Do games help the learning of probability?

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Timothy Angus Burgess

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Abstract

The value of using games to assist students' learning of probability concepts was investigated, primarily using qualitative methods. Games, although generally useful in mathematics for helping children learn, may not automatically be as useful in helping students develop normative probability concepts, particularly because of the nature of randomness.

Sixteen students (Years 7 and 8) participated in the study. The students initially completed a written questionnaire which was designed to explore their understanding of probability. The students' misconceptions were categorized according to various types of probability reasoning. Two games were played by the students in groups of four, over two successive days; a game of chance on the first day, and a game of strategy and chance on the second. The game sessions, each lasting about 45 minutes, were audio-taped and video-taped. Group interviews were conducted during and following the playing of the games.

The study found differences between students in their levels of involvement in and discussion about the games, and differences between the two types of games in the degree of interaction within groups, all of which influenced the games' effectiveness in developing probability learning by the students. There was evidence of inconsistencies in some students' probability reasoning and understanding. When the empirical results from the games conflicted with their ideas, the students were not necessarily aware of such conflict so consequently did not adjust their thinking.

For the use of such games to maximize the opportunities for the students' learning of probability, some implications of the study for classroom teachers are suggested. Consequently the role of and knowledge and understanding of the teacher is critical to ensure effective learning of probability concepts by students.

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Chapter 1 Introduction

1.1 Statistics and the school curriculum

Throughout the world, there has been a growing emphasis on Statistics within mathematics education, with New Zealand leading the way to some extent, in relation to the coverage of statistics at the primary school level (Begg, 1995). This is a reflection of the increasing importance of statistics in our technological society, where the use of data in most areas of life, such as business, politics, sports, and medical testing and diagnosis, is critical. The importance of students developing an understanding of statistics and probability is recognized in the national curriculum statement for Mathematics, *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992) (hereafter referred to as *MINZC*), and is encapsulated in a general aim for mathematics education, being "to help students to achieve the mathematical and statistical literacy needed in a society which is technologically oriented and information rich" (p.8). Similarly, in the US, the National Council of the Teachers of Mathematics claim that: "A thorough grounding in statistics and probability provides tools and ways of thinking that will be useful throughout students' lives" (National Council of the Teachers of Mathematics, 1998, p.71).

In the world as it currently is, and will be in the future, concepts involving chance are pervasive. For instance, gambling, insurance, the weather, stocks and shares, and politics all have a somewhat unpredictable nature. Thus an understanding of chance and randomness is desirable so that we can make the best of the probabilistic decisions that face us in everyday situations.

1.2 Probability in the school curriculum

In different parts of the world, different terms are used in mathematics education for the part of the school curriculum which covers concepts of probability. The term 'stochastics' is used in some countries, and it refers to both probability and statistics; others refer to 'chance and data'; and in New Zealand, 'statistics' includes probability concepts.

Although probability concepts are an integral part of life, they have not always been included in the school curriculum at the primary school level. This is changing however, to a greater extent in some countries than others. Prior to the 1992 national curriculum statement (*MINZC*) being published in New Zealand, probability concepts were first encountered in the school syllabus at the year 7 and 8 level. It was dealt with only briefly and at a superficial level, through the use of random generating devices such as coins and spinners, and moved quickly into developing a formal interpretation of probability. *MINZC* has raised the profile of probability in the school curriculum, through a broad achievement aim of the Statistics 'strand': "The mathematics curriculum intended by this statement will provide opportunities for students to develop the ability to estimate probabilities and to use probabilities for prediction" (Ministry of Education, p. 169). Consequently, probability is introduced from the earliest school level at year 1, and concentrates initially on developing an understanding of probability through language, specifically the understanding and use of subjective terms such as certain, possible and impossible. Similarly, in US, the National Council of Teachers of Mathematics recommendations include probability to be dealt with from the earliest years at school: Zawojewski (1991, p. 56) contends that "exploring data and chance in school should be as natural as it is in everyday life".

1.3 Some challenges for the teaching of probability

Because there is the trend to introduce probability learning at earlier levels than has happened in the past, teachers need ways to help the learning of probability be as effective as possible. However, it is readily acknowledged that many of the ideas behind probability are "difficult to learn and therefore hard to teach" (Ahlgren & Garfield, 1991, p. 112). The National Council of Teachers of Mathematics (NCTM) in the US recognizes some of the difficulties that learners have with probability compared with other areas of mathematics, particularly in regard to the understanding that learners acquire prior to formal learning:

> *Because some things children learn in school seem to them pre-determined and rule-bound, it is critical that they also learn that some problems involve solutions that depend on assumptions and have some degree of uncertainty. The kind of reasoning used in probability and statistics is not always intuitive, and so it will not be developed in young children if it is not included in the curriculum. Students will benefit from the ability to deal intelligently with variation and uncertainty.*
>
> (National Council of Teachers of Mathematics, 1998, p.71)

The intuitive ideas of probability that students develop are known to be firmly held, and particularly resistant to change (Hope & Kelly, 1983; Konold, 1995). This is problematic because probability concepts are often counter-intuitive. In other areas of mathematics, the use of equipment, teaching experiments, and practical activities can be used to demonstrate various concepts. From the specific examples and results obtained, students are encouraged to look for patterns and generalize, thereby developing their mathematical understanding. This process of learning through induction may not be as relevant and useful in probability, because of the nature of randomness and random events. If this is the case, the teacher would need alternative strategies to help students learn probability concepts.

Constructivist ideas about learning are important to consider, given the widespread acceptance by mathematics educators of constructivism. The factors that affect learning and the strategies that enhance it also need to be considered, since it is well-known by teachers that students do not necessarily learn what was intended by the teacher. How teachers explore the ideas that students have, and progress from there, is critical in the learning process.

1.4 Contexts for the learning of mathematics

Another trend in mathematics education worldwide has been to encourage the use of contexts in learning, in order to make the mathematics more relevant and meaningful to the students. Some examples are:

from New Zealand

> *Students learn mathematical thinking most effectively through applying concepts and skills in interesting and realistic contexts which are personally meaningful to them. Thus, mathematics is best taught by helping students to solve problems drawn from their own experience.*
>
> (Ministry of Education, 1992, p. 11)

from US

> *We should begin in a world of experience and contexts that are relevant and familiar to our students.*
>
> (Zawojewski, 1991, p.56)

from England and Wales

> *The application of mathematics in contexts which have relevance and interest is an important means of developing pupil's understanding and appreciation of the subject and of those contexts... Mathematics is a powerful tool with great relevance to the real world. For this to be appreciated by pupils, they must have direct experience of using mathematics in a wide range of contexts throughout the curriculum.*
>
> (Department of Education, 1991, p. F1)

The aim of using contexts and drawing on the students' experiences and understanding is to link with and thereby enhance the students' learning of mathematics. It is known however, that some experiences contribute to the development of 'inadequate' types probability reasoning, such as the 'availability heuristic' (Kahneman & Tversky, 1972). This reasoning, and numerous other types of 'inadequate' reasoning that learners develop, are referred to as 'misconceptions', to differentiate them from the commonly accepted correct, or normative concepts. The extensive research on probability 'misconceptions' is generally more relevant to the tertiary level than the school level (and in particular the primary school level). So embedding the learning of probability in contexts that students can identify with has the aim of overcoming some of these misconceptions.

1.5 The use of games for learning?

Games are commonly used classroom mathematics activities in the primary school. Numerous resources for teachers give details of games which can be used in the classroom, for a variety of purposes. Many games are for practising and consolidating skills, but of interest in this study are those which help students learn new concepts. Since games involving chance are a common part of many cultures and therefore part of the real-life experiences of many children, it would seem sensible to use games as a context for learning mathematics, and in particular probability, in the classroom. There is, however, a limited amount of research available on the use of games as they relate to the learning of probability concepts for the younger students. It is known that there are problems with the use of contexts interfering with mathematics learning, and that often there is a lack of transfer of mathematics skills from the classroom to situations outside of the classroom (Ahlgren et al, 1991). Also, the intuitive knowledge that students bring to the classroom can advantage the learning process, but unfortunately, it can also impede the learning (Borovcnik & Bentz, 1991), and this appears to be especially pronounced in probability learning, because of the often strong intuitions that students develop prior to any formal learning in the classroom (Fischbein, 1987). So although games have a use in the classroom, there is also a question about their value in some areas of learning.

1.6 Research Questions

Since there are aspects of probability that set it apart from other areas of mathematics learning, the choice of teaching strategies that suit the learning of probability need to be investigated. Teachers need to have available instructional strategies that will overcome the difficulties that students have with the learning of probability.

This research focuses on the use of games for helping students learn probability concepts and whether they provide an effective strategy for this.

Specifically, this research was designed to investigate the following questions:

1. Does the use of games encourage students' thinking in relation to specific probability concepts as well as any misconceptions that the students may have?

2. Is there a difference between the use of a game involving only chance compared with a game involving chance and strategy on students' thinking about probability, and on the interaction between students while they are playing the games?
3. Are there particular teaching strategies that may enhance the use of games for learning probability concepts?

1.7 Overview

The next chapter examines the research literature, beginning with ideas on the theoretical basis of how children learn, including the effect on learning of intuition and intuitively-held concepts. These are discussed in relation to both mathematics learning generally, and probability learning in particular, before looking in detail at the common misconceptions that exist in probability. The history of games is briefly looked at, and their use in the classroom is examined, including the role of the teacher. Chapter 3 outlines the research design of this study. This is followed by a discussion of the results in Chapter 4. Chapter 5 draws together the conclusions from the results, and gives suggestions and recommendations for future studies.

Chapter 2 Literature Review

2.1 Constructivism

Introduction

How do children learn? What are the explanations for the commonly occurring situations where some children do learn while others do not, and others learn something quite different from what the teacher intended? There are many theories of learning, and therefore a variety of explanations for what is happening or not happening in these situations.

It has, in the past, been commonly accepted that knowledge could be transmitted from the mind of an expert (the teacher) to the mind of the learner. But this model of learning could not adequately explain why learners either did not learn, or learned the 'wrong' things. Orton (1992), when discussing this transmission model of learning, indicates that teachers know very well that the knowledge received by the learner is not always an exact copy of what was transmitted by the teacher.

Definition of constructivism

Currently, it is accepted by many mathematics educators that constructivism, as a theory of learning (Fosnot, 1996), best explains how children learn. Although there are many differing forms of constructivism, Noddings (1990, p.10) lists the features of constructivism that she believes would be generally agreed upon:

1. *All knowledge is constructed. Mathematical knowledge is constructed, at least in part, through a process of reflective abstraction.*
2. *There exist cognitive structures that are activated in the processes of construction. These structures account for the construction; that is, they explain the result of cognitive activity in roughly the way a computer program accounts for the output of a computer.*
3. *Cognitive structures are under continual development. Purposive activity induces transformation of existing structures. The environment presses the organism to adapt.*
4. *Acknowledgment of constructivism as a cognitive position leads to the adoption of methodological constructivism.*
   - *a) Methodological constructivism in research develops methods of study consonant with the assumption of cognitive constructivism.*
   - *b) Pedagogical constructivism suggests methods of teaching consonant with cognitive constructivism.*

Wood, Cobb, and Yackel (1990) describe this further in terms of learners being "active participants who construct knowledge by reorganizing their current ways of knowing".

As such, learning is a personal construction that is unique to the individual (Noddings, 1990), a product of the learner's own cognitive acts:

> *We construct our understanding through our experiences and the character of our experience is influenced profoundly by our cognitive lenses.*
>
> (Confrey, 1990, p.108)

Social Effects on Learning

Although construction of knowledge is personal to the learner, the influence of others on the learning process is being increasingly acknowledged (for example, Cobb, Wood, & Yackel, 1990; Fosnot, 1996; Leder, 1993; von Glasersfeld, 1995). The interactions with others may be with peers, the teacher, or both. The theory of social constructivism has developed in recognition that there is socially constructed knowledge "created and constrained by the shared experience of the underlying physical reality" (Ernest, 1995, p.480). Vygotsky, in particular, has had an influential part to play in emphasizing the importance of interaction on learning (Driver, 1995). Through the process of communication, "ideas are accepted as truth insofar as they make sense to the community and thus rise to the level of taken-as-shared" (Fosnot, 1996, p.30).

Interaction with peers

The influence of peers on children's learning has a noticeable effect on the shaping and challenging of the constructions of the individuals in the group (Leder, 1993; von Glasersfeld, 1995).

> *What we see others do, and what we hear them say, inevitably affects what we do and say ourselves. More importantly, it reflects upon our thinking. If one takes seriously the idea that the others we experience are the others we construct, it follows that whenever they prove incompatible with our model of them, this generates a perturbation of the ideas we used to build up the model. These ideas are our ideas, and when they are perturbed by constraints, we may be driven to an accommodation. Socially oriented constructivists speak of "the negotiation of meaning and knowledge". This is an apt description of the procedure because, as a rule, it takes a sequence of small reciprocal accommodations to establish a modicum of compatibility.*
>
> (von Glasersfeld, 1995, p.191)

The growing acceptance of social constructivism suggests an increasing "pedagogical emphasis on discussion, collaboration, negotiation, and shared meanings" (Ernest, 1995, p.485). The important aspect of the social interaction is that the individual's learning benefits through the collaboration and negotiation. Fosnot (1996) encourages dialogue within the mathematics community to help enhance learning. The use of small groups of children working together encourages 'powerful' constructions as opposed to 'weak' constructions (Noddings, 1990). Driver (1995), describing the interactions of a small group of pupils working in a science context, argues that "pupils, if motivated and given the opportunity, can bring ideas and prior experiences together to take their thinking forward" (p.394).

The use of language and social interaction is central to individual knowledge construction (Jaworski, 1994). Discussion provides a forum for individuals to clarify their own ideas as well as to build on each other's ideas: previously implicit ideas can be made "explicit and available for reflection and checking" (Driver, 1995, p.394). It may even happen that understanding increases even though the group discussion may not reflect progress: "progress in understanding is brought about not so much through the scaffolding offered by other children's ideas as the opportunity for each individual to reorganize his or her own ideas through talk and listening" (p.395). In a similar way, Wood, Cobb, and Yackel (1995) contend that through the interaction and shared understandings that arise as a consequence, it is possible for children to develop understandings that are more sophisticated than the original individual ideas. Often the sharing of ideas challenges individual constructions and leads to "disequilibrium" and subsequent modification of previously accepted ideas (Leder, 1993).

Another strategy for eliminating misconceptions through 'conflict-discussion' involves small groups discussing an individual's response to a question, followed by putting group conclusions to the whole class. The assumption is that the group work helps ensure that pupils' wrong ideas are brought out, expressed, and then subjected to challenge and criticism in an "unthreatening situation" (Orton, 1992). The concept of disequilibrium will be further discussed in the section dealing specifically with the learning of probability concepts, and the common misconceptions that learners develop.

Role of the teacher

The importance of the teacher in the process of learning is widely acknowledged. Teachers, within a constructivist framework, need to operate in interactive lessons:

> *Teachers both model and elicit, but they model by asking questions, following leads, and conjecturing rather than presenting faultless products. Teaching in this way requires considerable mathematical knowledge as well as pedagogical skill.*
>
> (Noddings, 1990, p.17)

Confrey (1990, p.108) refers to the facilitative role of the teacher:

> *The teacher must form an adequate model of the student's ways of viewing an idea and s/he then must assist the student in restructuring those views to be more adequate from the student's and from the teacher's perspective.*

However, since learning is taken to be an individual construction, a difficulty arises when considering the appropriate role of the teacher – how does the teacher gain an understanding of the constructions of the individuals? Cobb, Wood and Yackel (1990, p. 128) suggest that it is necessary for the teacher to

> *construct models of her students' mathematical understandings as she interacts with them...She would then use these models to generate conjectures about the*

children's potential mathematical constructions and, on this basis, select instructional activities and interact with them in ways that might give rise to opportunities to construct mathematical knowledge....She would test and when necessary, revise her interpretations of children's mathematical understandings.

So students are not the only ones who are constructing knowledge. Teachers are also constructing and modifying their own interpretations of children's understanding (Jaworski, 1994). However, the teacher must be aware of the need to examine the children's solutions from the perspective of the children, and from the position that the children are currently at (Wood et al, 1990), rather than from an adult view. "Students perceive their environment in ways that may be very different from those intended by the educators" (von Glasersfeld, 1996, p.7). In order to gain an insight into the understanding of the children the teacher must "prompt students to justify what they say or do and thus reveal their thinking and logic. In order to expose different levels understanding, tasks need to be used which allow for varying levels of response" (Pirie & Kieren , 1992, p.509).

In considering the position that children are currently at, it is recognized that learning has already been constructed. These prior experiences and prior knowledge therefore affect the future learning. Noddings (1990, p.9) asserts that:

active construction implies both a base structure on which to construct (a structure of assimilation) and a process of transformation or creation which is the construction. It implies, also, a process of continual revision of structure (a process of accommodation).

Because learning is based on prior experiences, what the learner learns (constructs) might not be what the teacher intended (Noddings, 1990). Sometimes this is due to the gap between the student's existing (maybe informal) knowledge and formal instruction (Baroody & Ginsburg, 1990; Pirie & Kieren, 1992). Since prior experience and existing knowledge plays an important part in any new learning, the next section discusses intuition and its place in the learning process in relation to subsequent learning, and in particular its place in the learning of probability concepts.

2.2 Intuition and the place of inductive learning

Definition and features of intuition

The role of intuition in the learning of mathematics is very important. Learners intuitively develop many concepts through their experiences in the world, prior to, as well as independently of, formal instruction. Intuition can be defined as "the direct perception of truths, facts etc., independently of any reasoning process; pure, untaught, non-inferential knowledge" (Macquarie dictionary, 2nd revised edition, 1988).

Fischbein (1987) equates 'intuition' with 'intuitive knowledge' and suggests that these terms can be and are used interchangeably. He lists the defining characteristics of intuition as follows:

- self-evident, self-justifiable, self-explanatory;
- intrinsic certainty – which determines the robustness of intuitions;
- perseverance;
- coerciveness – which contributes to the perpetuation of incorrect interpretations, even when these have been disproved;
- has "theory" status;
- extrapolativeness – intuitions can exceed the available data, with a certainty, not just a guess;
- globality – which distinguishes intuitive from analytical thinking;
- implicitness – intuitions perceive situations as whole in an uncritical way, and resist analysis.

Fischbein also distinguishes two types of intuitions: primary and secondary. Primary intuitions are those which individuals develop independently of any systematic instruction as a result of their personal experience. Secondary intuitions are those that are created as a result of systematic, instructional influences. Fischbein claims that the intuitions which develop during the concrete operational stage, remain as stable acquisitions for the whole of one's life. Even though the development of formal capacities may result in precision of the intuitions, the primary intuitions remain basically the same.

Intuition and learning

Borovcnik and Bentz (1991, p.75) developed a model (see Figure 2-1) showing the link between intuitions and 'official' mathematics, and illustrating the difficulties that learners have with learning mathematics. They argue that conventional teaching establishes too few links between primary intuitions and the mathematical model. The barriers that can hinder the linking of learners' intuitions (both primary and secondary) with 'official mathematics', as far as probability learning is concerned, may be related to the lack of direct experiences which would help learners link their intuitions with the 'accepted' probability theory.

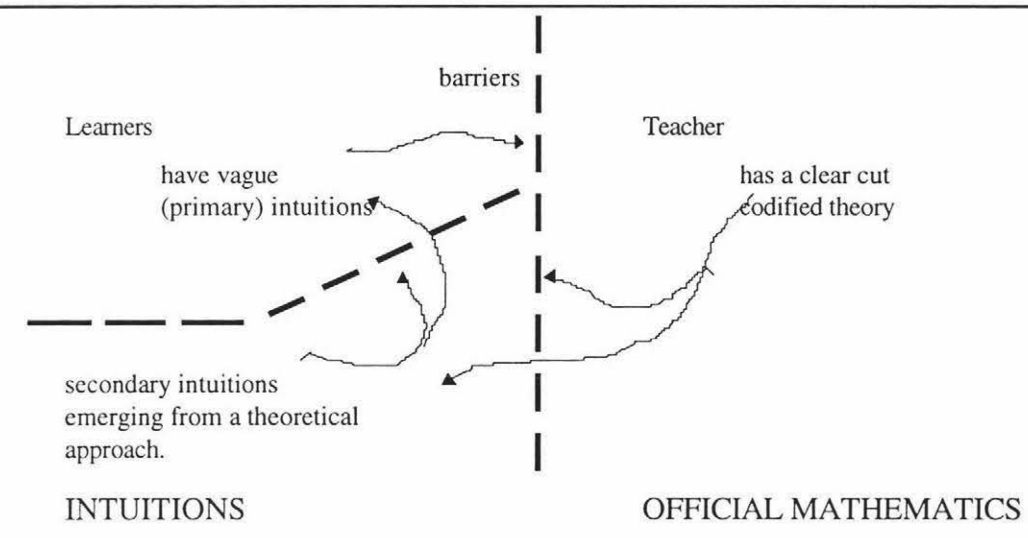

***Figure 2-1**: Sparse relations between intuitions and official mathematics*

If the learner is to understand mathematics and apply it to other situations, teaching has to start from the learner's intuitions and to change them gradually. The learner needs to see how mathematics will reconstruct his/her intuitions.

Conflicts can and do arise between intuitive interpretations and formal ones which have been acquired by instruction. "Such contradictory interpretations may co-exist. But very often the intuitive representation is stronger and tends to annihilate the formal conception" (Fischbein, 1987, p.205). Fischbein recommends that the student should be made aware of these mental conflicts (that is, have 'disequilibrium' induced) in order to strengthen the control of the taught conceptual structures over the primary intuitive ones. Likewise, Freudenthal (1973, p.606) acknowledges the difficulty in changing these understandings despite the learner's possible awareness of conflict:

> *Students recognize that their method was wrong again, but I am convinced that in their hearts, the majority do not believe what logic tells them – these are the reasons of the brain which the heart does not know (to invert a famous saying of Pascal).*

In probability learning, there are numerous examples of conflicts between intuitive and formal concepts, some of which will be examined in greater detail in the section on specific misconceptions in probability. Freudenthal (1973), when discussing the types of problems that arise in probability which are particularly difficult for learners to grasp, indicates that pure mathematics does not know of similar analogies; it appears that probability learning may be different from learning in other areas of mathematics. Borovcnik and Peard (1996, p.244) explain:

> *Trial and error is a basic type of learning which accompanies us from our earliest age. It is very popular when using computers; one tries some solution, gets immediate feedback, and tries again until one comes up with the solution. This trial and error is also applicable to some mathematical concepts, at least in the beginning stages. Starting from concrete operations upon a material representation of a concept, the individual gets the required feedback, leading to a revision of existing but vague primary ideas. For example, concrete counting leads to the structure of the natural numbers and establishes clear ideas about them. Counterintuitive results are not met until the student has reached high levels of abstraction... The same cannot be said about probability concepts.*
>
> *In other fields of mathematics, modeling usually involves producing a first representation of the real problem, getting more data as feedback of its fit, and then refining the model to make it fit better to reality. This implies that there is a comparably simple pattern to interpret feedback from reality. However the very basic concepts in probability reveal characteristics which defy such direct feedback. In a sense, probability is much more of a heuristic, a strategy, to explore reality in something like a phase space, yielding only possible scenarios of the real world.*

This difference between pure mathematics and probability is exemplified in the following situation with children:

> *Furthermore a check of success is easy with natural numbers where the calculation of 2 + 2 yields the immediately testable result of 4. Such feedback is complicated with probability; the extremely small probability of winning in a state lottery is countervailed by the fact that people win every week.*
>
> (Borovcnik & Bentz, 1991, p.74)

Borovcnik and Bentz (1991) suggest that such an example illustrates the gap between intuition and mathematical theory, given that stochastic reasoning has no empirical control to revise inadequate strategies. Thus, the nature of randomness and probability is difficult for students to grasp (Kapadia & Borovcnik, 1991), in contrast to much of the mathematics they learn using a deterministic approach. Borovcnik and Bentz (1991) suggest that our desire for deterministic explanations works against the development of an adequate understanding of randomness, which is inextricably linked to an understanding of probability. Steinbring (1991, p.334) also refers to the difficulty of understanding the concept of chance:

> *Chance may be seen as the only legitimate pattern of justification for the difference between theoretical predictions and empirical results. Chance no longer expresses a positive conceptual idea suitable for exploring, understanding, and solving a stochastic problem. It has degenerated into a substitute for justification, which serves to deny the importance of the difference between theory and empirical facts in probability.*

Chance is thus seen as the reason why something did not occur as predicted, rather it was just chance the way it happened.

Conflicts, paradoxes, and fallacies which occur in probability can, however, be useful for analyzing and helping to understand concepts better, for interpreting results more effectively, and to educate probability intuition and reasoning firmly (Borovcnik et al, 1991). With probability the paradoxes and counterintuitive ideas occur "at the very heart of the subject, in the definition, and subsequently in relatively simple applications" (Kapadia & Borovcnik, 1991, p.2), unlike geometry for example, where the difficulties and paradoxes occur at a rather deep level which need not influence school mathematics. In comparison with geometric or visual intuitions, probabilistic intuitions are generally quite poorly developed (Kapadia & Borovcnik, 1991). Borovcnik et al (1991, p.67) claim that:

> *Probabilistic intuitions are one of the poorest among our natural and developed senses. Perhaps this is a reflection of the desire for deterministic explanation. We have great difficulty in grasping the origins and effects of chance and randomness: we search for pattern and order even amongst chaos.*

Too often, the entrenched nature of intuitions makes them resistant to change through analysis (Hope & Kelly, 1983) and students' conceptions are difficult to alter (Konold, 1995). Teachers need to be aware of the intuitions that learners have, and then select and structure activities that will allow the learner to confront and evaluate their intuitions (Konold, 1991).

Simulations or experiments are commonly used in classrooms as activities to explore probability. It is important, however, to be aware of the differences between probability experiments and scientific experiments (Borovcnik & Bentz, 1991). Even though probabilistic experiments are replicable under the same conditions, as are experiments in science, they show very distinct features. Results of probabilistic experiments will still vary, even when the conditions are perfectly reproduced, unlike scientific experiments. This obviously creates a difficulty for the learning of probability concepts. When an experiment is to be used to illustrate certain concepts, the outcomes of the experiment may not occur as expected, in which case a conflict arises. How does the learner make sense of the result, when they do not know that the result may, in fact, be a very unlikely one, which does not illustrate the concept that the teacher was wanting to highlight? Understanding the concept of chance variation and being able to rationalize this against the empirical data is problematic for learners. The teacher must know how to capitalize on this situation, rather than ignore or excuse it. Students must be enabled to know when to use probabilistic thinking compared with deterministic thinking, and to be comfortable doing so (Pfannkuch & Brown, 1996).

A different source of conflict which may mediate against successful learning of probability relates to ideas of probability and randomness and the authoritative nature of school (Falk, Falk & Levin, 1980). Children develop the impression that uncertainty and ambiguity are not acceptable in school, which therefore hinders the development of an adequate understanding of probability. The learning environment in the classroom must be structured in such a way that this potential influence against probability is minimized.

Inductive learning

Inductive constructions have been explained by von Glasersfeld (1995, p.70) in terms of the assimilation of new experiences and the resulting restoration of equilibrium:

> *By empirical abstraction, sensory particulars that recur in a number of experiential situations are retained and coordinated to form more or less stable patterns. These patterns are considered viable insofar as they serve to assimilate new experiences in a way that maintains or restores equilibrium. This simple form of the principle of induction, namely 'to retain what has functioned successfully in the past', can be abstracted and turned upon itself: because the inductive procedure has been a successful one, it may be advantageous to generate situations in which it could be employed. Consequently, a thinking subject that has reached this point by reflective abstraction and, for the time being, is not under pressure to cope with an actual problem, can imaginatively create material and generate reflective abstractions from it that may become useful in some future situation. This may involve material actually found in experience or it may take the form of a thought experiment with imaginary material.*

Inductive learning, whereby generalization from specific examples and results can occur and be transferred to some other situation, is a powerful learning device. However, the 'extrapolativeness' feature of intuitions may make inductive learning less useful in the probability area. An intuition can be induced (or 'extrapolated') from too small an amount of data. Fischbein (1987, p.210) describes the development of anticipatory intuitions (conjectures associated with a feeling of total confidence) in relation to inductive processes:

> *Though mathematics is a deductive system of knowledge, the creative activity in mathematics is a constructive process in which inductive procedures, analogies and plausible guesses play a fundamental role. The effect is very often crystallized in anticipatory intuitions.*

Fischbein suggests that, rather than trying to eliminate intuitions which would be an impossible task, the problem is to develop new, adequate, intuitive interpretations together with developing the formal structures of reasoning.

> *This may be done especially through appropriate practical activities and not through mere verbal explanations. Intuitions are in our opinion by their function and their nature behaviorally, practically oriented.*
>
> (ibid, p.211)

2.3 Misconceptions in probability learning

Definitions

Some researchers argue against using the term 'misconception', as they believe it suggests a deficit view of learning which judges the constructions of the individual learner. Their preference, in line with a constructivist view of learning, is to refer to the student's ideas as alternative conceptualizations to those which have been conventionally agreed, or as an incomplete understanding (Lidster, Pereira-Mendoza, Watson & Collis, 1995). The term 'misconception' is commonly used in research literature however, and so is used in this study to refer to the types of reasoning and understanding which differ from the accepted concepts, that is, from the 'normative' views of probability. There appears to be no single term that could be used instead to represent the same ideas.

Interpretations of probability

In order to make sense of the difficulties that occur in the learning of probability, it would first be helpful to consider the various ways in which probability can be interpreted. Probability may be thought of as the "mathematical approach to the quantification of chance, just as rulers measure distances" (Kapadia & Borovcnik, 1991, p.2). Konold (1991) says that on the one hand, probability is a degree of subjective belief in the truth of some proposition; on the other, it refers in a precise way to an objective property (frequency of occurrence) of certain types of events.

There are, according to some writers (for example, Fine, 1973; Hawkins & Kapadia, 1984; Kapadia & Borovcnik, 1991; Konold, 1991), four different interpretations for probability. Firstly, the classical interpretation (or *a priori* probability) is based on equally-likely outcomes. It is this type that many students encounter first, through the use of such devices as spinners, coins, and dice. The probability of an event is defined to be the number of desirable outcomes divided by the total number of possible outcomes, where all outcomes are equally likely. This approach allows the calculation of probabilities before any trial is made (Fine, 1973). However, this definition of probability raises some problems: with the definition of the probability of an event referring to the need for outcomes to have equal probability, a circular definition has arisen. How can one ensure that the outcomes are equally likely in order to calculate the probability of the event being considered? This issue appears to be conveniently overlooked by those who use this interpretation as a starting point for the development of probability concepts with students.

The frequentist interpretation (or empirical interpretation) requires repeated identical trials in order to establish long-run relative frequencies. These long-run frequencies are then taken to be the probability of the event in question. The observer who counts the trials and records the outcomes must consider the trials to be of the same type. This may not be easy, so a difficulty arises with the frequentist interpretation not being so easy to apply. For instance, is the throwing action involving a dice the same from trial to trial, or has there been a change in the action which may have resulted in the probability of particular outcomes changing from the earlier trials to the later? Also, with the frequentist interpretation, it is meaningless to talk of the probability of an event in a single trial, that will not or could not be repeated (such as the probability of one competitor winning a race).

Subjective or intuitive probability measures the belief in the truth of a proposition. Probabilities are then used as a belief system – the stronger the belief, the higher the number, and vice versa. In fact, a probability close to 100% indicates 'yes it *will* happen', close to 0% means that 'no, it *won't* happen'. For a subjectivist, there are two categories of information, namely prior information which is independent of any empirical data in a subject's mind, and empirical data which amounts to frequencies in repeated experiments (Fine, 1973). Both types of data are combined in Bayes' formula to give a new probability of the event in question. This updating of probabilities is sometimes called "learning from experience". It is acknowledged that subjectivist interpretations can be applied to a wide range of phenomena; and also that to formalize the subjective interpretation, theorists have adopted adjustment mechanisms (such as Bayes' theorem) that lead to revision of the initial probabilities, given new information such as results of actual trials (Konold, 1991).

The fourth interpretation is formal or theoretical probability. In this case, there is no justification for the numerical values obtained for probabilities in any case of application. Rather, it is an abstract system of rules and procedures.

Classifying misconceptions

The reasons for the difficulties in learning probability are numerous and not always obvious. When examining difficulties that children have, reference can be made to the various ways of interpreting probability, which helps put the difficulties into context. Randomness itself is difficult to understand, as Borovcnik and Bentz (1991) indicate when outlining some of the pioneering work in probability theory. The fact that the randomness property can only be checked and clarified by probability itself presents a problem with a circular approach.

Some of the common misconceptions that arise in the learning of probability have been classified by various researchers (Kahneman & Tversky, 1972; Konold, 1989; 1991; 1995; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993). Kahneman and Tversky (1972) have defined and described the heuristics (or reasoning strategies) that people use to arrive at probabilistic judgments, whereas Konold's approach has been to explore how people interpret probability questions or a particular probability value.

Representativeness

One explanation for how people arrive at probabilistic judgments is that they do not follow the principles of probability theory in judging the likelihood of uncertain events, but rely on a limited number of heuristic principles which reduce the complex tasks of assessing probabilities and predicting values to simpler judgmental operations (Kahneman & Tversky, 1972; Tversky & Kahneman, 1974). The use of the representativeness heuristic indicates how people estimate the likelihood of events based on how well an outcome represents some aspect of its parent population, and on how well it reflects the salient features of the process by which it is generated. An event A is judged to be more probable than event B whenever A appears more representative of the population than B. For instance, when considering the sequence of outcomes for 10 successive tosses of a coin, a person applying the representativeness heuristic would choose HHTHTTTHHT as being more likely than HHHHTTTTTH, because it reflects the random process which generated the results (even though both sequences have five heads and five tails); also HHTHTTHHTH would be believed to be a more likely sequence than HHHHHHHHTT because it represents the 'parent population' better than the second sequence (where the 'parent' population of coin outcomes would tend to have about 50% heads and 50% tails). Representativeness is used because people want things to balance out quickly. The representativeness heuristic is modified slightly when people assign greater probability to a result which exhibits some perturbation, that is, the result should not be too regular nor too irregular. For instance, when considering the three sequences of coin outcomes, HTHTHTHTHT, HHTHTTTHHT, and HHHHTTTTTH, someone using the representativeness heuristic would reason that the first one is too regular, the third sequence has too long a pattern of Hs then of Ts, therefore the second sequence must be the most likely of the three.

The representativeness heuristic is also the basis by which people explain that, for example, after a sequence of nine heads with a coin, there is a greater chance of a tail than a head on the next throw. This is also known as the 'gambler's fallacy', and it relates to a belief in 'local' representativeness. This version of the gambler's fallacy is also known as the 'negative recency effect', whereby something that has not happened for some time has a greater chance of occurring. Another variation of the gambler's fallacy is the 'positive recency effect', whereby there is believed to be a higher probability of a current trend continuing. For example, a shop that sells lottery tickets, and advertises the fact that there has been a major prize won by a person who bought a ticket from their shop, is attempting to take advantage of people's belief in the gambler's fallacy – "If it happened to them, it could happen to me". The business is hoping that people's belief in the positive recency effect will result in an increase in sales for their shop.

A representative sample is believed to be one in which the characteristics of the parent population are represented globally in the whole sample as well as locally in each of its parts. This situation relates to the belief that the law of large numbers applies to small sample sizes as well. The law of large numbers ensures that very large samples are highly representative of the populations from which they are drawn. Nisbett, Krantz, Jepson and Kunda (1983) indicate that people have an intuitive appreciation of the law of large numbers in that they readily accept, in some contexts at least, that more evidence is better than less. Also, it would appear that the more heterogeneous that events are, that is the more they differ from one to another or the more unpredictable are the results, the larger the sample should be. Application of the law of large numbers to small samples as well implies that small samples are also expected to represent the parent population. When there is belief in local representativeness, the gambler's fallacy is no longer seen to be fallacious (Kahneman & Tversky, 1972).

Availability

Another heuristic described by Kahneman and Tversky (1972) is the 'availability' heuristic. This is when people estimate the likelihood of events on the basis of the ease with which they can call to mind particular instances of the event. This can result in significant bias because of the narrow experience or personal perspective of an individual. For instance, because a person cannot recall having seen a particular sequence of numbers as a Lotto result, then that sequence is deemed less likely to occur than some other sequence. Our impressions of the relative frequency of events are biased, because even a single occurrence of an event can take on an inflated significance when it happens to us (Shaughnessy, 1993).

Outcome approach

Konold (1995) explored the way that people interpret probability values or view probability questions. He identified a specific type of commonly-used reasoning as the 'outcome approach':

> *Asked for a probability of some event, people reasoning according to the outcome approach do not see their goal as specifying probabilities that reflect the distribution of occurrences in a sample, but as predicting results of a single trial.*

> *Given the desire for predictions, outcome oriented individuals translate probability values into yes/no decisions.*
>
> (para.6)

Thus the focus of the task becomes a prediction of the next outcome, rather than an estimation of the relative likelihood of an outcome. A probability value of 50% is interpreted as not enough information about the event to make a prediction, whereas a probability of significantly above 50% means that "yes it will happen", and "no it won't happen" for values below 50%. For example, a student, when asked for an interpretation of probabilities given in weather forecasts, suggested that

> *If I heard there was a 90% chance of rain, I'd think yuk, I won't plan anything, but 70%, well, it's not going to be a shocking day. If it was 5% I'd think it was going to be a ripper day.*
>
> (Burgess, 1996, p.11)

Here the probability indicates to the student whether it will rain on the day, as well as the extent or severity of the rain.

The outcome approach is at odds with a frequentist interpretation of probability: a person using the outcome approach to evaluate the prediction of a particular outcome would not be concerned about the relative frequencies over a large number of cases (Konold, 1989). Also, a person who uses the outcome approach would feel quite comfortable about evaluating their prediction after just one trial; they would conclude that they either were right or were wrong with their prediction.

M-L switch

Students' responses on a particular type of problem, as investigated by Konold et al (1993), show some inconsistencies in the reasoning used. Konold et al gave the students two versions of the same problem - one asked the students to identify the least likely of a number of sequences of coin results; and the next to identify the most likely of the same sequences. Konold et al postulated that the students' inconsistent answers between the two parts indicated reasoning referred to as the M-L switch. One type of reasoning was used with a '<u>M</u>ost likely' question, and a completely different type of reasoning was used with a '<u>L</u>east likely' question. The students concerned were not aware of the inconsistency in their answers, nor that they had switched the type of reasoning from one question to the other. Konold et al believed that the reasoning in both cases was based on the outcome approach. They interpreted the students' responses in terms of the students thinking more about the nonpredictability of coin flipping, rather than about the probability of the various sequences. Kahneman and Tversky (1972) would, however, give a different interpretation on this situation, based instead on the representativeness heuristic.

Equiprobability bias

Lecoutre (1992, cited in Sharma, 1997) describes the equiprobability bias whereby people tend to assume that random events are equiprobable by nature. This reasoning is commonly used for events governed by symmetrical random generators (such dice and coins). Amir and Williams (1994) found that 11 and 12 year olds commonly exhibited this bias in their reasoning. Interestingly, Nisbett et al (1983, p.344), in their work with children of a similar age, claim just the opposite:

> *By the age of 11 or so, many children have – in addition to a clear conception both of fully deterministic systems and of random generating devices – a good understanding of non-uniform probability distributions. These are partially random systems in which causal factors are at work making some of the possible outcomes more likely than others. The child comes to learn that even though individual events are uncertain in such a system, aggregate events may be highly predictable. In such a probabilistic system, the child grasps the relevance to prediction of the base rate, that is, the distribution and relative frequency of the various outcomes.*

Young children and language

Konold (1991) states that long before their formal introduction to probability, students have dealt with countless situations involving uncertainty and have learned to use in common discourse words such as probable, random, independent, lucky, chance, fair, unlikely. They have an understanding that permits them to use these words in sentences that are comprehensible to other language users in everyday situations. It is into this web of meanings that students attempt to integrate and thus to make sense of their classroom experience. However, Fischbein, Nello and Marino (1991) point out that children do not spontaneously fully understand the meanings of probability terms such as impossible, possible and certain. They argue that children must be trained to distinguish between rare and impossible events, and between highly frequent and certain. Children's language development and competency is also important for assessing their mathematical understanding, because of the discrepancies that occur between children's operational achievement and verbal achievement (Falk et al, 1980).

Belief systems

Sometimes probability judgments are not made using any of the typical probability heuristics, but rather on some personal belief system. One of the most commonly observed beliefs in young children is that the six on a dice is the hardest to throw (Amir & Williams, 1994; Green, 1983). This can be attributed to the children's memory of playing games where the six is needed to start or some other significant requirement in the game, and the length of time they had to wait until they had thrown a six. Rather than a belief about the dice, it could be that the children are using the availability heuristic.

It is common for children, in particular, to believe that the outcome can be controlled when a random generator (coin, dice, or spinner) is used (Amir & Williams, 1994; Fischbein et al, 1991). This causal approach to probability comes from the belief that if the generator is thrown, held, spun, or shaken in a particular way, then the chance of the desired outcome improves. Also, some people believe that there are certain outcomes that they are more likely to get, for example, "I always get a 3". The implication is that they know how to do it, to control the outcome. Again, in the face of evidence to the contrary when the desired outcome does not eventuate, the belief is still maintained.

Another belief system which affects probability thinking deals with the influence of God, superstitions, or luck. These often have a cultural base (Amir & Williams, 1994; Sharma, 1997). The difficulty with resolving these beliefs with a normative view is that often there are situations for which a degree of belief is the only possible evaluation for the event being considered (Falk et al, 1980). This corresponds to a subjective interpretation of probability, which cannot be readily evaluated nor confirmed. For example, will the telephone ring in the next hour? or, will the plane land on time?

Strategies for overcoming misconceptions

Commonly advocated methods for helping overcome misconceptions and assisting development of normative understandings include the use of practical activities and real-life contexts, along with discussion and interaction of ideas. Konold (1991) advocates a multifaceted approach to instruction designed to confront the students' misconceptions. In this approach, students are encouraged to evaluate their current beliefs and how well they fit against: 1) the beliefs of others; 2) their other, related beliefs; and 3) their own observations. The first aspect involves discussion and Konold describes the ways in which teachers can facilitate this discussion by helping to keep the conversation going, and focusing the discussion. However, according to Konold (1991), the most important role of the teacher is to create "an atmosphere in which it is acceptable to articulate an opposing view and to challenge what one does not understand or believe" (p. 152). As students try to convince one another of their views, they "explore in greater depth the implications of and interconnections among their own beliefs" (p. 153). The teacher, through the use of probing questions and prompts, encourages the students to analyze the consistency and completeness of their own beliefs – both related to the misconception being considered as well as other related beliefs. Sometimes, students will realize that there are inconsistencies in their views. The teacher can assist with this self-checking aspect through the use of verbal probes, such as "Why do you think that...", or "How does that relate to what you said earlier?"

Overall, the discussions achieve a number of purposes. The students are required to express their views clearly before they know what may be considered the 'expert' view; and the discussion encourages personal involvement of the learner and therefore interest and motivation in the subject. In addition, the teacher has the opportunity to gain an insight into the thinking of the students, which is helpful in planning further learning experiences and evaluating student progress.

However, strategies which assist the students to examine their beliefs and confront any inconsistencies which they may recognize, are not sufficient on their own:

> *One reason that misconceptions are so difficult to alter is that they tend to comprise a coherent, self-consistent framework. To the extent that a person's beliefs are self-consistent, they are impervious to this type of challenge.*
>
> (Konold, 1991, p. 154)

The third criterion involves the use of empirical data to check the validity of their beliefs, and is crucial, according to Konold. Empirical observations are important, but are often weakened in their value through two aspects: people do not keep accurate records; and they "do not look for data that would be inconsistent with, and thus controvert, a belief that they hold" (p. 154). Classroom demonstrations must emphasize these two features so that students' beliefs will be adequately challenged.

Barnes (1998) lists a similar sequence for the structure of classroom activities suitable for learning related to probability. It involves: motivating the students through posing an interesting question; getting them to commit themselves to an intuitive answer on paper; small group discussions about the answers they have given; devising experiments to check answers and collating data from all groups, so that the results cover a large number of trials; analyzing the combined data and comparing this with the initial ideas; and if appropriate, discussing and working out theoretical probabilities, and comparing these with the experimental results. Barnes discusses how simulations may be used during the experiment phase, but stresses the importance of 'real' experiments first, as opposed to simulations. This is because the students may not otherwise make the link between the real and the simulated experiment and therefore not find the simulation convincing.

Shaughnessy (1977) also designed and trialed a teaching experiment for tertiary level students to confront their misconceptions. This teaching experiment was based on a very similar approach to those outlined above. His experimental group showed gains in probability concepts that the control group (traditionally taught) did not. However, Shaughnessy warns that there were some students who held on to their intuitive conceptions, in spite of the empirical evidence. This finding has also been found by other researchers, and consequently would appear to be a feature of probability learning.

One significant finding related to the use of practical activities is that the empirical evidence is not always as conclusive as the teacher would hope. When dealing with random events, there is always the chance that the result obtained may have a very low probability of occurring. For example, Burgess (1996), when interviewing tertiary students, obtained two successive outcomes which had a probability of occurrence of 0.000 000 06. Naturally, such a rare result conflicts with what is expected, and students are therefore faced with experimental evidence that does not illustrate the concept, and furthermore, they are unlikely to realize this. The danger is that students will try to rationalize this evidence with the misconception that they have, which could well consolidate that misconception, or even create a new misconception in the students' mind. Rather than the student recognizing that randomness is being exemplified when unusual results (in probability terms) are obtained, the student is more likely to ascribe the result to an overall pattern and view the outcome deterministically. This unusual result then does nothing to help the understanding that the teacher is hoping to encourage, even though it does help illustrate the nature of randomness. How this conflict is resolved, or how it is used by the teacher to continue with the experiment, can be difficult. In spite of this difficulty, it is essential that the teacher utilizes it in some appropriate way to aid the development of a normative understanding of probability. One way to minimize the potential difficulty is to ensure that practical activities involve many trials:

> *[Since] there is no guarantee that a correct choice would always be followed by success (or that a wrong choice would be followed by failure), it is important to expose the child to many repeated trials. Experiencing many repetitions will give the child a chance to learn the probabilities with which he is being reinforced, and moreover will let him realize that correct choices will make him win in the long run.*
>
> (Falk et al, 1980, p.200)

2.4 Games

History of games

Games involving chance have been in existence since about 3500 BC, discovered through the evidence of both archaeological evidence of artifacts that have been unearthed and through tomb-paintings depicting such games (David, 1962). Analysis of games of chance from different cultures shows a probabilistic basis for the choices and point values (scores) gained during the games. These point values must have been assigned to the games from experience accrued from repeated playing of them, rather than from any theoretical analysis. So experience was used to create the games as they were, and are now, played (Ascher, 1991).

The early, major development of probability theory evolved through the correspondence between Pascal and Fermat concerning a problem related to gambling and the division of stakes, which had been posed by the Chevalier de Mere (David, 1962). This correspondence during the 1650s, and the subsequent development of probability ideas by Huygens in the late 1650s, was sparked by consideration of the outcomes of gambling. The initial question posed by de Mere was based on his observations of many games of chance, and the patterns of the outcomes. Through playing many games, de Mere developed a number of questions regarding pay-outs from gambling. Pascal and Fermat, through trying to answer de Mere's questions, also used empirical observations to begin formalizing a coherent theory of probability.

It can be seen that games (and gambling) led to the use of empirical data, followed by the development of the theory surrounding probability. The theories were subsequently modified, due to the availability of more empirical evidence.

Games in the classroom

As far back as Plato, games have been advocated for assisting learning (Johnson, 1958). Although games are both extensively used and widely advocated as suitable for classroom use (e.g. Ainley, 1990; Biehler, 1991; Bright, 1980; Inbar & Stoll, 1970; Szendrei, 1996), the research justification for the classroom use of games to enhance learning is not as extensive. Since, historically, games involving chance brought about the development of probability theory, it has been suggested that games of chance could profitably be used in the classroom to help children learn probability concepts (Biehler, 1991). It is important therefore, in considering how this might occur, to firstly define what constitutes a game, and to explore the general benefits of using games in the classroom. The specific use of games to assist with the learning of probability will also be examined.

Definition of a game

In discussing the use of games in the classroom, Bright, Harvey and Wheeler (1985, p.5) distinguish them from other classroom activities by the following seven criteria:

1. *A game is freely engaged in.*
2. *A game is a challenge against a task or an opponent.*
3. *A game is governed by all of the procedures for playing the game, including goals sought; in particular the rules are structured so that once a player's turn comes to an end, that player is not permitted to retract or to exchange for another move the move made during that turn.*
4. *Psychologically, a game is an arbitrary situation clearly delimited in time and space from real life activity.*
5. *Socially, the events of the game situation are considered in and of themselves to be of minimal importance.*
6. *A game has a finite state-space. The exact states reached during play of the game are not known prior to beginning of play.*
7. *A game ends after a finite number of moves within the state-space.*

Bright et al based these criteria on work by Inbar and Stoll (1970), but developed the last two criteria to effectively exclude related activities or other structures, such as play and puzzles.

Learning opportunities with games in the classroom

There are various ways in which teachers can plan for the use of games in the classroom as a part of the educational process: for practice and consolidation of skills; to introduce and develop new concepts; to develop perceptual abilities; and to provide opportunities for logical thinking or problem solving (Burnett, 1993; Smith & Backman, 1975).

Bright et al (1985, p.9) describe instructional games (that is, those which have a pre-determined set of instructional objectives) and categorize them into three specific levels related to their cognitive effect:

> *They are: pre-instructional level - students have not received instruction on the instructional objectives of the game and will not receive instruction on those objectives other than that provided by playing the game; post-instructional level - prior to the beginning of play of the game, the students have received instruction designed to produce mastery of the instructional objectives of the game; and co-instructional level - the game playing is part of an instructional package designed to produce student mastery.*
>
> *Of paramount importance is the assumption that games can be used to enhance students' achievement of mathematics skills, concepts, and processes where the measure of achievement is attainment of the particular instructional objectives of a game rather than improvement of general tests of mathematics achievement.*

Inbar and Stoll (1970) suggested that it seems inevitable to conclude that the impact of a game, because of it being a total activity, would be fundamental and pervasive. They describe some definite benefits for the learner from the use of games:

> *they provide feedback on the consequence of actions; ...they offer an opportunity for discovering or uncovering a series of interconnected relationships. These situations encourage the learner to ask questions and receive responses, and, hence, place an emphasis on making fresh deductions and inductions.*
>
> (Inbar & Stoll, 1970, p.58)

Ainley (1988) questions the claims made by others that games help with the acquisition and development of concepts, as well as help to develop problem solving strategies. In particular, she discusses a claim made by Ernest (1986) as to how games can teach mathematics:

> *Whatever the nature of the games, Ernest's phrase that 'games **teach** mathematics' seems to be misleading; at most, games can help children to learn mathematics. This is not simply a linguistic quibble. If teachers use games in the hope that **the games** will teach their pupils particular pieces of mathematics they will be sadly disappointed. Children will certainly learn from playing games, but **what** they learn will vary enormously, just as what children learn when working from a textbook varies. A well-designed game may create a good environment for learning some mathematics, but it will not ensure that the children learns the mathematics, and more importantly it will not replace the teacher. The teacher's role in stimulating mathematical learning during the playing of a game, and monitoring the learning which is going on, is vital".*
>
> (p. 243)

Thus, although games are strongly advocated because of the way that they assist with the mathematical development of children, they may be inadequate on their own: it is the teacher's planning of the educational process which is essential (Szendrei, 1996). Inbar and Stoll (1970, p.58) point out that "because they [the games] are self-judging, games remove teachers from the role of judge to free them to coach the activity rather than focus on evaluating it". Games also have the advantage of providing a natural situation for the teacher to be able to question children about their understanding in a less formal, and potentially threatening, atmosphere experienced in some other types of assessment procedures (Ainley, 1990, p.90).

Burnett (1993) supports the use of games as a fundamental part of children's learning, in particular because of the mathematical language development which occurs when children play games. But she also notes that game itself does not provide the teaching. Instead, it is the discussion which arises from the game, as well as the social context in which it is played, that facilitates learning. This view is supported by others (for example, Booker, 1997; Ellingham, Gordon & Fowlie, 1998) who contend that games encourage students to listen to the viewpoints of others and make sense of those interpretations.

Games and probability learning

Biehler (1991) contends that games can be a major teaching and learning activity for developing probability concepts. He uses the analogy that "the statistician's role [has] been defined as 'playing games against nature' " (p.200). Similarly, instead of just using games in the classroom as a motivation for, or an introduction to, the concept of chance, Steinbring (1991) suggests that games should be used throughout the process of developing the concept, since they can provide a concrete context for students' experiences with chance. "Games of chance served as a link between intuition and developing concepts as well as a tool to structure real phenomena" (Borovcnik et al, 1991, p.31).

Games can be used in a "pre-instructional" sense, prior to any formally-structured teaching, to help students' learning of probability concepts. Bright, Harvey and Wheeler (1980) claimed that their research was the first known instance which showed effective probability learning occurred from the "pre-instructional" use of a game. Hirst (1977) asserts that because board games with dice are a part of students' everyday experience, they can lead students "towards an understanding of such concepts as equal likelihood and frequency" (p.6). Through the course of playing many games, children acquire experience with random processes, which should help develop the potential for the children's understanding of probability (Falk et al, 1980).

Limitations of games for probability learning and strategies for overcoming these

Bright (1980) however, expressed caution about the use of games for probability learning because of the possibility that, as the students become more familiar with a game, they may become more skillful in the strategy required in the game without necessarily increasing their understanding of probabilities related to the dice outcomes. He indicated the importance of carefully examining the players' moves during the game, in order to determine how much is attributable to practice at playing the game compared with real understanding of probability. Falk et al (1980) also referred to the possibly passive role of children during the playing of games involving only chance. They suggested that to overcome this, games involving some decision-making from the players should preferably be used. This would force the children to "cope with the probabilities which determine the course of the game" (p.199) and therefore ensure that probability concepts are more likely to be explicitly thought about during the game. Also, the game should be played a number of times, since, because of the nature of random events, less likely outcomes can occur, which in small trials would not necessarily help with gaining an understanding of probability.

When choosing the type of game for use in the classroom, particular care needs to be taken to ensure that the game will encourage the sort of thinking necessary to improve the understanding of probability. Eade (1988) recommends 'games of conflict'. By this, he refers to games that may induce 'cognitive conflict', where the players' intuitions about the game and its outcomes conflict with the empirical results, thereby challenging the students' understanding. These games encourage the students "to realize that a particular way of thinking does not produce an acceptable answer thus encouraging the pupil to examine his approach to the problem" (p.41). Through this, it is hoped that the conflict would be resolved with an improved (normative) understanding.

Eade (1988) gives some reasons why activities designed to improve the understanding of probability are often not successful. Firstly, students may be instructed to predict an outcome but they do not do this, so they are not committed to the activity in the way intended. As a consequence, they are not surprised by an unexpected result; in fact results are neither expected nor unexpected because the students have not thought sufficiently about the activity. To overcome this, students should be required to write down their prediction, having been convinced of the usefulness of doing so, in relation to learning. Eade does acknowledge the potential difficulty of ensuring that students do not look on their predictions, particularly incorrect ones, as an indicator of their ability. Rather, students need to be encouraged to view an incorrect prediction as a challenge for them to overcome and improve in the future. This approach will, Eade believes, result in improved understanding.

The second reason for 'failure' of an activity is that once students realize their conceptions are inadequate, they are not given sufficient time to discover the reasons for this inadequacy nor given the opportunity to reconstruct their ideas. The teacher sometimes just explains that here is the new way of doing the problem. This method "disregards the mounting evidence that primitive notions are not easily replaced and can only be altered if the new notion can be seen to make sense in terms of the primitive one" (Eade, 1988, p.41). This cause of failure to result in meaningful learning can be remedied by the teacher organizing the activity to ensure that: (a) there is opportunity for discussion between pupils as well as between pupils and the teacher; (b) more than one activity is provided, and over a period of time, to give time for meaningful learning to occur; and (c) the pupils are involved in evaluation of the activity as it is occurring as well as after the completion of the activity. These 'conditions' for meaningful learning from using a game can be met if the game is structured in such a way as to accommodate the requirements and, importantly, the teacher is actively involved.

Sometimes when students are using games for learning, the players may be able to complete a game without being challenged or required to consider the concepts for which the teacher has chosen the game (Ellingham et al, 1998). They suggest that when the educational aspects and the gaming components are not be fully integrated, students treat it merely as a game rather than a learning experience. Active involvement along with the intention of learning is required to ensure maximum benefit.

Bright et al (1980) contended that the type of grouping (homogenous or heterogeneous in relation to mathematical ability) used in the classroom when children are playing games had no significant effect on the learning - similar gains were made by children in both types of grouping. The generalizability of this finding may be questioned in relation to the interaction that occurs within the group, which is seen as a significant factor in the learning process. The 'quality' of the discussion that occurs in the group could well depend on the nature of the group in relation to the ability of the individuals in the group. From the social constructivist perspective, the group understanding that develops within a group must depend on the contributions to the group by the various individuals.

An advantage of games as an activity relates to the defining characteristics of games, in the way that they are distinguished from other activities:

> *....the activity is self-motivating. They are cut off from the serious consequences of everyday life, for the activities within them can be enjoyed for their own sake. Thus it is possible to "make a fool of one's self", guess outrageously, or behave in very risky ways.....the activities are visible, and often players can talk to one another about their respective plans and moves.*
>
> (Inbar & Stoll, 1970, p. 58)

But since playing a game involves some aspect of competition, there are psychological aspects involved which are related to competition, as well as social aspects of game playing, which may have an influence of the value of the game as a learning activity:

> *It has never been demonstrated that the participants in educational games feel free from the consequences of their acts; impressionistic evidence would suggest that there is often great ego-involvement and the accompanying fear of failure.*
>
> (Inbar & Stoll, 1970, p.60)

Although games have the feature that the social events in the game have minimal importance, these factors can influence what the learner gains from the game situation.

2.5 Summary

Relevant literature to this study has been reviewed, examining the theoretical basis of learning, specifically the theory of constructivism. One important aspect of this theory is that learning, as construction of ideas, is personal to the learner. The implications of this for teaching, such as the influence on learners from others, peers and teachers, are important. Peer influence is believed to be significant, therefore the use of groups so that learners develop shared understanding of concepts needs to be considered. Teachers, in order to help students learn, are believed to also be constructing ideas about their students' learning simultaneously with that learning. It was suggested that teachers can facilitate learning through encouraging and being involved in the classroom discussions.

Intuitions, and how they influence learning, were discussed as they are part of the experiences which students bring into the classroom and thus influence future learning. Some intuitions are believed to be powerful, which can be problematic for learning if they are not the correct concepts. Ways of helping students become aware of these intuitions, confront them, and hopefully change them, were considered, as was the use of inductive learning in relation to probability.

In order to further help understand probability learning, the various interpretations for probability were described and the common misconceptions that learners develop were classified and defined. Some strategies for helping students overcome these misconceptions were considered.

The final section of the Review examined what games are and how they can be used in the classroom. The focus then was on the use of games in relation to probability learning, their usefulness and some of the possible limitations of their use.

Chapter 3 Research Design

3.1 Introduction

Qualitative research methods, which are prevalent in educational research, include (among others) naturalistic observation, case studies, ethnography, interviews, and narrative reports. The main aim of the qualitative researcher to examine what people are doing and how they interpret what they are doing (Morse, 1994), and to make sense of the meaning that people construct from their experiences (Merriam, 1998). In this regard, qualitative research is in harmony with a constructivist perspective of knowing and learning.

Quantitative research methods, which may be concerned with measurement, experimental methods, statistical analysis, and mathematical models, cannot be dismissed as inappropriate in education. Linn (1990) describes how the two forms of research are often intertwined with blurred boundaries. A researcher using qualitative research methods will often find that quantitative summaries and classifications are a useful part of the research, while those employing a quantitative approach will have to make many qualitative decisions, such as questions to pose, analytical procedures to use, and interpretations to stress. In fact Linn recommends that the results from research which uses both categories of methods will contribute to understanding and therefore have value. Silverman (1993) agrees that often the researcher will want to combine the approaches. It is believed that mixing the methods is desirable, in order to take advantage of the various strengths, while reducing the weaknesses found in single design methods (Jick, 1979).

In this study, the main focus is on the interpretation of students' understanding of concepts in probability. Because of factors related to learning, which were discussed in the Literature Review, qualitative methods are seen as the most appropriate for exploring the students' understanding. However, some quantitative methods can usefully be employed to organize data and examine it for trends and patterns.

3.2 Data Collection

Given the nature of misconceptions in probability learning, and the need to evaluate the effectiveness of games for learning, in terms of the interactions between students along with any changes in understanding that may occur, it was decided that a mixture of data collection methods would be necessary for this research. This process, of using two or more methods of data collection, known as triangulation, has the advantages of:

(a) reducing any bias that may arise from the use of only one method; and (b) contributing to verification and validation of qualitative analysis (Merriam, 1998).

Duit, Treagust & Mansfield (1996) describe types of interviews that can be carried out, for example clinical Piagetian interviews, interviews about instances, interviews about events, and teaching experiments. Because of the limitation of a single interview on its own, they suggest:

> *it can be advantageous to merge interviewing with questionnaire methods to allow the investigation of conceptions held by large numbers of students. In one strategy, an interview is followed by an experiment for which students write down their predictions and explain them. Subsequently, the experiment is carried out for students to observe and comment on whether their predictions were right.*
>
> (Duit et al, 1996, p.24)

The methods chosen for this research followed those recommendations. The methods included a written questionnaire, observation of the students in a game-playing situation, and open-ended interviews to further explore the apparent understandings of the students.

Questionnaire

The questionnaire (see Appendix) was based on the objectives from the national curriculum statement, *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992), that would be appropriate for students of this school level (years 7 and 8). The questions were adapted from questions that had been used by a number of other researchers in the area of probability misconceptions (for example, Fischbein et al, 1991; Fischbein & Schnarch, 1997; Konold, 1995; Konold et al, 1993).

Observation and Interviews

Observations

The difficulty of making judgments about children's understanding from the results of written, and even oral tests is well researched (Ainley, 1990). Ainley suggests that when children are playing games, their thinking is much more transparent than in tests, because their actions reveal much about their thinking strategies. The game can be used as an unobtrusive assessment tool, "without the potentially intimidating atmosphere of a more obvious assessment" (p.248). The researcher in this study was able to observe the strategies and 'thinking' of the students through using a game-playing session, and the observations were backed up by the video recording of the sessions.

Games selection

Since the research was concerned with comparing the effect of using games involving chance only with that of using chance and strategy, two simple games were chosen for the students to play. They needed to be relatively short in duration, so that within the allocated time for the games sessions, a number of repetitions of the games could occur. The repetitions of a short game allowed random outcomes to be observed as well as having the potential for showing the 'longer term' trends of the random outcomes. This suited the nature of the various probability concepts being examined, including those that were probed in the questionnaire.

The advantages to a learning activity of recording the successive outcomes of a random process have been discussed in Chapter 2, so the selected games (see Chapter 4 for a more detailed discussion) were adapted to enable a type of 'record' to be kept of the outcomes through the use of the counters on the game board. For example, the first game (involving the difference between the numbers on two dice) has been used as a learning activity but without the use of a board with counters (for example, Dice Difference, by C. Lovitt). In that version of the activity, the participants would need to rely on their memory of the outcomes to gain some understanding of the (lack of) fairness in the game. This reliance on memory is known to be unsatisfactory in relation to probability-based events (see discussion in Chapter 2 on the availability heuristic of Kahneman and Tversky).

In this research, two versions of the second game were used, the first of which was based on the sums of two dice and the second was based on dice differences (as was the first game mentioned above). The two versions of this second game used some strategy, through players choosing particular numbers to play. Once the numbers had been chosen and the counters placed on the board, the remainder of the game involved the throwing of dice, the 'chance' phase of the game.

When trying to determine the probability understanding of children, care must be taken with the design of the equipment (Falk et al, 1980), because of the potential for children to make decisions based, not on probability type judgments but, on other factors such as number or position. The games needed to be carefully chosen and the game boards designed in such a way to minimize any such possible effect. The researcher believes that the use of dice, and the relatively simple layout of the board, satisfied this requirement, and that the boards would not have contributed any 'contamination' to the understandings that were the focus of the research.

Interviews

Interviews were selected as a data collection method, because of their potential for exploring student conceptions and yielding a rich source of data for analysis. During interviews, students are required to articulate their ideas, which in itself can be valuable for the students themselves. Duit et al (1996) describe how the interview situation requires students to create ideas to "make some sense of their intuitive thoughts" (p.18). The researcher, in posing questions for the students to consider, can provide,

> *the opportunity or even the necessity for students to investigate their own existing views, to construct or reconstruct new explanations for the views they hold, or to extend existing views or create new ones to account for the structure and subtleties of the context the interviewer is inviting them to consider for the first time.*
>
> (Duit et al, 1996, p.18)

Additionally, group interviews enable the interaction processes between students to be studied (Duit et al, 1996). Since the students in this research were involved as a group in playing a game, the group interview could be easily integrated into this observation. Group interviews involving a practical component, such as the playing of a game, have been referred to as 'interviews about events' (Carr, 1996). The stimulus for the interview questions is the activity carried out by the students – namely the game.

The researcher - limitations and strengths

Given that learners construct their own knowledge and understanding, one of the problems for the researcher is to determine the students' understandings. As discussed in the section dealing with constructivism, there are difficulties in "getting inside the student's head". Duit et al (1996, p.18) acknowledge this difficulty:

> *What researchers call "student conceptions" are actually their own conceptions of students' conceptions. Much care is necessary in planning, carrying out research, and especially in interpretation, in order to remain sensitive to one's own conceptions, ideas, beliefs, and prejudices about others' conceptions.*

The game-playing phase of the research could not be considered a 'natural' game setting, because of the researcher's use of question probes to explore the students' ideas. The questioning process requires the students to think about and explain aspects of their understanding and knowledge that they may not have otherwise considered. Therefore the researcher, as well as gaining an insight into the students' existing ideas, could be responsible for the further construction and refinement of ideas and understanding: some learning or reshaping of their cognitive 'schema' may occur as a result of the interview (Spilka, 1993). So, unlike experimental research in which it is important to rule out contaminants such as respondent-researcher effect, qualitative research finds it critical to identify the 'contaminants' and assess the interaction (Morse, 1994).

The researcher is an experienced teacher who has taught at all levels in both primary and secondary schools. He is currently working in a university College of Education, teaching mathematics education courses in the pre-service teacher education programmes (both primary and secondary). His interest in mathematics education has developed from the time that he finished a mathematics degree and decided that teaching was his interest. Following teacher training, he commenced his teaching career at the primary school level.

3.3 Phases of the research

Two intermediate schools (years 7 and 8 students) were approached to be involved in the research. Following permission and agreement from each principal, one class was selected by each principal. The respective classroom teachers were asked to select eight children representing a range of mathematical abilities. The selected children were given an information sheet and consent form, and those who agreed to take part in the research indicated this by returning the consent form, signed by themselves as well as at least one parent or caregiver.

The questionnaire was administered to the students in the first school on day one of the research, with the game sessions taking place on the two following days. The researcher collated the questionnaire data before the games sessions, and from this initial 'reading' of the data, the two groups of four students were selected by the researcher for the game sessions, so that in each group, there was a mix of students in relation to the probability understandings shown within the questionnaires. Within each group for the game sessions, two pairs were formed, each pair consisting of one boy and one girl. The pairs competed against each other in the game sessions. The game sessions were audio-taped as well as video-taped, and lasted for about 45 minutes. The plan for the second school was to be the same the following week. However, the second game session in this school was delayed because of a planned class trip, and instead occurred a few days later than originally planned.

The researcher transcribed the audio-tapes, and used the video-tapes as a back-up when it was unclear from the audio-tape who was speaking, or what had happened. This was particularly useful for checking any non-verbal interactions that may have occurred, and checking the involvement of students who may not have been speaking very much.

Analysis of the data consisted of searching the questionnaire responses for identifiable trends in relation to the misconceptions that are known to commonly exist in students. These responses included the objective answers to questions, many of which were multi-choice, as well as the subjective justifications that the students were asked to provide on most of the questions. The questionnaire results were analyzed in two ways. Firstly, each question was examined, to show the understandings of the participants as a group in relation to the particular concept being focused on. Secondly, each individual's responses to all questions were searched for patterns or inconsistencies in that student's reasoning and understanding.

The interview and observation data from the games sessions were analyzed in a number of different ways, in relation to the research questions. The analysis included looking at: the interactions between the students and the development of possible 'shared meanings'; any misconceptions that showed up as a result of the game strategies or the discussions that took place; and inconsistencies shown either by individuals or within groups.

3.4 Limitations

Reliability, which is concerned with giving the same result consistently under the same conditions, and validity, which is concerned that an assessment or judgment measures what it is supposed to measure, can be problematic in qualitative research. For instance, in a questionnaire, consistent results (reliability) may be obtained from a group, but interviewing may reveal misconceptions in the reasons given for the otherwise correct answers (lack of validity). The best way to overcome these potential problems is through triangulation (Burns, 1997) – the combination of data collection methods which allow for cross-matching of data, to verify the interpretations and inferences taken from any one source of data.

There are obvious limitations of this study which temper the conclusions that can be drawn:

- The small sample size (16 students) makes it difficult to generalize the results.
- The study took place outside of the classroom, and the results therefore reflect the 'un-natural' environment that the students were in. Again, one must be wary about generalizing to what may happen in the classroom, where the teacher does not have just one group of four students to manage, but a whole class.
- The students' willingness to contribute depended on how comfortable they felt both with the researcher, who was previously unknown to them, and with the other students in the particular group they were placed in. Although the students were from the same class, they may not have worked as closely with each other previously. The students may have reacted quite differently if they were working with a different group, or with their classroom teacher rather than the researcher.

- The researcher did not know the students other than what he was able to learn within the components of the study: from the questionnaire and the interactions in the game sessions – for example, their understanding of mathematics in general and specifically probability; their ability with communicating their ideas; their level of cooperation when working in small groups; and their attitudes both to mathematics and to the other students in the selected groups.
- The research was not linked to what the students were currently studying in their class mathematics programme, so they came into the study 'cold'. Naturally, in any change of topic in the normal class programme, this same situation occurs, so it is unknown how significant this was to the study.
- In the second game session, the students may have been more willing to contribute than the first session, because of knowing the researcher a bit better by that stage, and also knowing more of what was expected from them, having experienced one game session already. So any differences between the first session and the second may be due to this extra familiarity.
- One of the versions of the game used in the second game session (namely the differences between the numbers on the dice) was the same as the first game session, but with the added component of choosing the numbers for the counters. The extra familiarity with the dice outcomes gained from the first game may have had an effect on the way the second game was played, compared with how they may have played had they not encountered it previously. Some effect in the second game session could therefore have been cumulative from the first game session.

3.5 Ethics considerations

For this research, ethics considerations were contained within the following points:

1. access to students was gained through a letter to the Principal seeking permission to use the school for the research. The permission came from the Board of Trustees and the Principal of each school;
2. the classroom teacher selected eight children, across a range of mathematical abilities, to be involved;
3. informed consent was sought from the parents and the students, through a letter which was sent home after the teacher had selected the children to be involved. The consent form outlined all the rights of the students participating in the research;
4. anonymity and confidentiality of those involved was assured;
5. no potential harm to the participants was envisaged.

The Massey University Human Ethics Committee considered the application and approved the research study .

Chapter 4 Results and Discussion

4.1 Introduction

The results and discussion of the research are presented in two main parts - analysis of the questionnaire data, and of the interactions and questioning during the game sessions. The discussion identifies individual students as S1 to S16. Students S1 to S8 were from one school, and S9 to S16 from the second school. Details about each of these students, such as mathematical ability and school year level, are included in Table 4-2 (which summarizes their responses to the questionnaire) on page 49.

4.2 Questionnaire

The written questionnaire was administered first, on a day separate from the game and interview sessions. This enabled the researcher to obtain a preliminary impression of each student's understanding of probability prior to the interview sessions, along with any possible misconceptions that they may have had. The students were given the instruction that after answering each question (which was on a separate page), they were not to go back to that question again. This instruction was given because some of the later questions would have given cues and clues for answering the earlier questions, so it ensured, as much as possible, that the students' responses matched what may have been their primary intuitions regarding each particular concept.

The results of the questionnaire will be discussed firstly question-by-question, comparing the students' responses and identifying trends across the whole group of students. In the second part of the discussion, each individual student's responses to all questions will be analyzed to highlight the range of misconceptions or inconsistencies which that student may exhibit.

Question 1 and 2 : Certain, possible, likely, and impossible

These questions explored the area of subjective probability:

1. Certain, Possible, Likely, or Impossible?

Describe a future event that is :

certain to occur
possible that it will occur
likely that it will occur
unlikely that it will occur
impossible that it will occur

2. Possible, impossible, or certain?
Put a ring around the word that best matches the statement.
If I roll a dice, to get

an even number is	impossible	possible	certain
a number smaller than 7 is	impossible	possible	certain
a number bigger than 6 is	impossible	possible	certain
a number bigger than 0 is	impossible	possible	certain
a 5 is	impossible	possible	certain

Overall, the students showed a good understanding of these terms. There was one student who only described two events, to match the certain and the possible categories in the first question, yet in the second question, was able to accurately match each of the events with the subjective descriptors. Another appeared to have some difficulty with 'possible', leaving it blank in the first question, but was then able to answer all but one of the parts in question 2 correctly. One student left all but one of the parts in question 1 blank (he only described a 'possible' event) and in question 2, he matched all parts correctly except for two related to 'certain' events. This student is a relatively new immigrant to New Zealand and may have had some difficulty with the language terms. One student gave an answer of 'possible' for both a 'certain' event (although she then matched another of the 'certain' events) as well as an 'impossible' one. She correctly matched the other parts.

Question 3 : Coins

Put a ring around the letter of the statement that you think is best.
If I hope to get 2 heads from throwing coins, which should I do?
a) Throw 2 coins at the same time.
b) Throw 1 coin, then throw the second coin.
c) Throw 1 coin, write down the result, then throw the same coin again.
d) It makes no difference - all the methods give the same chance of getting 2 heads.
Explain your answer.

This question was designed to explore the students' belief systems – did they believe that they could control the dice? – as well as their understanding of random outcomes and independence.

Eleven of the 16 students answered correctly (option d), while two chose option b and three chose option c. The explanation of S4 (who chose option b): "So it would be more likely to throw heads twice", which gives little indication as to his understanding. The second student (S5) whose answer was b, suggested to "Throw one because you could get a tail first so then you'll know to carry on". He interpreted the question to indicate that, although two heads was the desired outcome, it would be possible to carry on throwing the dice until two heads had been obtained; a tail as the first result would require more throws. Of the three students who chose option c, two did not give any explanation for their answer. The one who did give a reason suggested that: "Because then you know your last result and you could compare it". By comparing it, she may have thought that the chance of getting the result that she wanted for the second throw would be somehow improved.

A number of those who answered correctly exhibited the outcome approach in their reasons. For example:

- *Because it's all chance. You never know what you are going to get.*
- *It doesn't matter how you throw it because you still wouldn't know which it would land on.*

Other responses were unclear in the reasoning that was being used, such as:

- *Because it's luck if you get two heads*
- *The coin that you use is the same - they all have one head and one tail so it makes no difference.*

The student who referred to luck being involved may have a belief system or may be indicating some other understanding about random events – it is unclear which option applies.

A correct response on its own can be very misleading as has been illustrated above. An examination of the written reason helps to determine the understanding of the student, but the student's facility with written language can affect the response and the interpretation of the teacher (or researcher). It is possible to form an interpretation that does not accurately reflect what the student knows and understands for a particular situation.

Question 4 : Tossing a coin

This question asked students to indicate which statement they thought was best for the fifth toss of a coin, when each of the previous four tosses had all been heads:

a) Another head is more likely than a tail.
b) A tail is more likely than a head.
c) The two outcomes (head or tail) are equally likely.
Explain your answer.

Fifteen of the sixteen students gave the correct answer to the question. The one student who differed, along with one student who gave the correct answer, justified their choices by the use of the gamblers' fallacy :

- *Since this coin has landed four times in a row as a head, then I feel it is much more likely to land on a head than a tail.* (positive recency form of gamblers' fallacy)

- *Because it is likely it will change, and there is no way of telling for sure.* (negative recency form of gamblers' fallacy)

This last student, as well as using the gamblers' fallacy, may also be using the outcome approach in his reasoning He acknowledged that it is not possible to be sure of the next outcome, that it is not accurately predictable.

The 15 students who gave the correct answer had a variety of reasons for their choice. Only five of them could be classified as having used normative reasoning (that is, reasoning which fits the accepted, 'correct' theory). For instance:

- *When you toss a coin, there is 50% chance it will land on a head or a tail.*
- *Because you have a 50-50 chance of getting either.... unless the coin is rigged.*
- *There's a 50% chance either way.*

Six of the correct responses reflected an outcome approach – they indicated that it is not possible to predict what the next result would be, therefore the two results are possible, which is taken as the two outcomes being equally likely. For example:

- *Well, if you throw a coin, you can't guarantee you will get a head, it is likely you could get either.*
- *They're both the same because you can't get heads every time and you can't get tails every time so they're both possible.*
- *You could get either. You never know what you could get, because there are two sides.*
- *Because it's a possibility.*
- *Because it's all chance.*
- *It was all luck to throw four heads.*

Questions 5 and 6 : Coins - most and least likely results

These two questions explored the students' understanding of the most and the least likely result from tossing a coin six times. It was designed to check whether the students used the representativeness heuristic to justify their choices or, at a more specific level, displayed the M-L switch misconception (using contradictory reasoning about the most likely and least likely events).

Question 5 asked for the <u>most</u> likely result, and question 6 asked for the <u>least</u> likely result (with a justification for each), of the following possibilities:

a) H H H T T T
b) T H H T H T
c) T H T T T T
d) H T H T H T
e) All are equally likely.

Eleven of the students answered question 5 correctly, and only eight students answered question 6 correctly: thus there were three who, having answered question 5 correctly that all the sequences were equally likely, proceeded to nominate one of the sequences as being the least likely.

For the students who answered either part incorrectly, Table 4-1 gives a summary of their responses, including the classification of the type of reasoning that appears to have been used.

Table 4-1 Responses to questions 5 and 6, and suggested type of reasoning used.

Student	*Question 5 - most likely sequence*	*Justification*	*Question 6 - least likely sequence*	*Justification*	*Type of reasoning*
S7	THHTHT	I don't think it would have any more than 3 of the same in a row. But it is just luck.	HHHTTT	I don't think you would get 3 heads in a row then 3 tails because it doesn't seem right.	representativeness
S11	THHTHT	blank.	HHHTTT	blank	??
S9	THHTHT	Because when you usually flip a coin, you don't get heads or tails all in a row.	THTTTT	Because you usually get a more even amount.	representativeness
S8	HTHTHT	I don't really think the coin will land the same way a lot.	HHHTTT	It won't land the same way 3 times -it's very unlikely.	representativeness
S10	HTHTHT	blank.	THTTTT	blank	??
S5	equal	They're both equally likely because you can't ask for the order they're in.	HHHTTT	Because 3 heads then 3 tails - and you can't do that.	outcome approach (Q.5) representativeness (Q.6)
S6	equal	You can get a head or a tail, either way.	HTHTHT	Because it keeps getting the same pattern again and again - it's not very likely to happen.	outcome approach (Q.5) representativeness (Q.6)
S12	equal	It's all chance.	HTHTHT	blank.	outcome approach (Q.5) representativeness (Q.6)

The representativeness heuristic was the most common explanation of the misconceptions displayed. Students use this to explain a result being either too regular, or too irregular with long runs, and therefore not as likely. Of the eight students who answered both parts correctly, there were only two who appear to have a normative understanding regarding these outcomes. For example:

- *It doesn't matter because there are thousands more combinations and they all have an equal chance.*
- *Flipping a coin is a game of chance. You can get all heads or all tails or randomly.*

The other responses could be classified as involving the outcome approach, as it is not possible to predict which outcome would occur, so therefore all are equally likely.

Two of the students (S5 and S6) displayed an inconsistency in their reasoning for the two related questions - in Q.5 their reasoning is based on the outcome approach (even though their answers were correct) and in Q.6, they have used the representativeness heuristic. This inconsistency, of which they obviously were not aware, is referred to as the M-L switch (Konold et al, 1995).

Question 7 : Throwing one dice

This question gave a number of different statements about the expected outcome when throwing one dice. It was to explore the belief systems related to the causal approaches to probability, that research has shown commonly affects the judgments made by students of this age. The students were asked to answer true or false to each of the following statements:

When one dice is thrown,

a) all the numbers have the same chance of coming up.
b) Some numbers have more chance than others of coming up.
c) Some numbers have less chance than others of coming up.
d) It is easier to get some numbers than others.
e) It is harder to get some numbers than others.
f) I can control the throw of the dice to give myself more chance of getting the number I want.

Eight students answered all parts correctly. The remaining students gave inconsistent answers. For example, although one student agreed that all numbers have the same chance of occurring, she stated that it is easier to get some numbers than others, and harder to get some than others. This student does not equate the 'chance' of certain outcomes with how 'easy' or 'hard' it is to obtain those outcomes. Four of the students agreed that all the numbers have the same chance of coming up, but then affirmed that some numbers have more chance of coming up, while others have less chance of coming up. Four of the students believed they are able to control the dice – for these students, probability has a causal basis – although three of them correctly answered the other parts, indicating that all outcomes have the same chance and there are no numbers that are either easier or harder to get than others..

Two students had similar ideas, but the reverse of each other: one claimed that some numbers are harder to get than others, but none are easier to get; while the other took the opposite view – some numbers are easier to get than others, while none are harder to get.

It was not possible to follow up on these questionnaire responses, and so it was not known how the students would have explained the discrepancies in their reasoning, if it had been possible to challenge them.

Questions 8 and 9 : Totals from two dice

These dealt with throwing two dice and comparing the probabilities of the possible totals of the numbers on the dice.

8. A pair of dice
Put a ring around T if you think it is true, or F if you think it is false.
When 2 dice are thrown, and the numbers added, which of the following is true?
a) All the possible totals have the same probability. T or F
Explain or give an example.
b) Some totals are more likely than others T or F
Explain or give an example.
c) Some totals are less likely than others T or F
Explain or give an example.

9. Betting on 2 dice
Put a ring around the total that has a better chance of coming up when 2 dice are thrown fairly, and the numbers on the dice are added. Or do they have the same chance?
a) total of 3 or total of 6 or the same chance
Explain:
b) total of 7 or total of 10 or the same chance
Explain:
c) total of 2 or total of 12 or the same chance
Explain:
d) total of 3 or total of 11 or the same chance
Explain:

The differences between the questions 8 and 9 were language-based (same probability, more likely, less likely, better chance, same chance) as well as focusing on particular outcomes. As well as the language aspect, the 'equiprobable bias' was another focus.

Two students answered all but one part correctly, showing that their reasoning did not succumb to the equiprobable bias. One of these students listed and counted the possible outcomes to justify her answers, while the other did this partially, only giving the possibilities for the preferred answer. It appears that while this justification was incomplete, she understood that the outcomes are not equiprobable.

Of the remaining students, a number of different misconceptions were displayed. Five students claimed that although the totals had the same probability, some were more likely than others, and some were less likely than others. This appears to point to a problem with the language, in that the expression "same probability" was interpreted to mean something different from the mathematical meaning. Also, it may indicate that the expressions of "more likely" and "less likely" are considered in relation to the students' experiences and what they can recall, rather than from a probability perspective. If this is so, the misconception could be classified as use of the 'availability heuristic'.

One of the students responded that the totals have the "same chance", but it is "not as likely" to get two ones compared with a total of 12, or to get two fives compared with a total of 7. This suggests that the doubles are less likely than other outcomes, although there is an obvious oversight that when comparing two ones with a total of 12, the only way to get a 12 is by two sixes. It is clear however, from the explanation, that the understanding that doubles are harder to get than other outcomes is not based on normative reasoning. A similar explanation in relation to doubles occurred with another student, who stated that the totals have the same probability, and none are more likely nor less likely, yet there is a better chance of getting a three compared with a six because it is easier not getting doubles – maybe he only thought of six as a double three. He then stated, consistently with the first part of his answer, that for each of the pairs of totals given (7 and 10, 2 and 12, and 3 and 11), both totals have an equal chance of occurring.

Another two students, having stated that the totals do not have the same probability, then claimed that none are more or less likely than any other. One of these continued then to state that the even totals (2,4,6,...12) have a better chance of occurring than the odds (3,5,7, ...11), and each of the even totals have the same chance.

One student reasoned correctly about the relative likelihood of different totals by referring to the ways that each total could be obtained, but she inadvertently suggested that there were more ways to get a total of 10 than a total of 7. It is possible that she forgot that the highest number on a dice is six, but was thinking of all the pairs of numbers that total to 10. Another student referred to her personal experience, such as "I always get three" compared to six, so the total of three is more likely. This belief, based on personal experience of "what I get" or "what is easy for me to get" is relatively common in children. The availability heuristic appears to have been used in this instance. One student, who used normative reasoning on some parts of the questions, suggested that double ones are harder to get in practice than double sixes, therefore less likely.

Four students who were incorrect in all their answers used the equiprobable bias as the basis for their answers and their reasoning.

Question 10 : Two dice and the numbers on each dice

10A. 2 dice

Put a ring around the letter of the statement that you think is best.

A. I throw 2 dice.

a) There is a better chance of getting a 5 and a 6 than getting a 6 and a 6.
b) There is a better chance of getting a 6 and a 6 than getting a 5 and a 6.
c) Both have the same chance.

Explain.

10B. 2 dice

Put a ring around the letter of the statement that you think is best.

I throw 2 dice.

a) There is a better chance of getting a 1 and a 6 than getting a 6 and a 6.
b) There is a better chance of getting a 6 and a 6 than getting a 1 and a 6.
c) Both have the same chance.

Explain.

10C. 2 dice

Put a ring around the letter of the statement that you think is best.

I throw 2 dice.

a) There is a better chance of getting a 3 and a 4 than getting a 3 and a 3.
b) There is a better chance of getting a 3 and a 3 than getting a 3 and a 4.
c) Both have the same chance.

Explain.

Only two students correctly answered the three parts to this question. For one of those students, it is not possible to conclude why he was correct: his justification for each part was that "You cannot judge it", which is rather inconclusive as to his understanding related to the question. It may have been that in each case he chose the first option which, coincidentally, was the correct answer. The other student claimed, as one of the reasons, that there were two numbers so there are two chances of getting it. A third student answered two of the parts correctly but missed the third. She suggested that a five and a six would be more likely than a six and a six because "it is very hard to get a six". Again, this is an example of a personal belief that impedes a normative understanding of probability from developing.

All the other students chose the option that the two different outcomes have the same chance of occurring. Among the reasons given for their choices were:

- *Anything can happen.*
- *You cannot tell.*
- *Because they both can come up".*

None of the students, including maybe the ones who answered the question correctly, appeared to have the understanding that, for example, a five and a six can happen two ways so therefore is more likely than a six and a six.

It is acknowledged that this question may have been misleading for the students because of possible ambiguity in interpreting the question. The students, rather than interpreting the question as intended (the order of the outcomes is unimportant), may have taken it to mean that the order of the outcomes was relevant. If they had interpreted the question in this way, then most of the responses would have been correct.

Students' understanding: a summary

In this section, the questionnaire responses of each student have been analyzed to determine the types of reasoning used by each student and their possible misconceptions. Table 4-2 (on page 49) summarizes the findings. A wide range of misconceptions were demonstrated by students, even in situations where they have answered the questions correctly. The misconceptions included the representativeness heuristic (including both the negative and positive recency versions of the gamblers' fallacy), the outcome approach, the M-L switch, the equiprobability bias, various belief systems, and language difficulties. There were also numerous instances where the written explanations for answers were insufficient to determine the understanding that the students held regarding the probability concept under investigation.

Table 4-2 also includes for each student a suggested dominant type of reasoning used, where it could be determined from the responses. For some students, there was no obvious dominant type of reasoning, as indicated by "??" in the table. Overall, the students' written explanations suggest that the predominant type of reasoning used is the outcome approach – all students used it in some form, mainly in relation to the belief that they could not predict exactly what would be the result of a random event.

Difficulty with language and terminology, or lack of understanding of the terms, was also quite common, in the questions that asked about the *chance* of something occurring, *how likely* was it to occur, the *probability* of it occurring, or *how easy or hard* was it to get a particular result. These were not interpreted by the students to be equivalent questions. For instance, five students indicated that the totals from throwing two dice have the same probability, but in a subsequent question gave some totals a better chance of occurring than other totals.

It is not surprising that, in a small sample of students, there is no obvious pattern in the misconceptions shown in relation to the level of mathematical ability of the students, nor their school year level.

Table 4-2 Summary of responses, reasoning, and misconceptions of students

Student: year level	Math.ability: gp. for game	Responses, reasoning, and misconceptions demonstrated	Dominant reasoning
S1: 8	above ave.:1	Satisfactory responses Q. 1-7. Outcome approach; Equiprobability bias; equiprobability bias plus outcome approach; normative reasoning.	outcome approach
S2: 7	average:2	Satisfactory responses Q. 1,2,5,6. Outcome approach (cannot predict outcome); language - easier or harder to get but not more or less likely; belief - evens have better chance than odds; not equal probability but not more or less likely; equiprobability bias with outcome approach.	outcome approach
S3: 8	above ave:2	Satisfactory responses Q. 1,2,3,5,6. Gamblers' fallacy (negative recency); causal (can control dice outcome); Equiprobability bias with outcome approach.	equiprob. bias
S4: 7	average:2	Satisfactory responses Q. 1,2,5,6,7. Many reasons not given or unclear. Equiprobability bias with outcome approach.	outcome approach
S5: 7	below ave.:1	Satisfactory responses Q. 1,2,7. Causal; outcome approach; representativeness; equiprobability bias; language - outcomes have equal probability, none more or less likely but some have better chance; normative reasoning.	outcome approach
S6: 7	average:1	Satisfactory responses Q. 1-4, 7-9. Normative; outcome approach; representativeness; equiprobability bias.	normative reasoning
S7: 8	average:1	Satisfactory responses Q. 2,3. Outcome approach; representativeness; language - equal probability but some have more or less chance than others; inconsistency with higher numbers most likely, and lower total easier because of small numbers on dice; 2 and 12 same chance but 2 not as likely (language problem); availability (based on numbers I always or hardly ever get).	??
S8: 8	above ave.:2	Satisfactory responses Q. 1,2. Outcome approach; gamblers' fallacy (positive recency); representativeness; causal (can control the dice); normative; equiprobability bias.	??
S9: 7	below ave.:4	Satisfactory responses Q. part of 1 and 2. Language difficulties; outcome approach; representativeness; causal (can control the dice); availability ("From my experience..."); equiprobability bias; language - (equal chance outcomes plus some more and less likely).	??
S10: 7	above ave.:3	Satisfactory responses Q. 1-4,7-9. Outcome approach; representativeness; normative reasoning; equiprobability bias?.	normative reasoning
S11: 7	average:3	Satisfactory responses Q.1,2. Outcome approach; representativeness; language - equal chance and equal probability outcomes but some more and less likely; some normative reasoning; equiprobability bias.	??
S12: 7	average:4	Satisfactory responses Q. 3,7. Language difficulties; outcome approach; representativeness; equiprobability bias.	outcome approach
S13: 8	average:4	Satisfactory responses Q.1-6. normative reasoning; language - equal chance outcomes but some more and less likely; some easier to get but none harder to get; probability term not understood; equiprobability bias; outcome approach.	normative reasoning
S14: 8	above ave.:3	Satisfactory responses Q. 1-6,8,9. Normative reasoning; outcome approach; causal (can control the dice); equiprobability bias; probability term not understood.	normative reasoning
S15: 8	average:4	Satisfactory responses Q.1-3,5-7. Normative reasoning; equiprobability bias; belief system (luck?).	equiprob. bias
S16: 8	average:3	Satisfactory responses Q. 1, part of 2, 3,5-7. Subjective terms not understood; outcome approach.	outcome approach

4.3 Game playing - Observation and Interviews

Introduction

Further evidence of the students' thinking and understanding was provided by the game playing interactions and interview questions, which are discussed in the subsequent sections. Two game playing sessions, each using a different game, were conducted at the respective students' school, in a room separate from the classroom. Each group of four students, chosen by the researcher, had two game playing sessions that each lasted about 45 minutes on different days. The students played in teams of two, again chosen by the researcher so that each team consisted of one boy and one girl. Each group of four students was kept the same for both sessions, but the teams changed from the first session to the second. In both sessions however, each team consisted of a boy and a girl. At each session, after the game was explained, the teams played the game a number of times. In the discussion which follows, the term 'trial' is used to denote one play of the game, so in any one session, there were a number of 'trials' of that game. Although the researcher asked some questions during the playing of the game, most of the questioning took place at the end of each trial, and prior to the next trial.

First game session - game of chance

Description

The first game involved chance only. The game board (see Appendix) consisted of twelve spaces. The first team to get their counter from the start to the finish (12 moves) would win. Each time the pair of dice were thrown, the difference between the numbers on the dice determined which team moved – if the difference was 0, 1, or 2, then the red team moved their counter forward one space; if the difference was 3, 4, or 5, then the yellow team moved their counter forward one space. The two teams alternated the throwing of the dice, but either team could move as determined by the difference. Each group had four or five trials of the game during their session.

Initial ideas about the fairness of the game

Once the game had been explained, the researcher asked the students if they thought it was a fair game (which it is not – the probability on any one throw of the red team moving is two-thirds, while for the yellow team it is one-third). Three of the four groups who played the game gave an initial response that it was fair. Only two students gave reasons to support their claim and these reasons indicate use of the equiprobable bias:

- *Because there are two dice.*
- *Both teams have got three* [outcomes on which to move].

The two students who disagreed, claiming that the game was unfair, had however, opposing reasons: one student said, "No, it's easier to get 0, 1, or 2" (but gave no further supporting reason); while the other was of the opinion that "No, it's easier to get 3, 4, or 5".

Group 1

As a group, they moved towards a normative explanation for the game's lack of fairness, although it took four trials of the game to gradually come to a shared understanding. There were attempts at explanations which, although rather inadequate, helped move the group as a whole towards a reasonable understanding. Their discussion at one stage indicated a causal-type understanding of chance, through a suggestion that the way the dice are thrown could make a difference to the outcome. This however was soon overtaken by other views. Another belief that was mentioned was "beginners' luck". This arose from the teams swapping colours of counters, and the new pair with yellow gaining a higher score on the subsequent trial – they still lost by a reasonable amount though not as much as the other team in the previous trial – than the first team to have had yellow for two trials. However, the idea of beginners' luck did not gain credence as a significant factor with the group, and it was not mentioned again.

The group's opinion about the fairness or otherwise of the game was reasonably confirmed by the second trial, and consolidated further during the subsequent trials. They were quite convinced by the evidence from the four trials that the game was not fair. For the fourth trial the group decided that in order to make the game more fair, yellow could be given a head start. When asked how much of a head start yellow should be given, it was suggested by one student, "Halfway". Another agreed, saying that "It's half the chance". At this stage, the group had not completely evaluated the relative probabilities for red or yellow scoring on one throw of the dice, so it was the empirical evidence from the previous trials that led this student to make the suggestion about "half the chance". Another student clarified this by stating: "Yes, because when we finished last time, they were about half way, that's where they were averaging". As a result of this observation, they decided to put the yellow counter on the half-way point for the next trial. The result was that red won again, but yellow ended up only one square behind. This convinced them that their ideas regarding the relative chance of each colour winning were correct.

So from the start, where there were different ideas as to whether the game was fair and the possible suggestions as to which colour had the advantage, the group moved to an accepted understanding. This position was attained through the members of the group expressing their own ideas, listening to others, and modifying their ideas from the discussion and from the evidence gained from the actual results of the trials.

Group 2

This group initially believed that the game was fair. The first trial was close with yellow only two spaces back (a relatively unlikely result - with a probability of about 0.05). Consequently, they remained satisfied that the game was fair. Each pair predicted that they would win the next trial. The second trial was a more convincing win to red (yellow only got to the fourth space). The reaction from one (S3) of yellow pair was that the dice were biased ("they're rigged"), and consequently he wanted the dice changed for the next trial. S3, an above average student, dominated the discussion following the second trial. Once red had won the third trial, there were some interesting ideas expressed. Student S8 claimed:

- *I think yellow is very unlucky for them, because the board is yellow, one dice is yellow, and their counter is yellow. And they're not winning at all.*

One of the pair decided then that they would exchange their counter for a blue one. Two of the players indicated that they thought that by changing the colour of the counter, there would be a better chance for that team, although one of them conceded, upon being challenged by the researcher about this view, that it would make no difference to the game. S3 continued to maintain however that the dice were 'rigged', although there is some doubt that he genuinely held this view (based on the way he said it, and his sense of humour that had been demonstrated at times through the session). He also claimed that it was just his unlucky day. The beliefs that were expressed within this group were partly pointing towards a causal basis of probability.

The following trial was even closer in its result than the first trial – this time, yellow was only one space behind red at the end. The probability of this occurring is only 0.03 – an even more unlikely event than the result of the first trial – for every hundred games, the result of red winning with yellow only one space behind would occur, on average, only three times. However, at this stage, there was a belief that red had a better chance of winning (even though there had been two trials with a winning margin of only one or two spaces). When there was a possibility of the teams changing colours (and therefore changing the chances of winning), the red team did not really want to, whereas the yellow team were quite happy to do so. When asked which team would win the next trial, the 'new' red team indicated that they would, "because we're red". Red won again, this time by four spaces, and the student, who had believed quite strongly that the dice were biased, now was of the opinion that "The dice aren't rigged at all. The colour red is just lucky." On being questioned about this statement, he made it clear that, instead of red having a better chance of winning, red was just a lucky colour.

One player commented, after the group was asked whether they had learnt anything from playing this game, that:

- *We can't learn much, because it's ... not a game of skill, it's more chance.*

The group did not move towards a normative understanding of probability, possibly because the results in the four trials were not conclusive enough to convince them of the greater chance of the red team winning, and possibly because of the strong influence exerted on the group by one student in particular.

Group 3

This group started with the belief that the game was fair. Part way into the first trial, after only six throws of the dice, red had moved four spaces while yellow had only moved two spaces. This led S14 to suggest (unprompted by the researcher) that it was not a fair game. She was part of the pair who were 'yellow' in the game, and were 'behind' at this point, and her partner suggested, and she agreed, that it was unfair because they were losing. However, very shortly afterwards, before the first trial had finished, she again suggested that it was not fair and backed-up this statement with a certain amount of normative-type reasoning:

- *Because there's more chance of getting a one and a two than having a higher number like a three or a four or a five.*

Her reasons were supported and refined by one of the students in the opposing pair. The trial finished with yellow only two spaces behind red (another close and unexpected result – probability of this result being 0.05). S14 again continued to refine her argument for the game being unfair. This student had used a reasonable amount of normative reasoning in the questionnaire, but had also demonstrated the range of other typical misconceptions.

During the second trial, which was won by red by a large margin, there was little discussion, unlike the first trial. On completion of this trial, student S10 discussed the possibility of God being involved – "Maybe God's giving them luck" – a belief system which affects the understanding of probability. On being questioned about this by the researcher, S10 replied to the effect that it really doesn't make a difference to the outcomes, although another student, by this stage, had already suggested that God makes no difference to the outcomes. Whether S10 did believe that God makes no difference, or whether she had changed her response because of the influence of the other student, is not known.

Following this exchange, the student who had originally argued that the game was not fair, continued with her reasoning, which by this stage was becoming more clear. She was firmly of the view that the game was not fair because there of the higher chance of red winning than of yellow winning.

It was decided by the group that for the third trial, the pairs would swap colours. The researcher asked about the throwing of the dice and whether that influenced the result, because some of the students had started blowing on the dice before they were thrown. S14, who had previously exhibited normative reasoning, indicated that: "If you control the dice, then you can get what you want". This causal approach to probability was quite different to the normative reasoning that she had been using up to that point. The suggestion that blowing on the dice could make a difference was rejected categorically by the remainder of the group, although there had been a number of turns when various students blew on the dice before they threw.

The third trial was convincingly won by red again, and so it was reiterated that the game was not fair. The group as a whole had now accepted this viewpoint. On being asked how the game could be made more fair, it was suggested by one student that one pair would move if the difference was 0, 2, or 4, while the other team would move if the difference was 1, 3, or 5. This would indeed result in a fair game - the probability for each colour winning on a throw would be 0.5. The reason for her suggestion was not given nor asked for by the researcher, so it is not known whether the suggestion was based on normative reasoning, or whether possibly these numbers gave a more even spread for each team (for instance, 1, 3, and 5 versus 0, 2, and 4 as compared with 0, 1, and 2 versus 3, 4, and 5). The final trial proceeded under the new rules. After six throws, the scores were even, and at that point it was decided that the game was fair, even with those few results on which to base their conclusion.

There was much more interest and comment generated during this trial, which red won, but by only one space from yellow. This increased level of interest may have been caused by a number of factors: the closeness of the scoring all the way through the trial; a greater personal interest because the game was being played to their own rules and hence a personal 'investment' in the outcome of the trial. Following the trial, the researcher questioned the students about their belief in the fairness of this game, and how it was that they were convinced when the result was so close (only one space separating them at the end), whereas in the first trial (under the original rules), red had won by only two spaces. In spite of the closeness of that first trial, the students formed the belief part-way through that trial that it was not fair and this had not changed at the end of that trial. The response from one of the students was that the closeness or otherwise of the counters throughout the trial, rather than specifically the end result, had convinced her the most. This viewpoint was supported by one of the other students as well.

Group 4

This group started with the belief that it was a fair game, and as discussed earlier, two students had offered reasons why they believed this to be the case (see p.50). One reason conformed to the equiprobability bias – because each team had three outcomes that they could score on, then the game must be fair. The other reason was unclear, but it related in some way to the use of two dice. Trial one was won by red, with yellow onto the ninth space. One player suggested that it was a fair game, even though he had lost, but he also suggested an improvement: "It would be better if they had 0, 4, and 2 and we had 3, 1, and 5." This was immediately countered by an opposing player who said that she still wanted one [as one of her numbers to score on] because "it is the one that usually gets done". Two of the students agreed that the suggested change would be better, because it would be all chance then.

The second trial was won easily by red, with yellow only getting to the fourth space. One of the red team claimed that it was a fair game, while her partner disagreed. The researcher probed further into the idea of the game being fair. The responses from within the group were:

- *Chance.*
- *It's all luck.*
- *It's just how lucky you are.*

Against these viewpoints, the remaining student tried to explain her idea. It was towards a normative view, although she could not explain it adequately for the others to be able to think about it before being interrupted:

- *Somebody gets a two and a six and that's fair, but if you get two and something closer then that's .....*

Two of the other students instead talked about chance, one referring to Lotto and something she had seen on television about a man who had kept records of the results, attempting to show that it is not all chance, but rather that some numbers are occurring more often. The other student commented that he is "quite lucky at rolling 6s with rolling one dice". For the next trial, the group decided that for each turn, rather than teams alternating the throw of the dice, each team would throw one dice. The result on this trial was red 12, yellow 7.

The researcher reminded the students of the results from the three previous trials, and enquired whether the teams still thought the game was fair. Two of the students started to explore and explain their ideas about why the red was "more likely to get there". Their reasoning was approaching a normative understanding. S9, who had earlier expressed views about red being a lucky colour, was asked if it would make a difference if they were given a different coloured counter, such as blue or green (but the rules otherwise staying the same). Her response was that it would, and that she would like a blue counter.

The next trial elicited some comments from the students about the dice outcomes not being fair, but this was not followed up by the other students. The conclusion formed by one student after this trial was that blue was a sad colour while red was a happy colour. Another student, realizing that red had won every trial (even though the teams had swapped colours), suggested that it may be something to do with the size of the dice (one was smaller than the other). Two of the others disagreed that it would make any difference to the results. The student, who had suggested that colours may make a difference then wondered about the possibility of kissing the dice before throwing them. This idea gained no credence with the others, and was not followed up. The number of comments made by this student about some possible 'outside influence' on the outcomes of throwing dice strongly suggested that she had a causal view of probability.

Two of the students continued to express normative ideas and they refined their ideas in a collaborative way. They had some difficulty however expressing the ideas with sufficient clarity for the other members of the group to make sense of them, but they progressed their shared understanding of the reasons for the game not being a fair one. In spite of this, one member of group maintained her view of it all being just to do with chance, and that really anything could happen – an outcome approach to probability.

Second game session - game of strategy and chance

Description

For the second game session, each group of four students from the first game session was kept the same, but the pairs were changed, so that no individual played with the same partner from the first session. (Note that in the discussion which follows, the group and student numbers used are consistent with those used in the discussion on game one.) This session used two different versions of a game, each involving both chance and strategy. With version A, the total of the points on two dice determined the play. The board (see Appendix) consisted of two sets of the numbers from 1 to 12 spaced out along the board, one set to be used by each team. The rules and aim of the game were explained to the teams. The teams were given 10 counters, and they had to decide on which numbers they would place their counters, with more than one counter able to be placed on any one number. Following each throw of the dice, one counter could be removed from the board if the number it was placed on was the same as the total of the two dice. The teams alternated the throwing of the dice, but either team could remove a counter from the board, irrespective of which team threw the dice. The aim of the game was to be the first team to remove all their counters from the board The teams were given some time to decide on which numbers they would place their counters. This was the strategy part of the game. Once the counters were placed on the board, and the dice throwing started, the outcomes were "determined by chance" – the choice of numbers helped improve (or otherwise) the chance of winning the game. This version of the game was played twice by three groups, with the other group playing it three times.

Following the version A trials, the board was changed for version B (see Appendix), which followed exactly the same rules as version A except that the play was determined by the difference between the two dice, rather than the total. The board for version B had the numbers from 0 to 6 spaced out along the board for each team.

Group 1

This group played most of the first trial of version A (based on dice 'totals') in silence, apart from the occasional sign of pleasure at a good outcome for their team. The team with the counters less spread to the extreme numbers won the trial, so for the next trial, the other team chose to put five counters on the 7. This trial was eventually drawn, because both teams were left with counters on the same numbers, so they realized that neither team could win.

This group, collectively, accepted a normative explanation about the relative probabilities of the various outcomes. Only one student had not shared the same understanding initially, but he appeared convinced by the reasoning given by the others. They decided that if the game was to have been played again, they would have put their counters on the numbers 6 - 9, or maybe 6 - 10.

For version B of the game (which used dice 'differences'), the teams decided at the beginning not to put a counter on 6, as this difference was impossible. The placement of the counters by the two teams was very similar, and the trial was drawn. The next two trials were won by the same team, even though for one of the trials, the other team had chosen a combination of numbers with a better chance of winning.

The explanations at the end of the trials acknowledged that there were numbers that had a greater chance of occurring, although there was some indecision about what were the best numbers to choose, in terms of the chances of those numbers occurring.

Group 2

In the first trial of the version A game, one team put a counter on number 1, which is impossible to obtain when taking the total of the dice. It was not until some time into the trial that this team realized that they could not win the trial. They still carried on for a while from this stage, but then decided that there was no point in continuing. For the second trial, both teams chose a set of numbers for their counters which gave them an improved chance of success compared with their choices for the first trial.

For the second trial, the team with the better choice of numbers (that is, the set of numbers with the higher probability of winning than the second team's choice) won. The students noticed that some numbers came up more often than others, for example 6 and 8 occurred more than 2 or 3. One student commented that a one on each dice hardly ever came up. At this point, however, one of the students maintained that "they still have the same chance as the others". This student (S8), in the questionnaire responses, had displayed the equiprobability bias, which was again being demonstrated by her comment here.

When the group was asked about where they would place their counters if they were to play the game again, S8 commented that she would not use the 2 or the 12, but would put them more around the middle numbers. She reasoned that there was only one way of getting the numbers 2 or 12, as there is with the number 3, whereas the number 4 has more than one way. Two of the students agreed that the best place for the counters is from 4 to 9 because those numbers come up more often. The students were asked that if those numbers come up more often, does this mean that they have the same chance of happening? The responses from the students indicated some indecision. It appears from this exchange that a frequentist interpretation of probability (that is, the relationship between empirical results and the concept of chance or probability) is not part of the understanding of these students.

This group then moved onto version B of the game, placed their counters, and after one throw of the dice, one student came to the realization that they could not win the game because they had a counter on 6 (which is an impossible difference to obtain). The trial was immediately conceded by this team. For the second trial this team naturally improved their choice of numbers compared with the first trial, and the second team kept their counters on the same set of numbers. The students were asked which team they thought would win by looking at the sets of numbers chosen, and the one student to respond predicted that his team would win. Despite the first team having a higher probability of success in this second trial compared with the second team, they lost the trial. The student's prediction (of a win for his team) was satisfactory from a normative perspective, but the empirical result from this single trial did not match the theoretical result.

After placing the counters for the next trial, one of the players commented that their choice was "bad", because they had two counters on 5; and the only way of getting 5 was with a one and a six. Her team lost the next trial. For the final trial, one student suggested that all their counters be placed on 3, but after being reminded that throwing one 3 would not be sufficient to win (by removing all the counters in one turn), the counters were spread more. It was this team that won this final trial.

The students were questioned about the numbers that had been coming up the most often, and their responses did not actually match the reality. They commented that 3 and 4 were common, when in fact they had not. This 'distorted' memory of what had occurred previously matches the availability heuristic (as discussed in the Chapter 2). It was suggested by one of the students that if the trials were to continue they would improve their chances of winning because they would get to know which numbers came up more often. At this stage one student reminded the others that any number can occur, and that it's just luck. She had not moved from her outcome view of probability and equiprobability bias through playing the game a number of times (four trials with at least 15 throws of the dice for each trial).

Group 3

For version A of the game, both teams started with their counters spread out along the board, with only one team putting two counters on one number, namely 7. This resulted in a draw, without much comment from the students. In the second trial, neither team improved their game strategy, and the team which won this trial had chosen a set of numbers with a lower probability of winning than the other team. At one point during the trial, there had been three 7s in row thrown, and one student said that maybe they should have put all their counters on 7. When the group was asked if all the numbers have the same chance of occurring, one student answered no; he reasoned that 2, 3, 12, and 11 each only have one way of being obtained. The group eventually accepted that the best choice for placement of the counters would be on 4 up to 10, but again not much discussion occurred.

For version B of the game, one player immediately realized at the start that 6 was impossible to obtain and so no counters were placed on that number. Apart from that, the strategy of both teams was to have their counters reasonably spread along the board. The first trial resulted in a draw, while for the second trial, one team shifted only one counter while the other team moved just two. The team which had the lower probability of winning won the trial however. They indicated in the discussion that the lower numbers had a better chance of occurring, yet the teams had not placed more counters on these numbers.

The final part of the discussion was about games played at home and some of the students talked about playing board games such as Monopoly. Although it was acknowledged that sometimes when you want to land on a particular number, the dice cannot be controlled, one student commented that dice can be controlled through the way that they are rolled. A causal explanation in relation to probability still existed with this student, in spite of: the experiences resulting from playing the various games; and being questioned about their ideas and required to justify them to the other students in the group and the researcher.

Group 4

This group consisted of only three students, as the fourth one was absent from school. Before the first trial got underway, S12 talked about the numbers that had more chance of occurring, and used this to help him decide where to put the counters. His argument was normatively based. After one throw of the dice, he realized that the other team had put a counter on 1 but they carried on for a while longer until that team conceded that there was no point in the trial continuing. S12 made the further observation that the big numbers work better and have more chance. His partner (S9) disagreed. He referred to the fact that there's only one way to get 3 but many ways to get 8, at which point S9 made a counter-suggestion with 12:

- *A 12 doesn't work as well because it's hard to get two sixes.*

The discussion continued further, and, following another suggestion that there are more ways to get the higher numbers, S9 then agreed that there are more ways to get the higher numbers. This appeared to contradict her earlier example of the 12 that it's hard to get two sixes. It may be however that her reasoning about the 12 was based on the belief of the difficulty of getting two sixes, even though later she thought that there were more possible dice combinations for getting these higher numbers. As she started to list the combinations for getting 10, one of the other students corrected her when she gave 3 and 7 as a possibility. She realized at this point that there are only two ways to get 10. The group accepted this as evidence that there are not necessarily more possibilities for getting higher numbers.

S13 questioned whether the two dice should be the same size, because by having one dice smaller than the other, it can roll more, so maybe the dice should be the same size for it to be a fair game. This causal approach to probability was the first such indication from this particular student; she gave no questionnaire responses that indicated this type of understanding.

For the second trial, each team had only one counter on each number, and the only difference between the teams' placements being that one used the 11 and the other used the 12 instead. Part way through the trial, the fact that the 'middle' counters had been removed and only the 'extreme' ones remained, was commented on by S13. The placement of the counters for the next trial saw one team decide against two counters on 8 but opt for two on 4 (a lower probability of occurring). The change was made after listening to the justification from one student, who had previously demonstrated much less normative understanding than her partner, who acceded to her wish. This team won the trial however, assisted by the selection of numbers by the opposing player, who had chosen a less likely combination for winning.

In the subsequent discussion, it was agreed by the group that the middle numbers, from 4 to 8, occurred most often. But then the observation was made by one of the students that in the various trials that they had played, the small numbers occurred first for removing the counters, leaving the bigger numbers "waiting", while in another trial the opposite had occurred.

Before the first trial of version B of the game, one player suggested not putting counters on 6 because you can't get this difference. His partner commented though that in the previous trials they had put counters on 6 and they always got that number. The explanation from the other two students in the group convinced her of the reason why the change in the game made the 6 impossible. The two teams selected almost identical numbers and the team won that had the choice that was more likely to win. However, neither team chose a selection that reflected the probabilities of the various differences. For each trial, the counters were reasonably spread over all the numbers, rather than concentrating them on the lower numbers. The comments made about the numbers that occur most often was not followed up in the selection of numbers on the board.

4.4 Levels of participation by students

Although the students had been encouraged to talk and discuss their ideas throughout the trials, the level of involvement of the students varied, as measured by the number of times each spoke. Some of this variation may have been due to: discussion of ideas not being a natural part of playing games; the researcher not being well-known to the students; the presence of both an audio-recorder as well as a video-recorder may have made some of the students self-conscious about contributing; or their willingness to contribute reflected their attitude to and/or confidence in mathematics.

It was difficult to accurately count the number of times each student spoke, as there were a number of occasions when more than one student spoke at the same time. Estimates were obtained from the dual source of audio-tapes and video-tapes. Inaccuracies may have resulted from the occasions when there were a number of students speaking at the same time, although careful examination, where possible, of the video-tapes helped determine who was speaking and what was said.

The data in Table 4-3 below is an estimate of the number of contributions from each of the participants (excluding the times where it was not possible to determine who was speaking because more than one student responded at the same time). The data includes both prompted responses and 'spontaneous' contributions.

Table 4-3 Number of contributions by each student to discussions.

Student	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15	S16
No. of contributions	37	24	33	16	33	41	21	24	37	12	10	41	35	31	19 *	12
Math. ability	3	2	3	2	1	2	2	3	1	3	2	2	2	3	2	2

* *This student was absent for the second game session*

Statistically, there is no significant difference between the mean number of contributions for each of the three ability groups. Overall, the mean number of contributions per person (for the 15 students who were present for both game sessions) is about 27, but there is considerable variation in the number of contributions.

The types of reasons given in response to prompts and questions varied from student to student, and these affected the contribution of each individual to the discussion. The personality of each student also had an effect on the extent to which his or her contribution was listened to and critically examined by the others. For example, some students were more assertive in expressing their viewpoints, while other less assertive students allowed their comments to be dominated by others, even in situations where these other comments were less sophisticated mathematically, or less normative. Consequently, the nature of the groups varied, depending on the particular students in each of the groups, and the quality of the group interactions varied.

4.5 Comparison of the effects of the two games

'Spontaneous' discussions

To determine the level of involvement of each group of students during each of the game sessions, the number of comments made by students during a 'spontaneous interaction' were counted. A 'spontaneous interaction' was defined as being an interaction (one or more comments made by members in the group) which was unprompted by any question or comment from the researcher. For instance, often there was one comment made by a student that did not result in any other comments from the others – this was counted as one comment for that interaction. At other times, a comment from one student may have resulted in as many as nine comments altogether from the members of the group. The extent of the discussion indicated to the researcher the degree to which the students were stimulated and challenged by the initial comment and subsequent comments. Each 'spontaneous interaction' was identified from the transcript, and the number of comments made during that interaction were counted. The spontaneous interaction was concluded either by a significant pause in conversation while the game continued, the next comments being unrelated to the previous discussion, or intervention by the researcher with a question that was prompted by a comment made during the interaction.

An example of one 'spontaneous interaction' is:

S12: *[while placing the counters] - Big numbers, that works better. Big numbers have got more chance.*
S9: *No they don't.*
S12: *I think they do. Because if you want to get a 3, there's only 1 and 2, and you can't get another one. If you want to get an 8, that's 4 and 4, or 3 and 5,...*
S9: *A 12 doesn't work as well because it's hard to get two 6s.*
S13: *But on each dice, you've got all sorts of numbers up to 6. So if you throw them, you can get any number. It's just a game of chance. If you get two ones, that's 2, it's not a high number.*
S12: *That one is more likely....*

At this point, the researcher asked a question about one comment, so the spontaneous interaction was taken as complete, and consisted of six comments.

Only those spontaneous interactions were included that involved comments related to the game or to concepts relevant to probability. The data was collated for each of the four groups of students, over both game sessions, and is shown in Table 4-4 below.

Table 4-4 Number of comments per spontaneous interaction (unprompted) during games.

	Game 1				Game 2			
	Group 1	Group 2	Group 3	Group 4	Group 1	Group 2	Group 3	Group 4 *
Average number of comments per spontaneous interaction	2.4	2.0	1.8	2.2	2.5	1.7	1.7	2.1
Number of 'spontaneous interactions' during game session	32	24	25	24	21	21	13	29

* *Note: Group 4 consisted of only three students for game session 2, which may have affected these results.*

From the table above, it can be seen that game one (which involved only chance) generated 105 different spontaneous interactions overall, whereas game two (chance and strategy) only generated 84 such interactions. But since the average number of comments from students during each of those interactions is almost the same for each game (2.1 for game one, compared with 2.0 for game two), the greater number of spontaneous interactions during game one must have been shorter in duration than those in game two. For three of the groups, there was a decrease in the number of interactions from game one to game two. One particularly noticeable decrease in number of interactions (for group 3, from 25 down to 13 in game 2) may possibly be attributed to one of the students being 'grudgingly' involved in the second game session, as it meant that he was going to miss some of his class's Physical Education lesson that was due to start part way through the game session. His level of contribution dropped to almost nothing in game session two, even though he had participated quite well in session one. Group 4 was the only group to increase in the number of interactions from game 1 to game 2, even though one of the members from the first game session was absent. The reasons for this increase are unknown (a random fluctuation maybe?).

The quantitative analysis of the interactions for each of the games, as discussed above, is not sufficient in itself for drawing any definite conclusions. It is important also to consider the quality of the interactions, in relation to the concepts, and how well any group understanding of probability concepts may have developed. Therefore, the transcripts were examined to identify the concepts and misconceptions that were expressed by the students throughout each of the games, and how these may have been modified through the discussions of the groups.

Types of reasoning expressed during interactions

Group 1

Group 1 consisted of students covering the full range of mathematical ability (one rated above average, two average, and one below average). From the analysis of the questionnaire data, all the misconceptions were identified, and the suggested dominant understandings and types of probabilistic reasonings exhibited (see Table 4-2) were the outcome approach for two students, normative reasoning for one student, and the fourth student appeared not to have one dominant misconception – she used a variety of reasonings to the various problems, some of which were inconsistent or contradictory.

Group 1 - Game 1

During game one, the discussions revealed some normative reasoning, the outcome approach, and a causal approach, but gradually the dominant 'theme' of the discussion centred around normative ideas. Two of the students were dominating the discussions through most of the first three 'trials' of the game, but the student referred to above, who did not display any tendency towards a consistent understanding of probability, joined in the discussion, adding to the normative reasoning that had been discussed up to that stage. The fourth student, who had not contributed at all to the discussion up to this point (and incidentally was the weakest mathematically of the group), then joined in, and added a normative-based comment.

During the early part of the next trial of the game, beginners' luck was mentioned by the student with the strongest normative understanding. This may have indicated a belief in relation to probability concepts, although this is unlikely, because of her understanding that had been so clearly expressed. Her comment was not followed up on by any others in the group. Another student in the group, who had also been 'influential' in the discussion in relation to the normative understanding, expressed a view that the game was not fair, not because of the results or the discussion of the reasons related to fairness, but because one team (ie. colour) was always losing. However, the discussion that followed quickly resorted to consideration of the relative frequencies of the results and the possible outcomes of the dice, showing the differences that were more likely and those that were less likely. Together, this group (led by two students in particular), decided to change the rules to try and make the game more fair, which they did successfully!

The discussion of the group had moved quite strongly towards a normative understanding, even though there were instances when they moved 'off-track' because of various comments (such as the mention of beginners' luck, and aspects of fairness or otherwise). These did not unduly influence the group, and the role that two of the students had played in leading the discussions was quite obvious.

Group 1 - Game 2

During the early part of this series of games, normative-type understanding was being expressed within the group, again led by the two students who had been more assured of their views in the first game. One of the others expressed an idea related to the equiprobability bias, but this was refuted by one who explained some of the ways that certain numbers could occur. The other students listened and consequently agreed that the numbers were not equiprobable. The group had quite clearly developed an understanding together through the discussions and particularly because of the clear explanations given by two of the students.

Group 2

This group had two students whose mathematical ability was judged to be above average and two who were average. In their questionnaire responses, they exhibited the range of ideas and misconceptions about probability (see Table 4-2). Two 'preferred' the outcome approach in their reasoning, one the equiprobability bias, and one had no dominant misconception (but had demonstrated a variety).

Group 2 - game 1

Throughout the trials of game 1, this group did not progress towards a shared understanding of a normative type. From the outset, when it was agreed by the group that the game was fair, a variety of types of reasoning were exhibited. The large winning margin of the first trial was explained by causal-type reasoning, with a suggestion (made rather flippantly) that there may have been magnets inside the dice that caused them to be 'rigged'. Later, there were personal beliefs expressed, (such as red being a luckier colour than yellow, because the other team had a yellow counter, the board was yellow as well as one of the counters). Following the completion of another trial, there was causal explanation of the result, in relation to the way that the dice were thrown and how it may advantage one team. To overcome this, it was suggested that each team should throw one dice each time, rather than the teams alternating the throwing of both dice. However, some of the group disagreed with this, stating that this would make no difference to the outcomes of the dice. The equiprobability bias was also demonstrated – even though one team had won three times in a row, one student (S8) said that it was a fair game because:

- *everyone has got one thing* [3 possibilities for winning].

The variety of types of reasoning demonstrated as the game progressed did not show any pattern of moving towards a shared meaning within the group. The dominance of two students in the discussions (S3 and S4) and the nature of some of their responses (that appeared rather flippant), prevented real progress in ideas. The lack of any normative reasoning at any stage meant that there was no base for the group to work from in developing their ideas.

Group 2 - game 2

Despite the lack of normative reasoning from any student during game 1, the interactions during game 2 showed some definite progress towards normative reasoning. Initially, discussion involving three members of the group centred around the understanding that some numbers tended to occur more often than others, and they had identified some of the numbers that were more likely to occur. However, S8 maintained that "they still have the same chance as the others". Her use of the equiprobability bias was quite strong, but she agreed that the totals from 4 up to 9 tended to come up more often than the other totals.

During the 'difference' version of the game, the types of reasoning used in the discussions did not vary much from the other games. Some normative reasoning was expressed by student S2, but following on from this, student S8 (who had a strong tendency towards the equiprobability bias as mentioned above) continued with her reasoning using the outcome approach. She also reiterated that although some numbers come up more often, that it was because of luck.

The discussion had not really progressed the understanding of the students, partly because this student (S8) had such a strong intuitive belief in the equiprobability bias that was not shifted at all, either by the results of the games, or by the viewpoints of the other students in the group.

Group 3

This group consisted of two above average students, and two who were average in their mathematical ability. In the questionnaire, two students (S10 and S14) displayed normative reasoning; one (S16) used reasoning in accordance with the outcome approach, and the fourth student did not display a 'preference' in his reasoning.

Group 3 - game 1

After expressing the view that the game was fair, normative reasoning was used only part way into the first game by student S14. This was supported by another student (S10) who pointed out that to get a five, there is only one way, namely by throwing a one and a six. The group progressed their understanding by the explanations offered for the game not being fair.

However, student S10 who had used normative reasoning in her arguments, then expressed a possible belief in God being responsible. This view was not supported by the others, and she also moved away from this reasoning. Causal-type factors were mentioned as a possible reason for the red team winning each time, which included a discussion on whether blowing on the dice prior to throwing them made any difference. Although it was denied that this did make any difference, a number of players blew on the dice each time they threw. Again the discussion returned to normative reasoning, as it did after the representativeness heuristic was used to support the view that the new variation of the game that they played was fair, even though only six throws had occurred.

The types of reasoning, other than normative, used in the discussion did not progress at all, as they were not taken up by any student following the initial expression of the ideas. Normative reasoning was consistently returned to, as the members of the group developed their understanding throughout the game and the discussion which occurred.

Group 3 - game 2

Throughout the playing of this game, the predominant reasoning used was normative, even though there was not much discussion generated. When asked whether all the numbers (in the 'total' version of the game) have the same chance of coming up, S14 acknowledged that the 2, 3, 11, and 12 each have only one possible outcome. This was supported by another student, who suggested that the numbers around the middle have the best chance.

In the 'difference' version of the game, it was noted by the students that certain lower numbers had a better chance of coming up, but the discussion was minimal.

One player stated that by playing the games he had learnt not to bet on the high numbers, while another knew that some numbers have a better chance of occurring than others. One student disagreed about the game being unfair , and made the comment that for this game (version B):

- *It was closer, because it's not uneven chances... It all depends on how you choose your numbers.*

Here the student appears to using the equiprobability bias, which he had also used in the questionnaire, and had not "been swayed away from", in spite of discussion and results to the contrary.

Group 4

This group did not have any above average students – one was rated below average and the other three average. Like the other groups, this group had used a diverse range of probability reasoning, as determined from the questionnaire. One had a reasonably normative approach (S13), one exhibited an outcome approach as the dominant type of reasoning (S12), and another 'preferred' the equiprobability bias (S15). The fourth student, S9, the weakest mathematically, along with S12, had not shown any normative reasoning during the questionnaire. S9, because of the misconceptions exhibited, could be considered to have the least developed probability understanding.

Group 4 - game 1

Throughout this game, a variety of misconceptions were expressed, but interspersed with some normative reasoning. At the outset, there were equiprobable-type reasons given to support the claim that the game was fair, but following the first trial, there was the realization that the game may not be fair. At first, it was suggested that it may be not fair because the student making the claim was on the losing side. However, he backed up his claim with some normative-type reasoning. The group then took up the idea of "it's all chance", that is, reasoning using the outcome approach. It returned however to normative reasoning, until S9 expressed a belief that it may have something to do with the colour of the counter, and so she would like a blue counter instead. Another student had suggested a causal understanding, related to how the dice was thrown and her luck with throwing a six with a dice.

There was a return again to normative reasoning, with one student explaining how some numbers were less likely to occur because of the fewer ways of getting them. He then suggested that to prove his explanation, he would throw the dice twice to show how it was harder for yellow to get the point. Unfortunately, his experiment was not supportive of his argument, but this was brushed aside.

Even at the end of this game, although there had been some good normative reasoning put forward, the group finished off with some comments about the colour of the counters, suggesting a causal understanding of probability. It was decided that the colours made no real difference to the outcome of the game, but S9 then talked about some colours being happy and others sad. Whether this related to the outcomes of the games, it is unclear, as the discussion was finished at this stage without anyone adding to the comment. There were suggestions of a causal approach to probability, through the possibility that the size of the dice may make a difference to the outcomes, even though this was refuted by two of the students. Also, the possibility of kissing the dice to affect the outcome was raised, but this suggestion was not taken up by the others. The discussion came back again to some normative reasoning from S12, who was contributing the most at this stage. The final comment however came from S9 who summed it up with

- *I think that it has just to do with chance.*

Group 4 - game 2

S12, who had used normative reasoning in the previous session, started off in the 'total' version of the game with normative reasons as to which numbers would be better for placing the counters. There was progress in understanding shown during this game with a tendency towards normative reasoning from S9, who up until now had not demonstrated any such reasoning. She started to list, for example, the ways of getting 10. When she inadvertently suggested a three and a seven to obtain a total of 10, this was pointed out by another student as being impossible. But even so, she was moving in the right direction in her understanding. A causal understanding from S13 was illustrated by the comment that the size of the dice may be a factor:

- *After our last session, I thought about it. Because we had two dice, one small and one big, I think you'd have more chance with the smaller one because it's smaller, it rolls more. You should have two dice the same if you want it to be a fair game.*

This view was not supported by the other students.

S9, who had shown little or no normative reasoning, agreed that some numbers around the middle were better for placing counters on, rather than the extreme totals. All the students finally agreed that some numbers come up more often than others.

In the 'difference' part of this game session, although some areas of misconception were used in the reasoning, again the predominant reasoning used was normative. A suggestion from S9 that one of the other students may be hypnotizing the dice was not followed up by the others. All three students (one being absent for this session) moved towards a normative view when agreeing that some numbers were more likely to come up than others, and they were able to give some examples.

Strategies used in game 2

The strategies of the placement of the counters in the successive trials of this game generally showed improvement in relation to the relative probabilities of success. However, although a pair may have improved their chance of winning compared with the previous game, this increased chance did not necessarily result in a win, either because the other pair may have had a 'better' choice of numbers, or, as happened a number of times, the opposing pair won in spite of having a lower chance of success with their choice of numbers. Such is probability!

In most trials that were played, the students did not appear to make a decision on the placement of their counters in relation to the placement of counters by the other pair of students. That is, they did not use the other's placement to improve their own chances of winning by strategically placing their own counters. There was a realization within the group about the numbers that were more likely to occur, and the placement of the counters tended to reflect this, although there was variation.

Chapter 5 Conclusions and Implications

5.1 Conclusions

The influence of games on group understanding

The student interactions while they were playing the games were analyzed and it is clear that some, although not all, of the groups developed a shared understanding of probability concepts that was more sophisticated than the understanding that the individuals had demonstrated prior to the game sessions. Group members were influenced in their viewpoints and understanding initially by their peers and in some cases by the researcher, as the students responded to a question or prompt designed to challenge their ideas.

The negotiation of meaning within the groups was not straightforward however. For instance, the influence of some rather flippant comments by particular individuals could not be underestimated, as these tended to cause some of the less assertive students in the group to contribute less to the discussions. Also, some comments could be considered influential, in that they altered the 'path' of the discussion, leading the group away from developing the normative understanding that had been occurring. This change of direction was in some cases only temporary, but at other times significant. This was particularly noticeable with those primary intuitions which had been identified through either comments from individuals that were expressed on a number of occasions, or from questionnaire responses. These primary intuitions tended to be dominant in the students' ideas. For instance, when one group was developing a clear, shared understanding that the 2-dice differences were not equiprobable, one student reiterated her belief that these differences "still have the same chance as the others". She had been actively involved in the discussion, contributing to the shared meaning, but this comment (which had previously been expressed a number of times) showed that her primary intuition about dice outcomes as being equiprobable (as identified in the questionnaire responses) was strong and dominant. For this group, the empirical results from the trials of the game were quite clearly helping the students develop a new understanding. However, the group took notice of this comment from the student, and subsequently moved from a normative understanding. The discussion did not regain the same position that it had earlier reached.

Do games encourage active involvement from the students?

Games were claimed to be useful for encouraging students to ask questions and reflect on responses, hence to make new deductions and inductions. From a social constructivist perspective, this is considered to be major factor in learning. There was significant variation between the level of contribution to the interactions by the individuals, and in the amount of interaction within the groups. Some groups played the games with minimal verbal interaction, while the amount of discussion in other groups was significant. The level of contribution by individual students also varied considerably.

Student interest in and enjoyment of the games also varied considerably, at an individual level and at a group level. Some students and some groups were quite subdued, while the excitement in others was obvious. Personality factors were probably the most significant reasons for the variation. Whether this influenced the learning however is not known, although a lack of interest and enthusiasm could well be problematic for learning.

Do the results of games influence students' thinking?

Nisbett et al (1983) and Amir and Williams (1994) had opposing views about the ideas that 11 and 12 year old children would have about dice outcomes. This study quite clearly supports the views of Amir and Williams, that students of this age generally have misconceptions about dice. It was common for students to believe that the various totals or differences possible from two dice were equiprobable (or in the terminology of Nisbett et al, these students did **not** have an understanding of non-uniform probability distributions). The relative frequencies of the outcomes, which varied significantly, should have alerted the students to the fact that outcomes have different probabilities of occurring. These results were not conclusive to the students as evidence to contradict their beliefs or intuitions. The games could be said, for these students, to be ineffective in encouraging and altering their thinking about this concept.

Comparison between the games of chance and of strategy and chance

The passive role of some students during the first game, involving chance only, supports the claim of Falk et al (1980) that there is greater potential for students to be passive during a game of chance, compared with a game of strategy and chance. As a solution, they recommended using games that needed some decision making from the students. The second game session in this research used such games, but with varying amounts of success in relation to how active or passive the students were. One group in particular were quite passive in this second game, because of the influence of one player who made it clear that he was concerned in case he missed some of his class's gymnastics lesson. Two other groups also reduced the number of 'spontaneous' interactions during this game compared with the first game. The reasons for the reduction with these two groups is unclear. It would appear therefore that this particular game of strategy and chance did not necessarily induce a greater level of participation from the students, when measuring their participation by the amount of verbal interaction. Some comments from individual students suggest however that they were more involved in this second game:

- *In these games, you are thinking of what move you are going to make, or what numbers you need.*
- *It has made us think more than in a game like Monopoly, where you just think about how much money you have to hand out or something.*
- *In the other game, we didn't get to choose the numbers. So it didn't make you think too much about it.*

However one student claimed to have been more involved in game one (chance only), because:

- *I think it made me think harder because it was more of a competition, sort of. And you wanted to win, and you realized it was an unfair game.*

So the level of involvement varied in contrast to the views of Falk et al (1980), but for a number of different reasons.

Games of strategy and chance, which have been suggested as possibly being more useful to the learning of probability than games of chance only, showed some interesting aspects in regard to the strategies used by the students. The students tended to improve their chances of winning from one trial to the next through the choices they made, but rarely did a team take notice of what the opposition had chosen, in order to further optimize their chances. They tended to focus only their own choices. This may have been because they believed it to be cheating to take too much notice of what the other team were choosing. Once the choices had been made, and the result of the game was dependent on chance, the nature of the random events meant that a well-chosen strategy did not necessarily guarantee success. The random outcomes did not always favour the team with the better choice of numbers. Correct choices do not guarantee success, nor do incorrect choices guarantee failure. This contrasts with the claim that games give feedback on "consequence of actions" (Inbar et al, 1970).

Students' contradictory ideas, and transfer of knowledge

There was evidence of students holding contradictory beliefs within their questionnaire responses as well as across the questionnaire and game sessions. As claimed within research literature, the students were not aware that some of their beliefs were contradictory. This may occur because the beliefs were expressed in different contexts, in which case the students would consider the problems to be different. For effective learning to take place, the students should be made explicitly aware by the teacher of these contradictions, to have cognitive conflict induced. This would enable the students to consider their conflicting ideas and have an opportunity to resolve them.

Students do not necessarily transfer their learning and understanding from one context to another. Most groups concluded, as a result of playing game one, that the dice differences were not equiprobable. But they did not transfer this knowledge into the second game that used dice differences, as their choices for placement of the counters in the second game did not reflect the understanding that they had developed previously. Leading questions from the researcher, similar to a classroom teacher, did not appear to influence the students' thinking significantly in regard to their contradictory ideas.

5.2 Implications for teaching

Language

The language difficulty of terms being used in different ways to mean different things is always problematic in mathematics. The term 'fair', example, is used in a variety of ways by children long before the probability concept of fairness is examined. This makes it all the more difficult to establish a mathematical meaning, when there is a well-established meaning that the children already understand. Similarly, words like 'chance', 'possible', 'probable', 'likely' are used in everyday life in ways and in contexts that may be quite different from the use in the mathematics classroom. It is important therefore that teachers use the correct terminology, but are aware of the various possible interpretations of the terminology. Listening to students and discussing their understanding with them is important, as it is only through this that teachers can become aware of any misuse or misunderstanding of terms by students.

Obtaining sufficient results to show trends

The difficulty with understanding probability and random events compared with other areas of mathematics, as discussed in Chapter 2, can only be partly overcome in learning activities through the use of repeated trials. It is intended that the results of these will show long-term trends in the results. There were occasions when the empirical evidence (of the results of trials) was useful in changing the ideas of some students, such as in the first game when their initial idea that the game was fair was altered when the results obviously showed something different. But this was not always the case, particularly when the results of trials were unlikely (that is, had a low probability of occurring). At times like this, the teacher's role is crucial to help students work through the results and clarify their misconceptions.

The tendency for students to generalize from a small amount of data (that is, apply the law of large numbers to situations involving small numbers) could be accommodated through repeating the game many times to obtain more data. However, the smaller the differences between the probabilities (such as between the two teams' choices of numbers in the second game), the larger the number of trials needed to reveal those differences, and it is possible that the students would become bored with playing the game sufficiently long enough to obtain conclusive results. Therefore the games need adapting, or other strategies found, so that cumulative results are easily recorded, preferably as part of the game, so that they are available for students to refer to. Simulations have been suggested as one such approach, but the lack transfer of ideas from the 'real' game to the simulation is known to be problem with simulations.

In the difference game, out of 12 games played by the various groups, 3 of these were won by the team with a lower chance of winning. Although it was too difficult to quantify the probabilities of each team winning, it is believed that the differences between the probability of the teams winning would indicate that the 3 successes out of 12 by the team with the lower probability is not significant. It is instead just another example of random variation that must be faced in a probability learning activity.

Teacher involvement and listening to students

It is important that students are not left on their own to play such probability games, as it may happen that without the influence and input of the teacher, a group does not become as actively involved as is desirable. Also, the importance of the teacher listening to the students to gain an understanding of their concepts is obvious when comparing the results of the questionnaire with the views expressed by the students during the discussions. The questionnaire results were limited by not being able to follow up on what the student meant with some of the responses, but through being involved in the discussions, the researcher could easily seek clarification if necessary.

Teachers need knowledge of probability concepts so that they are able to adequately facilitate learning, especially in those instances when the results do not conform to what is expected. They need to know to sort of questions to ask, the strategies to suggest so that more data is obtained, to help the students overcome the cognitive-conflict that is probably occurring.

5.3 Limitations of the study

Some aspects of the limitations of the study in regard to the research design have been covered in Chapter 3. In this section, the limitations relevant to specific conclusions are discussed.

A language difficulty arises when exploring children's understanding if questions are interpreted differently from what was intended. This was noticeable in Question 10 of the questionnaire that related to the outcomes on two dice. It is possible that the students did not interpret the question in the way that was intended. When teachers are exploring children's ideas, they must be particularly wary of such situations, as often it is only through the students' explanations that we can construct an understanding of their understanding. If the interpretations are different, then the teacher's or researcher's construction of the student's understanding may well be not accurate.

As discussed in the Literature Review, the interpretation of students' responses by the researcher is merely the researcher's subjective interpretation of the response, the researcher's construction of the student's construction. Another researcher may well interpret the responses differently.

Design of the games was intended to provide a type of record of the results, as these are known to be important so that students can see long-term trends, rather than relying on memory (which is when students are prone to use the availability heuristic). The design of the games did not meet this requirement sufficiently, as there were instances of inadequate recall of the results by the students, indicating use of the availability heuristic. The design of the games would need further refinement to overcome this problem.

5.4 Suggestions for further research

This study has indicated that games have a place in the learning of probability, but teachers need to be aware of the limitations of using games. The games appear to not be the unqualified success that some people would suggest.

- The integration of games into a probability unit of learning in a classroom mathematics programme needs to be investigated.
- Since the role of the teacher in posing questions and challenging the students' viewpoints and understanding was important, the management of group games with a whole class may be difficult. How should the teacher organize the use of games, while ensuring that s/he has the amount of involvement that is necessary to gain maximum benefit from the use of such a game? What are the practical considerations that are needed for using games with a whole class?
- How could the design of the games be changed to enable a better record to be kept of the cumulative results, so that the long-term trends become obvious? How could the game be adapted into various equivalent forms so that students are encouraged to play a game long enough so that sufficient results are gathered to show the long-term trends? How could the students be encouraged to view the different variations of a game as equivalent, so that accumulation of the results in order to look for patterns would be feasible?
- In the classroom, how can students be encouraged to transfer probability concepts from one activity to another?

5.5 The final word

A thorough understanding of probability must be planned for by the teacher, taking into account the students' prior experiences, using well-designed games as one of the learning activities, and employing teaching methods that acknowledge and capitalize on the benefits of using small group interactions. The role of the teacher in the process is absolutely vital, through questioning, challenging, explaining, supporting, and encouraging the students.

The learning of probability concepts cannot be left to chance. To do so would be dicey.

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Appendix A Questionnaire

1. Certain, Possible, Likely, or Impossible?

Describe a future event that is :

a) certain to occur
b) possible that it will occur
c) likely that it will occur
d) unlikely that it will occur
e) impossible that it will occur

2. Possible, impossible, or certain?

Put a ring around the word that best matches the statement.
If I roll a dice, to get

a) an even number is	impossible	possible	certain
b) a number smaller than 7 is	impossible	possible	certain
c) a number bigger than 6 is	impossible	possible	certain
d) a number bigger than 0 is	impossible	possible	certain
e) a 5 is	impossible	possible	certain

3. Coins

Put a ring around the letter of the statement that you think is best.
If I hope to get 2 heads from throwing coins, which should I do?

a) Throw 2 coins at the same time.
b) Throw 1 coin, then throw the second coin.
c) Throw 1 coin, write down the result, then throw the same coin again.
d) It makes no difference - all the methods give the same chance of getting 2 heads.

Explain your answer.

4. More coin tossing

Put a ring around the letter of the statement that you think is best.
A fair coin is tossed 4 times, each time landing with Heads up. What is the most likely outcome of the coin is tossed a 5th time?

a) Another head is more likely than a tail.
b) A tail is more likely than a head.
c) The two outcomes (head or tail) are equally likely.

Explain your answer.

5. Coins again

Put a ring around the letter of the statement that you think is best.
Which of the following is the **most** likely result from 6 flips of a fair coin?

a) H H H T T T
b) T H H T H T
c) T H T T T T
d) H T H T H T
e) All are equally likely.

Explain your answer.

6. More coins

Put a ring around the letter of the statement that you think is best.
Which of the following is the **least** likely result from 6 flips of a fair coin?

a) H H H T T T
b) T H H T H T
c) T H T T T T
d) H T H T H T
e) All are equally likely.

Explain your answer.

7. Dice - True or False

Put a ring around T if you think it is true, or F if you think it is false.

When one dice is thrown,

a) all the numbers have the same chance of coming up. T or F

b) Some numbers have more chance than others of coming up. T or F

c) Some numbers have less chance than others of coming up. T or F

d) It is easier to get some numbers than others. T or F

e) It is harder to get some numbers than others. T or F

f) I can control the throw of the dice to give myself more chance of getting the number I want. T or F

8. A pair of dice

Put a ring around T if you think it is true, or F if you think it is false.

When 2 dice are thrown, and the numbers added, which of the following is true?

a) All the possible totals have the same probability.
T or F *Explain or give an example.*

b) Some totals are more likely than others
T or F *Explain or give an example.*

c) Some totals are less likely than others
T or F *Explain or give an example.*

9. Betting on 2 dice

Put a ring around the total that has a <u>better chance</u> of coming up when 2 dice are thrown fairly, and the numbers on the dice are added. Or do they have the same chance?

a) total of 3 or total of 6 or the same chance
Explain:

b) total of 7 or total of 10 or the same chance
Explain:

c) total of 2 or total of 12 or the same chance
Explain:

d) total of 3 or total of 11 or the same chance
Explain:

10A. 2 dice

Put a ring around the letter of the statement that you think is best.

A. I throw 2 dice.

a) There is a better chance of getting a 5 and a 6 than getting a 6 and a 6.

b) There is a better chance of getting a 6 and a 6 than getting a 5 and a 6.

c) Both have the same chance.

Explain.

10B. 2 dice

Put a ring around the letter of the statement that you think is best.

I throw 2 dice.

a) There is a better chance of getting a 1 and a 6 than getting a 6 and a 6.

b) There is a better chance of getting a 6 and a 6 than getting a 1 and a 6.

c) Both have the same chance.

Explain.

10C. 2 dice

Put a ring around the letter of the statement that you think is best.

I throw 2 dice.

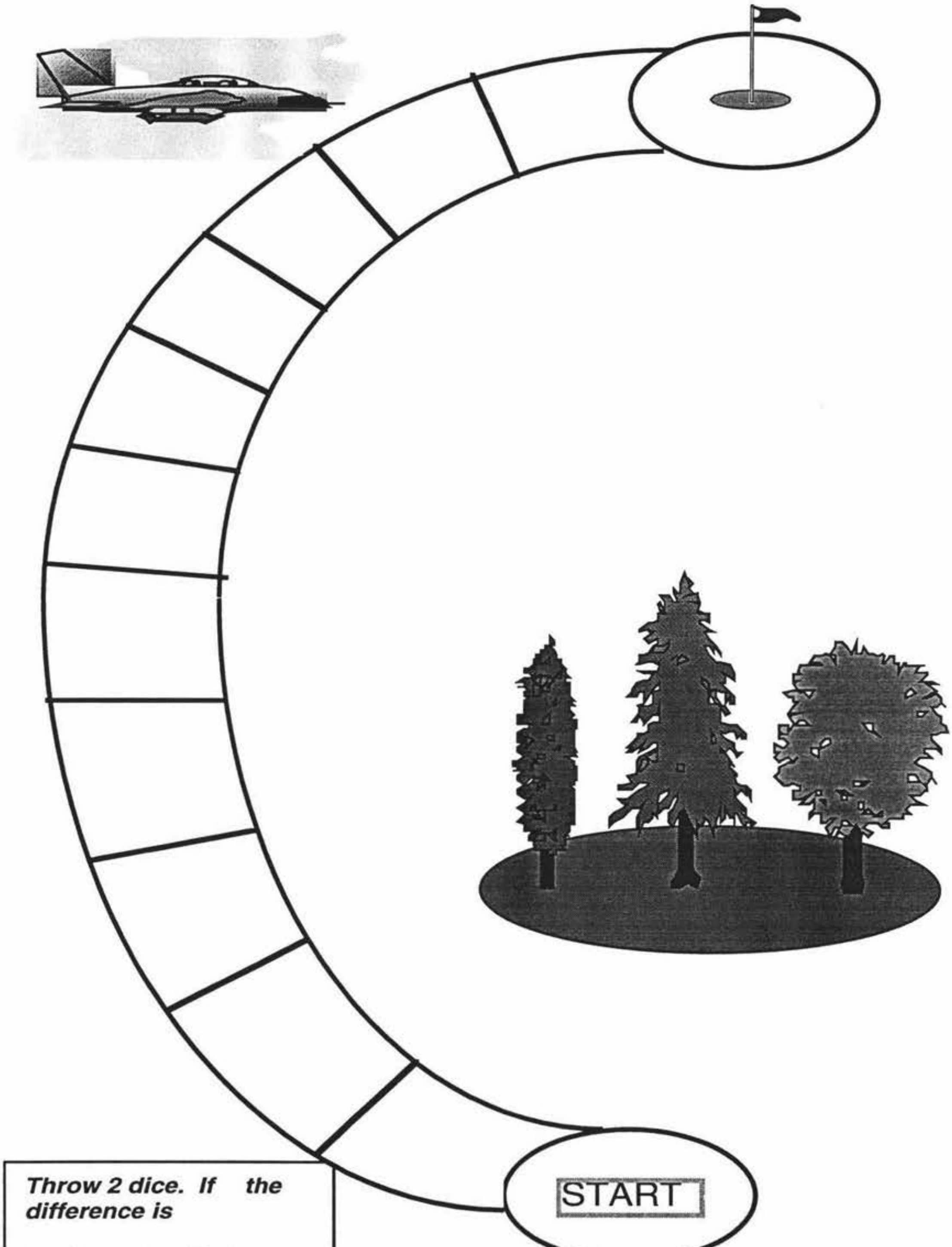
a) There is a better chance of getting a 3 and a 4 than getting a 3 and a 3.

b) There is a better chance of getting a 3 and a 3 than getting a 3 and a 4.

c) Both have the same chance.

Explain.

Appendix B Board for Game 1



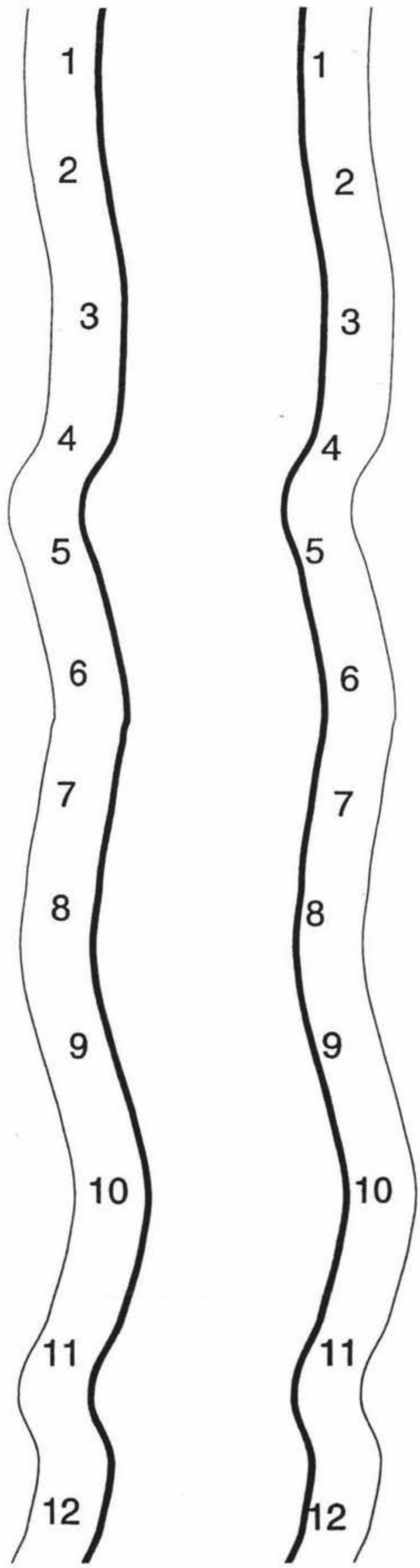
Throw 2 dice. If the difference is

0, 1, or 2 - Red moves 1 space

3, 4, or 5 - Yellow moves 1 space

Appendix C Boards for game 2

Version A - sums



Version B - differences