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The Conformal Group & Einstein Spaces

A thesis presented for the degree of
Doctor of Philosophy
in Mathematics at Massey University

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January, 1977.

THE CONFORMAL GROUP
&
EINSTEIN SPACES

ABSTRACT

This thesis presents

(a) a survey of the use of the conformal group from its beginnings to the present time, and

(b) a determination of those algebraically special vacuum Einstein space-times with an expanding and/or twisting congruence of null geodesics, which locally possess a homothetic symmetry as well as a Killing symmetry (isometry).

Unless the space-time is Petrov type N with twist-free geodesic rays, one can restrict attention to one proper homothetic motion plus the assumed Killing motion(s).

The formalism developed to undertake the systematic search for such vacuum space-times is an extension of the tetrad formalism used by Debney, Kerr & Schild[1] and by Kerr & Debney[2].

The spaces which admit one homothetic Killing vector (HKV) plus 2,3 or 4 Killing vectors (KVs) are completely determined. There are 9 such metrics (12 with 3 degeneracies) - one admitting 4 KVs, one with 3 KVs, and seven with 2 KVs. Those spaces which admit one HKV plus one KV are not completely determined owing to the field equations not being solved in some cases. However, 9 metrics are found, many of which appear to be new.

Petrov type N vacuum spaces with expansion and/or twist which admit a homothety are possible when one KV of special type is also present, or when the homothety alone is of special type.

An extensive bibliography is given.

References: (1) G.C. Debney, R.P. Kerr & A. Schild, J.Math.Phys. 10, 1842 (1969).

(2) R.P. Kerr & G.C. Debney, J.Math.Phys. 11, 2807 (1970).

ACKNOWLEDGEMENT

It is with the greatest of pleasure that I acknowledge the encouragement and guidance which I have received from my supervisor, Professor Roy Kerr. His deep insights into relativity and differential geometry have been a great source of inspiration. My thanks are due to Roy for the opportunity to undertake this study, which has been an enlightening and enjoyable experience in mathematics.

I want to mention here the debt which I owe my wife Anne and our children Deanne, Sherryn and Leon. Demands made on them were great, and without their ready cooperation this research would not have been possible. *Memor gratusque*.

Many thanks are also due to Lyn Stening who typed this thesis. Her patience and skill has been much appreciated.

PREFACE

The amount of interest in the use of the conformal group in physics has increased a great deal in the last decade. Most of the current activity appears to be in the microphysical arena, with attention being given to such matters as the breaking of conformal invariance in quantum field theory. But the study of conformal motions as an external symmetry in the theory of gravitation and cosmology is also developing. This thesis presents

(a) a survey of the use of the conformal group from its beginnings to the present time, and

(b) a determination of those algebraically special vacuum Einstein space-times with an expanding and/or twisting congruence of null geodesics, which locally possess a homothetic symmetry as well as a Killing symmetry.

Chapter 1 provides a brief introduction to the mathematical structure of the conformal group. Besides their group theoretic properties, the place of conformal motions within the hierarchy of collineations is discussed.

Chapter 2 is a survey of the mathematical development of the conformal group and its application to relativity and gravitation, cosmology, and other physical theories, notably quantum field theory.

With the background of the first two chapters, the scene is set in Chapter 3 for the task (b) above.

Chapter 4 sets up the formalism which is used throughout the rest of the work. It is an extension of the tetrad formalism used by Kerr and Debney to determine vacuum Einstein spaces which possess isometries.

Chapters 5 and 6 contain the bulk of the work involved in determining those spaces which admit one homothetic Killing vector plus 2, 3 or 4 Killing vectors (Chapter 5) or just one Killing vector (Chapter 6).

The possibility of Petrov type N vacuum spaces admitting a homothety is considered separately in Chapter 7.

There follows a Conclusion, a list of Appendices, and an extensive Bibliography containing over 460 references.

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