Editors' Suggestion

Effective Theory for Strongly Attractive One-Dimensional Fermions

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We study a one-dimensional system of two-component fermions in the limit of strong attractive particle-particle interactions. First, we analyze scattering in the corresponding few-body problem, which is analytically solvable via Bethe ansatz. This allows us to engineer effective interactions between the system's effective degrees of freedom: fermions and bosonic dimers (tightly bound pairs of fermions). We argue that, although these interactions are strong, the resulting effective problem can be mapped onto a weakly interacting one, paving the way for the use of perturbation theory. This finding simplifies studies of many-fermion systems under confinement that are beyond reach of state-of-the-art numerical methods. We illustrate this statement by considering an impurity atom in a Fermi gas.

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Introduction—Understanding physical systems with strong electron-electron correlations, e.g., itinerant ferromagnets [1], requires exploration of interacting spin-$\frac{1}{2}$ fermionic models. Typically, these models can be solved only approximately or within certain limits, motivating the search for exact solutions as foundational elements for further developments. One technique for generating such solutions is the mapping of a strongly interacting system onto a weakly interacting one. The celebrated Bose-Fermi mapping [2], which maps strongly repulsive one-dimensional (1D) continuum models [3] onto noninteracting fermions, is an example relevant for this Letter. Here, we address a two-component system in the opposite limit of strong attractions, ultimately demonstrating that it too can be mapped onto a weakly interacting problem.

In particular, we argue that strongly attractive systems can be linked to a weakly interacting mixture of fermions and dimers (bound states between spin-up and spin-down fermions). Our focus is on trapped spin-imbalanced systems, whose solution is, as a rule, particularly difficult [4]. As a limit, this Letter also includes the known spin-balanced model—a reference point for BEC-BCS studies [5,6] as well as for corresponding few-body problems [7].

Framework—Our goal is to relate a trapped system of $N = N_\downarrow + N_\uparrow$ fermions of mass m:

$$H = \sum_{i=1}^{N} [t_{x_i} + v_{\text{ext}}(x_i)] + g \sum_{i<j}^{N} \delta(x_i - x_j), \tag{1}$$

in the limit of "large" negative values of g to an effective weakly interacting model. Here, $x_{i \leq N_\downarrow}$ ($x_{i>N_\downarrow}$) are the coordinates of the spin-down (spin-up) fermions; $t_{x_i} = -\hbar^2 \partial_{x_i}^2/(2m)$ is the kinetic energy operator and v_{ext} describes an external trap. The Hamiltonian in Eq. (1) with a harmonic confinement is of particular interest in cold-atom physics where it can be realized experimentally [3,8,9]. Its strongly repulsive limit is well understood by now [3,10]. Here, we discuss the limit of strong attractions, which received less attention.

Without loss of generality, we assume that $N_\uparrow \geq N_\downarrow$. In the strongly attractive regime, opposite-spin fermions form deeply bound dimers whose binding energy $B_{\text{D}} \simeq \hbar^2 \tilde{g}^2/(4m)$ (see, e.g., Ref. [8]) is much larger than any other energy scale of the problem [11], here $\tilde{g} \coloneqq mg/\hbar^2$. At the same time, the size of the dimer, $r_{\text{D}} = -1/\tilde{g}$ [12], is much smaller than any other length scale [13]. This provides a separation of scales for constructing an effective theory with $N_\downarrow$ dimers and $M = N_\uparrow - N_\downarrow$

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unpaired fermions [14]:

$$h = \sum_{i=1}^{N_\downarrow}\left[\frac{t_{y_i}}{2} + 2v_{\rm ext}(y_i)\right] + \sum_{i=1}^{M}[t_{z_i} + v_{\rm ext}(z_i)] + w, \qquad (2)$$

where the first term describes the dimers that have mass $2m$ and feel a stronger external potential due to their composite nature; w is the effective interaction derived below by considering Eq. (1) for three and four fermions on a line (i.e., with $v_{\rm ext} = 0$). The low-energy spectrum of the Hamiltonian H plus the dimer binding energies $N_\downarrow B_{\rm D}$ gives the spectrum of h.

Three and four fermions on a line—In the absence of confinement ($v_{\rm ext} = 0$), the problem is solvable using the Bethe ansatz [19–22], allowing us to extract the scattering information for the effective degrees of freedom (dimers and fermions). For distinguishable particles, scattering properties in 1D follow from the wave function

$$\Psi(x_{\rm rel}) = e^{ikx_{\rm rel}} + \left(f_{\rm e} + f_{\rm o}\frac{|x_{\rm rel}|}{x_{\rm rel}}\right)e^{ik|x_{\rm rel}|}, \qquad (3)$$

where $|x_{\rm rel}| \to \infty$ describes the relative distance between the particles [33]; k is the corresponding momentum; $f_{\rm e}$ and $f_{\rm o}$ are, respectively, the even- and odd-channel scattering amplitudes. They enjoy the following effective range expansions as $k \to 0$ [34]:

$$\frac{1}{f_{\rm e}} \simeq -1 - ika_{\rm e} - i\frac{r_{\rm e}k^3}{2}, \qquad \frac{1}{f_{\rm o}} \simeq -1 - i\frac{a_{\rm o}}{k} - i\frac{r_{\rm o}k}{2}, \quad (4)$$

with the scattering length $a_{\rm e/o}$ and the effective range $r_{\rm e/o}$. By comparing the Bethe ansatz solution with Eqs. (3) and (4), we extract parameters for fermion-dimer (FD) and dimer-dimer (DD) scattering [22]:

$$\text{FD:}\; a_{\rm e}^{\rm FD} = r_{\rm o}^{\rm FD}/2 = 3r_{\rm D}; \quad r_{\rm e}^{\rm FD} = 0; \quad a_{\rm o}^{\rm FD} = 0; \qquad (5)$$

$$\text{DD:}\; a_{\rm e}^{\rm DD} = r_{\rm D}; \qquad r_{\rm e}^{\rm DD} = 0; \qquad f_{\rm o} = 0. \qquad (6)$$

As the dimers obey bosonic statistics, their scattering occurs solely in the even channel. By contrast, the fermion-dimer system can interact in both channels. In the strongly interacting limit $r_{\rm D} \to 0$, the dimers fully reflect and become impenetrable. The fermion-dimer scattering becomes fully transparent, yet with a π phase shift (the signs of the transmitted and incoming waves are different). The parameters for dimer-dimer scattering are known [35], and from now on we mainly focus on the fermion-dimer scattering to establish w for Eq. (2).

In a homogeneous setting, scattering in odd and even channels can be treated independently, because parity is conserved. This is also the case for external potentials that change weakly on the length scale given by $r_{\rm D}$. In light of this, we introduce parity-conserving effective interactions that reproduce the calculated effective-range parameters. For the even channels, we employ the δ function

$$V_{\rm e}^{\rm FD} = \frac{g}{2}\delta(x_{\rm rel}) \Leftrightarrow \partial_{x_{\rm rel}}\Psi|_{x_{\rm rel}=0^-}^{x_{\rm rel}=0^+} = \frac{g\mu_{\rm FD}}{\hbar^2}\Psi(0),$$
$$V_{\rm e}^{\rm DD} = g\delta(x_{\rm rel}) \Leftrightarrow \partial_{x_{\rm rel}}\Psi|_{x_{\rm rel}=0^-}^{x_{\rm rel}=0^+} = \frac{2g\mu_{\rm DD}}{\hbar^2}\Psi(0), \qquad (7)$$

where the right-hand sides represent the interactions as boundary conditions; $\mu_{\rm FD} = 2m/3$ and $\mu_{\rm DD} = m/2$ are the relevant reduced masses. By inserting a piecewise-defined plane wave for $x_{\rm rel} < 0$ and $x_{\rm rel} > 0$, one can show that the effective interactions lead to Eq. (5). One remark is in order here: g is negative in our system, and naively Eq. (7) seem to suggest the presence of bound states in dimer-dimer and fermion-dimer systems. In reality, these bound states are not present and should be eliminated as we exemplify below. This omission of bound states is natural in our effective model, because these states contain momenta that are beyond our low-energy description; see also Ref. [35], where dimer-dimer bound states are discussed.

Compared to the rather simple form of Eq. (7), the odd-channel interaction of the fermion-dimer system has a more complicated structure. Indeed, scattering in this case is dominated by the nonvanishing effective range $r_{\rm o}^{\rm FD}$ [36], beyond the standard odd-channel zero-range potentials [38,39]. Arguably, the simplest finite-range potential [40] to model the odd-channel FD scattering is an attractive square well whose strength is $-V_0$ if $|x_{\rm rel}| < r_{\rm o}^{\rm FD}$ and zero otherwise. The parameter $V_0 = \pi^2\hbar^2/[8\mu_{\rm FD}(r_{\rm o}^{\rm FD})^2]$ is tuned such that $a_{\rm o}^{\rm FD} = 0$ [22].

Note that the size of the well vanishes in the limit $\tilde{g} \to -\infty$ [$r_{\rm o}^{\rm FD} \simeq -6/\tilde{g}$; see Eq. (5)]; it cannot be resolved for low energies, $kr_{\rm o}^{\rm FD} \to 0$. This allows us to reframe the problem in terms of a boundary condition that fixes the wave function outside the well:

$$V_{\rm o}^{\rm FD} \Leftrightarrow \partial_{x_{\rm rel}}^2\Psi|_{x_{\rm rel}=0^-}^{x_{\rm rel}=0^+} = \frac{g\mu_{\rm FD}}{\hbar^2}\partial_{x_{\rm rel}}\Psi(x_{\rm rel}=0). \qquad (8)$$

Although this potential has vanishing range by construction, which is at odds with the Wigner lower limit [42], it leads to the correct energies of the fermion-dimer system in the limit $1/g \to 0$ as we illustrate below [22]. Additionally, this boundary condition will allow us to map the effective theory onto a weakly interacting system [43]. To understand the limits of validity of our effective theory formulation, we consider a three-body problem below. An analysis of a four-body problem yields similar conclusions [22].

Three fermions on a ring—First, we consider three fermions on a ring of length L. This system is one of the simplest instances of the Gaudin-Yang (GY) model [19,20], providing us with the exact results for benchmarking the performance of $V_{\rm e}^{\rm FD}$ and $V_{\rm o}^{\rm FD}$. For simplicity, we work with vanishing total momentum, $P = 0$, here [22].

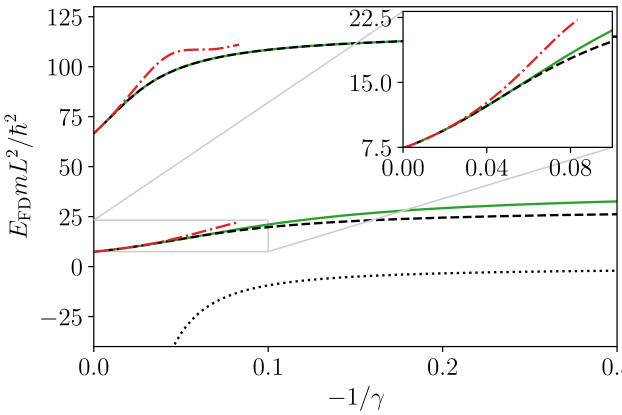

FIG. 1. Energy of the effective fermion-dimer system on a ring. Green solid curves show the corresponding energies of the GY model. Red dash-dotted curves are for a square-well potential. For $\gamma = -12$, the size of this potential approaches the length of the ring causing strong finite-size effects. Black dashed curves show the energy within the effective theory [solution of Eq. (9)]. The lowest (dotted) curve is an artifact of the effective model and should be omitted. The inset presents the region of strong interactions for the ground state.

Using the boundary conditions (7) and (8), we find the equation for the energies of the fermion-dimer system [22]:

$$3kL\ \tan(kL/2) = \gamma, \tag{9}$$

where $\gamma = \tilde{g}L$ is the dimensionless interaction strength and k is the relative momentum. We plot the effective fermion-dimer energy $E_{\rm FD} = \hbar^2 k^2/2\mu_{\rm FD}$ in Fig. 1 together with the exact three-body energy calculated using the GY model, according to the prescription $E_{\rm FD} = E^{\rm GY}_{\text{3-body}} + B_{\rm D}$. Note that Eq. (9) is applicable for both even and odd channels. The underlying physical reason for the degeneracy of the energy levels is the equivalence of the left and right directions in a ring geometry. We obtain another solution if we change the sign of momenta in the Bethe ansatz wave function [22].

Disregarding the deep bound state, which does not exist in the GY model and should be omitted, Fig. 1 demonstrates that Eq. (9) describes the exact energies for $-1/\gamma \lesssim 0.04$ well (the largest relative deviation in this region is less than one percent [45]). In Fig. 1, we also present the energies for an attractive square well in place of the fermion-dimer potential. For large values of $|\gamma|$, the corresponding energy spectrum is described by Eq. (9). For $-1/\gamma \simeq 0.1$, the size of the square well becomes comparable to the length of the ring, and the finite-size effects cannot be neglected [22].

Three fermions in a harmonic trap—The fact that the boundary conditions (8) work well in a ring might appear natural, because we used a homogeneous system to build them. Therefore, as a next illustration, we work with three particles in a harmonic trap, i.e., with $v_{\rm ext}(x) = m\omega^2 x^2/2$. This potential sets the length scale $l = \sqrt{\hbar/(m\omega)}$, suggesting the following definition for a dimensionless interaction strength: $\gamma^{\rm HO} = g/(\hbar\omega l)$.

The solution of the fermion-dimer model in a harmonic oscillator potential is straightforward after one notices that the center-of-mass motion can be decoupled from the relative one (a general feature of harmonically interacting systems [47]). The energy of the system is

$$E^{\rm HO}_{\rm FD} = \hbar\omega(2\nu_j + n_{\rm COM} + 1), \tag{10}$$

where $n_{\rm COM}$ is an integer that determines the center-of-mass dynamics. The quantum numbers ν_j for the relative motion are found from the equations [22]

$$\frac{l_{\rm FD}}{l} = -\frac{\gamma^{\rm HO}\Gamma(-\nu_{\rm e})}{4\Gamma(-\tilde{\nu}_{\rm e})}, \qquad \frac{l_{\rm FD}}{l} = \frac{\gamma^{\rm HO}\Gamma(-\tilde{\nu}_{\rm o})}{(4\nu_{\rm o}+1)\Gamma(-\nu_{\rm o})}, \tag{11}$$

where $\tilde{\nu}_j = \nu_j - 1/2$ and $l_{\rm FD} = \sqrt{\hbar/(\mu_{\rm FD}\omega)}$. The trap breaks translational invariance of the problem and, thus, lifts the degeneracy between even and odd solutions [48], which was present in the GY model; see Eq. (9).

To benchmark Eq. (11), we diagonalize the Hamiltonian in Eq. (1) for a three-body system numerically. In spite of a small number of particles, diagonalization of H is virtually impossible using standard methods already for intermediate interaction strengths (e.g., $\gamma^{\rm HO} \simeq -5$) [52,53]. Therefore, we resort to a transcorrelated method where the leading-order singularity due to the boundary condition is removed via a similarity transformation [54,55]. The computations were performed using the open-source package Rimu.jl [22,56]. In Fig. 2, we compare the energy $E^{\rm TCM}_{\text{3-body}} + B_{\rm D}$ with the prediction of Eq. (10). The overall agreement for

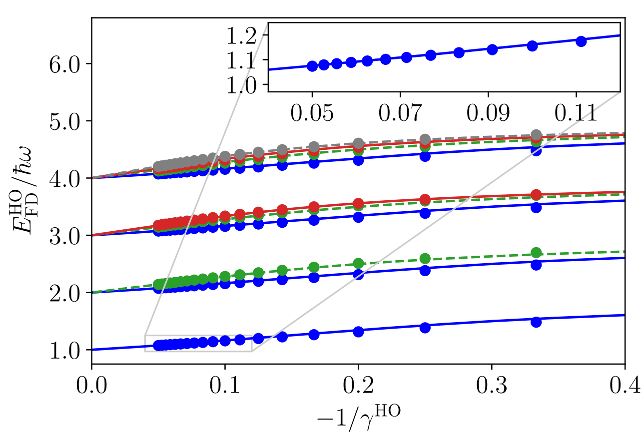

FIG. 2. Energy of the effective fermion-dimer system in a harmonic trap. Results of Eq. (11) are shown for odd (solid curves) and even states (dashed curves). Markers present outcomes of the numerical transcorrelated method. Curves with identical colors represent states with the same relative motion. The inset presents the region of strong interactions for the ground state.

the considered values of $\gamma^{\rm HO}$ for the ground as well as low-lying excited states provides a further validation of the proposed effective theory. Note that the energy spectrum for $1/\gamma^{\rm HO} = 0$ coincides with that of two noninteracting particles; the physical reason for that is explained below.

Many-body problem in a trap—The discussion above establishes the Hamiltonian from Eq. (2) as a viable framework for studying strongly attractive 1D fermions. Further progress can be made by mapping Eq. (2) onto a weakly interacting model. To implement this mapping, we rely on the boundary conditions from Eqs. (7) and (8). Let us assume that we have access to an eigenfunction Ψ of h for a given external potential. This function is defined on orderings of effective degrees of freedom, e.g., $y_1 < y_2 < \cdots < z_M$. Now, for a fixed ordering of fermions $z_1 < z_2 < \cdots < z_M$, we construct a new function $\phi = (-1)^{{\rm sgn}(P)}\Psi$, where ${\rm sgn}(P)$ is the parity of the permutation P in a set of coordinates of dimers and fermions. For example, $\phi = \Psi$ for the ordering $y_1 < y_2 < \cdots < z_M$, whereas $\phi = -\Psi$ for the ordering where the first two dimers are exchanged, i.e., $y_2 < y_1 < \cdots < z_M$. The function ϕ now can be trivially extended to any ordering of $\{z_1, \ldots, z_M\}$ using fermionic symmetry [57].

Since we use boundary conditions to define the effective interactions w, the function ϕ is (by construction) an eigenstate of the operator $h - w$ from Eq. (2) for every ordering of particles just as Ψ. However, boundary conditions for ϕ differ from Eqs. (7) and (8). For the odd channel, they are constructed following Refs. [38,39] from the rules for even-channel scattering in the original picture. For the dimer-dimer and fermion-dimer scattering, we have

$$\phi|^{x_{\rm rel}=0^+}_{x_{\rm rel}=0^-} = \frac{2\hbar^2}{\mu_{\rm DD} g}\partial_{x_{\rm rel}}\phi(x_{\rm rel}=0),$$
$$\phi|^{x_{\rm rel}=0^+}_{x_{\rm rel}=0^-} = \frac{4\hbar^2}{\mu_{\rm FD} g}\partial_{x_{\rm rel}}\phi(x_{\rm rel}=0). \quad (12)$$

For the even-channel fermion-dimer scattering, instead of Eq. (8), the following condition must be satisfied:

$$\partial_{x_{\rm rel}}\phi|^{x_{\rm rel}=0^+}_{x_{\rm rel}=0^-} = \frac{4\hbar^2}{\mu_{\rm FD} g}\partial^2_{x_{\rm rel}}\phi(x_{\rm rel}=0). \quad (13)$$

In the limit $g \to -\infty$, the wave function features no peculiarities and corresponds to a noninteracting mixture of two mass-imbalanced Fermi gases [58].

One important advantage of working with a weakly interacting system is the opportunity to use perturbation theory. The ground state of h is nondegenerate; its energy in the $1/g$th order reads

$$E_N = E^{\rm F}_M + E^{\rm F}_{N_\downarrow} + \langle\phi|V_{\rm FD} + V_{\rm DD}|\phi\rangle, \quad (14)$$

where the first two terms on the right-hand side are the noninteracting energies for M unpaired fermions and $N_\downarrow$ dimers. The last term represents the first-order correction due to the interaction. It can be easily calculated, as it represents a sum of two-body corrections [22].

To illustrate this, we consider a problem of a single impurity in a Fermi gas (i.e., $N_\downarrow = 1$) confined in a harmonic trap. For repulsive interactions, this problem was extensively studied numerically [60–63] and realized experimentally in few-atom systems [64]. In the limit $g \to -\infty$, this problem transforms into another impurity problem: a dimer interacting with $M = N_\uparrow - 1$ fermions. The corresponding ground-state energy is given by Eq. (14) with $\langle\phi|V_{\rm DD}|\phi\rangle = 0$ and $\langle\phi|V_{\rm FD}|\phi\rangle = \sum_{n=0}^{M}\Delta E_2(n)$, where $\Delta E_2(n)$ is the energy shift due the interaction in a two-body problem assuming that the noninteracting system has a fermion in the nth one-body energy level of the trap [22]. Figure 3 shows this perturbative result [22]. We rescale the energy in the units of the Fermi energy $E_{\rm F}$ for a faithful comparison of systems with different values of M. The dimensionless interaction strength is given by $\gamma_{\rm scaled} = \gamma^{\rm HO}\pi/\sqrt{2M}$.

The energies in Fig. 3 quickly approach the thermodynamic limit (i.e., $M \to \infty$ with fixed $E_{\rm F}$), where the problem is often called the Fermi polaron [3]. Our framework allows one to calculate this approach and together with the known results for repulsive interactions [64,65] provides an intuitive picture of a few-to-many-body crossover in strongly interacting 1D systems with impurities. Note that we can study this crossover with a handful fermions. Indeed, for $M \gtrsim 5$, we calculate $\lim_{|\gamma_{\rm scaled}|\to\infty}|\gamma_{\rm scaled}|\Delta E_M/E_{\rm F} \simeq 2.6$. This equation determines the energy as well as the universal tail of the

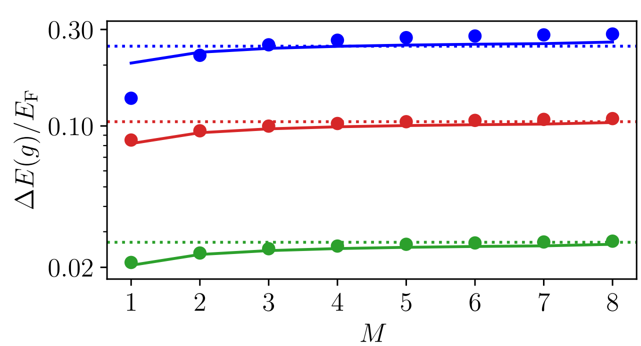

FIG. 3. Energy of the dimer confined within a harmonic trap, $\Delta E(g) = E(g) - E(g \to -\infty)$, as a function of the number of unpaired fermions $M = N_\uparrow - 1$ for different interaction strengths $\gamma_{\rm scaled} \in \{-10, -25, -100\}$ (blue, red, and green, respectively). Symbols show numerical results, while solid lines correspond to a perturbative calculation, both within effective theory; see the text for details. Note that we use the Fermi energy $E_{\rm F} = (M + 1/2)\hbar\omega$ as a unit of energy here; $\gamma_{\rm scaled} = \gamma^{\rm HO}\pi/\sqrt{2M}$. Dotted lines are the many-body energies calculated using the Bethe ansatz; see Ref. [22] for details.

momentum distribution (contact parameter) [66,67]. For $M \to \infty$, the role of the trap is negligible, and we derive from the Bethe ansatz [22,68]: $\lim_{|\gamma_{\text{scaled}}|\to\infty} |\gamma_{\text{scaled}}| \Delta E_M / E_F = 8/3$, in agreement with our few-body result.

To illustrate the finite-range effects on the energies, Fig. 3 also presents results of a direct numerical diagonalization of the effective model in Eq. (2) with Eq. (7) and a square-well potential in place of an odd-channel fermion-dimer interaction [22]. This energy agrees with the perturbative result of Eq. (14) for $\gamma_{\text{scaled}} = -25$ and $\gamma_{\text{scaled}} = -100$. The deviation between the two methods at $\gamma_{\text{scaled}} = -10$ signals a departure from the leading order in $1/g$. We observed that this deviation becomes less pronounced as the number of particles increases. Our interpretation is that, by fixing E_F, we effectively increase the size of the trap when increasing M. This reduces the finite-size effects, which dominate physics for the corresponding values of g; see Fig. 1.

Summary—We studied a two-component fermionic system with strong attractive particle-particle interactions. We showed that this model can be mapped onto a weakly interacting mass-imbalanced fermionic mixture, which can be analyzed using many-body perturbation theory. Our results rely on the separation of scales, $B_D/(\text{kinetic energy}) \gg 1$, making them applicable to dilute samples at zero or low temperatures. This condition is also satisfied for low-energy collective excitations, so our effective model can be directly used to study real-time dynamics of strongly attractive particles. The proposed framework could be extended to other systems, in particular, SU(N) fermionic mixtures that can be realized in a cold-atom laboratory [69]. These systems are Bethe ansatz solvable in the homogeneous case. Hence, we conjecture that, in the limit of strong attractive interactions, such systems can be mapped onto weakly interacting mass-imbalanced models whose constituents are the available bound states. For SU(3), e.g., the constituents would be trimers, dimers, and fermions [70].

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Data availability—The data that support the findings of this Letter are openly available [75]; embargo periods may apply.

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[12] We define r_D as the length scale that contains most of the probability to find two fermions, forming a bound dimer, close to each other, i.e., $\int_{-r_D}^{r_D} \Phi_D(z)^2 \, dz = 1 - e^{-1} \approx 0.63$, where $\Phi_D \propto \exp\{-|\tilde{g}z|/2\}$ is the wave function of the dimer.
[13] More precisely, a strongly attractive regime in our Letter implies that the size of the dimer, and, hence, the effective range is unresolved in scattering, i.e., $kr_D \ll 1$ for all relevant values of the momentum k.
[14] Other possibilities appear impossible because identical fermions cannot form bound states with more than two particles. This follows from few-body calculations [15,16] or distribution of momenta in the GY model [17,18]. Although these considerations are based on a homogeneous geometry, the conclusion holds also for the external potentials v_{ext} that change slowly on the length scales given by the size of the dimer state.
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[22] See Supplemental Material at http://link.aps.org/supplemental/10.1103/8mnc-x42q for additional information and results on the following topics: Bethe ansatz solution, boundary conditions as limit of a square-well potential, solution of our effective theory in different confinements, the transcorrelated methods, many-body calculations, and results. References in Supplemental Material: Refs. [23–32].

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[43] We remark here that the boundary conditions introduced in Eq. (8) do not correspond to the odd-channel interaction δ' introduced in Ref. [38]. Indeed, the potential δ' connects the derivative of the wave function to the discontinuity of the wave function itself, whereas Eq. (8) connects the derivative of the wave function to the discontinuity of the second derivative of the wave function. It is worth noting, however, that in the limit $1/g = 0$ both Eq. (8) and δ' demand that the derivative of the wave function vanishes, which leads to the important conclusion that strongly interacting odd channel can be mapped onto a weakly interacting even channel [44].

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