The Conformal Group

&

Einstein Spaces

A thesis presented for the degree of
Doctor of Philosophy
in Mathematics at Massey University

WILLIAM DEAN HALFORD
January, 1977.

THE CONFORMAL GROUP

&

EINSTEIN SPACES

ABSTRACT

This thesis presents

(a) a survey of the use of the conformal group from its beginnings to the present time, and

(b) a determination of those algebraically special vacuum Einstein space-times with an expanding and/or twisting congruence of null geodesics, which locally possess a homothetic symmetry as well as a Killing symmetry (isometry).

Unless the space-time is Petrov type N with twist-free geodesic rays, one can restrict attention to one proper homothetic motion plus the assumed Killing motion(s).

The formalism developed to undertake the systematic search for such vacuum space-times is an extension of the tetrad formalism used by Debney, Kerr & Schild(1) and by Kerr & Debney(2).

The spaces which admit one homothetic Killing vector (HKV) plus 2,3 or 4 Killing vectors (KVs) are completely determined. There are 9 such metrics (12 with 3 degeneracies) - one admitting 4 KVs, one with 3 KVs, and seven with 2 KVs. Those spaces which admit one HKV plus one KV are not completely determined owing to the field equations not being solved in some cases. However, 9 metrics are found, many of which appear to be new.

Petrov type N vacuum spaces with expansion and/or twist which admit a homothety are possible when one KV of special type is also present, or when the homothety alone is of special type.

An extensive bibliography is given.

References:

(1) G.C. Debney, R.P. Kerr & A. Schild, J.Math.Phys. 10, 1842 (1969).

(2) R.P. Kerr & G.C. Debney, J.Math.Phys. 11, 2807 (1970).

ACKNOWLEDGEMENT

It is with the greatest of pleasure that I acknowledge the encouragement and guidance which I have received from my supervisor, Professor Roy Kerr. His deep insights into relativity and differential geometry have been a great source of inspiration. My thanks are due to Roy for the opportunity to undertake this study, which has been an enlightening and enjoyable experience in mathematics.

I want to mention here the debt which I owe my wife Anne and our children Deanne, Sherryn and Leon. Demands made on them were great, and without their ready cooperation this research would not have been possible. *Memor gratusque*.

Many thanks are also due to Lyn Stening who typed this thesis. Her patience and skill has been much appreciated.

PREFACE

The amount of interest in the use of the conformal group in physics has increased a great deal in the last decade. Most of the current activity appears to be in the microphysical arena, with attention being given to such matters as the breaking of conformal invariance in quantum field theory. But the study of conformal motions as an external symmetry in the theory of gravitation and cosmology is also developing. This thesis presents

(a) a survey of the use of the conformal group from its beginnings to the present time, and

(b) a determination of those algebraically special vacuum Einstein space-times with an expanding and/or twisting congruence of null geodesics, which locally possess a homothetic symmetry as well as a Killing symmetry.

Chapter 1 provides a brief introduction to the mathematical structure of the conformal group. Besides their group theoretic properties, the place of conformal motions within the hierarchy of collineations is discussed.

Chapter 2 is a survey of the mathematical development of the conformal group and its application to relativity and gravitation, cosmology, and other physical theories, notably quantum field theory.

With the background of the first two chapters, the scene is set in Chapter 3 for the task (b) above.

Chapter 4 sets up the formalism which is used throughout the rest of the work. It is an extension of the tetrad formalism used by Kerr and Debney to determine vacuum Einstein spaces which possess isometries.

Chapters 5 and 6 contain the bulk of the work involved in determining those spaces which admit one homothetic Killing vector plus 2, 3 or 4 Killing vectors (Chapter 5) or just one Killing vector (Chapter 6).

The possibility of Petrov type N vacuum spaces admitting a homothety is considered separately in Chapter 7.

There follows a Conclusion, a list of Appendices, and an extensive Bibliography containing over 460 references.

CONTENTS

CHAPTER 1 *The Conformal Group*

1.1 Definitions.

Let M be a differentiable n-dimensional manifold and consider the point mapping

$$\varphi : p \to p' \tag{1.1}$$

on M. Suppose there is a geometric object field Ω on M, and pull back the object $\Omega(p')$ at p' to p by the mapping φ^{-1}. Then we have a geometric object $\Omega'(p)$.

Suppose that the mapping (1.1) is a 1-parameter local transformation of coordinates (x^α) on M

$$\varphi : x^\alpha \to x'^\alpha = f^\alpha(x^\beta;t), \quad (\alpha,\beta=1,2,\dots,n), \tag{1.2}$$

generated by a vector field X at p. Geometrically, this means that φ takes the point p a parameter distance t along the integral curves of X, with initial point p, given by the set of differential equations

$$dx^\alpha/dt = X^\alpha. \tag{1.3}$$

The Lie derivative of the object Ω with respect to X is defined at p to be

$$\mathcal{L}_X\Omega = \lim_{t\to 0} \frac{1}{t}\{\Omega'(p) - \Omega(p)\}. \tag{1.4}$$

It follows (see e.g. [1] - [3]) that, if we adopt a coordinate basis $\{\partial/\partial x^\mu\} \equiv \{\partial_\mu\}$ at p so that $X = X^\mu\partial_\mu$, the Lie derivative of a function f on M is

$$\mathcal{L}_X f = Xf = X^\mu\partial_\mu f. \tag{1.5}$$

The Lie derivative of a vector field Y on M is

$$\mathcal{L}_X Y = [X,Y] = -[Y,X] \equiv XY - YX \tag{1.6}$$

or, in local coordinates,

$$(\mathcal{L}_X Y)^\alpha = X^\mu\partial_\mu Y^\alpha - Y^\mu\partial_\mu X^\alpha. \tag{1.7}$$

For any vector fields X and Y,

$$\mathcal{L}_{[X,Y]} = [\mathcal{L}_X,\mathcal{L}_Y], \tag{1.8}$$

where the bracket is defined in (1.6).

The Lie derivative of a tensor field S of type (1,2), say, with respect to X is given by

$$(\mathcal{L}_X S)(Y,Z) = [X,S(Y,Z)] - S([X,Y]Z) - S(Y,[X,Z]) \tag{1.9}$$

or, in local coordinates,

$$(\mathcal{L}_X S)^\alpha_{\beta\gamma} = X^\mu\partial_\mu S^\alpha_{\beta\gamma} - S^\mu_{\beta\gamma}\partial_\mu X^\alpha + S^\alpha_{\mu\gamma}\partial_\beta X^\mu + S^\alpha_{\beta\mu}\partial_\gamma X^\mu. \tag{1.10}$$

For a Riemannian manifold with symmetric metric tensor g and affine connection Γ (1.10) gives

$$\pounds_X g_{\beta\gamma} = X^\mu \partial_\mu g_{\beta\gamma} + g_{\mu\gamma}\partial_\beta X^\mu + g_{\beta\mu}\partial_\gamma X^\mu. \tag{1.11}$$

The local components of the Lie derivative of Γ are given by

$$(\pounds_X \Gamma)^\alpha_{\beta\gamma} \equiv \pounds_X \{{}^{\alpha}_{\beta\gamma}\} = X^\alpha_{;\beta\gamma} + R^\alpha_{\mu\beta\gamma} X^\mu \tag{1.12}$$

$$= \frac{1}{2} g^{\alpha\mu}\{(\pounds_X g_{\gamma\mu})_{;\beta} + (\pounds_X g_{\beta\mu})_{;\gamma} - (\pounds_X g_{\beta\gamma})_{;\mu}\}, \tag{1.13}$$

where $\{{}^{\alpha}_{\beta\gamma}\}$, $R^\alpha_{\mu\beta\gamma}$ are the local components of the metrical connection and the curvature tensor respectively, and the semi-colon denotes covariant differentiation with respect to the connection.

Two Riemannian manifolds M, $\tilde{M}$ endowed with metrics g, $\tilde{g}$ are said to be conformally related iff

$$\tilde{g} = e^{2\phi} g \tag{1.14}$$

for some non-zero function ϕ. This relationship is called homothetic if ϕ is a non-zero constant. In the degenerate case $\phi = 0$, the relationship is isometric.

In terms of smooth local maps $\varphi : M \to M$, the 1-parameter infinitesimal transformations (1.2) generated by a vector field X on M are said to form a (local) group of conformal motions on M iff

$$\pounds_X g = \psi g$$

for some positive function ψ on M; in local coordinates

$$\pounds_X g_{\mu\nu} = \psi g_{\mu\nu} . \tag{1.15}$$

If ψ is a non-zero constant the conformal motion is homothetic. In case $\psi = 0$ the infinitesimal transformation (1.15) is a motion (isometry).

Using (1.11), we can express (1.15) in the form

$$X_{\mu;\nu} + X_{\nu;\mu} = \psi g_{\mu\nu}. \tag{1.16}$$

For $\psi \neq 0$ these are the conformal Killing equations, and a vector X which satisfies them is called a conformal Killing vector (CKV). When $\psi = 0$ they degenerate to Killing's equations [4] for Killing vectors (KVs) X.

The finite transformation equations of the group G_1 of 1-parameter transformations (1.2) are obtained in the usual way by exponentiation. For the infinitesimal transformation generated by a vector field X, the corresponding finite transformation equations are

$$x^{*\mu} = e^{tX} x^{\mu} . \tag{1.17}$$

t is referred to as the canonical parameter. This result generalizes naturally to a group G_r of transformations

$$x^{*\alpha} = f^{\alpha}(x^{\beta};\ a^1,\ldots,a^r) \tag{1.18}$$

in r essential parameters, generated by r linearly independent vector fields X_i, where

$$X_i = X_i^{\ \mu}\partial_{\mu} \qquad (i=1,\ldots,r)$$

in a local coordinate basis $\{\partial_{\mu}\}$. The commutation relations between the group generators are

$$[X_i,X_j] = C^k_{\cdot ij}X_k, \tag{1.19}$$

where the structure constants $C^k_{\cdot ij}$ obey the constraints

$$C^k_{\cdot ij} = -C^k_{\cdot ji},$$

$$C^r_{\cdot ij}C^s_{\cdot kr} + C^r_{\cdot jk}C^s_{\cdot ir} + C^r_{\cdot ki}C^s_{\cdot jr} = 0. \tag{1.20}$$

This last relation, the Jacobi identity, may be expressed in terms of the commutators as

$$[X_i,[X_j,X_k]] + [X_j,[X_k,X_i]] + [X_k,[X_i,X_j]] = 0. \tag{1.21}$$

If a Riemannian manifold admits a vector field X which possesses the property (1.16) we say that the manifold has a symmetry (conformal or isometric). In general, a Riemannian manifold will not have any symmetries. However, the manifold may admit r linearly independent vector fields X_i generating a local r-dimensional Lie group of symmetries. Then the relations (1.19) and (1.20) obtain, and the set of all symmetry vectors X_i on such a manifold forms a Lie algebra of dimension r, the product of the algebra being the Lie bracket [,]. It is well known (see e.g. [5],[6]) that the maximal group of motions of a Riemannian n-space has dimension $\frac{1}{2}n(n+1)$, while for conformal motions the maximal group dimension is $\frac{1}{2}(n+1)(n+2)$. If these maximal cases occur, the space is flat, conformally flat respectively. The full group of symmetries of a manifold may include some discrete ones, such as space-time reflections, which are not generated by Killing or conformal Killing vectors. In what follows we shall be concerned only with groups which do not include discrete symmetries i.e. the symmetry groups will be continuously connected to the identity.

1.2 Integrability conditions.

Substituting (1.15) into (1.12) we find

$$\mathcal{L}_X\{^{\alpha}_{\beta\gamma}\} = \tfrac{1}{2}(\delta^{\alpha}_{\beta}\psi_{,\gamma} + \delta^{\alpha}_{\gamma}\psi_{,\beta} - g_{\beta\gamma}\psi^{,\alpha}), \tag{1.22}$$

where $\psi_{,\alpha} \equiv \partial_{\alpha}\psi$ and $\psi^{,\alpha} = g^{\alpha\nu}\psi_{,\nu}$. If we put (1.22) into the result [6]

$$(\mathcal{L}_X\{^{\alpha}_{\beta\gamma}\})_{;\mu} - (\mathcal{L}_X\{^{\alpha}_{\mu\gamma}\})_{;\beta} = \mathcal{L}_X R_{\mu\beta\gamma}{}^{\alpha}, \tag{1.23}$$

we obtain

$$\mathcal{L}_X R_{\mu\beta\gamma}{}^{\alpha} = \tfrac{1}{2}(\delta^{\alpha}_{\beta}\psi_{\gamma;\mu} - \delta^{\alpha}_{\mu}\psi_{\gamma;\beta} + g_{\mu\gamma}\psi^{\alpha}{}_{;\beta} - g_{\beta\gamma}\psi^{\alpha}{}_{;\mu}), \tag{1.24}$$

where we have written $\psi_{\alpha} \equiv \psi_{,\alpha}$. Defining the Ricci tensor and curvature scalar by

$$R_{\beta\gamma} = R_{\alpha\beta\gamma}{}^{\alpha} \quad , \quad R = g^{\beta\gamma}R_{\beta\gamma} \ ,$$

contraction of (1.24) gives

$$\mathcal{L}_X R_{\beta\gamma} = -\tfrac{1}{2}(n-2)\psi_{\gamma;\beta} - \tfrac{1}{2}\, g_{\beta\gamma}\psi^{\mu}{}_{;\mu} \tag{1.25}$$

and

$$\mathcal{L}_X R = -\psi R - (n-1)\psi^{\mu}{}_{;\mu} \ . \tag{1.26}$$

Making use of (1.24), (1.25) and (1.26) we get

$$\mathcal{L}_X C_{\mu\beta\gamma}{}^{\alpha} = 0, \tag{1.27}$$

where $C_{\mu\beta\gamma}{}^{\alpha}$ is the <u>Weyl conformal curvature tensor</u> defined by

$$C_{\mu\beta\gamma}{}^{\alpha} = R_{\mu\beta\gamma}{}^{\alpha} + \frac{1}{n-2}(\delta^{\alpha}_{\beta}R_{\mu\gamma} - \delta^{\alpha}_{\mu}R_{\beta\gamma} + g_{\mu\gamma}R^{\alpha}_{\beta} - g_{\beta\gamma}R^{\alpha}_{\mu})$$
$$+ \frac{R}{(n-1)(n-2)}(\delta^{\alpha}_{\mu}g_{\beta\gamma} - \delta^{\alpha}_{\beta}g_{\mu\gamma}). \tag{1.28}$$

When written out in full, (1.27) reads (see [5], p.285, or [7])

$$X^{\nu}C_{\mu\beta\gamma\alpha;\nu} - X_{\nu;\rho}(\delta^{\rho}_{\alpha}C^{\nu}{}_{\gamma\mu\beta} - \delta^{\rho}_{\gamma}C^{\nu}{}_{\alpha\mu\beta} + \delta^{\nu}_{\mu}C^{\rho}{}_{\beta\gamma\alpha} + \delta^{\rho}_{\beta}C^{\nu}{}_{\mu\gamma\alpha}) = 0. \tag{1.29}$$

Equations (1.27) or (1.29) are the integrability conditions of

$$X_{\alpha;\beta\gamma} + X_{\mu}R^{\mu}{}_{\gamma\alpha\beta} - \tfrac{1}{2}(g_{\alpha\beta}\psi_{\gamma} + g_{\alpha\gamma}\psi_{\beta} - g_{\beta\gamma}\psi_{\alpha}) = 0 \tag{1.30}$$

which follow from (1.15) and the Ricci identities

$$X_{\alpha;\beta\gamma} - X_{\alpha;\gamma\beta} = X_{\mu}R^{\mu}{}_{\alpha\beta\gamma}\ . \tag{1.31}$$

Defining
$$L_{\beta\gamma} = \frac{1}{2(n-1)}\, g_{\beta\gamma}R - R_{\beta\gamma} \tag{1.32}$$

and
$$C_{\alpha\beta\gamma} = \frac{1}{n-2}\,(L_{\beta\gamma;\alpha} - L_{\alpha\gamma;\beta}), \tag{1.33}$$

further integrability conditions can be obtained [6]. Thus we have the following

Theorem 1.1: In order that a Riemannian n-space admit a group of infinitesimal conformal transformations generated by the vector field X, it is necessary and sufficient that the equations

$$\pounds_X g_{\alpha\beta} = \psi g_{\alpha\beta}, \tag{1.15}$$

$$\pounds_X C_{\alpha\beta\gamma}{}^{\mu} = 0, \tag{1.27}$$

$$\begin{aligned}\pounds_X(C_{\alpha\beta\gamma}{}^{\mu}{}_{;\nu}) = &- \psi_{\nu}C_{\alpha\beta\gamma}{}^{\mu} + \tfrac{1}{2}\,\delta^{\mu}_{\nu}C_{\alpha\beta\gamma}{}^{\rho}\psi_{\rho}\\ &- \tfrac{1}{2}(C_{\nu\beta\gamma}{}^{\mu}\psi_{\alpha} + C_{\alpha\nu\gamma}{}^{\mu}\psi_{\beta} + C_{\alpha\beta\nu}{}^{\mu}\psi_{\gamma} + C_{\alpha\beta\gamma\nu}\psi^{\mu})\\ &+ \tfrac{1}{2}\,\psi^{\rho}(g_{\nu\alpha}C_{\rho\beta\gamma}{}^{\mu} + g_{\nu\beta}C_{\alpha\rho\gamma}{}^{\mu} + g_{\nu\gamma}C_{\alpha\beta\rho}{}^{\mu}),\end{aligned} \tag{1.34}$$

$$\pounds_X C_{\alpha\beta\gamma} = - \tfrac{1}{2}\, C_{\alpha\beta\gamma}{}^{\mu}\psi_{\mu}, \tag{1.35}$$

$$\begin{aligned}\pounds_X(C_{\alpha\beta\gamma;\nu}) = &- \frac{1}{n-2}(\pounds_X L_{\nu\rho})C_{\alpha\beta\gamma}{}^{\rho} - \tfrac{1}{2}\,\psi_{\rho}C_{\alpha\beta\gamma}{}^{\rho}{}_{;\nu} - \tfrac{3}{2}\,\psi_{\nu}C_{\alpha\beta\gamma}\\ &- \tfrac{1}{2}(C_{\nu\beta\gamma}\psi_{\alpha} + C_{\alpha\nu\gamma}\psi_{\beta} + C_{\alpha\beta\nu}\psi_{\gamma})\\ &+ \tfrac{1}{2}\,\psi^{\rho}(g_{\nu\alpha}C_{\rho\beta\gamma} + g_{\nu\beta}C_{\alpha\rho\gamma} + g_{\nu\gamma}C_{\alpha\beta\rho}),\end{aligned} \tag{1.36}$$

and further higher order conditions

$$\pounds_X(C_{\alpha\beta\gamma}{}^{\mu}{}_{;\nu_1\nu_2}\ldots) = \ldots\ ,$$

$$\pounds_X(C_{\alpha\beta\gamma;\nu_1\nu_2}\ \ldots\ ) = \ldots\ ,$$

be algebraically consistent with respect to ψ, ψ_α, X^α and $X_{\alpha;\beta}$. If there exist among equations (1.27), (1.34), (1.35), (1.36), ... exactly s equations which are linearly independent among themselves and linearly independent of (1.15), then the Riemannian n-space admits a group of conformal transformations of dimension $\frac{1}{2}(n+1)(n+2)-s$.

#

Theorem 1.1 reduces to the corresponding one for motions when $\psi = 0$ (see [8], p.62). Collinson & French [9] have expressed the conformal Killing equations (1.15) and the integrability conditions given by the equations of Theorem 1.1 explicitly using a null tetrad and the Newman-Penrose formalism [10].

1.3 Group structure.

O(p,q) is the (real) non-compact orthogonal group which leaves invariant the quadratic form

$$(x^1)^2 + (x^2)^2 + \ldots + (x^p)^2 - (x^{p+1})^2 - \ldots - (x^{p+q})^2.$$

E(p,q) is the corresponding pseudo-euclidean group i.e. O(p,q) plus translations.

The Lorentz group O(3,1) of rotations and "boosts" has six generators $M_{\mu\nu}$ which form a Lie algebra with the Lie bracket [,] of generators as the product. If we add to these the four translation generators P_μ we obtain the Poincaré, or inhomogeneous Lorentz, group E(3,1). The conformal group $\mathbb{C}$ is obtained by adding five generators K_μ and D to the Poincaré group. The 4-vector K_μ generates the special conformal transformations (SCT), and D is the generator of dilations. The conformal group $\mathbb{C}$ is a non-compact 15-parameter simple Lie group.

If (x^α) are local coordinates on a Riemannian 4-space (space-time), the action of the 15 generators of the conformal group $\mathbb{C}$ is as follows (μ, ν = 1,2,3,4):

$$
\begin{aligned}
P_\mu &: x^\mu \to x'^\mu = x^\mu + b^\mu, \\
M_{\mu\nu} &: x^\mu \to x'^\mu = a_{\mu\nu}x^\nu, \qquad a_{\mu\nu} = -a_{\nu\mu}, \\
D &: x^\mu \to x'^\mu = \lambda x^\mu \quad , \qquad \lambda \text{ a scalar} > 0, \\
K_\mu &: x^\mu \to x'^\mu = (x^\mu - c^\mu x^2)/\Delta, \\
&\qquad \Delta = 1 - 2c^\mu x_\mu + c^2x^2, \quad x^2 = x^\mu x_\mu, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad c^2 = c^\mu c_\mu .
\end{aligned} \tag{1.37}
$$

The SCT generators and translation generators are related by

$$N\,P_\mu\,N = K_\mu,$$

where N is the inversion operator

$$N\!: x^\mu \to x'^\mu = -x^\mu/x^2.$$

The SCTs depend upon the four parameters c^μ and form an Abelian subgroup which is continuously connected to the identity (c=0). However, inversion is a discrete operation on points of the manifold, so N is not an element of the connected group.

Writing the generators in the form

$$
\begin{aligned}
P_\mu &= \partial_\mu, \\
M_{\mu\nu} &= x_\mu\partial_\nu - x_\nu\partial_\mu, \qquad (\mu \neq \nu) \\
D &= x^\mu\partial_\mu, \\
K_\mu &= x_\mu x^\nu\partial_\nu - \tfrac{1}{2}\,x^\nu x_\nu\partial_\mu,
\end{aligned} \tag{1.38}
$$

where

$$x_\mu = g_{\mu\nu}x^\nu, \qquad\qquad g = \mathrm{diag}\{-, +, +, +\},$$

the Lie algebra is given by

$$
\begin{aligned}
[M_{\mu\nu},M_{\lambda\rho}] &= g_{\mu\rho}M_{\nu\lambda} + g_{\nu\lambda}M_{\mu\rho} - g_{\mu\lambda}M_{\nu\rho} - g_{\nu\rho}M_{\mu\lambda}, \\
[M_{\mu\nu},P_\lambda] &= g_{\nu\lambda}P_\mu - g_{\mu\lambda}P_\nu, \\
[M_{\mu\nu},K_\lambda] &= g_{\nu\lambda}K_\mu - g_{\mu\lambda}K_\nu, \\
[P_\mu,K_\nu] &= g_{\mu\nu}D - M_{\mu\nu}, \\
[P_\mu,D] &= P_\mu, \\
[K_\mu,D] &= -\,K_\mu,
\end{aligned} \tag{1.39}
$$

all other commutators vanish.

(Note: It is conventional for most physicists to take x^{μ} and ∂_{μ} as conjugate variables in the sense that ∂_{μ} is replaced by $i\partial_{\mu}$, introducing a factor i on the right hand side of (1.39), where $i^2 = -1$.)

O(p,q) groups are defined as the real linear transformation groups on a (p+q)-dimensional real space. Unitary U(p,q) groups are defined as the complex linear transformation groups on a (p+q)-dimensional complex space; with indefinite metric they arise naturally in the study of the geometry of hermitian spaces (see e.g. [11]). It has long been known [12] that SU(2,2) is the group of Dirac spinor space, a four-dimensional complex vector space equipped with a metric of signature 0. The conformal group $\mathbb{C}$ is realized linearly in six dimensions as O(4,2) and in four dimensions by 4x4 complex matrices as SU(2,2). Its non-linear realization in four dimensions is given by the real transformation equations (1.37) above. As mentioned earlier we restrict our attention to the group component which is continuously connected to the identity, so it is the special orthogonal and unitary groups which are involved. Thus we have the following local isomorphisms:

$$\mathbb{C} \simeq SO(4,2) \simeq SU(2,2).$$

See also [13].

Among all locally isomorphic groups there is one which is simply-connected; it is the universal covering group. For the conformal group on Minkowski space M^4, the universal covering group is $\widetilde{SU}(2,2)$. According to Klotz [14], SU(2,2) is the connected component of the symmetry group of twistor space, and twistors can be thought of as the spinors appropriate to compactified Minkowski space M^4_c. The groups SU(2,2) and SO(4,2) are respectively 4:1 and 2:1 homomorphic to the conformal group on M^4_c ([14],[15]).

The conformal group is the smallest Lie group containing a subgroup isomorphic to the Poincaré group. All connected analytic subgroups of the conformal group are not yet known. However, Belinfante & Winternitz [16] in 1971 classified the 1-parameter subgroups of U(p,q) and SU(p,q) by using canonical forms for Hermitian linear operators on finite-dimensional vector spaces with indefinite metric. In an outstanding sequence of papers Patera, Winternitz, Zassenhaus & Sharp [17] have more recently

(i) constructed explicitly all (q+1) maximal solvable subalgebras of the algebra of SU(p,q) for $p \geq q > 0$ over the field of real numbers;

(ii) determined all conjugacy classes of the maximal solvable subgroups of SO(p,q) and O(p,q) for $p \geq q \geq 0$, $p+q \leq 6$, with some more general results;

(iii) presented a general method for reducing the problem of finding all continuous subgroups of any Lie group G, with a non-trivial continuous invariant subgroup N, to that of classifying the subgroups of N and the subgroups of the factor group G/N. They applied the method to find all classes of continuous subgroups of the Poincaré and Weyl groups with respect to conjugation under the groups themselves i.e. with respect to their inner automorphism groups;

(iv) listed all isomorphisms between different subgroups of the Poincaré group, and for each isomorphism class of the corresponding subalgebras found all invariants;

(v) given the invariants of all real Lie algebras of dimension ≤ 5 and of all real nilpotent algebras of dimension 6;

(vi) classified all subgroups of the de Sitter group O(4,1), which is a subgroup of the conformal group, into equivalence classes with respect to inner automorphisms of the group and found all invariants of the corresponding subalgebras. This systematic development has paved the way for a complete analysis of the continuous subgroup structure of the full conformal group.

The subgroups of the Lorentz group have been known for some time [18].

Unitary analytic representations of the conformal group and its subgroups have been considered, for example, in [19].

1.4 Collineations.

We can also view the group of conformal motions in the perspective of a classification of Riemannian space-times by means of collineations.

A Riemannian space M with curvature tensor $R^{\mu}{}_{\alpha\beta\gamma}$ is said [20] to admit a symmetry called a curvature collineation (CC) if

$$\pounds_X R^{\mu}{}_{\alpha\beta\gamma} = 0 \tag{1.40}$$

for some vector field X on M. A subclass of the CCs is the set of special curvature collineations (SCCs) on M defined by

$$(\pounds_X\{^{\alpha}_{\beta\gamma}\})_{;\mu} = 0, \tag{1.41}$$

where ; denotes covariant differentiation with respect to the metrical connection $\{^{\alpha}_{\beta\gamma}\}$ on M.

The affine collineation (AC) on M is defined by [6]

$$\pounds_X\{^{\alpha}_{\beta\gamma}\} = 0. \tag{1.42}$$

ACs are special cases of projective collineations (PCs) [21] defined by

$$\pounds_X\{^{\alpha}_{\beta\gamma}\} = \delta^{\alpha}_{\beta}\,\psi_{;\gamma} + \delta^{\alpha}_{\gamma}\,\psi_{;\beta}, \tag{1.43}$$

where

$$(n+1)\psi_{;\beta} = X^{\mu}{}_{;\mu\beta}.$$

A special projective collineation (SPC) is a PC for which $\psi_{;\beta\gamma} = 0$. The Weyl projective collineation (WPC) on M is defined by

$$\pounds_X W^{\mu}{}_{\alpha\beta\gamma} = 0, \tag{1.44}$$

where

$$W^{\mu}{}_{\alpha\beta\gamma} = R^{\mu}{}_{\alpha\beta\gamma} - (n-1)^{-1}(\delta^{\mu}_{\gamma}\,R_{\alpha\beta} - \delta^{\mu}_{\beta}\,R_{\alpha\gamma}).$$

A conformal motion (Conf M) has been defined in (1.15), and a special conformal motion (S Conf M) is (1.15) with $\psi_{;\alpha\beta} = 0$. These form subclasses of the conformal collineations (Conf C) given by

$$\pounds_X\{^{\alpha}_{\beta\gamma}\} = \tfrac{1}{2}(\delta^{\alpha}_{\beta}\,\psi_{;\gamma} + \delta^{\alpha}_{\gamma}\,\psi_{;\beta} - g_{\beta\gamma}\,\psi^{;\alpha})$$

which is just (1.22), and the special conformal collineations (S Conf C) for which $\psi_{;\alpha\beta} = 0$ in (1.22). All of these are contained within the Weyl conformal collineations (W Conf C) defined by

$$\pounds_X C^{\mu}{}_{\alpha\beta\gamma} = 0,$$

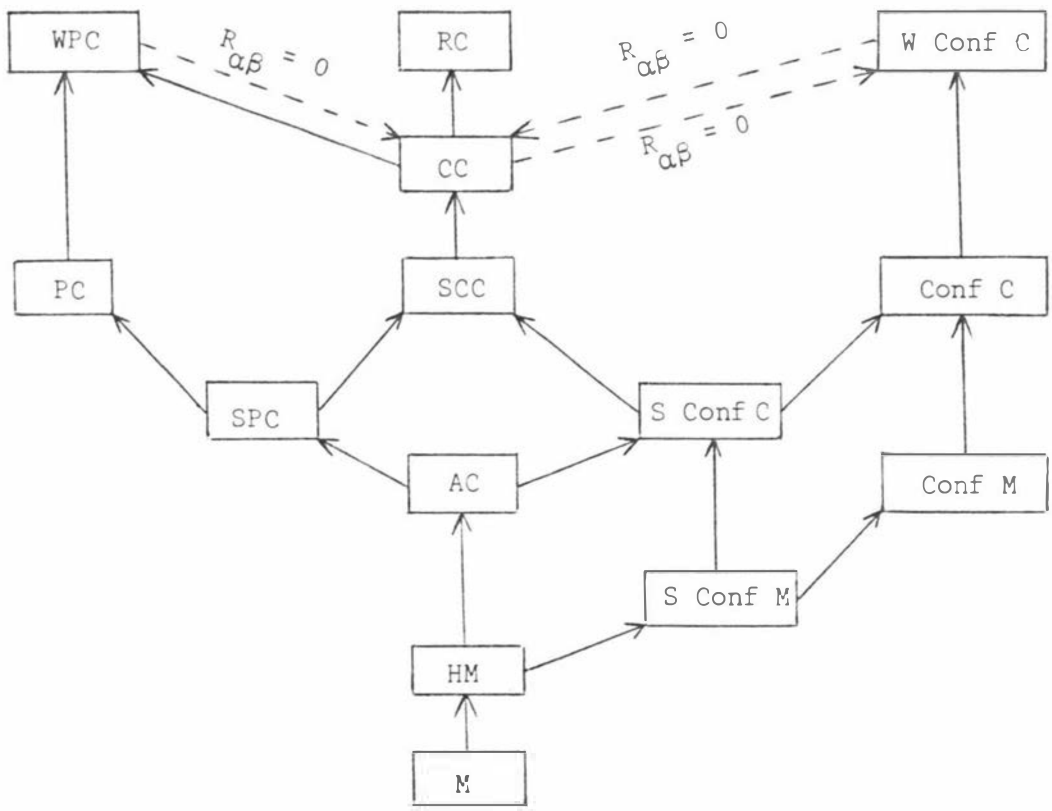
which is just (1.27).

Motions (M) and homothetic motions (HM) as defined earlier form subclasses of the ACs and S Conf Ms.

CCs belong to a more general class of Ricci collineations (RC) given by

$$\mathcal{L}_X R_{\alpha\beta} = 0. \tag{1.45}$$

The hierarchy of collineations is displayed in the following diagram due to Katzin, Levine & Davis [20]:



The arrows indicate direction of increasing generality, linking subclass to class. The broken arrows indicate a relationship only when $R_{\alpha\beta} = 0$.

The significance of these collineations is that

(a) they represent an invariant classification of Riemannian space-times on the basis of symmetry groups, which includes and extends the Petrov classification [8];

(b) they serve as a source of new field conservation laws in general relativity e.g. the Komar identity [22] appears as a necessary condition for the existence of a CC - see also the paper by Trautman [23];

(c) geometrically, the existence of a SCC implies the existence of a cubic first integral of a geodesic particle trajectory in space-time;

(d) every space-time with an expansion-free, shear-free, rotation-free, geodesic congruence admits groups of CCs, in particular pp-waves do.

Further details are given in [20].

CHAPTER 2 *Survey*

There is a vast literature on the conformal group, nearly all of it in the form of original research papers, although the basic facts are contained within the books of Eisenhart [5], Yano [6], Petrov [8] and Schouten [21] in particular. I have attempted to give here a fairly extensive bibliography, but it is not claimed to be exhaustive owing to library resources available to me being limited. I have chosen to present this account in several sections:
(1) the conformal group in general; (2) the use of the conformal group in physics; (3) its application to relativity and gravitation in particular; (4) cosmological applications. Naturally there is a great deal of overlap among the sections since many authors have in a single work dealt with matters pertaining to more than one area. Further, this survey is "local" in nature i.e. the global properties of the conformal group are mentioned only briefly; for a recent account of a global nature, see e.g. Yano [24].

2.1 General.

In this section I shall comment briefly on the papers in chronological order. The bibliographical reference is [26]. Special importance is attached to the work of Brinkmann.

As long ago as 1847 Liouville was using the idea of conformal mapping of spaces. It is now well known that every analytic function of a single complex variable is a conformal mapping in some domain of the plane. However, these mappings do not form a Lie group, because no finite-dimensional algebra can be associated with them (see Guggenheimer [25]). In higher dimensions the situation is different. If one extends Euclidean 3-space by the addition of the point at infinity, one can give this extended space the coordinate structure of the 3-sphere S^3 in Euclidean 4-space. Then inversion in a 2-sphere in Euclidean 3-space reduces to orthogonal geometry on S^3 in Euclidean 4-space (Guggenheimer). Liouville proved that the conformal mappings in S^3 form a Lie transformation group generated by similitudes (Euclidean motions + dilations) and inversions.

Although Klein (1872) was aware of the local isomorphism between the group of quadric collineations in flat 5-dimensional projective space and the conformal group of point transformations in a 4-dimensional space imbedded in the projective space, it was not until the work of Lie & Engel at the end of the 19th century that the study of the conformal group as a continuous group of transformations began systematically. Most of the work has been carried out in the last 40 years.

Bianchi (1902) announced that "the most general conformal map of a Euclidean n-space upon itself for $n > 2$ is obtained as the product of inversions with respect to a hypersphere, motions and transformations of similitude".

Fubini (1903) showed that two infinitesimal conformal transformations of a Riemannian n-space ($n > 2$) cannot have the same paths. Another of his theorems is: If a group G_r of conformal transformations of a Riemannian n-space admits minimum invariant varieties of order m, the G_r may be reduced by means of a transformation of variables to a group on m variables with only m linearly independent transformations.

Kasner (1913) began the development of the conformal geometry of sets of curves.

Weyl (1918,1921) characterized conformally flat spaces of dimension > 3 by the vanishing of the tensor (1.28) which bears his name, and published his unified field theory which made use of a subgroup of the conformal group.

Kasner (1921) proved a special case of a theorem of Brinkmann (Theorem 2.1 below).

Schouten (1921) obtained a necessary and sufficient condition for a given Riemann space to be conformally flat (see Section 2.3, p.37), and Schouten & Struik (1921) proved that if an Einstein n-space ($n > 3$) is conformally flat it must be of constant curvature.

Brinkmann (1923,1924) proved that a conformally flat Riemannian n-space can always be imbedded in a Euclidean (n+2)-space, and considered the question: "When can a Riemann space be mapped conformally on some Einstein space?" He determined (1925) all

Einstein spaces which can be conformally mapped non-trivially (i.e. non-homothetically) on Einstein spaces. His main results appear in the following three theorems:

Theorem 2.1:

The only Einstein 4-spaces which can be conformally but non-homothetically mapped on (possibly different) Einstein spaces are spaces of constant curvature.

Theorem 2.2:

An Einstein space V_n can be conformally but non-homothetically mapped on another Einstein space iff its metric takes the form

$$ds^2 = f^{-1}\, d\varphi^2 + f\, d\sigma^2,$$

where f is given by

$$f = K\varphi^2 + 2A\varphi + B,$$

A,B,K constants and

$$K = R/n(n-1),$$

R being the curvature scalar such that

$$R_{\alpha\beta} = (R/n)g_{\alpha\beta}.$$

Also $d\sigma^2 = \overset{*}{g}_{\alpha\beta}\, dx^\alpha\, dx^\beta$,

where $\overset{*}{g}_{\alpha\beta}$ is independent of φ and is the metric of an Einstein space $\overset{*}{V}_{n-1}$ with curvature scalar

$$\overset{*}{K} = BK^2 - A^2.$$

In order to achieve this form of ds^2, $g^{\alpha\beta}\varphi,_\alpha \varphi,_\beta \neq 0$ identically.

Theorem 2.3:

The most general form of the metric of an Einstein V_4 which admits a conformal but non-homothetic map on some (possibly different) Einstein 4-space is

$$ds^2 = 2\, dx\, dy + 2\, d\varphi\, d\theta + 2\, f(x,\varphi)\, dx\, d\varphi + 2\, g(y,\varphi)\, dy\, d\varphi.$$

If $f_{x\varphi} \neq 0$ and $g_{y\varphi} \neq 0$ identically, the most general Einstein 4-space $\bar{V}_4$ conformal to V_4 is given by

$$d\bar{s}^2 = (a\varphi + b)^{-2}\, ds^2, \qquad a,b \text{ constants}.$$

If $f_{x\varphi} \neq 0$ identically but $g_{y\varphi} \equiv 0$, new coordinates can be chosen so that ds^2 takes the form

$$ds^2 = 2\,dx\,dy + 2\,d\varphi\,d\theta + 2\,f(x,\varphi)dx\,d\varphi,$$

and the most general Einstein 4-space $\bar{V}_4$ conformal to V_4 is given by

$$d\bar{s}^2 = (ax + b\varphi + c)^{-2}\,ds^2, \qquad a,b,c \text{ constants.}$$

If $f_{x\varphi} \equiv 0$ and $g_{y\varphi} \equiv 0$ the Einstein 4-space is flat.

The homothetic case was dismissed by Brinkmann as being "without interest". This is, in fact, far from being true, and we shall take up this point later.

A result of Thomas (1925) is: At corresponding points of two Riemannian n-spaces whose metrics are conformally related by $\tilde{g}_{\alpha\beta} = e^{2\phi}\,g_{\alpha\beta}$, the tensor

$$K^{\alpha}_{\beta\gamma} = \left\{{\alpha \atop \beta\gamma}\right\} - \frac{1}{n}\left(\delta^{\alpha}_{\beta}\left\{{\nu \atop \nu\gamma}\right\} + \delta^{\alpha}_{\gamma}\left\{{\nu \atop \nu\beta}\right\} - g^{\alpha\mu}g_{\beta\gamma}\left\{{\nu \atop \nu\mu}\right\}\right)$$

has the same values.

Knebelman (1930) showed that if a Riemannian n-space V_n admits an intransitive group of motions G_r, there exists (n-r) functionally independent spaces conformal to V_n admitting the same group G_r of motions. This result was generalized by Yano (1951).

Modesitt (1938) considered some singular properties of conformal transformations between Riemann spaces.

Fialkow (1939-1945) investigated the properties of curves in a Riemannian space V_n which are invariant under conformal transformations of the space, and studied a V_n imbedded in a conformally flat V_{n+1}.

Coburn (1941) found a sufficient condition for two Riemann spaces to be conformally related in terms of the components of their affine connections and principal directions in the spaces. He also gave a theorem for two unitary spaces to be conformally related.

Following on Kasner's work, Coleman (1942) further developed the conformal geometry of 1-parameter families of curves.

A leading member of the Roumanian school of differential geometers, Vranceanu (1943) studied the geometry of spaces defined by a conformal connection.

Groups of conformal point transformations on even-dimensional spaces with a non-singular skew-symmetric metric tensor received the attention of Lee (1945).

De Cicc (1946) showed that the Kasner measure of horn angle (in the Kasner plane) is conformally invariant, and Kasner & De Cicco (1947) proved that the only groups in the set of harmonic transformations of the real cartesian plane are the conformal group, the affine group, and the subgroups of these two.

Rozenfel'd (1948) studied the conformal differential geometry of m-spheres and m-planes in metric n-spaces.

Su (1949) generalized the concept of Lie derivation to attack the problem of whether a space with a normal conformal connection admits infinitesimal transformations.

A concircular transformation of a Riemannian n-space is a conformal transformation of the space which maps geodesic circles onto geodesic circles. The Japanese school has done considerable work in this area. See e.g. Yano & Adati (1949).

Kuiper (1950) treated the conformal group as a subgroup of the general (n+1)-dimensional projective group leaving invariant a quadric hypersurface of n dimensions in the projective (n+1)-space.

Homothetic correspondences between Riemannian spaces V_n were investigated by Shanks (1950). Amongst his results was: A V_n is flat iff it admits a group of homothetic transformations of dimension $\frac{1}{2}n(n+1)+1$. He further proved that if a V_n admits a 1-parameter group H_1 of homothetic transformations, the complete group of homothetic transformations consists of H_1 plus the complete group of motions on V_n.

Debever (1950,1955) showed that with a compact semi-simple Lie group there is naturally associated a space with conformal connection; if the order of the group exceeds 3, the holonomy group of this space is the conformal group which leaves invariant a fixed imaginary hypersphere. He also gave a new proof of Segre's (1949) theorem: A Riemannian V_n represented conformally on Euclidean space S_n so that geodesics of V_n correspond to curves with constant curvature in S_n, is itself of constant curvature.

Yano (1951) showed how, with the use of the Lie derivative, the results of Shanks (1950) may be obtained more easily. Yano also proved that an Einstein n-space given by $R_{\mu\nu} = (R/n)g_{\mu\nu}$ with $R \neq 0$ cannot admit a proper (i.e. non-isometric) homothetic motion, thus strengthening Brinkmann's (1925) theorem. He also generalized a theorem of Knebelman (1930), obtaining: If a Riemannian space V_n admits an r-parameter group G_r of homothetic transformations $(r < n)$ with generators X^{μ}_{r}, then there exists (n-q) functionally independent spaces conformal to V_n and admitting the same homothetic transformation group G_r, where q is the generic rank of the matrix (X^{μ}_{r}). Some of the results obtained by Levine [90] (1936,1939) were generalized by Yano.

The topology of the conformal group in Euclidean 3-space was investigated by Pernet (1951).

Stellmacher (1951) considered the conformal geometry of a Riemannian V_n as a projective geometry of the (n-1)-dimensional isotropic hypersurface generated by all the geodesic null lines which emanate from a fixed point of V_n.

Clark (1951, 1956) developed a conformal theory of a general metric space and derived the conformal Killing equations for such a space. He also discussed the conformal equivalence of Riemannian manifolds carrying, in addition to the metric, a certain type of exterior form.

The invariant construction of the geometry of a hypersurface of a conformal space by means of the subgroups of the conformal group associated with the hypersurface was the subject of a paper by Akivis (1952).

Dzavadov (1952) generalized to pseudo-Euclidean spaces a result due to Study (~ 1909), namely, that the conformal transformations of a Euclidean space of 3 or 4 dimensions can be represented by linear fractional transformations of quaternions.

Kurita (1953,1955) did some work on projective and conformal correspondences between Riemann spaces. In particular, he re-derived certain results on conformally flat spaces of imbedding class one due to Schouten (1921), Matsumoto (1951), and Verbickii (1952).

Kobayashi (1954) proved that in certain important cases the group of transformations which leaves invariant an infinitesimal connection in a differentiable fibre bundle is a Lie group. These cases include the affine, projective and conformal groups.

Takano & Imai (1954) contributed a theorem on the conformal invariance of coefficients of the connections in subspaces of a Riemannian space.

The canonical forms of the metrics of Riemannian spaces which admit r independent infinitesimal conformal transformations whose paths constitute normal or geodesic congruences were determined by de Vries (1954). It is well known (see e.g. Eisenhart [5], p.223) that the coordinates of a Riemannian n-space V_n can be chosen so that the contravariant components of the generators of 1-parameter infinitesimal transformations are $X^{\mu} = \delta^{\mu}_{1}$. It seems that de Vries was the first to use this result to give the following general result:

A canonical form of the metric of a V_n which admits an infinitesimal conformal transformation generated by (i) the non-null vector X, (ii) the null vector X, with components X^{μ} in local coordinates (x^{α}) is

$$ds^2 = e^{2\phi} h_{\alpha\beta} dx^{\alpha} dx^{\beta},$$

where $\phi = \phi(x^{\alpha})$, $h_{\alpha\beta} = h_{\alpha\beta}(x^{i})$, $(\alpha,\beta = 1, \ldots, n;\ i = 2,\ldots,n)$,

and (i) $h_{11} = \pm 1$,

(ii) $h_{11} = 0$, $h_{12} = \pm 1$,

respectively, and $X^{\mu} = \delta^{\mu}_{1}$ in both (i) and (ii).

His results include those of Brinkmann (1925) and Shanks (1950).

Ishihara & Obata (1955) showed that if the group of all conformal transformations of a Riemann space is transitive and if there exists a point p which is left invariant by all homothetic transformations of the group, then the space is conformally flat.

Kano (1956) did some groundwork on the conformal geometry of a special Kawaguchi space with a specified metric.

Blum (1956) considered the equations which the Weyl tensor satisfies as a result of the Bianchi identities of any Riemannian n-space.

Brickell & Clark (1962) constructed conformal connections using methods due to E. Cartan.

Pirani & Schild (1966) set up a geometry of conformal space which parallels the usual Riemannian treatment. For example, the procedures which in Riemannian geometry yield the Riemann curvature tensor are modified so that in conformal geometry they yield the Weyl tensor.

Grgin (1968) used a projective geometric formalism for the description of the conformal compactification of Minkowski space and of its invariance groups.

The classification of a Riemannian V_4 according to groups of projective motions was undertaken by Aminova (1971), who showed *inter alia* that the complete group of projective motions is determined by the group of homothetic motions and one (non-homothetic) projective motion (cf. Kuiper (1950) and Shanks (1950)).

De Cicco & Anderson (1971) defined conformally equivalent Riemannian n-spaces, conformal Christoffel symbols, conformal covariant derivative, and the Weyl tensor in terms of the relative conformal metric tensor $g^{*}_{\alpha\beta} = g^{-1/n} g_{\alpha\beta}$, $g = \det(g_{\alpha\beta})$, and obtained some results, many of which do not appear to be new.

Lovkov (1971) determined the generators of a group of homothetic transformations acting simply transitively on a Riemannian V_4 with metric of the form

$$ds^2 = -dx^2 + e^{-2z}dy^2 + 2e^{-z}dy\,dt - x^2dz^2.$$

In addition, he determined all Riemannian 4-spaces admitting this homothetic group.

Egorov (1971) and Egorova (1971) studied homothetic motions in Riemannian spaces with metrics of the form

$$\text{(i)}\quad ds^2 = 2dx_1dx_2 + \sin^2 x_1(e_3dx_3^2 + \ldots + e_{m+2}dx_{m+2}^2) + \sinh^2 x_1(e_{m+3}dx_{m+3}^2 + \ldots + e_n dx_n^2)$$

and

$$\text{(ii)}\quad ds^2 = 2dx_1dx_2 + \cos^2 x_1(dx_3^2 + \ldots + dx_m^2) + \cosh^2 x_1(dx_{m+1}^2 + \ldots + dx_n^2)$$

and proved that these groups have dimension

$$\text{(i)}\quad n + \tfrac{1}{2}m(m-1) + \tfrac{1}{2}(n-m-1)(n-m-2) + 1,$$

$$\text{(ii)}\quad n + \tfrac{1}{2}(m-1)(m-2) + \tfrac{1}{2}(n-m)(n-m+1) + 1,$$

respectively.

Suguri & Ueno (1972) gave an incomplete summary of known results. However, included was the result, first proved by Yano (1951), that we shall use later: the commutator of two infinitesimal homothetic Killing vectors is a Killing vector. Another significant result proved in this paper is:

If $\tilde{g}$ and g are conformally related metrics on a Riemannian space M, and M is not conformally flat, then the complete isometry group on $(M,\tilde{g})$ is the complete conformal group on (M,g). This result, apparently first obtained by Defrise (1969) (see under Defrise-Carter (1975) below), may be restated as: the conformal group on (M,g) is conformally isometric. Yano (1951) had previously proved the theorem for the case when the conformal group is homothetic. Suguri & Ueno also proved some theorems regarding conformally flat manifolds.

Sigal (1973) showed that for Petrov type N vacuum spaces the Lie derivative in the principal null direction of the Weyl tensor and its dual is equal to a conformal transformation and a duality rotation of the original tensor.

Mavryčev (1973,1974) showed that the relation between analytic functions and conformal transformations of the complex plane does not generalize to a complex matrix representation of space-time (cf. Guggenheimer [25], pp.223-224), and applied conformal transformations to the equations of relativistic mechanics.

A topological description of Killing vector fields and conformal Killing vector fields, aimed at a unified qualitative (Poincaré - Bendixson) theory of stationary and conformally stationary timelike 2-surfaces, has been given by du Plessis (1974), who also discussed horizons for these two types of vector field. He also extended the 2-surfaces analytically.

Hori, Sakomoto & Sato (1974) presented a non-linear realization of conformal symmetry, which is linear under the Poincaré transformations, using a method based on the technique developed by Coleman, Wess & Zumino [59] in connection with chiral symmetry (see also Borisov & Ogievetskii [58]).

Loewner & Nirenberg (1974) made a useful contribution to mathematical physics with a paper on partial differential equations invariant under conformal or projective transformations.

Tilgner (1974) gave a description of the full collineation group and the conformal group of arbitrary finite-dimensional pseudo-orthogonal vector spaces over the real field. He argued that sets of (finite) rational transformations like special collineations and SCTs form a groupoid, rather than a group, since there are non-zero denominators. With every such groupoid there is associated a unique Lie group.

Using the modern theory of bundles Schmidt (1974) showed that a conformal structure on a manifold M defines in a natural way two principal bundles over M and a parallelisation on one of these, which then can be used to define a boundary of M in analogy to the b-boundary which is intrinsically defined by the Lorentz metric (Schmidt 1971, Penrose 1969, and Hawking & Ellis [1]). A similar construction is possible for a projective structure on M. The general mathematical background is the theory of prolongations of G-structures (see e.g. Kobayashi 1972) but this theory is not used by Schmidt.

Fediščenko (1975) considered special conformal mappings of Riemannian spaces. No details are available to me.

Lord (1975) applied generalized quaternion methods in a paper on conformal geometry.

Mayer (1975) considered the compactified Minkowski space M^4_c as a closed subspace of a 5-dimensional projective space, and investigated the transformation properties of vector and tensor fields on M^4_c under the conformal group. See also Go, Kastrup & Mayer (1974).

A Riemannian n-space ($n > 3$) is conformally symmetric iff its Weyl tensor obeys $C^{\mu}{}_{\alpha\beta\gamma;\nu} = 0$, and admits a special quadratic first integral (SQFI) defined by the symmetric tensor $a_{\mu\nu}$ iff $a_{\mu\nu;\lambda} = 0$. Every conformally flat n-space ($n > 3$) is conformally symmetric, but the converse is not true. Roter (1975) proved some theorems about conformally symmetric spaces with indefinite metric tensors which admit SQFIs and applied his results to conformally flat spaces, extending the work of Levine & Katzin [92].

The r independent generators X_i of a conformal group C_r acting on a Riemannian space (M,g) with metric g will also generate a conformal group $\tilde{C}_r$ acting on $(M, \tilde{g} = e^{2\phi}g)$. Thus, in certain cases, it will be possible, with an appropriate choice of ϕ, to find a space $(M,\tilde{g})$ for which the conformal group C_r is reduced to a group of either (i) isometries, or (ii) homotheties. In these cases C_r is said to be (i) conformally isometric, or (ii) conformally homothetic, respectively. Yano [6] proved that C_r is conformally isometric when rank $(X_i{}^{\mu}) = r$. This implies that C_r is simply transitive and hence excludes all conformal groups with isotropy subgroups. Defrise-Carter (1975) showed that C_r is conformally isometric under much more general conditions.

If a scalar concomitant F of the metric g is such that $\tilde{F} = e^{2p\phi}F$, where p is a constant, then F is said to be a conformal scalar of weight p (du Plessis, 1969). F is proper if $p \neq 0$; a conformal invariant if $p = 0$. A vector concomitant Y_μ is said to a conformal vector of weight p iff $\tilde{Y}_\mu = Y_\mu + 2p\phi,_\mu$ and is proper when $p \neq 0$. Y_μ is a conformal gradient iff $Y_{\mu,\nu} = Y_{\nu,\mu}$. Defrise-Carter (1975) proved that a conformal group C_r acting on a space (M,g) which admits (i) a proper conformal scalar, (ii) a proper conformal gradient, is respectively (i) conformally isometric, (ii) conformally homothetic. These results include the one obtained independently by Suguri & Ueno (1972) and the special case proved by Yano (1951). Defrise-Carter went on to prove the converse of these results for spaces with positive definite metrics, and conjectured that the converses hold in general. In particular, she proved that a Lorentzian 4-space with no proper conformal scalar is conformally equivalent to the pp-waves, or else is conformally flat. The pp-waves admit a proper conformal gradient and the conformal group is reduced to a group of homothetic motions of dimension 6 (= 5 isometries + 1 homothetic transformation), or 7 (= 6 isometries + 1 homothety) if the pp-wave metric takes a special form.

Boyer & Kalnins (1976) have given a detailed discussion of the infinitesimal symmetries of the Hamilton-Jacobi equation. The group of point transformations locally isomorphic to the conformal group O(3,2) is studied. They show that the separation of variables of the corresponding Hamilton-Jacobi equation in the form of a sum is related to orbits in the Schrödinger subalgebra of o(3,2).

Recently Berger (1976) announced results relating to the Cauchy development of homothetic symmetries in space-like hypersurfaces of empty and non-empty space-times. Constraints which must be satisfied by the spatial metric and extrinsic curvature in the presence of a homothetic Killing vector are obtained by projecting Killing's equations into a space-like hypersurface.

2.2 Use of the Conformal Group in Physics.

Chronology will not be preserved in this and the remaining sections, and we revert to separate references in the bibliography.

The conformal group has appeared in physics both as an (external) symmetry group of space-time coordinate transformations and as an (internal) dynamical group of transformations of rest-frame states of a quantum system having internal degrees of freedom. Bateman & Cunningham (1910) [27] appear to have been the first to explicitly link the conformal group with physics. They showed that Maxwell's equations are invariant under the 4-parameter Abelian subgroup of special conformal transformations (SCTs), and therefore under the full 15-parameter conformal group; it was already known that the Maxwell equations are invariant under dilations and the Poincaré group (translations and rotations). Mayer (1975) [28] has recently given an account of this covariance working on compactified Minkowski space M_c^4 instead of the usual Minkowski space M^4.

In their 1962 paper [29], as well as giving a brief history of conformal transformations in physics and a discussion of conformal invariance in quantum mechanics, Fulton, Rohrlich & Witten distinguished between three transformation groups, all bearing the name "conformal group". These are: (1) the group C_0 of conformal transformations in flat Minkowski space; (2) the group C_g of conformal transformations of the metric tensor $g_{\mu\nu}$ of a (non-flat) Riemannian space; and (3) the group C, which consists of conformal transformations of the metric plus equations which characterize the tensor nature of $g_{\mu\nu}$. They called C the "extended conformal group"; it has C_0, C_g and the group of all coordinate transformations as proper subgroups, and so includes Einstein's theory of general relativity within its domain of applicability. The group C_0 is defined as the set of those transformations in C that transforms flat space into flat space; either C_0: Minkowski → Minkowski, or C_0: Minkowski → conformally Minkowski.

Haantjes [30] showed in 1937 that every element of C_0 can be composed of motions and inversions only.

Fulton et al. [29] showed that Maxwell's equations and the conservation of charge and of energy-momentum are invariant under C. This appears to have been noticed first by Schouten & Haantjes [31] in 1934. The C_0-invariance of Maxwell's equations, discovered by Bateman & Cunningham, is a special case of this result.

The conformal group is the lowest-dimensional Lie group containing the Poincaré (inhomogeneous Lorentz) group as a subgroup. The Poincaré group is important as the symmetry group underlying the kinematics of any (special) relativistic theory. It is natural that many attempts have been made (see e.g. [32]) to generalize physical symmetries in terms of the conformal group. One might then adopt the definition that a physical theory is conformally invariant if physical laws do not change in form under coordinate transformations of the conformal group. This is not the only possible definition of conformal invariance in physics (see e.g. Sigal [79]).

Weyl [33] published his unified field theory in 1918, the underlying group of that theory being the 11-parameter group of Poincaré transformations plus dilations, to which his name is given. His theory was not fully conformally-invariant, of course, but represented an attempt to incorporate the conformally invariant Maxwell equations into general relativity. The attempt itself had lasting impact, even though the theory was found to be physically unsatisfactory.

The conformal group is the largest group which preserves null line elements [32], so there arises the possibility that this group is an exact symmetry group for massless particles (photons, neutrinos) in space-time. This conformal symmetry is broken, however, in the presence of massive particles [34] and, as Barut & Bornzin [35] have remarked, "limits the use of the conformal group to the high energy domain" where the rest-masses of the particles are negligible in comparison. This situation is in keeping with the general observation that in physics symmetry breaking often reduces the symmetry group to one of its subgroups (in this case, the 11-parameter Weyl group). Barut & Haugen [36] have argued that by replacing the usual rest-mass by a scale factor times a new, conformally-invariant mass (first considered by Schouten & Haantjes [37] in 1936), and by interpreting conformal transformations as a space- and time-dependent change of scale, conformally invariant equations of motion can be written for massive particles as well.

It is known that relativistic wave equations become conformally invariant when the mass term is removed [32],[34]. In fact, Dirac [32] in 1936 gave such wave equations for the electromagnetic field and for spin-½ fields, thereby placing the conformal group firmly in the quantum arena. Pauli [38] considered the invariance of the Dirac equation under the conformal group C defined by Fulton et al. (see above). The next success in this area appears to be that of Hoffmann [39] who in 1948 provided a conformal treatment of the equations of a meson field, but restricted his consideration to invariance under homothetic motions. Gürsey [32] developed a 2x2 matrix formalism in view of obtaining a synthetic expression for conformal transformations as well as for spinor wave equations of Dirac type.

Among the first to consider the application of the conformal group to elementary particle physics was Ingraham [40] in 1954. Regarding the conformal group as a group of projective transformations on a 5-dimensional manifold of spheres in Minkowski space [41] (point transformations of the conformal group are identified with null spheres, but non-null spheres are also required to form the domain of definition of physical fields), he develops a theory in which free elementary particles maintain a state of uniform velocity under all motions of the group. This contradicts Einstein's relativity theory, and Ingraham suggests that one may therefore design an experiment to choose between the conformal and Minkowski geometries for physics.

Over the last fifteen years an increasing amount of attention has been given to the use of the conformal group as a dynamical (internal symmetry) group in quantum physics, with activity in this area currently being quite intense. In reference [42] we give a list of some of the original and review papers not already referred to explicitly in this section. The books by Ferrara, Gatto & Grillo [43] and by Barut & Brittin [13] provide fairly recent summaries of the use of the conformal group in quantum physics. Both books contain fairly extensive bibliographies. An earlier paper by Kastrup [44] reports critically on the applications of the conformal group up to 1962.

As mentioned before, global considerations are not developed in this account and, apart from their appearance in the discussion on causality to follow, will remain outside the scope of this dissertation. We remark in passing, however, that a very recent treatment of zero rest-mass wave equations under the action of global conformal transformations has been given by Post [45].

Causality.

The notion of causality has been made rigorous by Hawking & Ellis [1] who define local causality as follows: If U is a convex normal neighbourhood of a connected C^{∞} space-time (Hausdorff, paracompact) manifold M endowed with a Lorentz metric g and if p,q are points in U, then a signal can be sent in U between p and q if and only if p,q can be joined by a C^1 curve lying entirely in U, whose tangent vector is everywhere non-zero and is either timelike or null. This definition determines the metric g at p up to a conformal factor [1]. The conformal factor may be determined if one assumes that energy and momentum are conserved locally i.e. in the presence of matter fields there exists an energy-momentum tensor $T^{\mu\nu}$ on the open set $U \in M$ such that $T^{\mu\nu}{}_{;\nu} = 0$.

For a global treatment, Hawking & Ellis lay down a causality condition: M is causal if and only if there are no closed non-spacelike curves in M. They assume that M is non-compact, and in Chapter 6 of reference [1] give a number of conditions which rule out violations or near violations of causality. A well-known example of an acausal space-time is Gödel's universe [46].

In a very recent paper [47] Hawking, King & McCarthy introduce a new topology P for strongly causal space-times. The homeomorphism group of P is the group of conformal diffeomorphisms of Minkowski space. They claim P is more intuitively appealing, manageable, and physical than the topology proposed by Zeeman [48]. However, Göbel [49] introduces a finer topology which is even more physical than the P-topology of Hawking, King & McCarthy.

Zeeman [50] showed that the largest connected group of space-time transformations which preserves the causal relation between pairs of events is the Weyl group (= Poincaré group + dilations). Thus the conformal group violates causality, since a SCT can always be found which changes the timelike or spacelike separation of an arbitrary pair of events into a spacelike or timelike separation respectively. Recently Nanda [51] has given a geometrical proof of Zeeman's theorem.

Rosen [52] gives two examples in which reversal of the temporal ordering of events is achieved by a SCT. He argues that causality violation should not in itself invalidate the conformal group as a symmetry group of physics. This view is based on his interpretation of space-time transformations as leaving both the manifold and the coordinate system unaffected, the transformations serving only to map the world-lines and events of a physical process under consideration. According to this non-conformist view, violation of causality means that the causal relations among events of a physical process might differ from the causal relations among corresponding events of a conformally-transformed physical process.

Schroer & Swieca [53] have resolved the conflict with causality in quantum field theory by showing that one is, in general, dealing there with representations of the universal covering group of the conformal group.

Causality is essentially of topological character, and the only satisfactory way to interpret the conformal group physically in the large is with respect to the topological properties of the manifold on which it acts. The causality violating property of the conformal transformations is usually avoided by restriction to infinitesimal transformations. To put this on a more rigorous footing, Go [54] pointed out that SCTs can be considered on Minkowski space M^4 only as a local group of local diffeomorphisms. They can be defined as a global transformation group on compactification of Minkowski space [55], but then no global causal structure can be defined [56]. However, by considering the universal covering space $\widetilde{M}$ of M^4_c (compactified M^4), the local causal structure on M^4_c can be made global. That is, the four-fold universal covering group $\widetilde{SU}(2,2)$ of the conformal group of M^4 acts on $\widetilde{M}$ as a transitive group of transformations preserving the causal ordering of events, and allows a conformal metric $\widetilde{g}$ on $\widetilde{M}$ which is invariant under $\widetilde{SU}(2,2)$. It turns out that the reason for the causality violating property of the (global) finite special conformal mapping on M^4 is that this mapping is not a homeomorphism from M^4 onto M^4. The space $(\widetilde{M},\widetilde{g})$ has no constant curvature and so admits only 7 Killing vector fields, in contrast to Minkowski space M^4 which admits 10. In the above sense $\widetilde{M}$ is a physically acceptable non-compact Lorentz 4-manifold having the conformal group as its infinitesimal and global transformation group.

2.3 Conformal Group in Relativity & Gravitation.

The general relativity theory of Einstein is invariant under the infinite-parameter group of general coordinate transformations

$$x'^{\mu} = f^{\mu}(x^{\alpha}),$$

also called the general covariance group. A remarkable recent development was the recognition by Ogievetsky [57] in 1973 that the action of this group can be reduced to alternating actions of its two finite-parameter subgroups, namely, the special linear group SL(4,R) and the conformal group. Specifically, he proved that any generator of the general covariance group can be represented as a linear combination of repeated commutators of the generators of SL(4,R) and the conformal group. That is, the infinite-dimensional algebra of the general covariance group is the closure of the finite-dimensional algebras of the special linear and conformal groups.

In a more recent paper [58] Borisov & Ogievetsky deduced Einstein's field equations from the requirement of invariance under the affine and conformal groups realized non-linearly (the realization becoming linear on the Poincaré subgroup), in the same way that the equations of SU(2)xSU(2) chiral dynamics are derived in the theory of non-linear realizations of chiral symmetry [59]. They maintain that the analogy between the theory of the gravitational field and the essentially simpler theories of non-linear realizations of internal symmetries (unitary, chiral, etc.) suggests new ways of searching for connections between the theory of gravitation and the theory of elementary particles, although they do not discuss any of these "new ways".

Freund [60] has generalized these arguments to the case of a conformal graded Lie algebra (orthosymplectic algebra) with a non-linear realization over the imbedding superspace. This has possible applications in elementary particle physics.

In 1936 Page [61] developed a "new relativity" theory which was invariant not only under coordinate transformations between two observers moving relative to each other with constant velocity, but also with constant acceleration. He did not realize that this was just an extension of Einstein's special relativity in flat Minkowski space, based on the conformal group instead of the Poincaré group. This was pointed out by Engstrom & Zorn [62] who showed that Page's theory involved the determination of all coordinate transformations which preserve the null line element $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = 0$, a problem solved by Lie [[illegible]] long ago.

Page's theory was physically meaningful if all measurements were local; distances and times were to be determined by the radar principle (attributed principally to Milne [63]), and differed from point to point in the space-time manifold. Rest masses were no longer constant in Page's theory. Robertson [64] claimed that Page's kinematical theory could be regarded as a special case of his own (Robertson's) more general theory [65], and that Page's treatment should lead to the usual classical expression for the ponderomotive force. In reply, Page pointed out that he and Adams [61] had shown that the "new relativity" may lead to a "very different electrodynamical equation of motion". But just what this difference is is not very clear. Robertson also demonstrated that the relation between two equivalent observers in Page's theory is formally the same as that between two free observers in general relative motion in the de Sitter universe of general relativity theory. Bourgin [66] has also criticised Page's theory.

Hill [67] joined the fray in 1945 when he considered the problem of determining those coordinate transformations for which the uniformly accelerated motion (hyperbolic motion) of a particle is transformed into another motion of the same type, for both Newtonian and special relativistic mechanics. The symmetry group of the characteristic differential equation of hyperbolic motion is the conformal group of Minkowski space. Hill also applied the theory of uniformly accelerated motions based on the conformal group to the so-called "clock problem" of general relativity and found two solutions, both at variance with the usual theory.

The kinematical properties of the conformal transformation group have continued to cause problems in interpretation. As recently as 1971 Laue [68] gave examples to show that one of the SCTs transforms world lines of massive particles at rest into world lines of massive particles with different constant accelerations.

It has been noted earlier (Section 2.2) that conformal symmetry is broken in the presence of massive particles unless one hypothesizes that mass transforms under dilations and SCTs according to

$$m_0 = f(x)m_c,$$

where m_0 is the (usual) rest mass of the particle, m_c is a new conformally invariant mass, and $f(x)$ is a scale factor representing the conformal transformation. Some authors have interpreted m_0 as a rest energy (rather than rest mass) containing the potential energy associated with the position (x) of the particle in an apparent gravitational field ("apparent" because we are talking about conformal transformations in flat Minkowski space). Fulton, Rohrlich & Witten [29] asserted that such apparent gravitational fields are constant and homogeneous. This assertion is false because of the property of SCTs which takes world-lines of massive particles at rest into world-lines of particles with different constant accelerations, as emphasized by Laue. An earlier example of Rohrlich [69] supports this conclusion.

A belief which had become firmly established (see e.g. [70]) was that no radiation field is generated by the linear uniformly-accelerated motion of a point charge. This was repudiated by Rohrlich [71]. Under a SCT the coulomb field of a charge at rest is mapped into the field of a uniformly accelerated charge, as shown by Haantjes [72]. This latter field contains radiation.

By restricting the interpretation of the term "equivalent observers" in general relativity to mean those observers whose (local) coordinate systems are connected by a group of transformations which correspond to physical reality, e.g. Lorentz transformations, Ueno & Takeno [73] considered two kinds of "equivalent observer". In their first paper, the class of observers of the "second kind" is characterized by their coordinate systems being connected only by the three-dimensional rotation group when they are at relative rest; when these observers are in relative motion, it was shown that the group of transformations connecting them contained the SCTs. An extension of this theory was given by Ueno & Takeno in the second paper.

Meksyn [74] reminded us that transformations from rest frames to accelerated frames are, in general, singular on certain surfaces. There is a misprint in his paper, so the transformation of his example is spelt out here. The frame (x_0, y_0, z_0, t_0) is transformed to the frame (x, y, z, t) moving with constant relative acceleration α by

$$\begin{aligned} wx_0 &= (1+2wx)^{\frac{1}{2}} \cosh wt - 1, \\ y_0 &= y, \\ z_0 &= z, \\ wt_0 &= (1+2wx)^{\frac{1}{2}} \sinh wt, \end{aligned}$$

where $w = \alpha/c^2$ and c is the velocity of light. This transformation becomes singular on the surface $x = -c^2/2\alpha$, as can be seen by computing the Jacobian determinant of the transformation.

The interpretation of the SCTs as connecting coordinate frames with constant relative accelerations has been contested by Kastrup [42] on quantum mechanical grounds. The main objection appears to be that the group velocity of the wave packets formed by the eigenfunctions of the Hermitian operators (of the Lie subalgebra of SCTs) has the same form as that of plane waves, whereas the phase velocity shows the hyperbolic structure usually related to accelerated motions. In quantum mechanics it is the group velocity, not the phase velocity, which describes the motion of particles. Thus if, as he desires, the conformal group is to span both microscopic and macroscopic physics, Kastrup suggests that the interpretation of an accelerated motion be dropped. Another objection is that the Schouten-Haantjes postulate [37], that mass transforms locally under dilations and SCTs as on page 30 of this chapter, is at variance with the quantum mechanical view that 4-momenta, and therefore masses, are non-local quantities. Kastrup argues that "the formal local transformation of a non-local quantity has the consequence that no conservation law exists, at least not in the usual sense, as one can see from the Klein-Gordon equation with non-vanishing rest mass".

This apparent conflict over the kinematical interpretation of the conformal group seems unresolvable at present. As long as one believes that relativity can be quantized, there should be agreement over physical interpretation of the transformations of the conformal group if this group is to be accepted as a symmetry group of physics.

Imbedding techniques have been studied extensively in relativity theory (a fairly recent discussion is given in [75]). We note in passing that among the best results available are:

(1) Friedman's theorem [76]: An n-dimensional Riemannian space with line element $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ can always be locally and isometrically imbedded in an m-dimensional pseudo-Euclidean space with line element

$$ds^2 = (dz^1)^2 + \ldots + (dz^p)^2 - (dz^{p+1})^2 - \ldots - (dz^{p+q})^2,$$

where $m = p+q = \frac{1}{2}n(n+1)$ and neither p nor q is less than the number of positive or negative eigenvalues of $g_{\mu\nu}$ respectively. In the case of the Riemannian spaces of general relativity ($n = 4$ and $g_{\mu\nu}$ has signature -2) it is known that the imbedding pseudo-Euclidean space (a) has dimension 10 at most, (b) must be

at least 6-dimensional to imbed a non-flat vacuum space,
(c) must have at least 5 dimensions to imbed a non-vacuum solution.

(2) Nash's theorem [77]: A compact space-time can be globally imbedded in a pseudo-Euclidean space of at most 46 dimensions. The present ceiling is higher for a non-compact space-time.

Imbedding techniques have been used, for example by Sigal and Ingraham, in the application of the conformal group to relativity. See also Maia [78].

According to Sigal [79] a field theory is conformally invariant if solutions of the field equations are mapped into solutions of the field equations by the conformal group, assuming that the field equations and fields are tensor densities under a general change of coordinates. This assumption puts restrictions on the transformation properties of terms in the equations under the Minkowski space conformal group. In particular, it implies that all terms in the equations must have the same scale dimension. Sigal claims that it also makes questionable the introduction of conformally invariant masses (à la Schouten & Haantjes [37]). To support his claim he demonstrates that, under his definitions and assumptions, the massive Klein-Gordon equation is not conformally invariant.

Making use of the local isomorphism between SU(2,2) and SO(4,2), Sigal examines conformal invariance (in his sense) by studying the relationships between geometrical structures and field equations in the flat 6-space and an arbitrary 4-space conformal to Minkowski space. He then studies the imbedding of Minkowski space equations in the flat 6-space. He shows that, in terms of 6-space structures, there are only two ways to break conformal invariance, corresponding to the addition of mass-like terms and to coupling terms between the covariant derivatives of each geometry.

Ingraham's [41],[80] "conformal relativity" demands that physical laws be invariant in form under the full conformal group. He treats the conformal group as a group of projective transformations in "sphere space". The 4-dimensional space-time events are represented as null spheres in a curved 5-dimensional manifold, described by six homogeneous coordinates. At each point of this manifold there is a local flat 5-dimensional projective geometry. Equivalent observers are defined by the property that they are related transformationally by the conformal group i.e. point transformations between observers carry null curves into null curves. (This is an extension of the concept of equivalent observers introduced by Page [61], Ueno & Takeno [73], and Hill [67].) In each local space there is a quadric, and the group of transformations preserving this quadric (the quadric group) induces a group of point transformations in the local 4-dimensional space-time via the imbedding equations. The intersection of this last group and the group of conformal point transformations in the 4-space is called the physical group of the theory. The quadric geometry and physical geometry of Ingraham's conformal relativity denote the set of all geometric definitions and properties invariant under the quadric group and physical group respectively. The principal theorem of his theory is: The quadric geometry and physical geometry coincide, and the physical geometry is the conformally flat metric 4-space. This result had been established very much earlier by Klein [26] in 1872, so one wonders what the force of Ingraham's theory is.

He claims that his (Ingraham's) theory is a unified field theory in the sense that it describes force fields which comprise several mesons along with the gravitational and electromagnetic fields. The Kaluza relativity [81] and Einstein's general relativity are subtheories, according to Ingraham, and he also extends his theory to what he terms "spinor relativity". But Taub [82] disagrees with Ingraham's claim because "the theory of a perfect electrically neutral fluid moving in its own gravitational field does not seem to be contained within this theory". Ingraham does not appear to have replied to this criticism.

Takeno [83] has studied conformal transformations which transform spherically symmetric (s.s.) space-times into s.s. space-times. By definition a space-time is s.s. when its metric tensor $g_{\mu\nu}$ is form-invariant under the 3-dimensional group of space rotations. The metrics discussed by Takeno do not necessarily satisfy Einstein's field equations. The main theorems are:

(a) Let S_0 be any s.s. space-time with metric tensor $g_{\mu\nu}$ which is not conformal to the space-time S_{IIA} with metric

S_{IIA}: $ds^2 = C(r,t)dt^2 - A(r,t)dr^2 - B(d\theta^2 + \sin^2\theta\, d\phi^2)$, where B is a constant. If $S^*(g^*_{\mu\nu})$ be related to S_0 by $g^*_{\mu\nu} = e^{2f(x^\alpha)} g_{\mu\nu}$, then S^* is s.s. <u>iff</u> $f = f(r,t)$.

(b) The metric of any conformally flat s.s. space-time is reducible to the form

$$ds^2 = A\{\lambda dt^2 - (dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2)\},$$

where $A = A(r,t)$ and $\lambda = [a(t) + r^2 b(t)]^2$, and a,b are arbitrary functions of t.

Takeno also determined the groups of infinitesimal coordinate transformations which leave any given s.s. metric conformally invariant. He obtained the specific forms of the conformal Killing vectors (CKVs) for the Robertson-Walker metric in the form

$$S(L): \quad ds^2 = dt^2 - f^2(dx^2 + dy^2 + dz^2), \quad f = e^{g(t)}(1 + r^2/4R^2)^{-1},$$

where $r^2 = x^2 + y^2 + z^2$, and R = constant. Now the Robertson-Walker metric is conformally flat, so the CKVs are just those of flat Minkowski space. But Takeno saw fit to obtain the CKVs directly from the R-W metric S(L). This was followed by a specific determination of CKVs in general s.s. space-times with metrics of the form

$$S_I(0): \quad ds^2 = C(r,t)dt^2 - A(r,t)dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2;$$

$$S_I(\Delta): \quad ds^2 = 2D(r,t)dr\, dt - B(t)(d\theta^2 + \sin^2\theta\, d\phi^2), \quad dB/dt \neq 0;$$

$$S_{II}: \quad ds^2 = C(r,t)dt^2 - A(r,t)dr^2 - B(d\theta^2 + \sin^2\theta\, d\phi^2), \quad B \text{ const.} > 0.$$

<u>Note</u>: At each point of the space-time S_{II} the tangent space is composed of two 2-dimensional subspaces $V_2(r,t)$ and $V_2(\theta,\phi)$ whose metrics are

$$V_2(r,t): \quad ds^2 = C\, dt^2 - A\, dr^2,$$
$$V_2(\theta,\phi): \quad ds^2 = - B(d\theta^2 + \sin^2\theta\, d\phi^2).$$

The 2-space $V_2(\theta,\phi)$ has constant curvature -1/B. When the 2-space $V_2(r,t)$ also has constant curvature, the space-time S_{II} is said to be of class S_{IIA}.

The theorems proved by Takeno will not be restated here. The following table, adapted from one given by Takeno, indicates in the final column the space-times of particular interest because they contain CKVs which are not Killing vectors. The notation is as defined above, plus

$$\left.\begin{array}{ll} S(J_1): & ds^2 = dt^2 - A(t)\, dr^2 \\ S(J_2): & ds^2 = C(r)\, dt^2 - dr^2 \end{array}\right\} \text{non-constant 2-curvature.}$$

Space-time type	Remarks	Dim. of conformal group	No. CKVs not Killing vect.
S(L)	Conformally flat	15	5
$S_I(0)$	(i) Conformally flat	15	5
	(ii) Conformal to $S_{II}A$	6	≤ 3
	(iii) Static, $A=A(r)$, $C=f(r)g(t)$	4	0
	(iv) $A = f(r)g(t)$, $C = r^2h(t)$	4	1
	(v) Others	3	0
$S_I(\Delta)$	(i) Conformally flat	15	5
	(ii) Conformal to S_{IIA}	6	≤ 3
§	(iii) $B/D = e^h/uv$	4	1
	(iv) Others	3	0
S_{II}	(i) Conformally flat	15	5
	(ii) S_{IIA} not conformally flat	6	0
	(iii) $S(J_1)$ or $S(J_2)$	4	0
	(iv) Others	3	0

§ In $S_I(\Delta)$ (iii) u and v are arbitrary functions of r and t, $h = h(W)$, $W = U+V$, $U = \int udr = U(r)$, $V = \int vdt = V(t)$.

Ignoring the conformally flat cases, we see that the only s.s. metrics admitting conformal transformations which are not motions are $S_I(0)$(ii), (iv) and $S_I(\Delta)$(ii), (iii). The exterior Schwarzschild space-time is of class $S_I(0)$ and is static, so possesses no CKVs other than Killing vectors.

Any s.s. geometry admits a shear-free null hypersurface. The same will be true of any conformally related geometry which is s.s., since the vanishing of shear is a conformally invariant property [118]. Derry, Isaacson & Winicour [84] showed that the only "regular" conformally s.s. solution to the vacuum Einstein field equations is the Schwarzschild solution. "Regular" here means asymptotically flat plus the following restriction: "The shear-free null hypersurfaces of the s.s. geometry form a regular (sic!) diverging family of hypersurfaces with topology $S^2 \times E^1$ such that consecutive members do not intersect in the neighbourhood of future null infinity". Their proof did not apply to Petrov type N vacuum space-times.

Conformally flat spaces. These admit the full 15-parameter conformal group. Well-known conformally flat spaces are:

(1) The Schwarzschild interior solution [85].

(2) The de Sitter space-time [86].

(3) The Einstein space-time [87].

(4) The Robertson-Walker metric [88].

(5) The Friedmann space-times [89].†

Conformally flat spaces are amongst the easiest to deal with on account of the vanishing of the Weyl tensor in these spaces (Weyl [26], 1918).

Schouten (1921), Schouten & Struik (1921), and Brinkmann (1923) [26] were early workers on questions of conformal flatness. In particular, Schouten proved that a Riemannian n-space is conformally flat iff

$$R_{\alpha\beta\gamma\delta} + g_{\beta\delta}M_{\alpha\gamma} + g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\delta}M_{\beta\gamma} = 0 \tag{1}$$

and

$$M_{\alpha\beta;\gamma} - M_{\alpha\gamma;\beta} = 0 \tag{2}$$

for some symmetric tensor $M_{\alpha\beta}$. For $n > 3$, (1) implies (2).

Levine (1936) [90] determined the necessary and sufficient condition for a conformally flat V_n to admit a group of motions, in terms of the rank of a certain matrix involving derivatives of the conformal factor in the metric of V_n. He also proved that every conformally flat space with metric of the form

$$ds^2 = e_\alpha (dx^\alpha)^2/f(R), \quad R = e_\alpha (x^\alpha)^2, \quad e_\alpha = \pm 1$$

admits the rotation group as a group of motions; if other groups of motions are admitted by such a space, then $f(R) = aR$, or the space has constant curvature and

$$f(R) = (aR + b)^2,$$

where a and b are constants. Other results given by Levine are:
If a conformally flat space admits a simply transitive group of translations, the space is flat.
A given simply transitive group G_r will be a group of motions of a conformally flat space iff G_r is a subgroup of the 15-parameter conformal group.

In a sequel Levine (1939) tabulated the 20 types of subgroups of the conformal group that can serve as a group of motions of a conformally flat Riemannian n-space into itself, and gave the form of the metric of each of the admissible spaces.

† In fact, any isotropic space-time is conformally flat [Penrose, in Relativity, Groups & Topology, p.580 (Gordon & Breach, 1963)]

Later (1950) Levine proved that a conformally flat, non-flat Riemannian space M can admit at most one (linearly independent) field of parallel vectors, which if non-null implies that the universal covering space $\tilde{M}$ of M is the product of a one-dimensional Riemannian space and a (n-1)-dimensional Riemannian space, and hence M has constant curvature (see also Yano & Nagano [91]).

In their 1969 paper Katzin, Levine & Davis [20] considered curvature collineations (see Section 1.3) and proved that if a conformally flat Riemannian n-space ($n > 2$), which is also a non-flat Einstein space, admits a curvature collineation (CC), then the CC must be a motion. In a subsequent paper these three authors (1970b) began the task of determining all conformally flat Riemannian n-spaces (not Einstein spaces) which admit a CC. This paper was devoted to CCs which are not conformal motions (Conf M). Levine & Katzin (1970) [20] continued the story for the case when a CC is a Conf M. They proved *inter alia* that there are no non-flat conformally flat spaces which admit a symmetry which is simultaneously a CC and a proper Conf M with $\psi_{;\mu}$ a non-null vector, where the function ψ is given by (1.15). They also determined the nature of the various non-flat conformally flat spaces which admit a S Conf M and a null field of parallel vectors for each of the three canonical forms of the metric [92]; the group of S Conf Ms was specified in each case.

Conformally flat spaces admitting special quadratic first integrals (SQFIs) i.e. covariant-constant symmetric tensor fields $h_{\mu\nu}$, were the subject of two papers by Levine & Katzin [92]. Among their results was the following: A conformally flat Riemannian n-space C_n which is (i) flat, or (ii) of non-zero constant curvature, admits (i) $\frac{1}{2}$ n(n+1), (ii) one SQFI respectively. If a C_n of non-constant curvature admits more than one linearly independent SQFI, then it admits exactly two. Canonical forms of the metric were obtained for these latter C_n.

Ruse [93] mentioned conformally flat 4-spaces as a very special case in his discussion of a quadratic complex of lines in an (n-1)-dimensional projective hypersurface of an n-dimensional Riemannian space.

Narlikar & Karmarkar [94] used 14 independent scalar invariants of the curvature tensor in a Riemannian V_4 to establish the necessary and sufficient conditions that a spherically symmetric V_4 be conformally flat.

Taub [7] gave the first proof of the theorem: A Riemannian V_n $(n \geq 3)$ admits a $\frac{1}{2}(n+1)(n+2)$-parameter group of infinitesimal conformal transformations iff the V_n is conformally flat. Sasaki [95] had previously established part of this result.

Adati [96] studied Riemannian spaces V_n admitting a family of totally umbilical hypersurfaces, when the hypersurfaces are conformally flat. Also included was the proof that when a conformally flat V_n $(n > 3)$ admits a torse-forming vector field $\sigma,_\alpha$ the hypersurfaces $\sigma(x^\mu)$ = const. are of constant curvature.

Matsumoto [26] (1951) proved that a positive definite conformally flat n-space $(n \geq 4)$ with non-constant curvature is of imbedding class one iff a certain matrix is of rank ≥ 2 and certain inequalities are satisfied; the matrix and the inequalities involve only $g_{\mu\nu}$ and $R_{\alpha\beta\gamma\delta}$. A new proof of a theorem of Brinkmann [26] (1923) was given, namely, the imbedding class of a conformally flat space is at most two.

Verbickii [97] developed a criterion, involving the second fundamental form of a positive definite Riemannian V_n $(n \geq 4)$, for the V_n to be conformally flat of imbedding class one.

Vranceanu [98] had proved that in a sufficiently small neighbourhood of a Riemannian subspace V_n, the metric of the enveloping Euclidean space E_N may be written in terms of the fundamental tensors of the first, second and third kinds and the torsion of the V_n. Blum [99] generalized this result to the case where the enveloping space is conformally Euclidean.

Using properties of an imbedding in a flat 6-dimensional space, Stephani [100] found all solutions of the Einstein equations for a perfect fluid or an electromagnetic field, which are conformally flat. These solutions include the metrics of Bertotti [101].

Space-times of imbedding class one have also been studied by Barnes [102] who showed that a class one perfect-fluid space-time has at least one of the properties (a) conformal flatness, (b) geodesic flow, (c) it admits a 3-dimensional group of isometries with 2-dimensional spacelike paths.

The conformally flat nature of gravitational fields in perfect fluids has also been discussed by Obozov [103] who proved that geodesic flow lines imply the conformal flatness of the space-time (cf. Barnes).

Sigal [79] generalized his discussion of conformal invariance to the case where the 4-space field equations are imbedded in a conformally flat 6-space (see also page 33 of this chapter).

Kloster, Som & Das [104] investigated the class of conformastationary vacuum metrics; these are the stationary metrics whose background 3-space is conformally flat. They found that there are only three such metrics, one of which is the NUT solution, which belong to the Papapetrou-Ehlers class [105]. Their proof that there cannot be any metrics outside of this class was faulty, and this possibility still remains open.

The initial-value problem of general relativity is that of constructing a complete set of Cauchy data on a spacelike hypersurface for Einstein's equations. These data are subject to initial constraints. O'Murchadha & York [106] treated the special case of conformally flat metrics on the initial spacelike hypersurface.

McLenaghan, Tariq & Tupper [107] employed the Newman-Penrose formalism to obtain a derivation of the most general conformally flat solution of the source-free Einstein-Maxwell equations for null electromagnetic fields. The metrics are of the form

$$ds^2 = 2q^2(u)z\bar{z}du^2 + 2\,du\,dr - 2\,dz\,d\bar{z},$$

which are the conformally flat members of the exact plane-wave family of solutions of the Einstein-Maxwell equations.

Peters [108] solved the equation of geodesic deviation in conformally flat space-times in a covariant manner. This enables one to express, say, the derivatives of the parallel propagator in terms of other geometrical quantities, independent of any particular coordinate system. The solution is given by Peters as an integral equation for general geodesics.

Zund [109] gave an example of a conformally flat space-time with recurrent curvature which represents pure radiation in the sense of Lichnerowicz [110]. Then Levine & Zund [111] extended this discussion and gave, in particular, the theorem: A conformally flat space-time in which there is a null parallel vector field which is a gradient represents pure radiation.

Hall [112] proved that a null electromagnetic field in a conformally flat space-time must necessarily be expansion-free and twist-free. This result also holds for null electromagnetic fields whose repeated principal null direction coincides with the repeated principal null direction of the Weyl tensor of a space-time of Petrov type N. This was an elegant extension of the work of Levine & Zund [111].

Based upon the paper of Dirac [32], a new form of conformally invariant wave equation was studied by Sokolik [113], and the gravitational interaction with respect to conformally flat space was investigated.

Pavelle [114] and Thompson [115] have discussed the Kilmister-Yang equations [116] (a K-Y space is a Riemannian V_n which satisfies locally $R_{\alpha\beta;\gamma} = R_{\alpha\gamma;\beta}$). Pavelle argued that conformally flat solutions should not be allowed (are unphysical). Thompson proved some theorems on conformally flat spaces, including:
(a) For $n \geq 4$, every conformally flat V_n with constant scalar curvature is a K-Y space. (b) The class of conformally flat solutions

of the K-Y equations is determined by solutions of $\Box^2 p = \frac{1}{6} Rp^3$, where R is the constant scalar curvature of V_4 with metric

$$ds^2 = p^2 \, d\sigma^2, \quad d\sigma^2 \text{ the Minkowski space metric,}$$

and $\Box^2$ is the D'Alembertian operator. When R = 0, the conformally flat spaces are determined by solutions of the wave equation.

Curvature collineations. A theorem due to Collinson [117] states that the only CCs admitted by an empty (Einstein) space-time, not of Petrov type N, are conformal motions. Collinson also found the CCs admitted by the plane-fronted gravitational waves, and this work showed that empty space-times of Petrov type N do admit CCs which are not conformal motions. He proved that the plane-wave metrics admit a 6-dimensional group of conformal motions consisting of a 5-dimensional group of motions and a 1-dimensional group of homothetic motions. In particular, he showed that the metric

$$ds^2 = 2 \, du \, dv - 2 \, dz \, d\bar{z}$$

is a plane-fronted gravitational wave which admits the homothetic motion given by the homothetic Killing vector (HKV)

$$\tilde{K} = 2u\partial_u + z\partial_z + \bar{z}\partial_{\bar{z}} \, .$$

Collinson appears to have a sign wrong in his expression for this HKV.

Katzin, Levine & Davis [20] (1970a) showed that every Riemannian V_n with an expansion-free, shear-free, rotation-free geodesic congruence admits groups of CCs. In particular, the plane-fronted gravitational waves with parallel rays (pp-waves) do.

Geometrical and physical properties of the Weyl tensor were outlined by Pirani & Schild [118] in the space-time of general relativity. In particular, they gave conformally invariant definitions of null geodesics and shear (see Chapter 4 and Sachs [204]) and gave an equation which related the propagation of shear along a null geodesic to the Weyl tensor.

The Goldberg-Sachs theorem ([209], see Chapter 4) is deservedly one of the most celebrated in general relativity theory. Very soon after its discovery the generalized Goldberg-Sachs theorem was established by Kundt & Thompson [119]. This states:

(i) Any two of the following properties imply the third:

(A) The Weyl tensor C_{abcd} is algebraically special, with Debever vector k^a.

(B) There exists a shear-free geodesic null congruence with k^a as tangent.

(C) $V^{ea}V^{bc}C^{d}{}_{abc;d} = 0$

or $V^{bc}C^{d}{}_{abc;d} = 0$ if C_{abcd} is Petrov type III,

or $V^{ea}C^{d}{}_{abc;d} = 0$ if C_{abcd} is Petrov type N,

where $V_{ab}k^{b} = 0$ and V_{ab} is a null complex bivector.

(ii) The properties (A), (B), (A)∪(C) are conformally invariant. This theorem was proved independently by Robinson & Schild [120] who published their proof about five months later. Kundt & Thompson used spinors, while Robinson & Schild used tensorial methods to prove the theorem. The fact (ii) of conformal invariance was fully established by the latter two authors, while the former two merely noted it.

Szekeres [121] studied conformal transformations of a subset of the following hierarchy of Riemannian 4-spaces:

(a) C-spaces, characterized by $C^{\mu}{}_{\alpha\beta\gamma;\mu} = 0$,

(b) J-spaces, $R^{\mu}{}_{\alpha\beta\gamma;\mu} = 0$,

(c) Einstein spaces, $R_{\mu\nu} = \lambda g_{\mu\nu}$, λ constant

(d) empty spaces, $R_{\mu\nu} = 0$,

(e) flat spaces, $R_{\mu\alpha\beta\gamma} = 0$.

where $R_{\mu\alpha\beta\gamma}$ and $C_{\mu\alpha\beta\gamma}$ are the Riemann and Weyl tensors. Szekeres obtained necessary and sufficient conditions for a space to be conformal to a space of type (a), (c), or (d), using spinors. J-spaces (b) were considered by Thompson [122]. [Note: C-spaces and J-spaces are special cases of conformally symmetric spaces (see p.22) and Cartan symmetric spaces [123] respectively.]

Debever [124] used the local isomorphism between the Lorentz group and the 3-dimensional orthogonal complex rotation group SO(3,C) to develop a vectorial representation of bivectors, in particular of the curvature form and connection form. The formalism was applied to conformal changes of the metric, and in particular to specify the choice of conformal parameter along isotropic geodesics. Included was a proof of the Pirani-Schild formula [118] on the propagation of the shear along a null geodesic.

The formalism developed by Debever was described more fully by Cahen, Debever & Defrise [125]. They showed, for example, how the formalism is related to the usual spinor representation. After a characterization of the Petrov types of the Weyl tensor, they introduced suitable canonical triads and computed the curvature invariants. The conformal theory developed in [124] was re-presented, and concise proofs of the Robinson theorem [126] for null electromagnetic fields and the Goldberg-Sachs theorem were given. All homogeneous 4-space solutions of the vacuum field equations were constructed; these are the spaces on which a group of isometries acts transitively. It was shown that there exist no homogeneous spaces of Petrov types II, D, and III. They also studied spaces admitting "large" groups of isometries (when the stability group at each point of the space is not reduced to the identity).

Mielke [127] has reviewed conformal techniques as applied to the initial-value problem of general relativity (see e.g. [106]) and has analyzed conformal vector fields on compact manifolds with constant scalar curvature.

Hansen & Winicour [128] found a conformal Killing vector field on the manifold of trajectories of the timelike Killing vector field of the Kerr solution [129], and showed that the conformal Killing vector field leads to a conserved quantity along certain null geodesics.

Collinson & French [9] wrote the conformal Killing equations (1.15) and their first order integrability conditions (1.27), (1.34) and (1.35) in the Newman-Penrose formalism and proved the following two important theorems:

(1) A conformal motion of non-flat empty space-time must be homothetic, unless the space-time is Petrov type N with hypersurface-orthogonal (twist-free) geodesic rays.

(2) For each Petrov type the maximum order of the group of conformal motions admitted in non-flat empty space-time is at most one greater than the maximum order of the group of isometries.

For non-flat empty space-times of Petrov type I a result stronger than (2) is available: the maximum orders of the groups of conformal motions and isometries are equal. These theorems are improvements on the results of Shanks [26] (1950) and Yano [26] (1951). Collinson & French also determined the maximal groups of isometries admitted by empty space-times of each Petrov type which possess hypersurface-orthogonal geodesic rays with non-vanishing divergence (i.e. expanding).

Einstein [130] used similarity solutions of his vacuum field equations (such a solution admits homothetic motions) to demonstrate that there are no gravitational solutions without singularities that represent particles of finite non-vanishing total mass.

Cahill & Taub [131] discussed similarity solutions of the Einstein equations for a spherically symmetric distribution of a self-gravitating perfect fluid. It was found that the metric coefficients of such solutions depend essentially on a single variable (the ratio of the radial coordinate and the time coordinate). The field equations then reduce to ordinary differential equations. They also treated the problem of fitting a similarity solution to another solution of the field equations across a shock wave (hypersurface).

In a paper for the Synge festschrift [132] Taub obtained similarity solutions for self-gravitating perfect fluids with a particular equation of state, possessing plane symmetry i.e. admitting a 3-parameter group of motions of the Euclidean plane. In contrast with the spherically symmetric case [131] Taub showed that a certain class of such similarity plane-symmetric space-times cannot be fitted to a static plane-symmetric space-time across a timelike shock. The results obtained were applied to similarity solutions of the equations of special relativistic hydrodynamics.

Godfrey [133] classified static, axisymmetric vacuum metrics (Weyl metrics) according to the homothetic motions they admit. Of all types of collineation (see Section 1.3) the Weyl metrics admit only the two simplest, namely, homothetic motions and isometries. Godfrey discovered two then unknown families of space-times, none of which is asymptotically flat. This would seem to make them physically uninteresting, but many possess interesting horizons which Godfrey has investigated in detail. The Weyl metric may be written in the form [134]

$$ds^2 = e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2\, e^{-2\lambda} d\phi^2 + e^{2\lambda} dt^2,$$

where λ and ν are functions of r and z. It admits two orthogonal commuting Killing vectors K_1 and K_2, one of which is spacelike and the other timelike. Godfrey determined all Weyl metrics admitting a group G_n of homothetic motions ($n \geq 3$) with this G_2 of isometries as a subgroup. He divided the metrics into three classes:

Class I. The symmetry group is G_3. There exists a homothetic Killing vector (HKV)

$$\tilde{K}_1 = \partial_z + 2a\phi\partial_\phi + 2(a-1)t\,\partial_t,$$

where a is a constant, besides the two Killing vectors

$$K_1 = \partial_z, \qquad K_2 = \partial_\phi.$$

The functions λ and ν take the form

$$\lambda = a \log r + z,$$
$$\nu = a^2 \log r + 2\,az - \frac{1}{2}r^2 + c_0, \qquad c_0 \text{ constant.}$$

(McIntosh [141] has quoted this metric, but there is a misprint in his paper.)

Class II. The symmetry groups are (i) G_3, (ii) G_4, (iii) G_5, (iv) G_{11}. Each is characterized by the presence of one HKV

$$\tilde{K}_2 = r\partial_r + z\partial_z + ab\phi\partial_\phi + \left(\frac{a-1}{b-1}\right)t\partial_t,$$

where a, b are constants. The dimensionality of the groups is made up by the Killing vectors. The functions λ and ν take the form

$$\lambda = b \log r + \frac{1}{2}(a-b)\log[z + (r^2+z^2)^{\frac{1}{2}}],$$
$$\nu = b^2\log r + \frac{1}{2}(a^2-b^2)\log[z + (r^2+z^2)^{\frac{1}{2}}] - \frac{1}{2}(a^2-b^2)\log(r^2+z^2) + c_0,$$

where c_0 is a constant.

(i) The symmetries are given by $\tilde{K}_2$, K_1 and K_2. The cases a = b, and a = 0,1 and b = 0,1 simultaneously are excluded. This class includes the C-metric (see Ehlers & Kundt [135]) given by a = 2, b = -1 (or a = -1, b = 2) admitting

$$\tilde{K}_2 = r\partial_r + z\partial_z - 2\phi\partial_\phi - \frac{1}{2}t\partial_t.$$

(ii) Besides K_1, K_2 and $\tilde{K}_2$ there is a third Killing vector

$$K_3 = \partial_z.$$

Here a = b, but $a \neq -1, 0, \frac{1}{2}, 1, 2$. The metrics are Levi-Civita's cylindrically symmetric static fields [136].

(iii) Besides K_1, K_2, K_3 and $\tilde{K}_2$ there is another Killing vector which takes different forms depending upon values of the constants a and b:

$$a = b = -1, \qquad K_{41} = \phi\partial_z - z\partial_\phi,$$
$$a = b = \frac{1}{2}, \qquad K_{42} = t\partial_\phi + \phi\partial_t,$$
$$a = b = 2, \qquad K_{43} = t\partial_z + z\partial_t.$$

All metrics in this case are Petrov type D and have been discovered by Levi-Civita [136], Petrov [8], and Kasner [137].

(iv) When a = b = 0 and a = b = 1 the metric is that of flat Minkowski space, which admits the maximal group of 10 Killing vectors and 1 proper HKV ($\tilde{Y}_2$).

Class III. The symmetry group is G_4 and comprises 4 Killing vectors. The metrics are the Schwarzschild metric (A1) and three others closely related to it (A2, B1, B2). All are Petrov type D. The notation in parentheses is that of Ehlers & Kundt [135]. The ray congruences of the metrics A1, A2 and the C-metric are hypersurface-orthogonal and diverging; those of B1 and B2 and hypersurface-orthogonal and non-diverging.

Godfrey also gives the Bianchi type for each of the metrics listed above.

Sigal [138] proved that the only vacuum Einstein space which admits a timelike proper (i.e. non-isometric) homothetic motion with hypersurface-orthogonal trajectories is flat.

Eardley [139] discussed the nature and uses of self-similarity (homothetic transformations of a space-time into itself) in general relativity. Amongst other results he showed that the evolution equations of the initial-value problem preserve a self-similarity of initial data; he seems to have been unaware of the work of York [140] in this area. His main result prior to an application of self-similarity to cosmological models (see Section 2.4) is:

Each space-time (M,g) with non-trivial (i.e. non-isometric) homothetic group H_n of dimension n and isometry subgroup $G_m \subseteq H_n$ has the properties:

(1) The commutator of two homothetic vectors in H_n is a Killing vector in G_m, and m = n-1.

(2) Either (a) (usual case) (M,g) is conformally related to another space-time $(M,\tilde{g})$ with isometry group $\tilde{G}_r$ such that $H_n \subseteq \tilde{G}_r$, and dim $G_m(p)$ = dim $H_n(p)$-1, where $H_n(p)$ denotes the orbit of $p \in M$ under the action of H_n, etc.;

or (b) (exceptional case) (M,g) is a (vacuum or non-vacuum) plane-wave space-time.

The part of this result due to Eardley appears to be his listing in an Appendix of the exceptional plane-wave space-times of (2)(b).

Part (1) is due to Yano [26] (1951) and Collinson & French [9], and (2) is due to Defrise-Carter [26] (1969, 1975).

McIntosh [141] studied properties of homothetic motions in general relativistic space-times with particular reference to vacuum and cosmological perfect-fluid space-times. Using the formalism employed by Debney [142] he corrected and extended Debney's results to the case when a homothetic bivector (HBV) with components $H_{[a;b]}$ formed from the homothetic vector $\underline{H} = H_a$ and interpreted as a test electromagnetic field for any Killing vector field, was present. He also showed that quite strong restrictions are placed on the nature of the proper homothetic motions admitted by vacuum space-times. These are summarized in his theorem:

Non-flat vacuum space-times can admit a non-trivial homothetic vector field $\underline{H}$ only if such a vector field is non-null. $\underline{H}$ has either

(a) a non-null HBV in which case $\underline{H}$ is not hypersurface-orthogonal, is not tangent to a geodesic, is shear-free and has constant expansion, or

(b) a null HBV in which case the space-time is necessarily Petrov type III or N.

Case (b) is contained within the result obtained by McIntosh in a second paper [141]: If a non-flat vacuum space-time admits a homothetic vector field $\underline{H}$ (trivial or non-trivial) with an associated null HBV, then the space-time is algebraically special.

Other contributors to the application of the conformal group in general relativity and gravitation are Buchdahl [143] on a set of equations derived from a conformal invariant which possess solutions which are conformally-Einstein spaces; Popovici [144] who obtained conformally-invariant gravitational and electromagnetic field equations from a variational principle based on a particular Hamiltonian; Erez & Rosen [145] who displayed explicitly a conformal mapping between a given static axially-symmetric metric and the Schwarzschild (exterior) solution; McLenaghan & Leroy [146] on conformally recurrent space-times; Scheurer [147] on a 5-dimensional space-time admitting the conformal group; Schnirman & Oliveira [148] on conformal invariance of the equations of motion in curved spaces; Soleimany [149] on generalized conformal transformations of space-times and the construction of boundaries for past and future timelike infinities; Agnese & Calvini [150] who investigated the consequences of conformal invariance of the matter Lagrangian; Englert, Gunzig, Truffin & Windey [151] on conformal invariance with a dynamical symmetry breakdown; Ross [152] who produced a scalar-tensor theory of gravitation with field equations "conformally equivalent to the vacuum Einstein equations"; and Barut & Komy [153] on conformally invariant action-at-a-distance electrodynamics.

2.4 Cosmological Applications.

By 1940 spatially isotropic, homogeneous relativistic cosmology had become well established, resting on the foundation laid by Einstein [87], and with important contributions from de Sitter [86], Friedmann [89], Lemaitre [154], Milne [63], Robertson, McCrea, Tolman, Eddington, and McVittie [155].

Robertson [155] introduced the metric, named for him and Walker, in 1929 and developed the kinematics of a cosmological theory with this space-time metric in 1935. In his critical appraisal of Page's relativity [61], Robertson showed [64] how his kinematical theory of 1935 could be seen to contain Page's relativity.

Wave geometry was the subject of attention of the Hiroshima school earlier this century, and was applied to cosmology. The basic ideas of wave geometry as developed by Sibata, Takeno, Itimaru & Iwatsuki [156] are the adoption of a "microscopic metric" involving a set of Dirac matrices γ_i and a spinor ψ (to be interpreted analogously to the Dirac wave functions) which is required to satisfy a certain constraint equation. The theory is applied to cosmology by requiring that this constraint equation for ψ be completely integrable. Sibata showed that under these conditions the constraint equation took one of two forms. Takeno obtained the solutions of each of these forms and showed that the only types of universe allowed by the wave geometry are the Einstein and de Sitter universes. Itimaru computed the mass of the universe in terms of the integral of the time component of the velocity vector of the cosmological fluid.

Later Takeno [83] (1955,1966) determined the infinitesimal conformal transformations admitted by the Robertson-Walker metric. This metric is conformally flat and so admits the full conformal group of Minkowski space-time. But Takeno calculated the conformal Killing vectors directly from the R-W metric.

Infeld & Schild [157] believed that cosmological theory could provide a link between special and general relativity, and adopted an approach which was different both in spirit and in detail to the state of the art as summarized very elegantly in Robertson's Rev. Mod. Phys. article of 1933 [155]. The Robertson model involved a metric expressed in comoving coordinates; the problem of the motion of the fundamental particles (galaxies) disappeared and the model was characterized by the curvature of the 3-spaces of constant cosmological time. In the Infeld-Schild cosmology (1945) the metric was interpreted as differing from the Minkowski metric only by a gauge factor which determined the behaviour of clocks and measuring rods from point to point. The metric was taken to be

$$ds^2 = F(t,r)(dt^2 - dx^2 - dy^2 - dz^2), \quad r^2 = x^2 + y^2 + z^2,$$

and the structure of a 3-space did not enter the picture; the type of motion of the fundamental particles became the characteristic of the model. They showed that three types of fundamental particle motion were possible: (I) oscillatory, (II) radially convergent-divergent motion, and (III) rest. To each allowable form of $F(r,t)$ there corresponded at least (and, in general, exactly) one kind of motion. The type (I) motions occurred in a model with a metric invariant under spatial rotations, type (II) $\longleftrightarrow$ Lorentz transformations, type III $\longleftrightarrow$ spatial translations and inversions. The correspondence between the Robertson and Infeld-Schild models is summarized as follows:

Robertson 3-space curvature	Infeld-Schild type of fundamental motion
1	I oscillatory
-1	II convergent-divergent
0	III rest.

In Robertson coordinates the velocity of light is a function of position and direction, in Infeld-Schild coordinates it is constant. Although the Infeld-Schild models are metrically equivalent to Robertson's, there is a topological difference.

Infeld & Schild appear to have been unaware that Schouten & Haantjes [31] noticed earlier that Maxwell's equations take the same form in both Minkowski and conformally-Minkowski space. In a second paper (1946) Infeld & Schild addressed themselves to the question of whether the form of Dirac's equation for an electron depended on the choice of their conformal metric function F(t,r); their conclusion was that it didn't.

Gürsey [158] produced a theory of gravitation in conformally flat space-time. This theory is well at odds with existing experimental evidence, because it predicts that there is no bending of light rays and that the perihelion advance is half that of general relativity. Gürsey used this theory to build an expanding steady-state universe with continuous creation of matter.

Littlewood [159] pointed out that the most general system in which Einstein's velocity-addition formula is valid is a Riemannian V_4 with signature ±2 subject to an arbitrary conformal transformation. If space-time is conformally flat with zero scalar curvature, Littlewood showed that the wave equation for the gravitational potential in Einstein's theory must be replaced by $\Box^2 e^{\phi} = 0$, where e^{ϕ} is the conformal factor of the metric. Littlewood applied his theory to a uniformly expanding world model and gave an upper limit to the maximum speed of recession of a galaxy as 7/8 the speed of light. Pirani [160] showed that Littlewood's theory leads to a perihelion advance 1/6 that of general relativity, with opposite sign. This makes the theory untenable.

In two later papers [159] (1955, 1956) Littlewood employed his method of conformal transformations to determine the properties of uniformly expanding world models with zero pressure, first when the perfect cosmological principle was assumed, and second when the restricted form of this principle was assumed. To conform with Einstein's theory an anisotropic cosmological term had to be added to the equations in the first case. No new solutions were obtained, the only interest being in the simple method used to solve the cosmological equations.

Bhattacharya [161] investigated an expanding conformally flat, zero pressure, perfect-fluid universe.

Tauber & Weinberg [162] discussed the significance of a statistical mechanical theory of gravitational equilibrium of masses in connection with possible general relativistic effects in white dwarf stars. The assumption of local dynamical isotropy restricts the velocity field v^μ of a perfect fluid. The flow pattern is such that successive 3-dimensional spacelike hypersurfaces normal to v^μ must differ essentially only by local constant magnification i.e. there exists a subgroup of homothetic motions transitive on these 3-spaces.

Ehlers, Geren & Sachs [163] examined the gravitational field generated by a gas whose one-particle distribution function obeys Liouville's equation (Boltzmann equation without collision term) assuming the distribution to be locally isotropic in momentum space with respect to some timelike velocity field v^μ, and the gas to be irrotational if the rest-mass of the particles is zero. Thus their methods and results are extensions of the work of Tauber & Weinberg [162]. They (Ehlers et al) showed that the model is either stationary or a Robertson-Walker model. Their conclusion is that in general relativistic cosmology the restricted cosmological principle and the Weyl geodesic postulate "can both be considered as consequences of the apparently weaker postulate of an isotropic distribution of peculiar velocities" of particles near each event in the universe. Among their results is the following, due originally to Ehlers [164]:

> Let v^μ be the tangent vector to a congruence of timelike curves, such that $v_\mu v^\mu = -1$. Then the curves are the orbits of a 1-dimensional (local) group of conformal mappings of space-time into itself iff the congruence is shear-free and satisfies
>
> $$V_{\mu,\nu} = V_{\nu,\mu}, \qquad V_\mu = v_{\mu;\lambda}v^\lambda - \frac{1}{3}\theta v_\mu, \qquad V_\mu = U_{,\mu}$$
>
> where $\tilde{g}_{\mu\nu} = e^{-2U}g_{\mu\nu}$ is the new metric under the conformal mapping, and θ is the expansion.
>
> The generators of the group are $X^\mu = e^U v^\mu$.
>
> If $\theta = 0$, the mappings are isometries.

Developing these investigations further, Trümper [165] showed that Liouville's equation implies that the locally ellipsoidal distribution function in momentum space depends only on a quadratic form in the 4-momenta, whose coefficients are a Killing tensor in the case of non-vanishing particle rest-mass, and a conformal Killing tensor in the case of rest-mass zero particles. He suggested that cosmological models of Bianchi type I can be described in terms of ellipsoidal momentum distribution functions whose ellipsoidal tensor is built out of the Killing vectors associated with the spatial homogeneity.

Hoyle & Narlikar [166] (1964,1966) developed a conformal theory of gravitation based on the idea of particle interaction "at-a-distance", the equations of motion (not geodesics, in general) and the gravitational equations being obtained from a variational mass-action principle. Unlike the Einstein equations which determine the 10 components of the metric tensor completely, the Hoyle-Narlikar gravitational equations, of which only 9 are independent, do not give a complete determination of $g_{\mu\nu}$ - one function is left undetermined. This function may be taken to be the conformal factor Ω in the metric mapping $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, consistent with their requirement of conformal invariance of the propagator (scalar wave) equation for the interaction between two particles, plus the adoption of the Schouten-Haantjes device [37] that mass transforms according to $m \rightarrow \tilde{m} = \Omega^{-1} m$. The Hoyle-Narlikar gravitational equations for a smooth fluid approximation reduce to Einstein's equations for a special choice of Ω. However, the smooth fluid approximation is not valid in the neighbourhood of a particle, as they showed explicitly.

Arguing strongly for physical theories to be conformally invariant, Hoyle & Narlikar [166] (1972a) applied their direct-particle conformal theory of gravitation to cosmology. They discussed the Friedmann models which have a Robertson-Walker metric with 3-spaces of constant curvature k. This metric is conformally flat, so by a suitable conformal transformation the geometry of these world models can be made Minkowskian. However, in the usual relativistic cosmology it is not possible to employ this geometrical simplification because the physics changes; Einstein's equations are not conformally invariant. Hoyle & Narlikar's argument is that, because their gravitational equations are conformally invariant, they can exploit the conformal transformation of the metric in these models. Doing so, they find that although the cases k = ±1 are spatially homogeneous in the Robertson-Walker frame they are not spatially homogeneous in the Minkowski frame. Spatial homogeneity is, however, preserved under a conformal transformation from the k = 0 Robertson-Walker frame to the Minkowski frame. More surprisingly, the singularity ("origin of the universe") of the Friedmann models in the Robertson-Walker frame does not arise physically in the present theory. Hoyle & Narlikar take the view that the singularity is only a mathematical construct, due to an unfortunate choice of the conformal frame; there is a second half to the universe which appears when the Minkowski frame is used. Both halves contribute to the mass function m of the theory, and so both appear to be required in this cosmology.

In another paper (1972b) Hoyle & Narlikar, instead of starting with the Friedmann models and proving their consistency with the direct-particle theory characterized by the mass function m(X), started by introducing a constant λ such that the individual particle mass was $\lambda m(X)$. Assuming that λ^2 is related to the number of particles giving rise to the mass field, a new conformal cosmological model was obtained which involved continuous creation of matter. The interesting thing about this new model is that the creation is concentrated in active localized "centres", rather than uniformly as in the now disfavoured Steady-State theory [167].

Segal [168] proposed a space-time $\widetilde{M}$ which is an extension of the one considered by Veblen [12]. Briefly, $\widetilde{M}$ denotes the 4-dimensional manifold of all pairs (t,u), where t is a real number and $u = (u^1,u^2,u^3,u^4)$ is a point on the sphere S in 4-dimensional Euclidean space. Minkowski space M^4 is imbedded in $\widetilde{M}$ in a causality-preserving manner. $\widetilde{M}$ admits the full 15-parameter conformal group which is causality-preserving since $\widetilde{M}$ is a covering space of compactified Minkowski space ([28],[54], and see Section 2.2). Locally $\widetilde{M}$ and M^4 are indistinguishable, but on the cosmological scale differences arise. Segal claims that his world model "provides a satisfactory explanation for the cosmological red-shift and eliminates the apparent need to hypothesize the expansion of the universe". He also claims that his theory resolves the controversy about the smallness in size of quasars relative to their energy output.

Barnes [169] investigated flows of a perfect fluid in which the flow lines form a timelike shear-free normal (twist-free) congruence. Such a flow restricts the space-time to be Petrov type I and either static or degenerate (Petrov type D or conformally flat). He proved that a non-degenerate perfect fluid field admits a conformal Killing vector (CKV) field parallel to the flow (cf. McIntosh [141]) Trümper [170] had proved that an expanding vacuum space-time which admits a hypersurface-orthogonal CKV is either conformally flat or static. Barnes arrived at an immediate generalization of this result:

A non-degenerate perfect fluid space-time in which the flow lines form a normal shear-free congruence is static.

Barnes listed all conformally flat prefect fluid space-times with such a flow; these include a Friedmann universe, the interior Schwarzschild solution, the Stepanyuk metric [171], the Einstein universe, and the de Sitter universe, plus models which contain no Killing vectors. All static conformally flat perfect fluid fields with non-negative density were shown to be spherically symmetric. The Petrov type D fields with the present flows admit at least a 1-parameter group of isometries with spacelike trajectories; they include the exterior Schwarzschild solution with cosmological constant, the Levi-Civita metrics [136], the expanding spherically symmetric shear-free perfect-fluid flows with uniform density considered by Thompson & Whitrow [172], the metrics of Faulkes [173] and Nariai [174], and generalizations of the vacuum B- and C-metrics of Jordan, Ehlers & Kundt [175]. Barnes showed that a Petrov type D vacuum metric which admits a shear-free normal congruence of timelike curves is static. His results include those of Godfrey [133].

Eardley [133] defined a spatially self-similar cosmological model to be a space-time which admits a similarity group H_3 (= group of homothetic transformations) transitive on certain spacelike hypersurfaces. Such a space-time generally admits only the isometry group $G_2 \subset H_3$ and is therefore spatially inhomogeneous (unless $G_3 \equiv H_3$). Closely parallelling the work of Taub, Heckmann & Schücking, Ellis & MacCallum, Collins & Hawking [176], Eardley constructed all spatially self-similar cosmological space-times, putting them into classes A and B (including Bianchi types I, II, VI_0, VII_0, VIII and IX) which are the homogeneous models ($H_3 \equiv G_3$), and classes C and D which admit one proper homothetic motion. Noteworthy is the fact that the homogeneous Bianchi type VIII and IX cosmologies do not admit homothetic motions. He pointed out that only the simplest kinds of matter are allowed in a self-similar space-time e.g. dust, electromagnetic field, photon gas; mixtures of these are not permissible, in general. This restriction can be somewhat relaxed by allowing, for example, matter to be "self-similarly shocked" (see [131] and [132]) - Taub's plane-symmetric similarity solutions [132] are, in fact, special cases of Bianchi types $_1$I and V_F.

Eardley discussed the homothetic nature of the following space-times:

(a) Minkowski. The similarity group is the Weyl group $H_{11} \supset G_{10}$.

(b) Robertson-Walker. (i) k = 0. The metric

$$ds^2 = R^2(t)(dx^2 + dy^2 + dz^2) - dt^2$$

admits a homothetic motion if $R \propto t^n$, n constant, given by the homothetic Killing vector

$$\tilde{K} = (1-n)(x\partial_x + y\partial_y + z\partial_z) - t\partial_t .$$

For $n \geq \frac{2}{3}$ this metric is the solution of the Einstein equations for hydrodynamic matter with equation of state $p = (\gamma-1)\rho$, $\gamma = 2/(3n)$. "Other equations of state generally break self-similarity e.g. the 'hot big-bang' model of the universe is asymptotically self-similar before and after onset of matter dominance, but not during". The symmetry group is $H_7 \supset G_6$, where H_7 is transitive on the 4-dimensional space-time.

McIntosh [141] has also remarked that for arbitrary R(t) the hypersurface of homogeneity t = const. admits the homothetic motion given by the homothetic Killing vector

$$\tilde{K} = x\partial_x + y\partial_y + z\partial_z.$$

These k = 0 Robertson-Walker models are special cases of self-similar cosmologies of Bianchi types ${}_f\mathrm{III}$, ${}_f\mathrm{V}$ and ${}_f\mathrm{VII}_h$.

(ii) $k \neq 0$. None of these Robertson-Walker models admit exact non-trivial homothetic motions, except for the unphysical equation of state $p = -\rho/3$. At sufficiently early times (close to the "big bang") the models are asymptotically self-similar. At large times the k = -1 models are close to being Minkowskian and so are asymptotically self-similar.

(c) Kasner. These vacuum space-times with metric

$$ds^2 = -dt^2 + \sum_i t^{2p_i}(dx^i)^2, \quad \sum_i p_i = 1 = \sum_i p_i^2,$$

where the p_i are constants, admit a homothetic group $H_4 \supset G_3$, the homothetic Killing vector being

$$\tilde{K} = t\partial_t + \sum_i (1-p_i)x^i\partial_i .$$

A wide class of space-time singularities, investigated by Belinskii, Khalatnikov & Lifshitz [177] are approximated by the Kasner metric sufficiently near the singularity. Such singularities are therefore asymptotically self-similar.

(d) Heckmann-Schücking. These anisotropic dust universes with metric

$$ds^2 = -dt^2 + \sum_i t^{2p_i}(t+t_0)^{4/3-2p_i}(dx^i)^2, \quad t_0 = \text{const.},$$

where the p_i are constants satisfying the same constraints as in the Kasner metric, do not admit homothetic motions - their symmetry group is a G_3 of isometries. However, for $t \gg t_0$ and $t \ll t_0$ they are approximated by Kasner and dust Robertson-Walker models respectively, and so are asymptotically self-similar.

Eardley also constructed the ADM [178] Hamiltonian action principle for vacuum self-similar cosmologies, and warned that such an action principle sometimes leads to wrong field equations, just as in the homogeneous cosmologies which have only a Killing symmetry. He proved that the correct field equations were obtained for a self-similar cosmology iff the space-time is of (his) class A or a subclass of (his) class D, thus generalizing a result of MacCallum & Taub [179].

Milton & Ng [180] considered a scalar-tensor theory proposed by Schwinger [181] and showed, by subjecting Schwinger's Lagrangian function to conformal transformations, that his theory was in many ways similar to the Brans-Dicke [182] theory. For example, both theories may be characterized by a time-varying gravitational "constant", and the theories coincide when the scalar field is weak. Applying Schwinger's theory to Friedmann world models, Minton & Ng obtained a scalar-dominated cosmology in which the acceleration parameter $q_0 \simeq 2$, and which differs considerably from the corresponding Brans-Dicke cosmology. Milton & Ng suggested that their model could provide a resolution of the cosmological "missing mass" problem. On the other hand, the matter-dominated model in the Schwinger-Milton-Ng cosmology gives predictions identical with the Brans-Dicke one.

Gürses & Gürsey [85] showed that the Schwarzschild interior solution is the only solution of Einstein's equations for a spherically-symmetric perfect-fluid distribution with non-negative pressure which is both static and conformally flat. It is remarkable that they did not acknowledge that Buchdahl [85], in a widely available article, had dealt fully with this matter some years before. The same criticism can be levelled at Rao & Patel [85]. Two points perhaps worth mentioning about the Gürses & Gürsey paper are: (i) they gave explicit conformal transformation equations for expressing the Schwarzschild interior solution in each of the Einstein and de Sitter forms; (ii) they remarked that a physical model of the interior of a star consistent with causality cannot have the Schwarzschild interior metric as a metric, and so cannot be conformally flat (see also [183]).

Chang & Janis [184] studied conformally invariant scalar radiation fields, electromagnetic fields, and non-conformally invariant Klein-Gordon and gravitational radiation fields in Friedmann cosmological backgrounds. Their purpose was to find under what conditions back-scattering of the waves (i.e. the presence of wave-tails) would not occur; the multipole radiation is then said to be "characteristic". They found that characteristically propagating waves are possible for both conformally invariant scalar fields and electromagnetic fields in any Friedmann universe. This confirmed expectations because the Friedmann geometry is conformally flat, and so the behaviour of such radiation in a Friedmann model should be similar to the behaviour of the corresponding radiation in a Minkowski world. On the other hand, characteristic propagation of Klein-Gordon and gravitational radiation fields was found to be possible only for special Friedmann worlds, which nonetheless include two physically important cases, namely, those world models in which $p = 0$ and $p = \rho/3$, where p is the pressure and ρ is the density. This result should be compared with the earlier work of Kundt & Newman [185] which suggested that the presence of matter would lead to the presence of wave-tails. Chang & Janis used perturbation methods employing the formalism of Newman & Penrose [10] and Hawking [186], achieving a simplification over previous methods used by Tauber [187].

Other contributors to the application of the conformal group in cosmology are Gürsey [188]; Lopez [189] who obtained "exact solutions to the conformally homogeneous model universes constructed by Edelen" (no reference available); Stephani [190] who used Debye potentials to get all solutions of the source-free Maxwell vacuum equations from a single scalar equation, with applications to conformally flat cosmological models, plus extensions; and Caves [191] who compared recent observations with cosmological theories which are either conformally flat or have conformally flat spacelike sections.

CHAPTER 3 Scenario

For the remainder of this work I shall consider only Einstein vacuum spaces of dimension 4, which satisfy the field equations

$$R_{\mu\nu} = 0. \qquad (3.1)$$

The purpose of the study is to see what conformal symmetries these spaces admit. As we shall see, the spaces selected for investigation are in fact a subclass of those represented by equations (3.1). I shall be concerned with those vacuum spaces which admit proper homothetic motions and which are algebraically special, possessing a diverging and/or twisting geodesic shear-free ray congruence. There are good reasons why one should study such special spaces, and this chapter is devoted to setting the scene for such a study.

3.1 The Backdrop.

The simplicity of the vacuum space-times of general relativity make them obvious candidates for a first look into the real world. A huge amount of effort has gone into the study of the groups of motions (isometries) which these space-times admit. Indeed, the exploitation of the group theoretical method has been a major reason for the success in finding large numbers of metrics satisfying Einstein's equations. Such successes have led many investigators, as we observed in the last chapter, to look for more general symmetries in the vacuum space-times, and this search has been spurred on by such deeply held convictions as the conformal invariance of physical theories (see also Section 3.3 below).

Collinson [117] proved that the only curvature collineations (CCs) admitted by a vacuum space-time not of Petrov type N are conformal motions. He also found that Petrov type N vacuum spaces do admit CCs which are not conformal motions i.e. they admit more general types of symmetry.

Brinkmann [26] (1925) determined all Einstein spaces which can be conformally mapped non-trivially (i.e. non-homothetically) on Einstein spaces. Indicating his scorn for the homothetic case, Brinkmann omitted the qualification "non-trivially" from his results, as did Ehlers & Kundt [135] in their version of Brinkmann's theorem:

A vacuum field can be mapped conformally on another vacuum field iff both fields admit a covariant constant vector field i.e. iff both fields are pp-waves.

Schouten ([21], p.314) rendered the following version of Brinkmann's theorem:

If a special Einstein n-space (i.e. one satisfying equations (3.1)) is not conformally Euclidean, then for n = 4 it is impossible to map it conformally on another special Einstein space,

and added the footnote:

It is impossible to map it (i.e. a special Einstein space) conformally on a non-special Einstein space.

Ehlers & Kundt [135] gave an example to show that Schouten's version is false. It is easy to see why Schouten's proof fails. For, in the case of a homothetic motion (ϕ = const. in equation (1.14)), the Ricci identities (1.31) for $X = \phi,_{\alpha}$ become

$$R^{\mu}{}_{\alpha\beta\gamma}\phi,_{\mu} = 0, \tag{3.2}$$

which is trivial since ϕ = const. Furthermore, in a vacuum $R^{\mu}{}_{\alpha\beta\gamma} = C^{\mu}{}_{\alpha\beta\gamma}$ and so (3.2) and the first order integrability condition (1.35) give

$$\pounds_X C_{\alpha\beta\gamma} = 0.$$

The whole of Schouten's argument, which hinges on the non-triviality of (3.2), breaks down at this point.

Thus it appears that the question of whether or not a vacuum Einstein space-time admits a homothetic motion remains wide open. We know that the connections in two homothetically related vacuum spaces are given by

$$\tilde{\Gamma}^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu}, \qquad \tilde{g}_{\mu\nu} = e^{2\phi}g_{\mu\nu} \quad (\phi \text{ const.}),$$

$$\tilde{R}^{\mu}{}_{\alpha\beta\gamma} = \tilde{C}^{\mu}{}_{\alpha\beta\gamma} = C^{\mu}{}_{\alpha\beta\gamma} = R^{\mu}{}_{\alpha\beta\gamma},$$

and

$$\tilde{R}_{\mu\nu} = 0 = R_{\mu\nu}. \tag{3.3}$$

On the face of it, investigation of the Riemann and Ricci tensors is fruitless in providing an answer to the question.

Suppose, however, that a vacuum Einstein space admits a homothetic motion. Then setting $\psi = 1$ and choosing a coordinate system so that the components of the generator X^μ of the homothetic motion are such that $X^\mu = \delta^\mu_4$ $(x^4 = t)$, equations (1.15) imply that

$$g_{\mu\nu} = e^t h_{\mu\nu}(x^1,x^2,x^3) \qquad (\mu,\nu = 1,2,3,4),$$

where $\quad h_{44} = \epsilon = \pm 1, \qquad \epsilon g_{\mu\nu} > 0.$

With the metric in this canonical form it is easily shown that Ricci tensor $R_{\mu\nu}$ is independent of t, and equations (3.1) can in principle be solved for the $h_{\mu\nu}$. This is, however, a formidable task, and an alternative procedure which adapts a coordinate system to the algebraically special ray congruence of the space-time is followed, as described in detail in the next chapter.

Fortunately, we are aided a great deal in our task by the results of Collinson & French [9] who proved

Theorem 3.1 A conformal motion of non-flat empty space-time must be homothetic, unless the space-time is Petrov type N with hypersurface-orthogonal geodesic rays.

Their proof was based on the use of the Newman-Penrose spin coefficients. A direct proof, which to my knowledge has not appeared before, is as follows:

Suppose a vacuum space-time is mapped conformally onto itself by

$$\tilde{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}, \qquad \phi = \phi(x^\alpha) \text{ in general.}$$

Then equations (3.3) hold iff

$$\phi_{;\mu\nu} - \phi_{,\mu}\phi_{,\nu} + K g_{\mu\nu} = L_{\mu\nu}, \tag{3.4}$$

where

$$K = \frac{1}{2} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \tilde{R}\, e^{2\phi}/2n(n-1) \tag{3.5}$$

and $L_{\mu\nu}$ is defined in (1.32). Now, in accordance with the theorem of Yano [26] (1951) (see p.18 of Chapter 2), an Einstein space with $R \neq 0$ cannot admit a proper homothetic motion (and *a fortiori* a proper conformal motion), so we are taking $R = 0 = \tilde{R}$. Then from (3.5) and (1.32) we have

$$K = \frac{1}{2} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$$

and

$$L_{\mu\nu} = 0.$$

From (3.4) and these last two results we obtain for a vacuum

$$\phi,_{\mu}C^{\mu}{}_{\alpha\beta\gamma} = 0 = \phi,^{\mu}C_{\mu\alpha\beta\gamma} . \tag{3.6}$$

If $\phi \neq$ constant, this is just the condition for the space-time to be Petrov type N or conformally flat (see Table 1, Chapter 4, p.76). The gradient $\phi,_{\mu}$ is the Debever vector which is tangent to a hypersurface-orthogonal congruence of geodesic rays, as may be seen from the argument given by Eisenhart ([5], p.114). The argument used in the above proof breaks down when ϕ is constant, leading us back to equation (3.2). Since a conformally flat vacuum space-time is flat, we have proved Theorem 3.1.

Another result due to Yano [26] (1951) and also given by Suguri & Ueno [26] (1972) and Eardley [139] is the following:

Theorem 3.2 The commutator of two homothetic Killing vectors is a Killing vector.

A short proof of this result, based on the theory of the Lie derivative, is as follows:

Let ψ_1, ψ_2 be two scalars corresponding to the homothetic motions generated in a space-time by the homothetic Killing vectors $\tilde{K}_1$, $\tilde{K}_2$. Then, by equation (1.15),

$$\pounds_{\tilde{K}_1} g_{\mu\nu} = \psi_1 g_{\mu\nu}, \qquad \pounds_{\tilde{K}_2} g_{\mu\nu} = \psi_2 g_{\mu\nu}.$$

Therefore

$$[\pounds_{\tilde{K}_1}, \pounds_{\tilde{K}_2}] g_{\mu\nu} = \pounds_{\tilde{K}_1}(\psi_2 g_{\mu\nu}) - \pounds_{\tilde{K}_2}(\psi_1 g_{\mu\nu})$$

i.e.

$$\pounds_{[\tilde{K}_1,\tilde{K}_2]} g_{\mu\nu} = \psi_2 \pounds_{\tilde{K}_1} g_{\mu\nu} - \psi_1 \pounds_{\tilde{K}_2} g_{\mu\nu}$$

$$= 0$$

and hence

$$[\tilde{K}_1, \tilde{K}_2] = \lambda K, \qquad \lambda \text{ a constant,} \tag{3.7}$$

where

$$\pounds_K g_{\mu\nu} = 0.$$

Theorem 3.2 asserts that any vacuum space-time admits at most one independent proper homothetic motion. This has also been pointed out by McIntosh [141]. Hence we need concern ourselves with only those space-times which admit one proper homothetic motion together with a group of isometries.

Theorems 3.1 and 3.2 together tell us that, in looking for higher symmetries of non-flat vacuum spaces, we can restrict our attention to those spaces which admit at most one proper homothetic motion, unless the space is Petrov type N with twist-free geodesic rays. Trümper [170] showed: (a) If an expansion-free vacuum field admits a non-null hypersurface-orthogonal conformal Killing vector (CKV), then the CKV is a Killing vector. (b) If an expanding vacuum field admits a hypersurface-orthogonal CKV, then the space-time is either static or conformally flat. (If static, it must admit a timelike Killing vector which is also hypersurface-orthogonal.) McIntosh [141] showed that if a vacuum field admits a proper homothetic vector field $\tilde{K}$, then $\tilde{K}$ must be non-null. If $\tilde{K}$ has a non-null homothetic bivector, then $\tilde{K}$ is not hypersurface-orthogonal, is not tangent to a geodesic, is shear-free and has constant expansion. If $\tilde{K}$ has a null homothetic bivector, then the space is algebraically special and, if non-flat, is necessarily Petrov type III or N.

Petrov type N vacuum spaces are perhaps the most interesting and mathematically tantalising of all. The only type N vacuum fields which admit proper conformal motions are the hypersurface-orthogonal plane-fronted parallel (pp-) waves (Brinkmann [26], Collinson [117], Ehlers & Kundt [135], Thompson [192]). A subclass of the plane-fronted gravitational waves, the pp-waves have all been determined by Kundt [193]; they are characterized by the presence of an expansion-free, twist-free, null geodesic shear-free ray congruence. Some pp-waves which admit homothetic motions are known e.g. McIntosh [141] has cited the pp-wave

$$ds^2 = 2\,du\,dv - 2U(\zeta,\bar{\zeta},u)du^2 - |d\zeta|^2,$$

where $U = |x^2 - y^2|, \quad U_{\zeta\bar{\zeta}} = 0, \quad U_{\zeta\zeta} \neq 0, \quad \sqrt{2}\zeta = x+iy,$

which admits the homothetic Killing vector

$$\tilde{K} = x\partial_x + y\partial_y + 2v\partial_v.$$

For type N vacuum spaces with expanding and/or twisting rays, the symmetries present depend on whether the ray congruence is (i) twisting, or (ii) twist-free. In case (i) Collinson [194] has shown that there exists at most one Killing vector in the space. He was unable to integrate the field equations. In case (ii) Held [195] proved that the metric admits at most two isometries - the Killing vectors are necessarily spacelike in the asymptotically flat region. Collinson & French [9] have given examples of such metrics.

The Weyl metrics which Godfrey [133] found to admit homothetic motions could be among the metrics to be obtained in later chapters, as could be the Kasner metric (Section 2.4, page 56).

3.2 The Plot.

Against the backdrop described above, we see that the type of vacuum space within which we can perform our conformal motions is very much restricted. Indeed, for Petrov type I spaces the conformal motions must be homothetic, and the dimensions of the maximal group of homothetic motions and the maximal group of isometries are equal. There is a little more latitude in the case of algebraically special spaces, and it is this case which will be investigated in the remainder of this work.

Specifically, a systematic search for algebraically special vacuum Einstein spaces with a diverging and/or twisting shear-free null geodesic ray congruence, which admit homothetic motions will be undertaken. The results of this research fill part of the gap in our knowledge of symmetries of vacuum Einstein spaces. Such a systematic search has not been done before.

There remains the problem of finding all Petrov type I vacuum spaces which admit homothetic motions, and the problem of determining those pp-waves which admit (a) proper conformal motions, and (b) homothetic motions. This last hole must be filled by future research.

3.3 The Players: their physical characteristics.

The principal players are the homothetic Killing vectors. They determine the symmetry, along with the Killing vectors (if any). The physical importance of homothetic motions has been noted by Göbel [49], Einstein [130], Cahill & Taub [131], Taub [132], Eardley [139], McIntosh [141], Winicour [196], and more generally by Hoyle & Narlikar [166] (1972a). A few points will be made briefly here.

It is well known that all space and time measurements can be made in the same kind of unit, say length L. All physical measurements are in terms of dimensionless numbers, which are formed by combining physical quantities with dimensionality L^n at some space-time point P. Changing the unit of length by a factor λ does not affect the formation of dimensionless numbers at P. The factor λ may depend upon the point P in the space-time manifold, or it may be a constant. Either way, physical measurements may be compared via dimensionless numbers at the same space-time point and the physics is conformally invariant.

The use of dimensionless variables in classical continuum mechanics to produce "similarity solutions" of a given problem is well known. The usual technique consists of exploiting the symmetry to reduce the system of partial differential equations to ordinary ones by assuming a solution in which the dependent variables are essentially functions of a single independent dimensionless variable. This same technique should be of use in obtaining solutions of Einstein's (vacuum or non-vacuum) equations which are conformally or, in particular, homothetically invariant. Eardley [139], in applying this technique to cosmology, has said: "One may hope to discover new facts about cosmology and singularities by building new models that presume self-similarity (homothetic invariance) from the outset".

Scale invariance i.e. invariance under a uniform change in length scale in space-time, is of considerable interest in elementary particle physics because of its relation to deep inelastic scattering, as we noted in Chapter 2. The Weyl group is the transformation group concerned, and it has also served as a gauge group for relativistic theories of gravitation (see e.g. [197]).

These are a few reasons why one can feel confident that an investigation of conformal and homothetic symmetries will help towards a better understanding of physical theories. It is to be hoped that the present work will contribute towards that end.

3.4 The Props.

The framework of the theory and the formalism used to develop the argument is an extension of the work of Debney, Kerr & Schild [198] and Kerr & Debney [199], and is the subject of the next chapter.

Before the performance begins, it is emphasized that the play and players are local. To enable the production to extend globally would first require it to win approval locally and then would necessitate finding more time and more durable props, at least.

Let the curtain rise.

CHAPTER 4 *Formalism*

This chapter sets out the formalism that will be used in the rest of the work. The main references are the papers of Debney, Kerr & Schild [198], Kerr & Debney [199], and the thesis of Debney [200]. Similar formalisms have been developed by Cahen, Debever & Defrise [125], Wilson [201], and others.

4.1 Tetrad Formalism.

The tetrad formalism has been used in general relativity for a long time (see e.g. [5], [10], [21], [202], [203]) so the following presentation is not completely detailed.

Take a 4-dimensional Riemannian manifold M with a metric g expressed in local coordinates x^μ (μ = 1,2,3,4) by the tensor $g_{\mu\nu}(x^\alpha)$, and with signature (+ + + -). Let T_p denote the tangent space at $p \in M$ and let T_p^* denote the dual (cotangent) space. In terms of the coordinates x^μ at p the holonomic (natural) basis for T_p is $\{\partial_\mu \equiv \partial/\partial x^\mu\}$. The dual basis for T_p^* is $\{dx^\mu\}$.

Let $\{e_a | a = 1,2,3,4\}$ be any anholonomic basis for T_p i.e. to each point $p \in M$ there is attached a tetrad of vectors e_a. Let $\{\epsilon^a | a = 1,2,3,4\}$ be its dual in T_p^*, defined by the inner product

$$(\epsilon^a, e_b) = \delta^a_b \; . \qquad (4.1)$$

If $x \in T_p$ and $f \in T_p^*$, we have

$$x = x^a e_a,$$

$$f = f_a \epsilon^a.$$

We adopt the convention that Latin indices a, b, c, ... will always refer to components of geometrical objects with respect to an anholonomic basis (tetrad), while Greek indices λ, μ, ν, ... will refer to components with respect to a holonomic (coordinate) basis. Vector symbols are not singled out typographically by underlining, unless confusion will result in this not being done.

Each of the bases $\{e_a\}$, $\{\epsilon^a\}$ may be expressed, in terms of local coordinates at p, by

$$e_a = e_a{}^\mu \partial_\mu, \qquad (4.2)$$

$$\epsilon^a = \epsilon^a{}_\mu dx^\mu, \qquad (4.3)$$

where the components $e_a{}^\mu$, $\epsilon^a{}_\mu$ are C^∞ functions on M.

We may rewrite the orthogonality relations (4.1) as

$$e_a{}^{\mu}\epsilon^b{}_{\mu} = \delta^b_a, \qquad e_a{}^{\mu}\epsilon^a{}_{\nu} = \delta^{\mu}_{\nu} \; . \tag{4.4}$$

The inner product of two vectors e_a and e_b is defined by

$$g_{ab} = (e_a, e_b) = g_{\mu\nu}e_a{}^{\mu}e_b{}^{\nu} = e_a{}^{\mu}e_{b\mu} \tag{4.5}$$

and inversely

$$g^{ab} = (\epsilon^a, \epsilon^b) = g^{\mu\nu}\epsilon^a{}_{\mu}\epsilon^b{}_{\nu} = \epsilon^a{}_{\mu}\epsilon^{b\mu} . \tag{4.6}$$

The g_{ab}, g^{ab} will change from one point of M to another, in general.

The tetrad components $T^{b\ldots}_{a\ldots}$ of any Tensor $T^{\nu\ldots}_{\mu\ldots}$ are computed as follows:

$$T^{b\ldots}_{a\ldots} = e_a{}^{\mu}\epsilon^b{}_{\nu} \;.\;.\;. \; T^{\nu\ldots}_{\mu\ldots} \tag{4.7}$$

and inversely

$$T^{\nu\ldots}_{\mu\ldots} = \epsilon^a{}_{\mu}e_b{}^{\nu} \;.\;.\;. \; T^{b\ldots}_{a\ldots} \; . \tag{4.8}$$

Tensor indices are raised and lowered by $g_{\mu\nu}$, $g^{\mu\nu}$ and tetrad indices by g_{ab}, g^{ab}.

The directional derivative of a tensor $T^{\cdots}_{\cdots}$ in the direction of the tetrad vector e_a is

$$T^{\cdots}_{\ldots,a} \equiv \partial_a T^{\cdots}_{\cdots} = e_a{}^{\mu}T^{\cdots}_{\ldots,\mu} \tag{4.9}$$

where the comma denotes partial differentiation. The tetrad components of the covariant derivative $T^{\nu\ldots}_{\mu\ldots;\gamma}$ are

$$T^{b\ldots}_{a\ldots;c} = e_a{}^{\mu}\epsilon^b{}_{\nu} \;.\;.\;. \; e_c{}^{\gamma}T^{\nu\ldots}_{\mu\ldots;\gamma} \tag{4.10}$$

$$= T^{b\ldots}_{a\ldots,c} - \Gamma^d_{ac}T^{b\ldots}_{d\ldots} - \;.\;.\;. + \Gamma^b_{dc}T^{d\ldots}_{a\ldots} + \;.\;.\;. \tag{4.11}$$

where the Γs are the Ricci rotation coefficients defined by

$$\Gamma^a_{bc} = - \epsilon^a{}_{\mu;\nu}e_b{}^{\mu}e_c{}^{\nu} \tag{4.12}$$

or, equivalently,

$$\Gamma_{abc} = - \epsilon_{a\mu;\nu}e_b{}^{\mu}e_c{}^{\nu}, \tag{4.13}$$

where

$$\Gamma_{abc} = g_{ad}\Gamma^d_{bc} \qquad \text{and} \qquad g_{ab}\epsilon^a{}_{\mu} = \epsilon_{b\mu} = e_{b\mu}.$$

Using (4.3) together with the rotation coefficients and the tetrad components $R^a{}_{bcd}$ of the curvature tensor, the connection forms $\omega^a{}_b$ and the curvature forms $\Theta^a{}_b$ are given by

$$\omega^a{}_b = \Gamma^a{}_{bc}\epsilon^c, \qquad \omega_{ab} = \Gamma_{abc}\epsilon^c, \tag{4.14}$$

$$\Theta^a{}_b = R^a{}_{bcd}\epsilon^c \wedge \epsilon^d, \tag{4.15}$$

where $\wedge$ denotes the wedge product of differential forms.

The Cartan structural equations

$$d\epsilon^a + \omega^a{}_c \wedge \epsilon^c = 0, \tag{4.16}$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2}\Theta^a{}_b \tag{4.17}$$

relate the forms $\omega^a{}_b$ and $\Theta^a{}_b$ to the vectors ϵ^a.

The metric on M is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{ab}\epsilon^a \wedge \epsilon^b \tag{4.18}$$

and this gives

$$dg_{ab} = \omega_{ab} + \omega_{ba} .$$

In the case of rigid tetrads for which $dg_{ab} = 0$, that is g_{ab} = covariant constant over the whole of M, this gives

$$\omega_{ba} = -\omega_{ab}. \tag{4.19}$$

so that there are only six independent connection forms and therefore only six independent curvature forms. We shall adopt a system of rigid tetrads for the present work. Definition (4.14) and property (4.19) give

$$\Gamma_{abc} = -\Gamma_{bac} = \Gamma_{a[bc]} + \Gamma_{b[ca]} - \Gamma_{c[ab]}, \tag{4.20}$$

where we have used the convention that round and square brackets denote symmetrization and skew-symmetrization respectively, thus:

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$$

$$T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$$

and then recursively

$$T_{(a_1 \ldots a_m)} = \frac{1}{m}\{T_{a_1(a_2 \ldots a_m)} + T_{a_2(a_1 a_3 \ldots a_m)} + \ldots + T_{a_m(a_1 \ldots a_{m-1})}\},$$

$$T_{[a_1 \ldots a_m]} = \frac{1}{m}\{T_{a_1[a_2 \ldots a_m]} - T_{a_2[a_1 a_3 \ldots a_m]} + \ldots + (-1)^{m-1}T_{a_m[a_1 \ldots a_{m-1}]}\}.$$

The $\Gamma_{a[bc]}$ are determined from definition (4.14) and equations (4.16).

Then the components of the curvature tensor $R^a{}_{bcd}$ are determined from (4.15) and (4.17), where for any vector u_a the Ricci identity

$$u_{a;[bc]} = \frac{1}{2} R^m{}_{abc} u_m \tag{4.21}$$

defines the curvature tensor.

We shall employ a complex null tetrad (see e.g. [200], p.6)

$$\{e_a\} = \{m, \bar{m}, n, k\}, \tag{4.22}$$

where m, $\bar{m}$ are complex conjugate null vectors and n,k are real null vectors. The bar above a symbol denotes complex conjugation. The vector m may be defined from a pair of real orthogonal unit spacelike vectors p and q by

$$\sqrt{2}\, m = p+iq.$$

The dual tetrad is

$$\{\epsilon^a\} = \{\bar{m}, m, k, n\} .$$

With

$$g_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = g^{ab} \tag{4.23}$$

the following orthogonality relations obtain:

$$\begin{aligned} k^\mu k_\mu = m^\mu m_\mu = n^\mu n_\mu = k^\mu m_\mu = m^\mu n_\mu = 0, \\ k^\mu n_\mu = \bar{m}^\mu m_\mu = 1. \end{aligned} \tag{4.24}$$

Also

$$g_{\mu\nu} = e_{a\mu} e_{b\nu} g^{ab} = 2m_{(\mu}\bar{m}_{\nu)} + 2n_{(\mu}k_{\nu)} \tag{4.25}$$

so that the metric on M in terms of this null tetrad is

$$\begin{aligned} ds^2 &= 2(m_\mu dx^\mu)(\bar{m}_\nu dx^\nu) + 2(n_\mu dx^\mu)(k_\nu dx^\nu) \\ &= 2\epsilon^1\epsilon^2 + 2\epsilon^3\epsilon^4 \\ &= 2\epsilon_1\epsilon_2 + 2\epsilon_3\epsilon_4 \\ &= 2m\bar{m} + 2nk, \end{aligned} \tag{4.26}$$

where $\epsilon_a = g_{ab}\epsilon^b$.

The set of proper (no space reflections), orthochronous (no time reversal) Lorentz transformations of the null tetrad which leave unchanged the direction of one of the null vectors, say $e_4 = k$, is,

$$m^* = e_1^* = e^{-iB}(e_1+\gamma e_4) = e^{-iB}(m+\gamma k), \tag{4.27a}$$

$$n^* = e_3^* = e^{-A}\,(e_3 - \bar{\gamma}e_1 - \gamma e_2 - \gamma\bar{\gamma}e_4)$$

$$= e^{-A}\,(n - \bar{\gamma}m - \gamma\bar{m} - \gamma\bar{\gamma}k), \tag{4.27b}$$

$$k^* = e_4^* = e^A e_4 = e^A k, \tag{4.27c}$$

where A and B are real numbers and γ is a complex number. If $\gamma = 0$ equations (4.27) describe a timelike rotation in the (n,k)-plane and a spacelike rotation in the (p,q)-plane, where $\sqrt{2}m = p+iq$. For A = B = 0 equations (4.27) describe a null rotation about the vector k. Thus the general transformation (4.27) is a non-commutative product of these two types of transformation.

Null tetrads have been widely used in general relativity. A more detailed description, together with discussions on null rotations, may be found e.g. in [10] and [204] - [208] and references cited therein.

Using the complex null tetrad, the six independent connection forms ω_{ab} from which the rotation coefficients are determined are ω_{42}, $\omega_{12} + \omega_{34}$, ω_{31} and their complex conjugates ω_{41}, $-\omega_{12} + \omega_{34}$, ω_{32}. The independent curvature equations (4.17) are

$$d\omega_{42} + \omega_{42} \wedge (\omega_{12} + \omega_{34}) = \tfrac{1}{2} R_{42ab}\varepsilon^a \wedge \varepsilon^b, \tag{4.28a}$$

$$d(\omega_{12} + \omega_{34}) + 2\omega_{42} \wedge \omega_{31} = \tfrac{1}{2}(R_{12ab} + R_{34ab})\varepsilon^a \wedge \varepsilon^b, \tag{4.28b}$$

$$d\omega_{31} + (\omega_{12} + \omega_{34}) \wedge \omega_{31} = \tfrac{1}{2} R_{31ab}\varepsilon^a \wedge \varepsilon^b. \tag{4.28c}$$

The independent components of the Ricci tensor $R_{ab} = R_{ba} = R^c{}_{abc}$ are

$$\begin{aligned} R_{22} &= 2\,R_{4223}\,, \\ R_{24} &= R_{4212} - R_{4234}\,, \\ R_{44} &= 2\,R_{4214}\,, \end{aligned} \tag{4.29a}$$

and

$$\begin{aligned} R_{12} &= R_{1212} + R_{3412} - 2\,R_{4231}\,, \\ R_{34} &= R_{1234} + R_{3434} - 2\,R_{4231}\,, \\ R_{33} &= -\,2\,R_{3132}\,, \\ R_{23} &= R_{1232} + R_{3432}\,, \end{aligned} \tag{4.29b}$$

where R_{44}, R_{12}, R_{34}, R_{33} are real and R_{11}, R_{14}, R_{13} are the complex conjugates of R_{22}, R_{24}, R_{23} respectively.

4.2 Congruences.

To expedite this study of algebraically special vacuum spaces we shall orient the null tetrad and choose coordinate systems bearing in mind the Goldberg-Sachs theorem [209]:

A source-free gravitational field is algebraically special iff it admits a shear-free null geodesic congruence.

The terms used in this theorem are defined below. Here we note two significant points: (1) we shall choose one of the tetrad vectors ($e_4 = k$) to be tangent to a null geodesic congruence, (2) the Goldberg-Sachs theorem applies to any field that is conformal to a vacuum field [120], [209].

Much of the present knowledge of null congruences and their properties can be attributed to Ehlers [210] and Sachs [204] and may be summarized in the Ehlers-Sachs theorem (below). Since the theory is well presented in the literature (see also [5] pp.97 ff., [207], and [211] - [213]), only an outline is given here for completeness.

Consider a space-time filling set of curves whose equations are $x^{\alpha} = x^{\alpha}(y^{\beta}, w)$, $\alpha = 1,2,3,$ where the individual curves are given by y^{β} = constant, and w is an affine parameter along each curve. Let the null vector

$$e_4{}^{\mu} = k^{\mu} = \partial x^{\mu}/\partial w, \qquad k^{\mu}k_{\mu} = 0$$

be tangent to one of these curves. Then

$$k_{\mu;\nu}k^{\nu} \equiv (\partial k^{\mu}/\partial w) + A^{\mu}_{\nu\sigma}k^{\nu}k^{\sigma} = 0,$$

where $A^{\mu}_{\nu\sigma}$ is the affine connection, is the condition for the curve to be a (null) geodesic. This condition is preserved under conformal transformations of the metric. Then the complex rotation coefficient Γ_{424} vanishes:

$$\Gamma_{424} \equiv -k_{\mu;\nu}\bar{m}^{\mu}k^{\nu} = 0, \tag{4.30}$$

which implies that its complex conjugate does too,

$$\Gamma_{414} = 0. \tag{4.31}$$

(The complex conjugate of a real geometrical object is obtained by performing the permutation 1,2,3,4 → 2,1,3,4 on the tetrad indices.)

The geometrical properties of a null geodesic congruence can be visualized as follows [204], [207]:

Think of the congruence as a bundle of light rays. Insert a small plane circular disk into the bundle at right angles to the rays. On a nearby plane screen, also perpendicular to the rays, the shadow of the disk appears as an ellipse, and all portions of the shadow hit the screen simultaneously. The shape,size and orientation of the shadow depend only on the location of the screen, and not on its world velocity. If the screen is an affine parameter distance Δw from the disk, then the shadow is expanded, rotated and sheared relative to the disk by the respective amounts $\theta\Delta w$, $\omega\Delta w$ and $|\sigma|\Delta w$, where

$$\text{rate of expansion } \theta = \tfrac{1}{2} k^{\mu}{}_{;\mu}$$

$$\text{rate of rotation (twist) } \omega = \{\tfrac{1}{2} k_{[\mu;\nu]} k^{\mu;\nu}\}^{\frac{1}{2}} \qquad (4.32)$$

$$\text{rate of shear } |\sigma| = \{\tfrac{1}{2} k_{(\mu;\nu)} k^{\mu;\nu} - \theta^2\}^{\frac{1}{2}}.$$

This is the Ehlers-Sachs theorem. θ, ω, σ are referred to as the optical scalars.

Let u = constant be a family of null hypersurfaces in M. The differential equation of the family is $g^{\mu\nu}u_{,\mu}u_{,\nu} = 0$. Therefore the vector $k_\mu = u_{,\mu}$ is null and is normal to these hypersurfaces. But, being null, it is self-orthogonal an so lies in the hypersurface to which it is normal. Thus a single null hypersurface $u = 0$ determines within itself a null geodesic congruence. Conversely, given a congruence of null geodesics, these curves lie in a 1-parameter family of null hypersurfaces u = constant to which k_μ is normal iff

$$k_{[\mu;\nu}k_{\lambda]} = 0 \quad \Leftrightarrow \quad k_\mu = f(x^\alpha)u_{,\mu}. \qquad (4.33)$$

A congruence of null geodesics with tangent vector k_μ satisfying condition (4.33) is called hypersurface-orthogonal or normal.

It can be shown that any congruence of null curves which is hypersurface-orthogonal is always a geodesic congruence. The result

$$k_{[\mu;\nu}k_{\lambda]} = \pm \tfrac{1}{3}\omega\, \eta_{\mu\nu\lambda\sigma}k^{\sigma},$$

where $\eta_{\mu\nu\lambda\sigma}$ is the alternating symbol, shows that (4.33) holds iff $\omega = 0$ i.e. a null geodesic congruence is hypersurface-orthogonal iff it is twist-free.

From (4.32) it follows that the optical scalars θ, ω, σ are determined by the null geodesic congruence (i.e. by specification of k^{μ}) alone, up to an ambiguity in sign of ω and an undetermined phase in σ. The vectors m, n which complete the tetrad do not play a part in determining the optical scalars.

The completeness relation (4.25) and the null geodesic property $k_{\mu;\nu}k^{\nu} = 0 = k_{\mu;\nu}k^{\mu}$ give

$$k_{\mu;\nu} = \delta^{\alpha}_{\mu}k_{\alpha;\beta}\delta^{\beta}_{\nu} = (m_{\mu}\bar{m}^{\alpha} + \bar{m}_{\mu}m^{\alpha} + k_{\mu}n^{\alpha} + n_{\mu}k^{\alpha})k_{\alpha;\beta}(m_{\nu}\bar{m}^{\beta} + \bar{m}_{\nu}m^{\beta} + k_{\nu}n^{\beta} + n_{\nu}k^{\beta}).$$

Then

$$k_{\mu;\nu} \equiv (\rho m_{\mu}\bar{m}_{\nu} + \sigma m_{\mu}m_{\nu} + (\alpha+\bar{\beta})k_{\mu}m_{\nu} + \tau\bar{m}_{\mu}k_{\nu} + \gamma k_{\mu}k_{\nu}) + \text{complex conjugate}$$

$$\begin{aligned}
\Leftrightarrow \quad \text{complex divergence} \quad \rho &= k_{\mu;\nu}\bar{m}^{\mu}m^{\nu} = -\Gamma_{421} = \theta + i\omega, \\
\text{complex shear} \quad \sigma &= k_{\mu;\nu}\bar{m}^{\mu}\bar{m}^{\nu} = -\Gamma_{422}, \\
\alpha+\bar{\beta} &= k_{\mu;\nu}n^{\mu}\bar{m}^{\nu} = -\Gamma_{432}, \\
\tau &= k_{\mu;\nu}m^{\mu}n^{\nu} = -\Gamma_{413}, \\
\gamma+\bar{\gamma} &= k_{\mu;\nu}n^{\mu}n^{\nu} = -\Gamma_{433} .
\end{aligned} \tag{4.34}$$

The notation used here is chosen to accord with that used by Newman & Penrose [10]. It can be seen from (4.34) that the optical scalars θ, ω, σ are invariant under a null rotation given by (4.27) with $A = B = 0$.

For a null geodesic congruence with $\Gamma_{424} = 0$, the connection form ω_{42} contains all the optical information:

$$\begin{aligned}
\omega_{42} = \Gamma_{42a}\varepsilon^{a} &= -\rho\varepsilon^{1} - \sigma\varepsilon^{2} + \Gamma_{423}\varepsilon^{3} \\
&= -\sigma\varepsilon_{1} - \rho\varepsilon_{2} + \Gamma_{423}\varepsilon_{4} .
\end{aligned} \tag{4.35}$$

We shall return to this equation when setting up a coordinate system.

4.3 Conformal tensor. Petrov classification.

Let $R_{\mu\nu\alpha\beta}$ be the Riemann tensor, $R_{\mu\nu} = R^{\alpha}{}_{\mu\nu\alpha}$ be the Ricci tensor, $R = R^{\alpha}{}_{\alpha}$ be the curvature scalar, $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$ be the traceless form of the Ricci tensor. The Weyl conformal tensor is defined by

$$C^{\mu\nu}{}_{\alpha\beta} = R^{\mu\nu}{}_{\alpha\beta} + 2\delta^{[\mu}_{[\alpha}S^{\nu]}_{\beta]} + \frac{1}{6}\,\delta^{[\mu}_{[\alpha}\delta^{\nu]}_{\beta]}\,R \tag{4.36}$$

and possesses the symmetries

$$C_{\mu\nu\alpha\beta} = C_{[\mu\nu][\alpha\beta]} = C_{\alpha\beta\mu\nu}\ , \quad C_{\mu[\nu\alpha\beta]} = 0 = C^{\mu\nu}{}_{\alpha\mu}\ . \tag{4.37}$$

These statements also apply to tetrad components. Integrability conditions for conformal transformations, involving the Weyl tensor, have been discussed in Chapter 1. Here we are concerned with types of this tensor.

The algebraic and geometric study of the Riemann and Weyl tensors has done much to clarify the structure of gravitational fields. In particular, it has advanced the understanding of gravitational radiation fields and has become a convenient tool in the search for new exact solutions of Einstein's field equations (see e.g. Ehlers & Kundt [135], Pirani [213]). Earlier work on the geometry of the Riemann tensor may be found in references [213] - [216]. Petrov [8], [217] gave, using matrix methods, the first systematic account of the algebraic classification of the Riemann tensor for Einstein spaces. The physical significance of the Petrov scheme for gravitational radiation was recognized by Pirani [218]. Kerr [219], Goenner & Stachel [220], Petrov and others have discussed the classification of the Weyl and Ricci tensors from the point of view of symmetry groups. Penrose [221], following Witten [222], developed a spinor method for classifying the Weyl tensor according to its Petrov type. Further contributions to the algebraic and geometric study of the Riemann and Weyl tensors and the use of the Petrov classification in general relativity up to 1971 may be found in references [175] and [223]-[227] and the literature cited therein. An exhaustive set of references, including more recent ones, is not given here.

For vacuum spaces $R_{\mu\nu} = 0$ and (4.36) gives $C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta}$, so the algebra and geometry of the Weyl and Riemann tensors is the same in such spaces. Debever [216] (see also Debney [200]) showed by tensor methods that in every vacuum space-time there exists at least one and at most four independent vectors $k^\mu \neq 0$ which satisfy the algebraic relations

$$k_{[\alpha}C_{\beta]\mu\nu[\gamma}k_{\delta]}k^\mu k^\nu = 0, \qquad k^\mu k_\mu = 0. \tag{4.38}$$

These k^μ are called Debever vectors and determine the principal null directions of the Weyl tensor. The Petrov type is given according to the partitioning of these Debever vectors; the higher the multiplicity of coincident Debever vectors, the greater the degree of specialization of Petrov type. Table 1 is adapted from that of Pirani's [207] and displays the relationship between Petrov types:

Debever partition	Petrov type	Algebraic relation
[1 1 1 1]	I	$k_{[\alpha}C_{\beta]\mu\nu[\gamma}k_{\delta]}k^\mu k^\nu = 0$
[2 1 1]	II	$C_{\beta\mu\nu[\gamma}k_{\delta]}k^\mu k^\nu = 0$
[2 2]	D	
[3 1]	III	$C_{\beta\mu\nu[\gamma}k_{\delta]}k^\nu = 0$
[4]	N	$C_{\beta\mu\nu\gamma}k^\gamma = 0$
-	0	$C_{\beta\mu\nu\gamma} = 0$

Table 1

Other notations for types D, N are I_d, II_d respectively. Type 0 corresponds to a conformally flat space. Type I is called the algebraically general type. Any space with a Weyl tensor of type II, D, III or N is called an algebraically special space. Penrose [221] gave the following scheme (the Penrose diagram) for the increasing order of specialization of Petrov types:

$$\begin{tikzcd}
 & & \mathrm{I} \arrow[dl] \arrow[dr] & & \\
 & \mathrm{II} \arrow[rr] \arrow[dl] \arrow[dr] & & \mathrm{D} \arrow[dl] \arrow[dr] & \\
\mathrm{III} \arrow[rr] & & \mathrm{N} \arrow[rr] & & 0
\end{tikzcd}$$

Sachs [204] pointed out the existence of five complex quantities which completely characterize the Weyl tensor. Referred to our null tetrad they are

$$
\begin{aligned}
C^{(5)} &= 2\,R_{4242},\\
C^{(4)} &= R_{4234} + R_{4212},\\
C^{(3)} &= R_{1234} + R_{3434} - R_{34} + \tfrac{1}{6}\,R,\\
C^{(2)} &= R_{1231} + R_{3431},\\
C^{(1)} &= 2\,R_{3131}\,.
\end{aligned}
\qquad (4.39)
$$

The necessary and sufficient condition for the tetrad vector $e_4 = k$ to be a Debever principal null vector is $C^{(5)} = 0$. The vacuum spaces are algebraically special as follows:

$$\text{Type II or D,}\quad C^{(5)} = C^{(4)} = 0,\quad C^{(3)} \neq 0, \qquad (4.40a)$$
$$\text{Type III,}\quad C^{(5)} = C^{(4)} = C^{(3)} = 0,\quad C^{(2)} \neq 0, \qquad (4.40b)$$
$$\text{Type N,}\quad C^{(5)} = C^{(4)} = C^{(3)} = C^{(2)} = 0,\quad C^{(1)} \neq 0, \qquad (4.40c)$$
$$\text{Type 0,}\quad C^{(5)} = C^{(4)} = C^{(3)} = C^{(2)} = C^{(1)} = 0. \qquad (4.40d)$$

Goldberg & Sachs [209] showed that a vacuum metric with $C_{abcd} \neq 0$ is Petrov type D iff there exist two independent shear-free null geodesic congruences, i.e. if $C^{(3)} \neq 0$, $C^{(1)} = C^{(2)} = 0$. Hence type D metrics are completely characterized by

$$\text{Type D,}\quad C^{(5)} = C^{(4)} = C^{(2)} = C^{(1)} = 0,\ C^{(3)} \neq 0. \qquad (4.41)$$

Under a proper, orthochronous Lorentz transformation (4.27) of the null tetrad which leaves unchanged the direction of $e_4 = k$, the $C^{(i)}$ transform as ([200], p.33)

$$
\begin{aligned}
C^{(5)*} &= e^{4(A+iB)}C^{(5)},\\
C^{(4)*} &= e^{2(A+iB)}[\gamma C^{(5)} + C^{(4)}],\\
C^{(3)*} &= \gamma^2 C^{(5)} + 2\gamma C^{(4)} + C^{(3)},\\
C^{(2)*} &= e^{-2(A+iB)}[\gamma^3 C^{(5)} + 3\gamma^2 C^{(4)} + 3\gamma C^{(3)}],\\
C^{(1)*} &= e^{-4(A+iB)}[\gamma^4 C^{(5)} + 4\gamma^3 C^{(4)} + 6\gamma^2 C^{(3)} + 4\gamma C^{(2)} + C^{(1)}],
\end{aligned}
\qquad (4.42)
$$

and under a conformal change of metric

$$g'_{\mu\nu} = e^{2\phi} g_{\mu\nu}, \qquad (4.43)$$

where $\phi = \phi(x^{\alpha})$ in general, the $C^{(i)}$ transform as [120]

$$C^{(5)\prime} = e^{-4\phi}C^{(5)},$$
$$C^{(4)\prime} = e^{-3\phi}C^{(4)},$$
$$C^{(3)\prime} = e^{-2\phi}C^{(3)}, \qquad (4.44)$$
$$C^{(2)\prime} = e^{-\phi}C^{(2)},$$
$$C^{(1)\prime} = C^{(1)}.$$

The transformation equations (4.44) show at once that algebraic degeneracy of the Weyl tensor is a conformally invariant property.

4.4 Coordinate system. Field equations.

The coordinate system used by Kerr & Debney [199], developed by Debney, Kerr & Schild [198], [200], will be used in this investigation of the homothetic symmetry of vacuum Einstein spaces.

The null tetrad $\{e_a\} = \{m,\bar{m},n,k\}$ is chosen so that $e_4 = k$ is tangent to a shear-free null geodesic congruence in the algebraically special vacuum. That such a congruence exists is guaranteed by the Goldberg-Sachs theorem. Then we have $\kappa = -\Gamma_{424} = 0$ (geodesic property) $\sigma = -\Gamma_{422} = 0$ (shear-free property). We further demand that $\Gamma_{421} \neq 0$ i.e. the rate of expansion θ and the rate of twist ω do not both vanish, $\rho = \theta + i\omega \neq 0$.

From (4.35) we then have

$$\omega_{42} = -\rho\epsilon_2 + \Gamma_{423}\epsilon_4. \qquad (4.45)$$

Under a transformation (4.27) of the null tetrad, Γ_{423} transforms as follows:

$$\begin{aligned}\overset{*}{\Gamma}_{423} &= -\overset{*}{e}_{4\mu;\nu}\overset{*}{e}{}_2^{\mu}\,\overset{*}{e}{}_3^{\nu}\\ &= -e^{A}k_{\mu;\nu}e^{iB}(\bar{m}^{\mu} + \bar{\gamma}k^{\mu})e^{-A}(n^{\nu} - \bar{\gamma}m^{\nu} - \gamma\bar{m}^{\nu} - \gamma\bar{\gamma}k^{\nu})\\ &= e^{iB}(\Gamma_{423} + \bar{\gamma}\rho),\end{aligned}$$

where we have used (4.24) and $k_{\mu;\nu}k^{\mu} = 0$. Choosing $\bar{\gamma} = \rho^{-1}\Gamma_{423}$ we get $\overset{*}{\Gamma}_{423} = 0$. Thus by a tetrad transformation we can make Γ_{423} vanish. In order to preserve this condition we are henceforth allowed only tetrad transformations with $\gamma = 0$; specifically

$$\overset{*}{m} = e^{-iB}m,$$
$$\overset{*}{n} = e^{-A}n, \qquad (4.46)$$
$$\overset{*}{k} = e^{A}k,$$

where A and B are real numbers.

Setting $\Gamma_{423} = 0$ equation (4.45) reduces to

$$\omega_{42} = -\rho\epsilon_2. \tag{4.47}$$

For vacuum spaces the field equations $R_{ab} = 0$, together with the degeneracy conditions $C^{(5)} = C^{(4)} = 0$ and (4.39), show that the only non-zero component of R_{42ab} is R_{4231} (see equation (4.29b)). Therefore (4.28a) becomes

$$d\omega_{42} + \omega_{42} \wedge (\omega_{12} + \omega_{34}) = \frac{1}{2} R_{4231}\, \epsilon^3 \wedge \epsilon^1. \tag{4.48}$$

Hence

$$\omega_{42} \wedge d\omega_{42} = 0. \tag{4.49}$$

This is the condition for the existence of two complex functions ζ, ψ such that

$$\omega_{42} = -e^{\psi} d\zeta\ . \tag{4.50}$$

Under a tetrad transformation (4.46) ω_{42} transforms as

$$\overset{*}{\omega}_{42} = \overset{*}{\Gamma}_{42a}\overset{*}{\epsilon}{}^{a} = e^{A}\Gamma_{421}e^{iB}\epsilon^1$$

$$= e^{A+iB}\,\omega_{42} \tag{4.51}$$

$$= -e^{A+iB+\psi} d\zeta, \tag{4.52}$$

remembering that $\Gamma_{422} = \Gamma_{423} = \Gamma_{424} = 0$. Taking $A+iB+\psi = 0$ we further restrict the allowed tetrad transformation, but this enable us to write, after dropping the asterisk,

$$\omega_{42} = -d\zeta\ . \tag{4.53}$$

Then (4.47) gives

$$\epsilon^1 = \epsilon_2 = \rho^{-1} d\zeta\ . \tag{4.54}$$

We take ζ, $\bar{\zeta}$ as two coordinates in the space-time, where $\bar{\zeta}$ is the complex conjugate of ζ.

The other two coordinates u,v of our coordinate system are both real and are introduced in the definitions

$$u_{,3} = 1, \qquad u_{,4} = 0, \tag{4.55}$$

$$\epsilon_4 = k = du + \Omega d\zeta + \bar{\Omega} d\bar{\zeta}, \tag{4.56}$$

and $$v = \mathrm{Re}(\rho^{-1}), \qquad \text{or} \qquad \rho = (v+\Delta)^{-1}, \tag{4.57}$$

where ρ is the complex divergence and Δ is defined by

$$\Delta = i\,\mathrm{Im}(\bar{D}\Omega) = -\bar{\Delta}. \tag{4.58}$$

The operator D and hence its complex conjugate $\bar{D}$ in the last equation is defined by

$$D \equiv \partial_\zeta - \Omega\partial_u . \tag{4.59}$$

Upon using $\rho = \theta + i\omega$ we find

$$\Delta = i\,\mathrm{Im}(\rho^{-1}) = -i\omega/(\theta^2+\omega^2).$$

Thus the Debever vector k is hypersurface-orthogonal iff $\Delta = 0$ (from the well known result on page 73). Hypersurface-orthogonal spaces belong to the Robinson-Trautman class [228].

In the $(\zeta,\bar{\zeta},u,v)$ coordinates the metric takes the form

$$d\tau^2 = 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) = 2(m\bar{m} + nk), \tag{4.60}$$

where the metric vectors are

$$\begin{aligned}
\epsilon_1 &= m = (v+\Delta)d\zeta, \\
\epsilon_3 &= n = dv - 2\mathrm{Re}\{[(v-\Delta)\dot{\Omega} + D\Delta]d\zeta\} \\
&\qquad + \mathrm{Re}(D\dot{\bar{\Omega}} + \mu\rho)\epsilon_4, \\
\epsilon_4 &= k = du + \Omega d\zeta + \bar{\Omega}d\bar{\zeta} .
\end{aligned} \tag{4.61}$$

The functions Ω, μ, Δ are independent of the coordinate v, and μ is referred to as the "complex mass". The dot used in (4.61) denotes differentiation with respect to u, thus:

$$\dot{\Omega} = \partial\Omega/\partial u \equiv \partial_u\Omega .$$

The reader is referred to the paper by Debney, Kerr & Schild [198] for details of the derivation of the field equations, which are

$$\bar{D}\mu = 3\dot{\bar{\Omega}}\mu, \tag{4.62a}$$

$$\mathrm{Im}(\mu - \bar{D}\bar{D}D\Omega) = 0, \tag{4.62b}$$

$$\partial_u(\mu - \bar{D}\bar{D}D\Omega) = |\partial_u D\Omega|^2. \tag{4.62c}$$

The independent components of the Weyl tensor are

$$C^{(3)} = \mu\rho^3, \tag{4.63a}$$

$$C^{(2)} = -(\bar{D}\partial_u D\Omega)\rho^2 + (\text{terms} = 0 \text{ if } C^{(3)} = 0), \tag{4.63b}$$

$$C^{(1)} = (\partial_u\partial_u D\Omega)\rho + (\text{terms} = 0 \text{ if } C^{(3)} = C^{(2)} = 0). \tag{4.63c}$$

The Weyl tensor therefore has the following Petrov types:

$$\text{Type II or D} \Leftrightarrow \mu \neq 0. \tag{4.64a}$$

$$\text{Type III} \Leftrightarrow \mu = 0,\ \bar{D}\partial_u D\Omega \neq 0. \tag{4.64b}$$

$$\text{Type N} \Leftrightarrow \mu = 0 = \bar{D}\partial_u D\Omega,\ \partial_u\partial_u D\Omega \neq 0. \tag{4.64c}$$

$$\text{Type 0} \Leftrightarrow \mu = \bar{D}\partial_u D\Omega = \partial_u\partial_u D\Omega = 0 \tag{4.64d}$$

It can be shown* that the conditions for Petrov type D are

$$\begin{aligned} \text{Type D}, \quad & 3\mu\partial_u\partial_u D\Omega = (\bar{D}\partial_u D\Omega)^2, \\ & 4\, D\mu\bar{D}\partial_u D\Omega = 3\mu D\bar{D}\partial_u D\Omega, \\ & 4(D\mu)^2 + 3\mu\bar{D}\partial_u D\Omega(DD\bar{\Omega} - \bar{D}D\Omega) \\ & \qquad = 3\mu DD\mu - 9\mu^2\partial_u D\Omega, \\ & D\mu(DD\bar{\Omega} - \bar{D}D\Omega) = \mu D(DD\bar{\Omega} - \bar{D}D\Omega). \end{aligned} \tag{4.65}$$

4.5 Group of Allowed Transformations.

We shall now adapt the coordinate system to the case of proper homothetic changes of the vacuum metric

$$g_{\mu\nu} \to g'_{\mu\nu} = e^{2\phi} g_{\mu\nu}, \tag{4.66}$$

where ϕ is a constant.

Consider a diffeomorphism φ from one connected manifold M to another $\tilde{M}$. Write $q = \varphi(p) \in \tilde{M}$ for each point $p \in M$. Let $\{e_a\}$ be a basis in $T_p(M)$, the tangent space at $p \in M$, and let $\{\tilde{e}_a\}$ be a basis in $T_q(\tilde{M})$, the tangent space at $q \in \tilde{M}$. φ defines the linear map φ_*,

$$\varphi_*: T_p(M) \to T_q(\tilde{M}).$$

Let $\{\epsilon^a\}$, $\{\tilde{\epsilon}^a\}$ be bases in the cotangent spaces $T^*_p(M)$, $T^*_q(\tilde{M})$ at $p \in M$, $q \in \tilde{M}$ respectively. φ induces the linear map φ^*,

$$\varphi^* : T^*_q(\tilde{M}) \to T^*_p(M)$$

by the requirement that the inner product (,) is preserved, thus:

$$(\epsilon^a, e_b)(p) = (\varphi^*\tilde{\epsilon}^a, e_b)(p) = (\tilde{\epsilon}^a, \varphi_* e_b)(q) = (\tilde{\epsilon}^a, \tilde{e}_b)(q) = \delta^a_b. \tag{4.67}$$

Let $\{\hat{e}_a\} = \{e^{-\phi}\tilde{e}_a\}$ be a new basis in $T_q(\tilde{M})$, and let $\{e'_a\}$ be a basis in $T_q(\tilde{M})$ obtained from $\{\hat{e}_a\}$ by a Lorentz transformation which leaves the direction of $\hat{e}_4$ unchanged. Then the transformation

* R.P. Kerr & G. Weir, Private communication.

equations (4.46) apply since we are demanding that $\Gamma_{423} = 0$ in the manifold. We have

$$\begin{aligned} e'_1 &= e^{-iB}\hat{e}_1 &= e^{-\phi}e^{-iB}\tilde{e}_1 \,, \\ e'_3 &= e^{-A}\hat{e}_3 &= e^{-\phi}e^{-A}\tilde{e}_3 \,, \\ e'_4 &= e^{A}\hat{e}_4 &= e^{-\phi}e^{A}\tilde{e}_4 \,, \end{aligned} \tag{4.68}$$

where A and B are real functions.

Also let $\{\hat{\epsilon}^a\} = \{e^{\phi}\tilde{\epsilon}^a\}$ be a new basis in $T^*_q(\tilde{M})$ and introduce the basis $\{\epsilon'^a\}$ in $T^*_q(\tilde{M})$ by

$$(\epsilon'^a, e'^b)(q) = \delta^a_b \,. \tag{4.69}$$

Define the commutator $[X,Y]$ of two vector fields X and Y on M by

$$[X,Y]f = X(Yf) - Y(Xf)$$

for any function f on M. In particular

$$[e_a,e_b]f = e_a(e_bf) - e_b(e_af) = (e_a{}^{\mu}e_b{}^{\nu}{}_{,\mu} - e_b{}^{\mu}e_a{}^{\nu}{}_{,\mu})f_{,\nu} \tag{4.70}$$

upon using (4.2).

The structure constants $C^c{}_{ab}$ are defined by

$$C^c{}_{ab} = (\epsilon^c,[e_a,e_b]) = -\,C^c{}_{ba}$$

which implies

$$[e_a,e_b] = C^c{}_{ab}e_c \,. \tag{4.71}$$

This last equation is equivalent to the Maurer-Cartan equations

$$d\epsilon^c = -\frac{1}{2}C^c{}_{ab}\epsilon^a \wedge \epsilon^b.$$

Equations (4.70) and (4.71) give

$$e_a{}^{\mu}e_b{}^{\nu}{}_{,\mu} - e_b{}^{\mu}e_a{}^{\nu}{}_{,\mu} = C^c{}_{ab}e_c{}^{\nu}.$$

Multiplying by $e_{d\nu}$ we get

$$(e_{d\nu,\mu} - e_{d\mu,\nu})e_a{}^{\mu}e_b{}^{\nu} = C^c{}_{ab}e_c{}^{\nu}e_{d\nu} = g_{cd}C^c{}_{ab}$$

i.e. $$e_{d[\mu,\nu]}e_a{}^{\mu}e_b{}^{\nu} = -\frac{1}{2}C_{dab} \,.$$

Now $\Gamma_{d[ab]} = - e_{d\mu;\nu}e_{[a}{}^{\mu}e_{b]}{}^{\nu} = - e_{d[\mu;\nu]}e_a{}^{\mu}e_b{}^{\nu}$

$$= - e_{d[\mu,\nu]}e_a{}^{\mu}e_b{}^{\nu} .$$

Hence the rotation coefficients and the structure constants are related by

$$\Gamma_{d[ab]} = \frac{1}{2} C_{dab} . \tag{4.72}$$

Using (4.23) we obtain

$$\Gamma_{abc} = \frac{1}{2}(C_{abc} + C_{bca} - C_{cab}). \tag{4.73}$$

Under the mapping φ we require

$$\varphi_*[e_a,e_b] = [\varphi_* e_a, \varphi_* e_b] .$$

Define, at $q \in \widetilde{M}$,

$$\begin{aligned}
\hat{C}^c{}_{ab}\hat{e}_c &= [\hat{e}_a,\hat{e}_b] \\
&= [e^{-\phi}\tilde{e}_a, e^{-\phi}\tilde{e}_b] = [e^{-\phi}\varphi_* e_a, e^{-\phi}\varphi_* e_b] \\
&= e^{-2\phi}[\varphi_* e_a, \varphi_* e_b] + e^{-\phi}\{(\varphi_* e_a)e^{-\phi}\}(\varphi_* e_b) \\
&\qquad - e^{-\phi}\{(\varphi_* e_b)e^{-\phi}\}(\varphi_* e_a) \\
&= e^{-2\phi}[\varphi_* e_a, \varphi_* e_b], \quad \text{since } \phi \text{ is constant,} \\
&= e^{-2\phi}\varphi_*[e_a,e_b] = e^{-2\phi}\varphi_*(C^c{}_{ab}e_c) \\
&= e^{-2\phi}\widetilde{C}^c{}_{ab}\tilde{e}_c ,
\end{aligned}$$

where $\varphi_* C^c{}_{ab} = \widetilde{C}^c{}_{ab}$. Therefore

$$\hat{C}^c{}_{ab}\hat{e}_c = e^{-\phi}\widetilde{C}^c{}_{ab}\hat{e}_c$$

and so, at $q \in \widetilde{M}$,

$$\hat{C}^c{}_{ab} = e^{-\phi}\widetilde{C}^c{}_{ab} . \tag{4.74}$$

If Γ'_{abc}, $\hat{\Gamma}_{abc}$, $\widetilde{\Gamma}_{abc}$ are the rotation coefficients on $\widetilde{M}$ defined with respect to the bases $\{e'_a\}$, $\{\hat{e}_a\}$, $\{\tilde{e}_a\}$ respectively, then using (4.73) and (4.74) at $q \in \widetilde{M}$, we have

$$\begin{aligned}
\hat{\Gamma}_{abc} &= \frac{1}{2}(\hat{C}_{abc} + \hat{C}_{bca} - \hat{C}_{cab}) \\
&= e^{-\phi}\widetilde{\Gamma}_{abc} .
\end{aligned} \tag{4.75}$$

Hence

$$\hat{\omega}_{ab}(q) = \hat{\Gamma}_{abc}(q).\hat{\epsilon}^c(q) = e^{-\phi}\tilde{\Gamma}_{abc}(q).e^{\phi}\tilde{\epsilon}^c(q)$$
$$= \tilde{\omega}_{ab}(q) \qquad = \varphi^{*-1}(\omega_{ab}(p)). \tag{4.76}$$

Using (4.53) we have, therefore,

$$\hat{\omega}_{42}(q) = \tilde{\omega}_{42}(q) = -\varphi^{*-1}(d\zeta(p))$$
$$= -d(\varphi^{*-1}\zeta(p)) = -d(\zeta \circ \varphi^{-1})(q)$$
$$= -d\tilde{\zeta}(q), \tag{4.77}$$

where $\tilde{\zeta} = \zeta \circ \varphi^{-1}$ is a differentiable function on $\tilde{M}$.

If ζ' is a differentiable function on $\tilde{M}$, we have also, by the reasoning of Section 4.4, that

$$\omega'_{42}(q) = -d\zeta'(q) \tag{4.78}$$

and

$$\omega'_{42}(q) = e^{A+iB}\hat{\omega}_{42}(q). \tag{4.79}$$

Therefore, from (4.77) and (4.79), we have

$$d\zeta' = e^{A+iB}d\tilde{\zeta}$$

which implies

$$\zeta' = \Phi(\tilde{\zeta}), \tag{4.80}$$

where

$$\Phi_{\tilde{\zeta}} \equiv \partial_{\tilde{\zeta}}\Phi = e^{A+iB}.$$

The function $\tilde{\zeta}$ is thus coupled to the tetrad through

$$|\Phi_{\tilde{\zeta}}| = e^A. \tag{4.81}$$

Define functions $\tilde{u}$, $\tilde{v}$, $\tilde{\Omega}$ on $\tilde{M}$ by

$$\tilde{u} = u \circ \varphi^{-1}, \qquad \tilde{v} = v \circ \varphi^{-1}, \qquad \tilde{\Omega} = \Omega \circ \varphi^{-1} \tag{4.82}$$

and also define

$$\tilde{\epsilon}_4 = d\tilde{u} + \tilde{\Omega}d\tilde{\zeta} + \bar{\tilde{\Omega}}d\bar{\tilde{\zeta}}. \tag{4.83}$$

Let u', Ω' be functions on $\tilde{M}$ such that

$$\epsilon'_4 = du' + \Omega' d\zeta' + \bar{\Omega}' d\bar{\zeta}'. \tag{4.84}$$

Then because of

$$\epsilon'_4 = e^A\hat{\epsilon}_4 = e^{\phi}e^A\tilde{\epsilon}_4$$

we have

$$du' + \Omega' d\zeta' + \bar{\Omega}' d\bar{\zeta}' = e^{\phi}|\Phi_{\tilde{\zeta}}|(d\tilde{u} + \tilde{\Omega}d\tilde{\zeta} + \bar{\tilde{\Omega}}d\bar{\tilde{\zeta}}). \tag{4.85}$$

Now

$$u' = u'(\tilde{\zeta}, \bar{\tilde{\zeta}}, \tilde{u}, \tilde{v})$$

and if we restrict the function u' by requiring

$$u'_{,3} = 1, \qquad\qquad u'_{,4} = 0 \tag{4.86}$$

we get

$$du' = \frac{\partial u'}{\partial \tilde{\zeta}} d\tilde{\zeta} + \frac{\partial u'}{\partial \bar{\tilde{\zeta}}} d\bar{\tilde{\zeta}} + \frac{\partial u'}{\partial \tilde{u}} d\tilde{u} .$$

Then (4.80) and (4.85) give

$$\frac{\partial u'}{\partial \tilde{u}} d\tilde{u} + \left(\frac{\partial u'}{\partial \tilde{\zeta}} + \Phi_{\tilde{\zeta}} \Omega'\right) d\tilde{\zeta} + \left(\frac{\partial u'}{\partial \bar{\tilde{\zeta}}} + \bar{\Phi}_{\bar{\tilde{\zeta}}} \bar{\Omega}'\right) d\bar{\tilde{\zeta}}$$

$$= e^{\phi} |\Phi_{\tilde{\zeta}}| (d\tilde{u} + \tilde{\Omega} d\tilde{\zeta} + \bar{\tilde{\Omega}} d\bar{\tilde{\zeta}}) \tag{4.87}$$

Hence

$$\frac{\partial u'}{\partial \tilde{u}} = e^{\phi} |\Phi_{\tilde{\zeta}}|$$

which has the solution

$$u'(q) = e^{\phi} |\Phi_{\tilde{\zeta}}| (\tilde{u} + \tilde{S})(q), \tag{4.88}$$

where $\tilde{S}(\tilde{\zeta}, \bar{\tilde{\zeta}})$ is a real function on $\tilde{M}$.

Equation (4.87) also gives

$$\Phi_{\tilde{\zeta}} \Omega' = e^{\phi} |\Phi_{\tilde{\zeta}}| \tilde{\Omega} - \frac{\partial u'}{\partial \tilde{\zeta}}$$

so that

$$\Omega'(q) = e^{\phi} |\Phi_{\tilde{\zeta}}| \Phi_{\tilde{\zeta}}^{-1} [\tilde{\Omega} - \tilde{S}_{\tilde{\zeta}} - \tfrac{1}{2} \Phi_{\tilde{\zeta}\tilde{\zeta}} \Phi_{\tilde{\zeta}}^{-1} (\tilde{u} + \tilde{S})](q), \tag{4.89}$$

where we have used (4.88)

The complex divergence relative to bases $\{e'_a\}$ and $\{\tilde{e}_a\}$ is given by

$$\rho'(q) = -\Gamma'_{421}(q) = \epsilon'_{4\mu;\nu}\, e'_2{}^{\mu}\, e'_1{}^{\nu}(q)$$

$$= (e^{\phi} e^{A} \tilde{\epsilon}_{4\mu;\nu})(e^{-\phi} e^{iB} \tilde{e}_2{}^{\mu})(e^{-\phi} e^{-iB} \tilde{e}_1{}^{\nu})(q)$$

$$= - e^{-\phi} e^{A}\, \tilde{\Gamma}_{421}(q)$$

$$= e^{-\phi} |\Phi_{\tilde{\zeta}}| \tilde{\rho}(q), \tag{4.90}$$

where $\tilde{\rho} = \rho \circ \varphi^{-1}$.

Let v' be a function on $\tilde{M}$ defined by

$$v' = \mathrm{Re}\, \rho'^{-1}.$$

Then, using (4.90),

$$v'(q) = \mathrm{Re}(e^{\phi}|\Phi_{\tilde{\zeta}}|^{-1}\,\tilde{\rho}^{-1})$$

$$= e^{\phi}|\Phi_{\tilde{\zeta}}|^{-1}\,\tilde{v}(q). \tag{4.91}$$

Let Δ' be a function on $\tilde{M}$ defined by

$$\rho' = (v' + \Delta')^{-1}.$$

Then

$$\Delta'(q) = (\rho'^{-1} - v')(q)$$
$$= e^{\phi}|\Phi_{\tilde{\zeta}}|^{-1}(\tilde{\rho}^{-1} - \tilde{v})(q)$$

$$\therefore \qquad \Delta'(q) = e^{\phi}|\Phi_{\tilde{\zeta}}|^{-1}\,\tilde{\Delta}(q), \tag{4.92}$$

where $\tilde{\Delta} = \Delta \circ \varphi^{-1}$ and we have used (4.90) and (4.91).

Let $\tilde{\mu} = \mu \circ \varphi^{-1}$, where $\mu(\zeta, \bar{\zeta}, u)$ is the complex mass function introduced in Section 4.4. Under the map φ we have

$$\varphi^{*}\tilde{C}^{(3)} = C^{(3)} = \mu\rho^{3}$$

which defines $\tilde{C}^{(3)}$

i.e. $$\tilde{C}^{(3)} = \tilde{\mu}\,\tilde{\rho}^{3}.$$

At $q \in \tilde{M}$ we have (see [198], equation (3.17))

$$\hat{C}^{(3)} = 2\hat{\rho}(\hat{\Gamma}_{123} + \hat{\Gamma}_{343})$$

$$= e^{-2\phi}.2\tilde{\rho}(\tilde{\Gamma}_{123} + \tilde{\Gamma}_{343}) \quad \text{from (4.75)},$$

$$= e^{-2\phi}\tilde{C}^{(3)}, \tag{4.93}$$

where $\hat{C}^{(3)}$ is defined relative to the basis $\{\hat{e}_a\}$ at $q \in \tilde{M}$. Alternatively, we could use (4.44). Also, by the theory of Section 4.4, we have

$$C^{(3)\prime} = \mu'\rho'^{3}. \tag{4.94}$$

Under a tetrad transformation (4.68) we find

$$C^{(3)\prime} = \hat{C}^{(3)}$$

(see also (4.42) with $\gamma = 0$). Hence

$$(\mu'\rho'^{3})(q) = e^{-2\phi}(\tilde{\mu}\,\tilde{\rho}^{3})(q)$$

which gives, upon using (4.90),

$$\mu'(q) = e^{\phi}|\Phi_{\tilde{\zeta}}|^{-3}\,\tilde{\mu}(q). \tag{4.95}$$

Now let us view φ as a mapping, not of one manifold into another, but of the manifold M into itself,

$$\varphi : M \to M.$$

Then we can interpret the transformation equations (4.80), (4.88), (4.89), (4.91), (4.92) and (4.95) in the following ways:

(i) φ = identity map ($q \equiv p$), $\phi = 0$. This corresponds to a change of coordinates at the same point p in M.

(ii) $\varphi \neq$ identity map (q distinct from p, in general), $\phi \neq 0$. This corresponds to a proper homothetic motion, where the coordinate system is "dragged along" by φ i.e. the same coordinates are used at p and q. The symbols with a tilde are to be identified with the symbols without.

Collecting together the group of allowed transformations under a homothetic change of metric (4.66) and a tetrad rotation, we have

$$\begin{aligned} \zeta' &= \Phi(\zeta), \\ u' &= e^{\phi}|\Phi_\zeta|(u + S), \\ v' &= e^{\phi}|\Phi_\zeta|^{-1}v, \end{aligned} \qquad (4.96)$$

where $S(\zeta,\bar{\zeta})$ is a real function, and

$$\begin{aligned} \Omega' &= e^{\phi}|\Phi_\zeta|\Phi_\zeta^{-1}[\Omega - S_\zeta - \tfrac{1}{2}\Phi_{\zeta\zeta}\Phi_\zeta^{-1}(u + S)], \\ \mu' &= e^{\phi}|\Phi_\zeta|^{-3}\mu, \\ \Delta' &= e^{\phi}|\Phi_\zeta|^{-1}\Delta. \end{aligned} \qquad (4.97)$$

The tetrad vectors transform as follows:

$$\begin{aligned} e_1' &= e^{-\phi}|\Phi_\zeta|\Phi_\zeta^{-1} e_1, \\ e_3' &= e^{-\phi}|\Phi_\zeta|^{-1} e_3, \\ e_4' &= e^{-\phi}|\Phi_\zeta| e_4. \end{aligned} \qquad (4.98)$$

When $\phi = 0$ the homothety becomes an isometry and equations (4.96), (4.97) and (4.98) reduce to those of Kerr & Debney [199].

4.6 Local one-parameter groups. Homothetic Killing vectors.

Let I_δ be an open interval $(-\delta,\delta)$ of $\mathbb{R}$ and let U be an open set of M. A local 1-parameter Lie group of local (infinitesimal) transformations of M is a mapping $(t,p) \to \varphi_t(p)$ of $I_\delta \times U$ into M which satisfies the following conditions [2]:

(i) For each $t \in I_\delta$ the map $\varphi_t : p \to \varphi_t(p)$ is a diffeomorphism of U onto the open set $\varphi_t(U)$ of M.

(ii) If t,s and t+s are in I_δ, and if p, $\varphi_s(p)$ are in U, then $\varphi_{t+s}(p) = (\varphi_t \circ \varphi_s)(p) = \varphi_t(\varphi_s(p))$.

The group properties are completed by noting that

$$(\varphi_t \circ \varphi_s)(p) = (\varphi_s \circ \varphi_t)(p), \quad (\varphi_t)^{-1}(p) = \varphi_{-t}(p),$$

and $\varphi_o(p)$ is the identity.

Each local 1-parameter group of local transformations φ_t induces a vector field X defined on $U \subset M$ as follows:
For every point $p \in U$, the vector X_p is tangent to the curve $\lambda(t) = \varphi_t(p)$ at $\lambda(0) = p$. This curve is the integral curve of X. If (x^α) are local coordinates so that the curve $\lambda(t)$ is given parametrically by $x^\alpha(t)$ and the vector X has components X^μ, then this curve is locally a solution of the set of differential equations

$$dx^\mu/dt = X^\mu(x^\alpha(t)).$$

Conversely, it can be proved [2] that every vector field X on M generates a local 1-parameter group of local transformations.

Geometrically, the diffeomorphism $\varphi_t : U \to M$ takes each point $p \in U$ a parameter distance t along the integral curves of X. The tangent vector field X to the integral curves $\lambda(t)$ of the group φ_t on M is defined by

$$X_p f = \left[\frac{d}{dt} f(\varphi_t(p))\right]_{t=0}, \qquad (4.99)$$

where f is a differentiable function on M.

If (x^α) are local coordinates at $p \in M$, and (x'^α) are local coordinates at $q = \varphi_t(p)$ on $\lambda(t)$, then

$$x'^\alpha = x'^\alpha(x^\beta,t), \qquad x'^\alpha(x^\beta,0) = x^\beta.$$

Putting $f = x^\alpha$ in (4.99) gives

$$X^\alpha(p) = (Xx^\alpha)(p) = \left[\frac{\partial}{\partial t} x'^\alpha\right]_{t=0} , \tag{4.100}$$

where $X^\alpha(p)$ denotes the components of X relative to a coordinate basis $\{\partial_\mu\}$ at p, so that

$$X_p = X^\mu(p)\partial_\mu = \left[\frac{\partial x'^\mu}{\partial t}\right]_{t=0} \partial_\mu . \tag{4.101}$$

A diffeomorphism $\varphi : M \to M$ is a conformal symmetry if the metric $g(e_a,e_b)$ at $p \in M$ is related to the mapped metric $\varphi_* g(\varphi_* e_a,\varphi_* e_b)$ at $q = \varphi(p) \in M$ by

$$\varphi_* g(\varphi_* e_a,\varphi_* e_b)(q) = e^{2\phi} g(\varphi_* e_a,\varphi_* e_b)(q) \tag{4.102}$$

for some non-zero differentiable function ϕ on M.

If the local 1-parameter group of diffeomorphisms φ_t generated by a vector field $\tilde{K}$ is a group of conformal motions (i.e. for each t, the transformation φ_t is a conformal symmetry), the vector field $\tilde{K}$ is a conformal Killing vector field.

Suppose now that $\tilde{K}$ is a homothetic Killing vector (HKV) i.e. the ϕ in (4.102) is constant, and that the local 1-parameter group φ_t generated by $\tilde{K}$ is defined by the coordinate transformation $x^\alpha \to x'^\alpha$, where $(x^\alpha) = (\zeta, \bar{\zeta}, u, v)$ and the $x'^\alpha(x^\beta,t)$ are given by

$$\begin{aligned} \zeta' &= \Phi(\zeta;t), \\ u' &= e^{\phi(t)}|\Phi_\zeta(\zeta;t)|[u + S(\zeta,\bar{\zeta};t)], \\ v' &= e^{\phi(t)}|\Phi_\zeta(\zeta;t)|^{-1}v, \end{aligned} \tag{4.103}$$

according to (4.96). That is, we are now visualizing the homothetic motion as a coordinate change $x^\alpha \to x'^\alpha$ rather than a point transformation which drags along the coordinate system. Since $x'^\alpha(t=0) = x^\alpha$, we have

$$\Phi(\zeta;0) = \zeta, \qquad \Phi_\zeta(\zeta;0) = 1, \qquad S(\zeta,\bar{\zeta};0) = 0 = \phi(0).$$

Then, by (4.100), the components of the HKV are

$$\tilde{K}^1 = \left[\frac{\partial\zeta'}{\partial t}\right]_{t=0} = \left[\frac{\partial\Phi}{\partial t}\right]_{t=0} \equiv \alpha(\zeta),$$

$$\tilde{K}^3 = \left[\frac{\partial u'}{\partial t}\right]_{t=0} = \left[u' \frac{\partial \phi}{\partial t}\right]_{t=0} + \left[e^{\phi(t)} \frac{\partial}{\partial t}\{|\Phi_\zeta|(u+S)\}\right]_{t=0}$$

$$= au + u\,\mathrm{Re}(\alpha_\zeta) + R,$$

$$\tilde{K}^4 = \left[\frac{\partial v'}{\partial t}\right]_{t=0} = \left[v' \frac{\partial \phi}{\partial t}\right]_{t=0} + \left[e^{\phi(t)} \frac{\partial}{\partial t}\{|\Phi_\zeta|^{-1}v\}\right]_{t=0}$$

$$= av - v\,\mathrm{Re}(\alpha_\zeta),$$

where $\left[\frac{\partial \phi}{\partial t}\right]_{t=0} \equiv a$ (real constant),

$$\left[\frac{\partial S}{\partial t}\right]_{t=0} \equiv R(\zeta,\bar{\zeta})$$

and we have used the result

$$\left[\frac{\partial}{\partial t}|\Phi_\zeta(\zeta;t)|\right]_{t=0} = \left[\frac{1}{2}|\Phi_\zeta(\zeta;t)|^{-\frac{1}{2}}(\Phi_{t\zeta}\bar{\Phi}_{\bar{\zeta}} + \Phi_\zeta\bar{\Phi}_{t\bar{\zeta}})\right]_{t=0} = \mathrm{Re}(\alpha_\zeta).$$

Hence, by (4.101), we have

$$\boxed{\tilde{K} = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + \mathrm{Re}(\alpha_\zeta)(u\partial_u - v\partial_v) + R\partial_u + a(u\partial_u + v\partial_v)} \tag{4.104}$$

as the form of a HKV which generates the allowed local 1-parameter group of local homothetic transformations, in the sense of Section 4.5.

The form of $\tilde{K}$ will change under an allowed transformation on M to

$$\tilde{K} = \alpha'\partial_{\zeta'} + \bar{\alpha}'\partial_{\bar{\zeta}'} + \mathrm{Re}(\alpha'_{\zeta'})(u'\partial_{u'} - v'\partial_{v'}) + R'\partial_{u'} + a(u'\partial_{u'} + v'\partial_{v'})$$

where ζ', u', v' are related to ζ, u, v by equations (4.98).

To find the transformation equations for α and R we need

$$\partial_\zeta = \Phi_\zeta\partial_{\zeta'} + e^\phi|\Phi_\zeta|S_\zeta\partial_{u'} + \frac{1}{2}\Phi_{\zeta\zeta}\Phi_\zeta^{-1}(u'\partial_{u'} - v'\partial_{v'}),$$

$$\partial_u = e^\phi|\Phi_\zeta|\partial_{u'},$$

$$\partial_v = e^\phi|\Phi_\zeta|^{-1}\partial_{v'}.$$

These are substituted in (4.104) and the result compared with the primed expression for $\tilde{K}$, giving

$$\alpha' = \Phi_\zeta\alpha, \tag{4.105}$$

and

$$R' = e^\phi|\Phi_\zeta|[R - (\mathrm{Re}(\alpha_\zeta) + a)S + \tilde{K}S]. \tag{4.106}$$

Using these transformation equations it is possible to write $\tilde{K}$ in simple form. For example, given a HKV $\tilde{K}$ with $\alpha \neq 0$ we can transform R to zero by solving

$$\tilde{K}S - (\mathrm{Re}(\alpha_{\zeta}) + a)S + R = 0$$

for S. If, further, we choose $\alpha' = 1$, we can solve $\Phi_{\zeta}\alpha = 1$ for Φ. The transformations (4.96) are then completely determined and put the HKV in the form

$$\tilde{K} = \partial_{\zeta} + \partial_{\bar{\zeta}} + a(u\partial_u + v\partial_v),$$

where we have dropped the primes on the new coordinates. On the other hand, if $\alpha = 0$ the form of $\tilde{K}$ in (4.104) reduces to

$$\tilde{K} = R\partial_u + a(u\partial_u + v\partial_v).$$

Introduce a new function u^* through

$$au^* = au + R.$$

Then, after dropping the asterisk, we have

$$\tilde{K} = a(u\partial_u + v\partial_v).$$

Since $a \neq 0$ for a proper homothetic motion and it is not possible to transform a non-zero α to zero, we thus arrive at the following two <u>canonical forms</u> of the HKV (we are at liberty to set $a = 1$):

$$\text{(i)} \quad \tilde{K}_1 = \partial_{\zeta} + \partial_{\bar{\zeta}} + u\partial_u + v\partial_v,$$
$$\text{(ii)} \quad \tilde{K}_2 = u\partial_u + v\partial_v. \tag{4.107}$$

These two forms are mutually exclusive. Moreover,

$$[\tilde{K}_1, \tilde{K}_2] = 0$$

in accordance with Theorem 3.2, page 62.

Of interest in later chapters is the form of the finite transformation equations (FTEs) corresponding to an infinitesimal homothetic motion generated by a HKV of the form

$$\tilde{K} = m\zeta\partial_{\zeta} + \bar{m}\bar{\zeta}\partial_{\bar{\zeta}} + pu\partial_u + qv\partial_v, \tag{4.108}$$

where m, p,q are constants. The FTEs are given by equation (1.17), and, for $\tilde{K}$ as in (4.108) are

$$\zeta^* = b^m\zeta, \qquad u^* = b^p u, \qquad v^* = b^q v, \tag{4.109}$$

where b $(=e^t)$ is a real constant. Consequently, a metric which admits the HKV (4.108) is homothetically invariant under the coordinate changes (4.109).

4.7 Homothetic Killing equations.

Expression (4.104) for $\widetilde{K}$ is the form which a HKV generating the local 1-parameter group of local transformations on M, consistent with a Lorentz rotation of the tetrad as well as the homothetic change of metric in a vacuum Einstein space M, must take. However, a vector of the form (4.104) is not necessarily a HKV; in order that it be so, it must satisfy the homothetic Killing equations [(1.15) with $X = \widetilde{K}$]

$$\pounds_{\widetilde{K}} g_{\mu\nu} = \psi g_{\mu\nu} ,$$

where $4\psi = \widetilde{K}^{\mu}{}_{;\mu}$ is a constant function on M.

It can be seen from (4.57), (4.58) and (4.61) that the only quantities appearing in the $g_{\mu\nu}$ are μ and Ω (and derivatives of Ω). We need, therefore, to find the homothetic Killing equations which will involve only the functions Ω and μ for a HKV of the form (4.104).

It is convenient to use the following definition [229]: Let $y_A(x)$ denote the components of the geometrical object at a point with local coordinates x^{μ}. Under a mapping φ of the manifold, let the transformed geometrical object be denoted by y'_A. A symmetry of the local geometrical object is defined by

$$y'_A(x') = y_A(x') \tag{4.110}$$

i.e. the difference between the original components $y_A(x')$ at the point x'^{μ} and the transformed components $y'_A(x')$ at a point that is mapped onto x'^{μ} by the mapping φ is zero.

Now let the manifold mapping be a local 1-parameter transformation φ_t generated by the HKV of (4.104). Then (4.110) defines a homothety and we have

$$\Omega'(x') = \Omega(x') \qquad \qquad (4.111)$$

and $$\mu'(x') = \mu(x'),$$

where the transformation equations (4.97) obtain. We have $x' = x'(x,t)$ and

$$\frac{\partial}{\partial t}\Big[\Omega'(x') - \Omega(x')\Big]_{t=0} = 0,$$

$$\frac{\partial}{\partial t}\Big[\mu'(x') - \mu(x')\Big]_{t=0} = 0.$$

Explicitly, these are

$$\frac{\partial}{\partial t}\Big[e^{\phi(t)}|\Phi_{\bar\zeta}(\zeta;t)|\Phi_{\bar\zeta}^{-1}(\bar\zeta;t)(\Omega(x) - S_{\zeta}(\zeta,\bar\zeta;t) - \tfrac{1}{2}\Phi_{\zeta\bar\zeta}(\zeta;t)\Phi_{\bar\zeta}^{-1}(\bar\zeta;t)\{u(x) + S(\zeta,\bar\zeta;t)\}) - \Omega(x')\Big]_{t=0} = 0 \tag{4.112}$$

and

$$\frac{\partial}{\partial t}\left[e^{\phi(t)}|\Phi_\zeta(\zeta;t)|^{-3}\mu(x) - \mu(x')\right]_{t=0} = 0. \qquad (4.113)$$

Using the results

$$\frac{\partial}{\partial t}\left[\Phi_\zeta(\zeta;t)\right]_{t=0} = \left[\Phi_{t\zeta}(\zeta;t)\right]_{t=0} = \alpha_\zeta ,$$

$$\frac{\partial}{\partial t}\left[|\Phi_\zeta(\zeta;t)|\right]_{t=0} = \mathrm{Re}(\alpha_\zeta),$$

$$\left[\frac{\partial\phi}{\partial t}\right]_{t=0} \equiv a \text{ (real constant)},$$

$$\frac{\partial}{\partial t}\left[S_\zeta(\zeta,\bar{\zeta};t)\right]_{t=0} = \left[S_{t\zeta}(\zeta,\bar{\zeta};t)\right]_{t=0} = R_\zeta \text{ , where } R = \bar{R},$$

$$\frac{\partial}{\partial t}\left[\Omega(x')\right]_{t=0} = \left[\frac{\partial x'^\mu}{\partial t}\ \frac{\partial\Omega}{\partial x'^\mu}\right]_{t=0} = \tilde{K}\Omega ,$$

$$\frac{\partial}{\partial t}[\mu(x')]_{t=0} = \tilde{K}\mu ,$$

the homothetic Killing equations (4.112) and (4.113) become

$$\text{(I)} \quad (\tilde{K} - a)\Omega + \tfrac{1}{2}(\alpha_\zeta - \bar{\alpha}_{\bar{\zeta}})\Omega + \tfrac{1}{2}\alpha_{\zeta\zeta}u + R_\zeta = 0 \qquad (4.114)$$

and

$$\text{(III)} \quad (\tilde{K} + 3\mathrm{Re}(\alpha_\zeta) - a)\mu = 0, \qquad (4.115)$$

where $\tilde{K}$ is given by (4.104). The references (I) and (III) are used to match the corresponding equations for isometries in the Kerr & Debney paper [199] to which (4.114) and (4.115) reduce on putting a = 0.

The integrability conditions on the homothetic Killing equations are, for a vacuum space (see Chapter 1),

$$\pounds_{\tilde{K}} R_{abc}{}^{d} = 0, \qquad \pounds_{\tilde{K}}(R_{abc}{}^{d}{}_{;e}) = 0, \ \ldots \qquad (4.116)$$

In terms of the formalism developed in this chapter, these integrability conditions are obtained by successive differentiation of (I) and (III) with respect to u, ζ and $\bar{\zeta}$ (Ω and μ are independent of v). However, not all of the conditions so obtained are independent.

The commutators

$$
\begin{aligned}
[\partial_u,\tilde{K}] &= (\mathrm{Re}(\alpha_\zeta) + a)\partial_u, \\
[\partial_{\bar\zeta},\tilde{K}] &= \bar{\alpha}_{\bar\zeta}\partial_{\bar\zeta} + R_{\bar\zeta}\partial_u + \frac{1}{2}\bar{\alpha}_{\bar\zeta\bar\zeta}(u\partial_u - v\partial_v), \\
[D,\tilde{K}] &= \alpha_\zeta D - \frac{1}{2}\alpha_{\zeta\zeta}\, v\partial_v
\end{aligned}
\tag{4.117}
$$

are useful in obtaining the integrability conditions.

Differentiating (4.114) with respect to u we get

$$\partial_u(\tilde{K}\Omega) - a\dot{\Omega} + \frac{1}{2}(\alpha_\zeta - \bar{\alpha}_{\bar\zeta})\dot{\Omega} + \frac{1}{2}\alpha_{\zeta\zeta} = 0,$$

where $\dot{\Omega} \equiv \partial_u\Omega$, and then using (4.117) this becomes

$$\text{(II)}\quad \tilde{K}\dot{\Omega} + \alpha\,\dot{\Omega} + \frac{1}{2}\alpha_{\zeta\zeta} = 0\ . \tag{4.118}$$

If we substitute (4.118) back into (4.114) we get (I) in the alternative form

$$\text{(I)}\quad (\tilde{K} - a)(\Omega - u\dot{\Omega}) + \frac{1}{2}(\alpha_\zeta - \bar{\alpha}_{\bar\zeta})(\Omega - u\dot{\Omega}) + R\dot{\Omega} + R_\zeta = 0. \tag{4.119}$$

Differentiating (4.118) with respect to u we get

$$\text{(IVd)}\quad (\tilde{K} + \alpha_\zeta + \mathrm{Re}(\alpha_\zeta) + a)\ddot{\Omega} = 0. \tag{4.120}$$

Differentiating (4.118) with respect to $\bar\zeta$ gives

$$\text{(IVb)}\quad (\tilde{K} + 2\,\mathrm{Re}(\alpha_\zeta))(\bar{D}\dot{\Omega}) = 0. \tag{4.121}$$

Differentiating (4.115) with respect to u gives

$$\text{(IVa)}\quad (\tilde{K} + 4\,\mathrm{Re}(\alpha_\zeta))\dot{\mu} = 0. \tag{4.122}$$

Differentiating (4.114) with respect to $\bar\zeta$ and using the fact that $R_{\zeta\bar\zeta}$ is real, we get

$$\text{(IVc)}\quad (\tilde{K} + \mathrm{Re}(\alpha_\zeta) - a)\Delta = 0. \tag{4.123}$$

We have now got all first order integrability conditions. Continuing in like manner would give higher order conditions. For easy reference, the full set of homothetic Killing equations and their first order integrability conditions are now collected together:

$$
\begin{aligned}
&\text{(I)} && (\tilde{K} - a)(\Omega - u\dot{\Omega}) + \frac{1}{2}(\alpha_\zeta - \bar{\alpha}_{\bar\zeta})(\Omega - u\dot{\Omega}) + R\dot{\Omega} + R_\zeta = 0, \\
&\text{(II)} && \tilde{K}\dot{\Omega} + \alpha_\zeta\dot{\Omega} + \frac{1}{2}\alpha_{\zeta\zeta} = 0, \\
&\text{(III)} && (\tilde{K} + 3\,\mathrm{Re}(\alpha_\zeta) - a)\mu = 0, \\
&\text{(IVa)} && (\tilde{K} + 4\,\mathrm{Re}(\alpha_\zeta))\dot{\mu} = 0, \\
&\text{(IVb)} && (\tilde{K} + 2\,\mathrm{Re}(\alpha_\zeta))(\bar{D}\dot{\Omega}) = 0, \\
&\text{(IVc)} && (\tilde{K} + \mathrm{Re}(\alpha_\zeta) - a)\Delta = 0, \\
&\text{(IVd)} && (\tilde{K} + \alpha_\zeta + \mathrm{Re}(\alpha_\zeta) + a)\ddot{\Omega} = 0.
\end{aligned}
\tag{4.124}
$$

The homothetic Killing equations can be expressed as a system of linear differential equations of the form

$$z^A{}_{,\mu} = Q^A{}_\mu(z;x) \tag{4.125}$$

in the coordinates x^μ, where $z^A = (\alpha, \bar{\alpha}, \alpha_\zeta, \bar{\alpha}_{\bar{\zeta}}, R, a)$ are unknown functions. The integrability conditions may then be expressed as

$$z^A{}_{,[\mu\nu]} = 0, \qquad \text{or} \qquad Q^A{}_{[\mu,\nu]} = 0. \tag{4.126}$$

Explicitly, equations (4.125) are

$$\begin{aligned}
z^1{}_{,1} &= z^3, & z^1{}_{,i} &= 0 \quad (i=2,3,4)\\
z^2{}_{,2} &= z^4 = \overline{z^3}, & z^2{}_{,i} &= 0 \quad (i=1,3,4)\\
z^3{}_{,1} &= -2\tilde{K}\dot{\Omega} - 2z^3\dot{\Omega}, & z^3{}_{,i} &= 0 \quad (i=2,3,4)\\
z^4{}_{,2} &= -2\bar{\tilde{K}}\dot{\bar{\Omega}} - 2z^4\dot{\bar{\Omega}} = \overline{z^3{}_{,1}}, & z^4{}_{,i} &= 0 \quad (i=1,3,4)\\
z^5{}_{,1} &= (\tilde{K} - a + \tfrac{1}{2}z^3 - \tfrac{1}{2}z^4)(u\dot{\Omega} - \Omega) - z^5\dot{\Omega}, & &\\
z^5{}_{,2} &= \overline{z^5{}_{,1}}, & z^5{}_{,i} &= 0 \quad (i = 3,4)\\
z^6{}_{,\mu} &= 0 \quad (\mu = 1,2,3,4), & &
\end{aligned}$$

which are all linear in the z^A. Then (4.126) gives, for example, $z^5{}_{,13} = z^5{}_{,31} = 0$ and when written out fully this is just (4.120).

Now equation (4.115) is of the form (4.126) of an integrability condition and may be treated as such. Also, the dimension of the group of homothetic motions is 6-s, where 6 is the number of unknowns z^A, and s is the number of independent integrability conditions including (4.115). Hence we have the following result, which is an extension of the Kerr-Debney Lemma 3.3 [199] to the case of homothetic motions:

Theorem 4.1 The dimension of the group of homothetic motions of an algebraically special Einstein vacuum space with non-zero complex divergence is at most 6. When the dimension is 6, all integrability conditions are zero identically and the space is conformally flat, and hence flat.

The last part of the theorem is valid because $\ddot{\Omega} = D\dot{\Omega} = \mu = \Delta = 0$ gives

$$\Omega = uf(\zeta,\bar{\zeta}) + g(\zeta,\bar{\zeta}),$$

and then (4.121) gives

$$(\widetilde{K} + 2\mathrm{Re}(\alpha_\zeta))f_{\bar{\zeta}} = 0$$

which is another integrability condition. Since all integrability conditions are identically zero, this implies $f_{\bar{\zeta}} = 0$.

Therefore
$$\Omega = uf(\zeta) + g(\zeta,\bar{\zeta}).$$

Then
$$\partial_u D\Omega = f_\zeta - f^2$$

so that

$$\bar{D}\partial_u D\Omega = \partial_u \partial_u D\Omega = 0$$

which, together with $\mu = 0$, implies conformally flat space, by equation (4.64d). Furthermore, a conformally flat vacuum space is flat.

According to the Kerr-Debney Lemma, the maximum order of the group G_m of isometries admitted by the algebraically special vacuum space M is 5. Theorem 4.1 above shows that the maximum order of the group H_n admitted by M is 6. Thus we have confirmed the Collinson - French theorem [9] that $G_m \subseteq H_n$ and $m = n-1$, as it applies to the space M.

Since by Theorem 3.2 M admits at most one independent proper homothetic motion, we need concern ourselves only with those spaces M which admit one HKV together with up to 5 Killing vectors. However, M with 5 Killing vectors is flat [199] so we shall investigate space M which admit one HKV and up to 4 Killing vectors, since the non-flat spaces are of greatest interest.

4.8 The $(\zeta, \bar{\zeta}, s, r)$ coordinate system.

In subsequent chapters we shall be examining algebraically special vacuum Einstein spaces which admit a Killing vector of the type $K = e^{-p}\partial_u$, where $p = p(\zeta,\bar{\zeta})$, as well as a HKV. Kerr & Debney [199] showed that such a Killing vector K is admissible iff

$$\ddot{\Omega} = \dot{\mu} = \dot{\Delta} = 0 \qquad p = p(\zeta,\bar{\zeta}). \tag{4.127}$$

Defining

$$s = e^{p}u, \qquad r = e^{-p}v, \tag{4.128}$$

local coordinates $(\zeta, \bar{\zeta}, s, r)$ can be introduced so that the metric (4.60), (4.61) is now

$$d\tau^2 = 2(r^2 + d^2)e^{2p}d\zeta d\bar{\zeta} + 2[dr + i(d_{\bar{\zeta}}d\bar{\zeta} - d_{\zeta}d\zeta)]\kappa + 2\{R^{(2)} + \mathrm{Re}[m/(r + id)]\}\kappa^2, \tag{4.129}$$

where

$$\begin{aligned} \kappa &= e^{p}\epsilon_4 = ds + \Lambda d\zeta + \bar{\Lambda}d\bar{\zeta}, \\ \Lambda &= e^{p}(\Omega - p_{\zeta}u), \\ d &= -i\Delta e^{-p} = e^{-2p}\mathrm{Im}(\Lambda_{\bar{\zeta}}), \\ m &= \mu e^{-3p}, \end{aligned} \tag{4.130}$$

and $R^{(2)}$ is the 2-curvature of the 2-metric $e^{2p}d\zeta d\bar{\zeta}$,

$$R^{(2)} = e^{-2p}p_{\zeta\bar{\zeta}} \ . \tag{4.131}$$

Now (4.127) and the equations corresponding to (4.114) and (4.115) for $K = e^{-p}\partial_u = \partial_s$ give

$$\dot{\Omega} = p_{\zeta}, \qquad \dot{p} = 0, \tag{4.132}$$

where the dot denotes $\partial/\partial u$. From (4.130) and (4.132) we see that

$$\Lambda = \Lambda(\zeta,\bar{\zeta}), \qquad d = d(\zeta,\bar{\zeta}) = \bar{d}$$

and so the metric (4.129) is independent of s, as it should be in the presence of the Killing vector ∂_s.

For a homothetic change of metric the relation (4.85) is, when expressed in the $(\zeta,\bar{\zeta},s,r)$ coordinates via (4.128),

$$e^{-p'}(ds' + \Lambda' d\zeta' + \bar{\Lambda}' d\bar{\zeta}') = e^{\phi}|\Phi_{\zeta}|e^{-p}(ds + \Lambda d\zeta + \bar{\Lambda}d\bar{\zeta}), \tag{4.133}$$

where

$$\zeta' = \Phi(\zeta). \tag{4.134}$$

Since Λ and Λ' are independent of s, (4.133) can be integrated to give

$$s' = e^{\phi}C_0(s+A), \tag{4.135}$$

where C_0 is a real constant and $A(\zeta,\bar{\zeta})$ is a real function. From (4.133) and (4.134) we have

$$e^{-p'}\left(\frac{\partial s'}{\partial s}\, ds + \Phi_{\zeta}\Lambda' d\zeta + \frac{\partial s'}{\partial \zeta}\, d\zeta + \bar{\Phi}_{\bar{\zeta}}\bar{\Lambda}' d\bar{\zeta} + \frac{\partial s'}{\partial \bar{\zeta}}\, d\bar{\zeta}\right)$$

$$= e^{\phi}|\Phi_{\zeta}|e^{-p}(ds + \Lambda d\zeta + \bar{\Lambda}d\bar{\zeta}). \tag{4.136}$$

From this equation, using (4.135) as well, there comes

$$e^{P'} = C_0|\Phi_\zeta|^{-1} e^P \tag{4.137}$$

and

$$\Lambda' = e^{\phi}C_0\Phi_\zeta^{-1}(\Lambda - A_\zeta). \tag{4.138}$$

Using (4.96), (4.137) in the definition $r' = e^{-P'}v'$ we get

$$r' = e^{\phi}C_0^{-1}r. \tag{4.139}$$

From (4.97) and (4.130) we have

$$m' = e^{\phi}C_0^{-3}m. \tag{4.140}$$

Also, from (4.97) and (4.137) we obtain

$$d' = e^{\phi}C_0^{-1}d. \tag{4.141}$$

Summarizing, we have the following allowed transformations under a homothetic change of metric and a Lorentz rotation of the tetrad, using the $(\zeta,\bar{\zeta},s,r)$ coordinate system:

$$\begin{aligned} \zeta' &= \Phi(\zeta), \\ s' &= e^{\phi}C_0(s+A), \\ r' &= e^{\phi}C_0^{-1}r, \end{aligned} \tag{4.142}$$

where $A = A(\zeta,\bar{\zeta})$ is a real function and C_0 is a real constant, and

$$\begin{aligned} \Lambda' &= e^{\phi}C_0\Phi_\zeta^{-1}(\Lambda - A_\zeta), \\ m' &= e^{\phi}C_0^{-3}m, \\ d' &= e^{\phi}C_0^{-1}d, \\ e^{P'} &= C_0|\Phi_\zeta|^{-1} e^P. \end{aligned} \tag{4.143}$$

Suppose next that the homothetic Killing vector $\tilde{K}$ generates a local 1-parameter group of local transformations given by the transformation $x^\alpha \to x'^\alpha = x'^\alpha(x^\beta,t)$, where

$$\begin{aligned} \zeta' &= \Phi(\zeta;t), \\ s' &= e^{\phi(t)}C_0(t)[s+A(\zeta,\bar{\zeta};t)], \\ r' &= e^{\phi(t)}C_0^{-1}(t)\, r. \end{aligned} \tag{4.144}$$

Since $x'^{\alpha}(t=0) = x^{\alpha}$, we must have

$$\Phi(\zeta;0) = \zeta, \qquad A(\zeta,\bar{\zeta};0) = 0 = \phi(0), \qquad C_0(0) = 1.$$

Then, by (4.100), the components of $\tilde{K}$ are

$$\tilde{K}^1 = \alpha(\zeta),$$

$$\tilde{K}^3 = (a + a_0)s + T,$$

$$\tilde{K}^4 = (a - a_0)r,$$

where $\left[\dfrac{\partial\Phi}{\partial t}\right]_{t=0} \equiv \alpha(\zeta), \qquad \left[\dfrac{\partial A}{\partial t}\right]_{t=0} \equiv T(\zeta,\bar{\zeta}) = \bar{T},$

and $\left[\dfrac{\partial\phi}{\partial t}\right]_{t=0} \equiv a, \quad \left[\dfrac{\partial C_0}{\partial t}\right]_{t=0} \equiv a_0$ are both real constants.

Hence, by (4.101), we have the following form of the HKV in the $(\zeta,\bar{\zeta},s,r)$ coordinate system:

$$\boxed{\tilde{K} = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + a_0(s\partial_s - r\partial_r) + T\partial_s + a(s\partial_s + r\partial_r)} \tag{4.145}$$

Under the allowed transformations (4.142) the functions α, T occurring in $\tilde{K}$ will transform to α', T', where

$$\alpha' = \Phi_\zeta \alpha \tag{4.146}$$

and

$$T' = e^{\phi}C_0[T - (a_0+a)A + \tilde{K}A], \tag{4.147}$$

where the form of $\tilde{K}$ in (4.147) is given by (4.145). These two transformation equations are obtained in exactly the same way as (4.105) and (4.106) were in the $(\zeta,\bar{\zeta},u,v)$ system of coordinates.

The two mutually exclusive canonical forms of $\tilde{K}$ in the present coordinate system are

$$\begin{aligned} &\text{(i)} \quad \tilde{K} = \partial_\zeta + \partial_{\bar{\zeta}} + a_0(s\partial_s - r\partial_r) + a(s\partial_s + r\partial_r), \\ &\text{(ii)} \quad \tilde{K} = a_0(s\partial_s - r\partial_r) + a(s\partial_s + r\partial_r). \end{aligned} \tag{4.148}$$

(cf. (4.107)).

For the homothetic motion represented by (4.144) we have

$$\frac{\partial}{\partial t}\left[\Lambda'(x') - \Lambda(x')\right]_{t=0} = 0,$$

$$\frac{\partial}{\partial t}\left[e^{P'}(x') - e^{P}(x')\right]_{t=0} = 0,$$

$$\frac{\partial}{\partial t}\left[m'(x') - m(x')\right]_{t=0} = 0,$$

from which follow equations (I), (II′) and (III) below. The other equations (II), (IVb) and (IVc) follow from these by differentiation. Alternatively, they can all be obtained directly from (4.124) by using the commutators

$$
\begin{aligned}
[\partial_s, \tilde{K}] &= (a_0 + a)\partial_s, \\
[\partial_\zeta, \tilde{K}] &= \alpha_\zeta \partial_\zeta + T_\zeta \partial_s, \\
[\partial_{\bar{\zeta}}, \tilde{K}] &= \bar{\alpha}_{\bar{\zeta}} \partial_{\bar{\zeta}} + T_{\bar{\zeta}} \partial_s
\end{aligned}
\tag{4.149}
$$

for the HKV given by (4.145). Thus we have the homothetic Killing equations and their first order integrability conditions in the $(\zeta, \bar{\zeta}, s, r)$ coordinate system:

(I) $(\tilde{K} + \alpha_\zeta - a_0 - a)\Lambda + T_\zeta = 0,$ (4.150)

(II) $(\tilde{K} + \alpha_\zeta)p_\zeta + \frac{1}{2}\alpha_{\zeta\zeta} = 0,$ (4.151)

(II′) $\tilde{K}p + \mathrm{Re}(\alpha_\zeta) = a_0,$ (4.152)

(III) $(\tilde{K} + 3a_0 - a)m = 0,$ (4.153)

(IVb) $(\tilde{K} + 2a_0)R^{(2)} = 0,$ (4.154)

(IVc) $(\tilde{K} + a_0 - a)d = 0,$ (4.155)

where $R^{(2)}$ is the 2-curvature of the 2-metric $e^{2p}d\zeta d\bar{\zeta}$,

$$R^{(2)} = e^{-2p}p_{\zeta\bar{\zeta}} \ .$$

The field equations (4.62) become, in the present coordinate system,

$$m_{\bar{\zeta}} = 0, \tag{4.156a}$$

$$R^{(2)}{}_{\zeta\bar{\zeta}} = 0, \tag{4.156b}$$

$$\mathrm{Im}(m) = e^{-2p}d_{\zeta\bar{\zeta}} - 2\,R^{(2)}d. \tag{4.156c}$$

CHAPTER 5 *Spaces with One HKV and 2, 3 or 4 Killing Vectors*

Although they could not solve some of the field equations, Kerr & Debney [199] found, in principle, all algebraically special vacuum Einstein spaces with non-vanishing complex divergence, which admit 2,3 or 4 Killing vectors, with 4 the maximum number for a non-flat space. Their results, summarized below, are used as the starting point in this chapter. The metric functions only are given.

Case I. (4 Killing vectors)

$\Omega = \Lambda = id_0\bar{\zeta}$,

$\mu = m = m_0$, non-zero real constant,

$\Delta = id_0$ = constant, d_0 real,

$p = 0, \qquad R^{(2)} = 0.$

Case II. (4 Killing vectors)

$\Lambda = -id_0\bar{\zeta}/R_0(\zeta\bar{\zeta} - R_0)$,

$m = m_0$, m_0 real constant $\longleftrightarrow$ Schwarzschild metric,

m_0 complex constant $\longleftrightarrow$ NUT metric,

$d = d_0 = i(m_0 - \bar{m}_0)/4R_0$ = real constant,

$e^{-P} = \zeta\bar{\zeta} - R_0, \qquad R^{(2)} = R_0$ = real constant.

Case III. (3 Killing vectors)

$\Lambda = m = d = 0$,

$e^{-2P} = \frac{2}{3}(\zeta + \bar{\zeta})^3, \qquad R^{(2)} = \zeta + \bar{\zeta}.$

Case IV. (2 Killing vectors)

$$\Lambda = i\bar{\zeta}\, e^{2P}[-\tfrac{1}{2}\,\mathrm{Im}(m_0)R^2 + C_1 + C_2(2\log R + R_0R^{-2}) + C_3(R^2 + R_0^2\, R^{-2})],$$

$m = m_0$ = complex constant, but we may take either

$\mathrm{Re}(m_0) = 1$ or $\mathrm{Im}(m_0) = 1$,

$d = e^{-2P}\mathrm{Im}(\Lambda_{\bar{\zeta}})$,

where $R = |\zeta|$ and R_0, C_1, C_2, C_3 are real constants,

and $e^{-P} = \zeta\bar{\zeta} - R_0 = R^2 - R_0, \qquad R^{(2)} = R_0.$

Case V. (2 Killing vectors)

$$\Lambda = i[c_0 x^{-5/2} \sinh \frac{\sqrt{13}}{2}(x-x_0) + \frac{3}{4}\,\mathrm{Im}(m_0)x^{-3}],$$

$$m = m_0 = \text{complex constant},$$

$$d = e^{-2p}\,\mathrm{Im}(\Lambda_{\bar\zeta}),$$

$$e^{-2p} = \frac{2}{3}x^3, \qquad R^{(2)} = x = \zeta + \bar\zeta,$$

where c_0 is an arbitrary real constant.

Case VI. (2 Killing vectors)

$$\Lambda = \lambda_0 \bar\zeta \zeta^{-1/\alpha_0} + m_0 \bar\zeta^2 \zeta^{-\frac{3}{\alpha_0}+1},$$

$$m = 2m_0(1 - \frac{3}{\alpha_0})\zeta^{-3/\alpha_0},$$

$$d = \mathrm{Im}(\Lambda_{\bar\zeta}),$$

$$p = 0, \qquad R^{(2)} = 0,$$

where $\mathrm{Re}(\alpha_0) = 1$ (α_0 invariant), λ_0 is a complex constant, and m_0 is a complex constant which can be made real if the remaining coordinate freedom is used.

The following cases were unsolved by Kerr & Debney:

Case VII. (2 Killing vectors)

$$\Lambda = i(\zeta + \bar\zeta)^{-1} L(\theta), \qquad \zeta = \rho e^{i\theta},$$

$$m = \mu_0 \zeta^{3/2},$$

$$d = (\zeta + \bar\zeta)^{\frac{1}{2}} D(\theta), \qquad D = \bar D,$$

$$e^{-2p} = \frac{2}{3}(\zeta + \bar\zeta)^3, \qquad R^{(2)} = \zeta + \bar\zeta,$$

where μ_0 is a complex constant. No solution was obtained for D and L, where

$$\frac{1}{2}\sin 2\theta (dL/d\theta) - L = e^{2p}D,$$

and

$$\mathrm{Im}[\mu_0(1 + 2e^{-2i\theta})^{-3/2}] = e^{-2P}\cos^2\theta \frac{d^2D}{d\theta^2} - (2 + \frac{1}{4}e^{-2p})D,$$

$$P = P(\theta),$$

$$e^{-2P}(\cos^2\theta \frac{d^2P}{d\theta^2} + \frac{3}{2}) = 1.$$

The only known solution is $e^{-2P} = \frac{2}{3}$.

Case VIII. (2 Killing vectors)

$$\Omega = \Omega(u),$$
$$\mu = \mu_0 \bar{\Omega}^{-3}.$$

No solution.

Case IX. (2 Killing vectors)

$$\Omega = \Omega(t), \qquad t = u/\mathrm{Im}(\zeta),$$
$$\mu = u^{-3}\nu(t).$$ (Note error in [199], p.2817).

No solution.

We shall now try to determine whether any of the spaces represented by the above cases also admit a HKV. In cases I - VII there is a Killing vector of the form

$$K = \partial_s = e^{-p}\partial_u$$

so that we may use the $(\zeta,\bar{\zeta},s,r)$ coordinate system to see whether the space will also admit a HKV of the form (4.145). If such a HKV is present, the additional equations to be solved i.e. satisfied by the functions Λ, m and d, are (4.131) and (4.150)-(4.155). The procedure is to use these equations to determine the functions $\alpha(\zeta)$, $T(\zeta,\bar{\zeta})$ and the constant a_0 of the HKV.

Case I. (4 Killing vectors)

$p = 0$, $\quad R^{(2)} = 0$ and (4.154) is trivial.

(i) Since $m = m_0$ (real constant), equation (4.153) gives for $m_0 \neq 0$

$$3a_0 - a = 0.$$

Since $d = d_0$ (real constant), equation (4.155) gives for $d_0 \neq 0$

$$a_0 - a = 0.$$

Hence $a = 0$, and there is no HKV for $m_0 \neq 0$, $d_0 \neq 0$.

(ii) If $m_0 \neq 0$ but $d_0 = 0$, then (4.155) is trivial. Equation (4.153) gives $\quad 3a_0 - a = 0.$

(4.152) implies $\alpha = \alpha_0\zeta + \beta$, $\quad \mathrm{Re}(\alpha_0) = a_0$,

where β is a constant. There is enough coordinate freedom left to transform α to

$$\alpha = \alpha_0\zeta, \qquad \mathrm{Re}(\alpha_0) = a_0.$$

Equation (4.151) is automatically satisfied, and (4.150) implies $T = T_0$ (real constant).

Hence we have succeeded in finding a HKV of the form

$$\tilde{K} = \alpha_0\zeta\partial_\zeta + \bar{\alpha}_0\bar{\zeta}\partial_{\bar{\zeta}} + \frac{4}{3}a\, s\partial_s + \frac{2}{3}a\, r\partial_r + T_0\partial_s,$$

where $3\,\mathrm{Re}(\alpha_0) = a$. But since a HKV is determined only up to a Killing vector, and because $K = \partial_s$ (and therefore $K = T_0\partial_s$) is already in the space, we may take the HKV in this case in the form

$$\tilde{K} = 3\alpha_0\zeta\partial_\zeta + 3\bar{\alpha}_0\bar{\zeta}\partial_{\bar{\zeta}} + 4a\, s\partial_s + 2a\, r\partial_r, \tag{5.1}$$

where $3\,\mathrm{Re}(\alpha_0) = a$.

The metric which admits this HKV is Petrov type D:

$$\frac{1}{2}d\tau^2 = r^2 d\zeta d\bar{\zeta} + dr ds + m_0 r^{-1} ds^2. \tag{5.2}$$

This metric can be written in the form

$$\frac{1}{2}d\tau^2 = d\eta d\bar{\eta} + d\omega dr + m_0 r^{-1}[d\omega + (\eta d\bar{\eta} + \bar{\eta}d\eta)r^{-1} - \eta\bar{\eta}r^{-2}dr]^2 \tag{5.3}$$

on putting $\eta = \zeta r$, $\omega = s - \zeta\bar{\zeta}r$. In these coordinates the HKV is

$$\tilde{K} = 3\eta\partial_\eta + 3\bar{\eta}\partial_{\bar{\eta}} + 4\omega\partial_\omega + 2r\partial_r\,. \tag{5.4}$$

This solution is, in fact, the Kerr-Debney solution ([199], equation (5.10)) with $d_0 = 0$. (Attention is drawn to a misprint in their equation (5.10)).

By writing $\sqrt{2}\,\omega = z+t$, $\sqrt{2}\,r = z-t$, $\sqrt{2}\,\eta = x+iy$ the metric (5.3) becomes

$$d\tau^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{m_0}{z-t}\left\{dz+dt - \frac{x^2+y^2}{(z-t)^2}(dz-dt) + \frac{2(xdx+ydy)^2}{z-t}\right\}^2, \tag{5.5}$$

which is manifestly of Kerr-Schild type [230]. This is also clear because $D\Omega = 0$. There is a plane of singularities $z-t = 0$ so the metric is that for a nullicle [231]. It is, in fact, one of the Debney-Kerr-Schild solutions ([198], page 1852) of the form

$$d\tau^2 = 2d\zeta d\bar{\zeta} + 2dudv + P^{-3}[m(Z+\bar{Z}) - \psi\bar{\psi}P^{-1}Z\bar{Z}] \quad (du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv)^2,$$

where $P = pY\bar{Y} + qY + \bar{q}\bar{Y} + c$,

$$F \equiv \phi(Y) + (qY+c)(\zeta - Yv) - (p\bar{Y}+\bar{q})(u+Y\bar{\zeta}) = 0,$$

$Z = -PP_Y^{-1}$,

ϕ, ψ are arbitrary functions of Y, $\psi = 0$ in vacuo,

m,p,c real constants, q complex constant.

Identification with the metric (5.3) is achieved through

$$\zeta \equiv \tau_1, \quad u \equiv \omega, \quad v \equiv r, \quad Y \equiv \eta/r,$$
$$\phi = 0, \quad Z = \bar{Z} = r^{-1}, \quad m \equiv m_0,$$
$$p = q = 0, \quad c = 1, \quad P = 1. \qquad (5.6)$$

Comparing (5.6) with [198], equation (5.80c), we find we have the Debney-Kerr-Schild case (c).

(iii) When $\underline{m_0 = 0 = d_0}$, the space is flat, as can be seen immediately from (5.5).

Case II. (4 Killing vectors)

(i) m_0, d_0 both non-zero constants.
Equations (4.153) and (4.155) imply a = 0.
Hence there is no HKV.

(ii) m_0 non-zero constant, $d_0 = 0$.
Equation (4.155) is the zero identity, while (4.154) gives $a_0 = 0$.
Then (4.153) implies a = 0.
Hence there is no HKV.
This result confirms the well known fact that the Schwarzschild metric does not admit a HKV, and also proves that the same is true of the NUT metric.

(iii) $m_0 = d_0 = 0$ gives $\Lambda = m = d = 0$, which implies that

$$\mu = 0 = \partial_u \partial_u D\Omega = \bar{D}\partial_u D\Omega$$

which is condition (4.64d) for a flat space.

Case III. (3 Killing vectors)

For $\Lambda = m = d = 0$, equations (4.155) and (4.153) are trivial. Since $R^{(2)} = \zeta + \bar{\zeta}$, equation (4.154) gives

$$\alpha + \bar{\alpha} + 2a_0(\zeta + \bar{\zeta}) = 0$$

with solution

$$\alpha = -2a_0(\zeta + ib_0), \; b_0 \text{ real constant.}$$

There is still a linear transformation in ζ left to transform α to the form

$$\alpha = -2a_0\zeta.$$

Then (4.152) with $e^{-2p} = \frac{2}{3}(\zeta+\bar{\zeta})^3$ is satisfied identically, as is (4.151). The remaining equation (4.150) gives

$$T = T_0 \text{ (real constant).}$$

Hence we have a HKV of the form

$$\tilde{K} = -2a_0(\zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}}) + a_0(s\partial_s - r\partial_r) + T_0\partial_s + a(s\partial_s + r\partial_r),$$

but because $K = \partial_s$ is already present in the space we take the HKV in the form

$$\tilde{K} = -2a_0(\zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}}) + (a_0 + a)s\partial_s - (a_0 - a)r\partial_r, \tag{5.7}$$

where a_0 and a are arbitrary real constants.

The metric which admits this HKV is

$$\frac{1}{2}\,d\tau^2 = \frac{3}{2}\,r^2(\zeta+\bar{\zeta})^{-3}d\zeta d\bar{\zeta} + drds + (\zeta+\bar{\zeta})ds^2, \tag{5.8}$$

which is of Petrov type III. The FTEs corresponding to this HKV can be read off from (4.109) and take the metric from $d\tau^2 \to d\tau^{*2}$, where

$$d\tau^{*2} = b^{2a}d\tau^2, \quad b \text{ real constant.}$$

Thus the choice of a_0 does not feature in the conformality, only a does - this is clear, anyway, because it is a which provides the essential difference between the HKV and a Killing vector.

By writing $8\zeta = 3(x+iy)$ we can write the metric (5.8) in the form

$$d\tau^2 = r^2x^{-3}(dx^2 + dy^2) + 2\,drds + \frac{3}{2}x\,ds^2, \tag{5.9}$$

which is the metric of Kerr & Debney ([199], equation (5.19) - but note the misprint in the line above their equation (5.19)).

In the real coordinates of (5.9), the HKV takes the form

$$\tilde{K} = -2a_0(x\partial_x + y\partial_y) + (a_0 + a)s\partial_s - (a_0 - a)r\partial_r, \tag{5.10}$$

where a_0 and a are arbitrary ($a \neq 0$).

Case IV. (2 Killing vectors)

The space is flat unless $m_0 \neq 0$.

(1) $m_0 \neq 0$, $R_0 \neq 0$.

Equation (4.154) gives $a_0 = 0$, which together with (4.153) implies a = 0. Therefore, there is no HKV.

(2) $m_0 \neq 0$, $R_0 = 0$.

Equation (4.154) is the zero identity, while (4.153) gives

$$3a_0 - a = 0. \tag{5.11}$$

With $e^{-P} = \zeta\bar{\zeta}$, equation (4.152) gives

$$(\alpha_\zeta + \bar{\alpha}_{\bar{\zeta}})\zeta\bar{\zeta} - 2(\alpha\bar{\zeta} + \bar{\alpha}\zeta) = 2a_0\zeta\bar{\zeta}$$

with solution

$$\alpha = \alpha_0\zeta^2 - a_0\zeta, \tag{5.12}$$

where α_0 is a complex constant.

Equation (4.151) is satisfied identically.

For $R_0 = 0$, and choosing $\mathrm{Im}(m_0) = 1$ as we may do, we have

$$\Lambda = i\bar{\zeta}\, e^{2P}[-\tfrac{1}{2}\zeta\bar{\zeta} + C_1 + C_2 \log(\zeta\bar{\zeta}) + C_3\zeta\bar{\zeta}] \tag{5.13}$$

so that

$$\Lambda_{\bar{\zeta}} = ie^{2P}[C_2 - C_1 - C_2 \log(\zeta\bar{\zeta})] \tag{5.14}$$

and

$$d = \mathrm{Im}(\Lambda_{\bar{\zeta}}) = e^{2P}[C_2 - C_1 - C_2\log(\zeta\bar{\zeta})]. \tag{5.15}$$

Then

$$\zeta d_\zeta = \bar{\zeta}d_{\bar{\zeta}} = e^{2P}[2C_1 - 3C_2 + 2C_2\log(\zeta\bar{\zeta})]$$

and so

$$\tilde{K}d = (\alpha_0\zeta + \bar{\alpha}_0\bar{\zeta} - 2a_0)\zeta d_\zeta.$$

Using this in equation (4.155) gives

$$C_2\log(\zeta\bar{\zeta})[2(\alpha_0\zeta + \bar{\alpha}_0\bar{\zeta}) - 5a_0 + a] + (2C_1 - 3C_2)(\alpha_0\zeta + \bar{\alpha}_0\bar{\zeta})$$
$$+ (a - 5a_0)C_1 + 2C_2(7a_0 - a) = 0, \tag{5.16}$$

which is satisfied when

either (i) $C_1 = C_2 = 0$,

or (ii) $\alpha_0 = C_2 = 0$, $a - 5a_0 = 0$.

Using (ii) with (5.11) gives $a = 0$. Thus there is no HKV.

When (i) obtains, we have from (5.13)

$$\Lambda = i\beta\zeta^{-1}, \qquad \beta = C_3 - \tfrac{1}{2}, \tag{5.17}$$

and from (5.14)

$$\Lambda_{\bar{\zeta}} = 0, \qquad \Rightarrow \qquad d = 0.$$

Then

$$\tilde{K}\Lambda = (a_0 - \alpha_0\zeta)\Lambda$$

and substituting in (4.150) gives

$$T_\zeta = (a_0 + a - \alpha_0\zeta)\Lambda = i\beta[(a_0+a)\zeta^{-1} - \alpha_0],$$

whence

$$T = i\beta[(a_0+a)\log(\zeta/\bar{\zeta}) - \alpha_0\zeta + \bar{\alpha}_0\bar{\zeta}].$$

Using (5.11) we get finally

$$T = i\beta[\tfrac{4}{3}\, a \log(\zeta/\bar{\zeta}) - \alpha_0\zeta + \bar{\alpha}_0\bar{\zeta}]. \tag{5.18}$$

Thus we have obtained a HKV which is, along with the two Killing vectors

$$K_1 = \partial_s, \qquad K_2 = i(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}), \tag{5.19}$$

a symmetry of the space with metric

$$\tfrac{1}{2}\, d\tau^2 = r^2(\zeta\bar{\zeta})^{-2}d\zeta d\bar{\zeta} + drds + i\beta(\zeta^{-1}d\zeta - \bar{\zeta}^{-1}d\bar{\zeta})dr + \mathrm{Re}(m_0)r^{-1}\,[ds + i\beta(\zeta^{-1}d\zeta - \bar{\zeta}^{-1}d\bar{\zeta})]^2, \tag{5.20}$$

where $\beta = C_3 - \frac{1}{2}$ is an arbitrary real constant, and m_0 is an arbitrary complex constant with $\mathrm{Im}(m_0) = 1$.

The HKV is

$$\tilde{K} = (3\alpha_0\zeta - a)\zeta\partial_\zeta + (3\bar{\alpha}_0\bar{\zeta} - a)\bar{\zeta}\partial_{\bar{\zeta}} + i\beta[\,4a\log(\zeta/\bar{\zeta}) - 3\alpha_0\zeta + 3\bar{\alpha}_0\bar{\zeta}]\partial_s + 4as\partial_s + 2ar\partial_r, \tag{5.21}$$

where β and a are arbitrary real constants, and α_0 is an arbitrary complex constant.

The metric (5.20) is Petrov type D and can be put in the form

$$\tfrac{1}{2}\, d\tau^2 = d\eta d\bar{\eta} + d\omega\, dr + i\beta(\eta/\bar{\eta})d(\bar{\eta}/\eta)dr + \mathrm{Re}(m_0)r^{-1}[d\omega + (\bar{\eta}d\eta + \eta d\bar{\eta})r^{-1} - \eta\bar{\eta}r^{-2}dr + i\beta(\eta/\bar{\eta})d(\bar{\eta}/\eta)]^2 \tag{5.22}$$

by writing

$$\eta = \zeta^{-1}r, \qquad \omega = s - (\zeta\bar{\zeta})^{-1}r.$$

This metric degenerates to the Kerr-Schild metric (5.3) when $\beta = 0$. However, (5.22) is not Kerr-Schild since $D\Omega = i\beta(\eta/\bar{\eta})$.

An alternative form of the above metric can be obtained by putting $\eta = \rho e^{i\theta}$. Then

$$\tfrac{1}{2}\, d\tau^2 = d\rho^2 + \rho^2 d\theta^2 + d\omega\, dr + 2\beta dr d\theta + \mathrm{Re}(m_0)r^{-1}[d\omega + 2r^{-1}\rho d\rho - \rho^2 r^{-2}dr + 2\beta\, d\theta]^2. \tag{5.23}$$

Case V. (2 Killing vectors)

$$R^{(2)} = \zeta + \bar{\zeta} = x, \qquad e^{-2p} = \frac{2}{3}x^3.$$

Since $\partial_u D\Omega = p_{\zeta\zeta} - p_\zeta^2 = -3/(4x^2)$, we have

$$\partial_u\partial_u D\Omega = 0, \quad \text{but} \quad \bar{D}\partial_u D\Omega = e^{2p} \neq 0,$$

so that, by (4.64d), the space cannot be flat. There are two possibilities: (i) $m = m_0 \neq 0$, (ii) $m = m_0 = 0$, where m_0 is a complex constant.

Equation (4.154) gives

$$\alpha + \bar{\alpha} + 2a_0(\zeta+\bar{\zeta}) = 0$$

with solution $\alpha = -2a_0(\zeta + ib_0)$, b_0 real constant.

But in the presence of the two Killing vectors

$$K_1 = \partial_s, \qquad K_2 = i(\partial_\zeta - \partial_{\bar{\zeta}}) \tag{5.24}$$

there is still a linear transformation available on ζ, so we use this freedom to write α in the form

$$\alpha = -2a_0\zeta. \tag{5.25}$$

Then, as in Case III, equations (4.151) and (4.152) are satisfied identically. Now

$$\Lambda_{\bar{\zeta}} = i\left[-\frac{5}{2}c_0x^{-7/2}\sinh\frac{\sqrt{13}}{2}(x-x_0) + \frac{\sqrt{13}}{2}c_0x^{-5/2}\cosh\frac{\sqrt{13}}{2}(x-x_0) - \frac{9}{4}\mathrm{Im}(m_0)x^{-4}\right].$$

Hence

$$\begin{aligned} d &= e^{-2p}\mathrm{Im}(\Lambda_{\bar{\zeta}}) \\ &= \frac{1}{6}x^{-\frac{1}{2}}\left\{2\sqrt{13}\,c_0x\cosh\frac{\sqrt{13}}{2}(x-x_0) - 10c_0\sinh\frac{\sqrt{13}}{2}(x-x_0) - 9\,\mathrm{Im}(m_0)x^{-\frac{1}{2}}\right\}, \end{aligned} \tag{5.26}$$

$$d_\zeta = d_{\bar{\zeta}} = \frac{1}{12}x^{-3/2}\left[26c_0x^3\sinh\frac{\sqrt{13}}{2}(x-x_0) - 8\sqrt{13}c_0x\cosh\frac{\sqrt{13}}{2}(x-x_0) + 10c_0\sinh\frac{\sqrt{13}}{2}(x-x_0) + 18\,\mathrm{Im}(m_0)x^{-\frac{1}{2}}\right].$$

Then

$$\tilde{K}d = -2a_0(\zeta d_\zeta + \bar{\zeta}d_{\bar{\zeta}}) = -2a_0(\zeta + \bar{\zeta})d_\zeta$$

so that

$$\tilde{K}d + (a_0-a)d = \frac{1}{6}x^{-\frac{1}{2}}\Big\{-a_0\Big[26c_0x^3\sinh\frac{\sqrt{13}}{2}(x-x_0) - 8\sqrt{13}\,c_0x\cosh\frac{\sqrt{13}}{2}(x-x_0) + 10c_0\sinh\frac{\sqrt{13}}{2}(x-x_0) + 18\,\mathrm{Im}(m_0)x^{-\frac{1}{2}}\Big] + (a_0-a)\Big[2\sqrt{13}\,c_0x\cosh\frac{\sqrt{13}}{2}(x-x_0) - 10c_0\sinh\frac{\sqrt{13}}{2}(x-x_0) - 9\,\mathrm{Im}(m_0)x^{-\frac{1}{2}}\Big]\Big\}.$$

Hence (4.155) is satisfied iff

$$a_0c_0 = 0, \qquad ac_0 = 0$$

and either (1) $a - 3a_0 = 0, \quad \mathrm{Im}(m_0) \neq 0,$

or (2) $a - 3a_0 \neq 0, \quad \mathrm{Im}(m_0) = 0,$

or (3) $a - 3a_0 = 0 = \mathrm{Im}(m_0).$

If $c_0 \neq 0$, then $a = 0$ and there is no HKV.

If $c_0 = 0$, then $\Lambda = i\gamma x^{-3}, \quad \gamma = \frac{3}{4}\mathrm{Im}(m_0)$ = real constant,

$m = m_0$ = complex constant,

$d = -2\gamma x^{-1}$.

(1) $a - 3a_0 = 0, \quad \mathrm{Im}(m_0) \neq 0.$

If $a_0 = 0$, then $a = 0$ and there is no HKV.

If $a_0 \neq 0$, then (4.153) is satisfied identically for $m_0 \neq 0$.

Equation (4.150) reduces to $T_\zeta = 0$ with solution

$$T = T_0 \text{ (real constant)}.$$

Thus we arrive at the metric

$$\frac{1}{2}d\tau^2 = \frac{3}{2}(r^2 + 4\gamma^2x^{-2})x^{-3}d\zeta d\bar{\zeta} + [dr - 2i\gamma x^{-2}(d\zeta - d\bar{\zeta})]\varkappa + \left\{x + \mathrm{Re}\left(\frac{m_0}{r-2i\gamma x^{-1}}\right)\right\}\varkappa^2, \tag{5.27}$$

where

$\varkappa = ds + ix^{-3}(\gamma d\zeta - \bar{\gamma}d\bar{\zeta}), \quad x = \zeta + \bar{\zeta},$

$\gamma \neq 0$ is an arbitrary real constant.

This metric is Petrov type II and admits the HKV

$$\tilde{K} = \zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} - 2s\partial_s - r\partial_r , \tag{5.28}$$

where we have taken account of the presence of $K_1 = \partial_s$ in the space. a_0 is an arbitrary real constant.

The FTEs corresponding to this HKV are read off from (4.109) and take $d\tau \to d\tau^{*}$, where

$$d\tau^{*2} = b^{-3}\, d\tau^{2}, \quad b \text{ real constant.}$$

If we put

$$2\zeta = x+iy, \qquad r = 2\omega, \qquad \gamma = A,$$

the metric (5.27) takes the form

$$d\tau^{2} = 3x^{-3}(\omega^{2} + A^{2}x^{-2})(dx^{2} + dy^{2}) + 4(d\omega + Ax^{-2}dy)\varkappa + \left\{2x + \mathrm{Re}\left(\frac{m_0 x}{\omega x - iA}\right)\right\}\varkappa^{2}, \tag{5.29}$$

$$\varkappa = ds - Ax^{-3}dy,$$

where $4A = 3\,\mathrm{Im}(m_0) \neq 0$ is arbitrary, real, constant.

The HKV admitted by (5.29) is

$$\tilde{K} = x\partial_x + y\partial_y - 2s\partial_s - \omega\partial_\omega . \tag{5.30}$$

(2) $\underline{\mathrm{Im}(m_0) = 0, \;\; a - 3a_0 \neq 0.}$

Now m_0 is a real constant, and equation (4.153) gives

$$(3a_0 - a)m_0 = 0,$$

so that $m_0 = 0$. Then

$$\Lambda = m = d = 0 \Rightarrow \Omega = -\frac{3}{2}x^{-1}u, \qquad \mu = \Delta = 0.$$

From (4.150) we get $T_\zeta = 0$, so that

$$T = T_0 \text{ (real constant).}$$

Thus we arrive at the Petrov type III metric

$$d\tau^{2} = 3r^{2}x^{-3}d\zeta d\bar{\zeta} + 2drds + 2xds^{2}, \tag{5.31}$$

which admits the HKV

$$\tilde{K} = 2a_0(\zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}}) - (a_0 + a)s\partial_s + (a_0 - a)r\partial_r, \tag{5.32}$$

where we have taken account of the fact that the Killing vector $K_1 = \partial_s$ is already in the space. The constants a_0 and a in (5.32) are subject to the restriction $a \neq 3a_0$, $a \neq 0$, but are otherwise arbitrary. This restriction is the only difference between Case III and the present case. Writing $8\zeta = 3(X+iY)$, we can put the metric (5.31) in the form of (5.9), with the HKV in the form

$$\tilde{K} = 2a_0(X\partial_X + Y\partial_Y) - (a_0+a)s\partial_s + (a_0-a)r\partial_r, \tag{5.33}$$

where $3a_0 \neq a \neq 0$.

(3) $a - 3a_0 = 0 = \mathrm{Im}(m_0)$.

Again m_0 is a real constant, and equation (4.153) gives

$$(3a_0 - a)m_0 = 0,$$

so that m_0 is arbitrary, but not zero because, if it were zero, we would have $\Lambda = m = d = 0$ and flat space.

Equation (4.150) now reduces to $T_\zeta = 0$ so that

$$T = T_0 \text{ (real constant).}$$

Thus we arrive at the metric (4.129) with

$$\Lambda = 0,$$

$$m = m_0, \text{ arbitrary non-zero real constant,}$$

$$d = 0,$$

$$e^{-2p} = \frac{2}{3} x^3, \qquad R^{(2)} = x = \zeta+\bar{\zeta}.$$

Written in full, this Petrov type II metric is

$$d\tau^2 = 3r^2x^{-3}d\zeta d\bar{\zeta} + 2drds + 2(x + m_0r^{-1})ds^2, \tag{5.34}$$

which admits the HKV

$$\tilde{K} = \zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} - 2s\partial_s - r\partial_r, \tag{5.35}$$

where we have accounted for the presence of the Killing vector $K_1 = \partial_s$ in the space.

Putting $x = \zeta+\bar{\zeta}$, $y = \zeta-\bar{\zeta}$ brings the metric (5.34) into the form

$$d\tau^2 = \frac{3}{4} r^2x^{-3}(dx^2 - dy^2) + 2drds + 2(x + m_0r^{-1})ds^2, \tag{5.36}$$

admitting the HKV

$$\tilde{K} = x\partial_x + y\partial_y - 2s\partial_s - r\partial_r. \tag{5.37}$$

Alternatively, by putting $8\zeta = 3(X+iY)$ we can write the metric in the form

$$d\tau^2 = r^2X^{-3}(dX^2 + dY^2) + 2drds + (\frac{3}{2} X + \frac{2m_0}{r})ds^2, \tag{5.38}$$

and the HKV becomes

$$\tilde{K} = X\partial_X + Y\partial_Y - 2s\partial_s - r\partial_r. \tag{5.39}$$

Case VI. (2 Killing vectors)

p = 0 and equation (4.152) give $\mathrm{Re}(\alpha_\zeta) = a_0$, with solution

$$\alpha = \beta_0\zeta + \gamma, \qquad \mathrm{Re}(\beta_0) = a_0.$$

If m_0 is taken to be a complex constant, there is enough coordinate freedom left on ζ to transform α to

$$\alpha = \beta_0\zeta, \qquad \mathrm{Re}(\beta_0) = a_0. \tag{5.40}$$

Since $m = 2m_0(1 - 3\alpha_0^{-1})\zeta^{-3/\alpha_0}$ and $p = 0$, we have $\partial_u D\Omega = 0$ so that $\partial_u\partial_u D\Omega = 0 = \bar{D}\partial_u D\Omega$ and the only non-flat possibility is $m \neq 0 \Leftrightarrow m_0 \neq 0$. Then (4.153) gives

$$(3a_0 - a)\alpha_0 - 3\beta_0 = 0. \tag{5.41}$$

Subtracting this from its complex conjugate gives

$$\bar{\alpha}_0\beta_0 - \alpha_0\bar{\beta}_0 = 0$$

so that

$$\beta_0 = \tfrac{1}{2}(\alpha_0 + \bar{\alpha}_0)\beta_0 = \tfrac{1}{2}\alpha_0(\beta_0 + \bar{\beta}_0) = \alpha_0 a_0,$$

where we have used $\mathrm{Re}(\alpha_0) = 1$ and $\mathrm{Re}(\beta_0) = a_0$. Substituting back in (5.41) gives $a = 0$, so there is no HKV.

Case VII. (2 Killing vectors)

Since $R^{(2)} = \zeta + \bar{\zeta} = x$, equation (4.154) gives

$$\alpha + \bar{\alpha} + 2a_0(\zeta + \bar{\zeta}) = 0.$$

Just as in Case III we obtain

$$\alpha = -2a_0\zeta. \tag{5.42}$$

Then (4.151) and (4.152) are satisfied identically.

For $m = \mu_0 \zeta^{3/2}$ there are two possibilities:
either (i) $m \neq 0 \Leftrightarrow \mu_0 \neq 0$, or (ii) $m = 0 \Leftrightarrow \mu_0 = 0$.

(i) $\underline{m \neq 0}$. Then (4.153) gives $a = 0$, so there is no HKV.

(ii) $\underline{m = 0}$. Equation (4.153) is satisfied identically, so we are left with equations (4.150) and (4.155) to determine d, Λ and T.

Now $\zeta = \rho e^{i\theta}$, so that

$$\zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} = \rho\partial_\rho, \qquad \zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}} = -i\partial_\theta.$$

Therefore

$$\tilde{K} = -2a_0\rho\partial_\rho + a_0(s\partial_s - r\partial_r) + T\partial_s + a(s\partial_s + r\partial_r).$$

Since it is known that

$$d = (\zeta+\bar{\zeta})^{\frac{1}{2}}D(\theta) = (2\rho\cos\theta)^{\frac{1}{2}}D(\theta)$$
$$= \rho^{\frac{1}{2}}f(\theta),$$

where $f(\theta) = (2\cos\theta)^{\frac{1}{2}}D(\theta)$, equation (4.155) gives

$$(\tilde{K} + a_0 - a)d = -a\rho^{\frac{1}{2}}f(\theta) = 0$$

so that

either $a = 0$ and there is no HKV,

or $f(\theta) = 0$, which implies $D(\theta) = 0$. Then the equation for $D(\theta)$, namely,

$$\mathrm{Im}[\mu_0(1+2e^{-2i\theta})^{-3/2}] = \frac{2}{3}\cos^2\theta\frac{d^2D}{d\theta^2} - (2 + \frac{1}{4}e^{-2P})D$$

is satisfied.

This leaves the equation in $L(\theta)$ to be solved:

$$(dL/d\theta) - 2L\operatorname{cosec}2\theta = 0. \tag{5.43}$$

The solution is

$$L = C\tan\theta, \qquad C \text{ real constant.}$$

Hence

$$\Lambda = i(\zeta+\bar{\zeta})^{-1}.L(\theta) = \frac{1}{2}iC\rho^{-1}\sec\theta\tan\theta$$
$$= C(\zeta-\bar{\zeta})(\zeta+\bar{\zeta})^{-2} \tag{5.44}$$
$$= -\bar{\Lambda}.$$

Equation (4.150) now becomes

$$T_\zeta - (a_0 + a)\Lambda = 0 \tag{5.45}$$

which, with its complex conjugate, implies

$$T_\zeta + T_{\bar{\zeta}} = 0$$

so that $T = T(\zeta-\bar{\zeta})$. Substituting back in (5.45) gives

$$T' = C(a_0+a)(\zeta-\bar{\zeta})(\zeta+\bar{\zeta})^{-2},$$

where the prime denotes differentiation with respect to $(\zeta-\bar{\zeta})$.
This last equation holds iff

(1) $a_0 + a = 0$, or (2) $C = 0$, or (3) $a_0 + a = 0 = C$,
and then we have

$$T = T_0 \quad \text{(real constant)}.$$

Taking these three possibilities in turn, we have:

(1) $a_0 + a = 0,\ C \neq 0$. Put $8\zeta = 3(X+iY)$. Then

$$\Lambda = \frac{4}{3}\, i\, C\, X^{-2}Y,$$

$$m = 0 = d,$$

$$e^{-2p} = \frac{9}{32}X^3, \qquad R^{(2)} = \frac{3}{4}X$$

and the metric (4.129) is

$$d\tau^2 = r^2X^{-3}(dX^2+dY^2) + 2drds - 2CX^{-2}YdY\,dr + \frac{3}{2}X\,(ds-CX^{-2}YdY)^2, \qquad (5.46)$$

where C is an arbitrary real, non-zero constant.

This Petrov type III metric admits the HKV

$$\tilde{K} = X\partial_X + Y\partial_Y + r\partial_r. \qquad (5.47)$$

(2) $a_0 + a \neq 0,\ C = 0$. The metric is (5.8) of Case III, admitting the HKV (5.7) but with the restriction $a_0 + a \neq 0$. So this is a degenerate case, which can be expressed in the form (5.9), (5.10) with $a_0 \neq -a \neq 0$.

(3) $a_0 + a = 0 = C$. Again a degenerate case with metric (5.9), but the HKV in this case is restricted to the form

$$\tilde{K} = x\partial_x + y\partial_y + r\partial_r. \qquad (5.48)$$

Cases VIII and IX. (2 Killing vectors)

These are the case II metrics of Kerr & Debney ([199], p.2817). They do not admit a Killing vector of type $\partial_s \equiv e^{-p}\partial_u$, so we must use the $(\zeta,\bar{\zeta},u,v)$ coordinate system and look for solutions $\alpha(\zeta)$, $R(\zeta,\bar{\zeta})$ of the equations (4.124) which are rewritten here for convenience, with the HKV in the form

$$\tilde{K} = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + \mathrm{Re}(\alpha_\zeta)(u\partial_u - v\partial_v) + R\partial_u + a(u\partial_u + v\partial_v). \qquad (5.49)$$

The homothetic Killing equations and their integrability conditions are

$$(\tilde{K} - a)(\Omega - u\dot{\Omega}) + \frac{1}{2}(\alpha_\zeta - \bar{\alpha}_{\bar{\zeta}})(\Omega - u\dot{\Omega}) + R\dot{\Omega} + R_\zeta = 0, \tag{5.50}$$

$$\tilde{K}\dot{\Omega} + \alpha_\zeta\dot{\Omega} + \frac{1}{2}\alpha_{\zeta\zeta} = 0, \tag{5.51}$$

$$(\tilde{K} + 3\,\mathrm{Re}(\alpha_\zeta) - a)\mu = 0, \tag{5.52}$$

$$(\tilde{K} + 4\,\mathrm{Re}(\alpha_\zeta))\dot{\mu} = 0, \tag{5.53}$$

$$(\tilde{K} + 2\,\mathrm{Re}(\alpha_\zeta))(\bar{D}\dot{\Omega}) = 0, \tag{5.54}$$

$$(\tilde{K} + \mathrm{Re}(\alpha_\zeta) - a)\Delta = 0, \tag{5.55}$$

$$(\tilde{K} + \alpha_\zeta + \mathrm{Re}(\alpha_\zeta) + a)\ddot{\Omega} = 0. \tag{5.56}$$

Case VIII. (2 Killing vectors)

The functions Ω and μ are known to the extent that they are functions of u only:

$$\Omega = \Omega(u), \tag{5.57}$$

$$\mu = \mu_0\bar{\Omega}^{-3} = \mu(u), \quad \mu_0 \text{ complex constant}, \tag{5.58}$$

where

$$\frac{d}{du}\left\{4\mu_0\bar{\Omega}^{-3} + 2\bar{\Omega}\frac{d}{du}\left[\bar{\Omega}\frac{d^2}{du^2}(\Omega^2)\right]\right\} = \left|\frac{d^2}{du^2}(\Omega^2)\right|^2, \tag{5.59}$$

and

$$\mathrm{Im}\left\{4\mu_0\bar{\Omega}^{-3} + 2\bar{\Omega}\frac{d}{du}\left[\bar{\Omega}\frac{d^2}{du^2}(\Omega^2)\right]\right\} = 0. \tag{5.60}$$

The two Killing vectors which give rise to equations (5.57) - (5.60) are

$$K_1 = \partial_\zeta + \partial_{\bar{\zeta}}, \qquad K_2 = i(\partial_\zeta - \partial_{\bar{\zeta}}). \tag{5.61}$$

If we can solve equations (5.50) - (5.56) for α and R subject to (5.57) - (5.60), with $a \neq 0$ in $\tilde{K}$, then we will have found a metric or metrics which admit a HKV as well as the two Killing vectors (5.61).

From Appendix 2 we see that $\tilde{K}$ can be present with K_1 and K_2 when, and only when,

(I) $\alpha = \zeta, \quad R = 0,$

$$\tilde{K} = \zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} + u\partial_u - v\partial_v + a(u\partial_u + v\partial_v), \tag{5.62}$$

$$= x\partial_x + y\partial_y + (a+1)u\partial_u + (a-1)v\partial_v, \quad \zeta = x+iy.$$

(II) $\alpha = i\zeta, \quad R = 0,$

$$\tilde{K} = i(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}) + a(u\partial_u + v\partial_v), \tag{5.63}$$

$$= x\partial_y - y\partial_x + a(u\partial_u + v\partial_v).$$

(III) $\alpha = \alpha_0$ (complex constant), $R = 0$,

$$\tilde{K} = \alpha_0\partial_\zeta + \bar{\alpha}_0\partial_{\bar{\zeta}} + a(u\partial_u + v\partial_v), \tag{5.64}$$

$$= p_0\partial_x + q_0\partial_y + a(u\partial_u + v\partial_v), \quad \alpha_0 = p_0+iq_0.$$

Taking each of these possibilities in turn, we have firstly

(I) Equations (5.50) and (5.51) give

$$(\tilde{K} - a)\Omega = 0.$$

With $\tilde{K}$ as in (5.62), this is

$$(a+1)u\,\frac{d\Omega}{du} - a\,\Omega = 0 \tag{5.65}$$

with two solutions, depending on the value of a:

<u>either</u> $\Omega = Cu^{a/(a+1)}$, $a \neq -1$, (5.66)

<u>or</u> $\Omega = 0$, $a = -1$, (5.67)

where C is a complex constant. But

$$\tilde{K}\mu = \tilde{K}(\mu_0\bar{\Omega}^{-3}) = -3\mu_0\bar{\Omega}^{-4}\,\tilde{K}\bar{\Omega} = -3\mu\bar{\Omega}^{-1}\,\tilde{K}\bar{\Omega}$$

so that (5.52) gives

$$(3\bar{\Omega}^{-1}\,\tilde{K}\bar{\Omega} + a - 3)\mu = 0. \tag{5.68}$$

<u>Either</u> (1) $\mu = 0$, so that

(i) $\mu_0 \neq 0$, $\Omega = 0$,

or (ii) $\mu_0 = 0$, $\Omega \neq 0$,

or (iii) $\mu_0 = 0 = \Omega$;

<u>or</u> (2) $\tilde{K}\bar{\Omega} = (1 - \frac{a}{3})\bar{\Omega}$, $\mu \neq 0$,

and substituting its complex conjugate back into $(\tilde{K} - a)\Omega = 0$ gives

$$a = \tfrac{3}{4},\ \Omega \neq 0,\ \mu \neq 0;$$

<u>or</u> (3) $\mu = 0 = (\tilde{K} - 1 + \frac{a}{3})\Omega$

so that, using $(\tilde{K} - a)\Omega = 0$,

$\mu = 0 = (1 - \frac{4a}{3})\Omega.$

Then (i) $\mu_0 \neq 0$, $\Omega = 0$, a arbitrary, non-zero,

or (ii) $\mu_0 = 0$, $\Omega \neq 0$, $a = \frac{3}{4}$.

Also, $2\Delta = \bar{D}\Omega - D\bar{\Omega} = 0$ for both (5.66) and (5.67), so that any metrics will have a <u>hypersurface-orthogonal</u> principal null ray congruence.

Case VIII (I)(1)(i). $\mu = 0 = \Omega$ $(\mu_0 \neq 0)$.

Condition (4.64d) is satisfied, so space is flat.

Case VIII (I)(1)(iii). $\mu_0 = 0 = \Omega$ $(\Rightarrow \mu = 0)$.

Likewise, flat space.

Case VIII (I)(1)(ii). $\mu = \mu_0 = 0$, $\Omega \neq 0$.

If $a = -1$, then (5.67) requires $\Omega = 0$, so we have a contradiction and there is no solution.

If $a \neq -1$, then $\Omega = Cu^{a/(a+1)}$, $C \neq 0$,

and $$\bar{\Omega} = \bar{C}C^{-1}\Omega,$$

so that the field equation (5.59) reduces to

$$2\{\Omega[\Omega(\Omega^2)^{\cdot\cdot}]^{\cdot}\}^{\cdot} = [(\Omega^2)^{\cdot\cdot}]^2. \tag{5.69}$$

Now

$$(\Omega^2)^{\cdot\cdot} = \frac{2a(a-1)}{(a+1)^2} C^2 \left(\frac{\Omega}{C}\right)^{-2/a}, \quad a \neq -1,$$

and substituting this into (5.69) gives

$$a(a-1)(2a-3) = 0, \quad a \neq -1,$$

so that $a = 1$ or $a = 3/2$, since $a \neq 0$ for a proper homothetic motion.

$a = 1$: $\Omega = Cu^{\frac{1}{2}}$, $C \neq 0$.

Condition (4.64d) is satisfied, so space is flat.

$a = \frac{3}{2}$: $\Omega = Cu^{3/5}$, $C \neq 0$.

The space is not flat and possesses the metric

$$d\tau^2 = 2v^2 d\zeta d\bar{\zeta} + 2[dv - \frac{3}{5} u^{-2/5} v(Cd\zeta + \bar{C}d\bar{\zeta}) + C\bar{C}u^{-4/5}\epsilon_4]\epsilon_4 \tag{5.70}$$

where

$\epsilon_4 = du + u^{3/5}(Cd\zeta + \bar{C}d\bar{\zeta})$,

and C is an arbitrary non-zero complex constant.

This Petrov type III metric admits the HKV

$$\tilde{K} = 2\zeta\partial_\zeta + 2\bar{\zeta}\partial_{\bar{\zeta}} + 5u\partial_u + v\partial_v \tag{5.71}$$

besides the two Killing vectors (5.61).

If we put $\zeta = x+iy$, $w = u^{1/5}$, $C = \frac{5}{2}(c_1+ic_2)$,

then the metric (5.70) becomes

$$d\tau^2 = 2v^2(dx^2 + dy^2) + 2[dv - 3w^{-2}v(c_1dx - c_2dy) + \tfrac{25}{4}(c_1^2 + c_2^2)w^{-4}\epsilon_4]\epsilon_4, \tag{5.72}$$

where

$$\epsilon_4 = 5w^3(wdw + c_1dx - c_2dy),$$

and c_1, c_2 are arbitrary real constants, not both zero.

The HKV is now

$$\tilde{K} = 2x\partial_x + 2y\partial_y + w\partial_w + v\partial_v \,. \tag{5.73}$$

Case VIII (I)(2). $\mu \neq 0$, $\Omega \neq 0$, $a = \frac{3}{4}$.

From (5.66) we have

$$\Omega = Cu^{3/7}, \quad C \neq 0.$$

Then $\bar{\Omega}/\bar{C} = \Omega/C$ and field equation (5.59) becomes

$$-12\mu_0 C^5\bar{C}^{-5}\Omega^{-4}\dot{\Omega} + 2\{\Omega[\Omega(\Omega^2)^{\cdot\cdot}]^{\cdot}\}^{\cdot} = [(\Omega^2)^{\cdot\cdot}]^2, \tag{5.74}$$

while field equation (5.60) becomes

$$\mathrm{Im}\{4\mu_0 C^3\bar{C}^{-3}\Omega^{-3} + 2C^{-2}\bar{C}^2\Omega[\Omega(\Omega^2)^{\cdot\cdot}]^{\cdot}\} = 0. \tag{5.75}$$

Calculating $\dot{\Omega}/C = \frac{3}{7}(\Omega/C)^{-4/3}$,

$$(\Omega^2)^{\cdot\cdot} = -\frac{6}{49}C^2(\Omega/C)^{-8/3}$$

and substituting these into (5.74), we get

$$343\mu_0 + 16C^2\bar{C}^5 = 0, \qquad C \neq 0.$$

Hence

$$\mu = \bar{\mu} = -\frac{16}{343}(C\bar{C})^2\,u^{-9/7}, \quad C \neq 0.$$

Field equation (5.75) is satisfied for this Ω and μ.

We have arrived at the Petrov type II metric

$$d\tau^2 = 2v^2 d\zeta d\bar{\zeta} + 2[dv - \tfrac{3}{7}u^{-4/7}v(Cd\zeta + \bar{C}d\bar{\zeta}) + \tfrac{4}{343}C\bar{C}u^{-9/7}(21u^{1/7} - 4C\bar{C}v^{-1})\epsilon_4]\epsilon_4, \tag{5.76}$$

where

$$\epsilon_4 = du + u^{3/7}(Cd\zeta + \bar{C}d\bar{\zeta}),$$

and C is an arbitrary non-zero complex constant.

This metric admits the HKV

$$\tilde{K} = 4\zeta\partial_\zeta + 4\bar{\zeta}\partial_{\bar{\zeta}} + 7u\partial_u - v\partial_v \tag{5.77}$$

besides the two Killing vectors (5.61).

Putting $\zeta = x+iy$, $w = u^{1/7}$, $C = \frac{7}{2}(c_1+ic_2)$,

the metric (5.76) can be written in the form

$$d\tau^2 = 2v^2(dx^2 + dy^2) + 2\{dv - 3vw^{-4}(c_1dx - c_2dy) + (c_1^2 + c_2^2)w^{-9}[3w - 7(c_1^2 + c_2^2)v^{-1}]\varepsilon_4\}\varepsilon_4, \tag{5.78}$$

where

$$\varepsilon_4 = 7w^3(w^3dw + c_1dx - c_2dy),$$

and c_1, c_2 are arbitrary real constants, not both zero.
The HKV is now

$$\tilde{K} = 4x\partial_x + 4y\partial_y + w\partial_w - v\partial_v. \tag{5.79}$$

Case VIII (I)(3)(i): $\mu = 0 = \Omega$, $\mu_0 \neq 0$, a arbitrary $\neq 0$.

Condition (4.64d) is satisfied so space is flat.

Case VIII (I)(3)(ii). $\mu = 0 = \mu_0$, $\Omega \neq 0$, $a = \frac{3}{4}$.

Equations (5.74) and (5.75) must hold if there is a solution. However, (5.74) gives $C = 0$, contrary to the condition $\Omega \neq 0$. Hence there is no solution.

(II) Equations (5.50) and (5.51) give

$$(\tilde{K} + i-a)\Omega = 0. \tag{5.80}$$

Since $\Omega = \Omega(u)$, and with $\tilde{K}$ as in (5.63), the last equation is

$$au\frac{d\Omega}{du} + (i-a)\Omega = 0 \tag{5.81}$$

with solution

$$\Omega = Cu^{(a-i)/a}, \tag{5.82}$$

where C is a complex constant.

From (5.58) we have $\tilde{K}\mu = -3\mu\bar{\Omega}^{-1}\tilde{K}\bar{\Omega}$

so that (5.52) gives

$$(3\bar{\Omega}^{-1}\tilde{K}\bar{\Omega} + a)\mu = 0. \tag{5.83}$$

Either (1) $\mu = 0$, so that

(i) $\mu_0 \neq 0, \quad \Omega = 0,$

or (ii) $\mu_0 = 0, \quad \Omega \neq 0,$

or (iii) $\mu_0 = 0 = \Omega;$

or (2) $\tilde{K}\bar{\Omega} = -\frac{a}{3}\bar{\Omega}, \quad \mu \neq 0,$
and substituting its complex conjugate back into (5.80) gives $3i - 4a = 0$. Since a is real, there is no solution in this case;

or (3) $\mu = 0 = (\tilde{K} + \frac{a}{3})\Omega$.
Putting $\tilde{K}\Omega = -\frac{a}{3}\Omega$ in (5.80) leads to only one possibility, namely, $\Omega = 0$.

Having dealt with (2), we are left with

(1) (i) Conditions (4.64d) are satisfied, so flat space.

(1) (iii) Likewise, flat space.

(1) (ii) Substituting (5.82) into field equation (5.59) gives $4ia + 2 = 0$. Since a is real, there is no solution.

(3) Condition (4.64d) is satisfied, so flat space.

(III) Equations (5.50) and (5.51) give

$$(\tilde{K} - a)\Omega = 0. \tag{5.84}$$

Since $\Omega = \Omega(u)$, and with $\tilde{K}$ as in (5.64), equation (5.84) is

$$u\frac{d\Omega}{du} - \Omega = 0$$

with solution

$$\Omega = Cu, \quad C \text{ complex constant.}$$

Hence

$$\partial_u\partial_u D\Omega = 0 = \bar{D}\partial_u D\Omega. \tag{5.85}$$

Now (5.58) and (5.52) imply

$$(3\tilde{K}\bar{\Omega} + a\bar{\Omega})\mu = 0. \tag{5.86}$$

Either (1) $\mu = 0$, and then (5.85) gives flat space;

or (2) $\tilde{K}\Omega = -\frac{a}{3}\Omega, \quad \mu \neq 0.$
Substituting in (5.84) gives $a = 0$, so there is no HKV;

or (3) $\mu = 0 = (\tilde{K} + \frac{a}{3})\Omega$, and then (5.85) gives flat space.

This concludes the discussion of Case VIII.

Case IX. (2 Killing vectors)

From Appendix 2 we see that the two Killing vectors

$$K_1 = \partial_\zeta + \partial_{\bar\zeta} = \partial_x,$$
$$K_2 = \zeta\partial_\zeta + \bar\zeta\partial_{\bar\zeta} + u\partial_u - v\partial_v \qquad (5.87)$$
$$= x\partial_x + y\partial_y + u\partial_u - v\partial_v,$$

where $\zeta = x+iy$, and the HKV

$$\tilde K = \alpha\partial_\zeta + \bar\alpha\partial_{\bar\zeta} + \mathrm{Re}(\alpha_\zeta)(u\partial_u - v\partial_v) + R\partial_u + a(u\partial_u + v\partial_v)$$

can all be present in the space when, and only when,

(I) $\alpha = \zeta - 1, \quad R = 0,$
$$\tilde K = K_2 - K_1 + a(u\partial_u + v\partial_v), \qquad (5.88)$$

(II) $\alpha = \zeta, \quad R = 0,$
$$\tilde K = K_2 + a(u\partial_u + v\partial_v), \qquad (5.89)$$

(III) $\alpha = 1, \quad R = 0,$
$$\tilde K = K_1 + a(u\partial_u + v\partial_v), \qquad (5.90)$$

(IV) $\alpha = 0 = R,$
$$\tilde K = u\partial_u + v\partial_v. \qquad (5.91)$$

<u>There is effectively only one case to consider</u>. For, since K_1 and K_2 are already present in the space, we can take the HKV to be (5.91), the constant a being absorbed into $\tilde K$ (equivalently, choose a = 1). We shall choose (IV) to represent the situation.

The homothetic Killing equations and their integrability conditions (5.50) - (5.56) are then:

$$(\tilde K - 1)(\Omega - u\dot\Omega) = 0, \qquad (5.92)$$
$$\tilde K\dot\Omega = 0, \qquad (5.93)$$
$$(\tilde K - 1)\mu = 0, \qquad (5.94)$$
$$\tilde K\dot\mu = 0, \qquad (5.95)$$
$$\tilde K(\bar D\dot\Omega) = 0, \qquad (5.96)$$
$$(\tilde K - 1)\Delta = 0, \qquad (5.97)$$
$$(\tilde K + 1)\ddot\Omega = 0. \qquad (5.98)$$

We shall also require the equations for K_1 and K_2 corresponding to (5.92), (5.93) and (5.94). They are:

$$K_1(\Omega - u\dot{\Omega}) = 0, \tag{5.99}$$
$$K_1\dot{\Omega} = 0, \tag{5.100}$$
$$K_1\mu = 0, \tag{5.101}$$

and

$$K_2(\Omega - u\dot{\Omega}) = 0, \tag{5.102}$$
$$(K_2 + 1)\dot{\Omega} = 0, \tag{5.103}$$
$$(K_2 + 3)\mu = 0. \tag{5.104}$$

Equations (5.99) and (5.100) give

$$K_1\Omega = 0 \tag{5.105}$$

while (5.102) and (5.103) give

$$K_2\Omega = 0 \tag{5.106}$$

and (5.92) and (5.93) give

$$(\widetilde{K} - 1)\Omega = 0. \tag{5.107}$$

Using (5.91) the last equation is $(u\partial_u - 1)\Omega = 0$, implying

$$\Omega = uf(y),$$

where we have used (5.105) as well. Substituting this in (5.106) gives $f = Cy^{-1}$, so that

$$\Omega = Cuy^{-1}, \tag{5.108}$$

where C is a complex constant.

Equations (5.91), (5.94) and (5.101) give $(u\partial_u - 1)\mu = 0$, and $\mu = ug(y)$.

Substituting this in (5.104) gives $g = Ay^{-4}$, and so

$$\mu = Auy^{-4}, \tag{5.109}$$

where A is a complex constant.

The constants A and C are related through the field equations (4.62). Equation (4.62a) becomes

$$A(2\bar{C} + i)uy^{-4} = 0. \tag{5.110}$$

Field equation (4.62c) is satisfied iff

$$A = \frac{1}{2}C(\frac{1}{2}i - C)(4i\bar{C} - 3). \tag{5.111}$$

Using this value of A the remaining field equation (4.62b) is satisfied identically. Equation (5.110) thus holds iff

$$\frac{1}{2}C(\frac{1}{2}i - C)(2\bar{C} + i)(4i\bar{C} - 3) = 0$$

$\Leftrightarrow \; C = 0, \frac{1}{2}i$ or $\frac{3}{4}i$.

C = 0: $\Omega = 0 = \mu$ and condition (4.64d) gives flat space.

$C = \frac{1}{2}i$: $\Omega = \frac{1}{2}i\,uy^{-1}$, $\mu = 0$ and condition (4.64d) gives flat space.

$C = \frac{3}{4}i$: $\Omega = \frac{3}{4}i\,uy^{-1}$, $\mu = 0$. Also $\Delta = \frac{1}{2}(\bar{D}\Omega - D\bar{\Omega}) = 0$.

Since $\bar{D}\partial_u D\Omega \neq 0$ the space is non-flat, and has the Petrov type III metric

$$d\tau^2 = 2v^2(dx^2 + dy^2) + 2[dv + \frac{3}{2}vy^{-1}dy - \frac{3}{8}y^{-2}(du - \frac{3}{2}uy^{-1}dy)] \times \times (du - \frac{3}{2}uy^{-1}dy) \qquad (5.112)$$

admitting the HKV (5.91).

If we put $2w = 3\log y$, the metric becomes

$$d\tau^2 = 2v^2(dx^2 + \frac{4}{9}e^{4w/3}dw^2) + 2[dv + vdw - \frac{3}{8}e^{-2w/3}(du - udw)] \times \times (du - udw). \qquad (5.113)$$

This concludes the discussion of Case IX.

Theorem 5.1 The non-flat algebraically special vacuum Einstein spaces with non-zero complex divergence which admit 2, 3 or 4 Killing vectors and one HKV are given in the following table:

Case	Metric	Petrov type	No. of Killing vectors	HKV
I(ii)	(5.3) Kerr-Schild	D	4	(5.4)
III	(5.9)	III	3	(5.10)
IV(2)	(5.22)	D	2	(5.21)
V(1)	(5.29)	II	2	(5.30)
*V(2)	(5.9)	III	3	(5.33)*
V(3)	(5.38)	II	2	(5.39)
VII(ii)(1)	(5.46)	III	2	(5.47)
*VII(ii)(2)	(5.9)	III	3	(5.10)*
*VII(ii)(3)	(5.9)	III	3	(5.48)*
VIII(I)(1)(ii)	(5.72)	III	2	(5.73)
VIII(I)(2)	(5.78)	II	2	(5.79)
IX	(5.112)	III	2	(5.91)

* degenerate cases

All of the above contain a hypersurface-orthogonal principal null congruence except Case V(1), metric (5.29).

CHAPTER 6 *Spaces with One HKV and One Killing Vector*

From Theorem 3.2 we know that the commutator of two HKVs is a Killing vector. This theorem and its proof can be specialized to give the result

$$[\tilde{K}, K] = \lambda K ,$$

where K is a Killing vector and $\tilde{K}$ is a HKV. This restriction on the geometry is discussed further in Appendix 1, where it is shown that for a given Killing vector the HKV must take a specific form. The purpose of this chapter is to determine those vacuum spaces which admit one HKV and one Killing vector. We shall consider the Killing vector in each of the canonical forms

$$\text{(i)} \quad K = \partial_s = e^{-p}\partial_u, \qquad \text{(ii)} \quad K = \partial_x = \partial_\zeta + \partial_{\bar{\zeta}}$$

in turn, using the results of Appendix 1 to write down the form of the associated HKV.

6.1 The space admits

$$K = \partial_s \tag{6.1}$$

and

$$\tilde{K} = a_0(s\partial_s - r\partial_r) + a(s\partial_s + r\partial_r), \tag{6.2}$$

where the $(\zeta,\bar{\zeta},s,r)$ coordinate system may be used owing to the presence of K in the form (6.1). Equation (4.152) gives

$$a_0 = 0.$$

Absorb the arbitrary constant a into $\tilde{K}$ (equivalently, choose a = 1) so that

$$\tilde{K} = s\partial_s + r\partial_r. \tag{6.3}$$

Equations (4.150), (4.153), (4.155) and field equation (4.156a) give

$$\Lambda = m = d = 0. \tag{6.4}$$

Equations (4.151) and (4.154) are satisfied identically, as is field equation (4.156c). By (4.130), $\Lambda = 0$ implies

$$\Omega = p_\zeta u, \qquad D\Omega = u(p_{\zeta\zeta} - p_\zeta^2),$$

so that Petrov type III metrics will exist if

$$\bar{D}(p_{\zeta\zeta} - p_\zeta^2) \neq 0,$$

where $p(\zeta,\bar{\zeta})$ is to be determined by the remaining field equation (4.156b). Otherwise the space is flat.

In discussing the field equation (4.156b) we shall consider separately the possibilities $R^{(2)} = 0$, $R^{(2)}$ = non-zero constant, $R^{(2)} \neq$ constant.

Case 6.1 (I): $R^{(2)} = 0$.

In this case $p_{\zeta\bar{\zeta}} = 0$. The coordinate freedom available is given in Appendix 1, equation (A1.11). In Appendix 3 we show that by an allowed coordinate transformation (4.134), (4.137) we can transform p to zero. In the new system of coordinates (4.130) gives $\Omega = \Lambda$ so that $\partial_u D\Omega = 0$. Hence, by (4.64d) the space is flat since $\mu = m = 0$ also.

Case 6.1 (II): $R^{(2)} = R_0$ (non-zero constant).

From (4.137) we find that $R^{(2)}$ transforms as

$$R^{(2)\prime} = C_0^{-2} R^{(2)} \qquad (6.5)$$

under the allowed group of transformations. By choosing $C_0^2 = |R_0|$ we can use (6.5) to make $R^{(2)} = \pm 1$. The 2-metric $e^{2p}d\zeta d\bar{\zeta}$ is then that for a sphere or pseudosphere and so the coordinates can be chosen (see Appendix 5) so that

$$e^{-p} = \zeta\bar{\zeta} - R_0. \qquad (6.6)$$

But then $p_{\zeta\zeta} - p_\zeta^2 = 0$ so that $\bar{D}(p_{\zeta\zeta} - p_\zeta^2) = 0$ and the space is flat.

Case 6.1 (III): $R^{(2)} \neq$ constant.

In this case $R^{(2)} = e^{-2p}p_{\zeta\bar{\zeta}} = 2\mathrm{Re}\{F(\zeta)\}$ for some analytic function F of ζ. Now (6.5) implies that, if we set $\zeta' = C_0^{-2}F(\zeta)$, then $R^{(2)\prime} = \zeta' + \bar{\zeta}'$.

Dropping the primes and working now in the new coordinates gives

$$e^{-2p}p_{\zeta\bar{\zeta}} = \zeta + \bar{\zeta}. \qquad (6.7)$$

The coordinate freedom left is, as may be seen from (6.5),

$$\zeta' = C_0^{-2}(\zeta + ie_0),$$
$$s' = C_0 s, \qquad (6.8)$$
$$r' = C_0^{-1} r,$$

where e_0 is an arbitrary real constant and we have used equation (A1.11) of Appendix 1 with $a_0 + a \neq 0$.

Differentiating (6.7) with respect to ζ gives

$$p_{\zeta\zeta\bar{\zeta}} - 2p_{\zeta}p_{\zeta\bar{\zeta}} = (\zeta+\bar{\zeta})^{-1}p_{\zeta\bar{\zeta}} = e^{2p}$$

and so for all p

$$\bar{D}(p_{\zeta\zeta} - p_{\zeta}^{2}) \neq 0.$$

Hence, if we can solve (6.7) for p, we have non-flat spaces admitting homothetic motions. Unfortunately, the only solution of (6.7) known is

$$e^{-2p} = \tfrac{2}{3}(\zeta+\bar{\zeta})^{3}. \tag{6.9}$$

Substituting (6.4) and (6.9) into (4.129), we have the Petrov type III hypersurface-orthogonal metric

$$d\tau^{2} = 3r^{2}(\zeta+\bar{\zeta})^{-3}d\zeta d\bar{\zeta} + 2drds + 2(\zeta+\bar{\zeta})ds^{2}.$$

Putting $8\zeta = 3(x+iy)$ this simplifies to the Kerr-Debney [193] metric

$$d\tau^{2} = r^{2}\,x^{-3}(dx^{2} + dy^{2}) + 2drds + \tfrac{3}{2}x\,ds^{2}, \tag{6.10}$$

which is the metric (5.9) of Chapter 5. It is, in fact, invariant not just under a G_1 of isometries but under a 3-dimensional group of isometries with Killing vectors

$$K_1 = \partial_s, \quad K_2 = \partial_y, \quad K_3 = -2(x\partial_x + y\partial_y) + s\partial_s - r\partial_r.$$

In this sense, we have here a degenerate case.

The HKV admitted here, though, is (6.3) which is (5.10) with the restriction $a_0 = 0$, $a = 1$.

Other solutions are theoretically possible, but await the discovery of further solutions of equation (6.7).

6.2 The space admits

$$K = \partial_s \tag{6.11}$$

and
$$\tilde{K} = \alpha\partial_{\zeta} + \bar{\alpha}\partial_{\bar{\zeta}} + a_0(s\partial_s - r\partial_r) + T\partial_s + a(s\partial_s + r\partial_r), \tag{6.12}$$

where $\alpha(\zeta)$ is non-zero. We could have chosen the canonical form (A1.7) of Appendix 1 for $\tilde{K}$, but this would have restricted the coordinate freedom available. At this stage we prefer to have the full coordinate freedom.

Since
$$[K,\tilde{K}] = (a_0 + a)K$$

we shall split the analysis according to whether $(a_0 + a)$ is zero or not, and whether the 2-curvature $R^{(2)}$ is constant or not.

$$\boxed{a_0 + a \neq 0, \qquad R^{(2)} = \text{constant} = R_0.}$$

From (4.154) we get $a_0 R_0 = 0$. We shall consider in turn the possibilities $a_0 \neq 0$, $a_0 = 0$.

Case 6.2 (I): $a_0 \neq 0$, $R_0 = 0$.

Just as in Case 6.1 (I) we can transform p to zero. The coordinate freedom left is a linear transformation in ζ, with complete freedom still in s and r (see (A3.3)).

The homothetic Killing equations and their integrability conditions (4.150) - (4.155) are now

$$(K + \alpha_\zeta - a_0 - a)\Lambda + T_\zeta = 0, \tag{6.13}$$

$$\alpha_{\zeta\zeta} = 0, \tag{6.14}$$

$$\mathrm{Re}(\alpha_\zeta) = a_0, \tag{6.15}$$

$$(\tilde{K} + 3a_0 - a)m = 0, \tag{6.16}$$

$$R_0 = 0, \tag{6.17}$$

$$(\tilde{K} + a_0 - a)d = 0. \tag{6.18}$$

The solution of (6.15) is

$$\alpha = \alpha_0(\zeta - \beta_0),$$

where α_0, β_0 are constants and $\mathrm{Re}(\alpha_0) = a_0$. Performing a coordinate transformation (A3.3)

$$\zeta' = C_0 e^{iA_0}(\zeta - \beta_0)$$

we can transform α to the form

$$\alpha = \alpha_0\zeta, \quad \mathrm{Re}(\alpha_0) = a_0, \tag{6.19}$$

where α_0 is an invariant (constant) and primes have been dropped from the new functions.

From field equation (4.156a) and equations (6.16), (6.19) we obtain

$$m(\zeta) = N\zeta^c, \qquad c = (a - 3a_0)\alpha_0^{-1}, \tag{6.20}$$

where N and c are complex constants.

The third field equation (4.156c) reduces to

$$m - \bar{m} = 2id_{\zeta\bar{\zeta}} . \tag{6.21}$$

Define

$$m(\zeta) = 2\beta_\zeta, \qquad \beta = \beta(\zeta). \tag{6.22}$$

Then the general solution of (6.21) is

$$2id = \Lambda_{\bar{\zeta}} - \bar{\Lambda}_\zeta = 2\bar{\zeta}\beta - 2\zeta\bar{\beta} + \varphi(\zeta) - \bar{\varphi}(\bar{\zeta}), \tag{6.23}$$

where φ is an analytic function of ζ. This implies

$$\Lambda_{\bar{\zeta}} = 2\bar{\zeta}\beta + \varphi + A_{\zeta\bar{\zeta}},$$

where $A(\zeta,\bar{\zeta})$ is a real function, and so

$$\Lambda = \bar{\zeta}^2\beta + \bar{\zeta}\varphi + A_{\zeta}. \tag{6.24}$$

We now have the option of <u>either</u> (a) transforming A_ζ to zero and finding a particular integral for Λ from (6.18) by using the coordinate freedom on s (equation (4.142)), and then determining $T(\zeta,\bar{\zeta})$ from (6.13); <u>or</u> (b) transforming $T(\zeta,\bar{\zeta})$ to zero by means of (4.147) and determining Λ from (6.13) and (6.18). <u>See Appendix 4</u>. We take option (a) in this case.

By means of a transformation (4.138) we can quit the last term of (6.24) and write

$$\Lambda = \bar{\zeta}^2\beta + \bar{\zeta}\varphi, \tag{6.25}$$

where we simply require a particular solution for φ from (6.18). Now (6.20) and (6.22) give

$$\beta(\zeta) = N_0\zeta^{c+1} + M_0,$$

where M_0, N_0 are complex constants and

$$2(c+1)N_0 = N.$$

Hence

$$\Lambda = N_0\bar{\zeta}^2\zeta^{c+1} + M_0\bar{\zeta}^2 + \bar{\zeta}\varphi$$
$$= N_0\bar{\zeta}^2\zeta^{c+1} + \bar{\zeta}\varphi + B_\zeta,$$

where $B = 2\,\mathrm{Re}(M_0\zeta\bar{\zeta}^2)$ and we have absorbed a term $(-\bar{\zeta}.2\bar{M}_0\zeta)$ into $\bar{\zeta}\varphi$. Now make a transformation $s \to s' = s + B$. After dropping the prime we are left with

$$\beta(\zeta) = N_0\zeta^{c+1} \tag{6.26}$$

and

$$\Lambda = N_0\bar{\zeta}^2\zeta^{c+1} + \bar{\zeta}\varphi. \tag{6.27}$$

It remains to determine φ. This is done by solving (6.18), which is

$$\alpha_0\zeta d_\zeta + \bar{\alpha}_0\bar{\zeta}d_{\bar{\zeta}} + (a_0 - a)d = 0$$

or, through (6.23) and (6.26),

$$\alpha_0\zeta\varphi_\zeta + (a_0 - a)\varphi = \text{complex conjugate},$$

remembering that $\mathrm{Re}(\alpha_0) = a_0$. This implies

$$\alpha_0\zeta\varphi_\zeta + (a_0 - a)\,\varphi = C \quad \text{(real constant)}$$

so that

$$\varphi(\zeta) = C\zeta(a_0 - a + \alpha_0)^{-1} + D\zeta^{(a-a_0)/\alpha_0},$$

where D is a complex constant. Remembering that we seek only a particular solution for φ, we choose D = 0. Then

$$\Lambda = N_0\bar{\zeta}^2\zeta^{c+1} + C(a_0 - a + \alpha_0)^{-1}\zeta\bar{\zeta}. \qquad (6.28)$$

Equation (6.13) now gives

$$T_\zeta = - C\zeta\bar{\zeta}.$$

This requires C = 0 since T is real, and hence

$$T = 0. \qquad (6.29)$$

Summing up, we have arrived at the metric (4.129) with

$$\begin{aligned} p &= 0 = R^{(2)}, \\ \Lambda &= N_0\bar{\zeta}^2\zeta^{c+1}, \\ m &= 2(c+1)N_0\zeta^c, \\ d &= i\zeta\bar{\zeta}(\bar{N}_0\bar{\zeta}^{\bar{c}} - N_0\zeta^c), \end{aligned} \qquad (6.30)$$

where α_0, N_0 are arbitrary complex constants with

$$\mathrm{Re}(\alpha_0) = a_0$$

and

$$\alpha_0 c = a - 3a_0,$$

where $a_0 + a \neq 0$ and $a \neq 0$. Since T = 0 we can absorb the constant factor a into the HKV i.e. effectively a = 1, and so

$$\alpha_0 c = 1 - 3a_0, \quad a_0 \text{ arbitrary},$$

and the HKV admitted by the metric (6.30) is

$$\tilde{K} = \alpha_0\zeta\partial_\zeta + \bar{\alpha}_0\bar{\zeta}\partial_{\bar{\zeta}} + (a_0 + 1)s\partial_s - (a_0 - 1)r\partial_r. \qquad (6.31)$$

Since $\Lambda = \Omega$ and $\partial_u D\Omega = 0$, condition (4.64d) tells us that space is flat iff m = 0, that is, iff $N_0 = 0$. Otherwise this metric is Petrov type II.

Case 6.2 (II): $a_0 = 0$ and either (i) $R_0 = 0$, or (ii) $R_0 \neq 0$.

(i) $a_0 = 0 = R_0$.

Again we can transform p to zero, leaving ζ free up to a linear transformation, with s and r completely free.

Choosing $a = 1$, the homothetic Killing equations and their integrability conditions reduce to

$$(\tilde{K} + \alpha_\zeta - 1)\Lambda + T_\zeta = 0, \tag{6.32}$$

$$\alpha_{\zeta\zeta} = 0, \tag{6.33}$$

$$\mathrm{Re}(\alpha_\zeta) = 0, \tag{6.34}$$

$$(\tilde{K} - 1)m = 0, \tag{6.35}$$

$$(\tilde{K} - 1)d = 0. \tag{6.36}$$

The solution of (6.34) is

$$\alpha = ib_0\zeta + \alpha_0, \tag{6.37}$$

where b_0, α_0 are real, complex constants respectively.

Either (A) $b_0 = 0$, $\alpha = \alpha_0$,

or (B) $b_0 \neq 0$.

In case (B) we can transform α to $\alpha = ib_0\zeta$, where b_0 is an invariant, by means of the transformation $\zeta' = b_0\zeta - i\alpha_0$. Without loss of generality we can choose $b_0 = 1$. The linearity restriction on the transformation of ζ prevents us from doing better than obtaining $\alpha = i\zeta$ for (B).

Case 6.2(II)(i)(A): $a_0 = 0 = R_0$, $\alpha = \alpha_0 \neq 0$.

$$\tilde{K} = \alpha_0\partial_\zeta + \bar{\alpha}_0\partial_{\bar{\zeta}} + T\partial_s + s\partial_s + r\partial_r.$$

Field equation (4.156a) and equation (6.35) give

$$m(\zeta) = N\, e^{\zeta/\alpha_0}, \tag{6.38}$$

where N is an arbitrary complex constant.

With $p = 0$ the field equation (4.156c) has the solution given by (6.22) and (6.23). This gives Λ as in (6.24). Again, just as described in the paragraph following equation (6.24), we choose option (a) of Appendix 4 to find a particular solution for Λ.

From (6.22) and (6.38) we get

$$\beta = N_0\, e^{\zeta/\alpha_0} + M_0,$$

where M_0, N_0 are complex constants and

$$2N_0 = \alpha_0 N.$$

Hence

$$\Lambda = N_0\bar{\zeta}^2\, e^{\zeta/\alpha_0} + \bar{\zeta}\varphi + B_\zeta,$$

where $B = 2\,\mathrm{Re}(M_0\zeta\bar{\zeta}^2)$ and φ is an analytic function of ζ.

Making an s-transformation $s \to s + B$ we obtain

$$\beta = N_0\, e^{\zeta/\alpha_0} \tag{6.39}$$

and

$$\Lambda = N_0\bar{\zeta}^2\, e^{\zeta/\alpha_0} + \bar{\zeta}\varphi, \tag{6.40}$$

where φ is to be determined by (6.36).

From (6.23) and (6.36) we get

$$\alpha_0\varphi_\zeta - \varphi + 2\bar{\alpha}_0\beta = \text{complex conjugate},$$

which implies

$$\alpha_0\varphi_\zeta - \varphi + 2\bar{\alpha}_0\, N_0\, e^{\zeta/\alpha_0} = -D \quad \text{(real constant)}.$$

This has solution

$$\varphi(\zeta) = D + (C - \bar{\alpha}_0 N\zeta)e^{\zeta/\alpha_0},$$

where C is a complex constant. We can eliminate D by means of the transformation $s \to s + D\zeta\bar{\zeta}$, so that

$$\varphi(\zeta) = (C - \bar{\alpha}_0 N\zeta)e^{\zeta/\alpha_0}.$$

Because we are seeking only a particular solution for φ, we choose $C = 0$. Then

$$\Lambda = N_0\bar{\zeta}(\bar{\zeta} - 2\alpha_0^{-1}\bar{\alpha}_0\zeta)e^{\zeta/\alpha_0}. \tag{6.41}$$

Equation (6.32) now gives

$$T_\zeta = N\bar{\alpha}_0^2\zeta\, e^{\zeta/\alpha_0}$$

so that

$$T = N\alpha_0\bar{\alpha}_0^2(\zeta - \alpha_0)e^{\zeta/\alpha_0} + \bar{N}\alpha_0^2\bar{\alpha}_0(\bar{\zeta} - \bar{\alpha}_0)e^{\bar{\zeta}/\bar{\alpha}_0} + E,$$

where E is a real constant which we can discard since it gives a term $E\partial_s$ in the HKV, and this is a constant multiple of the already present Killing vector. Hence

$$T = |\alpha_0|^2\{\bar{\alpha}_0 N\zeta e^{\zeta/\alpha_0} + \alpha_0\bar{N}\bar{\zeta}e^{\bar{\zeta}/\bar{\alpha}_0} - |\alpha_0|^2(N\, e^{\zeta/\alpha_0} + \bar{N}\, e^{\bar{\zeta}/\bar{\alpha}_0})\}. \tag{6.42}$$

Thus we have arrived at the metric (4.129) with

$$\begin{aligned}
p &= 0 = R^{(2)}, \\
\Lambda &= N\bar{\zeta}(\tfrac{1}{2}\,\alpha_0\bar{\zeta} - \bar{\alpha}_0\zeta)e^{\zeta/\alpha_0}, \\
m &= N\, e^{\zeta/\alpha_0}, \\
d &= \tfrac{1}{2}i\,(\bar{\alpha}_0\zeta - \alpha_0\bar{\zeta})(N\, e^{\zeta/\alpha_0} + \bar{N}\, e^{\bar{\zeta}/\bar{\alpha}_0}),
\end{aligned} \tag{6.43}$$

where α_0, N are arbitrary complex constants, but $\alpha_0 \neq 0$.

Since $\Lambda = \Omega$ and $\partial_u D\Omega = 0$, condition (4.64d) shows that the space will be flat iff $N = 0$, otherwise Petrov type II.

The HKV is

$$\tilde{K} = \alpha_0 \partial_\zeta + \bar{\alpha}_0 \partial_{\bar{\zeta}} + T\partial_s + s\partial_s + r\partial_r, \qquad (6.44)$$

where T is given by (6.42).

Case 6.2(II)(i)(B): $a_0 = 0 = R_0$, $\alpha = i\zeta$.

$$\tilde{K} = i(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}) + T\partial_s + s\partial_s + r\partial_r.$$

Equation (6.35) and field equation (4.156a) give

$$m(\zeta) = N\zeta^{-i}, \qquad (6.45)$$

where N is an arbitrary complex constant.

With $p = 0$ the field equation (4.156c) has the solution given by (6.22) and (6.23). This gives Λ as in (6.24). Again we choose option (a) of Appendix 4 to find a particular solution for Λ.

From (6.22) and (6.45) we get

$$\beta = N_0\zeta^{1-i} + M_0,$$

where

$$2(1-i)N_0 = N$$

and M_0 is a complex constant which we can transform to zero through $s \to s + 2\,\mathrm{Re}(M_0\zeta\bar{\zeta}^2)$. When this is done we have

$$\beta = N_0\zeta^{1-i} \qquad (6.46)$$

and

$$\Lambda = N_0\bar{\zeta}^2\zeta^{1-i} + \bar{\zeta}\varphi.$$

Using (6.23), equation (6.36) reduces to

$$\zeta\varphi_\zeta + \bar{\zeta}\bar{\varphi}_{\bar{\zeta}} + i(\varphi - \bar{\varphi}) = 0$$

which implies

$$\zeta\varphi_\zeta + i\varphi = iD,$$

where D is a real constant. This has solution

$$\varphi(\zeta) = C\zeta^{-i} + D,$$

where C is a complex constant. We can eliminate D through the transformation $s \to s + D\zeta\bar{\zeta}$. Remembering that under option (a) we need only a particular solution for φ, we choose $C = 0$.

Then

$$\varphi(\zeta) = 0$$

and

$$\Lambda = N_0\zeta^{1-i}\bar{\zeta}^2. \qquad (6.47)$$

From (6.32) we now obtain $T_\zeta = 0$, so that

$$T = T_0 \text{ (real constant)} \tag{6.48}$$

Thus we have arrived at the metric (4.129) with

$$\begin{aligned} p &= 0 = R^{(2)}, \\ \Lambda &= N_0\zeta^{1-i}\bar{\zeta}^2, \\ m &= 2(1-i)N_0\zeta^{-i}, \\ d &= i\zeta\bar{\zeta}(\bar{N}_0\bar{\zeta}^i - N_0\zeta^{-i}), \end{aligned} \tag{6.49}$$

where N_0 is an arbitrary complex constant. The space will be flat iff $N_0 = 0$, otherwise Petrov type II. Taking into account the fact that there is a Killing vector $T_0\partial_s$ already in the space, we have the following HKV admitted by this metric:

$$\tilde{K} = i(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}) + s\partial_s + r\partial_r. \tag{6.50}$$

By putting $\zeta^{-i} = e^z$ so that $i\zeta\partial_\zeta = \partial_z$, the metric components (6.49) become

$$\begin{aligned} p &= 0 = R^{(2)}, \\ \Lambda &= N_0 e^{(1+i)z - 2i\bar{z}}, \\ m &= 2(1-i)N_0 e^z, \\ d &= ie^{i(z-\bar{z})}[\bar{N}_0 e^{\bar{z}} - N_0 e^z] \end{aligned} \tag{6.51}$$

and the HKV is now

$$\tilde{K} = \partial_z + \partial_{\bar{z}} + s\partial_s + r\partial_r. \tag{6.52}$$

(ii) $a_0 = 0,\ R_0 \neq 0$.

As in Case 6.1 (II) we can choose coordinates so that

$$e^{-p} = \zeta\bar{\zeta} - R_0, \tag{6.53}$$

where $R_0 = \pm 1$. The coordinate freedom left is a bilinear transformation on ζ and complete freedom on s and r.

The HKV is of the form

$$\tilde{K} = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + T\partial_s + a(s\partial_s + r\partial_r)$$

so that, for p as in (6.53), equation (4.152) becomes

$$(\zeta\bar{\zeta} - R_0).\mathrm{Re}(\alpha_\zeta) = \alpha\bar{\zeta} + \bar{\alpha}\zeta$$

with solution

$$\alpha = \alpha_0\zeta^2 + ib_0\zeta - \bar{\alpha}_0 R_0, \tag{6.54}$$

where α_0 and b_0 are constants, b_0 real.

There are two cases to consider:

(A) $R_0 = -1$ (sphere), (B) $R_0 = 1$ (pseudosphere).

<u>Case 6.2(II)(ii)(A):</u> $R_0 = -1$.

The form (6.54) becomes

$$\alpha = \alpha_0\zeta^2 + ib_0\zeta + \bar{\alpha}_0. \tag{6.55}$$

Appendix 6 shows that it is always possible to reduce (6.55) to the form

$$\alpha = \alpha_0\zeta^2 + \bar{\alpha}_0. \tag{6.56}$$

Change coordinates to $(z,\bar{z},s,r)$, where

$$\zeta = (\bar{\alpha}_0/\alpha_0)^{\frac{1}{2}} \tan z. \tag{6.57}$$

Then

$$\partial_\zeta = (\alpha_0/\bar{\alpha}_0)^{\frac{1}{2}} \cos^2 z\, \partial_z$$

and

$$\alpha = \bar{\alpha}_0 \sec^2 z.$$

Now

$$\tilde{K} = |\alpha_0|(\partial_z + \partial_{\bar{z}}) + T\partial_s + a(s\partial_s + r\partial_r),$$

and we shall choose option (b) of Appendix 4 to transform T to zero. Absorbing the constant $|\alpha_0|$ we shall take

$$\tilde{K} = \partial_z + \partial_{\bar{z}} + a(s\partial_s + r\partial_r), \tag{6.58}$$

where we have redefined the constant a. Thus there is no loss of generality in taking $|\alpha_0| = 1$, so that

$$\alpha_\zeta = 2 \tan z, \qquad \zeta = \tan z, \qquad e^{-P} = 1 + \tan z \tan \bar{z}, \tag{6.59}$$

and the remaining equations (4.150) - (4.155) become

$$(\tilde{K} + 2 \tan z - a)\Lambda = 0, \tag{6.60}$$

$$(\tilde{K} - a)m = 0, \tag{6.61}$$

$$(\tilde{K} - a)d = 0. \tag{6.62}$$

The surviving field equations (4.156) are

$$m_{\bar{z}} = 0, \tag{6.63}$$

$$m - \bar{m} = 2i\ \cos^2(z-\bar{z})d_{z\bar{z}} + 4id. \tag{6.64}$$

In order to preserve the form (6.58) of $\tilde{K}$, equation (4.147) must be satisfied: $(\tilde{K} - a)A = 0$,

implying that

$$A = \bar{A} = (e^{az} + e^{a\bar{z}}).f(z - \bar{z}), \tag{6.65}$$

where $f = \bar{f}$ is an arbitrary function of $(z - \bar{z})$. Also (4.146) requires $\alpha' = 1 = \alpha$, implying that $z' = z + c$, where c is a complex constant. Hence the coordinate freedom left is

$$\begin{aligned} z' &= z + c, \\ s' &= C_0(s + A), \\ r' &= C_0^{-1}r, \end{aligned} \tag{6.66}$$

where C_0 is a real constant and $A(z,\bar{z})$ is as in (6.65).

Equation (6.60) with (6.58) gives

$$\Lambda = (e^{az} + e^{a\bar{z}})\cos^2 z.h(z - \bar{z}), \tag{6.67}$$

where h is an analytic function of $(z - \bar{z})$. Using the definition

$$2id = e^{-2P}(\Lambda_{\bar{\zeta}} - \bar{\Lambda}_{\zeta})$$

we get

$$2id = \cos^2\theta(e^{a\bar{z}}f - e^{az}\bar{f}), \qquad \theta = z - \bar{z}, \tag{6.68}$$

where

$$f(\theta) = ah - h_\theta + \bar{h}_{\bar{\theta}}.$$

Therefore

$$2id_z = -4id\tan\theta + \cos^2\theta(e^{a\bar{z}}f_\theta - ae^{az}\bar{f} + e^{az}\bar{f}_{\bar{\theta}}).$$

Substituting the last expression in (6.62) we find that equation (6.62) is satisfied identically.

From (6.61) and (6.63) we obtain

$$m = m_0 e^{az}, \tag{6.69}$$

where m_0 is an arbitrary complex constant. Calculating $2id_{z\bar{z}}$ and substituting in (6.64) gives

$$m_0 e^{az} - \bar{m}_0 e^{a\bar{z}}$$
$$= \cos^4\theta[2\tan\theta(2e^{az}\bar{f}_{\bar{\theta}} - ae^{az}\bar{f} - ae^{a\bar{z}}f + 2e^{a\bar{z}}f_\theta)$$
$$+ ae^{a\bar{z}}f_\theta - e^{a\bar{z}}f_{\theta\theta} - ae^{az}\bar{f}_{\bar{\theta}} + e^{az}\bar{f}_{\bar{\theta}\bar{\theta}}$$
$$+ 4\, e^{a\bar{z}}f - 4e^{az}\bar{f}]$$

so that

$$f_{\theta\theta} - (a + 4\tan\theta)f_\theta - (4-2a\tan\theta)f = \bar{m}_0\sec^4\theta. \tag{6.70}$$

Putting $a = 2$ (possible because $a_0 = 0$) and $x = \tan\theta$, this becomes

$$(1 + x^2)^2 f'' - 2(1 + x)(1 + x^2)f' - 4(1 - x)f = \bar{m}_0(1 + x^2)^2, \tag{6.71}$$

where the prime denotes d/dx.

I have no solution to (6.71) for $m_0 \neq 0$. If $m_0 = 0$, then $m = 0$. But for p as in (6.53) we have $\partial_u D\,\Omega = p_{\zeta\zeta} - p_\zeta^2 = 0$, so that <u>when $m = 0$</u> condition (4.64d) is satisfied and <u>space is flat</u>.

In this case, then, non-flat solutions are possible and derive from solutions of (6.71) when $m_0 \neq 0$. However, I have not been able to solve this equation.

<u>Case 6.2(II)(ii)(B):</u> $R_0 = 1$.

The form (6.54) becomes

$$\alpha = \alpha_0\zeta^2 + ib_0\zeta - \bar{\alpha}_0. \tag{6.72}$$

Appendix 6 shows that it is not always possible to express this quadratic in canonical form. We need to consider three cases separately:

(B1) $\alpha_0 \neq 0,\ b_0 = 0$ (B2) $\alpha_0 = 0,\ b_0 \neq 0$ (B3) $\alpha_0 \neq 0,\ b_0 \neq 0$.

<u>Case 6.2(II)(ii)(B1):</u> $\alpha_0 \neq 0,\ b_0 = 0$.

α can be reduced to the form

$$\alpha = \zeta^2 - 1. \tag{6.73}$$

Change coordinates to (w, w, s, r), where

$$w = \frac{\zeta - 1}{\zeta + 1} \qquad \Leftrightarrow \qquad \zeta = \frac{1 + w}{1 - w}\,. \tag{6.74}$$

It is stressed that this is a pure relabelling of the coordinates and is <u>not</u> an allowed transformation $\zeta \to \zeta' = \Phi(\zeta)$. For if (6.74) were to be interpreted as such a transformation, it would have to be subject to the constraint equations (A6.4) of Appendix 6 in order to leave the form of p unchanged.

Obviously these constraint equations are not satisfied when (6.74) is so interpreted.

Then $$\partial_\zeta = \tfrac{1}{2}(1 - w)^2\partial_w$$

and $$\alpha = 4w(1 - w)^{-2}$$

so that the HKV becomes

$$\tilde{K} = 2w\partial_w + 2\bar{w}\partial_{\bar{w}} + a(s\partial_s + r\partial_r) + T\partial_s.$$

Since there is still complete freedom on the coordinates s and r, we choose option (b) of Appendix 4 to transform T to zero. Choosing $a = 2$ and absorbing a factor 2, we may now take

$$\tilde{K} = w\partial_w + \bar{w}\partial_{\bar{w}} + s\partial_s + r\partial_r. \tag{6.75}$$

The remaining equations (4.150) - (4.155) become

$$[\tilde{K} + 2(1 + w)(1 - w)^{-1} - 2]\Lambda = 0, \tag{6.76}$$

$$(\tilde{K} - 2)m = 0, \tag{6.77}$$

$$(\tilde{K} - 2)d = 0. \tag{6.78}$$

The surviving field equations are

$$m_{\bar{w}} = 0, \tag{6.79}$$

$$m - \bar{m} = 2i(w + \bar{w})^2 d_{w\bar{w}} - 4id. \tag{6.80}$$

Equations (6.76) and (6.75) give

$$\Lambda = (1 - w)^2.f(\theta), \qquad \theta = w/\bar{w}, \tag{6.81}$$

where f is an arbitrary function of θ.

Using the definition $2id = e^{-2p}(\Lambda_{\bar{\zeta}} - \bar{\Lambda}_\zeta)$ we get

$$id = (w + \bar{w})^2(\bar{w}w^{-2}\bar{f}_{\bar{\theta}} - w\bar{w}^{-2} f_\theta). \tag{6.82}$$

Therefore

$$id_w = 2(w+\bar{w})^{-1}id - (w+\bar{w})^2(2w\bar{w}^{-3}\bar{f}_{\bar{\theta}} + \bar{w}^{-2} f_\theta + \bar{w}^2w^{-4}\bar{f}_{\bar{\theta}\bar{\theta}} + w\bar{w}^{-3} f_{\theta\theta})$$

and substituting in (6.78) we get

$$d = 0. \tag{6.83}$$

From (6.77) and (6.79) we obtain

$$m = m_0 w, \tag{6.84}$$

where m_0 is an arbitrary complex constant. Now (6.80) and (6.83) give $m = \bar{m}$, so with (6.84) this implies

$$m = 0. \tag{6.85}$$

But then condition (4.64d) is satisfied, so space is flat.

Case 6.2(II)(ii)(B2): $\alpha_0 = 0$, $b_0 \neq 0$.

Choosing $b_0 = 1$ we have

$$\alpha = i\zeta. \tag{6.86}$$

Then, using option (b) of Appendix 4 to transform T to zero, we may take the HKV to be

$$\tilde{K} = i(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}) + a(s\partial_s + r\partial_r). \tag{6.87}$$

The coordinate freedom left on s is

$$s' = s + A,$$

where $\zeta A_\zeta - \bar{\zeta}A_{\bar{\zeta}} + iaA = 0$, giving

$$A = (\zeta^{-ia} + \bar{\zeta}^{ia})g(x), \tag{6.88}$$

where $g = \bar{g}$ is a function of $x = \zeta\bar{\zeta} - 1$.

The remaining non-trivial equations (4.150) - (4.155) are

$$(\tilde{K} + i - a)\Lambda = 0, \tag{6.89}$$

$$(\tilde{K} - a)m = 0, \tag{6.90}$$

$$(\tilde{K} - a)d = 0, \tag{6.91}$$

and the surviving field equations (4.156) are

$$m_{\bar{\zeta}} = 0, \tag{6.92}$$

$$m - \bar{m} = 2i(\zeta\bar{\zeta} - 1)^2 d_{\zeta\bar{\zeta}} - 4id. \tag{6.93}$$

Equations (6.90) and (6.92) give

$$m = M\zeta^{-ia}, \tag{6.94}$$

where N is an arbitrary complex constant. Equation (6.89) is

$$\zeta\Lambda_\zeta - \bar{\zeta}\Lambda_{\bar{\zeta}} + (1+ia)\Lambda = 0$$

with solution

$$\Lambda = \zeta^{-1-ia}f(x), \qquad x = \zeta\bar{\zeta} - 1, \tag{6.95}$$

where f is a function to be determined by the remaining two equations. Of these, equation (6.91) is satisfied identically upon using the definition

$$2id = e^{-2p}(\Lambda_{\bar{\zeta}} - \bar{\Lambda}_\zeta) = x^2(\zeta^{-ia}f' - \bar{\zeta}^{ia}\bar{f}'), \tag{6.96}$$

where the prime denotes d/dx.

Substituting (6.94) and (6.96) into the field equation (6.93) gives

$$\zeta^{-ia}\{N - x^4[(1 - ia)f'' + (x+1)f''']$$
$$- 2x^3[(2-ia)f' + 2(x+1)f'']\} \tag{6.97}$$
$$= \text{complex conjugate.}$$

Denoting the expression in braces { } by $E(x)$, we can write (6.97) as

$$(x+1)^{-ia/2}\, E(x) = (x+1)^{ia/2}\, \bar{E}(x). \tag{6.98}$$

This equation is satisfied if

$$(x+1)^{-ia/2}\, E(x) = C \quad \text{(real constant).} \tag{6.99}$$

Put $H(x) = (1+ia)x^3 f' - N$. Then (6.99) reduces to

$$x(x+1)H'' + [(-1-ia)x - 2]H' + (1+ia)H$$
$$+ C(1+ia)(x+1)^{ia/2} = 0. \tag{6.100}$$

I do not have any solutions $H(x)$ when $C \neq 0$.

For $C = 0$, (6.100) is a hypergeometric equation with solution

$$H(x) = kF(-1, -1-ia, -2, -x) + \ell\, x^3\, F(2, 2-ia, 4, -x) \tag{6.101}$$

valid for $|x| < 1$, where k and ℓ are arbitrary (real) constants and F is the hypergeometric function. There is no solution expressible in finite terms (see Appendix 7) other than the trivial one $H = 0$. Then

$$(1+ia)x^3\, f'(x) = N$$

with solution

$$f(x) = B - \frac{1}{2} N x^{-2}(1+ia)^{-1},$$

where B is a complex constant. This gives

$$\Lambda = B\zeta^{-1-ia} - \frac{1}{2} N x^{-2} \zeta^{-1-ia}(1+ia)^{-1}.$$

We can transform the first term away by means of the transformation $s \to s + A$, where

$$A = (\zeta^{-ia} + \bar{\zeta}^{ia})B/ia,$$

in accordance with (6.88). Thus

$$\Lambda = - \frac{N\zeta^{-1-ia}}{2(1+ia)(\zeta\bar{\zeta}-1)^2} \; . \tag{6.102}$$

Summing up, the only solution found in this case is the metric (4.129) with

$$\begin{aligned} &e^{-P} = \zeta\bar{\zeta} - 1, \qquad R^{(2)} = 1, \\ &\Lambda \quad \text{as in (6.102),} \\ &m = N\zeta^{-ia}, \\ &d = -\frac{1}{2}\, i(1+a^2)^{-1}(\zeta\bar{\zeta}-1)^{-1}[(1-ia)N\zeta^{-ia} - (1+ia)\bar{N}\bar{\zeta}^{ia}], \end{aligned} \tag{6.103}$$

where N is an arbitrary complex constant. The space will be flat iff $N = 0$, otherwise Petrov type II. The HKV admitted by this metric is (6.87).

Other solutions are possible and are determined by solutions of (6.100) for $C \neq 0$, and by solving

$$(1+ia)x^3 f'(x) = N + H(x),$$

where $H(x)$ is given by (6.101). I do not have solutions of this sort.

Case 6.2(II)(ii)(B3): $\alpha_0 \neq 0, \quad b_0 \neq 0.$

In this case the best we can do with (6.72) is to reduce it to the form (see Appendix 6)

$$\alpha = \zeta^2 + ib_0\zeta - 1.$$

Choosing $b_0 = 2$ we can write this as

$$\alpha = (\zeta + i)^2. \tag{6.104}$$

Put $\zeta + i = -1/z$. Then

$$\partial_\zeta = z^2\partial_z,$$

$$\alpha = z^{-2},$$

and, after transforming T to zero as we may through option (b) of Appendix 4, we have

$$\tilde{K} = \partial_z + \partial_{\bar{z}} + a(s\partial_s + r\partial_r). \tag{6.105}$$

The remaining non-trivial equations (4.150) - (4.155) are

$$(\tilde{K} - 2z^{-1} - a)\Lambda = 0, \tag{6.106}$$

$$(\tilde{K} - a)m = 0, \tag{6.107}$$

$$(\tilde{K} - a)d = 0, \tag{6.108}$$

and the surviving field equations (4.156) are

$$m_{\bar{z}} = 0, \tag{6.109}$$

$$m - \bar{m} = 2i[1 + i(z-\bar{z})]^2 d_{z\bar{z}} - 4id. \tag{6.110}$$

Equations (6.107) and (6.109) give

$$m = m_0 e^{az}, \tag{6.111}$$

where m_0 is an arbitrary complex constant. From equations (6.106) and (6.105) we obtain

$$\Lambda = z^2 e^{az} f(z-\bar{z}), \tag{6.112}$$

where f is a function of $(z-\bar{z})$ to be determined by the remaining equations.

From the definition of d we find

$$2id = (1+i\theta)^2(e^{a\bar{z}}\,\bar{f}_{\bar{\theta}} - e^{az}f_\theta), \tag{6.113}$$

where $\theta = z-\bar{z}$. Using this result and

$$2id_z = -4(1+i\theta)^{-1}\,d - (1+i\theta)^2(ae^{az}f_\theta + e^{az}f_{\theta\theta} + e^{a\bar{z}}\bar{f}_{\bar{\theta}\bar{\theta}})$$

we find that (6.108) is satisfied identically.
Substituting in (6.110) we obtain

$$m_0 = (1+i\theta)^3\{2i(af_\theta + 2f_{\theta\theta}) + (1+i\theta)(af_{\theta\theta} + f_{\theta\theta\theta})\}.$$

Putting $x = 1+i\theta$, this may be written

$$m_0 = x^3\{2i(iaf' - 2f'') - x(af'' + if''')\},$$

where

$$if' = i(df/dx) = f_\theta.$$

The equation in f may be written as

$$[xf' + (2 - iax)f]'' = im_0x^{-3} \tag{6.114}$$

with solution

$$f = \left(\frac{C}{a^2} - \frac{2iB}{a^3} - \frac{m_0}{2a}\right)x^{-2} + \left(\frac{2B}{a^2} + \frac{iC}{a}\right)x^{-1} + \frac{iB}{a} + Dx^{-2}\,e^{iax} \tag{6.115}$$

where B, C, D are arbitrary complex constants. On putting $x = 1 + i(z-\bar{z})$ and substituting (6.115) into (6.112), we have an expression for Λ.

<u>Summing up</u>, we have the metric (4.129), which in $(z, \bar{z}, s, r)$ coordinates is

$$\tfrac{1}{2}\,d\tau^2 = (r^2 + d^2)[1 + i(z-\bar{z})]^{-2}dzd\bar{z} + [dr + i(d_{\bar{z}}d\bar{z} - d_zdz)]\varkappa$$
$$+ \{1 + \mathrm{Re}[m/(r+id)]\}\varkappa^2, \tag{6.116}$$

where

$$\varkappa = ds + e^{az}f\,dz + e^{a\bar{z}}\,\bar{f}\,d\bar{z},$$

and the functions m and f are given by (6.115) and (6.111) with $x = 1 + i(z-\bar{z})$. The function d is given by

$$2id = [1 + i(z-\bar{z})]^2(e^{a\bar{z}}\,\bar{f}_{\bar{z}} - e^{az}f_z). \tag{6.117}$$

This metric is <u>Petrov type II</u>, unless $m_0 = 0$ when it is the metric of flat space. The HKV is (6.105).

$$\boxed{a_0 + a \neq 0, \quad R^{(2)} \neq \text{constant}}$$

Case 6.2 (III):

We may take the 2-curvature $R^{(2)}$ in the form (6.7) of Case 6.1 (III), leaving the coordinate freedom (6.8), but now with full freedom on the s coordinate i.e. $s' = C_0(s + A)$, A arbitrary, according to the arguments employed in that case. Likewise, the only known solution for $p(\zeta,\bar{\zeta})$ is (6.9), namely,

$$e^{-2p} = \frac{2}{3} X^3, \quad X = \zeta + \bar{\zeta}.$$

Then

$$\tilde{K}(e^{-2p}) = -2 e^{-2p} \tilde{K}p = 2X^2\tilde{K}X = 2X^2(\alpha + \bar{\alpha})$$

for $\tilde{K}$ as in (6.12), so that

$$\tilde{K}p = -\frac{3}{2} X^{-1}(\alpha + \bar{\alpha}). \tag{6.118}$$

Equation (4.154) gives

$$\alpha + \bar{\alpha} + 2a_0(\zeta + \bar{\zeta}) = 0$$

with solution

$$\alpha = -2a_0(\zeta + ie_0),$$

where e_0 is a real constant. The coordinate transformation $\zeta' = \zeta + ie_0$, which is allowed by (6.8), brings α to

$$\alpha = -2a_0\zeta. \tag{6.119}$$

Substituting this in (6.118) gives $\tilde{K}p = 3a_0$. Then equations (4.151) and (4.152) are satisfied identically.

Since we still have complete freedom left on the s coordinate, we employ option (b) of Appendix 4 to transform T to zero.
Then (6.12) becomes

$$\tilde{K} = -2a_0(\zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}}) + a_0(s\partial_s - r\partial_r) + a(s\partial_s + r\partial_r), \tag{6.120}$$

The remaining equations (4.150) - (4.155) are

$$(\tilde{K} - 3a_0 - a)\Lambda = 0, \tag{6.121}$$

$$(\tilde{K} + 3a_0 - a)m = 0, \tag{6.122}$$

$$(\tilde{K} + a_0 - a)d = 0, \tag{6.123}$$

and the surviving field equations (4.156) are

$$m_{\bar{\zeta}} = 0, \tag{6.124}$$

$$m - \bar{m} = \frac{4}{3} i X^3 d_{\zeta\bar{\zeta}} - 4i X d. \tag{6.125}$$

Equations (6.122) and (6.124) give

$$m = m_0\zeta^c, \tag{6.126}$$

where m_0 is an arbitrary complex constant, and c is given by

$$2a_0c = 3a_0 - a.$$

Equation (6.121) gives

$$\Lambda = \zeta^b.f(\zeta/\bar{\zeta}) \tag{6.127}$$

where

$$-2a_0b = 3a_0 + a, \quad a_0(b-c) + 3a_0 = 0.$$

Using the definition of d we have then

$$2id = e^{-2p}(\Lambda_{\bar{\zeta}} - \bar{\Lambda}_{\zeta})$$
$$= \frac{2}{3}X^3(\zeta^{-2}\bar{\zeta}^{b+1}\bar{f}_{\bar{\theta}} - \bar{\zeta}^{-2}\zeta^{b+1}f_{\theta}), \quad \theta = \zeta/\bar{\zeta}. \tag{6.128}$$

We now find that (6.123) is satisfied identically.

Using (6.126) and (6.128), the field equation (6.125) is satisfied iff

$$f = C \quad \text{(complex constant)}, \quad \Rightarrow d = 0, \tag{6.129}$$

and
$$m = 0. \tag{6.130}$$

Now $\partial_u D\Omega = p_{\zeta\zeta} - p_{\zeta}^2 = -(3/4)X^{-2}$, so that $\partial_u\partial_u D\Omega = 0$ but $\bar{D}\partial_u D\Omega = (3/2)X^{-3} \neq 0$. Hence we have arrived at the non-flat metric (4.129) with

$$e^{-2p} = \frac{2}{3}X^3, \quad R^{(2)} = \zeta + \bar{\zeta} = X,$$
$$\Lambda = C\zeta^b, \tag{6.131}$$
$$m = 0 = d,$$

where C is a constant which can be made real by using the remaining coordinate freedom. In fact, from (4.138) we have, for $C = c_1 + ic_2$,

$$\Lambda' = C\zeta^b - A_{\zeta}$$
$$= c_1\zeta^b \Leftrightarrow A_{\zeta} = ic_2\zeta^b,$$

so that by taking $A = \frac{ic_2}{b+1}(\zeta^{b+1} - \bar{\zeta}^{b+1})$ we have transformed the imaginary part of C away. This transformation does not alter the zero value of T, as can be checked by inserting the function A just determined into equation (4.147).

The metric is

$$d\tau^2 = 3r^2(\zeta + \bar{\zeta})^{-3}\, d\zeta d\bar{\zeta} + 2\, drds + 2\, C\, dr(\zeta^b d\zeta + \bar{\zeta}^b\, d\bar{\zeta})$$

$$+ 2(\zeta + \bar{\zeta})[ds + C(\zeta^b d\zeta + \bar{\zeta}^b d\bar{\zeta})]^2, \qquad (6.132)$$

which is Petrov type III and admits the HKV (6.120). However, we can reduce the number of degrees of freedom by one by noting that we can absorb the non-zero constant a in $\tilde{K}$ (effectively, put a = 1). Then the metric is (6.132), but now

$$b = -(3a_0 + 1)/2a_0, \qquad (6.133)$$

where a_0 is an arbitrary real constant, and the HKV is

$$\tilde{K} = -2a_0(\zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}}) + (a_0 + 1)s\partial_s - (a_0 - 1)r\partial_r. \qquad (6.134)$$

When C = 0 the metric (6.132) degenerates into (6.10) [or (5.9)], but the HKV still differs from (6.3).

$$\boxed{a_0 + a = 0}$$

For proper homothetic motions to exist $a \neq 0$, and this is so when $a_0 + a = 0$ iff $a_0 \neq 0$. Then we have in the space

$$K = \zeta_s \qquad (6.135)$$

and

$$\tilde{K} = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + T\partial_s + 2ar\partial_r, \qquad (6.136)$$

where $\alpha(\zeta)$ is non-zero.

Equations (4.150) - (4.155) reduce to

$$(\tilde{K} + \alpha_\zeta)\Lambda + T_\zeta = 0, \qquad (6.137)$$

$$(\tilde{K} + \alpha_\zeta)p_\zeta + \frac{1}{2}\alpha_{\zeta\zeta} = 0, \qquad (6.138)$$

$$\tilde{K}p + \mathrm{Re}(\alpha_\zeta) = -a, \qquad (6.139)$$

$$(\tilde{K} - 4a)m = 0, \qquad (6.140)$$

$$(\tilde{K} - 2a)R^{(2)} = 0, \qquad (6.141)$$

$$(\tilde{K} - 2a)d = 0. \qquad (6.142)$$

There are two distinct cases to consider:

(i) $R^{(2)}$ = constant, (ii) $R^{(2)} \neq$ constant.

Case 6.2(IV)(i): $R^{(2)}$ = constant.

Equation (6.141) gives $P^{(2)} = 0$. Then, just as in Case 6.2(I), we can transform p to zero. We follow the analysis of that case step by step to deal with the present case. The details are omitted. We obtain the metric (4.129) with

$$\begin{aligned} p &= 0 = R^{(2)}, \\ \Lambda &= N_0\bar{\zeta}^2\zeta^{(\alpha_0-4a_0)/\alpha_0}, \\ m &= 2\alpha_0^{-1}(\alpha_0-4a_0)N_0\zeta^{-4a_0/\alpha_0}, \\ d &= i\zeta\bar{\zeta}(\bar{N}_0\bar{\zeta}^{-4a_0/\bar{\alpha}_0} - N_0\zeta^{-4a_0/\alpha_0}), \end{aligned} \tag{6.143}$$

where a_0, N_0, α_0 are arbitrary constants, a_0 real, with

$$\mathrm{Re}(\alpha_0) = a_0 \neq 0 \quad \text{(since a cannot be zero).}$$

If $N_0 = 0$ the space is flat, otherwise the metric is Petrov type II. The HKV is

$$\tilde{K} = \alpha_0\zeta\partial_\zeta + \bar{\alpha}_0\bar{\zeta}\partial_{\bar{\zeta}} - 2a_0 r\partial_r, \tag{6.144}$$

where we have accounted for the presence of a multiple of the Killing vector ∂_s. We now realize that a factor a_0 can be absorbed into $\tilde{K}$, that is, we may effectively put $a_0 = -1$. Then the metric is exactly that of (6.30) with the HKV of (6.31) with $a_0 = -a$. Thus the present case is a special case of Case 6.2(I).

Case 6.2(IV)(ii): $R^{(2)} \neq$ constant.

Just as in Case 6.2(III), page 170, we may take

$$R^{(2)} = \zeta + \bar{\zeta} = X, \qquad e^{-2p} = \frac{2}{3}X^3,$$

this being the only known solution for $p(\zeta,\bar{\zeta})$. Then, just as on page 143, we obtain from (6.141), remembering that now $a_0 + a = 0$,

$$\alpha = 2a\zeta \tag{6.145}$$

and the equations (6.138) and (6.139) are satisfied identically.

Using option (b) of Appendix 4 to transform T to zero, and choosing $a = \frac{1}{2}$, we have

$$\tilde{K} = \zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} + r\partial_r. \tag{6.146}$$

Equations (6.137), (6.140) and (6.142) become

$$(\tilde{K} + 1)\Lambda = 0, \tag{6.147}$$

$$(\tilde{K} - 2)m = 0, \tag{6.148}$$

$$(\tilde{K} - 1)d = 0, \tag{6.149}$$

and the surviving field equations are (6.124) and (6.125).

Equations (6.148) and (6.124) give

$$m = m_0\zeta^2, \tag{6.150}$$

where m_0 is an arbitrary complex constant. Equation (6.147) gives

$$\Lambda = \zeta^{-1}f(\zeta/\bar{\zeta}), \tag{6.151}$$

where f is a function of $(\zeta/\bar{\zeta})$. Using the definition of d,

$$2id = e^{-2p}(\Lambda_{\bar{\zeta}} - \bar{\Lambda}_{\zeta}) = \frac{2}{3}X^3(\zeta^{-2}\bar{f}' - \bar{\zeta}^{-2}f'), \tag{6.152}$$

where the prime denotes differentiation with respect to the argument of the function. But from (6.149),

$$d = Xg(\zeta/\bar{\zeta}),$$

where g is some function of $(\zeta/\bar{\zeta})$. So, if (6.152) is to be satisfied, then

$$f = C, \qquad d = 0, \tag{6.153}$$

where C is a complex constant. Then (6.125) and (6.150) imply $m_0 = 0$. Hence

$$m = 0. \tag{6.154}$$

Thus we have arrived at a non-flat metric (4.129) with

$$e^{-2p} = \frac{2}{3}X^3, \quad R^{(2)} = \zeta + \bar{\zeta} = X,$$

$$\Lambda = C\zeta^{-1}, \tag{6.155}$$

$$m = 0 = d,$$

where C is an arbitrary constant which can be made real by using the remaining coordinate freedom, by the same argument as follows (6.131), except here we take the function A to be $A = -ic_2\log(\zeta/\bar{\zeta})$.

The metric is

$$d\tau^2 = 3r^2(\zeta+\bar{\zeta})^{-3}d\zeta d\bar{\zeta} + 2\,drds + 2\,C\,dr(\zeta^{-1}d\zeta + \bar{\zeta}^{-1}d\bar{\zeta})$$
$$+ 2(\zeta+\bar{\zeta})[ds + C(\zeta^{-1}d\zeta + \bar{\zeta}^{-1}d\bar{\zeta})]^2, \tag{6.156}$$

where C is an arbitrary real constant. This metric is Petrov type III and admits the HKV (6.146). It degenerates to the metric (6.10) when C = 0, but even so the HKV is different from (6.3).

Putting $8\zeta = 3(x + iy)$, the metric (6.156) becomes

$$d\tau^2 = r^2 x^{-3}(dx^2 + dy^2) + 2\, dr ds + 2\, C\, dr.d[\log \tfrac{9}{64}(x^2 + y^2)] + \tfrac{3}{2} x\{ds + C\, d[\log \tfrac{9}{64}(x^2 + y^2)]\}^2 \tag{6.157}$$

with HKV

$$\tilde{K} = x\partial_x + y\partial_y + r\partial_r. \tag{6.158}$$

6.3 We turn now to the case in which the space admits a Killing vector of the form

$$K = \partial_\zeta + \partial_{\bar\zeta} = \partial_x, \tag{6.159}$$

where $\zeta = x + iy$. In Appendix 1 it is shown that the form of the HKV may be taken as

$$\tilde{K} = \zeta\partial_\zeta + \bar\zeta\partial_{\bar\zeta} + u\partial_u - v\partial_v + a(u\partial_u + v\partial_v).$$

Choose a = -1.(*) Then

$$\tilde{K} = \zeta\partial_\zeta + \bar\zeta\partial_{\bar\zeta} - 2v\partial_v = x\partial_x + y\partial_y - 2v\partial_v. \tag{6.160}$$

The coordinate freedom left is (see (A1.24) of Appendix 1)

$$\begin{aligned} \zeta' &= b\zeta + c, \\ u' &= b(u + 1), \\ v' &= b^{-1}v, \end{aligned} \tag{6.161}$$

where b, c are real, complex constants respectively.

The homothetic Killing equations and their integrability conditions are now

$$(\tilde{K} + 1)(\Omega - u\dot\Omega) = 0, \tag{6.162}$$

$$(\tilde{K} + 1)\dot\Omega = 0, \tag{6.163}$$

$$(\tilde{K} + 4)\mu = 0, \tag{6.164}$$

$$(\tilde{K} + 4)\dot\mu = 0, \tag{6.165}$$

$$(\tilde{K} + 2)(\bar{D}\dot\Omega) = 0, \tag{6.166}$$

$$(\tilde{K} + 2)\Delta = 0, \tag{6.167}$$

$$(\tilde{K} + 1)\ddot\Omega = 0. \tag{6.168}$$

(*) See note at end of this section.

The corresponding equations for the Killing vector K are

$$K(\Omega - u\dot{\Omega}) = 0, \tag{6.169}$$

$$K\dot{\Omega} = 0, \tag{6.170}$$

$$K\mu = 0, \tag{6.171}$$

$$K\dot{\mu} = 0, \tag{6.172}$$

$$K(\bar{D}\dot{\Omega}) = 0, \tag{6.173}$$

$$K\Delta = 0, \tag{6.174}$$

$$K\ddot{\Omega} = 0. \tag{6.175}$$

As usual, the dot denotes differentiation with respect to u.

Equations (6.162) and (6.163) give

$$(\tilde{K} + 1)\Omega = 0, \tag{6.176}$$

while (6.169) and (6.170) give $K\Omega = 0$. Using this result in (6.176), we get

$$(y\partial_y + 1)\Omega = 0, \qquad \Omega = \Omega(y,u). \tag{6.177}$$

Putting $\Omega = y^{-1}f(y,u)$ reduces (6.177) to $f_y = 0$, so that $f = f(u)$ and

$$\Omega = y^{-1}f(u). \tag{6.178}$$

Equations (6.164) and (6.171) give

$$(y\partial_y + 4)\mu = 0, \qquad \mu = \mu(y,u), \tag{6.179}$$

with solution

$$\mu = y^{-4}\, g(u). \tag{6.180}$$

The remaining equations (6.165) - (6.168) and (6.172) - (6.175) are all satisfied for this Ω and μ. The functions f(u) and g(u) must now be determined from the field equations.

For Ω and μ as in (6.178) and (6.180), the operator D becomes

$$D = -\frac{1}{2}\, i\partial_y - \Omega\partial_u$$

and field equation (4.62a) gives

$$\bar{f}\dot{g} + (3\dot{\bar{f}} + 2i)g = 0. \tag{6.181}$$

This cannot be integrated directly, but f and g can be expressed in terms of a new complex function h(u) as follows:

$$f = 2i/\dot{\bar{h}}, \qquad g = -\frac{1}{8}\, ie^{h}\, \dot{h}^{3}. \tag{6.182}$$

It can be shown, in fact, that the only occasion on which the field equation (6.181) <u>is</u> immediately integrable is when a = 0 i.e. for an isometry.

We next find

$$D\Omega = \frac{1}{2} y^{-2} E(u), \tag{6.183}$$

where

$$E(u) \equiv if - 2f\dot{f} = if + [(if)^2]^{\cdot}, \tag{6.184}$$

and

$$\partial_u D\Omega = \frac{1}{2} y^{-2} \dot{E}. \tag{6.185}$$

The field equation (4.62c) gives the ordinary differential equation

$$4\dot{g} - [3i(iE + \bar{f}\,\dot{E}) + 2\bar{f}(iE + \bar{f}\,\dot{E})^{\cdot}]^{\cdot} = |\dot{E}|^2, \tag{6.186}$$

while the remaining field equation (4.62b) gives

$$4(g-\bar{g}) = 3i(F + \bar{F}) + 2(\bar{f}\,\dot{F} - f\,\dot{\bar{F}}), \tag{6.187}$$

where

$$F \equiv iE + \bar{f}\,\dot{E}. \tag{6.188}$$

By substituting (6.182) in (6.184) and (6.186) we obtain a fifth order equation in the complex function h(u), subject to the constraint (6.187). The best I have been able to do with this situation is to spot some very special solutions which are rather uninteresting because they represent flat space. Noting that

$$\partial_u \partial_u D\Omega = \frac{1}{2} y^{-2} \ddot{E} \tag{6.189}$$

and

$$\bar{D}\partial_u D\Omega = -\frac{1}{2} y^{-3}(i\dot{E} + \bar{f}\,\ddot{E}), \tag{6.190}$$

we observe that, by (4.64d), the <u>space is flat</u> iff

$$g = 0 \text{ and } \ddot{E} = 0, \;\Rightarrow f(i - 2\dot{f}) = \text{constant}. \tag{6.191}$$

Hence

$$f = \text{constant}, \quad g = 0 \longleftrightarrow \text{flat space}.$$

[It can be shown that the field equations are <u>not</u> satisfied when

$$f = C\,u^n, \quad g = 0, \quad C \text{ complex constant.}]$$

<u>Note</u>: At the beginning of this case we chose a = -1. This is a special choice, to make the ensuing equations easier to handle. As we have seen, the mathematics is even then not in good shape. Keeping a as an arbitrary parameter seems to make things worse!

The results of this chapter are summarized in the following theorem:

Theorem 6.1 The non-flat algebraically special vacuum Einstein spaces with non-zero expansion and/or twist which admit one Killing vector and one HKV are given in the following table:

Case	Metric	Petrov ype	HKV	Hypersurface -orthogonal
*6.1(III)	(6.10)	III	(6.3)	Yes *
6.2(I)	(6.30)	II	(6.31)	No
6.2(II)(i)(A)	(6.43)	II	(6.44)	No
6.2(II)(i)(B)	(6.49)	II	(6.50)	No
6.2(II)(ii)(A)	?		(6.58)	No
6.2(II)(ii)(B2)	(6.103)+?	II	(6.87)	No
6.2(II)(ii)(B3)	(6.116)	II	(6.105)	No
6.2(III)	(6.132)	III	(6.134)	Yes
6.2(IV)(i)	(6.143)	II	(6.144)	No
6.2(IV)(ii)	(6.156)	III	(6.146)	Yes
6.3	?		(6.160)	?

* = degenerate case (admits 3 Killing vectors)

? = possible metrics but field equations unsolved.

CHAPTER 7 *Petrov Type N Spaces*

The type N vacuum spaces are treated separately because they are among the most interesting physically and mathematically. Only those type N spaces with the possibility of admitting homothetic motions are considered here. On account of the results of Chapter 3 the spaces can admit only one or two Killing vectors besides a proper HKV.

All algebraically special vacuum Einstein spaces with non-zero expansion and/or twist which admit 2 Killing vectors and one HKV are given in Chapter 5. From those results we have

Theorem 7.1 There are no Petrov type N vacuum spaces with non-zero expansion and/or twist which admit 2 Killing vectors and one HKV.

For spaces with only one Killing vector K we shall consider in turn the cases where K assumes one of the canonical forms

(i) $K = \partial_s = R(\zeta,\bar{\zeta})\partial_u$ (ii) $K = \partial_x = \partial_\zeta + \partial_{\bar{\zeta}}$,

and add a HKV of the appropriate kind.

For a type N space (4.64c) requires

$$\mu = 0 = \bar{D}\partial_u D\Omega\,, \tag{7.1}$$

while the field equations (4.62) reduce to

$$\bar{D}\bar{D}D\Omega = DD\bar{D}\bar{\Omega} \tag{7.2}$$

and

$$-\partial_u\bar{D}\bar{D}D\Omega = |\partial_u D\Omega|^2. \tag{7.3}$$

For K in form (i) we can use the $(\zeta,\bar{\zeta},s,r)$ coordinate system. Appendix 1 shows the allowed form for the HKV, but we do not need to concern ourselves with that here. For it is sufficient to note that, by (4.130),

$$\Omega = p_\zeta u + \Lambda e^{-p} = p_\zeta u + f(\zeta,\bar{\zeta})$$

so that $\partial_u\partial_u D\Omega = 0$. This result coupled with (7.1) guarantees that the space is flat, by (4.64d). Thus we arrive at

Theorem 7.2 There are no Petrov type N spaces which admit a HKV and a Killing vector of type $K = \partial_s = R(\zeta,\bar{\zeta})\partial_u$.

For K in form (ii), Appendix 1 shows that we may take the HKV in the form

$$\tilde{K} = \zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} + u\partial_u - v\partial_v + a(u\partial_u + v\partial_v).$$

By choosing a = -1 this simplifies to

$$\tilde{K} = x\partial_x + y\partial_y - 2v\partial_v, \qquad (7.4)$$

where $\zeta = x+iy$.

The homothetic Killing equations and their integrability conditions are now just (6.162) - (6.168), with their Killing equivalents (6.169) - (6.175).

Using (6.190), the type N condition (7.1) gives

$$i\dot{E} + \bar{f}\ddot{E} = 0, \qquad (7.5)$$

where E(u) is defined by

$$E = if - 2f\dot{f}, \qquad (7.6)$$

and

$$\Omega = y^{-1}f(u). \qquad (7.7)$$

Using (7.5) and $\mu = 0$, the field equation (7.3), which is just (6.186), becomes

$$\dot{E}(\bar{E} + i\bar{f} + 2\bar{f}\dot{\bar{f}})^{\cdot} = 0,$$

and by employing the definition (7.6) again, we see that this last equation is the zero identity. Thus there is only one field equation, namely (7.2), which can yield any information. It is just (6.187), which now reduces to

$$3E - \bar{f}\dot{E}(3i+2\dot{\bar{f}}) = 3\bar{E} - f\dot{\bar{E}}(-3i+2\dot{f}) \qquad (7.8)$$

or

$$3E - \dot{E}(2i\bar{f} - \bar{E}) = 3\bar{E} + \dot{\bar{E}}(2if + E).$$

This field equation together with the type N condition (7.5) must furnish all the information about f, and hence Ω. The only chance of success i.e. in finding non-flat solutions, is when, by (6.189), we have $\ddot{E} \neq 0$.

One possibility (not the general solution) is that the left side of (7.8) is a real constant. Then (7.5) and (7.6) reduce to a single ordinary differential equation in E, namely,

$$2\dot{E}^3\dddot{E} + \dot{E}^2\ddot{E}^2 - e(E+C)\ddot{E}^3 = 0, \qquad (7.9)$$

where C is a real constant. Putting z = E+C, this becomes

$$2\dot{z}^3\dddot{z} + \dot{z}^2\ddot{z}^2 - 3z\ddot{z}^3 = 0. \qquad (7.10)$$

I have found only two particular solutions of this equation:

Case 7(I): z = constant, $\Rightarrow E$ = real constant.

Then $\ddot{E} = 0$, so space is flat.

Case 7(II): $z = Ae^{ku}$, $\Rightarrow E = Ae^{ku} - C$, k and A real constant.

If $A \neq 0$, then $\ddot{E} \neq 0$ and we have a type N space.

Putting $w(u) = if$ in the definition (7.6), we must now solve

$$2w\dot{w} + w = Ae^{ku} - C. \tag{7.11}$$

According to Kamke [232], this equation is one of the class investigated by Lemke [233]. Solutions for $A \neq 0$ appear to be unknown in exact form. The differential equation (7.11) is also an Abel equation of the second kind (see Kamke [232], p.26), but this classification does not seem to help in finding a solution.

In summary, while I have no solution to offer as an example of a type N space which admits the Killing vector $K = \partial_x$ and the HKV (7.4), equation (7.11) ensures that there is a possibility that such a space exists. This possibility is strengthened by noting that equation (7.9) is only a particular solution of the field equation (7.8) and the type N condition (7.5).

In case the space admits just one HKV (and no Killing vectors) we shall consider each of the canonical forms

(i) $\tilde{K} = u\partial_u + v\partial_v$ (ii) $\tilde{K} = \partial_\zeta + \partial_{\bar{\zeta}} + u\partial_u + v\partial_v$.

The non-trivial equations (4.124) are

$$(\tilde{K} - 1)(\Omega - u\dot{\Omega}) = 0, \tag{7.12}$$

$$\tilde{K}\dot{\Omega} = 0, \tag{7.13}$$

$$\tilde{K}(\bar{D}\dot{\Omega}) = 0, \tag{7.14}$$

$$(\tilde{K} - 1)\Delta = 0, \tag{7.15}$$

$$(\tilde{K} + 1)\ddot{\Omega} = 0. \tag{7.16}$$

Case 7(III): $\tilde{K} = u\partial_u + v\partial_v$.

Equation (7.13) gives $\ddot{\Omega} = 0$, and then (7.16) is satisfied automatically. Equations (7.12) and (7.13) imply $\Omega = uf(\zeta,\bar{\zeta})$. Then $\partial_u\partial_u D\Omega = 0$, and this together with (7.1) ensures that the space is flat, by (4.64d). Hence we have

Theorem 7.3 There are no vacuum Einstein spaces of Petrov type N with non-zero expansion and/or twist which admit only a HKV of the form $\tilde{K} = u\partial_u + v\partial_v$.

Case 7(II): $z = Ae^{ku}$, $\Rightarrow E = Ae^{ku} - C$, k and A real constants.
For a type N space we require $\ddot{E} \neq 0$.
Substituting this expression for E into (7.5) we find that the condition $\ddot{E} \neq 0$ is not met. So the only possibility in this case is a flat space solution.

In summary, while I have no solution to offer as an example of a type N vacuum space which admits the Killing vector $K = \partial_x$ and the HKV (7.4), there is a possibility that such a space exists. For (7.9) is not the general solution of the field equation (7.8) and the type N condition (7.5).

Case 7(IV): $\tilde{K} = \partial_\zeta + \partial_{\bar{\zeta}} + u\partial_u + v\partial_v$.

The sole non-trivial field equation (7.2) and the type N condition (7.1) give a high order nonlinear partial differential equation which is very complicated. Other than some very special solutions representing flat spaces, it has proved too difficult to solve.

CONCLUSION

The first three chapters of this thesis provide an introduction to the structure of the conformal group, and a comprehensive survey of the use of the conformal group in mathematical physics; in particular, in relativity. These three chapters put together for the first time, as far as I am aware, the achievements of many people over the past 130 years. This review sets the stage for an application of a special case of conformal motions to relativity.

The application is specifically this: a systematic search is made for algebraically special expanding and/or twisting vacuum Einstein spaces which admit homothetic motions. To do this, a new extension of the formalism used by Kerr & Debney [199] is developed in Chapter 4.

The work of Chapters 5, 6 and 7 is all new. While many of the 23 metrics given in Theorems 5.1 and 6.1 have been discovered by others and have been known to admit a HKV, several are believed to be new, particularly the Case VIII and IX metrics of Chapter 5 and those Petrov type II metrics of Chapter 6. The list of metrics is incomplete in the sense that in two cases the field equations could not be solved completely. This is probably a characteristic of the coordinates used in the analysis. Another choice of coordinate system may have produced equations which were more readily soluble, or given metric forms which may have been simpler in some cases than those given here. However, the value of the system used throughout this work is that it <u>is</u> effective, and that it enables direct comparison with and the use of results obtained by Kerr & Debney.

Although not enunciated as a theorem, the result given in Chapter 5, that the NUT metric does not admit a HKV, is not without interest.

The work of Chapter 7 reduces the possibility of the existence of Petrov type N vacuum spaces which admit a HKV to a narrow class which also admit one Killing vector of a special type. There is also a possibility that type N spaces admit a HKV (of a special type) only.

Some problems for future research are indicated in Chapter 3. The work which is currently being done on conformal Killing tensors [234] should throw further light on the area of relativity studied in this thesis.

APPENDICES

Appendix 1

Given a Killing vector K in canonical form, the form of a HKV $\tilde{K}$ will be determined by a special case of Theorem 3.2, namely,

$$[K,\tilde{K}] = \lambda K, \quad \lambda \text{ a constant.} \tag{A1.1}$$

This is obtained from Theorem 3.2 by replacing $\tilde{K}_1$ in equation (3.7) by K and dropping the subscript on $\tilde{K}_2$.

Consider each of the canonical forms of K in turn:

(1) $$K = R(\zeta,\bar{\zeta})\partial_u = e^{-P}\partial_u \equiv \partial_s. \tag{A1.2}$$

Using the $(\zeta,\bar{\zeta},s,r)$ coordinate system, we wish to determine the functions $\alpha(\zeta)$ and $T(\zeta,\bar{\zeta})$ in

$$\tilde{K} = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + a_0(s\partial_s - r\partial_r) + T\partial_s + a(s\partial_s + r\partial_r) \tag{A1.3}$$

when the geometrical constraint (A1.1) is applied. This constraint is explicitly

$$(a_0 + a)\partial_s = \lambda\partial_s, \tag{A1.4}$$

which is valid for all α and T with $\lambda = a_0 + a$.

Under a coordinate transformation (4.142) with $\phi = 0$ which takes the Killing vector K into $K' = bK$, where b is a real constant, the functions α and T transform as

$$\alpha' = \Phi_\zeta\alpha, \tag{A1.5}$$

$$T' = C_0[T - a_0A + KA], \quad A = A(\zeta,\bar{\zeta}) = \bar{A}, \tag{A1.6}$$

where in this case

$$\alpha' = 0 = \alpha, \quad T' = b, \quad T = 1, \quad a_0 = 0.$$

Thus the coordinate transformations preserving the canonical form (A1.2) are (4.142) with $\phi = 0$ and arbitrary $\Phi(\zeta)$, $A(\zeta,\bar{\zeta})$ and C_0. Under these transformations we can always put the HKV into one or other of the two canonical forms

$$\tilde{K} = \partial_\zeta + \partial_{\bar{\zeta}} + a_0(s\partial_s - r\partial_r) + a(s\partial_s + r\partial_r), \tag{A1.7}$$

$$\tilde{K} = a_0(s\partial_s - r\partial_r) + a(s\partial_s + r\partial_r) \tag{A1.8}$$

by means of (4.146) and (4.147) with $\phi = 0$. Any subsequent transformation must preserve the canonical form of both K and $\tilde{K}$.

For $\tilde{K}$ as in (A1.7), $\tilde{K} \to b\tilde{K}$ if $\zeta' = b\zeta + c$, where b and c are real, complex constants respectively. So the coordinate freedom left in the presence of $K = \partial_s$ and $\tilde{K}$ as in (A1.7) is

$$\begin{aligned} \zeta' &= b\zeta + c, \\ s' &= C_0(s + A), \\ r' &= C_0^{-1} r, \end{aligned} \tag{A1.9}$$

where C_0 is an arbitrary real constant and A is a real function of the form

$$A = e^{(a_0+a)x} . f(y), \tag{A1.10}$$

where $\zeta = x+iy$ and f is an arbitrary real function.

For $\tilde{K}$ as in (A1.8), the coordinate freedom left in the presence of $K = \partial_s$ and $\tilde{K}$ is

$$\begin{aligned} \zeta' &= \Phi(\zeta), \\ s' &= C_0(s + A), \quad A = A(\zeta,\bar{\zeta}) = \bar{A}, \\ r' &= C_0^{-1} r, \end{aligned} \tag{A1.11}$$

where Φ is arbitrary and

(i) A is arbitrary when $a_0+a = 0$,

(ii) $A = 0$ when $a_0+a \neq 0$.

Of course, we may not wish to make T zero in (A1.3) i.e. the canonical form of $\tilde{K}$ is not desired, so that we have the full generality of $A(\zeta,\bar{\zeta})$ to play with in the transformation equations (4.142).

(2) $K = \partial_\zeta + \partial_{\bar{\zeta}} = \partial_x, \quad \zeta = x+iy.$ (A1.12)

In the $(\zeta,\bar{\zeta},u,v)$ coordinate system the HKV takes the general form

$$\tilde{K} = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + \mathrm{Re}(\alpha_\zeta)(u\partial_u - v\partial_v) + R\partial_u + a(u\partial_u - v\partial_v). \tag{A1.13}$$

The constraint (A1.1) is explicitly

$$\alpha_\zeta\partial_\zeta + \bar{\alpha}_{\bar{\zeta}}\partial_{\bar{\zeta}} + \frac{1}{2}(\alpha_{\zeta\zeta} + \bar{\alpha}_{\bar{\zeta}\bar{\zeta}})(u\partial_u - v\partial_v) + R_x\partial_u = \lambda\partial_x, \tag{A1.14}$$

which holds iff

$$R_x = 0, \Rightarrow R = R(y), \tag{A1.15}$$

and

$$\alpha_\zeta = \bar{\alpha}_{\bar{\zeta}} = \lambda, \ \alpha_{\zeta\zeta} = 0, \Rightarrow \alpha = \lambda\zeta + e. \tag{A1.16}$$

where λ, e are real, complex constants respectively.

Under a coordinate transformation (4.96) with $\phi = 0$ the Killing vector $K = \partial_x$ goes to $K' = b\partial_x$ provided the transformation equations

$$\alpha' = \Phi_\zeta \alpha, \qquad (A1.17)$$

$$R' = |\Phi_\zeta|[R - \mathrm{Re}(\alpha_\zeta)S + KS], \qquad (A1.18)$$

where $S = S(\zeta,\bar{\zeta}) = \bar{S}$, are satisfied for

$$\alpha' = b, \quad \alpha = 1, \quad R' = 0 = R.$$

This requires

$$\Phi = \zeta' = b\zeta + c, \qquad (A1.19)$$

$$S = S(y), \qquad (A1.20)$$

where b, c are real, complex constants respectively.

The form of the HKV is already restricted to (A1.13) with α and R as in (A1.15) and (A1.16). Any further coordinate transformation (4.96) made to simplify this form of the HKV is restricted by (A1.19) and (A1.20). By means of a transformation

$$\zeta' = \zeta + (e/\lambda),$$
$$u' = u + S(y), \qquad (A1.21)$$
$$v' = v,$$

the transformation equations (4.105) and (4.106) with $\phi = 0$ for α and R give

$$\alpha' = \lambda\zeta',$$
$$R' = R(y) - (\lambda+a)S(y) + (\lambda y + \mathrm{Im}\, e)\dot{S}(y)$$
$$= R(y) - (\lambda+a)S(y) + \lambda y'\dot{S}(y),$$

where the dot denotes differentiation with respect to y. We can, if we wish, choose $R' = 0$ for we are always guaranteed a solution S(y) of

$$(\lambda y + \mathrm{Im}\, e)\dot{S} - (\lambda+a)S + R = 0 \qquad (A1.22)$$

when e and R(y) are known. Then, choosing $\lambda = 1$, we can put the HKV in the form

$$\tilde{K} = \zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} + u\partial_u - v\partial_v + a(u\partial_u + v\partial_v) \qquad (A1.23)$$

simultaneously with the canonical form (A1.12) of the Killing vector K. However, the choice of the HKV in the form (A1.23) severely restricts the allowable form of S(y) in the coordinate transformations (4.96). In fact, the coordinate freedom left in the presence of $K = \partial_x$ and $\tilde{K}$ as in (A1.23) is

$$\zeta' = b\zeta + c,$$
$$u' = b(u + y^{1+a}), \qquad (A1.24)$$
$$v' = b^{-1}v,$$

where b, c are real, complex constants respectively. It may be preferable to leave the HKV in the more general form

$$\tilde{K} = \zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} + u\partial_u - v\partial_v + R(y)\partial_u + a(u\partial_u + v\partial_v) \qquad \text{(A1.25)}$$

with complete freedom of choice of S(y) in the coordinate transformations.

Appendix 2

When there are two Killing vectors K_1, K_2 and one HKV $\tilde{K}$ in the space we must use the special form of Theorem 3.2, as we did in Appendix 1, to determine the form of the HKV associated with each of the given Killing vectors. Specifically, we must consider the following nine cases:

(A) $[K_1,\tilde{K}] = \lambda_1 K_1$, $\quad [K_2,\tilde{K}] = \lambda_2 K_2$, $\quad \lambda_1,\lambda_2 \neq 0$.

(B) $[K_1,\tilde{K}] = \lambda_2 K_2$, $\quad [K_2,\tilde{K}] = \lambda_1 K_1$, $\quad \lambda_1,\lambda_2 \neq 0$.

(C) $[K_1,\tilde{K}] = \lambda_1 K_1 = [K_2,\tilde{K}]$, $\quad \lambda_1 \neq 0$.

(D) $[K_1,\tilde{K}] = \lambda_2 K_2 = [K_2,\tilde{K}]$, $\quad \lambda_2 \neq 0$.

(E) $[K_1,\tilde{K}] = \lambda_1 K_1$, $\quad [K_2,\tilde{K}] = 0$, $\quad \lambda_1 \neq 0$.

(F) $[K_1,\tilde{K}] = 0$, $\quad [K_2,\tilde{K}] = \lambda_2 K_2$, $\quad \lambda_2 \neq 0$.

(G) $[K_1,\tilde{K}] = \lambda_2 K_2$, $\quad [K_2,\tilde{K}] = 0$, $\quad \lambda_2 \neq 0$.

(H) $[K_1,\tilde{K}] = 0$, $\quad [K_2,\tilde{K}] = \lambda_1 K_1$, $\quad \lambda_1 \neq 0$.

$[K_1,\tilde{K}] = 0 = [K_2,\tilde{K}]$.

Here we shall be concerned with K_1 and K_2 as in Cases VIII and IX of Chapter 5. Not all of the cases (A) - (I) will be worked in detail, since the same technique applies to all of them.

<u>Case VIII (Chapter 5)</u>: $K_1 = \partial_\zeta + \partial_{\bar{\zeta}} = \partial_x$, $\quad \zeta = x+iy$, (A2.1)

$$K_2 = i(\partial_\zeta - \partial_{\bar{\zeta}}) = \partial_y, \qquad \text{(A2.2)}$$

$$\tilde{K} = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + \mathrm{Re}(\alpha_\zeta)(u\partial_u - v\partial_v) + R\partial_u + a(u\partial_u + v\partial_v). \qquad \text{(A2.3)}$$

(A) From Appendix 1 we know that $[K_1,\tilde{K}] = \lambda_1 K_1$ allows the form (A1.23) for $\tilde{K}$, with coordinate freedom (A1.24). Under such a transformation (A1.24) we find the form of K_2 preserved, and $[K_2,\tilde{K}] = K_2$. So by choosing $\lambda_1 = 1$ in this case we have the result that in the presence of K_1 and K_2 the HKV $\widetilde{K}$ may take the form (A1.23), with coordinate freedom (A1.24) remaining, with $y^{1+a} \to k$, constant

(B) $[K_1,\widetilde{K}] = \lambda_2 K_2$ iff

$$R_x = 0, \quad \Rightarrow \quad R = R(y), \tag{A2.4}$$

$$\alpha_\zeta = \lambda_2 i, \ \alpha_{\zeta\zeta} = 0, \quad \Rightarrow \quad \alpha = i(\lambda_2\zeta + e), \tag{A2.5}$$

where e is a complex constant. Then

$$[K_2,\widetilde{K}] = -\lambda_2(\partial_\zeta + \partial_{\bar\zeta}) + \dot{R}\partial_u \qquad (\dot{} \equiv d/dy)$$

$$= \lambda_1 K_1 \quad \text{iff}$$

$\lambda_2 = -\lambda_1$ and $\dot{R} = 0$, $\Rightarrow$ $R = R_0$ (real constant).

Then $\widetilde{K}$ must be of the form

$$\widetilde{K} = i(\lambda_2\zeta + e)\partial_\zeta - i(\lambda_2\bar\zeta + \bar e)\partial_{\bar\zeta} + R_0\partial_u + a(u\partial_u + v\partial_v).$$

Now K_1, K_2 and this $\widetilde{K}$ are all preserved under the coordinate transformation

$$\begin{aligned} \zeta' &= b\zeta + c, \\ u' &= b(u + S), \quad S = S(y) = \bar S, \\ v' &= b^{-1}v. \end{aligned} \tag{A2.6}$$

By taking $b = \lambda_2 = 1$, $c = e$ in (A2.6) and employing (4.105) and (4.106) we can transform R_0 to zero, thereby reducing $\widetilde{K}$ to the form

$$\begin{aligned} \widetilde{K} &= i(\zeta\partial_\zeta - \bar\zeta\partial_{\bar\zeta}) + a(u\partial_u + v\partial_v) \\ &= x\partial_y - y\partial_x + a(u\partial_u + v\partial_v). \end{aligned} \tag{A2.7}$$

This coordinate freedom left is (A2.6) with $S = 0$, as can be seen by applying (4.106).

(C) From (A) above (see also Appendix 1, equations (A1.15) and (A1.16)), we have $[K_1,\widetilde{K}] = \lambda_1 K_1$ iff

$$\alpha = \lambda_1\zeta + e, \quad R = R(y).$$

Then $[K_2,\widetilde{K}] = i\lambda_1(\partial_\zeta - \partial_{\bar\zeta}) + \dot{R}\partial_u \qquad (\dot{} \equiv d/dy)$

$$\neq \lambda_1 K_1 \quad \text{when } \lambda_1 \neq 0.$$

Hence this case is impossible.

Proceeding in the same way, we find

(D), (E), (F), (G) and (H) are all impossible, and

(I) $[K_1,\tilde{K}] = 0 = [K_2,\tilde{K}]$ allows

$$\tilde{K} = \alpha_0\partial_\zeta + \bar{\alpha}_0\partial_{\bar{\zeta}} + a(u\partial_u + v\partial_v), \qquad \text{(A2.8)}$$

where α_0 is a complex constant. The coordinate freedom left is (A2.6) and $S(y) = Ne^{ky}$, where N and k are real constants, $k = a/\mathrm{Im}(\alpha_0)$.

Case IX (Chapter 5): $K_1 = \partial_\zeta + \partial_{\bar{\zeta}} = \partial_x$, (A2.9)

$$K_2 = \zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} + u\partial_u - v\partial_v, \qquad \text{(A2.10)}$$

$\tilde{K}$ as before in (A2.3).

(A) For $[K_1,\tilde{K}] = \lambda_1 K_1$ Appendix 1 allows $\tilde{K}$ to assume the form (A1.23), which may now be written

$$\tilde{K} = K_2 + a(u\partial_u + v\partial_v).$$

But then $[K_2,\tilde{K}] = 0$, contrary to the hypothesis $\lambda_2 \neq 0$. So this case is impossible.

It is also found that (B), (D), (F) and (G) are impossible.

(C) Using Appendix 1, where equations (A1.15) and (A1.16) hold for $[K_1,\tilde{K}] = \lambda_1 K_1$, we obtain

$$[K_2,\tilde{K}] = -e(\partial_\zeta + \partial_{\bar{\zeta}}) + (y\dot{R} - R)\partial_u \qquad (\dot{} \equiv d/dy)$$

$$= \lambda_1 K_1 \quad \text{iff}$$

$$e = -\lambda_1 \neq 0 \quad \text{and} \quad R(y) = ky, \quad k \text{ real constant.}$$

Choosing $\lambda_1 = 1$, $\tilde{K}$ may now be written as

$$\tilde{K} = \zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} + u\partial_u - v\partial_v - (\partial_\zeta + \partial_{\bar{\zeta}}) + R(y)\partial_u + a(u\partial_u + v\partial_v).$$

By means of the coordinate transformation $u' = u + S(y)$ we can now use (4.106) to eliminate $R(y)\partial_u$. Thus we may take

$$\tilde{K} = K_2 - K_1 + a(u\partial_u + v\partial_v). \qquad \text{(A2.11)}$$

The coordinate freedom left is (A1.24).

(E) It is easily seen that $\tilde{K}$ as in (A1.23) of Appendix 1, with coordinate freedom (A1.24), satisfies the constraints.
Thus
$$\tilde{K} = K_2 + a(u\partial_u + v\partial_v). \tag{A2.12}$$

(H) $[K_2,\tilde{K}] = (\zeta\alpha_\zeta - \alpha)\partial_\zeta + (\bar{\zeta}\bar{\alpha}_{\bar{\zeta}} - \bar{\alpha})\partial_{\bar{\zeta}} + \frac{1}{2}(\zeta\alpha_{\zeta\zeta} + \bar{\zeta}\bar{\alpha}_{\bar{\zeta}\bar{\zeta}})(u\partial_u - v\partial_v)$
$\qquad + \{(\zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}})R\}\partial_u - R\partial_u$

$= \lambda_1 K_1$ iff

$$(\zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}} - 1)R = 0, \quad \Rightarrow \quad R = xf(y/x), \tag{A2.13}$$
$$\zeta\alpha_\zeta - \alpha = \lambda_1, \quad \tfrac{1}{2}(\zeta\alpha_{\zeta\zeta} + \bar{\zeta}\bar{\alpha}_{\bar{\zeta}\bar{\zeta}}) = 0, \quad \Rightarrow \alpha = B\zeta - \lambda_1, \tag{A2.14}$$

where B is a complex constant. Then
$$[K_1,\tilde{K}] = B\partial_\zeta + \bar{B}\partial_{\bar{\zeta}} + R_x\partial_u$$
$$= 0 \quad \text{iff}$$
$R_x = 0$, $\Rightarrow R = ky$, and also $B = 0$, where k is a real constant.
Choosing $\lambda_1 = -1$ and using a coordinate transformation $u' = S(y)+u$ with (4.106) to send R to zero, we can finally take
$$\tilde{K} = \partial_\zeta + \partial_{\bar{\zeta}} + a(u\partial_u + v\partial_v) = K_1 + a(u\partial_u + v\partial_v).$$
The coordinate freedom left is (A2.6) with $S = 0$. (A2.15)

In similar manner we find
(I) $\tilde{K} = u\partial_u + v\partial_v$, (A2.16)

with the same coordinate freedom as in (H) above.

Appendix 3

$$p_{\zeta\bar{\zeta}} = 0 \quad \text{implies} \quad p = \mathrm{Re}\{F(\zeta)\}, \tag{A3.1}$$

where F is an analytic function of ζ. Under a coordinate transformation (4.142) with $\phi = 0$, the function $p(\zeta,\bar{\zeta})$ transforms according to (4.137). Putting $F(\zeta) = \log f(\zeta)$ this gives
$$e^{p'} = c_0|\Phi_\zeta|^{-1}\, e^p = c_0|f\Phi_\zeta^{-1}|. \tag{A3.2}$$
Choose $\Phi_\zeta = c_0 f(\zeta)$. Then $p' = 0$.

From (A3.2) we see that $p = 0$ is preserved under further coordinate transformations if $C_0|\Phi_\zeta|^{-1} = 1$, which implies $\Phi(\zeta) = e^{iA_0}(C_0\zeta + b)$. Thus, having made p zero, the coordinate freedom left is

$$\begin{aligned} \zeta' &= e^{iA_0}(C_0\zeta + b), \\ s' &= C_0(s + A), \\ r' &= C_0^{-1} r, \end{aligned} \qquad \text{(A3.3)}$$

where C_0, A_0 are real constants, b is a complex constant, and $A(\zeta,\bar{\zeta})$ is an arbitrary real function.

In case 6.1(I) of Chapter 6 the freedom on the s coordinate is restricted by $A = 0$, but in Case 6.2(I) the full freedom on s is available.

Appendix 4

In solving the system of equations (4.150) - (4.155) the freedom available on the s coordinate may be used (a) to reduce the task of finding Λ to that of obtaining a particular solution, or (b) to transform the function $T(\zeta,\bar{\zeta})$ in the expression for $\widetilde{K}$ to zero.

(a) Putting $\Lambda_{\bar{\zeta}} + \bar{\Lambda}_\zeta = 2F_{\zeta\bar{\zeta}}$, where $F(\zeta,\bar{\zeta})$ is a real function, enables the definition of the function $d(\zeta,\bar{\zeta})$, namely,

$$\Lambda_{\bar{\zeta}} - \bar{\Lambda}_\zeta = 2ide^{2p} \qquad \text{(A4.1)}$$

to be written as

$$\Lambda_{\bar{\zeta}} = ide^{2p} + F_{\zeta\bar{\zeta}} \,. \qquad \text{(A4.2)}$$

Integrating,

$$\Lambda = \int ide^{2p}d\bar{\zeta} + F_\zeta + h(\zeta), \qquad \text{(A4.3)}$$

where $h(\zeta)$ is an arbitrary function of ζ. Now, under an s-transformation (4.142), we have by (4.138) that $\Lambda \to \Lambda'$, where

$$\Lambda' = \Lambda - A_\zeta .$$

By the Cauchy-Kowaleski theorem [235] the equation

$$F_\zeta + h(\zeta) - A_\zeta = 0 \qquad \text{(A4.5)}$$

can be solved for the real function $A(\zeta,\bar{\zeta})$. Thus we have

$$\Lambda' = \int ide^{2p}d\bar{\zeta}, \qquad \text{(A4.6)}$$

which is a particular integral of (A4.3).

(b) On the other hand, if we choose to transform T to zero by means of (4.147), we need to preserve this value of T thereafter and this places the restriction

$$\tilde{K}A = (a_0 + a)A \tag{A4.7}$$

on the real function $A(\zeta,\bar{\zeta})$ in (4.142), thereby restricting the coordinate freedom on s.

It is clear that one cannot simultaneously perform (a) and (b).

Appendix 5

2-surfaces of constant curvature.

Consider the 2-metric

$$d\sigma^2 = e^{2p} d\zeta d\bar{\zeta}, \quad p = p(\zeta,\bar{\zeta}). \tag{A5.1}$$

The 2-curvature is

$$R^{(2)} = e^{-2p} p_{\zeta\bar{\zeta}} . \tag{A5.2}$$

Suppose $R^{(2)} = R_0$ = constant. Then (A5.2) gives

$$\partial_{\bar{\zeta}}(p_{\zeta\zeta} - p_\zeta^2) = 0$$

so that

$$p_{\zeta\zeta} - p_\zeta^2 = F(\zeta). \tag{A5.3}$$

Any coordinate transformation of the allowed form $\zeta' = \Phi(\zeta)$ preserves the form (A5.1) of the 2-metric provided that, by (4.137),

$$p' = p + \log\{C_0 |\Phi_\zeta|^{-1}\}.$$

Hence $F(\zeta)$ in (A5.3) transforms as

$$F'(\zeta') = \Phi_\zeta^{-2}[F(\zeta) - \tfrac{1}{2}(\log \Phi_\zeta)_{\zeta\zeta} + \tfrac{1}{4}\{(\log \Phi_\zeta)_\zeta\}^2]. \tag{A5.4}$$

Given $p(\zeta,\bar{\zeta})$ i.e. given $F(\zeta)$, the Cauchy-Kowaleski theorem guarantees that we can find a solution $\Phi(\zeta)$ which makes the right hand side of (A5.4) vanish. Thus we can always transform $F(\zeta)$ to zero. Any subsequent coordinate transformation $\zeta' = \Phi(\zeta)$ has to satisfy, by (A5.4),

$$2(\log \Phi_\zeta)_{\zeta\zeta} = \{(\log \Phi_\zeta)_\zeta\}^2,$$

which is just the vanishing of the Schwarzian derivative of Φ, that is,

$$\{\Phi,\zeta\} \equiv \frac{\Phi_{\zeta\zeta\zeta}}{\Phi_\zeta} - \frac{3}{2}\left(\frac{\Phi_{\zeta\zeta}}{\Phi_\zeta}\right)^2 = 0. \qquad (A5.5)$$

This equation has the solution

$$\Phi(\zeta) = \frac{c_1\zeta + c_2}{c_3\zeta + c_4}, \quad c_1c_4 - c_2c_3 \neq 0, \qquad (A5.6)$$

where the c_i are complex constants. Hence the coordinate transformations which preserve $F(\zeta) = 0$ form the group of bilinear transformations (A5.6).

Without loss of generality, then, we may take

$$(e^{-p})_{\zeta\zeta} = -e^{-p}(p_{\zeta\zeta} - p_\zeta^2) = 0$$

which implies, since p is real,

$$e^{-p} = a\zeta\bar{\zeta} + B\zeta + \bar{B}\bar{\zeta} + c, \qquad (A5.7)$$

where a, c, B are constants, B complex. Substituting this in (A5.2) gives

$$R_0 = B\bar{B} - ac. \qquad (A5.8)$$

Now apply a bilinear transformation (A5.6) and use (4.137) to transform (A5.7) to the form

$$e^{-p'} = \zeta'\bar{\zeta}' - R_0,$$

where R_0 is the same constant as in (A5.8). [One such transformation is $\zeta' = C_0 a\zeta + \bar{B}$, with C_0 the constant of (4.137).]

Since the primed coordinate system is used henceforth, we shall drop the prime and write

$$e^{-p} = \zeta\bar{\zeta} - R_0. \qquad (A5.9)$$

The coordinate freedom left is a bilinear transformation on ζ and complete freedom on s and r in (4.142). If $R_0 = 1$ the 2-surface is a pseudosphere, and if $R_0 = -1$ it is a sphere.

Appendix 6

Consider a bilinear transformation

$$\zeta = \frac{a\zeta' + b}{c\zeta' + d}\,, \quad ad - bc \neq 0. \tag{A6.1}$$

We choose to normalize this so that $ad - bc = 1$. Then

$$\Phi_\zeta = (c\zeta' + d)^2. \tag{A6.2}$$

Using (4.137) with p given by (A5.9), we find that this form for p is preserved iff

$$\begin{aligned}\zeta'\bar{\zeta}' - R_0 = C_0^{-1}\{&(a\bar{a} - R_0c\bar{c})\zeta'\bar{\zeta}' + (a\bar{b} - R_0c\bar{d})\zeta' \\ &+ (\bar{a}b - R_0\bar{c}d)\bar{\zeta}' + b\bar{b} - R_0d\bar{d}\}\end{aligned} \tag{A6.3}$$

$\Leftrightarrow$

$$\left.\begin{aligned}&C_0 = 1, \quad a\bar{a} - R_0c\bar{c} = 1, \quad b\bar{b} - R_0d\bar{d} = -R_0, \\ &a\bar{b} - R_0c\bar{d} = 0, \quad ad - bc = 1.\end{aligned}\right. \tag{A6.4}$$

Using matrices we can represent the above as follows:

Let $\zeta = \zeta_1/\zeta_2$, $\quad S = \begin{pmatrix}\zeta_1 \\ \zeta_2\end{pmatrix}$, $\quad S' = \begin{pmatrix}\zeta'_1 \\ \zeta'_2\end{pmatrix}$,

$$A = \begin{pmatrix} a & b \\ c & d\end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & -R_0\end{pmatrix}.$$

Then (A6.1) is $\quad S = AS'$, $\hfill$ (A6.1*)

while (A6.2) may be written as

$$\Phi_\zeta = (\zeta_2/\zeta'_2)^2. \tag{A6.2*}$$

(A6.3) becomes $\quad e^{-P'} = C_0^{-1}|\zeta'_2|^{-2}\,\bar{S}'^T(\bar{A}^TPA)S'$, $\hfill$ (A6.3*)

where $\det A = 1$ and

$$e^{-P} = |\zeta_2|^{-2}\,\bar{S}^TPS.$$

The requirement $\det A = 1$ ensures that $C_0 = 1$.

For $C_0 = 1$ and $P' = \bar{A}^TPA$, we have therefore a matrix A which keeps the matrix P in diagonal form.

We want now to see whether we can transform the quadratic

$$\alpha = \alpha_0\zeta^2 + ib_0\zeta - \bar{\alpha}_0R_0, \tag{A6.5}$$

where b_0, α_0 are real, complex constants respectively, to a canonical form using relations (A6.4). This will depend on the value of R_0 and so we have two cases to consider

(i) $R_0 = -1$ (sphere) (ii) $R_0 = 1$ (pseudosphere).

In matrix terms this amounts to expressing

$$Q = \begin{pmatrix} \alpha_0 & \frac{1}{2}\, ib_0 \\ \frac{1}{2}\, ib_0 & -\bar{\alpha}_0 R_0 \end{pmatrix}$$

in canonical form under the transformation matrix A, since we can write

$$\begin{aligned} c &= \zeta_2^{-2}\, S^T Q S, \\ &= \zeta_2^{-2}\, S'^T Q' S' \end{aligned} \tag{A6.5*}$$

if Q' is defined by

$$Q' = A^T Q A. \tag{A6.6}$$

So we wish to find, using (A6.4), a matrix A such that Q' is in canonical form.

Case (i): $R_0 = -1$ (sphere).

Equations (A6.4) are satisfied if we choose A to be

$$A = \begin{pmatrix} e^{ip}\cos\theta & -e^{iq}\sin\theta \\ e^{-iq}\sin\theta & e^{-ip}\cos\theta \end{pmatrix}. \tag{A6.7}$$

Then (A6.6) gives

$$\begin{aligned} \alpha_0' = \alpha_0 e^{2ip}\cos^2\theta + \bar{\alpha}_0\, e^{i(p-q)}\sin\theta\cos\theta \\ + \frac{1}{2}\, ib_0[e^{2ip}\cos^2\theta + e^{i(p-q)}\sin\theta\cos\theta] \end{aligned} \tag{A6.8}$$

and

$$\frac{1}{2}\, ib_0' = [\bar{\alpha}_0\, e^{-i(p+q)} - \alpha_0\, e^{i(p+q)}]\sin\theta\cos\theta + \frac{1}{2}\, ib_0(\cos^2\theta - \sin^2\theta). \tag{A6.9}$$

$\alpha_0 = 0,\ b_0 \neq 0.$ Choosing $\theta = \frac{\pi}{4}$, $p = -q = -\frac{\pi}{4} + \frac{i}{2}\log\left(\frac{1}{2}\, b_0\right)$ reduces (A6.8) and (A6.9) to $\alpha_0' = 1$, $b_0' = 0$ so that

$$Q' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \frac{1}{2}\begin{pmatrix} (1-i)(\frac{1}{2}b_0)^{-\frac{1}{2}} & -(1+i)(\frac{1}{2}b_0)^{\frac{1}{2}} \\ (1-i)(\frac{1}{2}b_0)^{-\frac{1}{2}} & (1+i)(\frac{1}{2}b_0)^{\frac{1}{2}} \end{pmatrix}. \tag{A6.10}$$

Then $$\alpha' = \zeta'^2 + 1. \tag{A6.11}$$

In the original coordinates $\alpha = ib_0\zeta$. The transformation which takes α into α' is, from (A6.1) and (A6.10),

$$\zeta = \frac{\zeta' - \frac{1}{4}(1+i)^2 b_0}{\zeta' + \frac{1}{4}(1+i)^2 b_0}. \tag{A6.12}$$

<u>$\alpha_0 \neq 0,\ b_0 = 0$.</u> Choosing $\theta = 0$, $p = \frac{1}{2}\, i \log \alpha_0$, q arbitrary, equations (A6.8) and (A6.9) reduce to $\alpha'_0 = 1$, $b'_0 = 0$ so that

$$Q' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} \alpha_0^{-\frac{1}{2}} & 0 \\ 0 & \alpha_0^{\frac{1}{2}} \end{pmatrix}. \tag{A6.13}$$

Then $$\alpha' = \zeta'^2 + 1. \tag{A6.14}$$

In the original coordinates $\alpha = \alpha_0\zeta^2 + \bar{\alpha}_0$. The transformation which takes α into α' is, from (A6.1) and (A6.13),

$$\zeta' = \alpha_0\zeta. \tag{A6.15}$$

<u>$\alpha_0 \neq 0,\ b_0 \neq 0$.</u> Choosing $\theta = \frac{\pi}{4}$, $3p = q = \frac{3}{8}\, i \log(\alpha_0/\bar{\alpha}_0)$. Then (A6.8) and (A6.9) reduce to

$$\alpha'_0 = \frac{1}{2}\,\alpha_0(\alpha_0/\bar{\alpha}_0)^{-\frac{1}{4}} + \frac{1}{2}\,\bar{\alpha}_0(\alpha_0/\bar{\alpha}_0)^{\frac{1}{4}} + \frac{1}{4}\, ib_0[(\alpha_0/\bar{\alpha}_0)^{-\frac{1}{4}} + (\alpha_0/\bar{\alpha}_0)^{\frac{1}{4}}],$$

$$b'_0 = 0,$$

so that

$$Q' = \begin{pmatrix} \alpha'_0 & 0 \\ 0 & \bar{\alpha}'_0 \end{pmatrix}, \qquad A = \frac{1}{\sqrt{2}}\begin{pmatrix} (\alpha_0/\bar{\alpha}_0)^{-1/8} & -(\alpha_0/\bar{\alpha}_0)^{-3/8} \\ (\alpha_0/\bar{\alpha}_0)^{3/8} & (\alpha_0/\bar{\alpha}_0)^{1/8} \end{pmatrix}. \tag{A6.16}$$

Then $$\alpha' = \alpha'_0\zeta'^2 + \bar{\alpha}'_0 . \tag{A6.17}$$

In the original coordinates $\alpha = \alpha_0\zeta^2 + ib_0\zeta + \bar{\alpha}_0$. The transformation which takes α into α' is, from (A6.1) and (A6.16),

$$\zeta = \frac{\zeta' - r^{-2}}{r^4\zeta' + r^2}, \tag{A6.18}$$

where $r = (\alpha_0/\bar{\alpha}_0)^{1/8}$.

Thus, in the case of the sphere, it is always possible to express Q in canonical form, as in (A6.10), (A6.13) and (A6.16).

Case (ii): $R_0 = 1$ (pseudosphere).

Equations (A6.4) are satisfied if we choose A to be

$$A = \begin{pmatrix} e^{ip}\cosh\theta & e^{iq}\sinh\theta \\ e^{-iq}\sinh\theta & e^{-ip}\cosh\theta \end{pmatrix}. \tag{A6.19}$$

Then (A6.6) is, in full,

$$\alpha_0' = \alpha_0 e^{2ip}\cosh^2\theta - \bar{\alpha}_0 e^{-2iq}\sinh^2\theta + \frac{1}{2}\, ib_0 e^{i(p-q)}\sinh 2\theta \tag{A6.20}$$

and

$$ib_0' = [\alpha_0 e^{i(p+q)} - \bar{\alpha}_0 e^{-i(p+q)}]\sinh 2\theta + ib_0\cosh 2\theta. \tag{A6.21}$$

$\alpha_0 = 0,\ b_0 \neq 0.$ Choosing $\theta = 0$ reduces (A6.20) and (A6.21) to $\alpha_0' = 0,\ b_0' = b_0$ ($b_0' \neq 0$ since $\cosh 2\theta \geq 1$), so that

$$Q' = \begin{pmatrix} 0 & \frac{1}{2}\, ib_0 \\ \frac{1}{2}\, ib_0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} e^{ip} & 0 \\ 0 & e^{-ip} \end{pmatrix}, \quad p \text{ arbitrary.} \tag{A6.22}$$

Then $\qquad \alpha' = ib_0\zeta'. \qquad$ (A6.23)

In the original coordinates $\alpha = ib_0\zeta$. The transformation which takes α into α' is, from (A6.1) and (A6.22),

$$\zeta = e^{2ip}\zeta', \qquad p \text{ arbitrary.} \tag{A6.24}$$

$\alpha_0 \neq 0,\quad b_0 = 0.$ Choosing $\theta = 0$ and $p = \frac{1}{2}\, i \log \alpha_0$, q arbitrary, (A6.20) and (A6.21) reduce to $\alpha_0' = 1,\ b_0' = 0$ so that

$$Q' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} \alpha_0^{-\frac{1}{2}} & 0 \\ 0 & \alpha_0^{\frac{1}{2}} \end{pmatrix}. \tag{A6.25}$$

Then $\qquad \alpha' = \zeta'^2 - 1. \qquad$ (A6.26)

In the original coordinates $\alpha = \alpha_0\zeta^2 - \bar{\alpha}_0$. The transformation taking α into α' is, from (A6.1) and (A6.25),

$$\zeta' = \alpha_0\zeta. \tag{A6.27}$$

$\alpha_0 \neq 0,\ b_0 \neq 0.$ In this case (A6.20) and (A6.21) show that b_0' and α_0' can never be zero. The best we can do is to choose $\theta = 0$, $p = \frac{1}{2} i \log \alpha_0$, q arbitrary, and then we obtain $\alpha_0' = 1$, $b_0' = b_0$ so that

$$Q' = \begin{pmatrix} 1 & \frac{1}{2} ib_0 \\ \frac{1}{2} ib_0 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} \alpha_0^{-\frac{1}{2}} & 0 \\ 0 & \alpha_0^{\frac{1}{2}} \end{pmatrix}. \tag{A6.28}$$

Then $$\alpha' = \zeta'^2 + ib_0\zeta' - 1. \tag{A6.29}$$

In the original coordinates $\alpha = \alpha_0\zeta^2 + ib_0\zeta - \bar{\alpha}_0$. The transformation taking α into α' is, from (A6.1) and (A6.28),

$$\zeta' = \alpha_0\zeta. \tag{A6.30}$$

Thus, in the case of the pseudosphere, it is not always possible to express Q in canonical form. Q may be diagonalized in two cases, namely, (A6.22) and (A6.25).

Appendix 7

The hypergeometric equation (6.100) (with $C = 0$) is put into standard form by writing $z = -x$, obtaining

$$z(z-1)\ddot{H} + [(\alpha + \beta + 1)z - \gamma]\dot{H} + \alpha\beta H = 0, \tag{A7.1}$$

where $H = H(z)$ and the dot denotes d/dz, and

$$\alpha+\beta = -2 - ia, \qquad \alpha\beta = 1 + ia, \qquad \gamma = -2$$

giving $$\alpha = 1, \quad \beta = -1 - ia, \quad \gamma = -2. \tag{A7.2}$$

Two independent particular solutions of (A7.1) are

$$H_1 = F(\alpha, \beta, \gamma, z),$$

$$H_2 = z^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2-\gamma, z)$$

for $|z| < 1$, leading to the solution (6.101).

Now it is known (see e.g. Forsyth [236]) that the quotient s of any two particular solutions of the equation

$$\ddot{y} + Iy = 0, \qquad y = y(z), \tag{A7.3}$$

satisfies the equation

$$\{s,z\} = 2I, \tag{A7.4}$$

where $\{s,z\}$ is the Schwarzian derivative of s defined by the left hand side of equation (A5.5) of Appendix 5. In the case of the hypergeometric series for $F(\alpha, \beta, \gamma, z)$ the value of I is

$$\frac{1}{4}\left[\frac{1-\lambda^2}{z^2} + \frac{1-\nu^2}{(z-1)^2} + \frac{\lambda^2-\mu^2+\nu^2-1}{z(z-1)}\right], \qquad (A7.5)$$

where $\lambda^2 = (1-\gamma)^2$, $\mu^2 = (\alpha-\beta)^2$, $\nu^2 = (\gamma-\alpha-\beta)^2$.

From (A7.4) and (A7.5) we have a differential equation for s(z). If this can be solved for s in finite terms, it follows that the hypergeometric series will be expressible in finite terms. There are only 15 separate cases in which this is possible, and unfortunately the present case represented by (A7.2) is not one of them. For we have

$$\lambda^2 = 9, \qquad \mu^2 = -a^2 = \nu^2,$$

and Schwarz's table [237] of the 15 special cases does not include these numbers, whatever value (zero not allowed) of the real constant a we choose.

BIBLIOGRAPHY

1. S.W. Hawking & G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge, 1973).

2. S. Kobayashi & K. Nomizu, *Foundations of Differential Geometry* (Wiley, 1963).

3. S. Helgason, *Differential Geometry & Symmetric Spaces* (Academic 1962).

4. W. Killing, J. reine ang. Math. 109, 121 (1892).

5. L.P. Eisenhart, *Riemannian Geometry* (Princeton, 1964).

6. K. Yano, *The Theory of Lie Derivatives & Its Applications* (North-Holland, 1955).

7. A.H. Taub, Bull. Amer. Math. Soc. 55, 85 (1949).

8. A.Z. Petrov, *Einstein Spaces* (Pergamon, 1969).

9. C.D. Collinson & D.C. French, J. Math. Phys. 8, 701 (1967).

10. E. Newman & R. Penrose, J. Math. Phys. 3, 566 (1962).

11. J. Dieudonné, *The Geometry of the Classical Groups* (Springer, 1963).
N. Jacobson, *Lectures in Abstract Algebra* (Van Nostrand, 1953).

12. O. Veblen, Proc. Nat. Acad. Sci. 19, 503 (1933).
W.A. Hepner, Nuovo Cim. 26, 351 (1962).

13. Y. Murai, Prog. Theor. Phys. 11, 441 (1954).
A.O. Barut & W.E. Brittin (eds), *De Sitter & Conformal Groups & Their Applications*, Lectures in Theoretical Physics, Vol.13 (Colorado, 1971).

14. F.S. Klotz, J. Math. Phys. 15, 2242 (1974).

15. T.H. Go, Comm. Math. Phys. 41, 157 (1975).

16. J.G.F. Belinfante & P. Winternitz, J. Math. Phys. 12, 1041 (1971).

17. J. Patera, P. Winternitz, & H. Zassenhaus,
J. Math. Phys. 15, 1378, 1932 (1974).
ibid. 16, 1597, 1615 (1975).
ibid. 17, 717 (1976).
J. Patera, R.T. Sharp, P. Winternitz & H. Zassenhaus,
J. Math. Phys. 17, 977, 986 (1976).

18. D. Finkelstein, Phys. Rev. 100, 924 (1955).
I. Friš & P. Winternitz, Sov. J. Nucl. Phys. 1, 636 (1965).
J. Segercrantz, Ann. Acad. Sci. Fennicae, Ser.A VI 233, 1 (1967).
I. Lukač, J.A. Smorodinsky & P. Winternitz, Sov. J. Nucl. Phys. 7, 139 (1968).
R. Shaw, Quart. J. Math. 21, 101 (1970).
I.V. Bel'ko & A.S. Fedenko, Dokl. Akad. Nauk BSSR 14, 393 (1970).

19. A.H. Clark, Ph.D. Thesis (Princeton, 1969).
T. Yao, J. Math. Phys. 12, 315 (1971).
S. Lagu & H. Laue, Nuovo Cim. A20, 217 (1974).
L.J. Boya, J.F. Carinena & M. Santander, J. Math. Phys. 16, 1813 (1975).
Y. Güler, J. Math. Phys. 17, 554 (1976).

20. G.H. Katzin, J. Levine & W.R. Davis, J. Math. Phys. 10, 617 (1969).
ibid. 11, 1578 (1970a).
Tensor (N.S.) 21, 51 (1970b).
J. Levine & G.H. Katzin, Tensor (N.S.) 21, 319 (1970).
G.H. Katzin & J. Levine, Tensor (N.S.) 22, 64 (1971).

21. J.A. Schouten, *Ricci Calculus* (Springer, 1954).
G.H. Katzin & J. Levine, Tensor (N.S.) 19, 162 (1968).

22. A. Komar, Phys. Rev. 113, 934 (1959).

23. A. Trautman, Bull. Acad. Pol. Sci. 4, 679 (1956).

24. K. Yano, in *"Differentialgeometrie in Großen"* (Ber. Math. Forschungsinst. Oberwolfach, Heft 4, Bibliographisches Inst., Mannheim, 1971), pp.339-351.
Integral Formulas in Riemannian Geometry (Dekker, 1970).

25. H. Guggenheimer, *Differential Geometry* (McGraw-Hill, 1963).

26. J. Liouville, J. Math. Pures Appl. 12, 265 (1847).
F. Klein, Math. Ann. 5, 257 (1872).
S. Lie & F. Engel, *Theorie der Transformationsgruppen* (Leipzig, 1893).
L. Bianchi, *Lezioni di Geometria Differenziale,* 2nd ed., Vol.1 (Spoerri, Pisa, 1902).
G. Fubini, Atti (Torino) 38, 404 (1903).
E. Kasner, Proc. 5th Inter. Congress Math. (Cambridge, 1913), pp.81-87.
H. Weyl, Math.Z. 26, 384 (1918).
Raum, Zeit, Materie (Springer, 1921).
Göttinger Nachr., pp.99-112 (1921).
E. Kasner, Amer. J. Math. 43, 20 (1921).
Math. Ann. 85, 227 (1922).
J.A. Schouten, Math.Z. 11, 58 (1921).
J.A. Schouten & D.J. Struik, Amer. J. Math. 43, 213 (1921).
H.W. Brinkmann, Proc. Nat. Acad. Sci. 9, 1, 172 (1923).
Math. Ann. 91, 269 (1924).
Math. Ann. 94, 119 (1925).
J.M. Thomas, Proc. Nat. Acad. Sci. 11, 257 (1925).
M.S. Knebelman, Amer. J. Math. 52, 280 (1930).
V. Modesitt, ibid. 60, 325 (1938).
A. Fialkow, Trans. Amer. Math. Soc. 45, 443 (1939).
ibid. 51, 435 (1942).
ibid. 56, 309 (1944).
Proc. Nat. Acad. Sci. 26, 437 (1940).
Amer. J. Math. 67, 583 (1945).
N. Coburn, Proc. Nat. Acad. Sci. 27, 57 (1941).
Trans. Amer. Math. Soc. 50, 26 (1941).
R. Coleman, Bull. Amer. Math. Soc. 48, 94 (1942).
G. Vranceanu, Bull. Math. Soc. Roumaine Sci. 45, 3 (1943).
H.C. Lee, Amer. J. Math. 67, 321 (1945).
J. De Cicco, Amer. Math. Monthly 53, 305 (1946).

26. continued ...

E. Kasner & J. De Cicco, Duke Math. J. 14, 327 (1947).
B.A. Rozenfel'd, Mat. Sbornik (N.S.) 23, 297 (1948).
B. Su, Acad. Sinica Sci. Record 2, 331 (1949).
K. Yano & T. Adati, Proc. Japan Acad. 25, 188 (1949).
B. Segre, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 12 (1949).
N.H. Kuiper, Indag. Math. 12, 505, 513 (1950).
E.B. Shanks, Duke Math. J. 17, 299 (1950).
R. Debever, III[e] Congrès National des Sciences, Bruxelles. Vol.2 (1950), pp.56-58.
Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 19, 26 (1955).
K. Yano, J. Indian Math. Soc. (N.S.) 15, 105 (1951).
R. Pernet, C.R. Acad. Sci. (Paris) 233, 1340, 1419 (1951).
K.L. Stellmacher, Math. Ann. 123, 34 (1951).
R.S. Clark, Proc. Lond. Math. Soc. 53, 294 (1951).
Indag. Math. 18, 198 (1956).
J. Lond. Math. Soc. 31, 83 (1956).
M. Matsumoto, J. Math. Soc. Japan 3, 306 (1951).
M.A. Akivis, Mat. Sbornik 31, 43 (1952).
M.A. Dzavadov, Dokl. Akad. Nauk SSSR (N.S.) 86, 653 (1952).
L.L. Verbickii, Trudy Sem. Vektor. Tenzor. Anal. 9, 146 (1952).
M. Kurita, Nagoya Math. J. 5, 1, 59 (1953).
J. Math. Soc. Japan 7, 13 (1955).
S. Kobayashi, C.R. Acad. Sci. (Paris) 238, 644 (1954).
K. Takano & T. Imai, Tensor (N.S.) 3, 108 (1954).
H.L. de Vries, Math. Z. 60, 328 (1954).
S. Ishihara & M. Obata, Proc. Japan Acad. 31, 426 (1955).
C. Kano, Tensor (N.S.) 5, 187 (1956).
R. Blum, Trans. Roy. Soc. Canada, Sect.III (3) 50, 13 (1956).
F. Brickell & R.S. Clark, Bull. Math. Soc. France 90, 15, 27, (1962).
F.A.E. Pirani & A. Schild, *Perspectives in Geometry and Relativity* (ed. Hoffmann), pp.291-309 (Indiana, 1966).
E. Grgin, J. Math. Phys. 9, 1595 (1968).
A.V. Aminova, Sov. Phys. - Doklady 16, 294 (1971).
J. De Cicco & R.V. Anderson, Tensor (N.S.) 22, 191 (1971).
A.A. Lovkov, Penz. Ped. Inst. Učen. Zap. 124, 70 (1971).
A.I. Egorov, ibid. 124, 51 (1971).
L.I. Egorova, ibid. 124, 67 (1971).
T. Suguri & S. Ueno, Tensor (N.S.) 24, 253 (1972).
R. Sigal, J. Math. Phys. 14, 1434 (1973).
Yu. S. Mavryčev, Izv. VUZ Fiz. (USSR) 5, 125 (1973).
ibid. 9, 133 (1974).
J.C. du Plessis, J. Math. Phys. 15, 329 (1974).
S. Hori, J. Sakomoto & A. Sato, Prog. Theor. Phys. 52, 1647 (1974).
C. Loewner & L. Nirenberg, *Contribution to Analysis* (ed. Ahlfors), pp.245-272 (Academic, 1974).
H. Tilgner, J. Math. Phys. 15, 360 (1974).
B.G. Schmidt, Comm. Math. Phys. 36, 73 (1974).
GRG 1, 269 (1971).
R. Penrose, *Battelle Recontres* (ed. De Witt & Wheeler), pp.121-235 (Benjamin, 1969).
S. Kobayashi, *Transformation Groups in Differential Geometry*, (Springer, 1972).
T. Go, H.A. Kastrup & D.H. Mayer, Rep. Math. Phys. 5, 43 (1974).
S.I. Fediščenko, Ukrain. Geometr. Sb. Vyp. 17, 134, 148 (1975).
E.A. Lord, Int. J. Theor. Phys. 13, 89 (1975).
D.H. Mayer, J. Math. Phys. 16, 884 (1975).

26. continued ...

W. Roter, Colloq. Math. 33, 225 (1975).
L. Defrise-Carter, Comm. Math. Phys. 40, 273 (1975).
L. Defrise, *Unpublished Ph.D. dissertation* (Brussels, 1969).
J.C. du Plessis, Tensor (N.S.) 20, 3 (1969).
C.P. Boyer & E.G. Kalnins, Waikato Uni. Research Rep. No.39 (1976).
B.K. Berger, Bull. Amer. Phys. Soc. 21, 38 (1976).
J. Math. Phys. 17, 1268 (1976).

27. E. Cunningham, Proc. Lond. Math. Soc. 8, 77 (1910).
H. Bateman, ibid. 8, 228, 469, (1910).

28. D.H. Mayer, J. Math. Phys. 16, 884 (1975).

29. T. Fulton, F. Rohrlich & L. Witten, Rev. Mod. Phys. 34, 442 (1962).
Nuovo Cim. 26A, 652 (1962).

30. J. Haantjes, Proc. Kon. Akad. Wetensch. 40, 700 (1937).

31. J.A. Schouten & J. Haantjes, Physica 1, 869 (1934).

32. E. Bessel-Hagen, Math. Ann. 84, 258 (1921).
P.A.M. Dirac, Ann. Math. 37, 429, 657 (1936).
H.J. Bhabha, Proc. Camb. Phil. Soc. 32, 622 (1936).
F. Gürsey, Nuovo Cim. 3, 988 (1956).
J.A. McLennan, Nuovo Cim. 3, 1360 (1956).
ibid. 5, 640 (1957).
F. Bopp, Ann. Phys. 4, 96 (1959)
H.A. Buchdahl, Nuovo Cim. 11, 496 (1959).
W.A. Hepner, Nuovo Cim. 26, 351 (1962).
E. Grgin, J. Math. Phys. 9, 1595 (1968).
H.A. Kastrup, Phys. Rev. 150, 1183 (1966).
L. Castell, Nuovo Cim. 46A, 1 (1966).

33. H. Weyl, S.b. Preuss. Akad. Wiss. Berlin, pp.465-480 (1918).

34. J. Wess, Nuovo Cim. 18 A, 1086 (1960).

35. A.O. Barut & G.L. Bornzin, J. Math. Phys. 15, 1000 (1974).

36. A.O. Barut & R.B. Haugen, Ann. Phys. (N.Y.) 71, 519 (1972).
Lett. Nuovo Cim. 7, 625 (1973).
Nuovo Cim. 18A, 495, 511 (1973).

37. J.A. Schouten & J. Haantjes, Proc. Kon. Akad. Wetensch, 39, 1059 (1936)

38. W. Pauli, Helv. Physica 13, 204 (1940).

39. B. Hoffmann, Phys. Rev. 73, 30, 1042 (1948).

40. R.L. Ingraham, Nuovo Cim. 12, 825 (1954).
Proc. Nat. Acad. Sci. 40, 237 (1954).

41. R.L. Ingraham, Nuovo Cim. 9, 886, (1952).
Y. Murai, Sci. Rep. Saitama Univ. Ser.A2, 7 (1955).

42. L. Garding, Proc. Camb. Phil. Soc. 41, 49 (1945).

42. continued ...

J.A. Schouten, Rev. Mod. Phys. 21, 421 (1949).
I.E. Segal, Duke Math. J. 18, 221 (1951).
J. Weyssenhoff, Acta Phys. Polon. 11, 273 (1953).
K. Goto, Prog. Theor. Phys. 12, 409 (1954).
M. Flato, D. Sternheimer & J.P. Vigier, C.R. Acad. Sci. (Paris) 260, 3869 (1965).
D. Bohm, M. Flato, D. Sternheimer & J.P. Vigier, Nuovo Cim. 38 A, 1941 (1965).
H.F. Gautrin, R. Prasad & J.P. Vigier, C.R. Acad. Sci. (Paris) 262, 1177 (1966).
H.A. Kastrup, Phys. Rev. 143, 1021 (1966).
L. Castell, Nuovo Cim. 49 A, 285 (1967).
F. Halbwachs, Nuovo Cim. 49 A, 517 (1967).
A. Clark, *Representations of the Special Conformal Group in High-Energy Physics*, Ph.D. Thesis (Princeton, 1968).
G.A. Mack & A. Salam, Ann. Phys. (N.Y.) 53, 174 (1969).
A. Salam & J. Strathdee, Phys. Rev. 184, 1760 (1969).
K. Wilson, Phys. Rev. 179, 1499 (1969).
D.G. Boulware, L.S. Brown & R.D. Peccei, Phys. Rev. D2, 293 (1970).
D.J. Gross & J. Wess, Phys. Rev. D2, 753 (1970).
G. Rosen, Phys. Rev. D3, 615 (1971).
D.G. Boulware, L.S. Brown & R.D. Peccei, Phys. Rev. D3, 1750 (1971).
P. Carruthers, Phys. Rev. C1, 3 (1971).
S.L. Adler, Phys. Rev. D6, 3445 (1972).
S. Ferrara, A.F. Grillo & R. Gatto, Phys. Lett. 42 B, 264 (1972).
A.F. Grillo, Riv. Nuovo Cim. 3, 146 (1973).
J. Kogut & L. Susskind, Phys. Rep. 8, 75 (1973).
Phys. Rev. D9, 690 (1974).
P. Ramond, Phys. Rev. D7, 449 (1973).
S.B. Berger, Lett. Nuovo Cim. 11, 89 (1974).
L.N. Chang & A. Chodos, Phys. Rev. D9, 355 (1974).
R.A. Brandt & W.C. Ng, Phys. Rev. D9, 373 (1974).
N. Christ, Phys. Rev. D9, 946 (1974).
S. Ciccariello & G. Sartori, Nuovo Cim. 19 A, 470 (1974).
B.G. Konopel'čenko & M.Ja. Pal'čik, Dokl. Akad. Nauk SSSR 214, 1052 (1974).
M. Lorente & P. Roman, J. Math. Phys. 15, 70 (1974).
J. Lukierski & K. Sienkiewicz, J. Math. Phys. 15, 344 (1974).
B. Schroer & J.A. Swieca, Phys. Rev. D10, 480 (1974).
M.A.B. Beg & S.S. Shei, Phys. Rev. D12 3092 (1975).
L. Castell, Nuovo Cim. 29 A, 445 (1975).
C. Fronsdal, Phys. Rev. D12, 3819 (1975).
S. Browne, J. Math. Phys. 17, 831 (1976).

43. S. Ferrara, R. Gatto & A.F. Grillo, *Conformal Algebra in Space-Time*, Tracts in Modern Physics, Vol.67 (Springer, 1973).

44. H.A. Kastrup, Ann. Physik 9, 388 (1962).

45. G. Post, J. Math. Phys. 17, 24 (1976).

46. K. Gödel, Rev. Mod. Phys. 21, 447 (1949).

47. S. Hawking, A.R. King & P.J. McCarthy, J. Math. Phys. 17, 174 (1976).

48. E.C. Zeeman, Topology 6, 161 (1967).

49. R. Göbel, J. Math. Phys. 17, 845 (1976).

50. E.C. Zeeman, J. Math. Phys. 5, 490 (1964).

51. S. Nanda, Proc. Camb. Phil. Soc. 79, 533 (1976).

52. J. Rosen, Ann. Phys. (N.Y.) 47, 468 (1968).

53. *See ref.* [42].

54. T.H. Go, Comm. Math. Phys. 41, 157 (1975).

55. R. Penrose, *see ref.* [26].

56. T.H. Go, H.A. Kastrup & D.H. Mayer, Rep. Math. Phys. 6, 395 (1974).

57. V.I. Ogievetsky, Lett. Nuovo Cim. 8, 988 (1973).

58. A.B. Borisov & V.I. Ogievetskii, Theor. & Math. Phys. 21, 1179 (1974).

59. S. Coleman, J. Wess & B. Zumino, Phys. Rev. 177, 2239 (1969).

60. P.G.O. Freund, J. Math. Phys. 17, 424 (1976).

61. L. Page, Phys. Rev. 49, 254, 946 (1936).
L. Page & I Adams, ibid. 49, 466 (1936).

62. H.T. Engstrom & M. Zorn, Phys. Rev. 49, 701 (1936).

63. E.A. Milne, *Relativity, Gravitation & World Structure* (Oxford, 1935).
Kinematic Relativity (Oxford, 1948).

64. H.P. Robertson, Phys. Rev. 49, 755 (1936).

65. H.P. Robertson, Ap. J. 82, 284 (1935); 83, 187 (1936).

66. D. Bourgin, Phys. Rev. 50, 864 (1936).

67. E.L. Hill, Phys. Rev. 67, 358 (1945); 72, 143, 236 (1947).

68. H. Laue, Nuovo Cim. 3 B, 55 (1971).

69. F. Rohrlich, Ann. Phys. (N.Y.) 22, 169 (1963).

70. W. Pauli, *Relativitätstheorie* (Leipzig, 1921).
E.L. Hill, Phys. Rev. 72, 143 (1947).

71. F. Rohrlich, *Classical Charged Particles* (Reading, 1965).

72. J. Haantjes, Proc. Ned. Akad. Wetensch. 43, 1288 (1940).

73. Y. Ueno & H. Takeno, Prog. Theor. Phys. 8, 291 (1952); 12, 129 (1954).

74. D. Meksyn, Nature 160, 834 (1947).

75. I. Robinson & Y. Ne'eman, Rev. Mod. Phys. 37, 201-230 (1965).
P. Szekeres, Nuovo Cim. 43 A, 1062 (1966).

76. A. Friedman, J. Math. Mech. 10, 625 (1961).

77. J. Nash, Ann. Math. 63, 20 (1956).

78. M.D. Maia, J. Math. Phys. 15, 420 (1974).

79. R.F. Sigal, Int. J. Theor. Phys. 11, 45 (1974).

80. R.L. Ingraham, C.R. Acad. Sci. (Paris) 232, 938, 1072 (1951).
Indag. Math. 13, 315 (1951).
Proc. Nat. Acad. Sci. 38, 921 (1952).
Nuovo Cim. 1, 82 (1955).
See also ref. [40].

81. Th. Kaluza, S. b. Preuss. Akad. Wiss. Berlin, p.966 (1921).

82. A.H. Taub, Math. Reviews 17, 305 (1956).

83. H. Takeno, J. Sci. Hiroshima Univ. Ser.A16, 299 (1952).
ibid. 16, 497 (1953).
Tensor (N.S.) 4, 141 (1955); 5, 23 (1955).
Sci. Rep. Hiroshima Univ., No.5 (1966).

84. L. Derry, R. Isaacson & J. Winicour, Phys. Rev. 185, 1647 (1969).

85. K. Schwarzschild, S.b. Preuss. Akad. Wiss. Berlin, pp.189, 424 (1916).
V.V. Narlikar & K.P. Singh, Philos. Mag. 41, 152 (1950).
H. Takeno, Sci. Rep. Hiroshima Univ., No.5, p.33 (1966).
H.A. Buchdahl, Am. J. Phys. 39, 158 (1971).
J.K. Rao & R.B. Patel, Current Science 41, 409 (1972).
K.P. Singh & Abdusattar, GRG 5, 115 (1974).
R.J. Adler, J. Math. Phys. 15, 727 (1974).
M. Gürses & Y. Gürsey, Nuovo Cim. 25 B, 786 (1975).

86. W. de Sitter, Mon. Not. Roy. Astron. Soc. 78, 3 (1917).

87. A. Einstein, S. b. Preuss. Akad. Wiss. Berlin, p.142 (1917).

88. H.P. Robertson, Ap. J. 82, 284 (1935).
A.G. Walker, Proc. Lond. Math. Soc. 42, 90 (1936).
See also ref. [1] pp.139-140.

89. A. Friedmann, Z. Phys. 10, 377 (1922).
ibid. 21, 326 (1924).

90. J. Levine, Bull. Amer. Math. Soc. 42, 418 (1936).
ibid. 45, 766 (1939).
Duke Math. J. 17, 15 (1950).

91. K. Yano & T. Nagano, Ann. Math. 69, 451 (1959).

92. J. Levine & G.H. Katzin, Tensor (N.S.) 19, 317 (1968).
ibid. 20, 251 (1969).

93. H.S. Ruse, Proc. Roy. Soc. Edinburgh A62, 64 (1944).

94. V.V. Narlikar & K.R. Karmarkar, Proc. Indian Acad. Sci. A29, 91 (1949).

95. S. Sasaki, Sci. Rep. Tôhoku Univ. (1) 29, 219 (1940).

96. T. Adati, Proc. Japan Acad. 27, 1, 49 (1951).

97. L.L. Verbickii, Dokl. Akad. Nauk SSSR (N.S.) 81, 133 (1951).

98. G. Vranceanu, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 11, 385 (1930).

99. R. Blum, Trans. Roy. Soc. Canada, Sect.III (3) 49, 1 (1955).

100. H. Stephani, Comm. Math. Phys. 5, 337 (1967).

101. B. Bertotti, Phys. Rev. 116, 1331 (1959).

102. A. Barnes, GRG 5, 147 (1974).

103. V.I. Obozov, Izv. VUZ Fiz. (USSR) 2, 12 (1974).
ibid. 11, 20 (1975).

104. S. Kloster, M.M. Som & A Das, J. Math. Phys. 15, 1096 (1974).
ibid. 17, 441 (1976).

105. A. Papapetrou, Ann. Physik 12, 309 (1953).
J. Ehlers, *Les Théories Relativistes de la Gravitation,* p.275 (Gauthier-Villars, 1952).

106. N. O'Murchadha & J.W. York, Phys. Rev. D10, 428, 437 (1974).
See also J.A. Isenberg, N. O'Murchadha & J.W. York, ibid. 13, 1532 (1976).

107. R.G. McLenaghan, N. Tariq & B.O.J. Tupper, J. Math. Phys. 16, 829 (1975).

108. P.C. Peters, J. Math. Phys. 16, 1780 (1975).

109. J.D. Zund, C.R. Acad. Sci. (Paris) A262, 1081 (1966).

110. A. Lichnerowicz, Ann. Mat. Pura Appl. (4) 50, 1 (1960).

111. J. Levine & J.D. Zund, C.R. Acad. Sci. (Paris) A264, 1029 (1967).
Ann. Mat. Pura Appl. (4) 80, 373 (1968).

112. G.S. Hall, Ann. Mat. Pura Appl. (4) 96, 175 (1972).

113. H. Sokolik, GRG 5, 599 (1974).
H. Sokolik & N.P. Konopleva, Dokl. Akad. Nauk SSSR 169, 539 (1966).

114. R. Pavelle, Phys. Rev. Lett. 34, 1114 (1975).

115. A.H. Thompson, Phys. Rev. Lett. 35, 320 (1975).

116. C.W. Kilmister & D.J. Newman, Proc. Camb. Phil. Soc. 57, 851 (1961).
C.N. Yang, Phys. Rev. Lett. 33, 445 (1974).

117. C.D. Collinson, J. Math. Phys. 11, 818 (1970).

118. F.A.E. Pirani & A. Schild, Bull. Acad. Pol. Sci. 9, 543 (1961).

119. W. Kundt & A. Thompson, C.R. Acad. Sci. (Paris) A254, 4257 (1962).

120. I. Robinson & A. Schild, J. Math. Phys. 4, 484, 1963).

121. P. Szekeres, Proc. Roy. Soc. A274, 206 (1963).

122. A. Thompson, *Thesis* (London, 1962).

123. *See e.g.* M. Cahen & R.G. McLenaghan, C.R. Acad. Sci. (Paris) 266, 1125 (1968).

124. R. Debever, in *"Perspectives in Geometry and Relativity"* (ed. Hoffmann), pp.96-104 (Indiana, 1966).

125. M. Cahen, R. Debever & L. Defrise, J. Math. Mech. 16, 761 (1967).

126. I. Robinson, J. Math. Phys. 2, 290 (1961).

127. E.W. Mielke, Bull. Amer. Phys. Soc. 19, 508 (1974).
Preprint (1975).

128. R.O. Hansen & J. Winicour, Phys. Rev. D13, 2705 (1976).

129. R.P. Kerr, Phys. Rev. Lett. 11, 237 (1963).
See also R. Geroch, J. Math. Phys. 12, 918 (1971).

130. A. Einstein, Univ. Nacional de Tucuman Rev. Ser.A2, 11 (1941).

131. M.E. Cahill & A.H. Taub, Comm. Math. Phys. 21, 1 (1971).

132. A.H. Taub, in *"General Relativity"* (ed. O'Raifeartaigh), pp.133-150 (Oxford, 1972).

133. B.B. Godfrey, GRG 3, 3 (1972).

134. H. Weyl, Ann. Physik 54, 117 (1919); 59, 185 (1919).
See also [135].

135. J. Ehlers & W. Kundt, in *"Gravitation: An Introduction to Current Research"*, p.80 (Wiley, 1962).

136. T. Levi-Civita, Rend. Accad. Lincei 26, 307 (1917).
ibid. 27, 3, 183, 220, 240, 283, 343 (1918).
ibid. 28, 3, 101 (1919).

137. E. Kasner, Trans. Amer. Math. Soc. 27, 160 (1925).

138. R. Sigal, GRG 5, 737 (1974).

139. D.M. Eardley, Comm. Math. Phys. 37, 287 (1974).

140. J.W. York, J. Math. Phys. 14, 456 (1973).

141. C.B.G. McIntosh, GRG 7, 199, 215 (1976).

142. G.C. Debney, J. Math. Phys. 12, 1088, 2372 (1971).
ibid. 13, 1469 (1972).

143. H.A. Buchdahl, Proc. Edinburgh Math. Soc. (2) 10, 16 (1953).

144. A. Popovici, Rev. Univ. "C.I. Parhon" Politehn. Bucuresti. Ser. Sti. Nat. 2, 78 (1953).

144. continued ...
A. Popovici, Acad. R.P. Romîne. Bul. Sti. Sect. Sti. Mat. Fiz. 6, 65 (1954).

145. G. Erez & N. Rosen, Bull. Res. Counc. Israel 8F, 47 (1959).

146. R.G. McLenaghan & J. Leroy, Proc. Roy. Soc. A327, 229 (1972).

147. P.B. Scheurer, Helv. Phys. Acta 45, 35 (1972).

148. E. Schnirman & C.G. Oliveira, Ann. Inst. H. Poincaré 17, 379 (1973).

149. A.K. Soleimany, J. Phys. A7, 223 (1974).

150. A.G. Agnese & P. Calvini, Phys. Rev. D12, 3804 (1975).

151. F. Englert, E. Gunzig, C. Truffin & P. Windey, Phys. Lett. B57, 73 (1975).

152. D.K. Ross, GRG 6, 157 (1975).

153. A.O. Barut & S. Komy, J. Phys. A9, 207 (1976).

154. G. Lemaitre, Ann. Soc. Sci. Bruxelles 47A, 49 (1927).
Mon. Not. Roy. Astron. Soc. 91, 483 (1931).

155. H.P. Robertson, Proc. Nat. Acad. Sci. 15, 822 (1929).
Rev. Mod. Phys. 5, 62 (1933).
See also [65].
W.H. McCrea, *see e.g.* Z. Astrophys. 9, 290 (1934/35).
E.A. Milne & W.H. McCrea, Quart. J. Math. 5, 73 (1934).
R.C. Tolman, *see e.g. Relativity, Thermodynamics & Cosmology* (Oxford, 1934).
A.S. Eddington, *see e.g. The Mathematical Theory of Relativity* (Cambridge, 1923).
Mon. Not. Roy. Astron. Soc. 97, 156 (1937).
G.C. McVittie, *see e.g. Cosmological Theory* (Methuen, 1937).

156. T. Sibata, J. Sci. Hiroshima Univ. Ser. A8, 199 (1938).
H. Takeno, ibid. 8, 223 (1938); 10, 173 (1940); 11, 201 (1942).
K. Itimaru, ibid. 10, 151 (1940).
T. Iwatsuki & T. Sibata, ibid. 10, 247 (1940).

157. L. Infeld & A. Schild, Phys. Rev. 68, 250 (1945).
ibid. 70, 410 (1946).

158. F. Gürsey, Proc. Camb. Phil. Soc. 49, 285 (1953).

159. D.E. Littlewood, Proc. Camb. Phil. Soc. 49, 90 (1953).
ibid. 51, 678 (1955).
ibid. 52, 88 (1956).

160. F.A.E. Pirani, Proc. Camb. Phil. Soc. 51, 535 (1955).

161. S. Bhattacharya, Prog. Theor. Phys. 11, 613 (1954).

162. G.E. Tauber & J.W. Weinberg, Phys. Rev. 122, 1342 (1961).

163. J. Ehlers, P. Geren & R.K. Sachs, J. Math. Phys. 9, 1344 (1968).

164. J. Ehlers, Akad. Wiss. Mainz. Math-Nat. Kl., No.11 (1961).

165. M. Trümper, GRG 5, 1 (1974).

166. F. Hoyle & J.V. Narlikar, Proc. Roy. Soc. A282, 191 (1964).
ibid. 294, 138 (1966).
Mon. Not. Roy. Astron. Soc. 155, 305 (1972a)
ibid. 155, 323 (1972b).

167. H. Bondi & T. Gold, Mon. Not. Roy. Astron. Soc. 108, 252 (1948).
F. Hoyle, ibid. 108, 372 (1948).
F.A.E. Pirani, Proc. Roy. Soc. A228, 455 (1955).

168. I.E. Segal, Astron. & Astrophys. 18, 143 (1972).
Proc. Nat. Acad. Sci. 71, 765 (1974).
See also Bull. Amer. Math. Soc. 77, 958 (1971).

169. A. Barnes, GRG 4, 105 (1973).

170. M. Trümper, Z. Phys. 168, 55 (1962).

171. N.M. Stepanyuk, Sov. Phys. JETP 26, 369 (1968).

172. I.H. Thompson & G.J. Whitrow, Mon. Not. Roy. Astron. Soc. 136, 207 (1967).
ibid. 139, 499 (1967).

173. M.C. Faulkes, Prog. Theor. Phys. 42, 1139 (1970).

174. H. Nariai, Prog. Theor. Phys. 40, 1013 (1969).

175. P. Jordan, J. Ehlers & W. Kundt, Akad. Wiss. Mainz. Math-Nat. Kl., No.2 (1960).

176. A.H. Taub, Math. Ann. 53, 472 (1951).
O. Heckmann & E. Schücking, in *Gravitation: An Introduction to Current Research* (ed. L. Witten), Chapter 11 (Wiley, 1962).
G.F.R. Ellis & M.A.H. MacCallum, Comm. Math. Phys. 12, 108 (1969).
M.A.H. MacCallum & G.F.R. Ellis, ibid. 19, 31 (1970).
M.A.H. MacCallum, ibid. 20, 57 (1971).
C.B. Collins & S.W. Hawking, Ap. J. 180, 317 (1973).

177. E.M. Lifshitz & I.M. Khalatnikov, Soviet Phys. Usp. 6, 495 (1964).
Advances in Physics 12, 185 (1963).
D. Eardley, E. Liang & R. Sachs, J. Math. Phys. 13, 99 (1972).

178. R. Arnowitt, S. Deser & C.W. Misner, in *Gravitation: An Introduction to Current Research,* Chapter 7 (Wiley, 1962).

179. M.A.H. MacCallum & A.H. Taub, Comm. Math. Phys. 25, 173 (1972).

180. K.A. Milton & Y.J. Ng, Phys. Rev. D10, 420 (1974).

181. J. Schwinger, *Particles, Sources & Fields, Vol.1, Sections 3-17* (Addison-Wesley, 1970).

182. C. Brans & R.H. Dicke, Phys. Rev. 124, 925 (1961).

183. C.W. Misner, K.S. Thorne & J.A. Wheeler, *Gravitation*, p.609 (Freeman, 1973).

184. S.C. Chang & A.I. Janis, J. Math. Phys. 17, 1432 (1976).

185. W. Kundt & E.T. Newman, J. Math. Phys. 9, 2193 (1968).

186. S.W. Hawking, J. Math. Phys. 9, 598 (1968).

187. G.E. Tauber, J. Math. Phys. 8, 118 (1967).

188. F. Gürsey, Ann. Phys. 19, 1 (1963).

189. R.J. Lopez, *Motion-preserving Conformally Homogeneous Model Universes*, Ph.D. Thesis (Purdue, 1970).

190. H. Stephani, J. Math. Phys. 15, 14 (1974).

191. C.M. Caves, Bull. Amer. Phys. Soc. 20, 98 (1975).

192. A. Thompson, Phys. Rev. Lett. 34, 507 (1975).

193. W. Kundt, Z. Phys. 163, 77 (1961).
Proc. Roy. Soc. A270, 328 (1962).

194. C.D. Collinson, J. Phys. A2, 621 (1969).

195. A. Held, J. Math. Phys. 17, 39 (1976).

196. J. Winicour, *Lecture Notes in Physics*, Vol.14, p.145 (Springer, 1972).

197. P.A.M. Dirac, Proc. Roy. Soc. A333, 403 (1973).
J.M. Charap & W. Tait, Proc. Roy. Soc. A340, 249 (1974).

198. G.C. Debney, R.P. Kerr & A. Schild, J. Math. Phys. 10, 1842 (1969).

199. R.P. Kerr & G.C. Debney, J. Math. Phys. 11, 2807 (1970).

200. G.C. Debney, Ph.D. Thesis (Texas, 1967).

201. W.B. Wilson, Private communication.

202. J.N. Goldberg & R.P. Kerr, J. Math. Phys. 2, 327 (1961).

203. F.A.E. Pirani, Bull. Acad. Pol. Sci. 5, 143 (1957).

204. R.K. Sachs, Proc. Roy. Soc. A264, 309 (1961).
ibid. A270, 103 (1962).

205. R.K. Sachs, in *Relativity, Groups & Topology*, p.530 (Gordon & Breach, 1964).

206. A. Trautman, in *Lectures in General Relativity*, Vol.1, p.57 (Prentice-Hall, 1964).

207. F.A.E. Pirani, in *Lectures in General Relativity*, Vol.1, Chapter 4 (Prentice-Hall, 1964).

208. S.L. Bazanski, J. Math. Phys. 6, 1201 (1965).

209. J.N. Goldberg & R.K. Sachs, Acta Phys. Polon. 22, 13 (1962).

210. P. Jordan, J. Ehlers & R.K. Sachs, Akad. Wiss. Lit. Mainz. Abh. Math-Nat. Kl., No.1 (1960).

211. R.K. Sachs, J. Math. PHys. 3, 908 (1962).

212. I. Robinson, J. Math. Phys. 2, 290 (1961).

213. F.A.E. Pirani, in *Gravitation: An Introduction to Current Research, Chapter 6* (Wiley, 1962).

214. R.V. Churchill, Trans. Amer. Math. Soc. 34, 784 (1932).

215. H.S. Ruse, Proc. Lond. Math. Soc. 50, 75 (1948).

216. J. Géhéniau & R. Debever, Bull. Acad. Belg. Cl. Sci.(5) 42, 114, 252, 313, 608 (1956).

217. A.Z. Petrov, Sci. Not. Kazan State Uni. 114, 55 (1954).

218. F.A.E. Pirani, Phys. Rev. 105, 1089 (1957).

219. R.P. Kerr, J. Math. Mech. 12, 33 (1963).

220. H. Goenner & J. Stachel, J. Math. Phys. 11, 3358 (1970).

221. R. Penrose, Ann. Phys. (N.Y.) 10, 171 (1960).

222. L. Witten, Phys. Rev. 113, 357 (1959).

223. L. Bel, C.R. Acad. Sci. (Paris) 247, 2096 (1958). ibid. 248, 1297, 2561 (1959).

224. A. Lichnerowicz, C.R. Acad. Sci. (Paris) 246, 893 (1958).

225. J. Ehlers & R.K. Sachs, Z. Phys. 155, 498 (1959).

226. J. Plebanski, Acta Phys. Polon. 26, 963 (1964).

227. R.S. Mishra, in *Relativity & Gravitation,* (ed. Kuper & Peres), p.239 (Gordon & Breach, 1971).

228. I. Robinson & A. Trautman, Proc. Roy. Soc. A265, 463 (1961).

229. J.L. Anderson, *Principles of Relativity Physics* (Academic, 1967).

230. R.P. Kerr & A.Schild, *Proceedings of Symposia in Applied Mathematics,* Vol.17, p.199 (Amer. Math. Soc., 1965).

231. P.A. Hogan, Int. J. Theor. Phys. 11, 419 (1974).

232. E. Kamke, *Differentialgleichungen: Lösungsmethoden und Lösungen,* Vol.1, p.562 (Chelsea, 1971).

233. H. Lemke, J. reine ang. Math. 142, 118 (1913).

234. M. Walker & R. Penrose, Comm. Math. Phys. 18, 265 (1970).
R. Geroch, J. Math. Phys. 11, 1955 (1970).
P. Sommers, J. Math. Phys. 14, 787 (1973).
I. Hauser & R.J. Malhiot, J. Math. Phys. 16, 150 (1975).
N.M.J. Woodhouse, Comm. Math. Phys. 44, 9 (1975).
G.J. Weir, Univ. of Canterbury Res. Rep. No.2 (1976).
J. Math. Phys. *(to be published).*
Private communication.

235. *See e.g.* P.R. Garabedian, *Partial Differential Equations* (Wiley, 1964).

236. A.R. Forsyth, *A Treatise on Differential Equations,* p.230 (Macmillan, 1956).

237. *See e.g.* A. Erdélyi (ed.), *Higher Transcendental Functions,* Vol. 1, p.97 (McGraw-Hill, 1953).

THE CONFORMAL GROUP AND EINSTEIN SPACES

W.D. Halford

ERRATA

Page 46, line 9: should read

(McIntosh [141] has quoted this metric. This points up an error in Godfrey's paper.)

Page 106, lines 8,9,10 and 11: replace by

but because of the presence of $K_1=\partial_s$ and $K_2 = 2(\zeta\partial_\zeta+\bar{\zeta}\partial_{\bar{\zeta}})-s\partial_s+r\partial_r$ in the space we may take the HKV in the form

$$\tilde{K} = s\partial_s + r\partial_r. \tag{5.7}$$

Page 106, lines 26 and 27: replace by

$$\tilde{K} = s\partial_s + r\partial_r. \tag{5.10}$$

Page 107, after line 25 add:

or (iii) $\alpha_0=C_1=C_2=0$, $a-5a_0=0$.

Page 107, line 26: after "Using (ii)" insert "and (iii)".

Page 108, after line 21 add:

But applying the theorem

$[\tilde{K},K_i] = mK_1+nK_2$ $(i=1,2)$ m,n constants

leads to $\alpha_0=0$. Hence the form of the HKV may be taken as

$$\tilde{K} = \zeta\partial_\zeta+\bar{\zeta}\partial_{\bar{\zeta}}-4s\partial_s-2r\partial_r+4i\beta\log(\bar{\zeta}/\zeta)\partial_s. \tag{5.21a}$$

Page 115, after line 22 insert:

However, it may be that $a=a_0$, when the matric (5.9) admits the HKV

$$\tilde{K} = x\partial_x+y\partial_y-s\partial_s.$$

Page 152, lines 29 and 30 (Theorem 7.2): should read

There are no Petrov type N vacuum spaces with non-zero expansion and/or twist which admit a HKV and a Killing vector of type $K = \partial_s = R(\zeta,\bar{\zeta})\partial_u$.

Page 160, second last line, after "preserved" add:

if $y^{1+a} \rightarrow k$, constant.