B(x,y), \eta\rangle$ (from definition 5.2.5)

$$= \langle B(X,Y), N\rangle(p)$$

$$= \langle \overline{\nabla}_X Y - \nabla_X Y, N\rangle(p) \quad \text{(using the Gauss formula)}$$

$$= \langle \overline{\nabla}_X Y, N\rangle(p) \quad (\text{since } \nabla_X Y \in \aleph(M) \text{ is normal to } N)$$

We know that $\langle N, Y\rangle = 0$, .

Taking derivative in the direction of X,

$$X\langle N, Y\rangle = 0,$$

$$\langle \overline{\nabla}_X N, Y\rangle + \langle \overline{\nabla}_X Y, N\rangle = 0, \quad (\text{since } \overline{\nabla} \text{ is the Riemannian connection on } \overline{M})$$

Therefore $\quad \langle S_\eta(x), y\rangle = -\langle \overline{\nabla}_X N, Y\rangle(p)$

$$\langle S_\eta(x), y\rangle = -\langle \overline{\nabla}_x\eta, y\rangle = -\langle (\overline{\nabla}_x\eta)^T, y\rangle \quad (\text{since } \langle(\overline{\nabla}_x\eta)^\perp, y\rangle = 0)$$

This is true for all $y \in T_pM$. So $S_\eta(x) = -(\overline{\nabla}_x\eta)^T$

5.2.8 Hypersurfaces in Euclidean Space

Now we specialize the preceding considerations to the case in which the codimension of the immersion is one. That is, f: $M^n \to \overline{M}^{n+1}$; $f(M) \subset \overline{M}$ is called a *hypersurface* in $\overline{M}^{n+1}$. At any point $p \in M$, we have seen that the shape operator S_η is a self-adjoint linear transformation on the tangent space T_pM . From elementary linear algebra, any such operator has real eigenvalues, $\lambda_1,\ldots,\lambda_n$, and there is an orthonormal basis of eigenvectors $\{e_1,\ldots,e_n\}$ of T_pM so that $S_\eta(e_i) = \lambda_i e_i$, $i = 1,\ldots,n$. In this situation, at each point of M there are exactly two unit normal vectors. If M is orientable, we can use an orientation to pick out a unique normal. Let $\eta \in (T_pM)^\perp$, $|\eta| = 1$. Then $\{e_1,\ldots,e_n,\eta\}$ is a basis for $T_p\overline{M}$. The eigenvalues of S_η are called the *principal curvatures* of M at p, and the corresponding unit eigenvectors are called the *principal directions.* $\operatorname{Det}(S_\eta) = \lambda_1\ldots\lambda_n$ is called the *Gauss Kronecker curvature* of M and $(\lambda_1+\ldots+\lambda_n)/n$ is called the *mean curvature* of M.

Next we are going to define a new operator, Hess f, acting on a tangent space of the hypersurface.

5.2.9 Example

Let f: $\overline{M}^{n+1} \to \Re$ be a differentiable function. Define the *Hessian*, Hess f, of f at $p \in \overline{M}$ as the linear operator Hess f : $T_p\overline{M} \to T_p\overline{M}$, (Hess f)Y = $\overline{\nabla}_Y$ (*grad f*), $Y \in T_p\overline{M}$, where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Let a be a regular value of

f and let $M^n \subset \overline{M}^{n+1}$ be the hypersurface in $\overline{M}$ defined by $M = \{p \in \overline{M} : f(p) = a\}$. Then we can prove that the following properties.

(a) The Laplacian Δf is given by

$$\Delta f = \text{trace}(\text{Hess}\, f)$$

(b) If $X, Y \in \aleph(\overline{M})$, then

$$\langle (\text{Hess}\, f)Y, X\rangle = \langle Y, (\text{Hess}\, f)X\rangle$$

(c) The mean curvature H of $M \subset \overline{M}$ is given by

$$nH = -\, \text{div}(grad\, f / |grad\, f|).$$

***Proof*:**

(a) We know from example 2.4.5, $\Delta f = \text{div}\ grad\, f,\ \ f \in D(\overline{M})$, where $grad\, f$ is a vector field on $\overline{M}$ and $D(\overline{M})$ is the set of real valued functions on $\overline{M}$.

Define the divergence of $grad\, f$ as a function div $grad\, f : \overline{M} \to \Re$ given by

div $grad\, f(p)$ = trace of the linear mapping $Y(p) \to \overline{\nabla}_Y\, (grad\, f)(p)$

But we are given that, Hess $f : Y(p) \to \overline{\nabla}_Y\, (grad\, f)(p)$

Therefore div $grad\, f$ = trace(Hess f) $\quad \Rightarrow \quad \Delta f$ = trace(Hess f).

(b) Using the gradient of f as a vector field $grad\, f$ on $\overline{M}$ defined by

$$\langle grad\, f, X\rangle = df(X) = X(f)\ \text{, where } X \in T_p\overline{M}$$

Taking the directional derivative in the direction of Y,

$$Y\langle grad\, f, X\,\rangle = YX(f)$$

$$\langle \overline{\nabla}_Y\, (grad\, f), X\,\rangle + \langle grad\, f,\ \overline{\nabla}_Y X\,\rangle\ = (YX)(f)$$

Interchanging X and Y,

$$\langle \overline{\nabla}_X (grad f), Y \rangle + \langle grad f, \overline{\nabla}_X Y \rangle = (XY)(f)$$

Subtracting these two equations,

$$\langle \overline{\nabla}_Y (grad f), X \rangle - \langle \overline{\nabla}_X (grad f), Y \rangle + \langle grad f, \overline{\nabla}_Y X - \overline{\nabla}_X Y \rangle = (YX)(f) - (XY)(f)$$

$$\langle \overline{\nabla}_Y (grad f), X \rangle - \langle \overline{\nabla}_X (grad f), Y \rangle + \langle grad f, [Y, X] \rangle = [Y, X](f) \qquad (5)$$

And also $[Y, X] \in \aleph(\overline{M})$, then $\langle grad f, [Y, X] \rangle = [Y, X](f)$

Substituting into (5),

$$\langle \overline{\nabla}_Y (grad f), X \rangle = \langle \overline{\nabla}_X (grad f), Y \rangle$$

$$\langle (\text{Hess} f)Y, X \rangle = \langle Y, (\text{Hess} f)X \rangle \Rightarrow \text{Hess} f \text{ is self-adjoint.}$$

We can also show that Hess f determines a symmetric bilinear form on $T_p\overline{M}$, $p \in \overline{M}$

given by $(\text{Hess} f)(X, Y) = \langle (\text{Hess} f)X, Y \rangle$, where $X, Y \in T_p\overline{M}$.

$$= \langle (\text{Hess} f)Y, X \rangle \text{ (since Hess } f \text{ is self-adjoint)}$$

$$= (\text{Hess} f)(Y, X)$$

Therefore Hess f is symmetric.

Consider $(\text{Hess} f)(aX_1 + bX_2, Y_1) = \langle (\text{Hess} f)(aX_1 + bX_2), Y_1 \rangle$, where $a, b \in D(\overline{M})$ and $X_1, X_2, Y_1 \in T_p\overline{M}$.

We know that $(\text{Hess} f)(aX_1 + bX_2) = \overline{\nabla}_{aX_1 + bX_2} (grad f)$

$$= a\overline{\nabla}_{X_1} (grad f) + b\overline{\nabla}_{X_2} (grad f)$$

$$= a(\text{Hess} f)\, X_1 + b(\text{Hess} f)\, X_2$$

Therefore, $(\text{Hess} f)(aX_1 + bX_2, Y_1) = \langle a(\text{Hess} f)X_1 + b(\text{Hess} f)X_2, Y_1\rangle$

$$= a\langle(\text{Hess} f)X_1, Y_1\rangle + b\,\langle(\text{Hess} f)X_2, Y_1\rangle$$

$$= a(\text{Hess} f)(X_1, Y_1) + b(\text{Hess} f)(X_2, Y_1)$$

Similarly, $(\text{Hess} f)(X_1, aY_1 + bY_2) = a(\text{Hess} f)(X_1, Y_1) + b(\text{Hess} f)(X_1, Y_2)$

Therefore Hess f is a symmetric bilinear form on $T_p\overline{M}$.

(c) Take an orthonormal frame $E_1, \ldots, E_n, E_{n+1} = grad\, f / |grad\, f| = \eta$ in a neighborhood of $p \in M$ in $\overline{M}$, where $|grad\, f| \neq 0$.

Since $|grad\, f|^2 = \langle grad\, f(p), grad\, f(p)\rangle = df_p(grad\, f(p)),\ p \in M$.

But $f(p) = a$ and a is a regular value of f, therefore $df_p(grad\, f(p)) \neq 0$.

Consider $S_\eta : T_pM \to T_pM$ as before.

The mean curvature H of $M = \dfrac{1}{n}\sum_{i=1}^{n}\langle S_\eta(E_i), E_i\rangle$

From proposition 5.2.7, $\langle S_\eta(E_i), E_i\rangle = -\langle\overline{\nabla}_{E_i}\eta, E_i\rangle$

$$\text{Therefore } nH = -\sum_{i=1}^{n}\langle\overline{\nabla}_{E_i}\eta, E_i\rangle$$

We have $\langle\eta,\eta\rangle = 1$

Taking the covariant derivative in the direction of E_{i+1},

$$E_{i+1}\langle\eta,\eta\rangle = 0 \Rightarrow \langle\overline{\nabla}_{E_{i+1}}\eta, \eta\rangle = 0$$

Then $$nH = -\sum_{i=1}^{n}\langle\overline{\nabla}_{E_i}\eta, E_i\rangle - \langle\overline{\nabla}_{E_{i+1}}\eta, \eta\rangle$$

$$nH = -\sum_{i=1}^{n+1}\langle \overline{\nabla}_{E_i}\eta, E_i\rangle \text{ (since } \eta = E_{i+1})$$

$$= -\,div_{\overline{M}}\,\eta = -\,div_{\overline{M}}\,(grad\, f/|grad\, f|).$$

A hypersurface with mean curvature identically equal to zero is called *minimal.*

Using the following theorem, we are now going to generalize the famous Theorem Egregium of Gauss, which says that the Gaussian curvature of $M^2 \subset \Re^3$ is an invariant under isometries.

5.2.10 Theorem

Let $p \in M$ and let x, y be orthonormal vectors in T_pM . That is $x, y \in T_pM \subset T_p\overline{M}$. Let $K(x, y)$ and $\overline{K}(x, y)$ be the sectional curvatures of M and $\overline{M}$, respectively, in the plane generated by x and y. Then

$$K(x, y) - \overline{K}(x, y) = \langle B(x,x), B(y,y)\rangle - |B(x,y)|^2$$

***Proof*:**

Let X, Y be local extensions of x and y respectively which are tangent to M.

$K(x, y) = \langle R(X,Y)X, Y\rangle(p)$ (from proposition 3.3.1 and $|x \wedge y|^2 = 1$)

And $\overline{K}(x, y) = \langle \overline{R}(X,Y)X, Y\rangle(p)$

So $K(x, y) - \overline{K}(x, y) = \langle \nabla_Y\nabla_X X - \nabla_X\nabla_Y X - (\overline{\nabla}_Y\overline{\nabla}_X X - \overline{\nabla}_X\overline{\nabla}_Y X), Y\rangle(p) +$

$$\langle \nabla_{[X,Y]}X - \overline{\nabla}_{[X,Y]}X, Y\rangle(p)$$

$$\overline{\nabla}_{[X,Y]}X = (\overline{\nabla}_{[X,Y]}X)^T + (\overline{\nabla}_{[X,Y]}X)^N \qquad \text{(since } \overline{\nabla}_{[X,Y]}X \in T_p\overline{M}\text{)}$$

$$\overline{\nabla}_{[X,Y]}X - \nabla_{[X,Y]}X = (\overline{\nabla}_{[X,Y]}X)^N \qquad \text{(since } (\overline{\nabla}_{[X,Y]}X)^T = \nabla_{[X,Y]}X\text{)}$$

Taking the inner product with Y at p,

$$\langle \overline{\nabla}_{[X,Y]}X - \nabla_{[X,Y]}X\ ,\ Y \rangle = \langle (\overline{\nabla}_{[X,Y]}X)^N\ ,\ Y\rangle = 0$$

Therefore, $K(x, y) - \overline{K}(x, y) = \langle\ \nabla_Y\nabla_X X - \nabla_X\nabla_Y X - \overline{\nabla}_Y\overline{\nabla}_X X + \overline{\nabla}_X\overline{\nabla}_Y X\ ,\ Y\rangle(p)$.

Let $E_1,\ldots,E_k$ be the local orthonormal fields which are normal to M,

where $k = \dim\overline{M} - \dim M$.

$$B(X,Y) = \sum_{i=1}^{k}\langle B(X,Y), E_i\rangle E_i \qquad \text{(since } B(X,Y) \text{ is normal to } M\text{)}$$

$$\langle B(X,Y),\ E_i\rangle = H_{E_i}(X,Y) \quad \text{(using the definition of second fundamental form)}$$

Write $\qquad H_{E_i}(X,Y) = H_i(X,Y)$

So $\qquad B(X,Y) = \sum_{i=1}^{k} H_i(X,Y)E_i$

Using the Gauss formula and replacing Y by X,

$$B(X,X) = \overline{\nabla}_X X - \nabla_X X$$

Therefore $$\overline{\nabla}_X X = \sum_{i=1}^{k} H_i(X,X)E_i + \nabla_X X \qquad (6)$$

Differentiating (6) in the direction of Y,

$$\overline{\nabla}_Y\overline{\nabla}_X X = \sum_{i=1}^{k}\{H_i(X,X)\overline{\nabla}_Y E_i + Y(H_i(X,X)E_i\} + \overline{\nabla}_Y\nabla_X X\ ,\ \text{at } p$$

Taking the inner product with Y,

$$\langle \overline{\nabla}_Y \overline{\nabla}_X X, Y \rangle = \sum_{i=1}^{k} H_i(X,X)\langle \overline{\nabla}_Y E_i, Y\rangle + \langle \overline{\nabla}_Y \nabla_X X, Y \rangle, \text{ at } p \tag{7}$$

Since $\langle E_i, Y \rangle = 0$, at p

Taking the covariant derivative in the direction of Y,

$$\langle \overline{\nabla}_Y Y, E_i \rangle + \langle Y, \overline{\nabla}_Y E_i \rangle = 0$$

$$\langle B(Y,Y) + \nabla_Y Y, E_i \rangle + \langle Y, \overline{\nabla}_Y E_i \rangle = 0 \qquad \text{(using the Gauss formula)}$$

$$\langle B(Y,Y), E_i \rangle + \langle Y, \overline{\nabla}_Y E_i \rangle = 0 \qquad \text{(since } \nabla_Y Y \in \aleph(M)\text{, then } \langle \nabla_Y Y, E_i \rangle = 0)$$

Therefore $\langle \overline{\nabla}_Y E_i, Y \rangle = -H_i(Y,Y)$

Also taking the Gauss formula and replacing X by Y and Y by $\nabla_X X$,

$$B(Y, \nabla_X X) = \overline{\nabla}_Y \nabla_X X - \nabla_Y \nabla_X X$$

Taking the inner product with Y,

$$\langle B(Y, \nabla_X X), Y \rangle(p) = \langle \overline{\nabla}_Y \nabla_X X - \nabla_Y \nabla_X X, Y \rangle(p)$$

Then, $\langle \overline{\nabla}_Y \nabla_X X - \nabla_Y \nabla_X X, Y \rangle(p) = 0$ (since $B(Y, \nabla_X X)$ is normal to Y at p)

Substituting above two results into (7),

$$\langle \overline{\nabla}_Y \overline{\nabla}_X X, Y \rangle = -\sum_{i=1}^{n} H_i(X,X) H_i(Y,Y) + \langle \nabla_Y \nabla_X X, Y \rangle, \text{ at } p$$

Similarly, $\langle \overline{\nabla}_X \overline{\nabla}_Y X, Y \rangle = -\sum_{i=1}^{k} H_i(X,Y) H_i(X,Y) + \langle \nabla_X \nabla_Y X, \mathrm{Y} \rangle$, at p

Therefore $K(x,y) - \overline{K}(x,y) = \sum_{i=1}^{n} H_i(X,X) H_i(Y,Y) - \sum_{i=1}^{k} H_i(X,Y) H_i(X,Y)$, at p

$$= \langle B(X,X), B(Y,Y) \rangle(p) - \langle B(X,Y), B(X,Y) \rangle(p)$$

$$K(x, y) - \overline{K}(x, y) = \langle B(x,x), B(y,y)\rangle - \langle B(x,y), B(x,y)\rangle$$

$$= \langle B(x,x), B(y,y)\rangle - |B(x,y)|^2 \qquad (8)$$

5.2.11 Remark

In the case of a hypersurface f: $M^n \to \overline{M}^{n+1}$, this theorem leads to a very important expression. Let $p \in M$ and $\eta \in (T_p M)^{\perp}$.

$$S_\eta(e_i) = \lambda_i e_i, i = 1, \ldots, n.$$

$\langle S_\eta(e_i), e_j\rangle = H_\eta(e_i, e_j) = \langle B(e_i, e_j), \eta\rangle$, where $e_i, e_j \in T_p M$.

Then $\quad \langle \lambda_i e_i, e_j\rangle = H_\eta(e_i, e_j) = \langle B(e_i, e_j), \eta\rangle$

Therefore, $B(e_i, e_j) = \langle \lambda_i e_i, e_j\rangle \eta$

$= 0, \quad$ if $i \neq j$

$= \lambda_i \eta$, if $i = j$

Then from (8), replace x by e_i and y by e_j, $i \neq j$

$$K(e_i, e_j) - \overline{K}(e_i, e_j) = \langle B(e_i, e_j), B(e_j, e_j)\rangle - |B(e_i, e_j)|^2$$

$$= \lambda_i \lambda_j$$

If $M = M^2$, a Riemannian manifold with dimension two, and $\overline{M} = \Re^3$,

then $\quad K(e_i, e_j) = \lambda_i \lambda_j \quad$ (since in Euclidean space $\overline{K} = 0$) $\qquad (9)$

Hence the equation (9) tells that the Gaussian curvature coincides with the sectional curvature of the surface and implies the famous theorem Egregium of Gauss.

5.3 The Fundamental Equations

The Gauss, Ricci and Codazzi's equations are the fundamental equations of the isometric immersion. The Gauss equation relates the curvature tensors of the tangent bundle with the second fundamental form of the immersion. Theorem 5.2.10 is a special case of the Gauss equation. The Ricci equation relates the curvature tensors of the normal bundle with the second fundamental form of the immersion. The codazzi's equation relates the curvature tensor of the vector bundle with the covariant derivative of the second fundamental form considered as a tensor.

5.3.1 Normal connection and normal curvature of the normal bundle

Given an isometric immersion f: $M^n \to \overline{M}^{m=n+k}$, we have at each $p \in M$,

$T_p\overline{M} = (T_pM)^T + (T_pM)^\perp$, we denote by $(T_pM)^\perp$ the normal space of M in $\overline{M}$ at p. That is, any vector $\xi \in (T_pM)^\perp$ is normal to M at p.

The set $TM^\perp = \coprod_{p \in M}(T_pM)^\perp$ has the structure of a vector bundle over M and is called the *normal bundle* of M in $\overline{M}$. From now on we shall use Latin letters X, Y, Z, etc., to denote differentiable vector fields tangent to M and Greek letters ξ, η, ζ, etc. to denote differentiable vector fields normal to M.

Given X and η, $\overline{\nabla}_X\eta$ can be written as,

$$\overline{\nabla}_X\eta = (\overline{\nabla}_X\eta)^T + (\overline{\nabla}_X\eta)^N$$

Then $(\overline{\nabla}_X\eta)^N = \overline{\nabla}_X\eta + S_\eta(X)$ (since $(\overline{\nabla}_X\eta)^T = -S_\eta(X)$, from proposition 5.2.7)

Denote $(\overline{\nabla}_X\eta)^N$ by $\nabla^\perp_X\eta$, and $\nabla^\perp$ is called the *normal connection* of the immersion.

Therefore $\nabla^\perp_X\eta = \overline{\nabla}_X\eta + S_\eta(X)$ (10)

$$\overline{\nabla}_X\eta = -S_\eta(X) + \nabla^\perp_X\eta,$$

where $-S_\eta(X)$ and $\nabla^\perp_X\eta$ are the tangential and normal components respectively. This is called the equation of Weingarten, after the mathematician who first obtained the equation for surfaces in Euclidean space.

5.3.2 Remark

It is easy to verify that the normal connection $\nabla^\perp$ has all of the properties of a connection. That is, (a) $\nabla^\perp_{fX+gY}\eta = f\nabla^\perp_X\eta + g\nabla^\perp_Y\eta$

(b) $\nabla^\perp_X(\xi+\eta) = \nabla^\perp_X\xi + \nabla^\perp_X\eta$

(c) $\nabla^\perp_X(f\eta) = f\,\nabla^\perp_X\eta + X(f)\eta$, where $f, g \in D(M)$

Proof:

(a) We know that $\nabla^\perp_{fX+gY}\eta = \overline{\nabla}_{fX+gY}\eta + S_\eta(fX+gY)$ (from (10))

But $S_\eta(fX+gY) = -(\overline{\nabla}_{fX+gY}\eta)^T$ (from proposition 5.2.7)

$= -(f\overline{\nabla}_X\eta + g\overline{\nabla}_Y\eta)^T$ (from (i) of definition 2.2.1)

$= -\{f(\overline{\nabla}_X\eta)^T + g(\overline{\nabla}_Y\eta)^T\}$

$= fS_\eta(X) + gS_\eta(Y)$

Then $\nabla^\perp_{fX+gY}\eta = f\overline{\nabla}_X\eta + g\overline{\nabla}_Y\eta + fS_\eta(X) + gS_\eta(Y)$

$= f(\overline{\nabla}_X\eta + S_\eta(X)) + g(\overline{\nabla}_Y\eta + S_\eta(Y))$

$= f\nabla^\perp_X\eta + g\nabla^\perp_Y\eta$ (using (8))

Hence the result.

(b)Using (8), we can write $\nabla^{\perp}_{X}(\xi+\eta)=\overline{\nabla}_{X}(\xi+\eta)+S_{(\xi+\eta)}(X)$

But $S_{(\xi+\eta)}(X)=-(\overline{\nabla}_{X}(\xi+\eta))^{T}$

$=-(\overline{\nabla}_{X}\xi+\overline{\nabla}_{X}\eta)^{T}$

$=-\{(\overline{\nabla}_{X}\xi)^{T}+(\overline{\nabla}_{X}\eta)^{T}\}$

$=S_{\xi}(X)+S_{\eta}(X)$

So $\nabla^{\perp}_{X}(\xi+\eta)=\overline{\nabla}_{X}(\xi+\eta)+S_{\xi}(X)+S_{\eta}(X)$

$=\overline{\nabla}_{X}\xi+\overline{\nabla}_{X}\eta+S_{\xi}(X)+S_{\eta}(X)$ (from (ii) of definition 2,2.1)

$=\overline{\nabla}_{X}\xi+S_{\xi}(X)+\overline{\nabla}_{X}\eta+S_{\eta}(X)$

$=\nabla^{\perp}_{X}\xi+\nabla^{\perp}_{X}\eta$ (using (8))

Hence the result.

(c) We know that $\nabla^{\perp}_{X}(f\eta)=\overline{\nabla}_{X}(f\eta)+S_{f\eta}(X)$ (from(8))

But $S_{f\eta}(X)=-(\overline{\nabla}_{X}(f\eta))^{T}$ (using proposition 5.2.7)

$=-(f\overline{\nabla}_{X}\eta+X(f)\eta)^{T}$ (from (iii) of definition 2.2.1)

$=fS_{\eta}(X)$ (since $(\eta)^{T}=0$, η is normal to M)

Therefore $\nabla^{\perp}_{X}(f\eta)=f\overline{\nabla}_{X}\eta+X(f)\eta+fS_{\eta}(X)$

$=f\{\overline{\nabla}_{X}\eta+S_{\eta}(X)\}+X(f)\eta$

$=f\nabla^{\perp}_{X}\eta+X(f)\eta.$

Hence the result.

As in the case of the tangent bundle, we can introduce the curvature of the normal bundle which is called the *normal curvature,* $R^{\perp}$, of the immersion and is defined by

$$R^{\perp}(X,Y)\eta = \nabla_Y^{\perp}\nabla_X^{\perp}\eta - \nabla_X^{\perp}\nabla_Y^{\perp}\eta + \nabla_{[X,Y]}^{\perp}\eta.$$

Using normal curvature, we will prove the Gauss equation.

5.3.3 Gauss Equation

$$\langle \overline{R}(X,Y)Z,T\rangle = \langle R(X,Y)Z,T\rangle - \langle B(Y,T),B(X,Z)\rangle + \langle B(X,T),B(Y,Z)\rangle.$$

***Proof*:**

$\overline{R}(X,Y)Z = \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_X\overline{\nabla}_Y Z + \overline{\nabla}_{[X,Y]}Z$, (using the definition of curvature)

From Gauss formula $B(X,Z) = \overline{\nabla}_X Z - \nabla_X Z$

So $\overline{\nabla}_X Z = B(X,Z) + \nabla_X Z$

Similarly, $\overline{\nabla}_Y Z = B(Y,Z) + \nabla_Y Z$

$$\overline{\nabla}_{[X,Y]}Z = B([X,Y],Z) + \nabla_{[X,Y]}Z$$

So $\overline{R}(X,Y)Z = \overline{\nabla}_Y(B(X,Z) + \nabla_X Z) - \overline{\nabla}_X(B(Y,Z) + \nabla_Y Z) + B([X,Y],Z) + \nabla_{[X,Y]}Z$

$$\overline{R}(X,Y)Z = \overline{\nabla}_Y(B(X,Z)) + \overline{\nabla}_Y\nabla_X Z - \overline{\nabla}_X(B(Y,Z)) - \overline{\nabla}_X\nabla_Y Z + B([X,Y],Z) + \nabla_{[X,Y]}Z \qquad (11)$$

Using Gauss formula, $B(Y,\nabla_X Z) = \overline{\nabla}_Y\nabla_X Z - \nabla_Y\nabla_X Z$

So $\overline{\nabla}_Y\nabla_X Z = B(Y,\nabla_X Z) + \nabla_Y\nabla_X Z$

Similarly $\overline{\nabla}_X\nabla_Y Z = B(X,\nabla_Y Z) + \nabla_X\nabla_Y Z$

And $$\overline{\nabla}_Y(B(X,Z)) = (\overline{\nabla}_Y(B(X,Z)))^N + (\overline{\nabla}_Y(B(X,Z)))^T$$

Since $B(X,Z)$ is normal to *M*, then using proposition 5.2.7,

$$\overline{\nabla}_Y(B(X,Z)) = \nabla_Y^{\perp}(B(X,Z)) - S_{B(X,Z)}(Y)$$

Similarly $$\overline{\nabla}_X(B(Y,Z)) = \nabla_X^{\perp}(B(Y,Z)) - S_{B(Y,Z)}(X)$$

Substituting all these results into (11),

$$\overline{R}(X,Y)Z = \nabla_Y^{\perp}(B(X,Z)) - S_{B(X,Z)}(Y) + B(Y,\nabla_X Z) + \nabla_Y\nabla_X Z - \nabla_X^{\perp}(B(Y,Z)) +$$

$$S_{B(Y,Z)}(X) - B(X,\nabla_Y Z) - \nabla_X\nabla_Y Z + B([X,Y],Z) + \nabla_{[X,Y]}Z$$

Since $$R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$$

$$\overline{R}(X,Y)Z = R(X,Y)Z + \nabla_Y^{\perp}(B(X,Z)) - \nabla_X^{\perp}(B(Y,Z)) - S_{B(X,Z)}(Y) + S_{B(Y,Z)}(X) +$$

$$B(Y,\nabla_X Z) - B(X,\nabla_Y Z) + B([X,Y],Z)$$

Taking the inner product with *T*, since the normal terms vanish, we get

$$\langle \overline{R}(X,Y)Z,T\rangle = \langle R(X,Y)Z,T\rangle - \langle S_{B(X,Z)}(Y), T\rangle + \langle S_{B(Y,Z)}(X), T\rangle \qquad (12)$$

But we have $\langle S_\eta(X), Y\rangle = \langle B(X,Y), \eta\rangle$ (from definition 5.2.5)

Replacing η by $B(X,Z)$, *X* by *Y* and *Y* by *T*,

$$\langle S_{B(X,Z)}(Y), T\rangle = \langle B(Y,T), B(X,Z)\rangle$$

Similarly $\langle S_{B(Y,Z)}(X), T\rangle = \langle B(X,T), B(Y,Z)\rangle$

Substituting these two results into (12),

$$\langle \overline{R}(X,Y)Z,T\rangle = \langle R(X,Y)Z,T\rangle - \langle B(Y,T), B(X,Z)\rangle + \langle B(X,T), B(Y,Z)\rangle .$$

Hence the result.

5.3.4 Ricci Equation

$$\langle \overline{R}(X,Y)\eta,\xi\rangle - \langle R^{\perp}(X,Y)\eta,\xi\rangle = \langle [S_\eta, S_\xi]X, Y\rangle,$$

where $[S_\eta, S_\xi]$ denotes the operator $S_\eta \circ S_\xi - S_\xi \circ S_\eta$.

Proof:

Using the definition of curvature tensor,

$$\overline{R}(X,Y)\eta = \overline{\nabla}_Y \overline{\nabla}_X \eta - \overline{\nabla}_X \overline{\nabla}_Y \eta + \overline{\nabla}_{[X,Y]}\eta \qquad (13)$$

From equation (10), $\overline{\nabla}_X \eta = -S_\eta(X) + \nabla_X^{\perp}\eta$

Taking the covariant derivative in the direction of Y,

$$\overline{\nabla}_Y \overline{\nabla}_X \eta = -\overline{\nabla}_Y S_\eta(X) + \overline{\nabla}_Y \nabla_X^{\perp}\eta \qquad (14)$$

$$\overline{\nabla}_Y \nabla_X^{\perp}\eta = (\overline{\nabla}_Y \nabla_X^{\perp}\eta)^T + (\overline{\nabla}_Y \nabla_X^{\perp}\eta)^N \quad (\text{since } \overline{\nabla}_Y \nabla_X^{\perp}\eta \in \aleph(\overline{M}))$$

Using proposition 5.2.7, since $\nabla_X^{\perp}\eta$ is normal to M

$$\overline{\nabla}_Y \nabla_X^{\perp}\eta = -S_{(\nabla_X^{\perp}\eta)}(Y) + \nabla_Y^{\perp}\nabla_X^{\perp}\eta$$

Then from (14),

$$\overline{\nabla}_Y \overline{\nabla}_X \eta = -\overline{\nabla}_Y S_\eta(X) - S_{(\nabla_X^{\perp}\eta)}(Y) + \nabla_Y^{\perp}\nabla_X^{\perp}\eta \qquad (15)$$

Using the Gauss formula and replacing X by Y and Y by $S_\eta(X)$, since $S_\eta(X)$ is tangent to M.

$$B(Y, S_\eta(X)) = \overline{\nabla}_Y S_\eta(X) - \nabla_Y S_\eta(X)$$

$$\text{So} \quad \overline{\nabla}_Y S_\eta(X) = B(Y, S_\eta(X)) + \nabla_Y S_\eta(X)$$

Substituting into (15),

$$\overline{\nabla}_Y \overline{\nabla}_X \eta = -B(Y, S_\eta(X)) - \nabla_Y S_\eta(X) - S_{(\nabla_X^{\perp}\eta)}(Y) + \nabla_Y^{\perp}\nabla_X^{\perp}\eta$$

Similarly, $\overline{\nabla}_X \overline{\nabla}_Y \eta = -B(X, S_\eta(Y)) - \nabla_X S_\eta(Y) - S_{(\nabla_Y^{\perp}\eta)}(X) + \nabla_X^{\perp}\nabla_Y^{\perp}\eta$

Also we know $\overline{\nabla}_{[X,Y]}\eta$ is tangent to $\overline{M}$.

Then it can be written as, $\overline{\nabla}_{[X,Y]}\eta = (\overline{\nabla}_{[X,Y]}\eta)^T + (\overline{\nabla}_{[X,Y]}\eta)^N$.

Using proposition 5.2.7 and the notation of the normal connection

$$\overline{\nabla}_{[X,Y]}\eta = -S_\eta([X,Y]) + \nabla^\perp_{[X,Y]}\eta$$

Substituting these three result into equation (13),

$$\overline{R}(X,Y)\eta = -B(Y,S_\eta(X)) - \nabla_Y S_\eta(X) - S_{(\nabla^\perp_X\eta)}(Y) + \nabla^\perp_Y\nabla^\perp_X\eta + B(X,S_\eta(Y)) +$$

$$\nabla_X S_\eta(Y) + S_{(\nabla^\perp_Y\eta)}(X) - \nabla^\perp_X\nabla^\perp_Y\eta - S_\eta([X,Y]) + \nabla^\perp_{[X,Y]}\eta$$

Using the definition of the normal curvature,

$$= R^\perp(X,Y)\eta - \nabla_Y S_\eta(X) + \nabla_X S_\eta(Y) - S_{(\nabla^\perp_X\eta)}(Y) + S_{(\nabla^\perp_Y\eta)}(X) - S_\eta([X,Y])$$

$$+ B(X,S_\eta(Y)) - B(Y,S_\eta(X))$$

Taking the inner product with ξ, where ξ is normal to M.

$$\langle \overline{R}(X,Y)\eta, \xi\rangle = \langle R^\perp(X,Y)\eta, \xi\rangle + \langle B(X,S_\eta(Y)), \xi\rangle - \langle B(Y,S_\eta(X)), \xi\rangle \quad (16)$$

Since $\nabla_Y S_\eta(X)$, $\nabla_X S_\eta(Y)$, $S_{(\nabla^\perp_X\eta)}(Y)$, $S_{(\nabla^\perp_Y\eta)}(X)$ and $S_\eta([X,Y])$ are tangent to M.

Therefore taking the inner product with ξ, all terms vanish.

Also we know that

$$\langle B(X,Y),\eta\rangle = \langle S_\eta(X), Y\rangle \quad (17)$$

Replacing Y by $S_\eta(Y)$ and η by ξ,

$$\langle B(X,S_\eta(Y)), \xi\rangle = \langle S_\xi(X), S_\eta(Y)\rangle$$

$$= \langle S_\eta(S_\xi(X)), Y\rangle \quad \text{(since } S_\eta \text{ is self-adjoint)}$$

$$\langle B(Y,S_\eta(X)), \xi\rangle = \langle B(S_\eta(X),Y), \xi\rangle \quad \text{(since } B \text{ is symmetric)}$$

$$= \langle S_\xi(S_\eta(X)), Y\rangle \quad \text{(from (17))}$$

Then substituting into (16),

$$\langle \overline{R}(X,Y)\eta, \xi\rangle = \langle R^\perp(X,Y)\eta, \xi\rangle + \langle S_\eta(S_\xi(X)), Y\rangle - \langle S_\xi(S_\eta(X)), Y\rangle$$

$$= \langle R^\perp(X,Y)\eta, \xi\rangle + \langle S_\eta(S_\xi(X)) - S_\xi(S_\eta(X)), Y\rangle$$

$$= \langle R^\perp(X,Y)\eta, \xi\rangle + \langle (S_\eta \circ S_\xi - S_\xi \circ S_\eta)X, Y\rangle$$

$$= \langle R^\perp(X,Y)\eta, \xi\rangle + \langle [[S_\eta, S_\xi]X, Y\rangle.$$

Hence the result

5.3.5 Remark

If $\langle R^{\perp}(X,Y)\eta,,\xi \rangle = 0$, we say that the normal bundle of the immersion is *flat*. Assume that the ambient space $\overline{M}$ has constant sectional curvature equal to K_0. Then using lemma 3.3.4,

$$\langle \overline{R}(X,Y)\eta, \xi\rangle = K_0\{\langle X,\eta\rangle\langle Y,\xi\rangle - \langle Y,\eta\rangle\langle X,\xi\rangle\}$$

$$= 0 \quad \text{(since } X \text{ and } Y \text{ are orthogonal to } \eta \text{ and } \xi)$$

Then from Ricci equation, $\langle R^{\perp}(X,Y)\eta,\xi \rangle = -\langle [S_\eta, S_\xi]X, Y \rangle$. It follows that if the normal bundle is flat then $[S_\eta, S_\xi]X = 0$, for all η, ξ, X

5.3.6 Remark

The second fundamental form of the immersion can be considered as a tensor of order three defined by

$$B: \aleph(M)\times\aleph(M)\times\aleph(M)^{\perp} \to \Re$$

$B(X,Y,\eta) = \langle B(X,Y),\eta\rangle$, where $\aleph(M)^{\perp}$ denotes the space of differentiable vector fields normal to M.

The *covariant derivative of B* relative to Z is a tensor of order 3 defined in the same way as definition 3.4.4.

$$(\overline{\nabla}_Z B)(X,Y,\eta) = Z(B(X,Y,\eta)) - B(\nabla_Z X,Y,\eta) - B(X,\nabla_Z Y,\eta) - B(X,Y,\overline{\nabla}_Z\eta) \quad (18(a))$$

Consider $B(X,Y,\overline{\nabla}_Z\eta) = \langle B(X,Y),\overline{\nabla}_Z\eta\rangle$

$$= \langle B(X,Y), (\overline{\nabla}_Z\eta)^T + (\overline{\nabla}_Z\eta)^N \rangle$$

$$= \langle B(X,Y), (\overline{\nabla}_Z\eta)^N\rangle \quad \text{(since } B(X,Y) \text{ is normal to } M)$$

$$= \langle B(X,Y), \nabla_Z^{\perp}\eta\rangle = B(X,Y,\nabla_Z^{\perp}\eta) \quad \text{(since } (\overline{\nabla}_Z\eta)^N = \nabla_Z^{\perp}\eta)$$

Then from (18(a)),

$$(\overline{\nabla}_Z B)(X,Y,\eta) = Z(B(X,Y,\eta)) - B(\nabla_Z X,Y,\eta) - B(X,\nabla_Z Y,\eta) - B(X,Y,\nabla_Z^{\perp}\eta) \quad (18(b))$$

5.3.7 Codazzi's Equation

$$\langle \overline{R}(X,Y)Z,\eta\rangle = (\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta).$$

***Proof*:**

Using (18(b)) and replacing Z by X, X by Y and Y by Z,

$$(\overline{\nabla}_X B)(Y,Z,\eta) = X(B(Y,Z,\eta)) - B(\nabla_X Y,Z,\eta) - B(Y,\nabla_X Z,\eta) - B(Y,Z,\nabla_X^{\perp}\eta) \quad (19)$$

Let $X(B(Y,Z,\eta)) = X\langle B(Y,Z),\eta\rangle$

$X(B(Y,Z,\eta)) = \langle \nabla_X^{\perp} B(Y,Z),\eta\rangle + \langle B(Y,Z),\nabla_X^{\perp}\eta\rangle$ ($B(Y,Z)$ and η are normal to M).

Therefore from equation (19),

$$(\overline{\nabla}_X B)(Y,Z,\eta) = \langle \nabla_X^{\perp} B(Y,Z),\eta\rangle - B(\nabla_X Y,Z,\eta) - B(Y,\nabla_X Z,\eta) \quad (20)$$

Similarly,

$$(\overline{\nabla}_Y B)(X,Z,\eta) = \langle \nabla_Y^{\perp} B(X,Z),\eta\rangle - B(\nabla_Y X,Z,\eta) - B(X,\nabla_Y Z,\eta) \quad (21)$$

Consider the expression in the proof of the Gauss equation,

$$\overline{R}(X,Y)Z = R(X,Y)Z + \nabla_Y^{\perp}(B(X,Z)) - \nabla_X^{\perp}(B(Y,Z)) - S_{B(X,Z)}(Y) + S_{B(Y,Z)}(X)$$

$$+ B(Y,\nabla_X Z) - B(X,\nabla_Y Z) + B([X,Y],Z)$$

Taking an inner product with η,

$$\langle \overline{R}(X,Y)Z,\eta\rangle = \langle \nabla_Y^{\perp}(B(X,Z)),\eta\rangle - \langle \nabla_X^{\perp}(B(Y,Z)),\eta\rangle + \langle B(Y,\nabla_X Z),\eta\rangle -$$

$$\langle B(X,\nabla_Y Z),\eta\rangle + \langle B([X,Y],Z),\eta\rangle$$

Since $R(X,Y)Z$, $S_{B(X,Z)}(Y)$, $S_{B(Y,Z)}(X)$ are tangent to M and $[X,Y] = \nabla_X Y - \nabla_Y X$

$$\langle \overline{R}(X,Y)Z,\eta\rangle = \langle \nabla_Y^{\perp}(B(X,Z)),\eta\rangle - B(\nabla_Y X, Z,\eta) - B(X,\nabla_Y Z,\eta) -$$

$$\{\langle \nabla_X^{\perp}(B(Y,Z)),\eta\rangle - B(\nabla_X Y, Z,\eta) - B(Y,\nabla_X Z,\eta)\}$$

Using equations (20) and (21),

$$\langle \overline{R}(X,Y)Z,\eta\rangle = (\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta).$$

Hence the result.

5.3.8 Remark

If the ambient space has constant sectional curvature, K_0, then $\langle \overline{R}(X,Y)Z,\eta\rangle$ is equal to zero. Since using lemma 3.3.4,

$$\langle \overline{R}(X,Y)Z,\eta\rangle = K_0\{\langle X,Z\rangle\langle Y,\eta\rangle - \langle Y,Z\rangle\langle X,\eta\rangle\}$$

$$= 0 \quad \text{(since } \eta \text{ is normal to } X \text{ and } Y\text{)}$$

$$\langle \overline{R}(X,Y)Z,\eta\rangle = 0$$

Then from Codazzi's equation, $(\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta) = 0$.

5.3.9 Remark

In the case of hypersurface, that is, the co-dimension of the immersion is one, the normal component of $\overline{\nabla}_Z\eta$ is zero. Therefore $\nabla_Z^{\perp}\eta = 0$.

Since $\langle\eta,\eta\rangle = 1$

Differentiating above inner product in the direction of Z,

$$\langle \overline{\nabla}_Z \eta, \eta \rangle = 0$$

$$\langle (\overline{\nabla}_Z \eta)^T + (\overline{\nabla}_Z \eta)^N, \eta \rangle = 0$$

$$\langle (\overline{\nabla}_Z \eta)^N, \eta \rangle = 0 \quad (\text{since } (\overline{\nabla}_Z \eta)^T, \eta \rangle = 0)$$

This implies that $(\overline{\nabla}_Z \eta)^N = 0$ (since $(\overline{\nabla}_Z \eta)^N$ does not belong to tangent space)

Then from equation (19),

$$(\overline{\nabla}_X B)(Y, Z, \eta) = X(B(Y,Z, \eta)) - B(\nabla_X Y, Z, \eta) - B(Y, \nabla_X Z, \eta)$$

$$= X\langle B(Y,Z), \eta \rangle - B(\nabla_X Y, Z, \eta) - B(Y, \nabla_X Z, \eta) \qquad (22)$$

But $\langle B(Y,Z), \eta \rangle = \langle S_\eta(Y), Z \rangle$ (from definition 5.2.5) (23)

Taking the covariant derivative in the direction of X,

$$X\langle B(Y,Z), \eta \rangle = X\langle S_\eta(Y), Z \rangle$$

$$= \langle \nabla_X (S_\eta(Y)), Z) \rangle + \langle S_\eta(Y), \nabla_X Z \rangle$$

Using equation (23) and replacing Y by $\nabla_X Y$,

$$\langle B(\nabla_X Y, Z), \eta \rangle = \langle S_\eta(\nabla_X Y), Z \rangle \qquad (\text{since } \nabla_X Y \text{ is tangent to } M)$$

Again using equation (23) and replacing Z by $\nabla_X Z$,

$$\langle B(Y, \nabla_X Z), \eta \rangle = \langle S_\eta(Y), \nabla_X Z \rangle \qquad (\text{since } \nabla_X Z \text{ is tangent to } M)$$

Substituting these results into equation (22),

$$(\overline{\nabla}_X B)(Y, Z, \eta) = \langle \nabla_X (S_\eta(Y)), Z \rangle - \langle S_\eta(\nabla_X Y), Z \rangle$$

Similarly, $(\overline{\nabla}_Y B)(X, Z, \eta) = \langle \nabla_Y (S_\eta(X)), Z \rangle - \langle S_\eta(\nabla_Y X), Z \rangle$

If the ambient space has constant sectional curvature then from remark 5.3.8,

$$(\overline{\nabla}_X B)(Y,Z,\eta) = (\overline{\nabla}_Y B)(X,Z,\eta)$$

Then $\langle\nabla_X(S_\eta(Y)),Z\rangle - \langle S_\eta(\nabla_X Y),Z\rangle = \langle\nabla_Y(S_\eta(X)),Z\rangle\rangle - \langle S_\eta(\nabla_Y X),Z\rangle$

So $\langle\nabla_X(S_\eta(Y)),Z\rangle - \langle\nabla_Y(S_\eta(X)),Z\rangle = \langle S_\eta(\nabla_X Y),Z\rangle - \langle S_\eta(\nabla_Y X),Z\rangle$

$$\langle\nabla_X(S_\eta(Y)) - \nabla_Y(S_\eta(X)),Z\rangle = \langle S_\eta(\nabla_X Y) - S_\eta(\nabla_Y X),Z\rangle$$

$$\langle\nabla_X(S_\eta(Y)) - \nabla_Y(S_\eta(X)),Z\rangle = \langle S_\eta(\nabla_X Y - \nabla_Y X),Z\rangle$$

$$= \langle S_\eta([X,Y]),Z\rangle$$

$$\nabla_X(S_\eta(Y)) - \nabla_Y(S_\eta(X)) = S_\eta([X,Y]), \qquad \text{for all } Z \in \aleph(M).$$

It is important to note that, in the case of the ambient space having constant sectional curvature, the Gauss, Codazzi and Ricci equations generalize the local theory of surfaces.

5.3.10 Connections of conformal metrics on a manifold

Let M be a differentiable manifold. Two Riemannian metrics g and $\overline{g}$ on M are *conformal* if there exists a positive function $\mu : M \to \Re$ such that $\overline{g}(X,Y) = \mu\, g(X,Y)$, for all X, $Y \in \aleph(M)$. Let ∇ be the Riemannian connection of g.

$$\text{If } \overline{\nabla}_X Y = \nabla_X Y + S(X,Y),$$

where $S(X,Y) = \dfrac{1}{2\mu}\{(X(\mu)Y + Y(\mu)X - g(X,Y)\mathit{grad}\,\mu\}$ and $\mathit{grad}\,\mu$ is calculated in the metric g, that is, $X(\mu) = g(X, \mathit{grad}\,\mu)$. then we can show that $\overline{\nabla}$ is the Riemannian connection of $\overline{g}$.

Proof: We are given that $\overline{\nabla}_X Y = \nabla_X Y + S(X,Y)$

Similarly $$\overline{\nabla}_Y X = \nabla_Y X + S(Y,X)$$

Subtracting these two equations,

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = \nabla_X Y - \nabla_Y X \quad (\text{since } S(X,Y) = S(Y,X))$$

$$= [X,Y] \qquad (\text{since } \nabla \text{ is symmetric})$$

Therefore $\overline{\nabla}$ is symmetric.

Next we are going to show that $\overline{\nabla}$ is compatible with $\overline{g}$.

That is, $$X(\overline{g}(Y,Z)) = \overline{g}(\overline{\nabla}_X Y, Z) + \overline{g}(Y, \overline{\nabla}_X Z) \qquad (24)$$

Consider the left-hand side of the equation (24),

$$X(\overline{g}(Y,Z)) = X(\mu g(Y,Z)) \quad (\text{from the definition of the conformal metrics})$$

$$= X(\mu)g(Y,Z) + \mu X(g(Y,Z))$$

$$= X(\mu)g(Y,Z) + \mu\{g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\}$$

And also taking the right hand side of the equation (24),

$$\overline{g}(\overline{\nabla}_X Y, Z) + \overline{g}(Y, \overline{\nabla}_X Z) = \mu g(\overline{\nabla}_X Y, Z) + \mu g(Y, \overline{\nabla}_X Z)$$

$$= \mu g(\nabla_X Y + S(X,Y), Z) + \mu g(Y, \nabla_X Z + S(X,Z))$$

$$= \mu g(\nabla_X Y, Z) + \mu g(S(X,Y), Z) + \mu g(Y, \nabla_X Z) + \mu g(Y, S(X,Z))$$

Comparing these results with equation (24), we can conclude that to prove (24) we need to show that

$$X(\mu)g(Y,Z) = \mu\{g(S(X,Y), Z) + g(Y, S(X,Z))\}$$

We are given that

$$S(X,Y) = \frac{1}{2\mu}\{(X(\mu)Y + Y(\mu)X - g(X,Y)grad\,\mu\}$$

Taking the inner product with Z,

$$g(S(X,Y),Z) = \frac{1}{2\mu}\{(X(\mu)g(Y,Z) + Y(\mu)g(X,Z) - g(X,Y)g(grad\,\mu, Z)\}$$

Similarly $g(S(X,Z),Y) = \frac{1}{2\mu}\{(X(\mu)g(Z,Y) + Z(\mu)g(X,Y) - g(X,Z)g(grad\,\mu, Y)\}$

Take $Y(\mu) = g(grad\,\mu, Y)$, and $Z(\mu) = g(grad\,\mu, Z)\}$

So $\mu\{g(S(X,Y), Z) + g(Y, S(X,Z))\} = X(\mu)g(Y,Z)$. Hence the result. $\overline{\nabla}$ is compatible.

5.3.11 Umbilic Hypersurface

Let $(\overline{M}^{n+1}, g)$ be a manifold with a Riemannian metric g and let ∇ be its Riemannian connection on M. We say an immersion f: $M^n \to \overline{M}^{n+1}$ is *(totally) umbilic* if for all $p \in M$, the second fundamental form B of f at p satisfies $\langle B(X,Y),\eta\rangle = \lambda(p)\langle X,Y\rangle$, where $\lambda(p) \in \Re$, for all $X, Y \in \aleph(M)$ and for a given unit field η normal to $f(M)$; here we are using $\langle\ ,\ \rangle$ to denote the metric g on $\overline{M}$ and the metric induced by f on M. If $\overline{M}^{n+1}$ has constant sectional curvature then we can show that λ does not depend on p, that is, λ is constant.

***Proof*:**

Let $X, Y, T \in \aleph(M)$

Decomposing the tangent space of M at p,

$$T_p\overline{M} = T_pM + (T_pM)^{\perp}, \text{ where } \eta \in (T_pM)^{\perp} \subset T_p\overline{M} \text{ so } X, Y, T, \eta \in T_p\overline{M}$$

Consider $\quad \nabla_X \eta = (\nabla_X \eta)^T + (\nabla_X \eta)^N \qquad$ (since $\nabla_X \eta \in T_p\overline{M}$)

$\nabla_X \eta = (\nabla_X \eta)^T \quad$ (since $(\nabla_X \eta)^N = 0$ in the hypersurface)

Then using proposition 5.2.7, $S_\eta(X) = -\nabla_X \eta$

So $\quad \langle B(X,Y), \eta\rangle = -\langle \nabla_X \eta, Y\rangle$

And we are given that $\langle B(X,Y), \eta\rangle = \lambda(p)\langle X,Y\rangle$

Therefore from these two equations,

$$-\langle \nabla_X \eta, Y\rangle = \lambda(p)\langle X,Y\rangle \tag{25}$$

Similarly, $\quad -\langle \nabla_T \eta, Y\rangle = \lambda(p)\langle T,Y\rangle$

Differentiating equation (25) in the direction of T,

$$-T\langle \nabla_X \eta, Y\rangle = T(\lambda(p)\langle X,Y\rangle)$$

$$-\langle \nabla_T \nabla_X \eta, Y\rangle - \langle \nabla_X \eta, \nabla_T Y\rangle = T(\lambda(p))\langle X,Y\rangle + \lambda(p)\langle \nabla_T X, Y\rangle + \lambda(p)\langle X, \nabla_T Y\rangle \tag{26}$$

Again taking equation (25) and replacing Y by $\nabla_T Y \in T_pM$,

$$-\langle \nabla_X \eta, \nabla_T Y\rangle = \lambda(p)\langle X, \nabla_T Y\rangle$$

Therefore from equation (26),

$$-\langle \nabla_T \nabla_X \eta, Y\rangle = T(\lambda(p))\langle X,Y\rangle + \lambda(p)\langle \nabla_T X, Y\rangle$$

Similarly $\quad -\langle \nabla_X \nabla_T \eta, Y\rangle = X(\lambda(p))\langle T,Y\rangle + \lambda(p)\langle \nabla_X T, Y\rangle$

Subtracting these two equations,

$$\langle \nabla_T \nabla_X \eta - \nabla_X \nabla_T \eta, Y\rangle = X(\lambda(p))\langle T,Y\rangle - T(\lambda(p))\langle X, Y\rangle + \lambda(p)\langle \nabla_X T - \nabla_T X, Y\rangle Y\rangle$$

$$= X(\lambda(p))\langle T,Y\rangle - T(\lambda(p))\langle X,Y\rangle + \lambda(p)\langle [X,T], Y\rangle \tag{27}$$

But $[X,T] \in T_pM$, then using (25) and replacing X by $[X,T]$,

$$-\langle \nabla_{[X,Y]}\eta, Y\rangle = \lambda(p)\langle [X,T], Y\rangle$$

Then from (27),

$$\langle \nabla_T \nabla_X \eta - \nabla_X \nabla_T \eta + \nabla_{[X,Y]}\eta, Y\rangle = X(\lambda(p))\langle T,Y\rangle - T(\lambda(p))\langle X,Y\rangle$$

$\langle R(T,X)\eta, Y\rangle = \langle X(\lambda(p))T - T(\lambda(p))X, Y\rangle$ (from the definition of curvature tensor)

If M^{n+1} has constant sectional curvature, K_0, then

$$\langle R(T,X)\eta, Y\rangle = K_0\{\langle T,\eta\rangle\langle X,Y\rangle - \langle X,\eta\rangle\langle T,Y\rangle\} \text{ (from lemma 3.3.4)}$$

$$= 0 \quad \text{(since } \langle T,\eta\rangle = 0 \text{ and } \langle X,\eta\rangle = 0\text{)}$$

That is, $\langle X(\lambda(p))T - T(\lambda(p))X, Y\rangle = 0$

That is, $X(\lambda(p))T - T(\lambda(p))X = 0$, where Y is an arbitrary vector.

(since $X(\lambda(p))T - T(\lambda(p))X \in T_pN$ and $Y \in T_pN$)

Because T and X can be chosen to be linearly independent, this implies that $X(\lambda(p))$ is equal to zero, for all $X \in \aleph(N)$; therefore λ = constant.

5.3.12 Remark

If we change the metric g to a metric $\overline{g} = \mu g$, conformal to g, where $\overline{\nabla}$ is the Riemannian connection of $\overline{g}$, the immersion $f: N^n \to (M^{n+1}, \overline{g})$, continues being umbilic, that is $\langle \nabla_X \eta, Y\rangle_g = -\lambda\langle X,Y\rangle_g$, then $\overline{g}(\overline{\nabla}_X(\frac{\eta}{\sqrt{\mu}}), Y) = \frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}}\,\overline{g}(X,Y)$.

***Proof*:**

$$\overline{\nabla}_X \left(\frac{\eta}{\sqrt{\mu}}\right) = X\left(\frac{1}{\sqrt{\mu}}\right)\eta + \frac{1}{\sqrt{\mu}} \overline{\nabla}_X \eta \text{ (from (iii) of definition 2.2.1)}$$

Taking an inner product with Y,

$$\overline{g}\left(\overline{\nabla}_X \left(\frac{\eta}{\sqrt{\mu}}\right), Y\right) = X\left(\frac{1}{\sqrt{\mu}}\right) \overline{g}(\eta, Y) + \frac{1}{\sqrt{\mu}} \overline{g}(\overline{\nabla}_X \eta, Y)$$

$$\overline{g}\left(\overline{\nabla}_X \left(\frac{\eta}{\sqrt{\mu}}\right), Y\right) = \frac{1}{\sqrt{\mu}} \overline{g}(\overline{\nabla}_X \eta, Y) \quad (\text{since } \overline{g}(\eta, Y) = 0) \tag{28}$$

Using the formula in 5.3.10 and replacing Y by η, since $Y, \eta \in T_pM$

$\overline{\nabla}_X \eta = \nabla_X \eta + S(X,\eta)$, where ∇ and $\overline{\nabla}$ are Riemannian connections of g and $\overline{g}$ respectively.

Substituting into the equation (28),

$$\overline{g}\left(\overline{\nabla}_X \left(\frac{\eta}{\sqrt{\mu}}\right), Y\right) = \frac{1}{\sqrt{\mu}} \overline{g}(\nabla_X \eta + S(X,\eta), Y)$$

$$= \frac{1}{\sqrt{\mu}} \{ \overline{g}(\nabla_X \eta, Y) + \overline{g}(S(X,\eta), Y) \} \tag{29}$$

Consider $S(X,\eta) = \frac{1}{2\mu} \{ (X(\mu)\eta + \eta(\mu)X - g(X, \eta) grad\, \mu \}$

So $\quad S(X,\eta) = \frac{1}{2\mu} \{ (X(\mu)\eta + \eta(\mu)X \}$ (since $g(X, \eta) = 0$, X is orthogonal to η)

Taking the inner product with Y using $\overline{g}$,

$$\overline{g}(S(X,\eta), Y) = \frac{1}{2\mu} \{ (X(\mu)\overline{g}(\eta, Y) + \eta(\mu)\overline{g}(X, Y) \}$$

$$\bar{g}(S(X,\eta),Y) = \frac{1}{2\mu}\eta(\mu)\bar{g}(X,Y) \qquad (\text{since } \bar{g}(\eta,Y) = \mu g(\eta,Y) = 0)$$

Next consider $\bar{g}(\nabla_X \eta, Y) = \mu g(\nabla_X \eta, Y)$ (using the definition of conformal metric)

$$= -\lambda\mu\, g(X,Y) \qquad \text{(from (25) for being umbilic)}$$

$$= -\lambda \bar{g}(X,Y)$$

Then from equation (29),

$$\bar{g}(\bar{\nabla}_X(\frac{\eta}{\sqrt{\mu}}),Y) = \frac{1}{\sqrt{\mu}}(-\lambda\bar{g}(X,Y)) + \frac{1}{\sqrt{\mu}}\frac{1}{2\mu}\eta(\mu)\bar{g}(X,Y)$$

$$= \frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}}\bar{g}(X,Y).$$

Hence the result.

5.4. Spaces of Constant Curvature

5.4.1 Introduction

Among the Riemannian manifolds, those with constant sectional curvature are the simplest. We can now compute the sectional curvature of our three families of model spaces of Riemannian geometry:- Euclidean space, spheres, and hyperbolic spaces.

5.4.2 Euclidean space

The simplest and most important model Riemannian manifold is of course Euclidean space, $\Re^n$. Since we have shown that the curvature tensor of $\Re^n$ is identically zero in Euclidean space, clearly all-sectional curvatures are zero. This is obvious geometrically, since each two-dimensional section is actually a plane, which has zero Gaussian curvature.

5.4.3 Spheres

Our second model space is the sphere of radius r in $\Re^{n+1}$, denoted by S_r^n, with the metric induced from the Euclidean space $\Re^{n+1}$. When $r = 1$, this is simply called the unit sphere S^n in $\Re^{n+1}$.

5.4.3.1 Curvature of S^n

Let $S^n = \{x \in \Re^{n+1};\ |x| = 1\}$ be the unit sphere in $\Re^{n+1}$. As in the case of surfaces (see [DC 1], page 136-137) we define the Gauss spherical mapping as follows.

Let $M^n \subset \overline{M} = \Re^{n+1}$ be an n-dimensional hypersurface with the metric induced from Euclidean space and N be a smooth unit normal vector field along M. At each point $p \in M$, $N_p \in T_p(\Re^{n+1})$ can be thought of as a unit vector in $\Re^{n+1}$ and therefore as a point in S^n. Thus each choice of normal vector field defines a smooth map $N : M \to S^n$, called the *Gauss map* of M.

It is clear that the Gauss map is differentiable. The differential dN_p of N at $p \in M$ is a linear map from T_pM to $T_{N(P)}S^n$. By parallel translating the normal vector N along M at p to the vector field $ON(p)$ along S^n at O, where O is the origin in $\Re^{n+1}$, we can observe that T_pM parallel translates to $T_{N(P)}S^n$. Therefore we can identify dN_p as a linear map on T_pM itself.

The linear map $dN_p : T_pM \to T_pM$ as follows. For each parametrized curve $c(t)$ in M with $c(0) = p$ and $c'(0) = v$, we consider the parametrized curve $N \circ c(t) = N(t)$ in the sphere S^n.

Then $dN_p(v) = \frac{d}{dt}(N \circ c(t))\big|_{t=0} = \overline{\nabla}_v N$

We have $\langle N, N\rangle = 1$

Taking the covariant derivative in the direction of v,

$\langle \overline{\nabla}_v N, N \rangle = 0$

So $\overline{\nabla}_v N = (\overline{\nabla}_v N)^T = -S_\eta(v)$, (from proposition 5.2.7)

where $\eta \in (T_p M)^N$ and N is a local extension of η normal to M.

Therefore $dN_p(v) = -S_\eta(v)$, it follows that $-S_\eta$ is the derivative of the Gauss map.

We orient S^n by the inward pointing unit normal $N(x) = -x \in \Re^{n+1}$, $|x| = 1$.

That is, $N(c(t)) = -c(t) \Rightarrow dN_q(c'(t)) = -c'(t)$, where $q = c(t) = x$. This implies that dN_q is the negative of the identity map of $T_q M$. It follows that S_η has all of its eigenvalues equal to 1. Then using the expression in remark 5.2.11,

$K(e_i, e_j) - \overline{K}(e_i, e_j) = \lambda_i \lambda_j$, where $\overline{K}(e_i, e_j) = 0$ in Euclidean space.

Therefore $K(e_i, e_j) = 1$, for all $i, j = 1, \ldots, n$.

That is, all sectional curvatures of S^n are equal to 1. Hence the sectional curvature of the unit sphere $S^n \subset \Re^{n+1}$ is a constant equal to 1.

5.4.4 Hyperbolic space

To describe the model space of constant sectional curvature equal to -1, we can give the following example.

Consider the half space of $\Re^n$ given by

$H^n = \{(x_1, \ldots, x_n) \in \Re^n ;\ x_n > 0\}$ and introduce the metric on H^n,

$g_{ij}(x_1, \ldots, x_n) = \dfrac{\delta_{ij}}{x_n^2}$, where H^n is simply connected and complete. The metric on H^n is conformal to the usual metric of Euclidean space $\Re^n$, since $\dfrac{1}{x_n^2}$ is a positive differentiable function on H^n. Then H^n is called the *hyperbolic space* of dimension n. Write $g^{ij} = F^2\delta_{ij}$ to denote the inverse matrix of g_{ij}, where $F^2 = x_n^2$, then $g_{ij}g^{ij} = (\delta_{ij})^2$.

Take $\log F = f$ and differentiate both sides with respect to x_j.

$$\frac{1}{F}\frac{\partial F}{\partial x_j} = \frac{\partial f}{\partial x_j}$$

$$= f_j$$

$$\Rightarrow \qquad \frac{\partial F}{\partial x_j} = Ff_j$$

We can write $g_{ik} = \dfrac{\delta_{ik}}{F^2}$

Differentiating with respect to x_j,

$$\frac{\partial g_{ik}}{\partial x_j} = \delta_{ik}\frac{\partial}{\partial x_j}(\frac{1}{F^2}) = \delta_{ik}(-\frac{2}{F^3}\frac{\partial F}{\partial x_j}) = -2\frac{\delta_{ik}}{F^2}f_j \tag{30}$$

To calculate the coefficients of the curvature, first we have to find out the Christoffel symbols,

$$\Gamma_{ij}^k = \frac{1}{2}\sum_m\{\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{mi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\}g^{mk} \quad \text{(from equation (12) of remark 2.3.7)}$$

$$= \frac{1}{2}\sum_m\{\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{mi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\}F^2\delta_{mk} \quad \text{(since } g^{mk} = F^2\delta_{mk}\text{)}$$

Using the equation (30),

$$\Gamma_{ij}^k = \frac{1}{2}\{(\frac{-2\delta_{jk}f_i}{F^2}) + (\frac{-2\delta_{ki}f_j}{F^2}) + (\frac{2\delta_{ij}f_k}{F^2})\}F^2, \text{ if } m = k.$$

$$= -\delta_{jk}f_i - \delta_{ki}f_j + \delta_{ij}f_k$$

If all three indices are distinct then $\Gamma_{ij}^{k} = 0$.

If two indices are equal and $i \neq j$ then either $k = i$ and $\Gamma_{ij}^{i} = -f_j$ or $k = j$ and $\Gamma_{ij}^{j} = -f_i$.

If $i = j$ and $k \neq i$ then $\Gamma_{ii}^{k} = f_k$. This can be written as $\Gamma_{ii}^{j} = f_j$ if $i \neq j$.

Therefore, if two indices are equal and $i \neq j$ then $\Gamma_{ij}^{i} = -f_j, \Gamma_{ii}^{j} = f_j, \Gamma_{ij}^{j} = -f_i$.

If all three indices are equal then $\Gamma_{ii}^{i} = -f_i$, where $i = j = k$.

Using remark 3.2.4, we have

$$R_{ijij} = \sum_l R_{iji}^{l} g_{lj} = \sum_l R_{iji}^{l} \frac{\delta_{lj}}{F^2}$$

$$R_{ijij} = R_{iji}^{j} \frac{1}{F^2}, \quad \text{if } l = j. \tag{31}$$

We also have from remark 3.2.4,

$$R_{iji}^{j} = \sum_l \Gamma_{ii}^{l}\Gamma_{jl}^{j} - \sum_l \Gamma_{ji}^{l}\Gamma_{il}^{j} + \frac{\partial \Gamma_{ii}^{j}}{\partial x_j} - \frac{\partial \Gamma_{ji}^{j}}{\partial x_i}, \tag{32}$$

where $\Gamma_{ii}^{l} = f_l$ if $l \neq i$, $\Gamma_{jl}^{j} = -f_l$ if $l \neq j$

Consider the first summation,

$$\sum_l \Gamma_{ii}^{l}\Gamma_{jl}^{j} = \sum_{l \neq i, l \neq j} \Gamma_{ii}^{l}\Gamma_{jl}^{j} + \Gamma_{ii}^{i}\Gamma_{ji}^{j} + \Gamma_{ii}^{j}\Gamma_{jj}^{j}$$

$$= \sum_{l \neq i, l \neq j} - f_l^2 + (-f_i)(-f_i) + f_j(-f_j)$$

$$= \sum_{l \neq i, l \neq j} - f_l^2 + f_i^2 - f_j^2$$

Consider the second summation,

$$\sum_l \Gamma_{ji}^{l}\Gamma_{il}^{j} = \Gamma_{ji}^{i}\Gamma_{ii}^{j} + \Gamma_{ji}^{j}\Gamma_{ij}^{j} = -f_j^2 + f_i^2 \quad (\text{since } \Gamma_{ij}^{l} = 0, \text{if } l \neq i, l \neq j)$$

And we know that $\frac{\partial}{\partial x_j}\Gamma_{ii}^j = \frac{\partial f_j}{\partial x_j} = f_{jj}$ and $\frac{\partial}{\partial x_i}\Gamma_{ji}^j = -f_{ii}$

Substituting into equation (32),

$$R_{iji}^j = \sum_{l\neq i, l\neq j} -(f_l)^2 + f_{jj} + f_{ii}$$
$$R_{iji}^j = \sum_l - f_l^2 + f_i^2 + f_j^2 + f_{jj} + f_{ii}$$

Then the sectional curvature with respect to the plane generated by $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$ is

$K_{ij} = \frac{R_{ijij}}{\left|X_i \wedge X_j\right|^2}$, where $\frac{\partial}{\partial x_i} = X_i$ and $\frac{\partial}{\partial x_j} = X_j$ are orthogonal, since

$\langle X_i, X_j\rangle = g_{ij} = \frac{\delta_{ij}}{F^2} = 0$, if $i \neq j$.

$\langle X_i, X_i\rangle = g_{ii} = \frac{1}{F^2} = \left|X_i\right|^2$, similarly $\left|X_j\right|^2 = \frac{1}{F^2}$

Therefore $\left|X_i \wedge X_j\right|^2 = \frac{1}{F^4}$, $\qquad K_{ij} = R_{ijij}F^4$

$$K_{ij} = R_{iji}^j F^2 \quad \text{(from equation (31))}$$

$$= (\sum_l - f_l^2 + f_i^2 + f_j^2 + f_{jj} + f_{ii})F^2$$

In the case of $F^2 = x_n^2$, $\log F = \log\ x_n = f$

Taking the derivative of f with respect to x_n, then $f_n = \frac{1}{x_n}$.

Consider the case if $i \neq n, j \neq n$. Then $f_i = 0, f_j = 0$, (since *f* is a function of x_n)

Therefore $f_{ii} = 0, f_{jj} = 0$, where i, j =1…,n-1.

So $K_{ij} = -f_n^2 F^2 = -\frac{1}{x_n^2}F^2 = -1$.

Similarly if $i = n, j \neq n$, then $K_{nj} = -1$.

And also if $i \neq j, j = n$, then $K_{in} = -1$.

Then we can conclude that the sectional curvature of H^n is a constant equal to -1.

5.5 Riemannian Submersion

A Riemannian submersion has been defined somewhat differently to the way of defining a Riemannian immersion.

5.5.1 Definition

Suppose $\overline{M}$ and M smooth manifolds. A differentiable mapping f: $\overline{M}^{n+k=m} \to M^n$ is called a *submersion* if f is surjective, and for all $\overline{p} \in \overline{M}$, $df_{\overline{p}} : T_{\overline{p}}\overline{M} \to T_{f(\overline{p})}M$ has rank n. In this case, for all $p \in M$, the fiber $f^{-1}(p) = F_p$ is a submanifold of $\overline{M}$.

If $\overline{M}$ has a Riemannian metric $\overline{g}$, at each point $\overline{p} \in \overline{M}$, the tangent space $T_{\overline{p}}\overline{M}$ decomposes into direct sum $T_{\overline{p}}\overline{M} = (T_{\overline{p}}\overline{M})^h + (T_{\overline{p}}\overline{M})^v$, where $(T_{\overline{p}}\overline{M})^h$ and $(T_{\overline{p}}\overline{M})^v$ denote the subspaces of horizontal and vertical vectors respectively.

A vector $\overline{X} \in (T_{\overline{p}}\overline{M})^h \subset T_{\overline{p}}\overline{M}$ is called *horizontal* if it is orthogonal to the fibre. If $X \in \aleph(M)$, the horizontal lift $\overline{X}$ of X is the horizontal field defined by $df_{\overline{p}}(\overline{X}(\overline{p})) = X(f(\overline{p})) = X(p)$. The horizontal lift of a vector field X on M is the unique vector field $\overline{X}$ on $\overline{M}$ which is horizontal and projects onto X.

A vector $\eta \in (T_{\bar{p}}\overline{M})^v \subset T_{\bar{p}}\overline{M}$ is called *vertical* if it is tangent to the fiber, where $(T_{\bar{p}}\overline{M})^v$ = Ker $df = \{\eta \in (T_{\bar{p}}\overline{M}) \ / \ df_{\bar{p}}(\eta) = 0\}$. Let g be a Riemannian metric on M. The mapping f is said to be a *Riemannian submersion* if $\bar{g}(\overline{X}, \overline{Y}) = g(df_{\bar{p}}(\overline{X}), df_{\bar{p}}(\overline{Y}))$.

5.5.2 Connection of a Riemannian submersion

Let ∇ and $\overline{\nabla}$ be the Riemannian connections of M and $\overline{M}$ recpectively. Then we can find a relationship between the connections in tems of a vertical vector. That is, if $\overline{X}, \overline{Y}$ are horizontal lift of X and Y respectively, where X and Y are tangent to M, then $\overline{\nabla}_{\overline{X}}\overline{Y} = (\overline{\nabla_X Y}) + \frac{1}{2}[X,Y]^v$, where $[X,Y]^v$ is the vertical component of $[X,Y]$.

Proof:

Let X, Y, $Z \in \aleph(M)$. Let $T \in \aleph(M)$ be a vertical field.

Let $\overline{X}, \overline{Y}$ and $\overline{Z}$ be horizontal lifts of X, Y and Z respectively.

Consider $\langle T, [\overline{X}, \overline{Y}] \rangle = \langle T, \ \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} \rangle$

$$= \langle T, \overline{\nabla}_{\overline{X}}\overline{Y} \rangle - \langle T, \overline{\nabla}_{\overline{Y}}\overline{X} \rangle$$

$$= \langle T, (\overline{\nabla}_{\overline{X}}\overline{Y})^v \rangle - \langle T, (\overline{\nabla}_{\overline{Y}}\overline{X})^v \rangle \qquad \text{(since } T \in (T_{\bar{p}}\overline{M})^V \text{)}$$

That is, $\langle T, (\overline{\nabla}_{\overline{X}}\overline{Y})^v - (\overline{\nabla}_{\overline{Y}}\overline{X})^v \rangle = \langle T, [\overline{X}, \overline{Y}]^v \rangle$ (33)

We know that $\langle \overline{X}, T \rangle = 0$.

Taking the covariant derivative in the direction of $\overline{Y}$,

$$\overline{Y}\langle \overline{X}, T \rangle = 0$$

$\langle \overline{\nabla}_{\overline{Y}} \overline{X}, T\rangle + \langle \overline{X}, \overline{\nabla}_{\overline{Y}} T\rangle = 0$

$\langle (\overline{\nabla}_{\overline{Y}} \overline{X})^v, T\rangle + \langle \overline{X}, (\overline{\nabla}_{\overline{Y}} T)^h\rangle = 0$ (since $\langle (\overline{\nabla}_{\overline{Y}} \overline{X})^h, T\rangle = 0, \langle \overline{X}, (\overline{\nabla}_{\overline{Y}} \overline{X})^v\rangle = 0$)

$\Rightarrow \langle (\overline{\nabla}_{\overline{Y}} \overline{X})^v, T\rangle = 0$ (since T is a vertical field, then $(\overline{\nabla}_{\overline{Y}} T)^h = 0$.)

Similarly $\langle (\overline{\nabla}_{\overline{X}} \overline{Y})^v, T\rangle = 0$

Adding these two results,

$\langle (\overline{\nabla}_{\overline{Y}} \overline{X})^v, T\rangle + \langle (\overline{\nabla}_{\overline{X}} \overline{Y})^v, T\rangle = 0$

$$\langle (\overline{\nabla}_{\overline{X}} \overline{Y})^v + (\overline{\nabla}_{\overline{Y}} \overline{X})^v, T\rangle = 0 \qquad (34)$$

Adding the equations (33) and (34),

$2\langle (\overline{\nabla}_{\overline{X}} \overline{Y})^v, T\rangle = \langle [\overline{X}, \overline{Y}]^v, T\rangle$

$(\overline{\nabla}_{\overline{X}} \overline{Y})^v = \frac{1}{2}[\overline{X}, \overline{Y}]^v$, for all $T \in (T_{\overline{p}} \overline{M})^v$

But $\overline{\nabla}_{\overline{X}} \overline{Y} = (\overline{\nabla}_{\overline{X}} \overline{Y})^h + (\overline{\nabla}_{\overline{X}} \overline{Y})^v$

$= \overline{\nabla_X Y} + \frac{1}{2}[\overline{X}, \overline{Y}]^v$. Hence the result

5.5.3 Curvature of a Riemannian submersion

Let R and $\overline{R}$ be the curvature tensors of M and $\overline{M}$ respectively. Then we can find the relationship between R and $\overline{R}$ in terms of vertical vectors as in the case of the Riemannian immersion. Using that relationship, we then find a result between the sectional curvatures of M and $\overline{M}$.

(a) $\langle \overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W} \rangle = \langle R(X,Y)Z,W\rangle - \frac{1}{4}\langle [\overline{X},\overline{Z}]^{v},[\overline{Y},\overline{W}]^{v}\rangle +$

$$\frac{1}{4}\langle [\overline{Y},\overline{Z}]^{v},[\overline{X},\overline{W}]^{v}\rangle - \frac{1}{2}\langle [\overline{Z},\overline{W}]^{v},[\overline{X},\overline{Y}]^{v}\rangle$$

(b) $K(\sigma) = \overline{K}(\overline{\sigma}) + \frac{3}{4}\left|[\overline{X},\overline{Y}]\right|^{2} \geq \overline{K}(\overline{\sigma})$, where σ is the plane generated by the orthonormal vectors X, $Y \in \aleph(M)$ and $\overline{\sigma}$ is the plane genarated by $\overline{X},\overline{Y}$.

Proof:

(a) First we show that $\overline{X}\langle\overline{Y},\overline{Z}\rangle = X\langle Y,Z\rangle$

Consider $\overline{X}\langle\overline{Y},\overline{Z}\rangle = \langle \overline{\nabla}_{\overline{X}}\overline{Y},\overline{Z}\rangle + \langle\overline{Y},\overline{\nabla}_{\overline{X}}\overline{Z}\rangle$

$$= \langle (\overline{\nabla}_{\overline{X}}\overline{Y})^{h},\overline{Z}\rangle + \langle \overline{Y},(\overline{\nabla}_{\overline{X}}\overline{Z})^{h}\rangle$$

$$= \langle \overline{\nabla_{X}Y},\overline{Z}\rangle + \langle \overline{Y},\overline{\nabla_{X}Z}\rangle$$

Using the definition of Riemannian submersion,

$$= \langle df_{\overline{p}}(\overline{\nabla_{X}Y}), df_{\overline{p}}(\overline{Z})\rangle + \langle df_{\overline{p}}(\overline{Y}), df_{\overline{p}}(\overline{\nabla_{X}Z})\rangle$$

$= \langle \nabla_{X}Y, Z\rangle + \langle Y,\nabla_{X}Z\rangle$ (from the definition of horizontal lift)

Then $\qquad \overline{X}\langle\overline{Y},\overline{Z}\rangle = X\langle Y,Z\rangle$

Replacing $\overline{Y}$ by $\overline{\nabla}_{\overline{Y}}\overline{Z}$ and $\overline{Z}$ by $\overline{W}$, we have

$$\overline{X}\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\rangle = X\langle\nabla_{Y}Z, W\rangle$$

$$\langle \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z}, \overline{W}\rangle + \langle \overline{\nabla}_{\overline{Y}}\overline{Z}, \overline{\nabla}_{\overline{X}}\overline{W}\rangle = \langle \nabla_{X}\nabla_{Y}Z, W\rangle + \langle \nabla_{Y}Z, \nabla_{X}W\rangle \qquad (35)$$

From 5.5.2 we know that $\overline{\nabla}_{\overline{Y}}\overline{Z} = \overline{\nabla_{Y}Z} + \frac{1}{2}[\overline{Y},\overline{Z}]^{V}$, and

$$\overline{\nabla}_{\overline{X}}\overline{W} = \overline{\nabla_X W} + \frac{1}{2}[\overline{X},\overline{W}]^V$$

Substituting into equation (35),

$$\langle \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z}, \overline{W}\rangle + \langle \overline{\nabla_Y Z} + \frac{1}{2}[\overline{Y},\overline{Z}]^V, \overline{\nabla_X W} + \frac{1}{2}[\overline{X},\overline{W}]^V\rangle = \langle \nabla_X\nabla_Y Z, W\rangle +$$

$$\langle \nabla_Y Z, \nabla_X W\rangle$$

$$\langle \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z}, \overline{W}\rangle + \langle \overline{\nabla_Y Z}, \overline{\nabla_X W}\rangle + \frac{1}{4}\langle [\overline{Y},\overline{Z}]^V, [\overline{X},\overline{W}]^V\rangle = \langle \nabla_X\nabla_Y Z, W\rangle +$$

$$\langle \nabla_Y Z, \nabla_X W\rangle$$

(since $\langle \overline{\nabla_Y Z}, [\overline{X},\overline{W}]^V\rangle = 0$ and $\langle \overline{\nabla_X W}, [\overline{Y},\overline{Z}]^V\rangle = 0$)

Using the definition of the Riemannian submersion,

$$\langle \overline{\nabla_Y Z}, \overline{\nabla_X W}\rangle = \langle\, df_{\overline{p}}(\overline{\nabla_Y Z}), df_{\overline{p}}(\overline{\nabla_X W})\rangle = \langle \nabla_Y Z, \nabla_X W\rangle$$

Substituting into the above expression,

$$\langle \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z}, \overline{W}\rangle = -\frac{1}{4}\langle [\overline{Y},\overline{Z}]^V, [\overline{X},\overline{W}]^V\rangle + \langle \nabla_X\nabla_Y Z, W\rangle \qquad \text{(A)}$$

Similarly,

$$\langle \overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z}, \overline{W}\rangle = -\frac{1}{4}\langle [\overline{X},\overline{Z}]^V, [\overline{Y},\overline{W}]^V\rangle + \langle \nabla_Y\nabla_X Z, W\rangle \qquad \text{(B)}$$

We can show that $\langle [T,\overline{X}], \overline{Y}\rangle = 0$, where T is a vertical field.

Since $\langle [T,\overline{X}], \overline{Y}\rangle = \langle \overline{\nabla}_T\overline{X} - \overline{\nabla}_{\overline{X}}T, \overline{Y}\rangle$

$= \langle (\overline{\nabla}_T\overline{X})^h, \overline{Y}\rangle - \langle (\overline{\nabla}_{\overline{X}}T)^h, \overline{Y}\rangle$ ($\overline{Y}$ is a horizontal vector field)

So $\langle [T,\overline{X}],\overline{Y}\rangle = 0$ (since $(\overline{\nabla}_T \overline{X})^h = 0,\ (\overline{\nabla}_{\overline{X}} T)^h = 0$)

Hence $\langle \overline{\nabla}_T \overline{X}, \overline{Y}\rangle = \langle \overline{\nabla}_{\overline{X}} T, \overline{Y}\rangle$ (36)

Also we have $\langle \overline{Y}, T\rangle = 0$

Differentiating in the direction of $\overline{X}$,

$$\overline{X}\langle \overline{Y}, T\rangle = 0$$

$$\langle \overline{\nabla}_{\overline{X}} \overline{Y}, T\rangle + \langle \overline{Y}, \overline{\nabla}_{\overline{X}} T\rangle = 0$$

That is, $\langle \overline{\nabla}_{\overline{X}} T, \overline{Y}\rangle = -\langle \overline{\nabla}_{\overline{X}} \overline{Y}, T\rangle$ (37)

Combining (36) and (37), we have

$$\langle \overline{\nabla}_T \overline{X}, \overline{Y}\rangle = -\langle \overline{\nabla}_{\overline{X}} \overline{Y}, T\rangle \tag{38}$$

Consider $\langle \overline{\nabla}_{[\overline{X},\overline{Y}]} \overline{Z}, \overline{W}\rangle = \langle \overline{\nabla}_{[\overline{X},\overline{Y}]^h + [\overline{X},\overline{Y}]^v} \overline{Z}, \overline{W}\rangle$

$$= \langle \overline{\nabla}_{[\overline{X},\overline{Y}]^h} \overline{Z}, \overline{W}\rangle + \langle \overline{\nabla}_{[\overline{X},\overline{Y}]^v} \overline{Z}, \overline{W}\rangle$$

Using the definition of Riemannian submersion and equation (38)

$$\langle \overline{\nabla}_{[\overline{X},\overline{Y}]} \overline{Z}, \overline{W}\rangle = \langle df_{\overline{p}}(\overline{\nabla_{[X,Y]} Z}), df_{\overline{p}}(\overline{W})\rangle + \langle \overline{\nabla}_{[\overline{X},\overline{Y}]^v} \overline{Z}, \overline{W}\rangle$$

$$= \langle \nabla_{[X,Y]} Z, W\rangle + \langle \overline{\nabla}_{[\overline{X},\overline{Y}]^v} \overline{Z}, \overline{W}\rangle$$

$$= \langle \nabla_{[X,Y]} Z, W\rangle - \langle \overline{\nabla}_{\overline{Z}} \overline{W}, [\overline{X},\overline{Y}]^v\rangle$$

$$= \langle \nabla_{[X,Y]} Z, W\rangle - \langle (\overline{\nabla}_{\overline{Z}} \overline{W})^v, [\overline{X},\overline{Y}]^v\rangle \tag{39}$$

From 5.5.2 we know $(\overline{\nabla}_{\overline{Z}} \overline{W})^v = \frac{1}{2}[\overline{Z},\overline{W}]^v$

Therefore from equation (39),

$$\langle \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z}, \overline{W} \rangle = \langle \nabla_{[X,Y]} Z, W \rangle - \langle \frac{1}{2}[\overline{Z},\overline{W}]^{v}, [\overline{X},\overline{Y}]^{v} \rangle \tag{C}$$

Using the definitions of the curvature tensor and the equations (A), (B) and (C),

$$\langle \overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W} \rangle = \langle \overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z},\overline{W} \rangle - \langle \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W} \rangle + \langle \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z},\overline{W} \rangle,$$

$$\langle R(X,Y)Z,W \rangle = \langle \nabla_Y \nabla_X Z, W \rangle - \langle \nabla_X \nabla_Y Z, W \rangle + \langle \nabla_{[X,Y]} Z, W \rangle$$

$$\langle \overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W} \rangle = \langle R(X,Y)Z,W \rangle - \frac{1}{4}\langle [\overline{X},\overline{Z}]^{v}, [\overline{Y},\overline{W}]^{v} \rangle + \frac{1}{4}\langle [\overline{Y},\overline{Z}]^{v}, [\overline{X},\overline{W}]^{v} \rangle - \frac{1}{2}\langle [\overline{Z},\overline{W}]^{v}, [\overline{X},\overline{Y}]^{v} \rangle.$$

Hence the result.

(b) Let $K(\sigma)$ and $\overline{K}(\overline{\sigma})$ be the sectional curvatures of M and $\overline{M}$ respectively.

Then $K(\sigma) = \dfrac{\langle R(X,Y)X,Y \rangle}{|X \wedge Y|^2}$, $\overline{K}(\overline{\sigma}) = \dfrac{\langle \overline{R}(\overline{X},\overline{Y})\overline{X},\overline{Y} \rangle}{|\overline{X} \wedge \overline{Y}|^2}$,

where X and Y are orthonormal, therefore $|X \wedge Y|^2 = 1$ and $|\overline{X} \wedge \overline{Y}|^2 = 1$.

Considering part (a),

$$\langle \overline{R}(\overline{X},\overline{Y})\overline{X},\overline{Y} \rangle = \langle R(X,Y)X,Y \rangle - \frac{3}{4}\langle [\overline{X},\overline{Y}]^{v}, [\overline{X},\overline{Y}]^{v} \rangle$$

$$\overline{K}(\overline{\sigma}) = K(\sigma) - \frac{3}{4}\left|[\overline{X},\overline{Y}]^{v}\right|^2$$

$$K(\sigma) = \overline{K}(\overline{\sigma}) + \frac{3}{4}\left|[\overline{X},\overline{Y}]^{v}\right|^2 \geq \overline{K}(\overline{\sigma}).$$

Hence the result.

Chapter 6

Conclusion

In this thesis we have progressed from the familiar notion of surface in $\Re^3$ to the general notion of manifold. Now we are in a position to reverse this process. An n-dimensional manifold M is a space that is locally like the Euclidean space, $\Re^n$. Every manifold has a calculus consisting of differentiable functions, tangent vectors, vector fields, mappings, etc. The simplest manifold of dimension n is $\Re^n$ itself. A two-dimensional manifold is called a surface, which generalizes the Euclidean plane by replacing the dot product on tangent vectors, by arbitrary inner products.

In Riemannian geometry, the length of a curve is a geometric notion of intrinsic distance directly generalizing the familiar Euclidean distance in the plane. In chapter 4, we defined that the geodesics are curves with acceleration zero. Geodesics are not only the straightest curves but also the shortest curves. This generalizes the simple Euclidean rule, which says that a straight line is the shortest distance between two points. Geodesics in an arbitrary surface generalize the straight lines in Euclidean geometry.

In chapter 2, we described two properties that determined a unique connection on any Riemannian manifold called Riemannian connection. Then we computed Christoffel symbols of the Riemannian connection and observed that those symbols are zero in Euclidean space. Next we proved that the covariant derivative coincides with the usual derivative in Euclidean space. In Riemannian space the covariant derivative differs from the usual derivative by terms which involve the Christoffel symbols.

In chapter 5, we focused on the special case of hypersurfaces in $\Re^{n+1}$ and showed how the second fundamental form is related to the principal curvatures and Gaussian curvature. We proved a generalization of the theorem Egregium of Gauss. This allowed us to relate the notion of curvature in Riemannian manifolds to the classical concept of Gaussian curvature for surfaces.

Finally we computed the sectional curvatures of our model Riemannian manifolds. The sectional curvatures of Euclidean space, unit sphere, S^n, and hyperbolic space are 0, 1 and -1 respectively. Comparing with the Gaussian curvature of model spaces of surfaces we can conclude that the model spaces of Riemannian manifolds are the natural generalization of the model spaces of surfaces. Concerning all above results we can conclude that Euclidean space is a special case of Riemannian space.

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