Contact Systems and Contact Integrators

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Abstract

This thesis is concerned with the study of contact systems, which are ordinary differential equations whose flow preserves a contact structure. We study contact systems from both an analytical and numerical point of view. The traditional point of view is to study the Reeb vector field of a contact form. However, if the contact Hamiltonian vanishes then its contact vector field is not the Reeb vector field of any contact form equivalent to the given one. In this thesis we study exactly this case, when the contact Hamiltonian vanishes on some submanifold of phase space. This submanifold is invariant under the flow and we study the flow on it, including its stability and fixed points.

The natural numerical method for a contact system is a 'contact integrator', a map that preserves the contact structure, which is suitable for exploring the long-time dynamics of contact systems. These have not been studied very much in geometric integration. In order to formulate our results and some consequences for contact integrators, we give a thorough development of the symplectification of a contact system and have found the integrable contact systems related to integrable homogeneous Hamiltonian systems via symplectification. We develop contact integrators by the splitting method, leading to an explicit contact integrator for any polynomial contact vector field. We also study how symplectic integrators for Hamiltonian systems and volume-preserving integrators for divergence-free systems are related to contact integrators for contact systems.

Acknowledgements

Life is filled with journeys. In my life, the longest journey was my PhD study. I commenced PhD study (Bezout's Theorem in Commutative Algebra and Algebraic Geometry) with Professor Wolfgang Vogel on June 1995. However, the sudden death of Professor Vogel during October 1996 changed my research topic. My research was suspended for 2 years, due to illness and to take care of my son. I was teaching at a NZ high School, Professor Robert McLachlan encouraged me to recommence my PhD study in Geometric Integration. I started work on this thesis in 2000. Before I could even organize this thesis, my father died, and I could not see him at the end due to the research.

I am happy to have achieved this milestone in my life. Even though it has brought me both joy and sorrow, it has become a part of my life.

This work was possible because of all the help from people around me, who encouraged me at all times. I thank everyone. Especially, I would like to thank my supervisor Professor McLachlan who has been unfailing in helping me get the resources and information I needed to complete this thesis. Dr. Kee Teo, Dr. Gillian Thornley and Dr. Tammy Smith have been most generous and helpful with their advice, recommendations and discussions and very kind in helping me over the past 8 years. Also I thank all the staff and graduate students in the Discipline of Mathematics. My warmest gratitude goes to my son Kyung-Il, my husband Joong-Ki Min, and to my mother for so much love and support. I couldn't have made it without them.

And last of all, I am really thankful to Professor Vogel who took care of my study until the last moment of his life. It was really sad that he was not able to see the end of my research. I dedicate this thesis to him and my father.

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Nomenclature

We list below standard notation of Differential Geometry (see, for instance, [66, 70]) used in this thesis. Entries are generally arranged in order of occurrence in the thesis.

C^∞	the set of infinitely differentiable functons
$\wedge$	exterior product
T_x^*M	the set of 1-forms at the point x of a manifold M, dual to T_xM
T_xM	the set of tangent vectors at the point x of a manifold M
T^*M	the cotangent bundle of M
TM	the tangent bundle of M, dual to T^*M
d	exterior derivative
f_*	the push-forward map derived from f
f^*	the pull-back map derived from f
dx	the gradient of the x coordinate function
$L_X\omega$	the Lie derivative of ω by the vector field X, ω is a symplectic form
$i_X\omega$	the interior product of the differential form ω by the vector field X
$\mathcal{X}(M)$	the set of vector fields on M
S^n	the surface of the sphere in $(n+1)$-dimensional Euclidean space
π	a projection map
$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$	a basis for the tangent space of $\mathbb{R}^3$
$[X, Y]$	the Lie bracket of the vector fields X and Y
$\{F, G\}$	the Poisson bracket of the functions F and G
$[f, g]$	the Jacobi bracket of the functions f and g
div	the divergence operator, as in $\mathrm{div}_\Omega X$

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Chapter 1

Introduction

1.1 Brief History

Numerical integrators for solving ordinary differential equations (ODEs) have a long history with significant interplay among the natural sciences and technology. They are extensively employed in mathematics, mechanics [51], astronomy [58, 47], physics [97], and in many problems of chemistry [96, 12] and biology. The objective laws governing certain phenomena can be written as differential equations. Therefore the equations themselves are a quantitative expression of these laws. The qualitative nature of solutions of differential equations can be determined by studying their geometry [54].

Geometric numerical methods for solving ODEs have been widely used in both pure and applied science and engineering. The most important examples of this approach were Hamiltonian systems or Lagrangian systems. During the last decade, there has been an increasing interest in numerical integration techniques for qualitatively correct simulation of ODEs. These structure-preserving algorithms for ODEs are known as geometric integrators. Geometric integration refers to numerical methods which preserve the qualitative and geometric features (the underlying structures) of

ODEs after discretization. Geometric integration has achieved remarkable successes in the numerical analysis of ODEs [22, 54]. Examples of geometric integrators that have been constructed for special classes of differential equations are symplectic and multisymplectic integrators for Hamiltonian or Poisson structures [40, 28, 107, 106, 69, 20], variational integrators for Lagrangian and canonical Hamiltonian systems [9, 79, 69], conservative integrators that preserve integrals or conservation laws [61, 62, 68, 94, 95, 63, 1], symmetric integrators that preserve the symmetries of the systems [21, 30], volume-preserving integrators [43, 85, 93], Lie group integrators for ODEs and discretizations of PDEs [57, 56], Lie-Poisson integrators [109, 15, 64], and isospectral integrators [23, 108, 24].

One approach to classifying geometric integrators is by using the Lie pseudogroups of local diffeomorphisms of a manifold. Any group of diffeomorphisms is a pseudogroup, but in general, the elements of a pseudogroup only need to be *local* diffeomorphisms. (See [76] for a precise definition.) The first classification of the so-called 'primitive' Lie pseudogroups; that is, those that do not leave any foliation invariant, was obtained by E. Cartan in 1913 [27]. It contains six classes of the diffeomorphism on any manifold; namely (i) Diff, the group of all diffeomorphisms of the manifold; (ii) Diff_Ω where Ω is a volume form, the group of volume preserving diffeomorphisms; (iii) Diff_ω where ω is a symplectic 2-form, the group of symplectic diffeomorphisms; (iv)Diff_α where α is a contact 1-form, the group of contact diffeomorphisms; (v) Diff^c_ω where c is a constant, the group of conformal symplectic diffeomorphisms; (vi) Diff^c_Ω, the group of conformal volume preserving diffeomorphisms. Strictly speaking, this is the classification in the complex case; in the real case these is a further minor refinement (see [77]). While classes (i)-(iii) are widely studied in the context of dynamical systems and class (v) has recently been utilized by McLachlan and Perlmutter [74], the contact group (iv) has not been extensively studied in this setting. This thesis is about contact structures, which are one of the Lie pseudogroup structures.

Contact geometry is an odd dimensional analog of symplectic geometry. The name 'contact' seems to be due to S. Lie when in 1872 [16], he introduced the notion of a contact transformation in his study of differential equations. Contact geometry has applications in many other areas, including non-holonomic mechanics [4], thermodynamics (this line of work was begun in 1873 by J.W Gibbs) [50] and hydrodynamics. S. Lie, E. Cartan, G. Darboux and many other great mathematicians devoted a lot of their work to this subject. Ever since the work of G. Reeb [87], on some properties of the trajectories of dynamical systems, the study of contact geometry has been considerably developed in the last fifty years. In the last 20 years or so, new important interactions with symplectic geometry and many other subjects were discovered. These include Hamiltonian mechanics, symplectic [35] and sub-Riemannian geometry [13], foliation theory [33, 65], complex geometry and analysis [49], topological hydrodynamics [36], the study of three-manifolds [32], and knot theory [89, 37]. Despite its long history and all the recent remarkable successes, contact geometry is still behind its symplectic counterpart [34].

1.2 General Summary of the Thesis

Consider the ODEs

$$X_K \begin{cases} \dot{x} = -K_y + xK_z \\ \dot{y} = K_x \\ \dot{z} = K - xK_x \end{cases}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}$. This is the (local) general form of a contact differential equation (DE) in $(2n+1)$-variables. The contact DEs are generated by the scalar function K called the contact Hamiltonian function. Contact DEs have not been studied much in dynamical systems. Usually, they are thought to be similar to Hamiltonian DEs. Furthermore, contact systems in geometric integration are also not widely discussed and not as well developed as Hamiltonian systems.

In this thesis, we consider two central questions:

1. What are the special features of these DEs?
2. How can we best solve them numerically?

We now provide a summary of this thesis.

- **Preliminaries**

  In Chapter 2 we state the basic tools of symplectic geometry. In particular, we present Hamiltonian systems which will be used when we discuss contact systems, and contact and symplectic integrators.

  In Chapter 3 we discuss the general background of contact geometry as required for this thesis. We cover contact structures, contact forms, and contact manifolds; Darboux's theorem in the contact case; the Reeb vector field of a contact form; and contact Hamiltonian functions and their associated contact vector fields. The last section of Chapter 3 covers a detailed development of the relationship between homogeneous Hamiltonian systems and contact systems, which will be needed for the construction of contact integrators.

- **Contact Systems**

  Chapter 4 and Chapter 5 contain an analytical study of contact systems. In Chapter 4 we first discuss the general features of contact systems on contact manifolds. We are particularly interested in the features of contact systems on the zero section of the contact Hamiltonian, which are presented in Section 4.2. This zero section is invariant under the flow, however the special nature of the flow on this zero section has not been recognized before. (Most authors study the Reeb vector field, whose contact Hamiltonian does not vanish.) In Section 4.3 we study the stability of the fixed points of a contact system and show some examples concerning the stability of a contact system.

One of the main goals of this thesis is to construct contact integrators for contact systems. In order to do this, we discuss in Chapter 5 the integrable contact systems, which are related to integrable homogeneous Hamiltonian systems. In Section 5.1 we recall some elementary definitions and results about integrals and integrable systems of DEs. In Section 5.2 we first review the notion of the Poisson bracket and integral for Hamiltonian systems, and then discuss some properties of the Jacobi bracket and integral for contact systems. In Section 5.3 we present our main results concerning integrable contact systems. To provide an insight to the integrability of contact systems, in this section we also discuss the integrability of homogeneous Hamiltonian systems.

- **Contact Integrators**

Contact integrators have been discussed in a single paper, due to Feng in 1998 [42]. In Chapters 6 and 7 we discuss how symplectic and volume integrators are related to contact integrators.

In Chapter 6 we discuss how contact integrators are constructed for contact systems. There are two main methods for constructing contact integrators. The first is known as symplectification, which relies on a change of variables to a canonical homogeneous Hamiltonian system. Each choice of a homogeneous symplectic integrator yields a contact integrator. The second is the splitting method, which relies on expressing the contact vector field as a sum of integrable contact vector fields. We show that (unlike the Hamiltonian case), one cannot rely on splitting the contact vector field into vector fields whose orbits are straight lines. However, symplectification allows us to describe large families of easily integrable contact systems, sufficient eventually to yield an explicit contact integrator for any polynomial contact Hamiltonian.

In Chapter 7 we discuss how contact systems are related to source-free systems and construct volume-preserving integrators for some contact systems.

(In contrast to the Hamiltonian case, not all contact systems are volume-preserving.) In Section 7.1 we review general source-free systems with arbitrary volume forms. In Section 7.2 we derive the source free system with a rescaled volume form from the Reeb vector field associated with a rescaled contact form. In Section 7.3 we survey the volume-preserving splitting method and apply this method to contact systems. We also apply three further methods (contact integrator, leapfrog splitting method and ODE45 in MATLAB).

- **Conclusion**

  In Chapter 8 we outline the main results obtained in this thesis, and suggest some topics for further studies.

1.2.1 Main Achievements

The main new results in this thesis are presented in Sections 4.1, 4.2, 4.3, 5.2.2, 5.3, 6.1.2, 6.2, 7.2 and 7.3. More specifically, the following are new

- Section 4.1

  Proposition 9, Proposition 10, Corollary 1, Proposition 11

- Section 4.2

  Proposition 12, Proposition 13, Proposition 15, Proposition 16, Example 8, Proposition 17

- Section 4.3

  Proposition 18, Proposition 19, Theorem 9, Example 9, Example 10, Example 11

- Section 5.2

  Proposition 21, Proposition 24, Proposition 25

- Section 5.3

  Theorem 11, Example 13, Example 14, Example 15, Theorem 12, Theorem 13, Theorem 14, Theorem 15, Theorem 16, Example 16

- Section 6.1.2

  Example 17, Example 18, Example 19

- Section 6.2

  Example 21, Example 22, Example 23, Theorem 17, Example 24, Theorem 18, Theorem 19, Theorem 20

- Section 7.2

  Proposition 27, Proposition 28, Proposition 29, Proposition 30

- Section 7.3

  Example 26

However we have given new proofs for some standard results. We use the convention that if no reference appears immediately after a theorem (Theorems 9, 11-20) or a proposition (Propositions 9, 11, 13, 15-19, 21, 24, 25, 28), then the proof is due to the author.

Chapter 2

Symplectic Preliminaries

Symplectic integration is a numerical method that has been used for integrating a Hamiltonian system based on canonical or symplectic transformations. We describe the basic mathematical structures needed to study Hamiltonian mechanics. From the definition of a symplectic structure on a symplectic manifold, the Hamiltonian vector field can be uniquely defined. In this chapter, we shall give some elementary properties of symplectic manifolds, symplectic diffeomorphisms and briefly describe the concepts of Hamiltonian vector fields and Hamiltonian flows. For more detailed discussion of symplectic geometry see [2, 3, 55, 66].

For the sake of simplicity, we generally assume that all the objects (manifolds, functions, diffeomorphisms, ...) are C^∞.

2.1 Symplectic Structures

Symplectic geometry is the geometry of a closed nondegenerate two-form on an even-dimensional manifold. The symplectic structure is fundamental in the study of Hamiltonian systems.

2.1.1 Symplectic Manifolds

Before dealing with symplectic manifolds, we first define the non-degeneracy of a skew-symmetric bilinear form on a real vector space.

A **symplectic form** ω on a finite dimensional vector space V over $\mathbb{R}$ is a function ω: $V \times V \longrightarrow \mathbb{R}$ which satisfies the following properties.

- bilinear: $\omega(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 \omega(x_1, y) + \lambda_2 \omega(x_2, y) \quad \lambda_1, \lambda_2 \in \mathbb{R}$,
- skew-symmetric: $\omega(x, y) = -\omega(y, x)$ for all $x, y \in V$,
- non-degenerate: $\omega(x, y) = 0$ for all $x \in V$ implies that $y = 0$.

In this case, (V, ω) is called a **symplectic vector space**.

Definition 1. Let P be an even (say $2n+2$) dimensional manifold. A **symplectic structure** on P is given by a differential 2-form ω satisfying the following two properties:

- $\forall z \in P$, ω is non-degenerate ($\forall X \in T_z P \backslash \{0\} \;\; \exists Y |\, \omega(X, Y) \neq 0$),
- ω is closed ($d\omega$=0).

In this case, we say (P, ω) is a **symplectic manifold**, and ω is a **symplectic form**.

Since the nondegeneracy condition on ω requires that P (or V) must have even dimension, symplectic forms can exist on P (or V) only if P (or V) is even dimensional.

2.1.2 Symplectic Forms

Example 1. The standard (canonical) symplectic form

Let P be the vector space $\mathbb{R}^{2n+2}$ with local coordinate system $(p, q) = (p_0, \dots, p_n, q_0, \dots, q_n)$. Then

$$\omega_0 = dp \wedge dq = \sum_{i=0}^{n} dp_i \wedge dq_i \tag{2.1}$$

is a symplectic structure on P and $(\mathbb{R}^{2n+2}, \omega_0)$ is a symplectic manifold.

Example 2. The symplectic form on a cotangent bundle

An example of a symplectic manifold is the cotangent bundle of the configuration space in the study of the classical mechanical problems. Let the manifold P be the configuration space of a mechanical system. Then TP is the disjoint union (over all points of P) of the tangent spaces and is called the tangent bundle. At each point $q \in P$ the tangent space T_qP is a vector space and hence there is an associated dual vector space T_q^*P. The disjoint union of all these dual spaces over all the points of the manifold is the cotangent bundle T^*P and it is a symplectic manifold with the symplectic structure defined below.

The cotangent bundle is defined by

$$T^*P{:=}\{\text{linear maps } f : T_qP \to \mathbb{R},\ q \in P\}.$$

Using the coordinates $(q, p) = (q_0, \dots, q_n, p_0, \dots, p_n)$ where $q = (q_0, \dots, q_n)$ is the local coordinate system on P and $p = (p_0, \dots, p_n)$ is the local coordinate system of T_q^*P, we define the 1-form on T_qP to be $\sum_{i=0}^{n} p_i dq_i$. In order to define a symplectic structure on T^*P we next define a distinguished 1-form θ on T^*P: For $z \in T^*P$ and $X \in T_zT^*P$ define a 1-form θ, by

$$\theta(X) = z(\pi_* X)$$

where

$$\pi : T^*P \longrightarrow P$$

is the projection map and $\pi_* : TT^*P \longrightarrow TP$. If $q_0, \ldots, q_n$ are the local coordinates on P, $p_i = q_i \circ \pi$ and coordinates $p_0, \ldots, p_n$ are the local coordinates on T^*P, then the 1-form θ has the local expression

$$\theta = p\, dq = \sum_{i=0}^{n} p_i dq_i.$$

Consequently this canonical 1-form gives us a symplectic form ω on T^*P:

$$\omega = d\theta = dp \wedge dq = \sum_{i=0}^{n} dp_i \wedge dq_i.$$

The 1-form θ on T^*P is called the **Liouville form** (or the **canonical 1-form**). The cotangent bundle of the configuration space may be thought of as the phase space of a dynamical system. There are very close relations between cotangent bundles of a manifold and contact manifolds. We will revisit Example 2 in the next chapter.

Darboux's Theorem for Symplectic Manifolds

The next result says that every symplectic manifold is, with respect to suitable local coordinates, a symplectic vector space.

Theorem 1. (Darboux's Theorem) [3] *Let (P, ω) be a $(2n+2)$-dimensional symplectic manifold. Then ω is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n+2}$,*

$$\omega_0 = dp \wedge dq = \sum_{i=0}^{n} dp_i \wedge dq_i$$

(the coordinates in $\mathbb{R}^{2n+2}$ are taken to be $(p_0, \ldots, p_n, q_0, \ldots, q_n)$).

The coordinates taken above are sometimes called symplectic coordinates.

2.2 Symplectic Diffeomorphisms

Definition 2. Let (P, ω) and (P', ω') be $(2n+2)$-dimensional symplectic manifolds. A diffeomorphism $f : P \to P'$ is called a **symplectic diffeomorphism** if

$$f^*\omega' = \omega, \tag{2.2}$$

where f^* is the pull-back map corresponding to f.

Note that a symplectic diffeomorphism preserves the symplectic structure and it is also called a **canonical transformation**. Recall that $\mathrm{Diff}(P)$ is the group of all diffeomorphisms from a manifold P. For any symplectic manifold (P, ω), the set

$$\mathrm{Diff}_\omega(P) = \{f \in \mathrm{Diff}(P) | f^*\omega = \omega\}$$

is a group, called **the group of symplectic diffeomorphisms** of (P, ω).

Here is an important example of symplectic diffeomorphisms.

Proposition 1. *Let P be an n-dimensional manifold, $f : P \longrightarrow P$ be a diffeomorphism and $\hat{f} : T^*P \longrightarrow T^*P$ its lifting to the cotangent bundles. Then $\hat{f}$ is a symplectic diffeomorphism of (T^*P, ω) onto (T^*P, ω).*

The importance of this construction is that $\hat{f}$ is guaranteed to be symplectic; it is often called a point transformation because it arises from a diffeomorphism on points in the configuration space. From the previous section, we know that if $z = (q, p)$, then $\hat{f}(z)$ is understood to be the 1-form $(f^{-1})^*(pdq)$ on $T^*_{f(q)}P$ (where $\hat{f} := (f^{-1})^* = f^{*^{-1}}$). So we have

$$\hat{f}^*\omega = \hat{f}^*d\theta = d(\hat{f}^*\theta) = d\theta = \omega.$$

Thus the symplectic diffeomorphism $\hat{f}$ preserves the symplectic form $\omega = d\theta$. In canonical coordinates,

$$\hat{f}(q,p) = \big(f(q), (Tf(q))^{-\mathrm{T}} p\big)\,.$$

2.3 Hamiltonian Systems

Hamiltonian systems are well-known from classical mechanics and are an important area of application for ODE numerical integrators. Newton's second law with potential forces is equivalent to a system of Hamiltonian equations. In this section, we study such systems.

2.3.1 Hamiltonian Vector Fields

Let (P, ω) be a symplectic manifold. The nondegeneracy condition of ω gives rise to an isomorphism between vector fields and 1-forms as follows:

$$\begin{array}{rcl} TP & \longrightarrow & T^*P \\ X & \longmapsto & i_X\omega \\ \text{vector fields} & \longmapsto & \text{one forms} \end{array}$$

In particular, ω associates with any smooth function $H : P \longrightarrow \mathbb{R}$ on the symplectic manifold (P, ω) a vector field X defined by the condition:

$$i_X\omega = -dH. \tag{2.3}$$

That is, for all $v \in T_zP$, we have the identity

$$\omega(X(z), v) = -dH(z) \cdot v.$$

In this case we write X_H for X. Such a vector field X_H is called a **Hamiltonian vector field** with Hamiltonian (or energy) function H.

Note that the nondegeneracy condition of the symplectic structure ω on the symplectic manifold (P, ω) implies that X_H exists and is uniquely defined.

Hamilton's equations are the evolution equations

$$\dot{z} = X_H(z).$$

The usual Hamiltonian equations in canonical coordinates are

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{2.4}$$

where $z = (p, q)$, $z_i = p_i$, $z_{n+1+i} = q_i$, for $i = 0, \ldots, n$. Then the associated Hamiltonian vector field is

$$X_H(z) := J \cdot dH \tag{2.5}$$

where J is the symplectic matrix:

$$J = \begin{bmatrix} 0 & -I_{n+1} \\ I_{n+1} & 0 \end{bmatrix}.$$

Thus we have

$$i_{X_H}\omega = \sum_{i=0}^{n} i_{X_H}(dp_i \wedge dq_i) = -dH.$$

Conservation of the Hamiltonian

With the help of Hamilton's equations we can compute the time-derivative of the

Hamiltonian function H.

- The Hamiltonian is conserved if it does not depend explicitly on time t. Using local coordinates (p, q), we have

$$\begin{aligned} \dot{H} &= \frac{dH(p,q)}{dt} \\ &= \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} \\ &= -\frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} \\ &= 0. \end{aligned}$$

  On (P, ω) we have

$$\begin{aligned} \dot{H} &= X_H(H) \\ &= i_{X_H} dH \\ &= -i_{X_H} i_{X_H} \omega \\ &= 0. \end{aligned}$$

  That is, the integral trajectories of a Hamiltonian system lie on the energy surface H=constant.

- The Hamiltonian is not conserved if it does depend on t. In this case

$$\begin{aligned} \dot{H} &= \frac{dH(p,q,t)}{dt} \\ &= \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial t} \\ &= -\frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t}. \end{aligned}$$

2.3.2 Hamiltonian Flows

Hamiltonian flows on a symplectic manifold provide nice examples of symplectic diffeomorphisms. A vector field X is called locally Hamiltonian if $i_X\omega$ is closed. This is equivalent to $L_X\omega = 0$, where $L_X\omega$ denotes the Lie derivative of ω along X, because

$$\begin{aligned} L_X\omega &= i_X d\omega + d(i_X\omega) \\ &= d(i_X\omega). \end{aligned}$$

Every Hamiltonian vector field is a locally Hamiltonian, because

$$d(i_X\omega) = -ddH = 0.$$

Conversely, if X is locally Hamiltonian, it follows from the Poincare Lemma [70] that there locally exists a function H such that $i_X\omega = -dH$, so locally $X = X_H$, and thus the terminology is consistent.

We denote the Lie algebra of locally Hamiltonian vector fields X to be

$$\mathcal{L}_\omega(P) = \{X \in \mathcal{X}(P) | L_X\omega = 0\},$$

where $\mathcal{X}(P)$ is the Lie algebra of smooth vector fields on P.

Definition 3. The flow φ_t of a Hamiltonian vector field

$$\dot{\varphi}_t = X_H \circ \varphi$$

is called the **Hamiltonian flow** of H.

We recall the following property [29]:

- The Hamiltonian flow (of a time-dependent Hamiltonian) preserves the symplectic form ω; that is,

$$(\varphi_t^H)^*\omega = \omega,$$

since

$$\frac{d}{dt}(\varphi_t^H)^*\omega = (\varphi_t^H)^* L_{X_H}\omega = 0.$$

Chapter 3

Contact Preliminaries

Contact geometry can be considered as the odd-dimensional analogue of symplectic geometry. An odd-dimensional manifold can not admit a symplectic structure. The contact structures on an odd-dimensional manifold are intimately related to the symplectic structures on an even-dimensional manifold. We will see that all problems of contact geometry can be reformulated as problems of symplectic geometry. This can be done via symplectification.

In this chapter, we begin to recall some elementary definitions and results about contact manifolds and contact diffeomorphisms. We define Reeb vector fields and the contact vector fields. Finally, we state a correspondence between contact systems and homogeneous Hamiltonian systems. Our main references for this chapter can be found in Arnold [5], and Banyaga [11], Libermann and Marle [66], Geiges [48], and J.B. Etnyre [34].

3.1 Contact Structures

Contact geometry is the geometry of a maximally nondegenerate field of tangent hyperplanes on an odd-dimensional manifold. A symplectic structure is defined to be a 2-form on an even-dimensional manifold, while a contact structure is defined as the null space of a 1-form on an odd-dimensional manifold.

3.1.1 Contact Manifolds

Definition 4. Let M be an odd (say $2n+1$) dimensional manifold. A **contact form** α on M is a 1-form such that $\alpha \wedge (d\alpha)^n \neq 0$ everywhere (i.e., $\alpha \wedge (d\alpha)^n \neq 0$ is a volume form on M). Such a contact form determines a contact structure. The **contact structure** D (often called the contact distribution) is a codimension one distribution which is determined by the kernel of a contact form α; that is,

$$D = \operatorname{Ker} \alpha.$$

We say that the pair (M, α) is a $(2n+1)$-dimensional **contact manifold**.

We shall denote the contact structure by

$$D = \cup_{x \in M} D_x \quad \text{where } D_x = \{X \in T_x M : \alpha_x(X) = 0\}. \tag{3.1}$$

More precisely, for each point $x \in M$, we have a hyperplane D_x in $T_x M$, varying smoothly with respect to x in such a way that the Frobenius condition ($[X, Y] \in D$, $\forall X, Y \in D$) [5] fails everywhere.

Choose a basis $\{X_1, \ldots, X_{2n}, E\}$ of $T_x M$ for each $x \in M$, such that

$$D = \operatorname{Ker} \alpha = \operatorname{span}\{X_1, \ldots, X_{2n}\}.$$

By applying the volume form to the basis, we obtain

$$\alpha \wedge (d\alpha)^n(X_1, \ldots, X_{2n}, E) = \alpha(E) \cdot (d\alpha)^n(X_1, \ldots, X_{2n}) \neq 0,$$

since $\alpha(X_i)$ is zero. Hence $(d\alpha)^n(X_1, \ldots, X_{2n}) \neq 0$. This means that the restriction of $d\alpha$ to D is nondegenerate (that is, $d\alpha$ induces a symplectic form on D) (see [10]). So this can be described as

- dim $D = 2n$ is even
- $(d\alpha)^n|_D \neq 0$ is a volume form on D
- $(D_x,\ d\alpha|_x)$ is a symplectic vector space.

Since there cannot be a nondegenerate field of hyperplanes (contact planes) on an even-dimensional manifold, a nondegenerate field of hyperplanes exists only on an odd-dimensional manifold. Thus we have

- any contact manifold (M, α) has odd-dimension
- $\alpha \wedge (d\alpha)^n$ is a volume form on M.

Throughout this thesis, we shall work on a $(2n+1)$-dimensional contact manifold (M, α).

3.1.2 Contact Forms

We note that the contact form is not unique; that is, different contact forms can define the same field of hyperplanes D.

Theorem 2. [6] *Let α be a contact form and f some non-vanishing function on (M, α). Then the form $\alpha' = f\alpha$ is also a contact form. Moreover the form $d(f\alpha)$ restricted to D is nondegenerate, and* $\operatorname{Ker}\alpha' = \operatorname{Ker}\alpha$.

This theorem gives us two contact forms α and $\alpha' = f\alpha$ which define the same contact structure. Two contact forms α and α' on a manifold M are equivalent if (M, α) is diffeomorphic to (M, α').

Darboux's Theorem for Contact Manifolds

A natural question is: when are two contact forms equivalent? Locally, this question has a simple answer as it does for symplectic forms. A contact form can be brought into a standard form. Thus we have the following fact which is a consequence of the Darboux Theorem for symplectic manifolds.

Theorem 3. (Darboux's Theorem) [6] *Let (M, α) be a $(2n+1)$-dimensional contact manifold. Then (M, α) is locally diffeomorphic to $(\mathbb{R}^{2n+1}, \alpha_0)$ where*

$$\alpha_0 = xdy + dz = \sum_{i=1}^{n} x_i dy_i + dz$$

(the coordinates of $\mathbb{R}^{2n+1}$ are taken to be $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$).

Example 3. On the odd dimensional Euclidian space $\mathbb{R}^3$ with coordinates (x, y, z), we consider the 1- form

$$\alpha = xdy + dz.$$

Since

$$\alpha \wedge (d\alpha) = (xdy + dz) \wedge (dx \wedge dy) = (dx \wedge dy) \wedge dz \neq 0,$$

α is a contact form on $\mathbb{R}^3$. This is known as the **canonical contact form** on $\mathbb{R}^3$. Similarly on $\mathbb{R}^{2n+1}$ the canonical contact form is

$$\alpha = \sum_{i=1}^{n} x_i dy_i + dz.$$

The contact structure can be spanned by $\{\frac{\partial}{\partial x}, (\frac{\partial}{\partial y} - x\frac{\partial}{\partial z})\}$ on $\mathbb{R}^3$; that is,

$$D = \operatorname{Ker} \alpha = c_1 \frac{\partial}{\partial x} + c_2 \left(\frac{\partial}{\partial y} - x \frac{\partial}{\partial z} \right).$$

Another example is the $(2n+1)$-dimensional manifold $M = T^*N \times \mathbb{R}$.

Example 4. [16] Let N be an n-dimensional manifold and T^*N its cotangent bundle. We can define the 1-form (Liouville form) θ by the local expression

$$\theta = pdq = \sum_{i=1}^{n} p_i dq_i.$$

Let t be the coordinate on $\mathbb{R}$ and $f : M \longrightarrow T^*N$ the projection to the first factor. Then

$$\alpha = dt + f^*\theta$$

is a contact form on M.

3.2 Contact Diffeomorphisms

Definition 5. Let (M, α) and (M', α') be $(2n+1)$-dimensional contact manifolds. A diffeomorphism $f : M \to M'$ is called a **contact diffeomorphism** if

$$f^*\alpha' = \mu\alpha, \tag{3.2}$$

for some nonvanishing function $\mu : M \longrightarrow \mathbb{R}$.

This means that the contact form is conformally preserved under f. In other words, a contact diffeomorphism can be defined as a map preserving the contact structure. More specifically, if there are two contact structures D and D' for α and α', respectively, then

$$f_*(D) = D'.$$

We will denote **the group of contact diffeomorphisms** of (M, α) by

$$\mathrm{Diff}(M, \alpha) = \{f \in \mathrm{Diff}(M) | f^*\alpha = \mu\alpha, \ \ \mu(x) \neq 0 \ \ \forall x \in M\}.$$

If $\alpha' = \rho\alpha$, $\rho \neq 0$, then we have

$$\mathrm{Diff}(M, \alpha') = \mathrm{Diff}(M, \alpha).$$

3.3 Contact Systems

3.3.1 Reeb Vector Fields

Throughout this section, we shall work on a (2n+1)-dimensional contact manifold (M, α). In 1952, Reeb proved the following theorem.

Proposition 2. [87] *There exists a unique vector field E on M which satisfies the following conditions:*

$$i_E\alpha = 1, \quad i_E d\alpha = 0, \tag{3.3}$$

$$i_E(\alpha \wedge (d\alpha)^n) = (d\alpha)^n.$$

This vector field E on M is called the **Reeb vector field** associated with the contact 1-form α. Sometimes it is called a **characteristic vector field**.

We give two examples of Reeb vector fields.

Example 5. In a Darboux coordinate system, the contact 1-form α on the Euclidean space $\mathbb{R}^{2n+1}$ may be written as

$$\alpha = xdy + dz = \sum_{i=1}^{n} x_i dy_i + dz.$$

Since

$$i_E(xdy + dz) = 1,$$

$$i_E(dx \wedge dy) = 0,$$

we observe that

$$E = \frac{\partial}{\partial z}. \tag{3.4}$$

The contact diffeomorphism generated by $\mathbb{R}$ is the translation

$$\varphi_t(x_1, \ldots, x_n, y_1, \ldots, y_n, z) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z + t).$$

Example 6. [66] Let (P_1, ω) be a symplectic $2n$-dimensional manifold, with $\omega = d\theta$. (The manifold is said to be exact.) Then the product

$$M = P_1 \times \mathbb{R}$$

carries the contact form:

$$\alpha = \pi_1^* \theta + \pi_2^* dt = \theta + dt,$$

where $\pi_1 : M \longrightarrow P_1$, $\pi_2 : M \longrightarrow \mathbb{R}$ are the projections and dt is the canonical 1-form on $\mathbb{R}$. Indeed we have

$$\alpha \wedge (d\alpha)^n = dt \wedge (d\theta)^n \neq 0$$

on M. Its Reeb vector field is

$$E = \frac{\partial}{\partial t},$$

and each trajectory may be identified with P_1.

Theorem 4. [66] *The tangent bundle TM of M may be decomposed as follows:*

$$TM = \operatorname{Ker} \alpha \oplus \operatorname{Ker} d\alpha,$$

where

(i) $\operatorname{Ker} d\alpha$ *is the 1-dimensional vertical bundle and is generated by the Reeb vector field* E*;*

(ii) $\operatorname{Ker} \alpha$ *is the* $2n$*-dimensional horizontal bundle.*

It means that $\operatorname{Ker} d\alpha$ is the line bundle over M spanned by E and the contact plane $D = \operatorname{Ker} \alpha$ carries the symplectic structure defined by $d\alpha$.

Proposition 3. [66] *Every vector field* X *on* (M, α) *may be decomposed into the sum of a vertical and a horizontal vector field:*

$$X = X_v + X_h,$$

(i) $X_v = (i_X\alpha)E$ *is vertical, i.e.,* $X_v \in \operatorname{Ker} d\alpha$ *;*

(ii) $X_h = X - X_v$ *is horizontal, i.e.,* $X_h \in \operatorname{Ker} \alpha$*.*

Each contact form induces the Reeb flow whose dynamical properties are important invariants.

Theorem 5. *Let* E *be the Reeb vector field on* (M, α)*. Then the flow of* E *preserves the contact 1-form* α *and the 2-form* $d\alpha$*.*

Proof. First of all,

$$L_E\alpha = d(i_E\alpha) + i_E d\alpha = 0.$$

Next,

$$L_E d\alpha = d(i_E d\alpha) = 0.$$

□

From the above result, we know that the flow of the Reeb vector field E preserves the contact 1-form. The flow generated by the Reeb vector field E is called the **Reeb flow** [33].

The subgroup $\mathrm{Diff}_\alpha M$ of diffeomorphisms preserving the contact form α is called the group of **strictly contact diffeomorphisms** [11]:

$$\mathrm{Diff}_\alpha M = \{f \in \mathrm{Diff}(M) | f^*\alpha = \alpha\}.$$

The corresponding Lie algebra is

$$\mathcal{L}_\alpha(M) = \{X \in \mathcal{X}(M) | L_X\alpha = 0\}.$$

The Reeb vector field can be regarded as the restriction of a time-independent Hamiltonian vector field to a constant energy surface, as in the following example.

Example 7. [66, 26] Let (x_1, y_1, x_2, y_2) denote the canonical coordinates of $\mathbb{R}^4$. Then the form θ is defined by

$$\theta = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)$$

and the form

$$\omega = d\theta = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$

is a canonical symplectic form of $\mathbb{R}^4$. Consider the Hamiltonian system

$$H = x_1^2 + y_1^2 + x_2^2 + y_2^2.$$

Then H=1 is the unit sphere S^3 in $\mathbb{R}^4$, where $\pi : M = S^3 \hookrightarrow \mathbb{R}^4$, and S^3 is equipped with the contact form

$$\alpha = \pi^*\theta, \qquad \alpha = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)|_{S^3}.$$

The Reeb vector field of the closed contact manifold induced by α is

$$E = 2\left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2}\right),$$

since

$$\begin{aligned} i_E\alpha &= x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1, \\ i_E d\alpha &= 0. \end{aligned}$$

3.3.2 Contact Vector Fields

A contact dynamical system on (M, α) is governed by a contact vector field. A vector field X on a contact manifold M is said to be a **contact vector field** if its flow preserves the contact structure $D = \operatorname{Ker}\alpha$. The following proposition gives an equivalent condition.

Proposition 4. [66] *A vector field X is contact if and only if there exists a smooth function $\lambda : M \longrightarrow \mathbb{R}$ such that*

$$L_X\alpha = \lambda\alpha. \tag{3.5}$$

In this case, the function λ is related to the function μ of (3.2) by

$$\mu(t) = \exp\int_0^t (\lambda \circ f)ds. \tag{3.6}$$

(Note that this condition is independent of the choice of 1-form α defining D.)

Proof. For $X \in D_x$, we have

$$Tf \cdot X \in D_{f(x)} \Longleftrightarrow f_t^*\alpha = \mu(t)\alpha \Longleftrightarrow L_X\alpha = \dot{\mu}\alpha = \lambda\alpha.$$

□

Definition 6. If X is a contact vector field, then the function

$$K = i_X\alpha \equiv \alpha(X) \tag{3.7}$$

is called the **contact Hamiltonian** defined by X.

By using formula (ii) of Proposition 3 we decompose X into

$$X = KE + X_h$$

where KE, with $K = i_X\alpha$, is the vertical component and X_h the horizontal component of X. By using the formula $L_X\alpha = i_X d\alpha + d(i_X\alpha) = \lambda\alpha$ on (M, α), moreover, we have the following relation:

Theorem 6. *Let (M, α) be a contact manifold. Then*

(i) $\lambda = i_E dK$,

(ii) $i_X d\alpha = \lambda\alpha - dK$,

where the vector field E is the Reeb vector field and X is a contact vector field.

Proof. (i)

$$\begin{aligned} L_X\alpha &= i_X d\alpha + d(i_X\alpha) \\ &= i_X d\alpha + dK \qquad (3.8) \\ &= \lambda\alpha. \end{aligned}$$

Applying (3.8) to E, we obtain

$$i_E i_{X_K} d\alpha + i_E dK = i_E \lambda\alpha.$$

Since $i_E d\alpha = 0$ and $i_E\alpha = 1$, we get

$$\lambda = i_E dK.$$

(ii) From (i), we have

$$i_X d\alpha = \lambda\alpha - dK.$$

□

Theorem 7. *For each given function $K : M \longrightarrow \mathbb{R}$, there is a unique contact vector field X with contact Hamiltonian function K.*

Proof. Let $\lambda = i_E dK$. From Theorem 6, we know that X must satisfy

$$i_X d\alpha = \lambda\alpha - dK, \tag{3.9}$$

and, if K is to be the contact Hamiltonian function of X, then we must have

$$i_X \alpha = K. \tag{3.10}$$

We show that (3.9) and (3.10) have a unique solution. From Proposition 3, any vector field X can be decomposed into its horizontal component X_h $(\in \operatorname{Ker}\alpha)$ and vertical component X_v $(\in \operatorname{Ker} d\alpha)$; that is, $X = X_h + X_v$. Equations (3.9) and (3.10) now become

$$i_{X_h} d\alpha = \lambda\alpha - dK, \tag{3.11}$$

$$i_{X_v} \alpha = K. \tag{3.12}$$

$\operatorname{Ker} d\alpha = \operatorname{span}\{E\}$ is one dimensional and $i_E d\alpha = 1$, so $X_v = KE$. Since $d\alpha$ is nondegenerate on $D = \operatorname{Ker}\alpha$, Equation (3.11) determines X_h uniquely. Finally X is a contact vector field because

$$L_X\alpha = i_X d\alpha + d i_X \alpha = \lambda\alpha = (i_E dK)\alpha.$$

□

Sometimes X in Theorem 7 is called the contact Hamiltonian vector field associated with the contact Hamiltonian K. From now on we will denote the contact Hamiltonian vector field by X_K instead of X.

With respect to canonical coordinates, the relationship between contact Hamiltonian field X_K with $\alpha = xdy + dz$ and the corresponding contact Hamiltonian function K may be expressed as follows:

Proposition 5. *Let (M, α) be a contact manifold, with the coordinates $(x, y, z) = (x_1, \dots, x_n, y_1, \dots, y_n, z)$ and contact form*

$$\alpha = xdy + dz = \sum_{i=1}^{n} x_i dy_i + dz.$$

Then the contact vector field X_K, which has K as its contact Hamiltonian, may be expressed as

$$X_K \begin{cases} \dot{x}_i = -K_{y_i} + x_i K_z \\ \dot{y}_i = K_{x_i} \\ \dot{z} = K - \sum_{i=1}^{n} x_i K_{x_i} \end{cases} \tag{3.13}$$

for $i = 1, \dots, n$, where

$$X_K = \sum_{i=1}^{n} \dot{x}_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \dot{y}_i \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z}.$$

Proof. We start with

$$i_{X_K} d\alpha = \lambda\alpha - dK \quad \text{and} \quad d\alpha = dx \wedge dy = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Since

$$i_{X_K} d\alpha = \sum_{i=1}^{n} \dot{x}_i dy_i - \sum_{i=1}^{n} \dot{y}_i dx_i \tag{3.14}$$

and

$$dK = \sum_{i=1}^{n} K_{x_i} dx_i + \sum_{i=1}^{n} K_{y_i} dy_i + K_z dz,$$

we have

$$\begin{aligned} i_{X_K} d\alpha &= \lambda\alpha - dK \\ &= \lambda(\sum_{i=1}^{n} x_i dy_i + dz) - \left(\sum_{i=1}^{n} K_{x_i} dx_i + \sum_{i=1}^{n} K_{y_i} dy_i + K_z dz\right) \\ &= (\lambda - K_z)dz - \sum_{i=1}^{n} K_{x_i} dx_i + \sum_{i=1}^{n}(\lambda x_i - K_{y_i})dy_i. \end{aligned}$$

Recall that $\lambda = i_E dK$ and $E = \frac{\partial}{\partial z}$, so $\lambda = K_z$. Hence we have

$$\begin{cases} \dot{x_i} = -K_{y_i} + x_i K_z \\ \dot{y_i} = K_{x_i} \end{cases}$$

for $i = 1, \ldots, n$. Since

$$K = \sum_{i=1}^{n} x_i \dot{y_i} + \dot{z},$$

we obtain

$$\dot{z} = K - x\dot{y} = K - \sum_{i=1}^{n} x_i K_{x_i}.$$

Thus the contact vector field X_K is given by

$$X_K \begin{cases} \dot{x_i} = -K_{y_i} + x_i K_z \\ \dot{y_i} = K_{x_i} \\ \dot{z} = K - \sum_{i=1}^{n} x_i K_{x_i} \end{cases}$$

for $i = 1, \ldots, n$. □

Let $\mathcal{L}(M, \alpha)$ be the Lie algebra of contact vector fields defined by

$$\mathcal{L}(M, \alpha) = \{X_K \in \mathcal{X}(M) | L_{X_K}\alpha = \lambda\alpha \text{ for some } \lambda \in C^\infty(M)\},$$

where $\mathcal{X}(M)$ denotes the Lie algebra of smooth vector fields on M. Then we have

- $\mathrm{Diff}_\alpha M \subset \mathrm{Diff}(M,\alpha)$,
- $\mathcal{L}_\alpha(M) \subset \mathcal{L}(M,\alpha)$.

In Chapters 4 and 5, we will study contact systems as dynamical systems.

3.3.3 Relationship between Contact and Homogeneous Hamiltonian Vector Fields

Now we want to establish a relationship between contact geometry and homogeneous symplectic geometry.

Recall that the symplectic structure in an even-dimensional manifold P is determined by a closed non-degenerate 2-form ω on this manifold.

Definition 7. [66] A manifold P is said to admit a **homogeneous symplectic structure** if there exists a symplectic form ω and P admits an action h of the multiplicative group $\mathbb{R}_* = \mathbb{R}_- \cup \mathbb{R}_+$ (here $\mathbb{R}_*$ denotes that set of non-zero real numbers: $\mathbb{R}_* = \{\lambda | \lambda \in \mathbb{R},\ \lambda \neq 0\}$, one may take instead $\mathbb{R}_+$); that is,

$$h_\lambda^* \omega = \lambda \omega$$

for any $\lambda \in \mathbb{R}_*$. We say that (P,ω) is a **homogeneous (Liouville) symplectic manifold**.

We shall denote the action of the group $\mathbb{R}_*$ on P by $h : \mathbb{R}_* \times P \longrightarrow P$, and h_λ will be the diffeomorphism from P to P defined by

$$h(\lambda, u) = h_\lambda(u).$$

We introduce the **Liouville vector field** by the formula

$$Z(u) = \frac{d}{d\lambda} h_\lambda(u)|_{\lambda=1}, \quad u \in P.$$

Proposition 6. [66] *Let (P, ω) be a homogeneous symplectic manifold, then the symplectic form ω is exact. More precisely,*

$$\omega = d\theta, \qquad \text{where} \quad \theta = i_Z \omega.$$

The form θ in the above proposition is called the Liouville form.

If the canonical symplectic space $P = \mathbb{R}^{2n+2}$ is homogeneous with homogeneous symplectic form $\omega = dp \wedge dq$, then h_λ is denoted by

$$h_\lambda(p, q) = (\lambda p, q)$$

and the Liouville vector field Z is

$$Z = p\frac{\partial}{\partial p} = \sum_{i=0}^{n} p_i \frac{\partial}{\partial p_i}.$$

Definition 8. A diffeomorphism $F : P \longrightarrow P$ of a homogeneous symplectic manifold (P, ω) is called a **homogeneous symplectic diffeomorphism** if it satisfies the following condition:

$$F^*\omega = \omega$$

$$h_\lambda \circ F = F \circ h_\lambda$$

for all $\lambda \in \mathbb{R}_*$.

Proposition 7. [66] *Let (P, ω) be a homogeneous symplectic manifold and Z its Liouville vector field and θ its Liouville form. The flow of the vector field X on P is a homogeneous symplectic diffeomorphism if and only if it satisfies the following conditions:*

(i) $L_X\omega = 0$; $(h_\lambda)_*X = X$ *for every* $\lambda \in \mathbb{R}_*$.

(ii) $L_X\theta = 0$; $[X, Z] = 0$.

(iii) *The function* H *defined by*

$$H = i_X\theta$$

is homogeneous of degree 1,

$$H \circ h_\lambda = \lambda H$$

and satisfy the condition

$$i_X\omega = -dH.$$

Symplectification

Let (P, ω) be a homogeneous symplectic manifold. Let M be the manifold of orbits of the action h. Then we have the projection

$$\pi : P \longrightarrow M.$$

Conversely, given a contact manifold (M, α), it is possible to construct a homogeneous symplectic manifold (P, ω) such that $P/h = M$. Thus (P, ω) is called the **symplectification** of (M, α).

Definition 9. [3] Let (M, α) be a contact manifold, and let $P \subset T^*M$ be the line bundle generated by

$$P = \{\lambda\alpha : \lambda \neq 0\} \cong M \times \mathbb{R}_*. \tag{3.15}$$

The symplectic form on T^*M restricts as a symplectic form $\omega = d\theta = d(\lambda\alpha)$ to P. The $(2n+2)$-dimensional symplectic manifold (P, ω) is called the **symplectification** of the $(2n+1)$-dimensional contact manifold (M, α).

In this case, h_λ is denoted by

$$h_\lambda : P \longrightarrow P$$

$$u \longmapsto \lambda u, \quad u \in P \subset T^*M.$$

Now we establish a relationship between contact diffeomorphisms and homogeneous symplectic diffeomorphisms.

Theorem 8. [3] *Any contact diffeomorphism $f : M \longrightarrow M$ lifts to homogeneous symplectic diffeomorphism $F : P \longrightarrow P$.*

Proof. If $f^*\alpha = \mu\alpha$ (see (3.2)), set

$$F(u, \lambda) = (f(u), \frac{\lambda}{\mu}). \tag{3.16}$$

We have F is symplectic:

$$F^*\omega = F^*d(\lambda\alpha) = dF^*(\lambda\alpha) = d(\frac{1}{\mu}f^*(\lambda\alpha)) = d(\lambda\alpha) = \omega.$$

□

Let $F : P \longrightarrow P$ be a homogeneous symplectic diffeomorphism, $f : M \longrightarrow M$ be a contact diffeomorphism and $\pi : P \longrightarrow M$ be the symplectification of the manifold M. Then the diagram

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d, "\pi"'] & P \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M \end{tikzcd} \tag{3.17}$$

commutes; that is,

$$\pi \circ F = f \circ \pi.$$

Conversely, any homogeneous symplectic diffeomorphism F projects under π to a contact diffeomorphism f.

We have an analogous correspondence between contact and homogeneous Hamiltonian vector fields; namely, any homogeneous Hamiltonian vector field projects to a contact vector field and any contact vector field can be lifted to a homogeneous Hamiltonian vector field as follows.

Remark. The lifting X (see Proposition 7) of a contact vector field X_K is called the symplectification of X_K [66]. Note that H is the Hamiltonian of X. Note that the vector field X has Z as a symmetry, however H is not quite invariant under h_λ; this is because h_λ is not symplectic.

Proposition 8. [66] *Let (M, α) be a contact manifold, let X be the vector field of a homogeneous symplectic manifold (P, ω), which is the lifting of contact vector field X_K on M. Then the vector field X may be expressed by*

$$X(u, \lambda) = X_K(u) - \lambda\rho(u)\frac{\partial}{\partial\lambda} \tag{3.18}$$

where ρ is the function defined by $L_{X_K}\alpha = \rho\alpha$, and $\frac{\partial}{\partial\lambda}$ is the vector field on $\mathbb{R}_*$ associated with its canonical coordinate λ.

The notions of homogeneous symplectic diffeomorphism and contact diffeomorphism give us a means for defining Hamiltonian and contact Hamiltonian vector fields. Recall that the Hamiltonian function corresponding to a Hamiltonian vector field X_H on a symplectic manifold P is a function H such that $i_{X_H}\omega = -dH$. We consider here a Hamiltonian system which is defined on the symplectification (P, ω) of (M, α). The symplectification of a contact vector field X_K is a Hamiltonian vector field [5]. In the homogeneous case, for each vector field X_H, there exists a Hamiltonian function which is homogeneous of degree 1 with respect to the action of the group $\mathbb{R}_*$, that is,

$$h_\lambda^* H = \lambda H$$

for $\lambda \in \mathbb{R}_*$. This homogeneous Hamiltonian function can be determined by the formula (see Proposition 7)

$$H = i_{X_H}\theta, \tag{3.19}$$

which satisfies the condition

$$i_{X_H}\omega = -dH.$$

We also have a nice relationship between the homogeneous Hamiltonian H of homogeneous Hamiltonian vector field X_H and the contact Hamiltonian K of the contact vector field X_K. And any Hamiltonian function H which is homogeneous of degree 1 is uniquely determined by its restriction K on M:

$$\begin{aligned} H(u,\lambda) &= (i_{X_H}\theta)(u,\lambda) \\ &= (i_{X_H}\lambda\alpha)(u,\lambda) \\ &= \lambda(i_{X_H}\alpha)(u,1) \\ &= \lambda(i_{X_K}\alpha)(u) \\ &= \lambda K(u) \end{aligned}$$

from (3.18, 3.19). That is

$$\lambda K(u) = H(u,\lambda)$$

or, identifying M with the $\lambda = 1$ level set of P,

$$K = H|_M.$$

Computation Formulas for the Canonical Case

The above assertion allows us to give a description of the relationship between contact Hamiltonian vector fields and Hamiltonian vector fields. Now we will find Hamiltonian vector fields in the canonical case [5] [40] [81].

Let (M, α) be a contact manifold with the contact form $\alpha = xdy + dz$ $(x, y \in \mathbb{R}^n,\ z \in \mathbb{R})$, and let

$$X_K = (-K_y + xK_z)\frac{\partial}{\partial x} + K_x\frac{\partial}{\partial y} + (K - xK_x)\frac{\partial}{\partial z}. \tag{3.20}$$

The Liouville form θ on the symplectification (P, ω) may be expressed as

$$\theta = \lambda\alpha = \lambda(xdy + dz).$$

The natural coordinates $(\mathbf{p}, \mathbf{q}) = ((x, \lambda), (y, z))$ are not canonical and so would be inconvenient for computational purpose and for constructing symplectic integrators. We therefore transform them to canonical variables as follows. Recall $\lambda \neq 0$, so $(p_0, p, q_0, q) \in \mathbb{R}^{2n+2}\backslash\{p_0 = 0\}$. On this space the coordinate transformation is globally defined. If we set

$$p_0 = \lambda, \qquad p = \lambda x,$$
$$q_0 = z, \qquad q = y,$$

then $\theta = \lambda(xdy + dz) = p_0 dq_0 + pdq$ is canonical. So we have that the canonical symplectic form is determined by ω with

$$\omega = d\theta = dp_0 \wedge dq_0 + dp \wedge dq.$$

The Liouville vector field is

$$Z = p_0\frac{\partial}{\partial p_0} + p\frac{\partial}{\partial p}$$

and hence, the corresponding contact form in $p_0 \neq 0$ is given by the equality

$$\alpha = \frac{1}{p_0}(i_Z\omega) = dq_0 + \frac{p}{p_0}dq.$$

Let K be a contact Hamiltonian function. The corresponding homogeneous Hamiltonian function H is then given as follows.

Let $H(p_0, p, q_0, q)$ be a homogeneous Hamiltonian function $H(p_0, p, q_0, q)$ of degree 1, which is defined for $p_0 \neq 0$ and satisfies the condition

$$\begin{aligned} H(p_0, p, q_0, q) &= p_0 H\left(1, \tfrac{p}{p_0}, q_0, q\right) \\ &= p_0 K\left(\tfrac{p}{p_0}, q_0, q\right) \end{aligned} \tag{3.21}$$

then the symplectification of the contact vector field X_K is a Hamiltonian vector field X_H (see (3.18)):

$$X_H = X_K - \lambda K_z \frac{\partial}{\partial \lambda}. \tag{3.22}$$

Therefore, the partial derivatives of H at the point

$$p_0 = 1, \quad p = x, \quad q_0 = z, \quad q = y$$

are related to the contact Hamiltonian function K at the point (x, y, z) by the relations:

$$\begin{aligned} \frac{\partial H}{\partial p_0} &= H_{p_0}(p_0, p, q, q_0) = \frac{\partial p_0}{\partial p_0} K\left(\frac{p}{p_0}, q, q_0\right) - (p_0)\frac{p}{p_0^2} K\left(\frac{p}{p_0}, q, q_0\right) \\ &= K\left(\frac{p}{p_0}, q, q_0\right) - \frac{p}{p_0} K_{\frac{p}{p_0}}\left(\frac{p}{p_0}, q, q_0\right) \\ &= K(x, y, z) - x K_x(x, y, z) \\ \frac{\partial H}{\partial p} &= H_p(p_0, p, q, q_0) = \frac{p_0}{p_0} K_p\left(\frac{p}{p_0}, q, q_0\right) = K_p\left(\frac{p}{p_0}, q, q_0\right) = K_x(x, y, z) \\ \frac{\partial H}{\partial q_0} &= H_{q_0}(p_0, p, q, q_0) = p_0 K_{q_0}\left(\frac{p}{p_0}, q, q_0\right) = p_0 K_z(x, y, z) = K_z(x, y, z) \\ \frac{\partial H}{\partial q} &= H_q(p_0, p, q, q_0) = p_0 K_q\left(\frac{p}{p_0}, q, q_0\right) = p_0 K_y(x, y, z) = K_y(x, y, z). \end{aligned}$$

Thus we have the following equations:

$$\begin{aligned} \dot{p}_0 &= \dot{\lambda} = -K_z \\ \dot{p} &= (\dot{\lambda x}) = \dot{x}\lambda + x\dot{\lambda} = -K_y \end{aligned}$$

$$\dot{q_0} = \dot{z} = K - xK_x$$
$$\dot{q} = \dot{y} = K_x.$$

From (3.20) and (3.22), the Hamiltonian vector field with Hamiltonian function H is given by:

$$X_H = \left(H_{p_0}\frac{\partial}{\partial q_0} - H_{q_0}\frac{\partial}{\partial p_0}\right) + \left(H_p\frac{\partial}{\partial q} - H_q\frac{\partial}{\partial p}\right).$$

That is,

$$X_H \begin{cases} \dot{p_0} = -H_{q_0} & \dot{q_0} = H_{p_0} \\ \dot{p} = -H_q & \dot{q} = H_p. \end{cases}$$

Chapter 4

Contact Dynamical Systems

The purpose of this chapter is to obtain some special features of the contact systems with contact Hamiltonian function K. We are particulary interested in the features of the contact systems on the zero section of K. One of the main questions is concerning the long-time behaviors of the contact system. In particular, we want to know whether solutions are stable or not. In this chapter, we will study the stability of the fixed points of a contact system.

4.1 Dynamics On the Contact Manifold

We recall that X_K is a contact vector field if and only if there exists a real valued function λ on (M, α) such that

$$L_{X_K}\alpha = \lambda\alpha \tag{4.1}$$

(see Proposition 4).

In this section, we consider some properties of contact systems on (M, α). By using Proposition 3, the contact vector field $X_K = X_v + X_h$ has the following properties:

- $i_{X_v}\alpha = K$ and $i_{X_v}d\alpha = 0$;
  that is, $L_{X_v}\alpha = di_{X_v}\alpha$ and $\lambda\alpha = dK$,
- $i_{X_h}\alpha = 0$ and $i_{X_h}d\alpha = \lambda\alpha$;
  that is, $L_{X_h}\alpha = i_{X_h}d\alpha$.

We prove the following proposition:

Proposition 9. *Let X_K be a contact vector field on (M,α) with $\lambda\alpha = dK$; that is,* $\operatorname{Ker}\alpha = \operatorname{Ker} dK$, *and λ a non-vanishing function on (M,α). Then*

$$i_{X_K}d\alpha = 0;$$

that is, $d\alpha$ is an absolute integral invariant under X_K.

Proof. Let X_K be a contact vector field. Then

$$L_{X_K}\alpha = i_{X_K}d\alpha + d\,i_{X_K}\alpha = \lambda\alpha$$

from (4.1). Since $i_{X_K}\alpha = K$, we have

$$i_{X_K}d\alpha = \lambda\alpha - dK. \tag{4.2}$$

By assumption $\lambda\alpha = dK$, we get

$$i_{X_K}d\alpha = 0.$$

Finally, we have that

$$\begin{aligned} L_{X_K}d\alpha &= i_{X_K}d\,(d\alpha) + d\,i_{X_K}d\alpha \\ &= 0. \end{aligned}$$

Thus $d\alpha$ is an absolute integral invariant under X_K. ☐

It is well known that, if α is a contact form on the contact manifold M, then for any non-vanishing function g on (M, α), $g\alpha$ is a contact form on M and $D = \operatorname{Ker}\alpha = \operatorname{Ker} g\alpha$ (see Theorem 2).

Proposition 10. *Let X_K be a contact vector field on the contact manifold $(M, g\alpha)$ where g is a non-vanishing function on M. We have*

$$L_{X_K}(g\alpha) = \tilde{\lambda}(g\alpha),$$

where $\tilde{\lambda} = \frac{i_{X_K}(gK)}{gK}$.

Proof.

$$\begin{aligned} L_{X_K} g\alpha &= gL_{X_K}\alpha + (L_{X_K} g)\alpha \\ &= g(\lambda\alpha) + (i_{X_K} dg)\alpha \\ &= g\frac{X(K)}{K}\alpha + X(g)\alpha \\ &= \left(\frac{X(K)}{K} + \frac{X(g)}{g}\right) g\alpha \\ &= \frac{X(gK)}{gK} g\alpha \\ &= \frac{i_{X_K}(gK)}{gK} g\alpha \\ &= \tilde{\lambda}(g\alpha) \end{aligned}$$

where $\tilde{\lambda} = \frac{i_{X_K}(gK)}{gK}$. □

From the above proposition, we have

Corollary 1. *If $\tilde{\alpha} = g\alpha$ is a contact form, then*

(i) $L_{X_K}\tilde{\alpha} = \tilde{\lambda}\tilde{\alpha} = \hat{\lambda}\alpha$, *where* $\hat{\lambda} = \frac{i_{X_K}(gK)}{K}$.

(ii) $i_{X_K} d\tilde{\alpha} = \tilde{\lambda}\tilde{\alpha} - d(gK)$.

Proof. (i)

$$\begin{aligned} L_{X_K}\tilde{\alpha} &= i_{X_K}d\tilde{\alpha} + di_{X_K}\tilde{\alpha} \\ &= i_{X_K}d\tilde{\alpha} + di_{X_K}(g\alpha) \\ &= i_{X_K}d\tilde{\alpha} + d(gi_{X_K}\alpha) \\ &= i_{X_K}d\tilde{\alpha} + d(gK) \\ &= \tilde{\lambda}\tilde{\alpha}. \end{aligned}$$

(ii) From (i), we have

$$i_{X_K}d\tilde{\alpha} = \tilde{\lambda}\tilde{\alpha} - d(gK).$$

□

We know that X_K is a contact vector field associated with contact form $\tilde{\alpha} = g\alpha$ on the contact manifold $(M, \tilde{\alpha})$.

Proposition 11. *Let X_K be a contact vector field on (M, α). We have*

(i) $L_{X_K}d\alpha = d(\lambda\alpha)$

(ii) $L_{X_K}d\alpha|_D = (\lambda d\alpha)\,|_D$ *on the contact structure D of M, where $d\alpha|_D$ is a non-degenerate symplectic form on D.*

Proof. (i) On the contact manifold (M, α), we have from (4.2)

$$\begin{aligned} L_{X_K}d\alpha &= d(i_{X_K})d\alpha + i_{X_K}d(d\alpha) \\ &= d(i_{X_K})d\alpha \\ &= d(\lambda\,\alpha - dK) \\ &= d(\lambda\,\alpha). \end{aligned}$$

(ii) On the contact structure $D = \operatorname{Ker}\alpha$, we have from (i)

$$\begin{aligned} L_{X_K} d\alpha \,|_D &= d(i_{X_K}) d\alpha \,|_D \\ &= d(\lambda\,\alpha)\,|_D \\ &= (d\lambda \wedge \alpha)\,|_D + (\lambda\,(d\alpha))\,|_D \\ &= (\lambda\, d\alpha)\,|_D, \end{aligned}$$

since $\alpha = 0$ on D.

□

Invariants of Contact Systems

One key instrument of studying analytically a given contact system is the invariants of the system. Now we are concerned with the derivative of K.

$$\begin{aligned} \frac{dK}{dt} &= L_{X_K} K \\ &= i_{X_K} dK \\ &= i_{X_K}\alpha\; i_E dK - i_{X_K} i_{X_K} d\alpha \\ &= K\; i_E dK \qquad (4.3) \\ &= \lambda K, \qquad (4.4) \end{aligned}$$

since $dK = (i_E dK)\alpha - i_{X_K} d\alpha$. Here (as before) we denote $\lambda = i_E dK$. In the canonical case, we have the derivative of K as follows:

$$\begin{aligned} \frac{dK}{dt} &= K_x\dot{x} + K_y\dot{y} + K_z\dot{z} \\ &= K_x(-K_y + xK_z) + K_y(K_x) + K_z(K - xK_x) \\ &= K_z K. \end{aligned}$$

from (3.13). Since $E = \frac{\partial}{\partial z}$ (see (3.4)) in the canonical case, we have $\lambda = K_z$. This means that the contact Hamiltonian function K is not an integral invariant.

However, we do have that

- If λ is a non-vanishing function on (M, α), then X_K is only tangent to the level surface $K = 0$. The level surface $K = 0$ is invariant. One also says that K is a weak integral of X_K.
- If $\lambda = 0$ on (M, α), then X_K is tangent to all level surfaces of K.

4.2 Dynamics On the Zero Section of K

In this section we will discuss some properties of the contact system on the zero section of K.

Definition 10. The **zero section** of $K : M \longrightarrow \mathbb{R}$ is the subset $S \subset M$ defined by

$$S := K^{-1}(0).$$

4.2.1 The Case $\lambda \in \mathcal{C}^{\infty}(M)$

Proposition 12. *Let X_K be a contact vector field on (M, α). On the zero section S, we have*

(i) $i_{X_K}\alpha = 0$,

(ii) $i_{X_K}d\alpha = \lambda\alpha$.

Proof. (i) If X_K is a contact vector field associated with the contact Hamiltonian function K, then

$$i_{X_K}\alpha = K.$$

On the zero section S, we have

$$i_{X_K}\alpha = 0.$$

(ii) By (i) and (4.1), we have

$$\begin{aligned} L_{X_K}\alpha &= di_{X_K}\alpha + i_{X_K}d\alpha \\ &= i_{X_K}d\alpha \\ &= \lambda\alpha. \end{aligned}$$

Thus we get

$$i_{X_K}d\alpha = \lambda\alpha.$$

□

Proposition 13. *Let X_K be a contact vector field on (M, α) and f a non-vanishing function on M. On the zero section S, we have*

$$L_{fX_K}\alpha = fL_{X_K}\alpha = (f\lambda)\alpha.$$

Proof. On the zero section S, we have

$$\begin{aligned} L_{fX_K}\alpha &= i_{fX_K}d\alpha + di_{fX_K}\alpha \\ &= i_{X_K}f(d\alpha) + d(fi_{X_K}\alpha) \\ &= fi_{X_K}d\alpha + df \wedge i_{X_K}\alpha + fdi_{X_K}\alpha \\ &= fL_{X_K}\alpha + df \wedge i_{X_K}\alpha \\ &= f(\lambda\alpha) \\ &= (f\lambda)\alpha, \end{aligned}$$

since $i_{X_K}\alpha = 0$. □

In general, the vector fX_K is not a contact vector field on (M, α), where f is a non-vanishing function on (M, α).

Proposition 14. [66] *Let (M, α) be a contact manifold, and K a contact Hamiltonian function such that, at every point x of the zero section $S = K^{-1}(0)$, the following inequality holds:*

$$(dK \wedge \alpha)|_x \neq 0.$$

Then

(i) *S is a submanifold of M of codimension one and its tangent plane and contact structure $D = \operatorname{Ker} \alpha$ are transversal at every point $x \in S$; that is,*

$$T_xM = T_xS + D_x, \quad \textit{for } x \in S.$$

(ii) *The restriction of the vector field X_K to S is a non-zero horizontal vector field.*

Proposition 15. *Let X_K be a contact vector field on (M, α). Let $x \in M$ obey $K(x) = 0$ and $(dK)(x) \neq 0$. Let S be zero section of K. Then*

$$\lambda(x) = 0 \quad \textit{if and only if} \quad (d\alpha|_S)(x) \quad \textit{is degenerate.}$$

Proof. Since $(dK)(x) \neq 0$, so we have that S is a submanifold in a neighborhood of x with tangent space

$$T_xS = \operatorname{Ker}(dK)(x).$$

Recall that the Reeb vector field E is defined by $i_E\alpha = 1$ and $i_E d\alpha = 0$; that is,

$$\operatorname{Ker} d\alpha = \operatorname{span}(E).$$

Then we have that

$$(d\alpha|_S)(x) \text{ is degenerate}$$

if and only if

$$\text{span}(E) = (\text{Ker}\, d\alpha)(x) \in T_xS = \text{Ker}(dK)(x),$$

which is equivalent to

$$(i_E dK)(x) = \lambda(x) = 0.$$

□

From Proposition 15, we see that $\omega = d\alpha|_S$ is a symplectic form on the submanifold S if $\lambda(x) \neq 0 \ \forall x \in S$. So (S, ω) is a symplectic manifold if λ is a non-vanishing function on M.

4.2.2 The Case $\lambda \neq 0$

We consider the contact vector field X_K with λ non-vanishing on the symplectic manifold $(S, \omega = d\alpha|_S)$. We will show that if λ is a non-vanishing constant then $L_{X_K}\omega = \lambda\omega$ where $\omega = d\alpha$. Before we proceed, we shall introduce the definition of a conformal vector field X and a conformal diffeomorphism on the symplectic manifold (S, ω).

Definition 11. [74] The vector field X is said to be **conformal** with parameter $c \in \mathbb{R}$ if

$$L_X\omega = c\omega. \tag{4.5}$$

The diffeomorphism φ is conformal if

$$\varphi^*\omega = c\omega.$$

The conformal diffeomorphisms φ^* form a Lie pseudogroup of local diffeomorphisms of a manifold, one of Cartan's six fundamental classes of Lie pseudogroups.

The following result deals with the submanifold S and its associated symplectic 2-form ω.

Proposition 16. *Let X_K be a contact vector field on (M, α) and λ a non-zero real constant. On the submanifold S of M, we have*

$$L_{X_K}\omega = \lambda\,\omega$$

where $\omega = d\alpha|_S$ is a symplectic form on S.

Proof. We have

$$\begin{aligned} L_{X_K}d\alpha &= di_{X_K}d\alpha \\ &= d(\lambda\alpha) \\ &= d\lambda \wedge \alpha + \lambda\, d\alpha \\ &= \lambda\, d\alpha \end{aligned}$$

since λ is a non-zero real constant; that is, $d\lambda = 0$. If we denote $\omega = d\alpha|_S$, which is a symplectic form, then we have

$$L_{X_K}\omega = \lambda\omega.$$

□

Proposition 16 implies that X_K is a conformal Hamiltonian vector field on the symplectic manifold (S, ω).

Example 8. Consider the contact vector field X_K on $(\mathbb{R}^{2n+1}, \alpha)$ with $\alpha = xdy + dz$, where the contact Hamiltonian function K has the form $K = cz + f(x, y)$ with

$c\ (\neq 0) \in \mathbb{R}$, and f is an arbitrary function of x, y. Then X_K is given by

$$X_K \begin{cases} \dot{x} = -f_y + cx \\ \dot{y} = f_x \\ \dot{z} = -xf_x. \end{cases}$$

on the submanifold S. In this case, $\lambda = K_z = c$ and

$$S = \{(x, y, z) : \ z = -\frac{f(x, y)}{c}\}$$

is a graph over (x, y). By taking (x, y) as coordinates the above defines a conformal Hamiltonian vector field.

If λ is not constant, we consider the question of whether rescaling the vector field X_K or the contact form α can produce a conformal vector field.

Proposition 17. *Let X_K be a contact vector field on (M, α) and assume λ is non-vanishing on (M, α). On the submanifold S of M, we have*

(i) $L_{\frac{X_K}{\lambda}} \alpha = \alpha$

(ii) $L_{\frac{X_K}{\lambda}} \omega = \omega$ *where $\omega = d\alpha|_S$ is a symplectic form.*

Proof. (i) By Proposition 13, we get

$$\begin{aligned} L_{\frac{X_K}{\lambda}} \alpha &= \frac{1}{\lambda} L_{X_K} \alpha \\ &= \frac{1}{\lambda} (\lambda\, \alpha) \\ &= \alpha. \end{aligned}$$

(ii) By (4.2), we get

$$\begin{aligned} L_{\frac{X_K}{\lambda}} d\alpha &= d\, i_{\frac{X_K}{\lambda}} d\alpha \\ &= d\, (\frac{1}{\lambda} i_{X_K} d\alpha) \\ &= d\, (\frac{1}{\lambda}(\lambda\alpha - dK)) \\ &= d\, (\alpha - \frac{1}{\lambda}(dK)) \\ &= d\alpha. \end{aligned}$$

So we have

$$L_{\frac{X_K}{\lambda}} \omega = \omega,$$

where $\omega = d\alpha|_S$, which is a symplectic form on S.

□

It means that the vector field $\frac{X_K}{\lambda}$ with a non-vanishing function λ on M is conformal symplectic on the symplectic manifold (S, ω), with conformal constant equal to 1.

Remarks.

1. We note that $\dot{K} = \lambda\, K$ (see (4.4)) where $\lambda = i_E dK$, implies that
    - when $\lambda(x) < 0$ for all $x \in S$, the zero section S is attracting,
    - when $\lambda(x) > 0$ for all $x \in S$, the zero section S is repelling.
2. The delicate case is when λ changes sign on S and the zero section S is neither attracting nor repelling, and nor is the dynamics on the zero section S globally conformal symplectic. (It is still locally conformal symplectic in neighborhoods where λ does not vanish, but if the flow does not stay in such a neighborhood, it is not clear if this places any constraints on the flow.)

4.3 Stability in Contact Systems

An orbit with initial condition near a stable fixed point will stay nearby or even return to the fixed point. The stability properties of the fixed points of a system tells us about the long-term behavior of some solutions. In this section, we will investigate the stability of each fixed point of a contact system with contact Hamiltonian function K. We will first analyze the linearized system around the fixed points of a contact system. Then we conclude whether the stability or instability can be determined by the eigenvalues of the matrix of the linearized system.

4.3.1 Stability On $\mathbb{R}^3$

Throughout this subsection, we shall work on $(\mathbb{R}^3, \alpha)$ in the canonical case.

Fixed Points of a Contact System

To analyze the dynamics, we first have to find all fixed points of the contact system with the contact Hamiltonian function K. The contact system considered is

$$\dot{u} = X_K(u)$$

$$\dot{u} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} X(x,y,z) \\ Y(x,y,z) \\ Z(x,y,z) \end{bmatrix} = \begin{bmatrix} -K_y + xK_z \\ K_x \\ K - xK_x \end{bmatrix}. \tag{4.6}$$

To find all fixed points of the contact system, we require all the solutions of $X_K(u) = 0$. Thus we get the fixed points as follows

$$K_y(u) = xK_z(u) \tag{4.7}$$

$$K(u) = K_x(u) = 0. \tag{4.8}$$

From (4.8), we know that the fixed points must be on the zero section $S = K^{-1}(0)$. Now we investigate the nature of a fixed point by examining its linear approximation. If $u = u^*$ is a fixed point, then a perturbation is $u = u^* + u'$ about the fixed point, where it is assumed that the magnitude of u' is small. Substitution into $X_K(u)$ leads to

$$\begin{aligned} \dot{u'} &= X_K(u^* + u') \\ &= X_K(u^*) + Fu' + o(\|u'\|) \\ &= Fu' + o(\|u'\|), \end{aligned}$$

where F is the 3×3 matrix of derivatives of X_K evaluated at the fixed point $u = u^*$; namely, the matrix with element given by

$$\begin{aligned} F(u^*) &= F(x^*, y^*, z^*) \\ &= \begin{bmatrix} \frac{\partial X}{\partial x}(x^*, y^*, z^*) & \frac{\partial X}{\partial y}(x^*, y^*, z^*) & \frac{\partial X}{\partial z}(x^*, y^*, z^*) \\ \frac{\partial Y}{\partial x}(x^*, y^*, z^*) & \frac{\partial Y}{\partial y}(x^*, y^*, z^*) & \frac{\partial Y}{\partial z}(x^*, y^*, z^*) \\ \frac{\partial Z}{\partial x}(x^*, y^*, z^*) & \frac{\partial Z}{\partial y}(x^*, y^*, z^*) & \frac{\partial Z}{\partial z}(x^*, y^*, z^*) \end{bmatrix}. \end{aligned}$$

The linear approximation is

$$\dot{u'} = F(u^*)u'. \tag{4.9}$$

In this contact system, the Jacobian matrix F is

$$F = \begin{bmatrix} -K_{yx} + K_z + xK_{zx} & -K_{yy} + xK_{zy} & -K_{yz} + xK_{zz} \\ K_{xx} & K_{xy} & K_{xz} \\ -xK_{xx} & K_y - xK_{xy} & K_z - xK_{xz} \end{bmatrix}.$$

The Eigenvalues of the Fixed Points

Next we want to find the eigenvalues of the fixed points of the contact system K on $(\mathbb{R}^3, \alpha)$.

Proposition 18. *Let* μ_0, μ_1 *and* μ_2 *be the eigenvalues of the fixed points* u^* *of* X_K *on a 3-dimensional contact manifold* $(\mathbb{R}^3, \alpha)$. *Then*

(i) *one of the eigenvalues, say* μ_0, *is* $\lambda = i_E dK = K_z$, *and*

(ii) $\mu_1 + \mu_2 = \lambda$.

Proof. The characteristic equation at each fixed point is a real cubic equation; namely,

$$\begin{aligned}\det(F - \mu I) &= \begin{vmatrix} -K_{yx} + K_z + xK_{zx} - \mu & -K_{yy} + xK_{zy} & -K_{yz} + xK_{zz} \\ K_{xx} & K_{xy} - \mu & K_{xz} \\ -xK_{xx} & K_y - xK_{xy} & K_z - xK_{xz} - \mu \end{vmatrix} \\ &= -\mu^3 + 2K_z\mu^2 - \mu(C + (Kz)^2) + CK_z \\ &= -(\mu - K_z)(\mu^2 - K_z\mu + C) \qquad (4.10) \\ &= 0,\end{aligned}$$

where

$$\begin{aligned}C =& 2xK_{zx}K_{yx} - 2xK_{xx}K_{zy} - xK_{zx}K_z + K_{xy}K_z - (K_{xy})^2 - x^2(K_{zx})^2 \\ &+ K_{xx}K_{yy} + x^2K_{zz}K_{xx}.\end{aligned}$$

Let μ_0, μ_1 and μ_2 be the solutions (the eigenvalues of the matrix F) of the above characteristic equation $\det(F(u^*) - \mu I) = 0$ at a fixed point. We see from (4.10) that the eigenvalues are

$$\mu_0 = K_z(u^*), \quad \mu_1 = \frac{1}{2}(K_z(u^*) + \sqrt{\Delta}) \text{ and } \mu_2 = \frac{1}{2}(K_z(u^*) - \sqrt{\Delta})$$

where $\Delta = [(K_z)^2 - 4C](u^*)$. The result follows. □

This result implies that the stability of the fixed points of the contact system is dependent on $\lambda = K_z = i_E dK$ in the 3-dimensional canonical case. Such an equation

(4.10) can either have three real solutions or one real and two complex solutions. Examining the order and signs of the real parts of the solutions (eigenvalues), we find that 5 cases are possible at $K_z(u^*) \neq 0$ and 2 cases at $K_z(u^*) = 0$ (we shall not consider the special case when two of the solutions (eigenvalues) of (4.10) are equal). These cases are distinguished by the sign of the **discriminant** Δ (we shall not consider the special cases when $\Delta = 0$). If $\Delta > 0$, then three eigenvalues are all real, and if $\Delta < 0$ then one is real and two complex (complex conjugates).

I. In the case $\lambda = K_z(u^*) \neq 0$

1. Three real eigenvalues ($\Delta > 0$)

   (a) Three positive real eigenvalues
   If $\lambda = K_z(u^*) > 0$ and $0 < \sqrt{\Delta} < K_z(u^*)$, then the eigenvalues have the order

   $$0 < \mu_2 < \mu_1 < \mu_0.$$

   The fixed point is unstable (**unstable node**).

   (b) Two positive and one negative real eigenvalues
   If $K_z(u^*) > 0$ and $0 < K_z(u^*) < \sqrt{\Delta}$, then the eigenvalues have the order

   $$\mu_2 < 0 < \mu_0 < \mu_1.$$

   The fixed point is unstable (**saddle**).

   (c) One positive and two negative real eigenvalues
   If $K_z(u^*) < 0$, then the eigenvalues have the order

   $$\mu_0 < \mu_2 < 0 < \mu_1, \quad \text{or} \quad \mu_2 < \mu_0 < 0 < \mu_1.$$

   The fixed point is unstable (**saddle**).

2. One real eigenvalue and two complex eigenvalues ($\Delta < 0$)

(a) Three eigenvalues with positive real parts

If $\lambda = K_z(u^*) > 0$, then the eigenvalues have the following property

$$0 < Re(\mu_1) = Re(\mu_2) < \mu_0.$$

The fixed point is unstable (**unstable spiral**).

(b) Three eigenvalues with negative real parts

If $\lambda = K_z(u^*) < 0$, then the eigenvalues have the order

$$\mu_0 < Re(\mu_1) = Re(\mu_2) < 0.$$

The fixed point is asymptotically stable (**stable spiral**).

II. In the case $\lambda = K_z(u^*) = 0$

We recall that a fixed point is called **hyperbolic** provided $Re(\mu) \neq 0$ for all eigenvalues μ of its linearization (4.9) [88]. If $\mu_0 = \lambda = 0$, then the fixed point is not hyperbolic. So the Hartman-Grobman theorem [88] does not apply: we can not conclude that the phase portrait of X_K near u^* is topologically conjugate to that of its linearization (4.9).

1. Three real eigenvalues ($\Delta > 0$)

   The eigenvalues have the order

   $$\mu_2 < \mu_0 = 0 < \mu_1.$$

   The fixed point is unstable (**saddle**).

2. One real eigenvalue and two complex eigenvalues ($\Delta < 0$)

   Complex eigenvalues are always complex conjugate, we have

   $$\mu_0 = Re(\mu_1) = Re(\mu_2) = 0.$$

It means that all three eigenvalues have zero real part. Stability can be very difficult to determine in this case because the center manifold is 3-dimensional; that is, we can not reduce the system at all.

4.3.2 Stability On the Contact Manifold

We shall now show that the properties stated in Proposition 18 for the 3-dimensional canonical case also hold in general in any contact system. We will investigate the stability of fixed points of a contact system on the $(2n+1)$-dimensional contact manifold (M, α). We have to find a fixed point of a contact system on (M, α).

Proposition 19. *Let X_K be a contact vector field on (M, α). Then u^* is a fixed point of X_K on (M, α) if and only if*

(i) $K(u^*) = 0$ *and* $\lambda\alpha(u^*) = dK(u^*)$.

(ii) *In the canonical case,* $K_{y_i}(u^*) = x_i K_z(u^*)$ *and* $K(u^*) = K_{x_i}(u^*) = 0$, *for all* $i = 1, \ldots, n$.

Proof. (i) We know that at each fixed point, u^* occurs as a solution of $X_K(u) = 0$. From $i_{X_K}\alpha = K$ and $i_{X_K}d\alpha = \lambda\alpha - dK$ (see Theorem 6), we have

$$K(u^*) = 0 \text{ and } (\lambda\alpha - dK)(u^*) = 0.$$

That is,

$$K(u^*) = 0 \text{ and } \lambda\alpha(u^*) = dK(u^*).$$

Conversely, if $K(u^*) = (\lambda\alpha - dK)(u^*) = 0$, then

$$i_{X_K(u^*)}\alpha = i_{X_K(u^*)}d\alpha = 0.$$

Thus from Proposition 3, we have

$$i_{X_v(u^*)}\alpha = i_{X_h(u^*)}d\alpha = 0;$$

that is,

$$X_K(u^*) = X_v(u^*) + X_h(u^*) = 0.$$

Hence u^* is a fixed point of X_K.

(ii) Let

$$X_K \begin{cases} \dot{x}_i = -K_{y_i} + x_i K_z \\ \dot{y}_i = K_{x_i} \\ \dot{z} = K - \sum_{i=1}^{n} x_i K_{x_i} \end{cases}$$

for all $i = 1, \ldots, n$, be the contact vector field in the canonical case. To find all fixed points u^* of the contact system, we require all the solutions of $X_K(u) = 0$. So we get the fixed points as follows

$$K_{y_i}(u^*) = x_i K_z(u^*)$$
$$K(u^*) = K_{x_i}(u^*) = 0$$

for all $i = 1, \ldots, n$.

□

Proposition 19 implies that a fixed point occurs on the zero section $S = K^{-1}(0)$ in the $(2n+1)$-dimensional contact manifold (M, α).

Theorem 9. *The eigenvalues $\mu_0, \ldots, \mu_{2n}$ of the fixed points u^* of X_K on a $(2n+1)$-dimensional contact manifold (M, α) can be ordered so that*

(i) $\mu_0 = \lambda = i_E dK$, *and*

(ii) $\mu_1 + \mu_2 = \lambda, \ldots, \mu_{2n-1} + \mu_{2n} = \lambda.$

Proof. (i) Consider any ODE

$$\dot{u} = f(u)$$

for which $\dot{K} = \lambda(u)K$ has a fixed point u^* such that $K(u^*) = 0$. Then

$$\dot{K} = \sum_{i=1}^{2n+1} \frac{\partial K}{\partial u_i}\dot{u}_i = \sum_{i=1}^{2n+1} \frac{\partial K}{\partial u_i} f_i = \lambda K.$$

Differentiating with respect to u_j, we obtain

$$\sum_{i=1}^{2n+1} \left(\frac{\partial K}{\partial u_i}\frac{\partial f_i}{\partial u_j} + \frac{\partial^2 K}{\partial u_i \partial u_j} f_i \right) = \lambda \frac{\partial K}{\partial u_j} + K \frac{\partial \lambda}{\partial u_j}.$$

At $u = u^*$, we have $f_i(u^*) = 0$ so

$$\sum_{i=1}^{2n+1} \frac{\partial K}{\partial u_i}\frac{\partial f_i}{\partial u_j} = \lambda \frac{\partial K}{\partial u_j}, \tag{4.11}$$

for all j; that is,

$$(\nabla K)^{\mathrm{T}} F = \lambda (\nabla K)^{\mathrm{T}}.$$

where $F_{ij} = \frac{\partial f_i}{\partial u_j}$ is the Jacobian derivative of f at u^*. Hence F has an eigenvalue λ with left eigenvector $(\nabla K)^{\mathrm{T}}$. (This does not allow one to determine a right eigenvector.)

We now give an alternative proof. At the fixed point u^*, we have

$$\begin{aligned} L_{X_K} dK &= d i_{X_K} dK \\ &= d(\lambda K) \\ &= \lambda(dK) + K(d\lambda) \\ &= \lambda dK \end{aligned}$$

since $K(u^*) = 0$. Since

$$dK = \sum_{i=1}^{2n+1} \frac{\partial K}{\partial u_i} du_i,$$

we have

$$\begin{aligned}
L_{X_K}dK &= \sum_{l=1}^{2n+1}\sum_{i=1}^{2n+1}(dK)_l\frac{\partial(X_K)_l}{\partial u_i}du_i \\
&= \sum_{l=1}^{2n+1}\sum_{i=1}^{2n+1}\frac{\partial K}{\partial u_l}\frac{\partial(X_K)_l}{\partial u_i}du_i \\
&= \sum_{i=1}^{2n+1}\lambda\frac{\partial K}{\partial u_i}du_i.
\end{aligned}$$

From (4.11) and $X_K(u) = f(u)$, we have the same conclusion as before.

(ii) Let $D = \operatorname{Ker}\alpha$. Then D is invariant under the flow of X_K. Thus at $u = u^*$, D is an invariant subspace of the (linearized) flow. We want to study the $2n$ eigenvalues of this linearization on D.

We show that the $2n$ eigenvalues, say $\mu_1, \ldots, \mu_{2n}$ can be arranged so that $\mu_{2i-1} + \mu_{2i} = \lambda$, $i = 1, \ldots, n$. Let X_K be a contact Hamiltonian vector field on (M, α). We have (see Proposition 4)

$$L_{X_K}\alpha = \lambda\alpha \tag{4.12}$$

where $\lambda : M \longrightarrow \mathbb{R}$ is given by $\lambda = i_E dK$. Then from (4.12), we get

$$\begin{aligned}
L_{X_K}d\alpha &= d i_{X_K}d\alpha + i_{X_K}d(d\alpha) \\
&= d(\lambda\alpha) \\
&= d\lambda \wedge \alpha + \lambda d\alpha.
\end{aligned}$$

Thus

$$\begin{aligned}
(L_{X_K}d\alpha)(u^*)|_D &= (d\lambda \wedge \alpha)(u^*)|_D + (\lambda d\alpha)(u^*)|_D \\
&= (\lambda\, d\alpha)(u^*)|_D
\end{aligned}$$

since $\alpha = 0$ on the contact structure D. That is, the linearized vector field is conformal symplectic at $u = u^*$ with symplectic structure $\omega = (d\alpha(u^*))|D$ and conformal constant $c = \lambda(u^*)$. (Recall that $d\alpha$ is nondegenerate on D, see Proposition 11.) That is,

$$L_{X_K}\omega = c\omega. \tag{4.13}$$

Choose coordinates on D and let F be the linearization of X_K on D and let J be the (invertible, antisymmetric) matrix of ω. Then (4.13) becomes

$$JF + F^{\mathrm{T}}J = cJ. \tag{4.14}$$

Now suppose μ is an eigenvalue of F. Then we have

$$\begin{aligned}
0 &= \det(F - \mu I) \\
&= \det((F - \mu I)^{\mathrm{T}}) \\
&= \det(F^{\mathrm{T}} - \mu I) \\
&= \det J^{-1} \det(F^{\mathrm{T}} - \mu I) \det J \\
&= \det(J^{-1}F^{\mathrm{T}}J - \mu I) \\
&= \det(J^{-1}(cJ - JF) - \mu I) \\
&= -\det(F - (c - \mu)I).
\end{aligned}$$

That is, $c - \mu$ is an eigenvalue of F. Thus the pair of eigenvalues $(\mu, c - \mu)$ sum to c. Thus besides the eigenvalue $c = \lambda = i_E dK$, the other pairs sum to λ; that is, $\mu_1 + \mu_2 = \lambda, \ldots, \mu_{2n-1} + \mu_{2n} = \lambda$.

□

Note that if $\lambda(u^*) \neq 0$ then $dK(u^*) \neq 0$ and the zero section $S = K^{-1}(0)$ is a submanifold in the neighborhood of the fixed point u^*; furthermore $D = T_{u^*}S$ in this case. However, the theorem is still true if $\lambda(u^*) = 0$ and S is not a submanifold.

Remark. In general we believe there is no simple formula for the (right) eigenvector associated to the eigenvalue λ. For example, an eigenvector of F corresponding to $\mu_0 = \lambda$ in the canonical case is

$$\begin{bmatrix} -K_{yz}K_{yx} + xK_{zz}K_{yx} + K_{yz}K_z - xK_{zz}K_x + K_{zx}K_{yy} - xK_{zx}K_{yx} \\ xK_{zx}^2 + K_{xx}K_{yz} - xK_{xx}K_{zz} - K_{yx}K_{zx} \\ -K_{xx}K_{yy} + xK_{xx}K_{yx} + K_{yx}^2 - K_{yx}K_z - xK_{yx}K_{zx} + xK_{zx}K_z \end{bmatrix},$$

which is very complicated.

4.3.3 Examples

Let us give some examples of contact systems on $(\mathbb{R}^3, \alpha)$.

Example 9. Consider the contact Hamiltonian $K(x, y, z) = x^2 + y^2 + z^2 - 1$. We have the contact vector field of K as follows:

$$X_K \begin{cases} \dot{x} = -K_y + xK_z = -2y + 2xz \\ \dot{y} = K_x = 2x \\ \dot{z} = K - xK_x = -x^2 + y^2 + z^2 - 1. \end{cases} \tag{4.15}$$

Fixed points occur where

$$-2y + 2xz = 0, \quad 2x = 0, \quad -x^2 + y^2 + z^2 - 1 = 0.$$

The fixed points are $(0, 0, 1)$ and $(0, 0, -1)$. The Jacobian of X_K is

$$F = \begin{bmatrix} 2z & -2 & 2x \\ 2 & 0 & 0 \\ -2x & 2y & 2z \end{bmatrix}.$$

We need to investigate the eigenvalues of $F(x, y, z)$ at each fixed point in order to decide the type of each linear approximation.

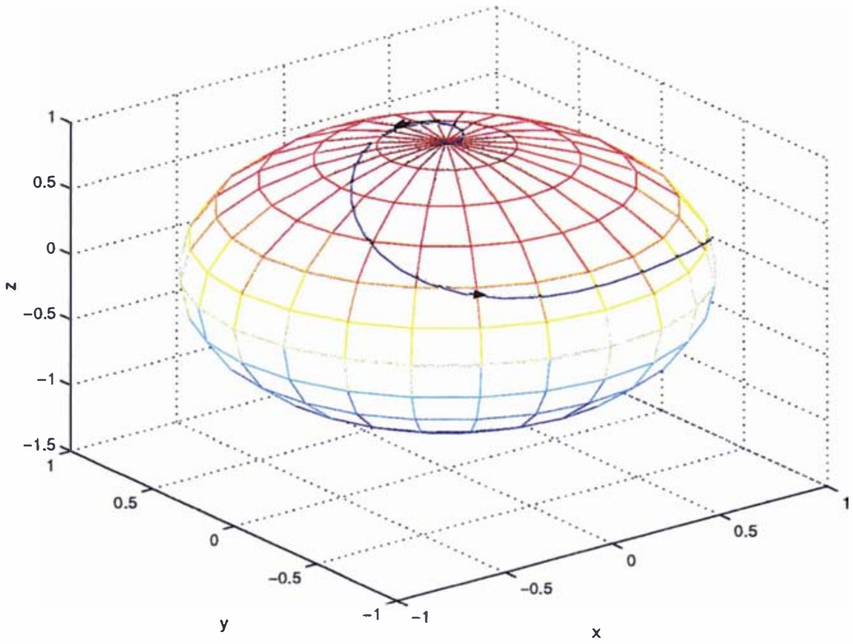

Figure 4.1: The flow of equation (4.15) on the sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ spirals away from the north pole $(0, 0, 1)$. Therefore the north pole is unstable.

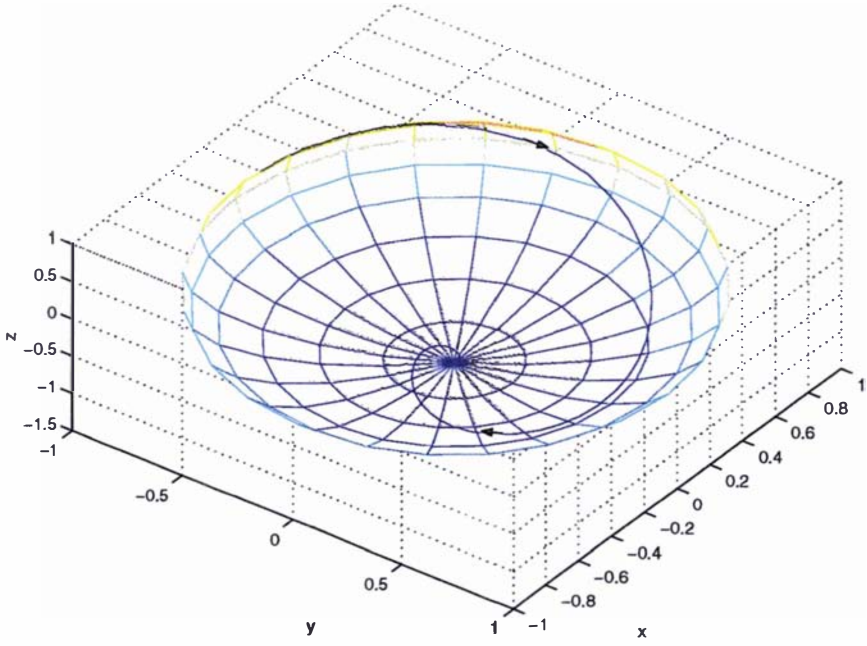

Figure 4.2: The flow of equation (4.15) on the sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ spirals toward the south pole $(0, 0, -1)$. Therefore the south pole is stable.

1. For the fixed point $(0,0,1)$:
   Here the eigenvalues of $F(0,0,1)$ are given by
   $$\det[F(0,0,1) - \mu I] = 0.$$
   So the eigenvalues are $\mu_0 = 2$, $\mu_1 = 1+\sqrt{3}i$ and $\mu_2 = 1-\sqrt{3}i$. Since all the eigenvalues have positive real parts, the fixed point $(0,0,1)$ is unstable.

2. For the fixed point on $(0,0,-1)$:
   The eigenvalues are $\mu_0 = -2$, $\mu_1 = -1+\sqrt{3}i$ and $\mu_2 = -1-\sqrt{3}i$. Since all the eigenvalues have negative real parts, the fixed point $(0,0,-1)$ is stable.

Because of the complex conjugate eigenvalues with positive real part, we expect that the flow on the sphere $K = 0$ spirals away from the fixed point $(0,0,1)$ and then spirals toward the fixed point $(0,0,-1)$. Therefore the fixed point $(0,0,1)$ is unstable but the antipodal point $(0,0,-1)$ is stable. One orbit is shown in Figure 4.1-4.2, computed with ODE45 in MATLAB. The initial condition is taken to be $(0.0001, 0.0001, 1)$ with time interval [0,100].

Note that $\lambda = 2z$ changes sign on S, so we are in the case when the flow on S is not conformal symplectic.

Example 10. Consider the contact Hamiltonian $K(x,y,z) = x^2+y^2-z$ with the contact system as follows:

$$X_K \begin{cases} \dot{x} = -K_y + xK_z = -2y - x \\ \dot{y} = K_x = 2x \\ \dot{z} = K - xK_x = -x^2 + y^2 - z. \end{cases} \tag{4.16}$$

Fixed points occur where

$$-2y - x = 0, \quad 2x = 0, \quad -x^2 + y^2 - z = 0.$$

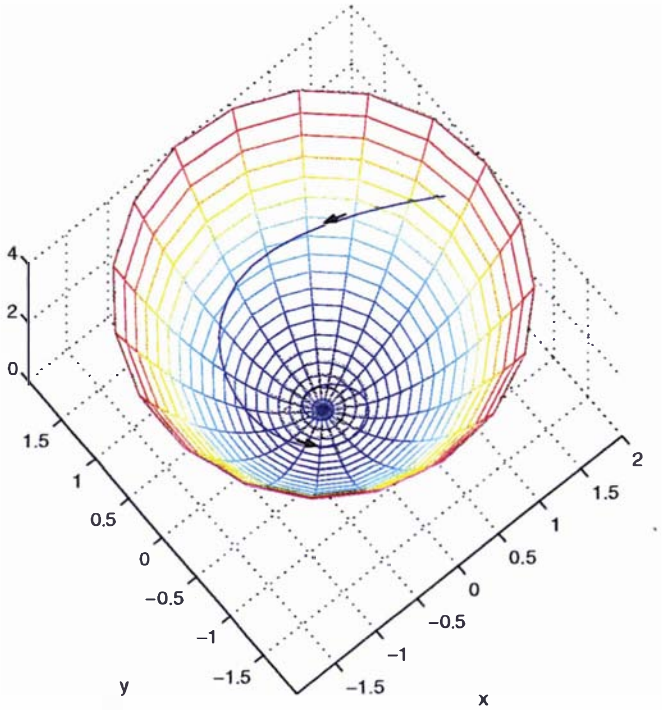

Figure 4.3: The flow of equation (4.16) on the paraboloid $z = x^2 + y^2$ spirals into the fixed point $(0,0,0)$. Therefore the fixed point $(0,0,0)$ is stable.

There is only one fixed point; namely, $(0,0,0)$. The Jacobian of X_K is

$$F = \begin{bmatrix} -1 & -2 & 0 \\ 2 & 0 & 0 \\ -2x & 2y & -1 \end{bmatrix}.$$

One check that the eigenvalues are $\mu_0 = -1$, $\mu_1 = \frac{-1+\sqrt{3}i}{2}$ and $\mu_2 = \frac{-1-\sqrt{3}i}{2}$. Since all eigenvalues have negative real parts, the fixed point $(0,0,0)$ is a stable spiral. One orbit is shown in Figure 4.3, computed with ODE45 in MATLAB. The initial condition is taken to be (2, 0, 4) with time interval [0,100].

Example 11. Consider the contact Hamiltonian $K(x,y,z) = x^2 + y^2 - z^2$. We have the contact vector field of K as follows:

$$X_K \begin{cases} \dot{x} = -K_y + xK_z = -2y - 2xz \\ \dot{y} = K_x = 2x \\ \dot{z} = K - xK_x = -x^2 + y^2 - z^2. \end{cases} \tag{4.17}$$

Fixed points occur where

$$-2y - 2xz = 0, \quad 2x = 0, \quad -x^2 + y^2 - z^2 = 0.$$

There is only one fixed point; namely, $(0,0,0)$. The Jacobian of X_K is

$$F = \begin{bmatrix} -2z & -2 & -2x \\ 2 & 0 & 0 \\ -2x & 2y & -2z \end{bmatrix}.$$

We need to investigate the eigenvalues of $F(x,y,z)$ at the fixed point in order to decide the type of the linear approximation. At the fixed point $(0,0,0)$, the eigenvalues are $\mu_0 = 0$, $\mu_1 = \sqrt{2}i$ and $\mu_2 = -\sqrt{2}i$. Since all the eigenvalues have zero real parts, the fixed point $(0,0,0)$ is non-hyperbolic. Furthermore we cannot use the center manifold reduction because the center manifold is 3-dimensional. In this example, however, we can determine the stability of the fixed point because $(0,0,z)$ is invariant, and on this line,

$$\dot{z} = -z^2.$$

Hence the fixed point $(0,0,0)$ is unstable.

We can also describe the flow of X_K on S. Let

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \pm r.$$

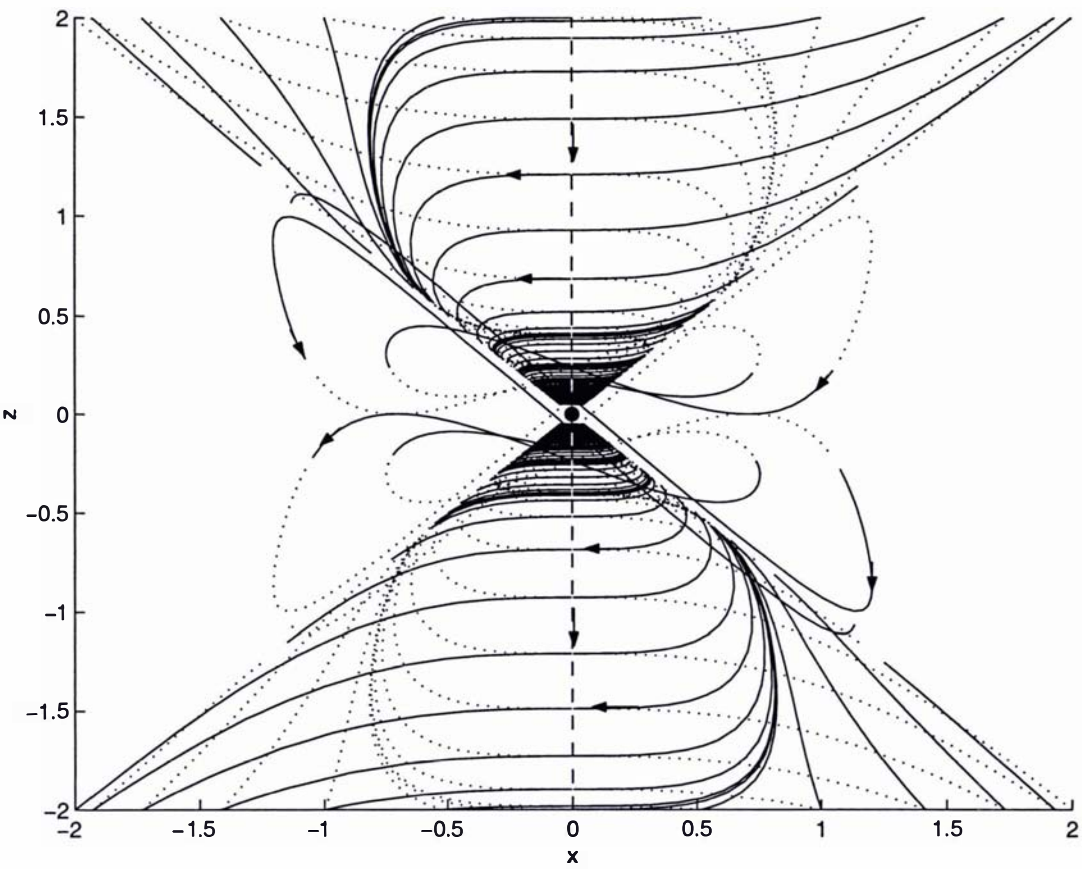

Figure 4.4: The phase portrait of contact system (4.17) is sketched, showing 24 orbits on the invariant cone $z = \sqrt{x^2 + y^2}$, 24 orbits on the invariant cone $z = -\sqrt{x^2 + y^2}$, two orbits on the invariant z-axis, and four other orbits intersecting the plane $z = 0$. The origin is a weakly unstable spiral. (Portions of orbits in $y > 0$ are shown as solid lines, portions of orbits in $y < 0$ are shown as dotted lines.)

Using the coordinate (r, θ), $X_K|_S$ is given by

$$\dot{\theta} = 2(1 \pm r \sin\theta \cos\theta)$$
$$\dot{r} = \mp 2r^2 \cos^2\theta.$$

When $z = r$, $\dot{r} < 0$ ($\cos\theta \neq 0$) and $\dot{r} = 0$ ($\cos\theta = 0$). But if $\cos\theta = 0$, then $\dot{\theta} > 0$. Hence $r(t)$ is a decreasing function of t and all orbits spiral into the origin. Similarly when $z = -r$, all orbits on S spiral away from the origin. The phase portrait is shown in Figure 4.4, calculated by ODE45 in MATLAB.

Chapter 5

Integrable Hamiltonian and Contact Systems

Hamiltonian theory is an important element of the theory of integrable systems, whether discrete, ordinary differential or partial differential equations [7, 19]. Integrable Hamiltonian systems form a very important class of dynamical systems. Over the last decades, they have appeared in many problems of classical dynamics, physics, control theory, etc [8]. They are also used in symplectic integration. The purpose of this chapter is to establish the integrability of contact systems and its relation to the integrability of homogeneous Hamiltonian systems.

We shall introduce the concepts of integral of Hamiltonian systems and contact systems. We shall show how integrable homogeneous Hamiltonian systems are related to integrable contact systems.

In Chapter 6 we shall obtain more information on integrable contact systems as used in contact integrators.

5.1 Integrals and Integrable Systems of DEs

5.1.1 What is an Integral?

Definition 12. Let M be a n-dimensional manifold. For the vector field X with local expression $X = X_i(x)\frac{\partial}{\partial x_i}$, one considers the system of differential equations which gives the flow of X:

$$\dot{x}_i(t) = \frac{dx_i}{dt}(t) = X_i(x_1(t), \dots, x_n(t)). \tag{5.1}$$

A solution $x_i(t) = c_i(t)$, $i = 1, \dots, n$ of this system of DEs is called an **integral curve** of X.

A function $F : M \longrightarrow \mathbb{R}$ is called an **integral** (or first integral, conserved quantity, or constant of motion, or invariant function) of X or of the system

$$\dot{x}_i = F(x(t))$$

if F is constant along the solution of (5.1); that is,

$$\frac{d(F \circ c)}{dt}(t) = 0,$$

for every integral curve $c(t)$ of X.

Example 12. The linear system

$$\dot{x} = y, \qquad \dot{y} = -x \tag{5.2}$$

has an integral $F = x^2 + y^2$. This ODE can be solved by separating the variables to obtain $x^2 + y^2$=constant. The derivative of F following the motion of (5.2) is

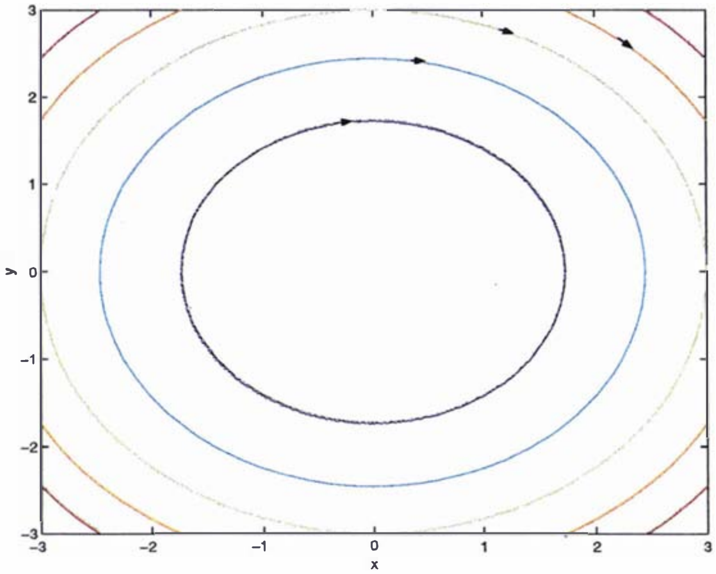

Figure 5.1: Solutions of the simple harmonic oscillator problem(5.2).

zero since

$$\dot{F} = F_x\dot{x} + F_y\dot{y} = 2x(y) + 2y(-x) = 0.$$

Thus F meets all the conditions for being an integral, and it is a conserved quantity. So we know that the orbits are circles enclosing the origin and F is constant on the integral curves of this system. Figure 5.1 shows the circles for several level sets of F, each of them is an orbit of this system.

5.1.2 What is an Integrable Systems of DEs?

Ever since the days of Newton (1687), one of the great dreams of 18th and 19th century analytical mechanics has been trying to solve differential equations describing mechanical systems by quadrature. For example, Newton himself solved the Kepler problem of calculating the orbit of a planet around the sun.

Definition 13. A system of differential equations is called an **integrable system** if it can be solved exactly with analytical methods by **quadrature** (integrations of functions of one variable).

Remark. The orbit of the system in Example 12 can be described in polar coordinates r, θ. Let

$$x = r\sin\theta, \qquad y = r\cos\theta.$$

Then $\dot{x} = y$ and $\dot{y} = -x$ if

$$\dot{r} = 0, \qquad \dot{\theta} = 1.$$

Since $\dot{r} = 0$, the function r is a first integral of the system. And since the solution of this system can be reduced to quadrature; that is,

$$r(t) = \text{constant}, \qquad \theta(t) = \int 1\, dt = t + \theta_0,$$

this system is integrable (see also [54]).

If a system of n-dimensional DEs has $n-1$ independent integrals, the solution of the system of differential equations can be reduced to quadrature. That is, the system is integrable.

For, let M be a n- dimensional manifold, and consider the system

$$\dot{x} = X(x), \quad x \in M$$

with

$$\begin{cases} I_1(x) = c_1 \\ \quad\vdots \\ I_{n-1}(x) = c_{n-1}. \end{cases}$$

Here $I_1, \ldots, I_{n-1}$ are first integrals of the system with n-dimensional DEs, so I_i is constant. Since the orbits are determined just by the integrals, so a joint level set is one-dimensional and thus determines an orbit.

5.2 Poisson Bracket and Jacobi Bracket

The set of all smooth vector fields on a manifold forms a Lie algebra under the Lie bracket. We introduce the Poisson bracket and the Jacobi bracket and show that Hamiltonian functions on a symplectic manifold form a Lie algebra under the Poisson bracket, and contact Hamiltonian functions on a contact manifold form a Lie algebra under the Jacobi bracket.

5.2.1 Poisson Bracket and Integrals for Hamiltonian Systems

Poisson Bracket for Hamiltonian Systems

Definition 14. Let $F, G : P \longrightarrow \mathbb{R}$ be Hamiltonian functions on the symplectic manifold (P, ω). We define the **Poisson bracket** of F and G as

$$\{F, G\} := -\omega(X_F, X_G) \tag{5.3}$$

where $i_{X_F}\omega = -dF$ and $i_{X_G}\omega = -dG$.

Then $\{F, G\}$ is the derivative of F in the direction of the Hamiltonian flow of G. Two functions F and G on P are said to be in **involution** if their Poisson bracket vanishes; that is,

$$\{F, G\} = 0.$$

We now give a property of the Poisson bracket:

$$\{F, G\} = L_{X_G}F = -L_{X_F}G. \tag{5.4}$$

Proposition 20. [5] *With respect to the symplectic coordinates* $(\mathbf{p}, \mathbf{q}) = (p_0, \ldots, p_n, q_0, \ldots, q_n)$ *we have*

$$\{F, G\} = \left(\frac{\partial F}{\partial \mathbf{q}}\frac{\partial G}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}}\frac{\partial G}{\partial \mathbf{q}}\right) = \sum_{i=0}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right). \tag{5.5}$$

Proof.

$$\begin{aligned}
\{F, G\} &= L_{X_G} F \\
&= i_{X_G} dF \\
&= (F_{\mathbf{p}} d\mathbf{p} + F_{\mathbf{q}} d\mathbf{q})\left(\dot{\mathbf{p}}\frac{\partial}{\partial \mathbf{p}} + \dot{\mathbf{q}}\frac{\partial}{\partial \mathbf{q}}\right) \\
&= F_{\mathbf{p}}\dot{\mathbf{p}} + F_{\mathbf{q}}\dot{\mathbf{q}} \\
&= F_{\mathbf{p}}(-G_{\mathbf{q}}) + F_{\mathbf{q}} G_{\mathbf{p}}.
\end{aligned}$$

□

First Integrals of Hamiltonian Systems

The notion of the Poisson bracket was generalized by S. Lie at the end of the 19th century. It predates the symplectic structure approach to Hamiltonian systems.

The Hamiltonian vector field X_H associated with H is

$$\begin{aligned}
\dot{\mathbf{p}} &= \{\mathbf{p}, H\} = -H_{\mathbf{q}} \\
\dot{\mathbf{q}} &= \{\mathbf{q}, H\} = H_{\mathbf{p}},
\end{aligned} \tag{5.6}$$

since

$$\dot{F} = F_{\mathbf{p}}\dot{\mathbf{p}} + F_{\mathbf{q}}\dot{\mathbf{q}} = \{F, H\}.$$

The Hamiltonian system (5.6) governs the evolution of the system. Solutions $\varphi(t)$ of (5.6) are the trajectories of the system, the Hamiltonian system (5.6) reduces to

$$\dot{\varphi} = X_H(\varphi).$$

First integrals are functions $F(\mathbf{p}, \mathbf{q})$ that are constant along the solutions $\varphi(t)$. A necessary and sufficient condition for F to be a first integral is $\{F, H\} = 0$. In particular, the Hamiltonian itself is a first integral [82].

We now consider the Poisson bracket of F and G which are homogeneous Hamiltonians of degree 1 in $\mathbf{p}$ on the symplectic manifold (P, ω) which is the symplectification of the contact manifold (M, α).

Proposition 21. *Let F and G be homogeneous Hamiltonian functions of degree 1 in $\mathbf{p}$ on the symplectic manifold (P, ω). Then the Poisson bracket of F and G is*

$$\{F, G\} = -i_{[X_F, X_G]}\theta, \tag{5.7}$$

where $\omega = d\theta$ and θ is the Liouville form.

Proof. We have

$$\{F, G\} = -L_{X_F}G.$$

Since G is a homogeneous Hamiltonian of degree 1, we have

$$i_{X_G}\theta = G. \tag{5.8}$$

Therefore

$$\{F, G\} = -L_{X_F}i_{X_G}\theta. \tag{5.9}$$

And we have

$$-i_{[X_F, X_G]}\theta = -L_{X_F}i_{X_G}\theta + i_{X_F}L_{X_G}\theta. \tag{5.10}$$

From (5.8), we obtain

$$L_{X_G}\theta = i_{X_G}d\theta + di_{X_G}\theta = -dG + dG = 0. \tag{5.11}$$

From (5.9) and (5.10), we have the Poisson bracket of two homogeneous Hamiltonian functions F and G of degree 1 in $\mathbf{p}$:

$$\begin{aligned}\{F, G\} &= -L_{X_F}i_{X_G}\theta \\ &= -i_{[X_F, X_G]}\theta.\end{aligned}$$

□

5.2.2 Jacobi Bracket and Integrals for Contact Systems

Jacobi Bracket for Contact Systems

Definition 15. [66] Let $f,\ g : M \longrightarrow \mathbb{R}$ be contact Hamiltonians on the contact manifold (M, α), and let X_f, X_g denote the contact vector fields of $f,\ g$. Then the **Jacobi bracket** of f and g is defined analogously to (5.7) by

$$\begin{aligned}[f, g] :&= -i_{[X_f, X_g]}\alpha \\ &= -L_{X_f}i_{X_g}\alpha + i_{X_g}L_{X_f}\alpha.\end{aligned} \tag{5.12}$$

The Jacobi bracket satisfies the relations

$$\begin{aligned}[f, g] &= -i_{[X_f, X_g]}\alpha \\ &= -L_{X_f}i_{X_g}\alpha + i_{X_g}L_{X_f}\alpha \\ &= -L_{X_f}g + i_{X_g}[(i_E df)\alpha] \\ &= -i_{X_f}dg + (i_E df)i_{X_g}\alpha\end{aligned}$$

$$\begin{aligned}
&= -i_{X_f}dg + gi_E df \\
&= -i_{X_f}[(i_E dg)\alpha - i_{X_g}d\alpha)] + gi_E df \\
&= i_{X_f}i_{X_g}d\alpha - fi_E dg + gi_E df \\
&= -d\alpha(X_f, X_g) - fi_E dg + gi_E df,
\end{aligned}$$

since $L_{X_f}\alpha = (i_E df)\alpha$ and $dg = (i_E dg)\alpha - i_{X_g}d\alpha$.

We next show a relationship between Poisson brackets and Jacobi brackets. The Poisson bracket $\{F, G\}$ of two homogeneous Hamiltonians of degree 1 in $\mathbf{p}$ on the symplectic manifold $(P \simeq M \times \mathbb{R}_*, \omega)$ corresponds to the Jacobi bracket $[f, g]$ over a contact manifold (M, α). Here the Hamiltonian F (respectively G) is the homogeneous function of degree 1 in $\mathbf{p}$ associated in the contact Hamiltonian f (respectively g) (see Section 3.3.3).

In the sense of (5.7) and (5.12), we have the following property:

Proposition 22. [66] *Let (P, ω) be the symplectification of the contact manifold (M, α). Then the Poisson bracket $\{F, G\}$ of two homogeneous Hamiltonians of degree 1 on (P, ω) and the Jacobi bracket of the corresponding contact Hamiltonians $[f, g]$ are related by*

$$\{F, G\}(u, \lambda) = \lambda[f, g](u). \tag{5.13}$$

We now prove an analogue of Proposition 20 for the Jacobi bracket:

Proposition 23. [5] *In contact coordinates $(x, y, z) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ we have*

$$\begin{aligned}
[f, g] &= \frac{\partial f}{\partial z}\left(g - x\frac{\partial g}{\partial x}\right) - \frac{\partial g}{\partial z}\left(f - x\frac{\partial f}{\partial x}\right) + \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right) \\
&= \frac{\partial f}{\partial z}\left(g - \sum_{i=1}^{n} x_i\frac{\partial g}{\partial x_i}\right) - \frac{\partial g}{\partial z}\left(f - \sum_{i=1}^{n} x_i\frac{\partial f}{\partial x_i}\right) + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}\right).
\end{aligned}$$

Proof. From the relationship between homogeneous Hamiltonians F, G and contact Hamiltonian f, g (see Section 3.3.3) and Proposition 22, we have

$$\begin{aligned}
\{F, G\} &= F_{q_0}G_{p_0} - F_{p_0}G_{q_0} + F_qG_p - F_pG_q \\
&= p_0 f_{q_0}\left(g - \frac{p}{p_0}g_{\frac{p}{p_0}}\right) - \left(f - \frac{p}{p_0}f_{\frac{p}{p_0}}\right)p_0 g_{q_0} + p_0 f_q g_p - p_0 f_p g_q \\
&= p_0\left[f_{q_0}\left(g - \frac{p}{p_0}g_{\frac{p}{p_0}}\right) - \left(f - \frac{p}{p_0}f_{\frac{p}{p_0}}\right)g_{q_0} + f_q g_p + f_p g_q\right] \\
&= p_0[f, g].
\end{aligned}$$

Therefore we can express the Jacobi bracket at the point

$$x = \frac{p}{p_0}, \quad y = q, \quad z = q_0$$

by

$$[f, g] = f_z(g - xg_x) - g_z(f - xf_x) + (f_y g_x - f_x g_y).$$

□

From the above result, we know that the Jacobi bracket for a contact system can be computed from the Poisson bracket for a homogeneous Hamiltonian system. If X_f and X_g are contact vector fields on the contact manifold (M, α), then we have

$$\begin{aligned}
L_{[X_f, X_g]}\alpha &= L_{X_f}(L_{X_g}\alpha) - L_{X_g}(L_{X_f}\alpha) \\
&= L_{X_f}[(i_E g)\alpha] - L_{X_g}[(i_E f)\alpha] \\
&= \alpha(L_{X_f} i_E g) + i_E g L_{X_f}\alpha - \alpha(L_{X_g} i_E f) - i_E f L_{X_g}\alpha \\
&= \alpha(L_{X_f} i_E g) + i_E g i_E f \alpha - \alpha(L_{X_g} i_E f) - i_E f i_E g \alpha \\
&= \alpha(L_{X_f} i_E g - L_{X_g} i_E f) \\
&= (X_f \cdot i_E g - X_g \cdot i_E f)\alpha.
\end{aligned}$$

We have the following Corollary.

Corollary 2. [80] *Let* $f,\ g : M \longrightarrow \mathbb{R}$ *be contact on the contact manifold* (M, α), *and let* X_f, X_g *denote the contact vector fields of* $f,\ g$. *Then*

$$X_{[f,g]} = -[X_f, X_g].$$

In particular, the contact vector fields form a Lie algebra.

Proposition 24. *Let* (M, α) *be a contact manifold. Then the Jacobi bracket defines a Lie algebra structure on* $C^\infty(M)$ *; that is, it is a bilinear map defined on* M

$$[\cdot\,,\cdot] : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

satisfying the following properties:

(i) *Bilinearity,* $[f, ag + bh] = a[f,g] + b[f,h]$

(ii) *Skew-symmetry,* $[f,g] = -[g,f]$

(iii) *Jacobi identity,* $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$

for $a, b \in \mathbb{R}$ *and* $f, g, h \in C^\infty(M)$.

Proof. (i) The Jacobi bracket is bilinear:

$$\begin{aligned}
[f, a\,g + b\,h] &= -L_{X_f} i_{X_{ag+bh}}\alpha + i_{X_{ag+bh}} L_{X_f}\alpha \\
&= -L_{X_f}(ag + bh) + (ag + bh)\, i_E df \\
&= -aL_{X_f}g - bL_{X_f}h + (ag + bh) i_E df \\
&= a(-L_{X_f}g + g i_E df) + b(-L_{X_f}h + h i_E df) \\
&= a[f,g] + b[f,h].
\end{aligned}$$

(ii) The Jacobi bracket is skew-symmetric:

$$\begin{aligned}[f,g] &= -i_{[X_f,X_g]}\alpha \\ &= i_{[X_g,X_f]}\alpha \\ &= -[g,f].\end{aligned}$$

(iii) We must show that the Jacobi bracket satisfies the Jacobi identity. Let f, g and h be three functions in $C^\infty(M)$. Then

$$\begin{aligned}[f,[g,h]] &= -L_{X_f}[g,h] + [g,h]i_E df \\ &= -L_{X_f}(-L_{X_g}h + hi_E dg) + (-L_{X_g}h + hi_E dg)i_E df\end{aligned}$$

$$\begin{aligned}[g,[h,f]] &= -[g,[f,h]] \\ &= L_{X_g}[f,h] - [f,h]i_E dg \\ &= L_{X_g}(-L_{X_f}h + hi_E df) - (-L_{X_f}h + hi_E df)i_E dg\end{aligned}$$

$$[h,[f,g]] = -[[f,g],h] = L_{X_{[f,g]}}h - hi_E d[f,g].$$

By addition, we obtain

$$\begin{aligned}&[f,[g,h]] + [g,[h,f]] + [h,[f,g]] \\ &= L_{X_f}L_{X_g}h - L_{X_g}L_{X_f}h + L_{X_{[f,g]}}h \\ &\quad - L_{X_f}h\, i_E dg - L_{X_g}h\, i_E df + L_{X_g}h\, i_E df + L_{X_f}h\, i_E dg \\ &\quad + hi_E dg i_E df - hi_E df\, i_E dg - hi_E d[f,g] \\ &= L_{X_f}L_{X_g}h - L_{X_g}L_{X_f}h + L_{X_{[f,g]}}h + h\, i_E d[f,g]\end{aligned}$$

Since $[X_f, X_g] = -X_{[f,g]}$ (see Corollary 2) and $i_E d[f,g] = 0$, the above sum is 0.

□

Note that the Poisson bracket also satisfies (i)-(iii) above. However, the Poisson bracket also satisfies the Leibniz product rule,

$$\{F, GH\} = \{F, G\}H + \{F, H\}G,$$

which the Jacobi bracket does not.

First Integrals of Contact Systems

We start on the Hamiltonian side. Since F is a homogeneous function of degree 1 in $\mathbf{p} = (p_0, p)$,

$$\begin{aligned} F(p_0, p, q, q_0) &= p_0 F\left(1, \frac{p}{p_0}, q, q_0\right) \\ &= p_0 f\left(\frac{p}{p_0}, q, q_0\right) \end{aligned}$$

we have the derivative of F as follows:

$$\begin{aligned} \frac{dF(p_0, p, q, q_0)}{dt} &= \frac{dp_0}{dt} F\left(1, \frac{p}{p_0}, q, q_0\right) + p_0 \frac{dF\left(1, \frac{p}{p_0}, q, q_0\right)}{dt} \\ &= \frac{dp_0}{dt} f\left(\frac{p}{p_0}, q, q_0\right) + p_0 \frac{df\left(\frac{p}{p_0}, q, q_0\right)}{dt}. \end{aligned}$$

We consider $\frac{dF}{dt}$ for the Hamiltonian H which corresponds to $\frac{df}{dt}$ for the contact Hamiltonian K. We can write this in the bracket form:

$$\begin{aligned} \frac{dF(p_0, p, q, q_0)}{dt} &= \{F, H\}(p_0, p, q, q_0) \\ &= p_0 \{F, H\}\left(1, \frac{p}{p_0}, q, q_0\right). \end{aligned}$$

Then we have

$$\begin{aligned}
\frac{dF\left(p_0\left(1,\frac{p}{p_0},q,q_0\right)\right)}{dt} &= p_0\frac{dF\left(1,\frac{p}{p_0},q,q_0\right)}{dt} - F\left(1,\frac{p}{p_0},q,q_0\right)p_0H_{q_0}\left(1,\frac{p}{p_0},q,q_0\right) \\
&= p_0\left(\frac{dF\left(1,\frac{p}{p_0},q,q_0\right)}{dt} - F\left(1,\frac{p}{p_0},q,q_0\right)H_{q_0}\left(1,\frac{p}{p_0},q,q_0\right)\right) \\
&= p_0\{F,H\}\left(1,\frac{p}{p_0},q,q_0\right) \\
&= p_0\left(\frac{df\left(\frac{p}{p_0},q,q_0\right)}{dt} - f\left(\frac{p}{p_0},q,q_0\right)K_{q_0}\left(\frac{p}{p_0},q,q_0\right)\right) \\
&= p_0[f,K]\left(\frac{p}{p_0},q,q_0\right).
\end{aligned}$$

By the relationship between the homogeneous Hamiltonian and contact Hamiltonian system we established earlier, we find that

$$\frac{dF(1,x,y,z)}{dt} - F(1,x,y,z)H_z(1,x,y,z) = \frac{df(x,y,z)}{dt} - f(x,y,z)K_z(x,y,z).$$

Since the relationship between the Poisson and Jacobi bracket is given by

$$\{F,H\}(1,x,y,z) = [f,K](x,y,z),$$

we can obtain

$$\frac{df(x,y,z)}{dt} - f(x,y,z)K_z(x,y,z) = [f,K](x,y,z); \tag{5.14}$$

that is,

$$\frac{df(x,y,z)}{dt} = [f,K](x,y,z) + f(x,y,z)K_z(x,y,z).$$

Proposition 25. *Let (M,α) be a contact manifold. A function f on (M,α) is a first integral of the contact Hamiltonian flow of K if and only if*

$$[f, K] + f i_E dK = 0.$$

Proof. If φ^t is the contact Hamiltonian flow for K then from (5.14)

$$(d/dt) f \circ \varphi^t = \varphi^{t^*} L_{X_K} f = \varphi^{t^*}([f, K] + f i_E dK) = 0$$

if and only if $[f, K] + f i_E dK = 0$. □

Contact Vector Field

As Hamiltonian vector fields are related to the Poisson bracket, we show how the contact vector fields are related to the Jacobi bracket. In $\mathbb{R}^3$, the contact Hamiltonian vector field associated with K is

$$X_K = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}.$$

Using the formula

$$\dot{f} = \frac{df}{dt} = [f, K] + f K_z,$$

we can find the contact vector fields as follows:

$$\begin{aligned}
\dot{x} &= [x, K] + xK_z = (-K_y + xK_z - xK_z) + xK_z = -K_y + xK_z \\
\dot{y} &= [y, K] + yK_z = (K_x - yK_z) + yK_z = K_x \\
\dot{z} &= [z, K] + zK_z = (K - xK_x - zK_z) + zK_z = K - xK_x.
\end{aligned}$$

That is, we have recovered the known formula for X_K,

$$X_K \begin{cases} \dot{x} = -K_y + xK_z \\ \dot{y} = K_x \\ \dot{z} = K - xK_x. \end{cases}$$

5.3 Integrable Hamiltonian and Contact Systems

In the classical theory (Lie, Liouville, etc.) of ODEs there are remarkable results which relate the property of integrability of ODEs in quadratures with the existence of first integrals [67, 59]. In order to solve the contact systems by quadrature, we study the integrable contact systems which can be reformulated as the integrable Hamiltonian systems in this section. We shall apply that the integrable contact systems to contact integrators in Chapter 6. In this section, we show the main result of this chapter; namely how integrable homogeneous Hamiltonian systems are related to integrable contact systems on the contact manifold (M, α).

5.3.1 Integrability for Hamiltonian and Contact Systems in the Liouville Sense

Hamiltonian systems form a special type of ODE that is an important element of integrable systems. If a Hamiltonian system can be solved analytically by quadrature to provide an exact solution, then it is possible to give a complete description of the phase space. In this section we shall describe integrable Hamiltonian and contact systems in the Liouville Sense.

Integrability for Hamiltonian Systems in the Liouville Sense

One of the central problems in Hamiltonian systems is to determine whether the Hamiltonian equations are integrable or not. The best known criterion of integrability goes back to Liouville's Theorem [67]. The usual definition of integrability for a Hamiltonian system is as follows (see for instance [54, 83, 105]).

Definition 16. [55] A Hamiltonian vector field X_H on a $(2n+2)$-dimensional symplectic manifold P is called **integrable** (in the Liouville sense) if there exist $(n+1) = \frac{1}{2}\dim P$ functions $F_j : P \longrightarrow \mathbb{R}$, $0 \leq j \leq n$ satisfying at every point of P the following conditions:

(i) $F_0, F_1, \dots, F_n$ are functionally independent

(ii) $\{F_i, F_j\} = 0$ for all i, j

(iii) $\{F_i, H\} = 0$ for all i.

Such Hamiltonian systems can be integrated by quadratures.

Theorem 10. [5, 52] *If X_K is integrable, then all F_i are first integrals of the Hamiltonian system with Hamiltonian F_j, and the flows $\varphi^t_{F_i}$ of these systems commute:*

$$\varphi^t_{F_i} \circ \varphi^s_{F_j} = \varphi^s_{F_j} \circ \varphi^t_{F_i}$$

for all i, j and all $t, s \in \mathbb{R}$. Moreover, compact connected components of regular invariant submanifolds

$$M_c = \{F_0 = c_0, \dots, F_n = c_n\},$$

where $c = c_i \in \mathbb{R}^{n+1}$, $i = 0, \dots, n$, are diffeomorphic to the $(n+1)$-dimensional Lagrangian torus.

Informally, an integrable system can be solved in terms of its first integrals.

Remark. In particular, some Hamiltonian systems have the property of integrability (see 5.1.2) on $(2n+2)$-dimensional symplectic manifold [38, 83, 78]. They possess $2n+1$ independent integrals, $n+1$ of them in involution for integrability in the Liouville sense. The orbits of Hamiltonian systems are determined by using first integrals. They can be integrated explicitly by quadratures in Liouville sense and these motions are restricted to a $(n+1)$-dimensional space. The motion on this space can be written in the form $\dot{x}$=constant, and hence is integrable. Therefore, any Hamiltonian system for which we can find $n+1$ first integrals is called integrable. Some familiar examples have been found important systems such as the Kepler problem, the free rigid body, and harmonic oscillator with rational frequencies [38, 78].

Integrability for the Contact Systems in the Liouville Sense

As in the case of Hamiltonian systems, we want to study integrability for contact systems in the Liouville sense.

Theorem 11. *Suppose X_K is a contact vector field on a (2n+1)-dimensional contact manifold (M, α). If there exist $n = \frac{1}{2}(\dim M - 1)$ functions $f_1, f_2, \ldots, f_n : M \longrightarrow \mathbb{R}$ satisfying the following conditions at every point of M:*

(i) *$K, f_1, f_2, \ldots, f_n$ are functionally independent*

(ii) *$[f_i, f_j] = 0$ for all i, j, and*

(iii) *$[f_i, K] = 0$ for all i,*

then X_K is integrable by quadratures.

Proof. Let $F_1, \ldots, F_n$ be the homogeneous Hamiltonian functions corresponding to $f_1, \ldots, f_n$ and let $F_0 = H$ be the homogeneous Hamiltonian function corresponding to K. Then $F_0, \ldots, F_n$ are functionally independent. By (ii), we have

$$\{F_i, F_j\} = p_0[f_i, f_j] = 0$$

for all i, j. And by (iii), we have

$$\{F_i, H\} = p_0[f_i, K] = 0$$

for all i. From Theorem 10, we have that X_H is integrable. The contact vector field X_K is the reduction of X_H by the symmetry Z (see Section 3.3), hence X_K is also integrable by quadratures. □

We shall call a contact system which satisfies the hypotheses of Theorem 11 Liouville integrable.

Note that, in contrast to the Hamiltonian case, the functions f_i are not necessary first integrals of the contact vector field X_K.

5.3.2 Examples

We show some examples of integrable homogeneous Hamiltonian and contact systems which can be integrated by quadrature.

	Homogenous Hamiltonian H	Contact Hamiltonian K
1	$H(p_0, p_1, \dots, p_n)$	$K(x_1, \dots, x_n)$
2	$H(p_0, q_1, \dots, q_n) = p_0 H(1, q_1, \dots, q_n)$	$K(y_1, \dots, y_n)$
3	$H(p_0, q_0) = p_0 H(1, q_0)$	$K(z)$
4	$H(p_0, q_1, \dots, q_n, q_0) = p_0 H(1, q_1, \dots, q_n, q_0)$	$K(y_1, \dots, y_n, z)$
5	$H(p_i, q_1, \dots, q_n) = p_i H(1, q_1, \dots, q_n)$	$x_i K(y_1, \dots, y_n) \quad i = 1, \dots, n$
6	$H(p_i, q_0) = p_i H(1, q_0)$	$x_i K(z) \quad i = 1, \dots, n$
7	$H(p_i, q_1, \dots, q_n, q_0) = p_i H(1, q_1, \dots, q_n, q_0)$	$x_i K(y_1, \dots, y_n, z) \quad i = 1, \dots, n$
8	$H(ap + bp_0 q + cp_0 q_0)$	$K(ax + by + cz)$

Table 5.1: The table contains some examples of homogeneous Hamiltonians H of degree 1 which are integrable by quadrature. These integrable homogeneous Hamiltonians H correspond to the contact Hamiltonians (see Section 3.3.3). Then the corresponding contact Hamiltonians are also integrable by quadrature.

We give three examples from Table 5.1.

Example 13. Consider the homogeneous Hamiltonian function on $\mathbb{R}^4$

$$H = p_0(q_1^2 + q_0^2).$$

The vector field of H is

$$X_H \begin{cases} \dot{p_0} = -H_{q_0} = -2p_0q_0 \\ \dot{q_0} = H_{p_0} = q_1^2 + q_0^2 \\ \dot{p_1} = -H_{q_1} = -2p_0q_1 \\ \dot{q_1} = H_{p_1} = 0. \end{cases}$$

Since $\dot{q_1} = 0$, the function q_1 is a first integral of the homogeneous Hamiltonian system. Then the solution of this homogeneous Hamiltonian system can be reduced to quadrature by determining in turn $q_0(t)$, $p_0(t)$ and $p_1(t)$, so this homogeneous Hamiltonian system is integrable.

Example 14. Consider the contact Hamiltonian function on $\mathbb{R}^3$

$$K = x(y^2 + z^2).$$

The contact vector field of K is

$$X_K \begin{cases} \dot{x} = -K_y + xK_z = -2xy + 2x^2z \\ \dot{y} = K_x = y^2 + z^2 \\ \dot{z} = K - xK_x = 0. \end{cases}$$

Since $\dot{z} = 0$, the function z is a first integral of the contact system. Then the solution of this contact system can be reduced to quadrature, by solving first for $y(t)$ and then $x(t)$ (which requires solving a Bernoulli equation). So this contact system is integrable.

Example 15. Consider the contact Hamiltonian function on $\mathbb{R}^3$

$$K = g(ax + by + cz).$$

The contact system of K is

$$X_K \begin{cases} \dot{x} = -bg' + cxg' \\ \dot{y} = ag' \\ \dot{z} = g - axg'. \end{cases}$$

If we set $f = ax + by + cz$, then we have

$$\dot{f} = a\dot{x} + b\dot{y} + c\dot{z} = cg(f).$$

So this reduced system can be integrated by quadrature to determine $f(t)$. We can then determine $x(t), y(t)$ and $z(t)$ by quadrature. This class of contact system is integrable.

In the following section we will develop a theory which in some sense explains and generalizes these simple examples.

5.3.3 Relationship between the Integrable Homogeneous and Contact Systems

We shall show how integrable homogeneous Hamiltonian systems are related to integrable contact systems.

Homogeneous Hamiltonian Systems with Related Homogeneous Hamiltonian Functions

If $F(p_0, p, q, q_0) = p_0 F\left(1, \frac{p}{p_0}, q, q_0\right)$ is homogeneous Hamiltonian of degree 1 in $\mathbf{p}$, then

$$F = \mathbf{p}\frac{\partial F}{\partial \mathbf{p}} = \sum_{i=0}^{n} p_i \frac{\partial F}{\partial p_i}. \tag{5.15}$$

We now consider a homogeneous Hamiltonian system with a homogeneous Hamiltonian function F. We are interested in trying to build new integrable systems from known, simple ones.

Theorem 12. *Let F be a homogeneous Hamiltonian of degree 1 in $\mathbf{p}$ and $g : \mathbb{R} \to \mathbb{R}$. Then $g(F)$ is a homogeneous Hamiltonian of degree 1 in $\mathbf{p}$ if and only if g is a homogeneous function of degree 1.*

Proof. Let F be a homogeneous Hamiltonian of degree 1 in $\mathbf{p}$. Then we have

$$\begin{aligned} g(F(\lambda \mathbf{p}, \mathbf{q})) &= g(\lambda F(\mathbf{p}, \mathbf{q})) \\ &= \lambda g(F(\mathbf{p}, \mathbf{q})) \Longleftrightarrow g \text{ is homogenous of degree 1.} \end{aligned}$$

□

From this theorem, we know that g is linear if and only if $g(F)$ is homogeneous of degree 1 in $\mathbf{p}$. We shall consider more general compositions of the form $H = g(F_1, \ldots, F_k) = g(F)$ where F_i are all homogeneous of degree 1 in $\mathbf{p}$ and g is homogeneous of degree 1 in each argument. Thus we have

$$\begin{aligned} H(\lambda\, \mathbf{p}, \mathbf{q}) &= g\left(F_1(\lambda\, \mathbf{p}, \mathbf{q}), F_2(\lambda\, \mathbf{p}, \mathbf{q}), \ldots, F_k(\lambda\, \mathbf{p}, \mathbf{q})\right) \\ &= g\left(\lambda F_1(\mathbf{p}, \mathbf{q}), \lambda F_2(\mathbf{p}, \mathbf{q}), \ldots, \lambda F_k(\mathbf{p}, \mathbf{q})\right) \qquad (5.16) \\ &= \lambda g\left(F_1(\mathbf{p}, \mathbf{q}), F_2(\mathbf{p}, \mathbf{q}), \ldots, F_k(\mathbf{p}, \mathbf{q})\right). \end{aligned}$$

Euler's Theorem states that under homogeneity of degree 1, a function $H = g(F_1, \ldots, F_k) = g(F)$ can be reduced to the sum of its arguments multiplied by their first partial derivatives. Let $\lambda = 1$ in (5.16), then we have the following formula

$$\begin{aligned} H(\mathbf{p}, \mathbf{q}) &= \mathbf{p}\frac{\partial F_1}{\partial \mathbf{p}}\frac{\partial g(F)}{\partial F_1} + \mathbf{p}\frac{\partial F_2}{\partial \mathbf{p}}\frac{\partial g(F)}{\partial F_2} + \cdots + \mathbf{p}\frac{\partial F_k}{\partial \mathbf{p}}\frac{\partial g(F)}{\partial F_k} \\ &= F_1\frac{\partial g(F)}{\partial F_1} + F_2\frac{\partial g(F)}{\partial F_2} + \cdots + F_k\frac{\partial g(F)}{\partial F_k} \end{aligned}$$

by (5.15). In general, from

$$i_{X_H}\omega = -dH = -\sum_{i=1}^{k} \frac{\partial g(F)}{\partial F_i} dF_i,$$

we have

$$X_H = \sum_{i=1}^{k} \frac{\partial g(F)}{\partial F_i} X_{F_i} \tag{5.17}$$

since $i_{X_{F_i}}\omega = -dF_i$ for all $i = 1, \ldots, k$.

Integrability of Homogeneous Hamiltonian Systems

In our application to integrals in Section 5.3.1, we have not dealt with very complicated integrable systems, whose flows may be too difficult to calculate. Instead, we shall describe some useful classes of integrable contact systems whose flows are relatively easy to calculate. To find the integrability of contact systems, we shall find the integrability of homogeneous Hamiltonian systems which are the symplectification of some contact systems.

Theorem 13. *Let (P, ω) be a homogeneous symplectic manifold. Let $F_1, F_2, \ldots, F_k$ be homogeneous Hamiltonian functions of degree 1 in* p *such that X_{F_i} is integrable for all i and $\{F_i, F_j\} = 0$ for all i and j. Then X_H is integrable where $H = g(F_1, F_2, \ldots, F_k) = g(F)$ is a homogeneous Hamiltonian of degree 1 in* p *and g is homogeneous of degree 1.*

Proof. Similar to the proof of Theorem 1 in [77]. Let $H = g(F_1, F_2, \ldots, F_k) = g(F)$ be a homogeneous Hamiltonian. From the chain rule for Poisson bracket, the homogeneous vector field X_H may be expressed as

$$X_H = \sum_{i=1}^{k} \frac{\partial g(F)}{\partial F_i} X_{F_i}. \tag{5.18}$$

Now

$$\begin{aligned}
\frac{d}{dt}\frac{\partial g(F)}{\partial F_i} &= \{\frac{\partial g(F)}{\partial F_i}, g(F)\} \\
&= \sum_{j=1}^{k} \frac{\partial^2 g(F)}{\partial F_i \partial F_j}\{F_j, g(F)\} \\
&= \sum_{j=1}^{k}\sum_{l=1}^{k} \frac{\partial^2 g(F)}{\partial F_i \partial F_j}\frac{\partial g(F)}{\partial F_l}\{F_j, F_l\} \\
&= 0.
\end{aligned}$$

Therefore the coefficients of X_{f_i} in (5.18) are constant along orbits. Furthermore

$$[X_{F_i}, X_{F_j}] = -X_{\{F_i,F_j\}} = 0,$$

so the flows of the vector fields X_{F_i} commute. The flow of X_H is given by the composition of the time $\frac{\partial g(F)}{\partial F_i}$ flows of the X_{F_i}. Therefore X_H for a homogeneous Hamiltonian H is integrable. That H is homogeneous was shown in Theorem 12. $\square$

Contact Systems with Related Contact Functions

Now we consider the relationship between the contact systems with contact Hamiltonian f and with $K = g(f)$. We have

$$i_{X_K}\alpha = g(f) = K,$$

and the contact vector field

$$X_K \begin{cases} \dot{x} = \frac{\partial g}{\partial f}(-f_y + x f_z) \\ \dot{y} = \frac{\partial g}{\partial f} f_x \\ \dot{z} = K - x\frac{\partial g(f)}{\partial f} f_x. \end{cases}$$

So in general there is no relationship between X_K and X_f. We look at two special cases:

- If g is linear
$$K = g(f) = f\frac{\partial g}{\partial f},$$
then we have
$$X_K = \frac{\partial g}{\partial f}X_f.$$

- On the hypersurface
$$S := \{(x,y,z) \in M : f(x,y,z) = 0 \text{ and } K(x,y,z) = 0\},$$
we have
$$X_K = \frac{\partial g}{\partial f}X_f.$$

In the following we will generalize the first case and consider more general compositions of the form $K = g(f_1,\dots,f_k) = g(f)$ where f_i are all contact Hamiltonian functions.

Theorem 14. *Let $K = g(f_1,\dots,f_k) = g(f)$ be the contact Hamiltonian function with the contact Hamiltonians f_i. If g is homogeneous of degree 1 in each argument, then we have the following relationship between X_K and X_{f_i}:*
$$X_K = \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i}X_{f_i}.$$

Proof. Let K be a contact Hamiltonian function with contact system
$$X_K = (X_K)_h + KE.$$

Then we have the following contact vector field:
$$X_K \begin{cases} \dot{x} = \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i}(-(f_i)_y + x(f_i)_z) \\ \dot{y} = \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i}(f_i)_x \\ \dot{z} = g(f) - x\sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i}(f_i)_x. \end{cases}$$

That is,

$$X_K = \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i}(X_{f_i})_h + g(f)E. \tag{5.19}$$

Since f_i is the contact Hamiltonian function, we have

$$X_{f_i} = (X_{f_i})_h + f_i E.$$

Therefore we have

$$(X_{f_i})_h = X_{f_i} - f_i E. \tag{5.20}$$

By (5.19) and (5.20), we obtain

$$\begin{aligned} X_K &= \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i}(X_{f_i} - f_i E) + g(f)E \\ &= \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i} X_{f_i} + E\left(g(f) - \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i} f_i\right). \end{aligned}$$

Since g is homogeneous of degree 1,

$$g(f) - \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i} f_i = 0.$$

Hence we have

$$X_K = \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i} X_{f_i}.$$

□

Relationship between the Integrable Homogeneous and Contact Systems

To study the integrability of the contact systems of $K = g(f_1, f_2, \dots, f_k) = g(f)$, we shall find a relationship between the homogeneous Hamiltonian function $H =$

$g(F_1, F_2, \ldots, F_k) = g(F)$ and the contact Hamiltonian function $K = g(f_1, f_2, \ldots, f_k) = g(f)$.

Because F_i are all homogeneous of degree 1 in $\mathbf{p}$ and g is homogeneous of degree 1 in each argument, we have

$$\begin{aligned} H(p_0, p, q, q_0) &= g(F_1(p_0, p, q, q_0), F_2(p_0, p, q, q_0), \ldots, F_k(p_0, p, q, q_0)) \\ &= g\left(p_0 f_1\left(\frac{p}{p_0}, q, q_0\right), p_0 f_2\left(\frac{p}{p_0}, q, q_0\right), \ldots, p_0 f_k\left(\frac{p}{p_0}, q, q_0\right)\right) \\ &= p_0 g\left(f_1\left(\frac{p}{p_0}, q, q_0\right), f_2\left(\frac{p}{p_0}, q, q_0\right), \ldots, f_k\left(\frac{p}{p_0}, q, q_0\right)\right) \qquad (5.21) \\ &= p_0 K\left(\frac{p}{p_0}, q, q_0\right). \end{aligned}$$

Since H is a homogeneous Hamiltonian of degree 1 in $\mathbf{p}$, it corresponds to $K = g(f_1, f_2, \ldots, f_k) = g(f)$. Since g is homogeneous of degree 1, we have

$$\begin{aligned} H(p_0, p, q, q_0) &= g(F_1(p_0, p, q, q_0), F_2(p_0, p, q, q_0), \ldots, F_k(p_0, p, q, q_0)) \\ &= F_1 \frac{\partial g(F)}{\partial F_1} + F_2 \frac{\partial g(F)}{\partial F_2} + \cdots + F_k \frac{\partial g(F)}{\partial F_k} \end{aligned}$$

and also

$$\begin{aligned} K\left(\frac{p}{p_0}, q, q_0\right) &= g\left(f_1\left(\frac{p}{p_0}, q, q_0\right), f_2\left(\frac{p}{p_0}, q, q_0\right), \ldots, f_k\left(\frac{p}{p_0}, q, q_0\right)\right) \\ &= f_1 \frac{\partial g(f)}{\partial f_1} + f_2 \frac{\partial g(f)}{\partial f_2} + \cdots + f_m \frac{\partial g(f)}{\partial f_k} \end{aligned}$$

and we have by Theorem 14,

$$X_K = \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i} X_{f_i}. \qquad (5.22)$$

By (5.17) and (5.22), we obtain the following relationship between X_H and X_K:

$$\begin{aligned}
X_H(p_0,p,q,q_0) &= \sum_{i=1}^{k} \frac{\partial g(F)}{\partial F_i}\left(1,\frac{p}{p_0},q,q_0\right) X_{F_i}(p_0,p,q,q_0)\\
&= \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i}\left(\frac{p}{p_0},q,q_0\right)\left(X_{f_i}\left(\frac{p}{p_0},q,q_0\right) - p_0\frac{\partial f_i}{\partial q_0}\frac{\partial}{\partial p_0}\left(\frac{p}{p_0},q,q_0\right)\right)\\
&= \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i} X_{f_i}\left(\frac{p}{p_0},q,q_0\right) - p_0\sum_{i=1}^{k}\frac{\partial g(f)}{\partial f_i}\frac{\partial f_i}{\partial q_0}\frac{\partial}{\partial p_0}\left(\frac{p}{p_0},q,q_0\right)\\
&= X_K\left(\frac{p}{p_0},q,q_0\right) - p_0\frac{\partial K}{\partial q_0}\frac{\partial}{\partial p_0}\left(\frac{p}{p_0},q,q_0\right)
\end{aligned}$$

since

$$X_{F_i} = X_{f_i} - p_0\frac{\partial f_i}{\partial q_0}\frac{\partial}{\partial p_0}.$$

That is, X_K is the contact vector field associated with the homogeneous Hamiltonian vector field X_H (see Section 3.3).

We summarize the above discussion in the following theorem.

Theorem 15. *Let $F = (F_1, F_2, \dots, F_k)$ be a homogeneous Hamiltonian function on the symplectic manifold (P,ω) and $f = (f_1, f_2, \dots, f_k)$ be a contact Hamiltonian function on the contact manifold (M,α), where P is the symplectification of M. Given a homogeneous function $g : \mathbb{R}^k \longrightarrow \mathbb{R}$, of degree 1, then the following diagram commutes:*

$$\begin{tikzcd}
P \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R}^k \arrow[r, "g"] \arrow[d, "id"] & \mathbb{R} \arrow[r] \arrow[d, "id"] & \mathcal{X}_{hom.Ham}(P) \arrow[d, "\pi"] \\
M \arrow[r, "f"] & \mathbb{R}^k \arrow[r, "g"] & \mathbb{R} \arrow[r] & \mathcal{X}_{contact}(M)
\end{tikzcd} \qquad (5.23)$$

We have the following relationship between the homogeneous Hamiltonian H and contact Hamiltonian K:

$$H = g \circ F$$

$$K = g \circ f.$$

Integrability of a Contact System

Consider a contact Hamiltonian $K = g(f_1, \ldots, f_k) = g(f)$. We have an analogous result of Theorem 13 concerning the integrability of the contact systems as following:

Theorem 16. *Let (M, α) be a contact manifold. Let $f_1, f_2, \ldots, f_k$ be contact Hamiltonian functions such that X_{f_i} is integrable for all i and $[f_i, f_j] = 0$ for all i and j. Then X_K is integrable for the contact Hamiltonian $K = g(f_1, f_2, \ldots, f_k) = g(f)$, where g is homogeneous of degree 1.*

Proof. Let $K = g(f_1, f_2, \ldots, f_k) = g(f)$ be a contact Hamiltonian where g is homogeneous of degree 1. By Theorem 14, the contact vector field may be expressed as

$$X_K = \sum_{i=1}^{k} \frac{\partial g(f)}{\partial f_i} X_{f_i}. \tag{5.24}$$

As in the proof of Theorem 13, we would like to establish that $\frac{\partial g(f)}{\partial f_i}$ is constant on orbits of X_K. However we found this difficult to prove directly. (Note in particular, that f_i are not first integrals of X_K (see Theorem 11).) Instead we have the following indirect argument: The function $K = g(f)$ is the contact Hamiltonian function on the contact manifold (M, α) associated the homogeneous Hamiltonian function $H = g(F)$ on the symplectic manifold (P, ω). We know from Theorem 13 that $\frac{d}{dt}\frac{\partial g(F)}{\partial F_i} = 0$ on orbits of X_H. Now g is homogeneous of degree 1, so $\frac{\partial g(F)}{\partial F_i}$ is homogeneous of degree 0 and

$$\begin{aligned}\frac{\partial g(F)}{\partial F_i}(p_0, p, q, q_0) &= \frac{\partial g(F)}{\partial F_i}(1, p, q, q_0)\\ &= \frac{\partial g(f)}{\partial f_i}(x, y, z).\end{aligned}$$

Orbits of X_H project to orbits of X_K, so $\frac{\partial g(f)}{\partial f_i}(x, y, z)$ is constant on orbits of X_K in (5.24); that is,

$$\frac{d}{dt}\frac{\partial g(f)}{\partial f_i} = 0.$$

(Note that F_i is an integral of X_H, but because it is homogeneous of degree 1, not degree 0, f_i is not necessary an integral of X_K. $F_i(p_0, p, q, q_0) = p_0 F_i(1, p, q, q_0)$ and $\dot{p}_0 \neq 0$ in general.) Furthermore

$$[X_{f_i}, X_{f_j}] = -X_{[f_i,f_j]} = 0,$$

so the flows of the vector field X_{f_i} commute. The flow of X_K is given by the composition of the time $\frac{\partial g(f)}{\partial f_i}$ flows of the X_{f_i}. Therefore X_K for contact Hamiltonian $K = g(f)$ is integrable. □

We will now apply this theory to illustrate and extend the integrable cases given in Table 5.1 of Section 5.3.2.

Example 16. We give in Table 5.2 some integrable homogeneous Hamiltonians $H = g(F_1, F_2, ..., F_k)$ of degree 1 in **p** and contact Hamiltonians $K = g(f_1, f_2, ..., f_k)$ corresponding to H, where g is an arbitrary homogeneous function.

Let us compare the entries $g(f_1, f_2, ..., f_k)$ in the following table with those in Table 5.1 of Section 5.3.2.

1. An arbitrary homogeneous function $g(1, x_1, \ldots, x_n)$ is equivalent to an arbitrary function $K(x_1, \ldots, x_n)$, which is item 1 of Table 5.1.

2. Similarly, $g(1, y_1, \ldots, y_n)$ is equivalent to $K(y_1, \ldots, y_n)$, which is item 2 of Table 5.1.

3. A homogeneous function of 1 variable, $g(z)$, must be linear, $g(z) = cz$, so this is merely a very special case of item 3 of Table 5.1.

4. $g(1, x_1, \ldots, \hat{x}_i, \ldots, x_n, y_i) = K(x_1, \ldots, \hat{x}_i, \ldots, x_n, y_i)$ is a new example.

5. $g(1, x_1y_1, \cdots, x_ny_n) = K(x_1y_1, \ldots, x_ny_n)$ is a new example.

	Homogeneous Hamiltonian H	Contact Hamiltonian K
1	$g(p_0,p_1,\dots,p_n)=p_0g(1,\frac{p_1}{p_0},\dots,\frac{p_n}{p_0})$	$g(1,x_1,\dots,x_n)$
2	$g(p_0,p_0q_1,\dots,p_0q_n)=p_0g(1,q_1,\dots,q_n)$	$g(1,y_1,\dots,y_n)$
3	$g(p_0q_0)=p_0g(q_0)$	$g(z)$
4	$g(p_0,p_1,\dots,\hat{p}_i,\dots,p_n,p_0q_i),\ i=1,\dots,n$	$g(1,x_1,\dots,\hat{x}_i,\dots,x_n,y_i)$
5	$g(p_0,p_1q_1,p_2q_2,\dots,p_nq_n)$	$g(1,x_1y_1,\dots,x_ny_n)$
6	$g(p_iq_1,\dots,\hat{p}_i\hat{q}_i,\dots,p_iq_n,p_iq_0),\ i=1,\dots,n$	$g(x_iy_1,\dots,\hat{x}_i\hat{y}_i,\dots,x_iy_n,x_iz)$
7	$g(p_0q_0,p_1,\dots,p_n)=p_0g(q_0,\frac{p_1}{p_0},\dots,\frac{p_n}{p_0})$	$g(z,x_1,\dots,x_n)$
8	$g(p_iq_0,p_1,\dots,p_n),\ i=1,\dots,n$	$g(x_iz,x_1,\dots,x_n)$
9	$g(ap+p_0(bq+cq_0))$	$g(ax+by+cz)$
10	$g(p_1-\frac{b_1}{c}p_0,\dots,p_n-\frac{b_n}{c}p_0,ap+p_0(bq+cq_0))$	$g(x_1-\frac{b_1}{c},\dots,x_n-\frac{b_n}{c},ax+by+cz)$

Table 5.2: The table contains some integrable homogeneous Hamiltonians $H = g(F_1, F_2, ..., F_k)$ and contact Hamiltonians $K = g(f_1, f_2, ..., f_k)$ corresponding to H, where g is an arbitrary homogeneous function. The notation $\hat{p}_i$ is used when the variable p_i is deleted from a list of variables. Here $\{F_i, F_j\} = 0$ for all arguments F_i of g and the homogeneous Hamiltonians F_i are all integrable.

6. $g(x_iy_1,\dots,\hat{x}_i\hat{y}_i,\dots,x_iy_n,x_iz) = x_ig(y_1,\dots,\hat{y}_i,\dots,y_n,z)$ is a special case of item 7 in Table 5.1. (The general case of item 7, $x_iK(y_1,\dots,y_n,z)$ is not covered by our general construction.)

7. $g(z,x_1,\dots,x_n)$ is a new example. Note that $K(z,x_1,\dots,x_n)$ (K not homogeneous) is not integrable.

8. $g(x_iz,x_1,\dots,x_n)$ is a new example.

9. $g(ax+by+cz)$ does not account for item 8, $K(ax+by+cz)$.

10. $g(x_1-\frac{b_1}{c},\dots,x_n-\frac{b_n}{c},ax+by+cz)$. Here we have found the complete set of commuting homogeneous integrals of $ap+p_0(bq+cq_0)$; namely, $p_1-\frac{b_1}{c}p_0,\dots,p_n-\frac{b_n}{c}p_0$, and added those to our set of Hamiltonians. This yields a new example but still does not include item 8 of Table 5.1.

Consider item 3 of Table 5.1. It is true that $g(p_0, p_0q_0)$ is integrable for all homogeneous Hamiltonian, with corresponding contact Hamiltonian $g(1, z) = K(z)$ (K arbitrary). However, this does not fall under Theorem 13 because p_0 and p_0q_0 do not commute-the integrability has to be checked directly. This is due to the fact that the system has only one degree of freedom.

Chapter 6

Contact Integrators

In this chapter, we are interested in constructing contact integrators to explore long-time dynamics for contact systems. Contact integrators were first introduced by Feng (1998) [42]. Feng's method for contact integrators are founded on the correspondence between the contact systems in $\mathbb{R}^{2n+1}$ and the homogeneous Hamiltonian systems in $\mathbb{R}^{2n+2}$ which we discussed in Section 3.3. We start with Feng's method, then we treat the splitting method for contact integrators. Finally we will investigate how any polynomial contact system can be split into a sum of integrable contact systems.

6.1 Feng's Methods

To construct contact integrators, Feng used the relationship between the contact systems in $\mathbb{R}^{2n+1}$ and the homogeneous Hamiltonian systems in $\mathbb{R}^{2n+2}$ (see Section 3.3). He derived the generating contact function theory for contact maps from the symplectic analog. In this section we will use the notation and definitions in Feng's papers [42, 41, 44]. Before discussing Feng's method, we introduce some symplectic integrators for Hamiltonian systems.

6.1.1 Symplectic Integrators for Hamiltonian Systems

Numerical integration methods for long-term integrations of Hamiltonian systems are called symplectic integrators. Many different symplectic integrators have been developed and discussed over the past ten years by both physicists and mathematicians (Ruth [90], Feng [39, 40], Channell and Scovel [28], Yoshida [107], McLachlan and Atela [73], Calvo and Sanz-Serna [25, 92], Blanes and Moan [17, 18], etc.):

- Runge-Kutta methods [60, 91]
- Generating functions [39, 40, 46, 28]
- Splitting methods [72, 31, 14, 77]
- Variational methods [100, 71, 103, 104]
- Characterization methods [53, 98].

We consider now Hamiltonian systems with Hamiltonian function

$$H(z) : \mathbb{R}^{2n+2} \longrightarrow \mathbb{R}$$

where $z = (p, q)^{\mathrm{T}}$, $z_i = p_i$, $z_{n+1+i} = q_i$, for $i = 0, \ldots, n$. Then the associated Hamiltonian vector field is

$$\frac{dz}{dt} = X_H(z) = J\nabla H(z) = (-H_q(z), H_p(z))^{\mathrm{T}}, \tag{6.1}$$

where

$$J = J_{2n+2} = \begin{bmatrix} 0 & -I_{n+1} \\ I_{n+1} & 0 \end{bmatrix}$$

and $\nabla H(z) = H_z = (H_{z_1}(z), \ldots, H_{z_{2n+2}}(z))^{\mathrm{T}}$ is the gradient of H. According to the general theory of ODEs, for a Hamiltonian system, the flow denoted by φ_H^t is a one-parameter (in t) group of symplectic transformations on $\mathbb{R}^{2n+2}$, which satisfies

$$\begin{aligned}
&\varphi_H^{t+s} = \varphi_H^t \circ \varphi_H^s, \quad \forall\, t,\ s\ \in \mathbb{R}, \\
&\varphi_H^0 = \text{identity} \\
&(\varphi_H^t)^* \omega = \omega.
\end{aligned}$$

If $z(0)$ is taken as the initial condition, then the solution of system (6.1) is generated by

$$z(t) = \varphi_H^t z(0).$$

Now we consider a **one-step numerical method** to approximate Hamiltonian system (6.1). When $z^0 = z(0)$ is given (z^0 can be taken as any initial point), we can then construct $z^1, z^2, \ldots$ by the method. Therefore this method in fact defines, a mapping on $\mathbb{R}^{2n+2}$ with the step-size, say h, as a parameter, denoted by Φ_H^h. Thus

$$z^{k+1} = \Phi_H^h z^k, \quad k \geqq 0, \quad \text{or} \quad \hat{z} = \Phi_H^h z.$$

Since we do not know the flow φ_H^h, we find maps Φ_H^h that approximate φ_H^h. The method is said to be of order r ($\geqq 1$) if

$$\Phi_H^h = \varphi_H^h + \mathcal{O}(h^{r+1}).$$

Symplectic integrators are numerical methods whose step transition mappings Φ_H^h preserve the symplectic structure of the phase space as applied to Hamiltonian systems; that is,

$$\Phi_H^h \omega = \omega,$$

or

$$(\Phi_H^h)_z^T(z) J (\Phi_H^h)_z(z) = J, \qquad \forall z \in \mathbb{R}^{2n+2}. \tag{6.2}$$

We call such Φ_H^h a **symplectic integrator** for the Hamiltonian system. Here we consider some symplectic integrators.

Some Elementary Symplectic Integrators

Our treatment follows the construction of symplectic integrators using generating functions due to Feng [39].

For the Hamiltonian systems (6.1), there is a symplectic approximation to the flow φ_H^h which can serve as the transition mapping of a symplectic method via generating function theory. Each choice of a symmetric matrix B_{2n+2} yields a symplectic integrator according to

$$z \longrightarrow \hat{z} = g_H^h z: \quad \hat{z} = z + J\nabla H(D\hat{z} + (I - D)z), \tag{6.3}$$

$$D = \frac{1}{2}(I_{2n+2} + JB_{2n+2}), \quad B = B^{\mathrm{T}}, \quad D^{\mathrm{T}}J + JD = J. \tag{6.4}$$

Note that the symplectic map induced by the generating function H from the relation (6.3) can be represented as the composition of the maps, non-symplectic generally, $z \rightarrow \bar{z}$ and $\bar{z} \rightarrow \hat{z}$, where

$$\bar{z} = z + DJ\nabla H(\bar{z}), \qquad \hat{z} = \bar{z} + (I - D)J\nabla H(\bar{z}).$$

We now give three basic symplectic integrators P, Q and C .

- P. Symplectic Euler method (P, q), where we take

  $$D = \begin{bmatrix} I_{n+1} & 0 \\ 0 & 0 \end{bmatrix},$$

  1 stage form: $\hat{z} = z + hJ\nabla H(\hat{p}, q)$, or

  $$\hat{p} = p - hH_q(\hat{p}, q)$$
  $$\hat{q} = q + hH_p(\hat{p}, q).$$

  This treats p-variable by the implicit and the q-variable by the explicit Euler method.

- Q. Symplectic Euler method (p, Q), where we take

  $$D = \begin{bmatrix} 0 & 0 \\ 0 & I_{n+1} \end{bmatrix},$$

  1 stage form: $\hat{z} = z + hJ\nabla H(p, \hat{q})$, or

  $$\begin{aligned} \hat{p} &= p - hH_q(p, \hat{q}) \\ \hat{q} &= q + hH_p(p, \hat{q}). \end{aligned}$$

  This treats q-variable by the implicit and the p-variable by the explicit Euler method.

- C. Centered Euler method (often called the symplectic midpoint rule), where we take

  $$D = \frac{1}{2} I_{2n+2},$$

  1 stage form: $\hat{z} = z + hJ\nabla H(\frac{1}{2}\hat{z} + \frac{1}{2}z)$,
  2 stage form: $\bar{z} = z + \frac{h}{2}J\nabla H(\bar{z})$, $\hat{z} = \bar{z} + \frac{h}{2}J\nabla H(\bar{z}) = 2\bar{z} - z$.

It should be noted the P and Q methods are of order 1 of accuracy and the C method is of order 2.

Remark. In order to construct contact integrators (Sections 6.1 and 6.2), we must apply an integrator to the homogeneous Hamiltonian vector field X_H which gives a homogeneous symplectic map (see Section 3.3). An integrator is homogeneous if and only if it preserves the symmetry Z in Section 3.3. A Runge-Kutta method preserves all linear symmetries, hence any Runge-Kutta method is homogeneous. Also, because $Z = p\frac{\partial}{\partial p}$ does not couple the q and p variables, any partitioned Runge-Kutta method (see [54]) is also homogeneous. In particular the above methods P, Q and C are homogeneous symplectic integrators.

6.1.2 Feng's Contact Integrators for Contact Systems

We use the following coordinates on $\mathbb{R}^{2n+1}$:

$$u = (x, y, z)^{\mathrm{T}}, \quad x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad z = (z).$$

We consider now contact systems with contact Hamiltonian function

$$K(u) : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}.$$

Then the associated contact vector field is

$$\begin{aligned} \frac{du}{dt} &= X_K(u) = J\nabla K(u) + e_{2n+1}K(u) \\ &= (-K_y(u) + xK_z(u),\ K_x(u),\ K(u) - x^{\mathrm{T}}K_x(u))^{\mathrm{T}} \end{aligned} \tag{6.5}$$

$$J = J_{2n+1} = \begin{bmatrix} O_n & -I_n & x \\ I_n & O_n & 0_n \\ -x^{\mathrm{T}} & 0_n^{\mathrm{T}} & 0 \end{bmatrix}, \qquad e_{2n+1} = \begin{bmatrix} 0_n \\ 0_n \\ 1 \end{bmatrix}$$

where $\nabla K(u) = K_u = (K_{u_1}(u), \ldots, K_{u_{2n+1}}(u))^{\mathrm{T}}$ is the gradient of K, x is the column vector of length n and $e_{2n+1} = E$ is the canonical Reeb vector field.

The flows φ_K^t of contact systems defined by contact Hamiltonian K on $\mathbb{R}^{2n+1}$ are contact transformations, satisfying

$$\begin{aligned} &\varphi_K^{t+s} = \varphi_K^t \circ \varphi_K^s, \quad \forall\, t,\ s\ \in \mathbb{R}, \\ &\varphi_K^0 = \text{identity} \\ &(\varphi_K^t)^*\alpha = \mu_{\varphi_K^t}\alpha \end{aligned}$$

for some everywhere non-vanishing function $\mu_{\varphi_K^t} : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$. Moreover, we have the following relation between $\mu_{\varphi_K^t}$ and the contact Hamiltonian K (see Proposition

4):

$$\mu_{\varphi_K^t} = \exp \int_0^t (K_z \circ \varphi_K^s) ds. \tag{6.6}$$

As for Hamiltonian systems, we can find maps Φ_K^h that approximate φ_K^h, such Φ_K^h for contact systems should be contact integrators which is a numerical method whose step transition mappings Φ_K^h preserve the contact structure of phase space as applied to contact Hamiltonian systems.

Definition 17. A **contact integrator** for a contact vector field X_K is a map Φ_K^h, that satisfies

$$(\Phi_K^h)^* \alpha = \mu_{\Phi_K^h} \alpha, \tag{6.7}$$

where $\mu_{\Phi_K^h}$ is some real function depending on Φ_K^h. The method is said to be of order r ($\geqq 1$) if

$$\Phi_K^h(u) = \varphi_K^h + \mathcal{O}(h^{r+1}).$$

Note that in general,

$$\mu_{\Phi_K^h} \neq \mu_{\varphi_K^t}.$$

In (6.2), we gave a formula in coordinates which a symplectic integrator satisfies. We now work out the analogous equations for the contact case. Suppose the contact form α is given in local coordinates as

$$\alpha = \sum_{i=1}^{2n+1} a_i(u) du_i. \tag{6.8}$$

Then we have

$$\begin{aligned} & (\Phi_K^h)^* \alpha = \mu_{\Phi_K^h} \alpha \\ \Longrightarrow\ & \alpha(\Phi_K^h(u))(T\Phi_K^h \cdot v) = \mu_{\Phi_K^h} \alpha(u)(v). \end{aligned}$$

for all $v \in \mathbb{R}^{2n+1}$. That is,

$$(T\Phi_K^h)^{\mathrm{T}}(a \circ \Phi_K^h) = \mu_{\Phi_K^h} a, \tag{6.9}$$

or

$$\sum_{i=1}^{2n+1} \frac{\partial(\Phi_K^h)_i}{\partial x_j} a_i(\Phi_K^h(u)) = \mu_{\Phi_K^h}(u) a_j(u)$$

for all $j = 1, \ldots, 2n+1$.

Linear Contact Systems

First we want to show a very simple example of a linear homogeneous Hamiltonian system on $(\mathbb{R}^4, \omega)$.

Example 17. Consider the homogeneous Hamilton system with the homogeneous Hamiltonian of degree 1 $H = pq - p_0q_0$ on $(\mathbb{R}^4, \omega)$,

$$X_H \begin{cases} \dot{p_0} = -H_{q_0} = p_0 & \dot{p} = -H_q = -p \\ \dot{q_0} = H_{p_0} = -q_0 & \dot{q} = H_p = -q. \end{cases}$$

Then the solution of the homogeneous Hamiltonian system is

$$\begin{cases} p_0(t) = p_0^0 e^t & p(t) = p^0 e^{-t} \\ q_0(t) = q_0^0 e^{-t} & q(t) = q^0 e^t. \end{cases}$$

The flow φ_H^t of this homogeneous Hamiltonian system satisfies

$$\begin{aligned} (\varphi_H^t)^* \omega\, (\varphi_H^t(z^0)) &= d(p_0^0 e^t) \wedge d(q_0^0 e^{-t}) + d(p^0 e^{-t}) \wedge d(q^0 e^t) \\ &= dp_0^0 \wedge dq_0^0 + dp^0 \wedge dq^0. \end{aligned}$$

Next we give an example of linear contact systems on $(\mathbb{R}^3, \alpha)$, which corresponds to Example 17, where $(\mathbb{R}^4, \omega)$ is the symplectification of $(\mathbb{R}^3, \alpha)$.

Example 18. Consider the contact system with the contact Hamiltonian $K = xy - z$ on $(\mathbb{R}^3, \alpha)$,

$$X_K \begin{cases} \dot{x} = -K_y + xK_z = -2x \\ \dot{y} = K_x = y \\ \dot{z} = K - xK_x = -z. \end{cases}$$

The solution of this contact system is

$$\begin{cases} x(t) = x^0 e^{-2t} \\ y(t) = y^0 e^t \\ z(t) = z^0 e^{-t}. \end{cases}$$

The flow φ_K^t of the contact system satisfies

$$\begin{aligned} (\varphi_K^t)^* \alpha \, (\varphi_K^t(u^0)) &= d(z^0 e^{-t}) + (x^0 e^{-2t}) \, d(y^0 e^t) \\ &= e^{-t}(dz^0 + x^0 \, dy^0). \end{aligned}$$

So we have

$$\mu_{\varphi_K^t} = \exp(-t),$$

and also the following relationship:

$$\begin{aligned} \mu_{\varphi_K^t} &= \exp \int_0^t (K_z \circ \varphi_K^s) ds \\ &= \exp \int_0^t (-1) ds \\ &= \exp(-t). \end{aligned}$$

In contrast, consider applying an integrator Φ_K^t to X_K, and suppose we obtain a linear map

$$(\Phi_K^t)_i(u) = \lambda_i(t) u_i.$$

Let α be the contact form

$$\alpha = \sum_{i=1}^{2n+1} a_i(u)du_i.$$

If we set $a = (0, x, 1)$, then (6.9) becomes

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 0 \\ \lambda_1 u_1 \\ 1 \end{bmatrix} = \mu \begin{bmatrix} 0 \\ u_1 \\ 1 \end{bmatrix};$$

that is,

$$\lambda_2\lambda_1 = \mu$$
$$\lambda_3 = \mu.$$

Then Φ_K^t is a contact integrator if and only if

$$\lambda_3(t) = \lambda_1(t)\lambda_2(t), \tag{6.10}$$

which is hard to satisfy. That is, the expansion in the z direction should balance the expansion in the xy plane. In the case that Φ_K^t is a Runge-Kutta method, with stability function R, a rational function, (6.10) becomes

$$R(-2\Delta t)R(\Delta t) = R(-\Delta t),$$

which is also hard to satisfy.

Consider a **linear contact system** with contact Hamiltonian function

$$K(x, y, z) = x^{\mathrm{T}}Ay + bz$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}$. Then the canonical system is linear

$$\dot{u}_i = L_i u_i \tag{6.11}$$

where $u_1 = x, u_2 = y$ and $u_3 = z$:

$$\begin{cases} \dot{x} = (-A + bI)\,x \\ \dot{y} = Ay \\ \dot{z} = bz. \end{cases}$$

The solution of (6.11) is

$$u_i(t) = (\varphi_K^t)_i u_i^0,$$

where $(\varphi_K^t)_i = e^{tL_i}u_i$. In contrast, consider applying an integrator Φ_K^t to (6.11), and suppose we get a linear map

$$(\Phi_K^t)_i(u) = L_i(t)u_i.$$

This is a contact integrator if and only if

$$L_3(t)I = L_1(t)L_2(t), \tag{6.12}$$

where I is the $n \times n$ identity matrix. That is, the expansion in the z direction should balance the expansion in the xy subspace. In the case that Φ_K^t is a Runge-Kutta method, with stability function R, a rational function, (6.12) becomes

$$R((-A + bI)\Delta t)R(A\Delta t) = R(bI\Delta t),$$

which has no nontrivial solution. (The exponential solution $R(z) = e^z$ is not rational.)

Choosing $n = 1, A = \alpha, b = \alpha + \beta$, shows that

$$R(\alpha)R(\beta) = R(\alpha + \beta)$$

is needed, which has no rational solutions. So no Runge-Kutta method is contact.

Feng's Contact Integrators

Under the correspondence between the contact geometry of $\mathbb{R}^{2n+1}$ and the homogeneous symplectic geometry of $\mathbb{R}^{2n+2}$ we mentioned earlier, we have the following relationships from Section 3.3 and the previous section:

1. The contact Hamiltonian $K(x, y, z)$ corresponds to the homogeneous Hamiltonian $H(p_0, p, q_0, q)$;
2. The contact system X_K corresponds to the homogeneous Hamiltonian system X_H; and
3. The contact map φ_K^t corresponds to the homogeneous symplectic map φ_H^t.

Feng constructed the contact integrators for the general contact systems. The contact integrators are analogues of homogeneous symplectic integrators for homogeneous Hamiltonian systems. That is, given a contact Hamiltonian K, we calculate X_H and then apply a homogeneous symplectic integrator (e.g. P, Q and C) to X_H, and calculate the corresponding contact map. We now give three methods $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{C}$ of Feng's analogous to the homogeneous symplectic methods P, Q, C. For more details of Feng's contact integrators, see [42, 41, 44].

- $\widetilde{P}$. Contact analog of the symplectic method P

  1 stage form: $\hat{u} = u + h(\hat{J}(x, \hat{x})\nabla K(\hat{x}, y, z) + e_{2n+1}K(\hat{x}, y, z))$

  where

$$\hat{J}(x, \hat{x}) = \begin{bmatrix} O_n & -I_n & x \\ I_n & O_n & 0_n \\ -\hat{x}^{\mathrm{T}} & 0_n^{\mathrm{T}} & 0 \end{bmatrix},$$

or

$$\begin{aligned}
\hat{x} &= x + h(-K_y(\hat{x}, y, z) + xK_z(\hat{x}, y, z)) \\
\hat{y} &= y + hK_x(\hat{x}, y, z) \\
\hat{z} &= z + h(K(\hat{x}, y, z) - \hat{x}^{\mathrm{T}} K_x(\hat{x}, y, z)).
\end{aligned}$$

This treats the x-variable by the implicit and the y-variable and z-variable by the explicit Euler method.

The integrator $\widetilde{P}$ can also be written in the 2 stage form:

$$\begin{aligned}
&\bar{x} = x + h(-K_y(\bar{x}, \bar{y}, \bar{z}) + xK_z(\bar{x}, \bar{y}, \bar{z})), \quad \bar{y} = y, \quad \bar{z} = z \\
&\hat{x} = \bar{x}, \quad \hat{y} = \bar{y} + hK_x(\bar{x}, \bar{y}, \bar{z})), \quad \hat{z} = \bar{z} + h(K(\bar{x}, \bar{y}, \bar{z}) - \bar{x}^{\mathrm{T}} K_x(\bar{x}, \bar{y}, \bar{z})).
\end{aligned}$$

- $\widetilde{Q}$. Contact analog of the symplectic method Q
  1 stage form: $\hat{u} = u + h(\hat{J}(\hat{x}, x)\nabla K(x, \hat{y}, \hat{z}) + e_{2n+1}K(x, \hat{y}, \hat{z}))$
  or

$$\begin{aligned}
\hat{x} &= x + h(-K_y(x, \hat{y}, \hat{z}) + \hat{x}K_z(x, \hat{y}, \hat{z})) \\
\hat{y} &= y + hK_x(x, \hat{y}, \hat{z}) \\
\hat{z} &= z + h(K(x, \hat{y}, \hat{z}) - x^{\mathrm{T}} K_x(x, \hat{y}, \hat{z})).
\end{aligned}$$

This treats the y-variable and the z-variable by the implicit and the x-variable by the explicit Euler method.

The integrator $\widetilde{Q}$ can also be written in the 2 stage form:

$$\begin{aligned}
&\bar{x} = x, \qquad \bar{y} = y + hK_x(\bar{x}, \bar{y}, \bar{z}), \quad \bar{z} = z + h(K(\bar{x}, \bar{y}, \bar{z}) - \bar{x}^{\mathrm{T}} K_x(\bar{x}, \bar{y}, \bar{z})) \\
&\hat{x} = \bar{x} + h(-K_y(\bar{x}, \bar{y}, \bar{z}) + \hat{x}K_z(\bar{x}, \bar{y}, \bar{z})), \quad \hat{y} = \bar{y}, \quad \hat{z} = \bar{z}.
\end{aligned}$$

- $\widetilde{C}$. Contact version of the centered Euler method C
  1 stage form: $\hat{u} = u + h((J\nabla K)(\frac{1}{2}\hat{u} + \frac{1}{2}u) - e_{2n+1}K(\frac{1}{2}\hat{u} + \frac{1}{2}u))$

2 stage form: $\bar{u} = u + \frac{h}{2}(\hat{J}(x,\bar{x})\nabla K(\bar{u}) - e_{2n+1}K(\bar{u}))$

$$\begin{aligned}
\bar{x} &= x + \frac{h}{2}(-K_y(\bar{x},\bar{y},\bar{z}) + xK_z(\bar{x},\bar{y},\bar{z}))\\
\bar{y} &= y + \frac{h}{2}K_x(\bar{x},\bar{y},\bar{z})\\
\bar{z} &= z + \frac{h}{2}(K(\bar{x},\bar{y},\bar{z}) - \bar{x}^{\mathrm{T}}K_x(\bar{x},\bar{y},\bar{z}))
\end{aligned}$$

$\hat{u} = \bar{u} + \frac{h}{2}(\hat{J}(\hat{x},\bar{x})\nabla K(\bar{u}) - e_{2n+1}K(\bar{u}))$

$$\begin{aligned}
\hat{x} &= \bar{x} + \frac{h}{2}(-K_y(\bar{x},\bar{y},\bar{z}) + \hat{x}K_z(\bar{x},\bar{y},\bar{z}))\\
&= (\bar{x} - \frac{h}{2}K_y(\bar{x},\bar{y},\bar{z}))(1 - \frac{h}{2}K_z(\bar{x},\bar{y},\bar{z}))^{-1}\\
\hat{y} &= \bar{y} + \frac{h}{2}K_x(\bar{x},\bar{y},\bar{z}) = 2\bar{y} - y\\
\hat{z} &= \bar{z} + \frac{h}{2}(K(\bar{x},\bar{y},\bar{z}) - \bar{x}^{\mathrm{T}}K_x(\bar{x},\bar{y},\bar{z})) = 2\bar{z} - z.
\end{aligned}$$

It should be noted the $\widetilde{P}$ and $\widetilde{Q}$ methods are of order 1 of accuracy and the $\widetilde{C}$ method is of order 2.

Example 19. Suppose we have the contact system with contact Hamiltonian $K = \frac{1}{2}(x^2 + y^2)$ on $(\mathbb{R}^3, \alpha)$. Then the corresponding contact system is

$$X_K \begin{cases} \dot{x} = -K_y + xK_z = -y \\ \dot{y} = K_x = x \\ \dot{z} = K - xK_x = \frac{1}{2}(y^2 - x^2). \end{cases} \tag{6.13}$$

So we have the solution of X_K

$$\begin{aligned}
x(t) &= x^0 \cos t - y^0 \sin t\\
y(t) &= x^0 \sin t + y^0 \cos t\\
z(t) &= z^0 + \tfrac{1}{2}((y^0)^2 - (x^0)^2)\sin t \cos t + x^0y^0(\sin^2 t).
\end{aligned} \tag{6.14}$$

Now we consider the flow of X_K; that is,

$$\begin{aligned} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \hat{z} &= z + \frac{1}{2}(y^2 - x^2)\sin t \cos t + xy(\sin^2 t). \end{aligned}$$

We verify directly by using (6.9) that this map is contact:

$$\begin{bmatrix} \cos t & \sin t & -x\sin t\cos t + y\sin^2 t \\ -\sin t & \cos t & y\sin t\cos t + x\sin^2 t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x\cos t - y\sin t \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ 1 \end{bmatrix}$$

where $\mu_{\Phi_K^h} = 1$.

Almost any sensible integrator applied to (6.13) will give a map of the form

$$\begin{aligned} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \hat{z} &= z + f(x,y). \end{aligned} \tag{6.15}$$

When is such an integrator contact? We make use of (6.9); that is, we show that

$$\begin{bmatrix} A_{11} & A_{21} & f_x \\ A_{12} & A_{22} & f_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ A_{11}x + A_{12}y \\ 1 \end{bmatrix} = \mu \begin{bmatrix} 0 \\ x \\ 1 \end{bmatrix}.$$

Equivalently, we must show

$$\begin{aligned} A_{21}A_{11}x + A_{21}A_{12}y + f_x &= 0 \\ (A_{22}A_{11} - 1)x + A_{22}A_{12}y + f_y &= 0 \end{aligned} \tag{6.16}$$

where $\mu = 1$. By the compatibility condition

$$f_{xy} = f_{yx},$$

we have

$$A_{21}A_{12} = A_{11}A_{22} - 1.$$

Thus the integrator (6.15) is contact if and only if

$$\det A = 1,$$

where

$$A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix},$$

and f satisfies (6.16). Now we apply Feng's method $\widetilde{P}$

$$\begin{aligned} \hat{x} &= x - hy \\ \hat{y} &= y + h\hat{x} \\ \hat{z} &= z + \frac{1}{2}h(y^2 - \hat{x}^2). \end{aligned}$$

That is,

$$\begin{aligned} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} &= \begin{bmatrix} 1 & -h \\ h & 1-h^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \hat{z} &= z + \frac{1}{2}h(y^2 - (x-hy)^2). \end{aligned}$$

Then we have

$$\begin{aligned} \begin{bmatrix} f_x \\ f_y \end{bmatrix} &= h \begin{bmatrix} -(x-hy) \\ y + h(x-hy) \end{bmatrix} \\ &= -A_{21} \begin{bmatrix} 1 & -h \\ h & 1-h^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

where $A_{21} = h$. Thus Feng's method $\widetilde{P}$ is contact. We illustrate this in Figure 6.1. As the time step we have selected $h = 0.01$ for 10000 steps. We compare it

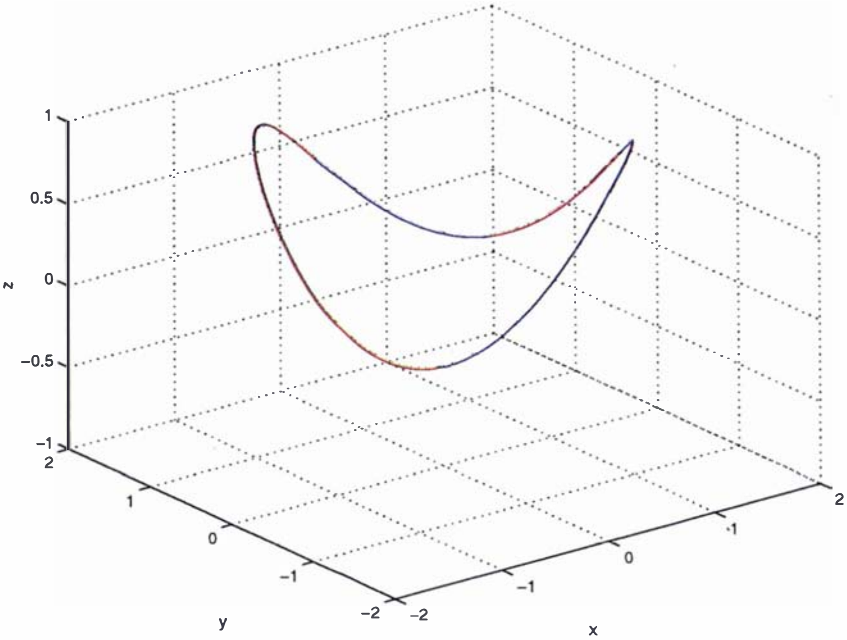

Figure 6.1: Solutions of the contact system (6.13) with $K = \frac{1}{2}(x^2 + y^2)$;Feng's method $\widetilde{P}$ with step size $h = 0.01$, 10000 steps and initial condition $(1, 1, 0)$. The numerical flow of Feng's method is shown in blue comparing with the exact flow (6.14) is shown in red.

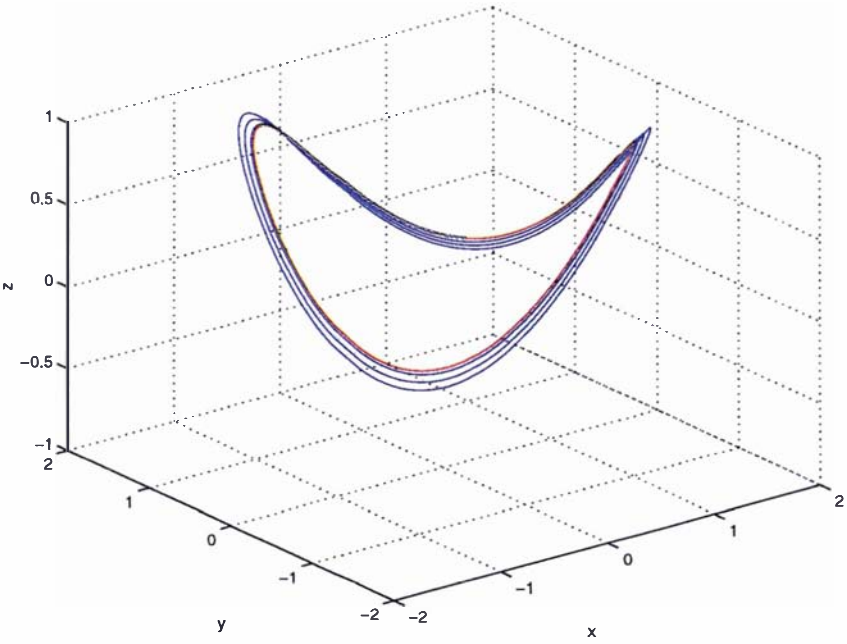

Figure 6.2: Euler's method with step size $h = 0.01$, 2000 steps and initial condition $(1, 1, 0)$. The numerical flow of Euler's method is shown in blue comparing with the exact flow (6.14) is shown in red.

with the periodic orbit of the exact solution (6.14) as shown in Figure 6.1 . We also compare the periodic orbit of the exact solution (6.14) with Euler method as shown in Figure 6.2. We see that Euler's method gives a completely wrong answer.

6.2 Splitting Methods for Contact Systems

6.2.1 Splitting into Two Contact Vector Fields

Consider the contact Hamiltonian system on a $(2n+1)$-dimensional manifold (M, α), and for $i = 1, \ldots, n$

$$X_K \begin{cases} \dot{x}_i = -K_{y_i} + x_i K_z \\ \dot{y}_i = K_{x_i} \\ \dot{z} = K - \sum_{i=1}^n x_i K_{x_i}. \end{cases}$$

Suppose the vector field X_K is split:

$$\dot{u} = X_K(u) = X_{K_1}(u) + X_{K_2}(u), \tag{6.17}$$

where

$$X_{K_1} \begin{cases} \dot{x}_i = -K_{1y_i} + x_i K_{1z} \\ \dot{y}_i = K_{1x_i} \\ \dot{z} = K_1 - \sum_{i=1}^n x_i K_{1x_i}, \end{cases} \qquad X_{K_2} \begin{cases} \dot{x}_i = -K_{2y_i} + x_i K_{2z} \\ \dot{y}_i = K_{2x_i} \\ \dot{z} = K_2 - \sum_{i=1}^n x_i K_{2x_i}. \end{cases}$$

Then we have the following:

1. X_{K_1} and X_{K_2} are contact vector fields such that $X_K = X_{K_1} + X_{K_2}$ on (M, α).
2. Let φ_{K_j} be the solutions of the system X_{K_j} for $j = 1, 2$. Then φ_{K_j} can be integrated explicitly from a given initial value $u(0) = u^0$.

3. Let φ be the solution of the system X_K. Then a contact integrator Φ_K is given by composing the solutions φ_{K_1} and φ_{K_2}, for example by

$$\Phi_K^h = \varphi_{K_1}^h \circ \varphi_{K_2}^h.$$

For this contact system, the flow of $\dot{u} = X_K(u)$ is given by

$$u(t) = \varphi_K^t := \exp(tX_K)u^0.$$

Next we introduce two simple splitting methods.

- Lie Trotter splitting (1959) [102]:

$$\begin{aligned}\Phi_K^h &= \varphi_{K_1}^h \circ \varphi_{K_2}^h \\ &= \exp(hX_{K_1})\exp(hX_{K_2}).\end{aligned}$$

  This method gives an approximation of order 1 to the solution of (6.17). By Taylor expansion we find that

$$(\varphi_{K_1}^h \circ \varphi_{K_2}^h)(u^0) = \varphi^h(u^0) + \mathcal{O}(h^2).$$

- Strang splitting (1968) [99]:

$$\begin{aligned}\Phi_K^h &= \varphi_{K_1}^{\frac{1}{2}h} \circ \varphi_{K_2}^h \circ \varphi_{K_1}^{\frac{1}{2}h} \\ &= \exp(\frac{1}{2}hX_{K_1})\exp(hX_{K_2})\exp(\frac{1}{2}hX_{K_1}).\end{aligned}$$

  This method is symmetric and of order 2.

6.2.2 Splitting into More than Two Contact Vector Fields

The oldest and simplest numerical method is the explicit Euler method. The explicit Euler method is generally non-symplectic for Hamiltonian systems and is also non-contact for contact systems. Feng constructed explicit symplectic integrators for separable Hamiltonian systems by composition of explicit Euler method [44, 45]. Using Feng's method, we will construct explicit contact methods. Especially, we want prove that any polynomial contact system can be split into a sum of integrable contact systems.

Explicit Hamiltonian Methods

Let

$$\hat{z} = z + hJH_z(z), \qquad E_H^h = 1 + JH_z \tag{6.18}$$

be the explicit Euler method for the Hamiltonian system (6.1).

Definition 18. [45] If a Hamiltonian H satisfies

$$\left(\frac{\partial c}{\partial z}\right) c = 0, \tag{6.19}$$

where

$$c := \dot{z} = J\nabla H(z) = JH_z;$$

that is,

$$JH_{zz}JH_z = 0 \quad \forall\, z \in \mathbb{R}^{2n+2}, \tag{6.20}$$

then H is said to be **nilpotent** of degree 2.

Usually, the explicit Euler method (6.18) is not symplectic for general Hamiltonian systems. But for a nilpotent Hamiltonian H of degree 2 (6.20), the explicit Euler method E_H^h is the phase flow of the Hamiltonian system (6.1) at $t = h$, therefore it must be symplectic [45].

Example 20. [44, 45] If

$$H(z) = H(p_0, p, q_0, q) = U(p_0, p), \quad \text{or} \quad H(z) = V(q_0, q),$$

where U and V are functions of $(n+1)$ variables, then $H(z)$ is nilpotent of degree 2. The corresponding explicit Euler methods are symplectic.

Before establishing explicit contact methods, we first give two examples of the homogeneous Hamiltonian systems.

Example 21. If

$$H(z) = H(p_0, p, q_0, q) = f(p_0, p), \quad \text{or} \quad p_0 g(q),$$

where $f(p_0, p)$ and $p_0 g(q)$ are functions of $(n+1)$ variables, then the homogeneous Hamiltonian function $H(z)$ is nilpotent of degree 2.

- For $H(p_0, p, q_0, q) = f(p_0, p)$,

$$\begin{aligned} H_{zz}JH_z &= \begin{bmatrix} f_{p_0p_0} & 0 & f_{p_0p} & 0 \\ 0 & 0 & 0 & 0 \\ f_{pp_0} & 0 & f_{pp} & O_n \\ 0 & 0 & O_n & O_n \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & O_n & -I_n \\ 0 & 0 & I_n & O_n \end{bmatrix} \begin{bmatrix} f_{p_0} \\ 0 \\ f_p \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} f_{p_0p_0} & 0 & f_{p_0p} & 0 \\ 0 & 0 & 0 & 0 \\ f_{pp_0} & 0 & f_{pp} & O_n \\ 0 & 0 & O_n & O_n \end{bmatrix} \begin{bmatrix} 0 \\ f_{p_0} \\ 0 \\ f_p \end{bmatrix} = \begin{bmatrix} \tilde{0} \\ \tilde{0} \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

- For $H(p_0, p, q_0, q) = p_0 g(q)$,

$$H_{zz} J H_z = \begin{bmatrix} 0 & 0 & 0 & g_q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & O_n & O_n \\ g_q & 0 & O_n & p_0 g_{qq} \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & O_n & -I_n \\ 0 & 0 & I_n & O_n \end{bmatrix} \begin{bmatrix} g \\ 0 \\ 0 \\ p_0 g_q \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & g_q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & O_n & O_n \\ g_q & 0 & O_n & p_0 g_{qq} \end{bmatrix} \begin{bmatrix} 0 \\ g \\ -p_0 g_q \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The corresponding explicit Euler method is the exact phase flow of the homogeneous Hamiltonian, and therefore symplectic.

Example 22. If

$$H(z) = H(p_0, p, q_0, q) = ph(q_0),$$

where $ph(q_0)$ is a function of 2 variables, then the homogeneous Hamiltonian function $H(z)$ is nilpotent of degree 2.

- For $H(p_0, p, q_0, q) = ph(q_0)$,

$$H_{zz} J H_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & ph_{q_0 q_0} & h_{q_0} & 0 \\ 0 & h_{q_0} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ ph_{q_0} \\ h \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & ph_{q_0 q_0} & h_{q_0} & 0 \\ 0 & h_{q_0} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -ph_{q_0} \\ 0 \\ h \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The corresponding explicit Euler method is the exact phase flow of the homogeneous Hamiltonian, and therefore symplectic.

Explicit Contact Methods

Definition 19. A contact Hamiltonian K is **nilpotent** of degree 2 if K satisfies

$$\left(\frac{\partial c}{\partial u}\right) c = 0, \tag{6.21}$$

where

$$c := \dot{u} = J\nabla K(u) + e_{2n+1}K(u).$$

We will show one example of a contact system corresponding to the homogeneous Hamiltonian system which is nilpotent of degree 2. From Section 5.3.2, we know that the homogeneous Hamiltonian $H(p_0, p, q_0, q) = f(p_0, p)$ on $\mathbb{R}^{2n+2}$ corresponds to the contact Hamiltonian $K(x, y, z) = f(x)$ on $\mathbb{R}^{2n+1}$ and also $H(p_0, p, q_0, q) = p_0 g(q)$ corresponds to the contact Hamiltonian $K(x, y, z) = g(y)$.

Example 23. Consider the contact Hamiltonian function

$$K(u) = K(x, y, z) = f(x), \quad \text{or} \quad g(y)$$

where $f(x)$ and $g(y)$ are functions of n variables. Then the contact Hamiltonian function $K(u)$ is nilpotent of degree 2.

- For $K(x, y, z) = f(x)$,

$$\left(\frac{\partial c}{\partial u}\right) c = \begin{bmatrix} 0 & 0 & 0 \\ f_{xx} & 0 & 0 \\ -xf_{xx} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ f_x \\ f - xf_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

  the explicit Euler method

$$\begin{cases} \hat{x} = x \\ \hat{y} = y + hf_x(x) \\ \hat{z} = z + h(f(x) - xf_x(x)) \end{cases}$$

is contact.

- For $K(x,y,z)=g(y)$,

$$\left(\frac{\partial c}{\partial u}\right)c=\begin{bmatrix}0 & -g_{yy} & 0\\ 0 & 0 & 0\\ 0 & g_y & 0\end{bmatrix}\begin{bmatrix}-g_y\\ 0\\ g\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

the explicit Euler method

$$\begin{cases}\hat{x}=x-h(g_y(y))\\ \hat{y}=y\\ \hat{z}=z+h(g(y))\end{cases}$$

is also contact.

Theorem 17. *If K is nilpotent of degree 2, then the explicit Euler method E_K^h is the exact flow of the contact Hamiltonian, and therefore contact.*

Proof. Let $u=u^0$. From the condition (6.21) it follows that

$$\ddot{u}(t)=0.$$

Therefore,

$$\dot{u}(t)=\dot{u}^0=J\nabla K(u^0)+e_{2n+1}K(u^0)$$

Hence

$$\begin{aligned}u(t)&=u^0+t[J\nabla K(u^0)+e_{2n+1}K(u^0)]\\ &=u+t[J\nabla K(u)+e_{2n+1}K(u)]\\ &=E_K^t(u),\end{aligned}$$

which is the explicit Euler method. This shows that for such systems, the explicit Euler method E^h_K is the exact flow, and therefore contact. □

We know that the homogeneous Hamiltonian $H(p_0, p, q_0, q) = ph(q_0)$ on $\mathbb{R}^4$ corresponds to the contact $K(x, y, z) = xh(z)$ on $\mathbb{R}^3$.

Example 24. We have seen in Example 22 that $H(p_0, p, q_0, q) = ph(q_0)$ is nilpotent of degree 2. Its associated contact Hamiltonian is

$$K(u) = K(x, y, z) = xh(z).$$

However, $K(u)$ is not nilpotent of degree 2.

- For $K(x, y, z) = xh(z)$

$$\left(\frac{\partial c}{\partial u}\right) c = \begin{bmatrix} 2xh_z & 0 & x^2 h_{zz} \\ 0 & 0 & h_z \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^2 h_z \\ h \\ 0 \end{bmatrix} = \begin{bmatrix} 2x^3 h_z^2 \\ 0 \\ 0 \end{bmatrix}.$$

Remark. As seen in Example 22 and Example 24, even if a homogeneous Hamiltonian is nilpotent of degree 2, the corresponding contact Hamiltonian may not be nilpotent of degree 2. However, the converse is true.

Theorem 18. *If the contact Hamiltonian is nilpotent of degree 2, then the corresponding homogeneous Hamiltonian is nilpotent of degree 2.*

Proof. If K is nilpotent of degree 2, then any orbit $(x(t), y(t), z(t))$ of X_K is a linear function of t. Therefore (under symplectification) $q(t), q_0(t)$ and $p(t)/p_0(t)$ are linear functions of t. Let $p(t) = p_0(t)(a + bt)$. Differentiating with respect to t gives $\dot{p} = (a + bt)\dot{p}_0 + bp_0$ or $-H_q = -(a + bt)H_{q_0} + bp_0$. Therefore either $b = 0$ or $H_{q_0} = 0$. If $H_{q_0} = 0$ then $p_0 =$ constant and p is a linear function of t, hence

H is nilpotent of degree 2. If, on the other hand, $b = 0$, then $dp/dp_0 = p/p_0$. But $dp/dp_0 = H_q/H_{q_0}$. The general solution of the PDE $H_q/H_{q_0} = p/p_0$ for H is $H = g(pq+p_0q_0)$. However, we already know that $q(t)$ and $q_0(t)$ are linear functions of t, which is only possible for this H if $g' = 0$, i.e. $g =$ constant and H is again (trivially) nilpotent of degree 2. □

Symplectically Separable Hamiltonian Systems

The Hamiltonian $H(z)$ is symplectically separable if

$$H(z) = \sum_{i=1}^{m} H_i(z) \tag{6.22}$$

where every $H_i(z)$ is a nilpotent Hamiltonian of degree 2 [45].

For example, if

$$H(p_0, \ldots, p_n, q_0, \ldots, q_n) = f(p_0, \ldots, p_n) + p_0 g(q_1, \ldots, q_n) + p_i h_i(q_0, q_1, \ldots, \hat{q}_i, \ldots, q_n) \tag{6.23}$$

for all $i = 1, \ldots, n$, where the notation $\hat{q}_i$ is used when the variable q_i is deleted from a list of variables, which is symplectically separable, then the composite methods are explicit and symplectic.

Contactically Separable Contact Systems

In contrast to the homogeneous Hamiltonian case, there are very few contact Hamiltonians which are nilpotent of degree 2. For example,

$$K(x_1, \ldots, x_n, y_1, \ldots, y_n, z) = f(x_1, \ldots, x_n) \quad \text{or} \quad g(y_1, \ldots, y_n).$$

As shown in (6.23), the following homogeneous Hamiltonian is nilpotent of degree

2:

$$p_i h_i(q_0, q_1, \ldots, \hat{q}_i, \ldots, q_n) \tag{6.24}$$

for all $i = 1, \ldots, n$, where the notation $\hat{q}_i$ is used when the variable q_i is deleted from a list of variables. But the corresponding contact Hamiltonian is not nilpotent of degree 2:

$$x_i h_i(z, y_1, \ldots, \hat{y}_i, \ldots, y_n) \tag{6.25}$$

for all $i = 1, \ldots, n$, where the notation $\hat{y}_i$ is used when the variable y_i is deleted from a list of variables.

Remarks.

- In Table 5.1 of Section 5.3.2, the homogeneous Hamiltonians 1, 2 and 6 are nilpotent of degree 2, items 1, 2 and 6 in Table 5.2 of Example 16 are also nilpotent of degree 2.
- In Table 5.1 of Section 5.3.2, items 1, 2 are the only contact Hamiltonian that are nilpotent of degree 2, items 1, 2 in Table 5.2 of Example 16 are also nilpotent of degree 2.

We have found that splitting is a more complicated process for contact systems than for Hamiltonian systems. Recall that we want to solve

$$K(u) = \sum_{i=1}^{m} K_i(u) \tag{6.26}$$

where each X_{K_i} is integrable, and form integrators from composition of their flows $\exp(hX_{K_i})$. In general we will have to proceed in three stages:

1. The easiest pieces K_i are nilpotent of degree 2. However there are very few of these.

2. The next simplest K_i are those whose associated homogeneous Hamiltonians are nilpotent of degree 2. There are more of these but we may not always be able to split a contact Hamiltonian, such as an arbitrary polynomial K, in terms of these. We need a third and harder class.

3. Those K_i which are integrable; usually, we can use those K_i which are solved directly by the solution of one-dimensional autonomous ODEs, for example, items 3-8 in Table 5.1 of Section 5.3.2, items 3-5 and 7-10 in Table 5.2 of Example 16.

We will now explore such more general splittings, finally showing that they allow us to split any polynomial K.

6.2.3 Polynomial Contact Systems

As stated in Chapter 5, some contact systems can be integrated by quadratures. We want to give a general splitting method for polynomial contact systems. Before dealing with general polynomial contact systems, we consider polynomial Hamiltonian systems.

Poisson System

We first introduce an interesting generalization of Hamiltonian systems, where J in Section 2.3.1 and Section 6.1.1 is replaced with a nonconstant matrix $B(z)$ [70, 54].

In local coordinates $(z_0, \dots, z_{2n+1})$, B is determined by its matrix elements $\{z_i, z_j\} = B_{ij}(z)$ and the Poisson bracket of F and G becomes

$$\{F, G\} = B_{ij}(z)\frac{\partial F}{\partial z_i}\frac{\partial G}{\partial z_j} \tag{6.27}$$

(or more compactly $\{F, G\} = \nabla F(z)^T B(z) \nabla G(z)$).

Proposition 26. [54] *The bracket defined in (6.27) is bilinear, skew-symmetric and satisfies Leibniz' rule as well as the Jacobi identity if and only if*

$$B_{ij}(z) = -B_{ji}(z) \quad \textit{for all} \quad i,\ j$$

and for all $i,\ j,\ k\ = 1, \ldots, n$ *(notice the cyclic permutations among* i, j, k*),*

$$\sum_{l=1}^{m} \left(\frac{\partial B_{ij}(z)}{\partial z_l} B_{ij}(z) + \frac{\partial B_{jk}(z)}{\partial z_l} B_{li}(z) + \frac{\partial B_{ki}(z)}{\partial z_l} B_{lj}(z) \right) = 0.$$

We recall that the Hamiltonian system (see Section 2.3.1)

$$\dot{z} = J \nabla H(z)$$

can be written as (see (5.6))

$$\dot{z}_i = \{z_i, H\}, \tag{6.28}$$

for $i = 0, \ldots, 2n+1$.

Definition 20. [54] If the matrix $B(z)$ satisfied of the properties of Proposition 26, formula (6.27) is said to represent a (general) **Poisson bracket**. The corresponding differential system, similar to (6.28),

$$\dot{z} = B(z) \nabla H(z),$$

is a **Poisson system**. We continue to call H a Hamiltonian.

A manifold P with a Poisson bracket is called a **Poisson manifold**, the bracket defining a **Poisson structure** on P. The notion of a Poisson manifold is slightly more general than that of a symplectic manifold [82]. Any symplectic manifold is a Poisson manifold.

Definition 21. [82] Let P be a Poisson manifold. A function $C : P \longrightarrow \mathbb{R}$ is called a **Casimir function** (or a distinguished function) if the Poisson bracket of C with any other function vanishes identically, that is,

$$\{C, H\} = 0$$

for all $H : P \longrightarrow \mathbb{R}$.

It means that C is constant along the flow of all Hamiltonian vector fields.

Polynomial Hamiltonian Systems

From Theorem 13, we know that the Hamiltonian function

$$H = g(a_1 p + b_1 q, \ldots, a_k p + b_k q)$$

is integrated exactly by Euler's method for all $F_i = a_i p + b_i q$, $i = 1, \ldots, k$ if and only if

$$a_i b_j - a_j b_i = 0 \tag{6.29}$$

for all i and j; that is, if and only if the new arguments commute. (Recall that $a_i, b_i \in \mathbb{R}^n$ and we write $a_i b_j = a_i \cdot b_j$.) However, it can easily happen that X_H is integrable even if (6.29) is not satisfied.

Theorem 19. *The Hamiltonian vector field* X_H *for* $H = g(F_1, \ldots, F_k)$, *where* $F_i = a_i p + b_i q$, $i = 1, \ldots, k$ *is integrable if the reduced system*

$$\begin{aligned} \dot{F} &= A \nabla_F g \\ A_{ij} &= a_i b_j - a_j b_i \end{aligned} \tag{6.30}$$

is integrable. The reduced system (6.30) is always integrable for $k = 2$ *and* $k = 3$.

Proof. From $\dot{F}_i = \frac{\partial g}{\partial F_i} X_{F_i}$ where $F_i = a_i p + b_i q$, we have the reduced system (6.30) as following:

$$\begin{aligned}
\dot{F}_i &= a_i \dot{p} + b_i \dot{q} \\
&= -a_i \frac{\partial H}{\partial q} + b_i \frac{\partial H}{\partial p} \\
&= \sum_{j=1}^{k} \left(-a_i \frac{\partial g}{\partial F_j} \frac{\partial F_j}{\partial q} + b_i \frac{\partial g}{\partial F_j} \frac{\partial F_j}{\partial p} \right) \\
&= \sum_{j=1}^{k} (-a_i b_j + b_i a_j) \frac{\partial g}{\partial F_j};
\end{aligned}$$

that is,

$$\begin{aligned}
\dot{F}_1 &= -\, a_1(b_1 g_1 + \cdots + b_k g_k) + b_1(a_1 g_1 + \cdots + a_k g_k) \\
\dot{F}_2 &= -\, a_2(b_1 g_1 + \cdots + b_k g_k) + b_2(a_1 g_1 + \cdots + a_k g_k) \\
&\vdots \\
\dot{F}_k &= -\, a_k(b_1 g_1 + \cdots + b_k g_k) + b_k(a_1 g_1 + \cdots + a_k g_k)
\end{aligned}$$

where $g_i = \frac{\partial g}{\partial F_i}$, i.e., derivative with respect to ith argument. This reduced system is a Poisson system with a constant Poisson structure A. In the case $k = 2$, it is equivalent to a conserved Hamiltonian system in the plane, hence integrable. In the case $k = 3$, it has a linear Casimir and hence the reduced system (6.30) is integrable. Having determined $F(t)$, $\dot{p}$ and $\dot{q}$ are known functions of t and $p(t)$ and $q(t)$ can be determined by quadrature. ☐

Polynomial Contact Systems

We now consider extending this result to the contact case, that is, we consider

$$K = g(a_1 x + b_1 y + c_1 z, \ldots, a_k x + b_k y + c_k z), \tag{6.31}$$

where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then the contact vector field X_K of K is

$$X_K \begin{cases} \dot{x} = -(b_1 g_1 + \cdots + b_k g_k) + x(c_1 g_1 + \cdots + c_k g_k) \\ \dot{y} = a_1 g_1 + \cdots + a_k g_k \\ \dot{z} = K - x(a_1 g_1 + \cdots + a_k g_k). \end{cases}$$

We have

$$\dot{K} = (c_1 g_1 + \cdots + c_k g_k)K.$$

If we set $f_i = a_i x + b_i y + c_i z$, for $i = 1, \ldots, k$, then

$$\begin{aligned} \dot{f}_1 = \;& -a_1(b_1 g_1 + \cdots + b_k g_k) + a_1 x(c_1 g_1 + \cdots + c_k g_k) + b_1(a_1 g_1 + \cdots + a_k g_k) \\ & -x c_1(a_1 g_1 + \cdots + a_k g_k) + c_1 K \\ \dot{f}_2 = \;& -a_2(b_1 g_1 + \cdots + b_k g_k) + a_2 x(c_1 g_1 + \cdots + c_k g_k) + b_2(a_1 g_1 + \cdots + a_k g_k) \\ & -x c_2(a_1 g_1 + \cdots + a_k g_k) + c_2 K \\ & \vdots \\ \dot{f}_k = \;& -a_k(b_1 g_1 + \cdots + b_k g_k) + a_k x(c_1 g_1 + \cdots + c_k g_k) + b_k(a_1 g_1 + \cdots + a_k g_k) \\ & -x c_k(a_1 g_1 + \cdots + a_k g_k) + c_k K \end{aligned} \tag{6.32}$$

- Case 1.

  If $c_i = 0$ for all i, then the system is simply a canonical Hamiltonian system in (x, y); this reduces to the known Hamiltonian case. And z can be determined by a single remaining quadrature.

- Case 2.

  If $c_i \neq 0$ for some i, the reduced equations are (say $c_k \neq 0$)

$$\begin{aligned} \dot{f}_1 &= -\, a_1(b_1 g_1 + \cdots + b_k g_k) + a_1 x(c_k g_k) + b_1(a_1 g_1 + \cdots + a_k g_k) \\ \dot{f}_2 &= -\, a_2(b_1 g_1 + \cdots + b_n g_n) + a_2 x(c_k g_k) + b_2(a_1 g_1 + \cdots + a_k g_k) \\ & \vdots \end{aligned}$$

$$\begin{aligned}\dot{f}_k =& - a_k(b_1 g_1 + \cdots + b_k g_k) + a_k x(c_k g_k) + b_k(a_1 g_1 + \cdots + a_k g_k)\\ &- x c_k(a_1 g_1 + \cdots + a_k g_k) + c_k K\end{aligned}$$

where $c_1 = \cdots = c_{k-1} = 0$. Because the equations depend on x, in general we find no new integrable cases (apart from known cases, e.g., $K = g(y, z)$, i.e., $a_i = 0$ for all i). Thus, the case at $k = 1$ (see Example 15) is quite special in the contact case and it is unlike the Hamiltonian case.

From the above discussion, we have the following theorem:

Theorem 20. *Any polynomial contact system can be split into a sum of integrable contact systems.*

Proof. For any polynomial K, there exists m and a_i, b_i, c_i $(i = 1, \dots, m)$ such that

$$K = \sum_{i=1}^{m} K_i(a_i x + b_i y + c_i z).$$

From Theorem 16, each contact vector field X_{K_i} is integrable. ☐

Chapter 7

Volume-Preserving Integrators

It was proved by Liouville that dynamical systems of classical mechanics preserve volume in the phase space. Later on it was noticed that the flow of a Hamiltonian system preserves volume in phase space. More generally, it was known that the volume structure is preserved by the flow of a source-free system. During the last 10 years, some people have devoted themselves to constructing volume-preserving integrators for solving source-free systems numerically [84, 41, 93, 85, 43, 75].

A contact manifold (M, α) carries a distinguished volume form, namely $\alpha \wedge (d\alpha)^n$. In the canonical case, $\alpha \wedge (d\alpha)^n = dx \wedge dy \wedge dz$ is the usual Euclidean volume form. However, the flow of a contact system may not preserve the general volume form $\alpha \wedge (d\alpha)^n$ in the contact manifold (M, α). It is interesting to ask: Is it possible that the flow of contact system preserves some volume form on the contact manifold? In this chapter, we will first show how a contact system is related to a source-free system on a contact manifold (M, α). In order to construct volume-preserving integrators for these source-free systems, we show that every $(2n+1)$-dimensional source-free vector field can be represented as a sum of $2n$ vector fields, where each vector field is a 2-dimensional Hamiltonian vector field.

7.1 Source-Free Dynamical Systems

In this section we define the concepts of a volume form, volume-preserving diffeomorphism and briefly describe source-free dynamical systems. Further information can be found in [5, 11, 54].

7.1.1 Volume Structures

Definition 22. Let (M, Ω) be an n-dimensional manifold with a differential form Ω on M; that is, for each $x \in M$

$$\Omega[x] : \underbrace{T_xM \times \cdots \times T_xM}_{n} \longrightarrow \mathbb{R}.$$

The differential form Ω is called a **volume form** and (M, Ω) a **volume manifold**.

Example 25. Let M be equal to $\mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$. Then

$$\Omega = dx_1 \wedge \cdots \wedge dx_n$$

is known as the standard volume form on $(\mathbb{R}^n, \Omega)$.

7.1.2 Volume-Preserving Diffeomorphisms

Definition 23. Let (M, Ω) and (M', Ω') be n-dimensional volume manifolds (that is, manifolds with volume elements). A diffeomorphism $f : M \to M'$ is called a **volume-preserving diffeomorphism** if it preserves the volume form. More specifically,

$$f^*\Omega' = \Omega. \tag{7.1}$$

The set

$$\mathrm{Diff}_\Omega(M) = \{f \in \mathrm{Diff}(M) | f^*\Omega = \Omega\}$$

is a group, called **the group of volume-preserving diffeomorphisms** on (M, Ω).

7.1.3 Source-Free Dynamical Systems

In this subsection, we will define source-free dynamical systems.

Definition 24. A vector field X on a volume manifold (M, Ω) which generates a 1-parameter group of volume-preserving diffeomorphism is called a **source-free vector field** (or **divergence-free vector field**).

Definition 25. The flow φ^t of a source-free vector field $\dot{\varphi}^t = X$ is called the **volume-preserving flow** of a source-free dynamical system.

The volume-preserving flow satisfies the following condition

$$(\varphi^t)^*\Omega = \Omega.$$

By differentiating this expression at $t = 0$, we get $\frac{d}{dt}(\varphi^t)^*\Omega|_{t=0} = 0$; that is,

$$L_X\Omega = 0.$$

This is equivalent to

$$\mathrm{div}_\Omega X = 0.$$

So we know that

- $\mathrm{div}_\Omega X = 0$ if and only if φ^t preserves Ω.

The volume-preserving property (7.2) says that the volume structure is preserved by the flow of differential equations with a source-free vector field. In terms of local coordinates $x_1, \ldots, x_n$ with

$$\Omega = \mu(x)\, dx_1 \wedge \cdots \wedge dx_n,$$

a map φ is volume-preserving if

$$\mu(\varphi(x))\, \det D\varphi(x) = \mu(x),$$

and a vector field is source-free if

$$\mathrm{div}_\Omega X = \nabla \cdot (\mu X) = \sum_{i=1}^{n} \frac{\partial(\mu X_i)}{\partial x_i} = 0.$$

In the Euclidean case $M = \mathbb{R}^n$ and $\mu = 1$, so a map φ is volume-preserving if $\det D\varphi = 1$ and the vector field X is source-free if $\nabla \cdot X = 0$.

7.1.4 Liouville's Theorem

One of the important properties of a Hamiltonian flow is that it preserves volume. More specifically if we take a region D on the phase space $\mathbb{R}^{2n+2}$ and let it flow for a finite time t then we get a new region $\varphi^t D$ with the same volume as that of D

$$(\text{volume of } \ \varphi^t D) = (\text{volume of } D).$$

That is known as Liouville's volume conservation theorem. Liouville's Theorem can be interpreted as the conservation of phase volume in Hamiltonian systems as shown in Figure 7.1. We investigate how a Hamiltonian vector field is related to the source free vector field on a symplectic manifold.

Let X_H be a locally Hamiltonian vector field on a $(2n+2)$-dimensional symplectic manifold (P, ω). Then

$$L_{X_H}\omega = 0,$$

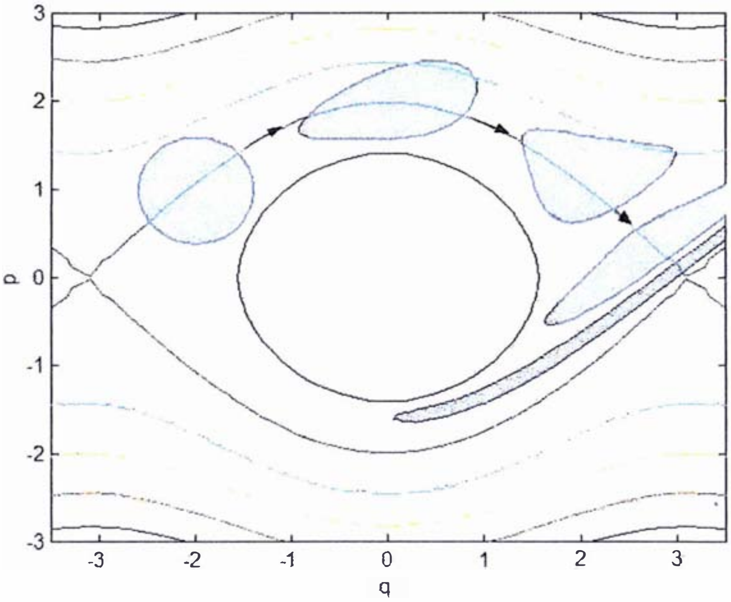

Figure 7.1: The numerical flow by the symplectic Euler method for a pendulum problem with the Hamiltonian $H = \frac{1}{2}p^2 - \cos q$ preserves volume. The divergence of the pendulum equations ($\dot{p} = -\sin q,\ \dot{q} = p$) is zero.

where (locally) $\omega = dp_0 \wedge dq_0 + \cdots + dp_n \wedge dq_n$. We have

$$L_{X_H}\Omega = 0.$$

Here $\Omega = dp_0 \wedge \cdots \wedge dp_n \wedge dq_0 \wedge \cdots \wedge dq_n$ is the volume form which resulted from

$$\Omega = \frac{(-1)^{\frac{n(n+1)}{2}}}{(n+1)!}\omega \wedge \omega \wedge \cdots \wedge \omega \ \ (n+1 \text{ times}).$$

(The scale factor is present to make Ω coincide with the Euclidean vector field in the canonical case.) This volume form is referred to as the Liouville form on (P, ω). The Liouville form Ω on (P, ω) is invariant under canonical transformations or symplectic diffeomorphisms.

If we consider the source-free vector fields on a 2-dimensional manifold $(\mathbb{R}^2, \omega)$, then

- the volume form $\omega = dp_0 \wedge dq_0$ is a symplectic form,

- the source-free vector field is a Hamiltonian vector field,
- the volume(area)-preserving diffeomorphism is a symplectic diffeomorphism.

7.2 Source-Free Vector Field for Contact Systems

As stated in Chapter 3, the flow of a contact vector field may not preserve the contact 1-form α but the flow of the Reeb vector field does. In this section we show that the flow of a contact system preserves some contact form when the contact Hamiltonian function K is nonzero. We will show how a contact vector field is related to a source-free vector field on a contact manifold (M, α).

7.2.1 Reeb flow for Contact Systems

We will construct a source-free vector field on the contact manifold (M, α). First we see how a contact vector field gives rise to a Reeb vector field. Note that if α is a contact form on a contact manifold M then

$$\tilde{\alpha} = \frac{\alpha}{K}$$

is also a contact form on M (see Theorem 2), where K is a nonvanishing contact Hamiltonian function on M. We will show in the next proposition that X_K is in fact a Reeb vector field associated with the contact form $\frac{\alpha}{K}$.

Proposition 27. *Let X_K be a contact vector field on the contact manifold (M, α). Then*

$$i_{X_K}\frac{\alpha}{K} = 1, \qquad i_{X_K}d(\frac{\alpha}{K}) = 0,$$

where $i_{X_K}\alpha = K$ and K is a nonvanishing contact Hamiltonian function on M.

Proof. First of all,

$$\begin{aligned} i_{X_K}\frac{\alpha}{K} &= \frac{1}{K}\cdot i_{X_K}\alpha \\ &= \frac{1}{K}\cdot K \\ &= 1. \end{aligned}$$

Next,

$$\begin{aligned} i_{X_K} d\left(\frac{\alpha}{K}\right) &= i_{X_K} d\left(\frac{1}{K}\right)\wedge\alpha + i_{X_K}\frac{1}{K}d\alpha \\ &= \left(i_{X_K} d\left(\frac{1}{K}\right)\right)\wedge\alpha - d\left(\frac{1}{K}\right)\wedge i_{X_K}\alpha + \frac{1}{K} i_{X_K} d\alpha \\ &= -\frac{1}{K^2}\cdot\lambda K\wedge\alpha + \frac{1}{K^2}dK\cdot K + \frac{1}{K}(\lambda\alpha - dK) \\ &= 0, \end{aligned}$$

since $d\left(\frac{1}{K}\right) = \frac{-dK}{K^2}$ and $i_{X_K}dK = \dot{K} = \lambda K$. □

As we have seen, the flow of the contact vector field X_K may not preserve a given contact 1-form α, but the flow of the Reeb vector field does. From Proposition 27, we know that the flow of the contact vector field X_K preserves the contact 1-form $\frac{\alpha}{K}$ and and 2-form $d(\frac{\alpha}{K})$. Therefore the contact vector field X_K associated with the contact form $\frac{\alpha}{K}$ is the Reeb vector field on the contact manifold $(M, \frac{\alpha}{K})$.

7.2.2 Source-Free Vector Field for Contact Systems

Consider the volume form $\alpha\wedge(d\alpha)^n$ on the contact manifold (M,α).

Question: Is it possible for a contact vector field X_K to preserve the volume form $\alpha\wedge(d\alpha)^n$ on the contact manifold (M,α)? (that is, can the contact vector field X_K be volume-preserving with respect to the volume form $\alpha\wedge(d\alpha)^n$?).

To answer the above question, we first consider the volume form on the 3-dimensional contact manifold $(\mathbb{R}^3, \alpha)$ in the canonical case. Let X_K be a contact vector field of contact Hamiltonian K and $\alpha = xdy + dz$ a contact form. Thus

$$X_K \begin{cases} \dot{x} = -K_y + xK_z \\ \dot{y} = K_x \\ \dot{z} = K - xK_x. \end{cases}$$

Now if we choose the volume-form to be

$$\Omega = \alpha \wedge d\alpha = (xdy + dz)(dx \wedge dy) = dx \wedge dy \wedge dz \neq 0,$$

then the divergence of the vector field X_K with respect to the volume form Ω is

$$\begin{aligned} \mathrm{div}_\Omega X_K &= \frac{\partial(-K_y + xK_z)}{\partial x} + \frac{\partial(K_x)}{\partial y} + \frac{\partial(K - xK_x)}{\partial z} \\ &= -K_{yx} + K_z + xK_{zx} + K_{xy} + K_z - xK_{xz} \\ &= 2K_z. \end{aligned}$$

We have

- If $\lambda = K_z \neq 0$, then $\mathrm{div}_\Omega X_K \neq 0$; that is,

  $$L_{X_K}(\alpha \wedge d\alpha) \neq 0.$$

  That is, X_K is not volume-preserving with respect to the volume form $\Omega = \alpha \wedge d\alpha$.

- If $\lambda = K_z = 0$, then $\mathrm{div}_\Omega X_K = 0$; that is,

  $$L_{X_K}(\alpha \wedge d\alpha) = 0.$$

  That is, X_K is volume-preserving with respect to the volume form $\Omega = \alpha \wedge d\alpha$.

□

From Proposition 28 and Proposition 29, we have

$$L_{X_K}\frac{\alpha\wedge(d\alpha)^n}{K^{n+1}} = 0 \Longleftrightarrow L_{\frac{X_K}{K^{n+1}}}\alpha\wedge(d\alpha)^n = 0, \tag{7.2}$$

and also have

$$\mathrm{div}_{\frac{\alpha\wedge(d\alpha)^n}{K^{n+1}}} X_K = 0 \Longleftrightarrow \mathrm{div}_{\alpha\wedge(d\alpha)^n}\frac{X_K}{K^{n+1}}.$$

By (7.2), we get

Proposition 30. *Let X_K be a contact vector field on a $(2n+1)$-dimensional contact manifold (M,α). Then the vector field*

$$\frac{X_K}{K^{n+1}}$$

is volume preserving with respect to the volume form $\alpha\wedge(d\alpha)^n$.

7.3 Volume-Preserving Integrators

The problem of constructing volume-preserving numerical integrators for every differential equation with volume-preserving flow was discussed by Thyagaraja and Haas [101], Feng and Wang [44], Quispel [85, 86], Shang [93]. In this section we will give a general method to construct volume-preserving integrators for a contact system by means of the essentially Hamiltonian decompositions of source-free vector fields.

Given a contact vector field X_K, we have the following options.

- Apply a contact integrator to X_K. However, we do not know if the integrator preserves $\Omega = \frac{\alpha\wedge(d\alpha)^n}{K^{n+1}}$. It seems unlikely because Ω is non-canonical. One can also ask if the contact integrator preserves any volume form. Since, by backward error analysis [54], the integrator is approximately equal to the flow of some contact vector field $\tilde{X}$ close to X_K. Consider the flow of a modified contact vector field $\tilde{X}$, with associated contact Hamiltonian $\tilde{K} = i_{\tilde{X}}\alpha$. The volume form $\frac{\alpha\wedge(d\alpha)^n}{\tilde{K}^{n+1}}$ is approximately preserved by the integrator. However, from the point of view of geometric integration such approximate preservation is not very valuable.

- Apply a volume-preserving integrator to X_K which preserves the volume form $\frac{\alpha\wedge(d\alpha)^n}{K^{n+1}}$. However, even in the canonical case $\alpha = xdy + dz$ this volume form is non-Euclidean. Algorithms are known which preserve an arbitrary volume form [86] but they are more difficult than in the canonical case.

- Re-scale the vector field in time and integrate $\frac{X_K}{K^{n+1}}$ preserving the volume form $\alpha \wedge (d\alpha)^n$. This is the approach we follow in the next section.

7.3.1 A 3-Dimensional Volume-Preserving Integrator

In this section we introduce the volume-preserving splitting method used by McLachlan and Quispel [75, 77]. We apply their method to construct volume preserving integrators for some contact systems.

We consider a source-free system on a 3-dimensional contact manifold $(\mathbb{R}^3, \alpha)$. Let

$$\dot{u} = f(u)$$

where $u = (x, y, z)$. Consider the vector field $\dot{u} = f(u)$ as follows:

$$\begin{cases} \dot{x} = K^{-2}(-K_y + xK_z) \\ \dot{y} = K^{-2}(K_x) \\ \dot{z} = K^{-2}(K - xK_x). \end{cases}$$

Then f is volume preserving with respect to the volume form $\alpha \wedge (d\alpha)$ on $(\mathbb{R}^3, \alpha)$. We split f as $f = \tilde{f}_1 + \tilde{f}_2$:

$$\tilde{f}_1 \begin{cases} \dot{x} = K^{-2}(-K_y) \\ \dot{y} = K^{-2}(K_x) = f_1 \\ \dot{z} = 0 \end{cases} \qquad \tilde{f}_2 \begin{cases} \dot{x} = K^{-2}(xK_z) \\ \dot{y} = 0 \\ \dot{z} = K^{-2}(K - xK_x) = f_2. \end{cases}$$

1. The vector fields $\tilde{f}_1$ and $\tilde{f}_2$ are volume-preserving with respect to the volume form $\alpha \wedge (d\alpha)$.

2. They each correspond to a 2-dimensional Hamiltonian system.

   - $\tilde{f}_1$ has Hamiltonian $S_1 := \int f_1 dx$; that is, $S_1 = -K^{-1}$

   $$\begin{cases} \dot{x} = K^{-2}(-K_y) = -\frac{\partial S_1}{\partial y} \\ \dot{y} = K^{-2}(K_x) = \frac{\partial S_1}{\partial x}. \end{cases}$$

   - $\tilde{f}_2$ has Hamiltonian $S_2 := \int f_2 dx$, that is, $S_2 = xK^{-1}$

   $$\begin{cases} \dot{x} = K^{-2}(xK_z) = -\frac{\partial S_2}{\partial z} \\ \dot{z} = K^{-2}(K - xK_x) = \frac{\partial S_2}{\partial x}. \end{cases}$$

3. Each of them can either be integrated (solved) exactly or approximated with a volume-preserving integrator Φ_i, such as the midpoint rule.

A volume-preserving integrator of the source-free vector field f is given by, for example,

$$\Phi = \Phi_1 \circ \Phi_2.$$

Example 26. We consider the source-free system with

$$K = \frac{1}{2}(x^2 + y^2 + z^2) + 1$$

on $(\mathbb{R}^3, \alpha)$. Then

$$\begin{cases} \dot{x} = K^{-2}(-K_y + xK_z) = K^{-2}(-y + xz) \\ \dot{y} = K^{-2}(K_x) = K^{-2}(x) \\ \dot{z} = K^{-2}(K - xK_x) = K^{-2}(K - x^2) \end{cases} \tag{7.3}$$

is the source-free vector field. Now we split f as

$$\tilde{f}_1 \begin{cases} \dot{x} = K^{-2}(-y) \\ \dot{y} = K^{-2}(x) \\ \dot{z} = 0 \end{cases} \qquad \tilde{f}_2 \begin{cases} \dot{x} = K^{-2}(xz) \\ \dot{y} = 0 \\ \dot{z} = K^{-2}(K - x^2). \end{cases}$$

Then the vector fields $\tilde{f}_1$ and $\tilde{f}_2$ are source-free. We denote $(x, y, z) = u$. Then volume-preserving integrators for $\tilde{f}_1$ and $\tilde{f}_2$ are given by the implicit midpoint rule, $u \mapsto u''$, where

$$u' = u + h\tilde{f}_1\left(\frac{u + u'}{2}\right), \qquad u'' = u' + h\tilde{f}_2\left(\frac{u' + u''}{2}\right).$$

We have taken an initial condition u=(0, -2, 0), step size $h = 0.2$ and 200000 steps. We illustrate the resulting orbit in Figures 7.2 (x and y axis) and 7.3 (x and z axis).

Now we consider the corresponding contact system. The contact vector field is

$$X_K \begin{cases} \dot{x} = -K_y + xK_z = -y + xz \\ \dot{y} = K_x = x \\ \dot{z} = K - xK_x = \frac{1}{2}(-x^2 + y^2 + z^2) + 1. \end{cases} \tag{7.4}$$

The contact Hamiltonian function K is non-vanishing, X_K is in fact the Reeb vector field on $(\mathbb{R}^{2n+1}, \frac{\alpha}{K})$. Now we apply three methods to this contact system: the contact integrator $\tilde{P}$ (see Section 6.1.2), leapfrog method obtained by splitting (see Section 6.2) and the method ODE45 in MATLAB.

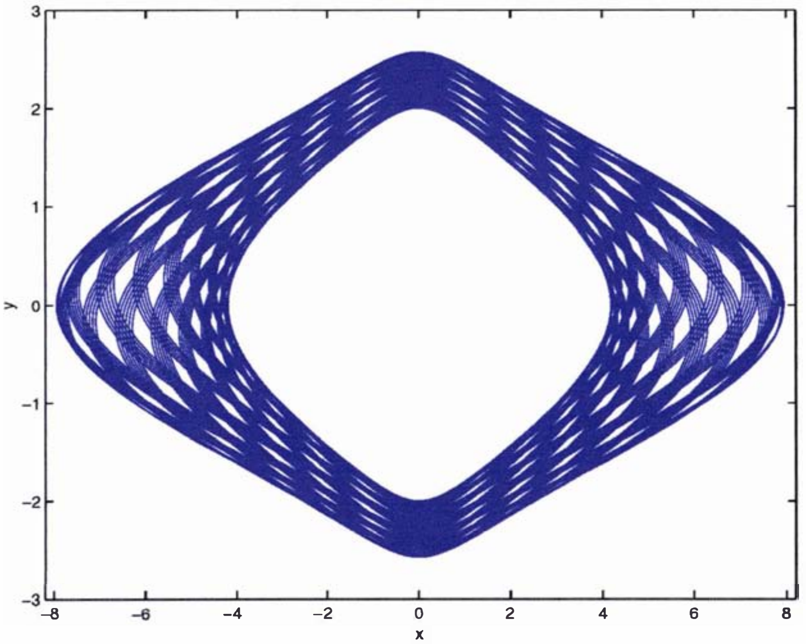

Figure 7.2: Numerical solution for the source-free system (7.3) with $K = \frac{1}{2}(x^2 + y^2 + z^2) + 1$; volume-preserving integrators (x, y axis) with step size $h = 0.2$, 200000 steps and initial condition (0,-2, 0).

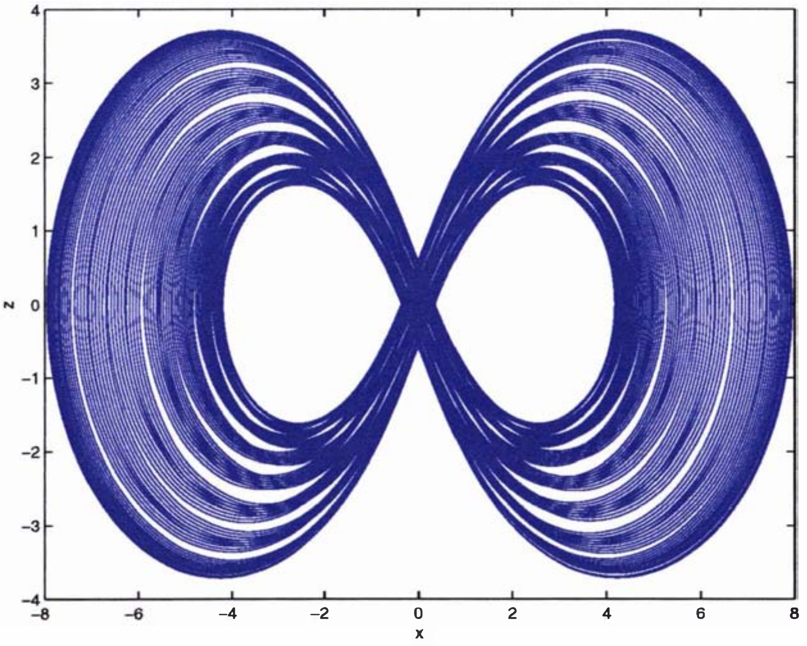

Figure 7.3: Volume-preserving integrator (x, z axis) with step size $h = 0.2$, 200000 steps and initial condition (0,-2, 0).

- Apply the contact integrator $\tilde{P}$. The first order contact integrator $\tilde{P}$ is

$$\begin{cases} \hat{x} = x + h(-y + xz) \\ \hat{y} = y + h(\hat{x}) \\ \hat{z} = z + \frac{1}{2}h(-\hat{x}^2 + y^2 + z^2 + 2). \end{cases}$$

For this particular K, the method $\tilde{P}$ is explicit. We have taken an initial condition (0, -2, 0), step size $h = 0.01$ and 50000 steps. The results of this contact integrator are shown in Figure 7.4 (x and y axis) and in Figure 7.7 (x and z axis).

- Apply the leapfrog method. Now we split $K = K_1 + K_2 + K_3$, where

$$K_1 = \frac{1}{2}x^2, \quad K_2 = \frac{1}{2}y^2 + 1, \quad \text{and} \quad K_3 = \frac{1}{2}z^2.$$

Then the corresponding contact vector fields are X_{K_1}, X_{K_2} and X_{K_3}, where

$$X_{K_1} \begin{cases} \dot{x} = 0 \\ \dot{y} = x \\ \dot{z} = -\frac{1}{2}x^2 \end{cases} \qquad X_{K_2} \begin{cases} \dot{x} = -y \\ \dot{y} = 0 \\ \dot{z} = \frac{1}{2}y^2 + 1 \end{cases} \qquad X_{K_3} \begin{cases} \dot{x} = xz \\ \dot{y} = 0 \\ \dot{z} = \frac{1}{2}z^2. \end{cases}$$

We apply the second order leapfrog method:

$$\hat{y} = y + 0.5h(x)$$
$$\hat{z} = z + 0.5h(-0.5x^2)$$

$$\hat{z} = \hat{z} + 0.5h(0.5\hat{y}^2 + 1)$$
$$\hat{x} = x + 0.5h(-\hat{y})$$

$$\hat{x} = 4\hat{x}/(h\hat{z} - 2)^2$$
$$\hat{z} = 2\hat{z}/(2 - h\hat{z})$$

$$\hat{z} = \hat{z} + 0.5h(0.5\hat{y}^2 + 1)$$
$$\hat{x} = \hat{x} + 0.5h(-\hat{y})$$

$$\hat{y} = \hat{y} + 0.5h(\hat{x})$$
$$\hat{z} = \hat{z} + 0.5h(-0.5\hat{x}^2)$$

$$e^{\left(0.5hX_{K_1}\right)}e^{\left(0.5hX_{K_2}\right)}e^{\left(hX_{K_3}\right)}e^{\left(0.5hX_{K_2}\right)}e^{\left(0.5hX_{K_1}\right)} = e^{hX_K} + \mathcal{O}(h^3).$$

This method is explicit and contact. The results of this leapfrog method are shown in Figure 7.5 (x and y axis) and in Figure 7.8 (x and z axis) with initial condition (0, -2, 0), step size $h = 0.01$ and 50000 steps.

- Apply ODE45 in MATLAB. The initial condition is taken to be (0, -2, 0) with time interval [0, 500]. We illustrate them in Figure 7.6 (x and y axis) and Figure 7.9 (x and z axis).

Remark. For this system, all three geometric integrators, (volume-preserving, leapfrog and contact), produce a quasi periodic orbit. They do show minor differences in frequency because $\tilde{P}$ is only first order while the volume-preserving and leapfrog methods are second order. However, the orbit produced by MATLAB is clearly not quasi periodic even though the integrator ODE45 is order 5 with built-in error control. We expect that for higher dimension systems, some differences between volume-preserving and contact integrator would become apparent.

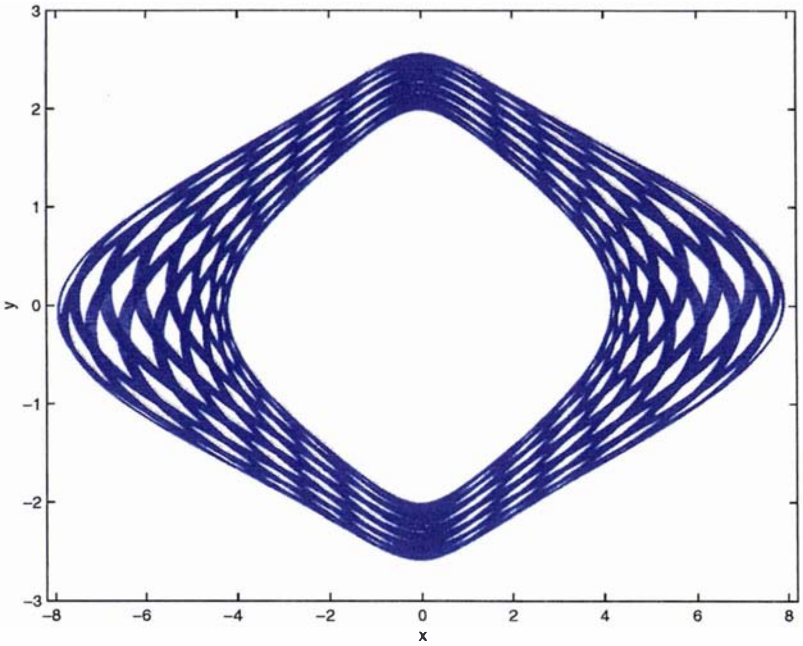

Figure 7.4: Numerical solution for the contact System (7.4) with $K = \frac{1}{2}(x^2 + y^2 + z^2) + 1$; contact integrator $\tilde{P}$ (x, y axis) with step size $h = 0.01$, 50000 steps and initial condition (0, -2, 0).

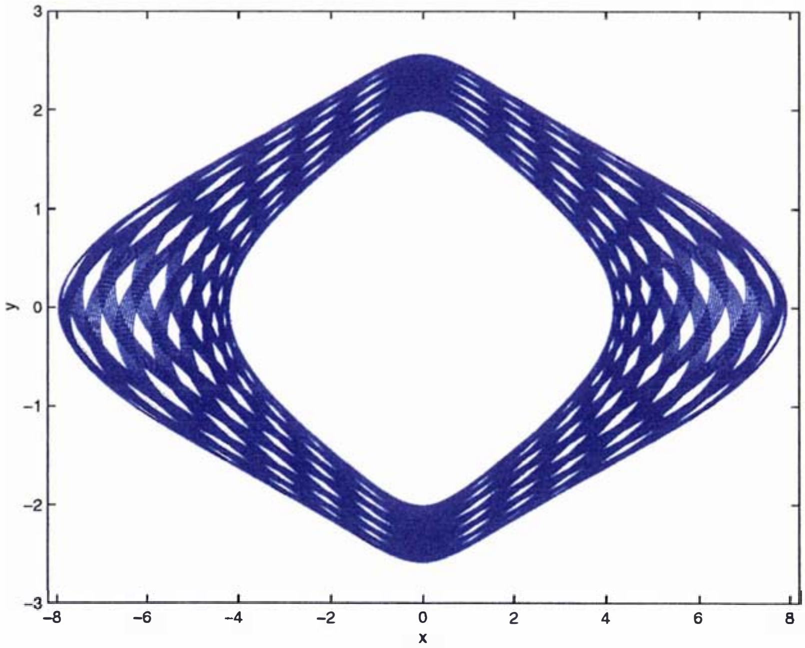

Figure 7.5: Leapfrog splitting method (x, y axis) with step size $h = 0.01$, 50000 steps and initial condition (0, -2, 0).

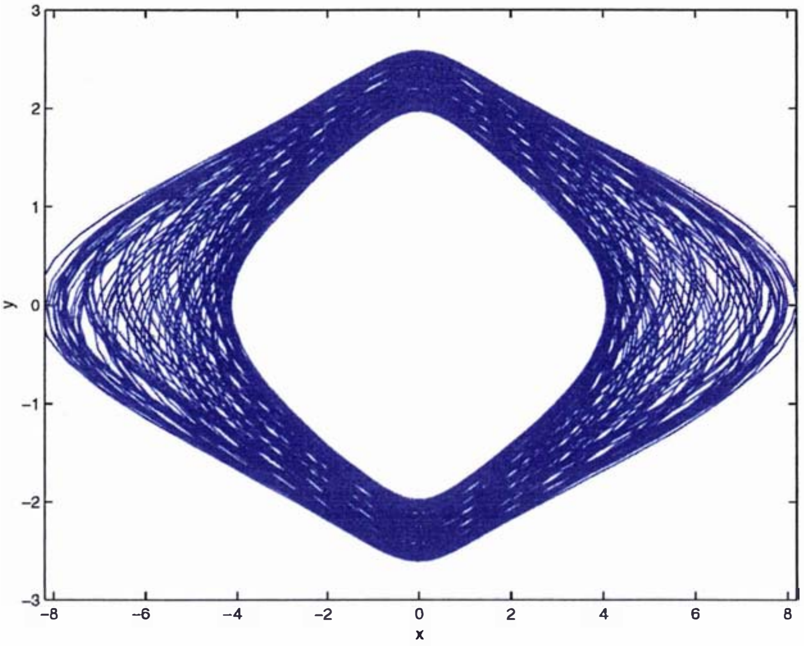

Figure 7.6: MATLAB ODE45 (x, y axis) with time interval [0, 500].

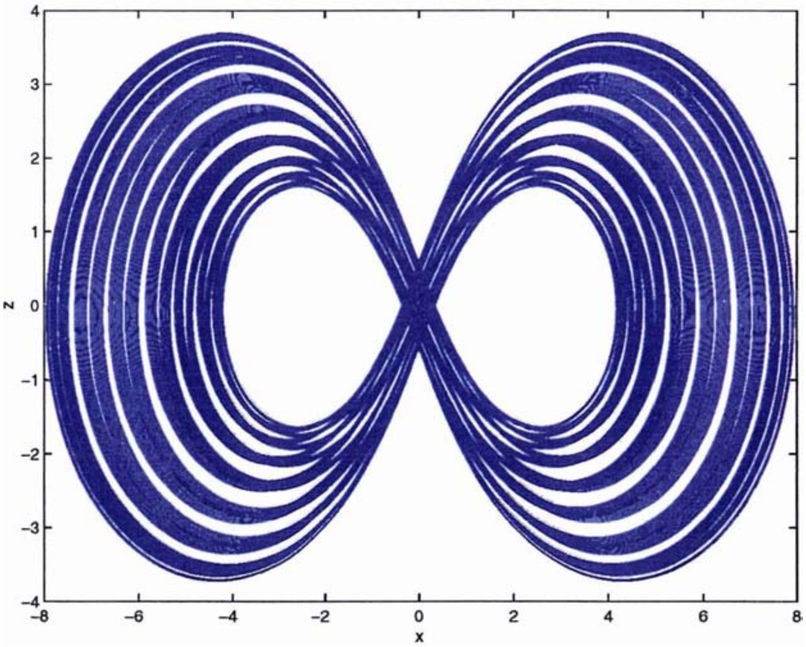

Figure 7.7: Contact integrator $\tilde{P}$ (x, z axis) with step size $h = 0.01$, 50000 steps and initial condition (0, -2, 0).

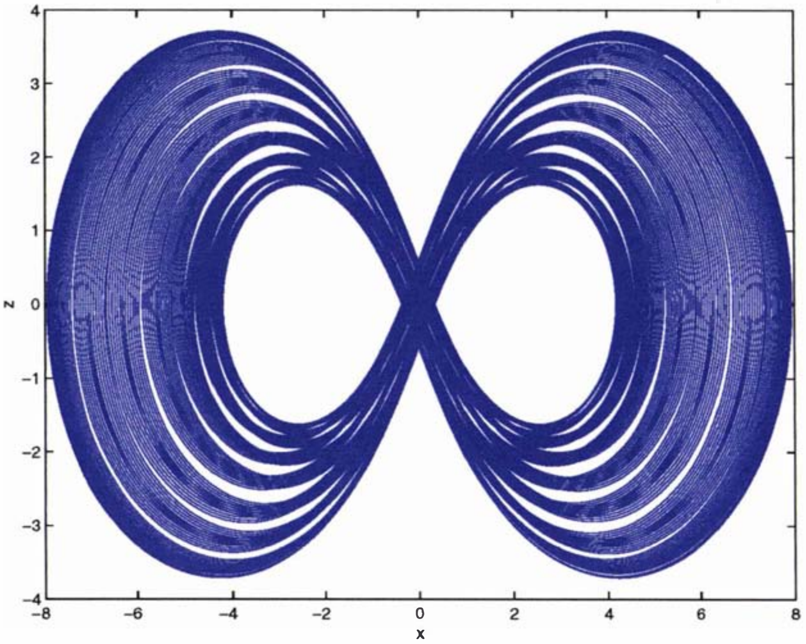

Figure 7.8: 5. Leapfrog splitting method (x, z axis) with step size $h = 0.01$, 50000 steps and initial condition (0, -2, 0).

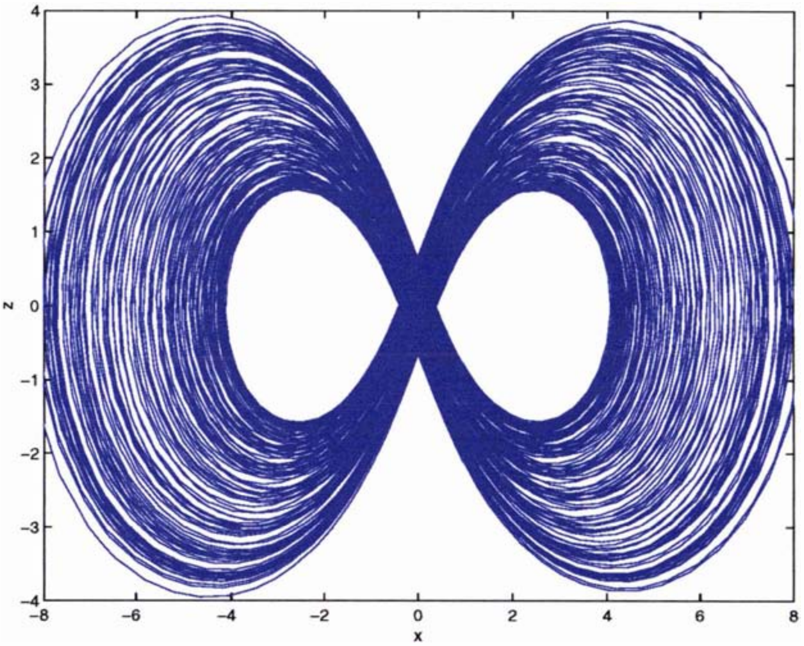

Figure 7.9: MATLAB ODE45 (x, z axis) with time interval [0, 500].

7.3.2 A (2n+1)-Dimensional Volume-Preserving Integrator

Now we consider the source-free system on a $(2n+1)$-dimensional contact manifold (M, α):

$$f \begin{cases} \dot{x}_i = K^{-(n+1)}(-K_{y_i} + x_i K_z) \\ \dot{y}_i = K^{-(n+1)}(K_{x_i}) \\ \dot{z} = K^{-(n+1)}(K - \sum_{i=1}^{n} x_i K_{x_i}) \end{cases}$$

for all $i = 1, \ldots, n$. Then f is volume preserving with respect to the volume form $\alpha \wedge (d\alpha)^n$ on (M, α). Let us split f as $\tilde{f}_1 + \tilde{f}_2 + \cdots + \tilde{f}_{2n}$, where

$$\tilde{f}_i \begin{cases} \dot{x}_i = K^{-(n+1)}(-K_{y_i}) \\ \dot{x}_j = 0 \quad j \neq i \text{ for all } j = 1, \ldots, n \\ \dot{y}_i = K^{-(n+1)}(K_{x_i}) = f_i \\ \dot{y}_j = 0 \quad j \neq i \text{ for all } j = 1, \ldots, n \\ \dot{z} = 0 \end{cases}$$

for $i = 1, \ldots, n$,

$$\tilde{f}_{n+i} \begin{cases} \dot{x}_i = K^{-(n+1)}(x_i K_z) \\ \dot{x}_j = 0 \quad j \neq i \text{ for all } j = 1, \ldots, n \\ \dot{y}_i = 0 \\ \dot{y}_j = 0 \quad j \neq i \text{ for all } j = 1, \ldots, n \\ \dot{z} = K^{-(n+1)}(\frac{K}{n} - x_i K_{x_i}) = f_{n+i} \end{cases}$$

for $i = 1, \ldots, n$.

1. Each of these $2n$ vector fields (i.e. $\tilde{f}_1, \ldots, \tilde{f}_{2n}$) is volume-preserving with respect to the volume form $\alpha \wedge (d\alpha)^n$.

2. They each correspond to a 2-dimensional Hamiltonian system.

- $\tilde{f}_i$ has Hamiltonian $S_i := \int f_i dx_i$; that is, $S_i = -\frac{1}{n}K^{-n}$

$$\begin{cases} \dot{x}_i = K^{-(n+1)}(-K_{y_i}) = -\frac{\partial S_i}{\partial y_i} \\ \dot{y}_i = K^{-(n+1)}(K_{x_i}) = \frac{\partial S_i}{\partial x_i} \end{cases}$$

  for $i = 1, \ldots, n$.

- $\tilde{f}_{n+i}$ has Hamiltonian $S_{n+i} := \int f_{n+i} dx_i$; that is, $S_{n+i} = \frac{1}{n}x_i K^{-n}$

$$\begin{cases} \dot{x}_i = K^{-(n+1)}(x_i K_z) = -\frac{\partial S_{n+i}}{\partial z} \\ \dot{z} = K^{-(n+1)}(\frac{K}{n} - x_i K_{x_i}) = \frac{\partial S_{n+i}}{\partial x_i} \end{cases}$$

  for $i = 1, \ldots, n$.

3. Each of them can either be integrated (solved) exactly or approximated with a volume-preserving integrator Φ_i, such as the midpoint rule.

A volume-preserving integrator for the source-free vector field f is given by, for example,

$$\Phi = \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_{2n}.$$

Chapter 8

Conclusions and Future Work

In this chapter we outline the main conclusions of this thesis and suggest some further questions that we leave till the future.

8.1 Summary and Conclusions

Now we present the main results of our work. We have used analytical, geometrical and numerical tools to study contact systems in this thesis.

Analytical Treatment

As far as we can tell, all other authors studying contact dynamics have only considered the case in contact geometry of the Reeb vector field. We have discussed both this case and the general features of contact systems on the contact manifold (M, α), and also have discussed contact systems in the case that the contact Hamiltonian vanishes on some submanifold of phase space- that is, non-Reeb contact vector fields.

- We have defined contact systems and the Reeb vector field. As we have shown in Proposition 27, the contact vector field X_K on (M, α) is the Reeb vector field on $(M, \frac{\alpha}{K})$, which has the same contact structure. Thus if K is non-vanishing, we may confine our attention to Reeb vector fields. The flow has a Hamiltonian-like property.

- However, if $K(x) = 0$ for some $x \in M$ then the contact vector field X_K is not in any sense a Reeb vector field. We have discussed contact systems on the zero section $S := K^{-1}(0)$. As we have discussed in Section 4.2, we have found conditions for the flow to be conformal symplectic on S. If λ $(= i_E dK)$ is a non-zero constant on S, then the flow of X_K is conformal symplectic with respect to the symplectic form $d\alpha|_S$. In fact (see for instance in Example 9) its dynamics can be dissipative. Also, if $\lambda \neq 0$ on S, the flow of $\frac{X_K}{\lambda}$ is conformal symplectic with respect to the symplectic form $d\alpha|_S$.

- If $\lambda = 0$ somewhere on S, we have no characterization of the dynamics. If λ changes sign on S, anything can happen (i.e., the zero section S is neither attracting nor repelling).

- We have stated the stability of each fixed point of a contact system with contact Hamiltonian function K. The fixed points can only occur on the zero section S and the stability of the fixed points on the contact system is dependent on λ. If $\lambda \neq 0$ at fixed points, then stability is as for conformal symplectic systems (Figures 4.1 and 4.3 illustrate this stability of the fixed points).

In order to construct our contact integrators, an important part of this thesis is devoted to integrable contact systems.

- We have explained the first integral of the contact Hamiltonian function and the relationship between the Jacobi bracket and contact vector fields. On the Hamiltonian side, if the Poisson bracket of F and H commutes; that is,

$\{F, H\} = 0$ then F is an integral of X_H. But on the contact side, even if the Jacobi bracket of f and K vanishes; that is, $[f, K] = 0$, we cannot say that f is an integral of X_K.

- In general, if $F_1, \dots, F_k$ commute and X_{F_i} is integrable for all $i = 1, \dots, k$, then X_H where $H = g(F_1, \dots, F_k)$ is integrable where g is any function. However, in the homogeneous Hamiltonian case, X_H of $H = g(F_1, \dots, F_k)$ is integrable when g is homogeneous of degree 1. We have studied the integrability of contact systems via symplectification. First, we have found the relationship between the homogeneous Hamiltonian $H = g(F_1, \dots, F_k)$ and contact Hamiltonian $K = g(f_1, \dots, f_k)$. Also we have found the relationship between X_H and X_K. Finally we proved that if $f_1, \dots, f_k$ commute and X_{f_i} is integrable for all $i = 1, \dots, k$, then X_K of $K = g(f_1, \dots, f_k)$ is integrable where g is homogeneous of degree 1.

Numerical Treatment

- As a Hamiltonian system projects to a contact system, a homogeneous symplectic integrator projects to a contact integrator. We have also pointed out which symplectic integrators, applied to homogeneous Hamiltonians, produce homogeneous maps. We have discussed contact integrators in Example 18 and Example 19.

- As discussed in Section 6.2, we have constructed contact integrators by the splitting method. Using the condition that the contact Hamiltonian function be nilpotent of degree 2, we have obtained the cases when the explicit Euler method becomes contact.

- We have discussed polynomial contact systems. We have found that any polynomial contact system can be split into a sum of explicitly integrable contact systems.

Finally, we have discussed volume-preserving integrators for some contact systems.

- In Section 7.2, we have applied volume-preserving integrators [75, 77] to the rescaled contact vector field $\frac{X_K}{K^{n+1}}$. We have shown that each of the (source-free) subsystems of the source-free system corresponds to a 2-dimensional Hamiltonian system. Figures 7.2 and 7.3 illustrate this volume-preserving integrator applied to a contact system. We have tested three further methods (the contact integrator $\tilde{P}$, leapfrog splitting method and ODE45 in MATLAB). The results of these three methods are shown in Figures 7.4-7.9.

8.2 Suggestions for Future Work

We close this chapter with some open problems.

- The flow of the contact vector field X_K is volume-preserving if $K(x, y, z)$ is non-vanishing ($K(x, y, z) \neq 0$ for all x, y and z). What can be said about volume-preservation for contact diffeomorphisms? By analogy with flows, we do not expect that all contact diffeomorphisms are volume-preserving. On the other hand, a contact integrator for X_K ($K \neq 0$) may or may not be volume-preserving.
    - Give necessary and sufficient conditions for a contact diffeomorphism to be volume-preserving.
    - Construct integrators which are contact and volume-preserving.

  By backward error analysis, any contact integrator for X_K is exponentially close to the flow of $X_{\tilde{K}}$ for some $\tilde{K} \approx K$. Thus we would expect that the integrator almost preserves a nearby volume element and almost preserves a nearby submanifold.

- The flow of X_K preserves the level set $K = 0$. So we can ask similar question as above.
    - When does a contact diffeomorphism have an invariant submanifold?

  - Construct contact integrators which also preserve $S := K^{-1}(0)$.

- If λ changes sign on $S := K^{-1}(0)$, anything can happen (that is, the zero section S is neither attracting nor repelling, and the flow of X_K on S is not conformal symplectic).

  - Clarify the nature of the flow on S when λ changes sign on this zero section.

- Concerning the dynamics on $S = K^{-1}(0)$, recall that $\lambda = i_E dK$. That is, $\lambda(x) = 0$ if and only if the Reeb vector field E is tangent to S at x. In the canonical case, therefore, $\lambda(x) \neq 0$ for all x if and only if S is a graph over (x, y), as in Example 8. On the other hand, if $\lambda \neq 0$ then S supports a conformal Hamiltonian vector field with conformal constant 1, hence it must be exact. In particular, S cannot be compact. This suggests making a general study of the relationship between the topology of the Reeb vector field and the topology of S.

- The stability of hyperbolic fixed points of contact systems is as for conformal symplectic systems.

  - Develop methods to determine the stability of non-hyperbolic fixed points of contact systems.
  - Investigate the stability of the zero section $S := K^{-1}(0)$.

- What are the relative advantages and disadvantages of contact integrators and volume-preserving integrators for contact systems?

- We have constructed contact integrators via symplectification. We dealt with only first and second order contact integrators (see Sections 6.1 and 6.2).

  - Develop and test higher order contact integrator using symplectification, for example, applying symplectic Runge-Kutta methods to X_H.

- Explore whether the splitting of Hamiltonian systems into integrable Poisson systems with linear Poisson structure (Theorem 19) can be useful in practice.

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