Parameters of the Two Generator Discrete Elementary Groups

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Abstract

Let f, g be elements of $\mathcal{M}$, the group of Möbius transformations of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. We identify each element of $\mathcal{M}$ with a 2×2 complex matrix with determinant 1. The three complex numbers,

$$\beta(f) = tr^2(f) - 4, \beta(g) = tr^2(g) - 4, \gamma(f, g) = tr[f, g] - 2,$$

define the group $\langle f, g \rangle$ uniquely up to conjugacy whenever $\gamma(f, g) \neq 0$; where $tr(f)$ and $tr(g)$ denote the traces of representive matrices of f and g respectively, $[f, g]$ denotes the multiplicative commutator $fgf^{-1}g^{-1}$. We call these three complex numbers the parameters of $\langle f, g \rangle$. This thesis is concerned with the parameters of discrete and elementary subgroups of $\mathcal{M}$.

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Contents

Introduction

Möbius transformations were studied by the German mathematician A. F. Möbius in the 19th century. F. Klein proved the group of Möbius transformations acting on Euclidean n-space is isomorphic to the group of isometries of hyperbolic $(n+1)$-space (see [17] , page 147). This discovery leads to a deeper understanding of hyperbolic space and relations between conformal geometry of spheres, the models of hyperbolic space they bound and n-dimension geometry. Relevant references can be found in the works of Beardon [1], Ratcliffe [17], Thurston [20], Gehring and Martin (see for example [5], [6], [9], [10]) and references therein. In recent years, the study of the 3-dimensional hyperbolic orbifolds, which can be represented as H^3/G where H^3 is hyperbolic 3-space (discussed in Chapter 1) and G is a discrete non-elementary orientation preserving subgroup of the group of the isometry group, has attracted much attention. We are concerned here with such discrete subgroups G. We shall assume a basic knowledge of group theory in our discussion.

Let $\mathcal{M}$ denote the group of Möbius transformations of the form:

$$f(z) = \frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{C}, ad - bc = 1, \tag{1}$$

which we associate with the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a, b, c, d \in \mathbb{C}, ad - bc = 1. \tag{2}$$

There are two basic types of discrete subgroup of $\mathcal{M}$: *elementary* and *non-elementary*, whose definitions are given in Chapter 2. The discrete non-elementary groups are known as Kleinian groups in memory of the Mathematician F. Klein. All the discrete elementary groups are known and classified (see [1]), hence the study of Kleinian groups are of interest. But the discreteness or otherwise of a Kleinian group is not easy to establish. Klimenko and Kopteva gave a criterion for discreteness of Kleinian groups with an invariant plane (see [3]). While for the Kleinian groups without invariant plane, we have only necessary or only sufficient conditions for the discreteness of such groups.

Theorem 5.4.2 of [1] states that a non-elementary subgroup G of $\mathcal{M}$ is discrete if and only if for each f and g in G, $\langle f, g\rangle$ is discrete. Thus the problem of deciding the discreteness or otherwise of G boils down to consideration of the two generator subgroups. We shall study the discreteness of two generator groups $\langle f, g\rangle$. The advantage of studying a two generator group is that for every such group $\langle f, g\rangle$, there are three complex numbers corresponding to it, and the necessary or sufficient condition(s) for non-elementary $\langle f, g\rangle$ to be discrete can sometimes be described in terms of these numbers. These three complex numbers are

$$\beta(f) = tr^2(f) - 4, \beta(g) = tr^2(g) - 4, \gamma(f,g) = tr[f,g] - 2,$$

where $tr(f)$ and $tr(g)$ denote the traces of representive matrices of f and g respectively, and $[f,g]$ denotes the multiplicative commutator $fgf^{-1}g^{-1}$, see [7]. These three numbers are called the *parameters* of the two-generator group $\langle f,g\rangle$ and we write

$$par(\langle f,g\rangle) = (\gamma(f,g), \beta(f), \beta(g)).$$

These parameters are independent of the choice of representive matrices for f and g and define $\langle f,g\rangle$ uniquely up to conjugacy whenever $\gamma(f,g) \neq 0$. See [7]. Two subgroups G_0 and G_1 of G are conjugate if for some h in G, $G_0 = hG_1h^{-1}$. Conjugate subgroups are the same from a geometric point of view. For example, if there exists a unique point fixed by all $g_0 \in G_0$, then there exists a unique point fixed by all $g_1 \in G_1$. The volumes of H^3/G_1 and H^3/G_0 are the same and so forth.

The study of the discreteness of two generator groups has a rich history, see all of our references except [15] and [16]. For example in [1], Beardon studies necessary conditions for a two generator Kleinian group by considering the displacement function

$$z \longmapsto \sinh \frac{1}{2}\rho(z, gz).$$

Gehring and Martin obtain conditions for $\langle f,g\rangle$ to be discrete by examining the distances of f, g from the identity element in $\mathcal{M}$ in [6]. They also obtain some sharp estimates for the distance between the axes of elliptic elements in a discrete group in [12]. Klimenko and Kopteva found criteria for discreteness of two generator Kleinian groups generated by a hyperbolic element and an elliptic element of even order with intersection axes in [3]. The most well-known necessary theorem in the subject is due to Jørgensen (see [1]):

Theorem 0.1. *(Jørgensen's inequality) Suppose that the Möbius transformations f and g generate a discrete non-elementary group with $\gamma(f,g) = \gamma$ and $\beta(f) = \beta$, then*

$$|\gamma| + |\beta| \geq 1. \tag{3}$$

This inequality was studied by Troels Jørgensen in [19]. He proved the inequality by the iteration of the relation

$$B_0 = B, \qquad B_{n+1} = B_n A B_n^{-1}$$

where A and B are the matrices representing f and g respectively. Another inequality was studied by Delin Tan in [2]:

Theorem 0.2. *Suppose that the Möbius transformations f and g generate a discrete group with $\gamma(f,g) = \gamma$ and $\beta(f) = \beta$. If $\gamma \neq -1$, then*

$$|\gamma + 1| + |\beta + 2| \geq 1. \tag{4}$$

If $\gamma = -1$ *and* $\beta \neq -2$, *then*

$$|\beta + 2| > \frac{1}{2}.$$

Tan used *Lemma 2* in his paper to prove (4); This *Lemma* was proved by the iteration of the relation

$$B_0 = B, \qquad B_{n+1} = [A_n, B_n],$$

which is essentially the same as Jøgensen's iteration scheme. Gehring and Martin proved (3) and (4) independently by investigating the two fixed points 0 and $\beta + 1$ of the polynomial trace $\gamma(f, gfg^{-1}) = \gamma(\gamma - \beta)$ in [4].

The inequalities (3) and (4) and Gehring and Martin's approach to them give a different perspective to look at the conditions for discreteness of $\langle f, g\rangle$. The fact is that in the space of two generator discrete groups, all two generator Kleinian groups form a closed set. This has essentially been proved by Jørgensen in [19]. We claim that all the elementary groups are isolated from the set of Kleinian groups in this space. This claim and precise bounds to describe this isolation in terms of geometric quantities as well as the complex parameters are investigated in this and future research. As we know that every two generator group $\langle f, g\rangle$ has three complex numbers as its parameters, we can therefore view $\langle f, g\rangle$ as a point in $\mathbb{C}^3$, the three dimensional complex space. Let D^3 be the subset of $\mathbb{C}^3$ which contains all the parameters of two generator discrete groups. We prove that whenever $(a, b, b') \in D^3$ corresponds to a discrete elementary group, it is isolated from the points corresponding to Kleinian groups. We establish the isolation of (a, b, b') by proving an inequality of the form

$$|\gamma + a| + |\beta + b| \geq c \tag{5}$$

where c is a real positive number and (γ, β, β') are the parameters for any Kleinian group. The reason that the isolation of (a, b, b') only depends on a, b will be explained in Chapter 5, but note here that (5) also implies immediately that

$$|\gamma + a| + |\beta' + b'| \geq c$$

by interchanging the order of the generators. Note also that the inequality (5) also indicates a necessary condition for a Kleinian $\langle f, g\rangle$ to be discrete. This is the main reason for looking at the isolation of discrete elementary groups.

The main concern of the first part of this thesis is to determine all the possible parameters for discrete elementary groups. These are the points in $\mathbb{C}^3$ that we shall show to be isolated. We then go on to give estimates on this isolation using some of the ideas discussed above (iteration). This recovers some known results and also generates some new ones. Baribeau and Ransford have given a general description of these parameters in [18]. Gehring and Martin have discussed some of them in many of their papers, see for example [4], 5], [13],or [14]. We consider all the parameters for all the discrete elementary groups more specified in this thesis. To this end we start with some preliminary topics such as the spherical and hyperbolic geometries, Möbius transformations, Triangle groups through Chapter 1 to Chapter 3. The results in these three Chapters are a matter of rewriting known facts. Our main results are stated in Chapter 4. In this Chapter we investigate the parameters under question using a combination of two methods: geometric and algebraic methods. We omit the lengthy but purely elementary computation process and state our final results in three tables. As we will see in these tables, for some elementary groups, we are able to find the exact parameters; while for others we can only give a general description similar to the results in [18] as there are parametrised families of these groups. For those whose exact parameters are known, we shall investigate their isolation from Kleinian groups by using the inequality of the form (5). For instance in Chapter 5, we consider several examples to show how to derive such inequalities for $(-1, -2, b_1)$, $(-2, -3, b_2)$, $(-1, -3, b_3)$.

Chapter 1

Spherical and Hyperbolic geometries

Euclidean geometry was the first geometry in human history, its foundations having been laid by Euclid in 300 B. C. The familiar model for an Euclidean n-space E^n is $\mathbb{R}^n$ with the Euclidean metric d_E where

$$d_E(x, y) = |x - y|, \qquad x, y \in \mathbb{R}^n.$$

We use $\mathbb{R}^n$ to denote E^n in this thesis. We assume the basic knowledge about the Euclidean geometry and metric space, and discuss mainly the spherical and hyperbolic geometries. Our discussions are based on [17] except where otherwise specified

1.1 Spherical geometry

Spherical geometry was the first geometry discovered after Euclidean geometry. It arose from the study of the starry heavens. Strictly speaking, spherical geometry is not one geometry since the geometries of two spheres with different radii are not identical. The different spherical geometries are distinguished by *Gaussian curvature*. A sphere of radius r has constant positive *(Gaussian) curvature* $1/r^2$. For convenience, we use the unit sphere S^n of $\mathbb{R}^{n+1}$, which has a constant curvature equal to 1, as our model for n-dimensional spherical geometry. S^n is defined by

$$S^n = \left\{x \in \mathbb{R}^{n+1} : |x| = 1\right\}.$$

The *Euclidean metric* d_E on S^n is defined by the formula

$$d_E(x, y) = |x - y|.$$

The Euclidean metric on S^n is sufficient for most purposes: for the notion of the *isometric sphere* of a Möbius transformation (see [1], page 41); for the group of isometries of S^n. But it is not intrinsic to S^n. An intrinsic metric on S^n follows from the Definition 1.1.1 and Theorem 1.1.2.

Definition 1.1.1. *let x, y be two points in S^n and let $\theta(x, y)$ be the Euclidean angle between x and y. The spherical distance between the rays $[0, x]$ and $[0, y]$ is defined to be the real* *number*

$$d_S(x, y) = \theta(x, y).$$

Note that

$$0 \leq d_S \leq \pi.$$

and $d_S(x, y) = \pi$ *if and only if* $y = -x$.

Theorem 1.1.2. *The spherical distance function* d_S *is a metric on* S^n, *topologically equivalent to the induced Euclidean metric.*

This intrinsic metric d_s of S^n plays a key role in determining the angles between axes of two generators in Chapter 4. There are some other key factors in the determination, such as the law of cosines and the cosine rules for the sides of a spherical triangle. To define a spherical triangle, we need the following definition

Definition 1.1.3. *A great circle of* S^n *is the intersection of* S^n *with a* 2*-dimensional vector subspace of* $\mathbb{R}^{n+1}$.

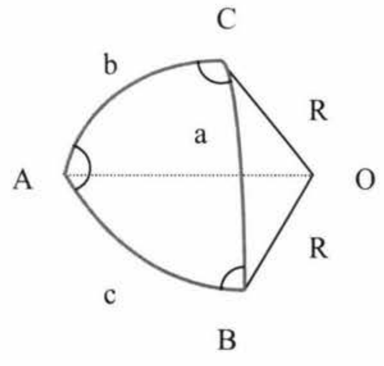
A great circle of S^n is indeed a unit circle with 0 as its center.

Definition 1.1.4. *Three points* x, y, z *of* S^n *are spherically collinear if and only if there is a great circle of* S^n *containing* x, y, z.

Now let A, B, C be three spherically noncollinear points of S^2, $S(A, B)$ be the unique great circle of S^2 containing A, B and Let $H(A, B, C)$ be the closed hemisphere of S^2 with $S(A, B)$ as its boundary and C in its interior. The *spherical triangle* with vertices A, B, C is defined to be

$$T(A, B, C) = H(A, B, C) \cap H(B, C, A) \cap H(C, A, B).$$

Let $[A, B]$ be the minor arc of $S(A, B)$ joining A to B. The sides of $T(A, B, C)$ are defined to be $[A, B], [B, C]$ and $[C, A]$. Let $a = \theta(B, C), b = \theta(C, A)$ and $c = \theta(A, B)$. Then a, b, c is the length of $[B, C], [C, A], [A, B]$ respectively. The following graph shows the spherical triangle $T(A, B, C)$ on a unit sphere centered at a point $O = (0, 0, 0)$ with $R = 1$.



Note that the sum of the internal angles of a spherical triangle is always greater than π.

We now state here without proof the results we need in later sections. We refer to [17] for a more detailed discussion of such results.

Theorem 1.1.5. *(The First Law of Cosines) If A, B, C are the angles of a spherical triangle and a, b, c are the lengths of the opposite sides, then*

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$
$$\cos B = -\cos A \cos C + \sin A \sin C \cos b,$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

Theorem 1.1.6. *(Second Law of Cosines) If A, B, C are the angles of spherical triangle and a, b, c are the lengths of the opposite sides, then*

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

1.2 Hyperbolic geometry

Hyperbolic geometry arose to prominence in the 19th century after many attempts, spanning over 2000 years, to disprove the independence of the Euclid's fifth postulate and the realisation that hyperbolic geometry is the canonical geometry of two dimension. In 1868, Eugenio Beltrami constructed a hyperbolic model in a Euclidean plane called the circle at infinity. In contrast to the spherical geometry, a hyperbolic plane has a negative constant curvature and the sum of internal angles of a hyperbolic triangle is always smaller than π. For convenience we shall take a sphere in $\mathbb{R}^{n+1}$ with constant curvature -1 as our model for hyperbolic n-space.

The existence of such a sphere depends on the *Lorentzian inner product* of $x, y \in \mathbb{R}^n$ with $n > 1$:

$$x \circ y = -x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

The Lorentzian inner product together with the vector space $\mathbb{R}^n$ is called the *Lorentzian space*, and is denoted by $\mathbb{R}^{1,n-1}$. The *Lorentzian norm* of a vector x in $\mathbb{R}^n$ is defined to be the complex number

$$\|x\| = (x \circ x)^{\frac{1}{2}}.$$

Note that $\|x\|$ is either positive, zero or positive imaginary. The *Lorentzian distance* between vectors x and y in $\mathbb{R}^n$ is defined to be the complex number

$$d_L(x, y) = \|x - y\|.$$

Notice here again that d_L is either positive, zero, or positive imaginary .

If $\|x\| = 0$, then x is said to be *light-like.* The set of all x in $\mathbb{R}^n$ with $\|x\| = 0$ is the hypercone C^{n-1}. It is defined by the equation

$$x_1^2 = x_2^2 + \ldots + x_n^2.$$

and called the *light cone* of $\mathbb{R}^n$. see Figure 1.

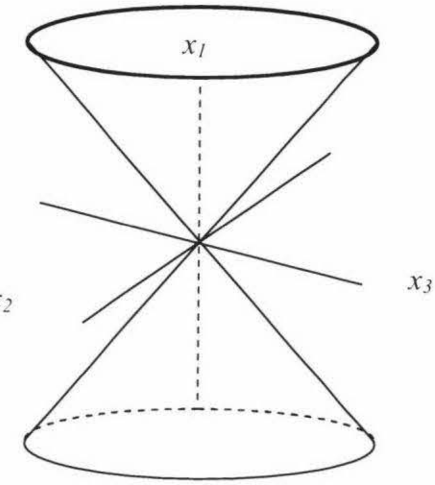


Figure 1: The light cone C^2 of $\mathbb{R}^{1,n-1}$

If $\|x\| > 0$, then x is said to be *space-like*. The exterior of C^{n-1} in $\mathbb{R}^n$ is the open subset of $\mathbb{R}^n$ consisting of all the *space-like* vectors which satisfy the inequality

$$x_1^2 < x_2^2 + \ldots + x_n^2.$$

If $\|x\|$ is imaginary, then x is said to be *time-like*. A time-like vector x is said to be *positive(negative)* if and only if $x_1 > 0$ $(x_1 < 0)$. The interior of C^{n-1} in R^n is the open subset of $\mathbb{R}^n$ consisting of all the time-like vectors which satisfy the inequality

$$x_1^2 > x_2^2 + \ldots + x_n^2.$$

Theorem 1.2.1. *Let x, y be positive (negative) time-like vectors in $\mathbb{R}^n$. Then $x \circ y \leq \|x\|\,\|y\|$ with equality if and only if x and y are linearly dependent.*

Let x and y be positive (negative) time-like vectors in $\mathbb{R}^n$. By Theorem 1.2.1, there is a unique nonnegative real number $\eta(x, y)$ such that

$$x \circ y = \|x\|\,\|y\| \cosh \eta(x, y).$$

$\eta(x, y)$ is defined as the *Lorentzian time-like angle* between x and y with $\eta(x, y) = 0$ if and only if x and y are positive scalar multiples of each other.

As "imaginary" distances are possible in Lorenzian $(n + 1)$-space. A sphere of unit imaginary radius in $\mathbb{R}^{n+1}$ with Lorentzian inner product is:

$$F^n = \{x \in \mathbb{R}^{n+1} : \|x\| = -1\}$$

The set F^n is a hyperboloid of two sheets defined by the equation

$$x_1^2 - (x_2^2 + \ldots + x_{n+1}^2) = 1.$$

The subset of all x in F^n such that $x_1 > 0$ ($x_1 < 0$) is called the positive (negative) sheet of F^n. see Figure 2.

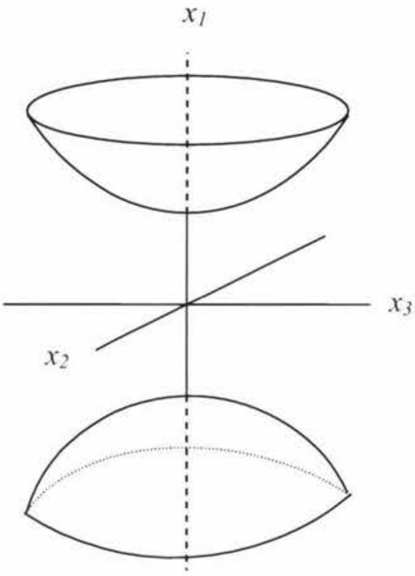


Figure 2 The hyperboloid F^2

The hyperboloid model H^n of hyperbolic n-space is defined to be the positive sheet of F^n. Let x, y be the vectors in H^n and let $\eta(x, y)$ be the Lorentzian time-like angle between x and y. Then the real number

$$d_H(x, y) = \eta(x, y)$$

is defined as the hyperbolic distance between x and y.

Theorem 1.2.2. *The hyperbolic distance function d_H is a metric on H^n.*

The metric d_H on H^n is called the *hyperbolic metric*. The metric space consisting of H^n together with its hyperbolic metric is called *hyperboloid model of hyperbolic n-space*.

We now redefine the Lorenzian inner product on $\mathbb{R}^{n+1}$ to be

$$x \circ y = x_1 y_1 + ... + x_n y_n - x_{n+1} y_{y+1}.$$

All the results discussed previously in this section remain true after reversing the order of the coordinates of $\mathbb{R}^{n+1}$. Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times 0$ in $\mathbb{R}^{n+1}$, the open unit ball B^n, where

$$B^n = \{x \in \mathbb{R}^n \times 0 : |x| < 1\},$$

is mapped onto H^n isometrically by the *stereographic projection* ζ that projects x in B^n away from $-e_{n+1}$ until it meets H^n in the unique point $\zeta(x)$. See Figure 3.
The explicit formula for $\zeta(x)$ is

$$\zeta(x) = \left(\frac{2x_1}{1 - |x|^2}, ..., \frac{2x_n}{1 - |x|^2}, \frac{1 + |x|^2}{1 - |x|^2} \right).$$

The map ζ is a bijection of B^n onto H^n and the inverse of ζ is given by

$$\zeta^{-1}(y) = \left(\frac{y_1}{1+y_{n+1}}, \dots, \frac{y_n}{1+y_{n+1}}\right)$$

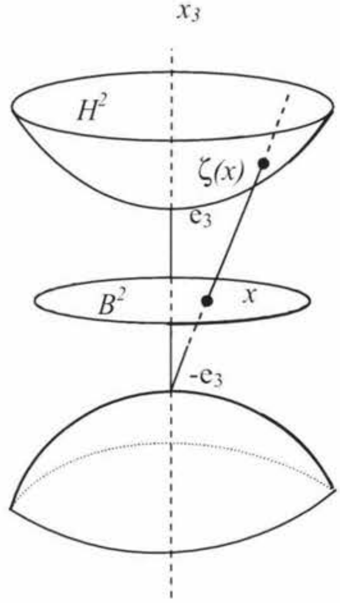


Figure 3 The stereographic projection ζ of B^2 to H^2.

which maps H^n to B^n.

Define a metric d_B on B^n by the formula

$$d_B(x, y) = d_H(\zeta(x), \zeta(y)).$$

The metric d_B is called the Poincaré metric on B^n. The unit ball B^n together with the metric d_B is called the *conformal ball model of hyperbolic n-space*. We also use the symbol Δ to denote B^n.

Theorem 1.2.3. *The metric d_B on B^n is given by*

$$\cosh d_B(x, y) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

There is also a transformation from B^n to U^n, where

$$U^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}.$$

Define $\phi = \psi\sigma$ where σ is the reflection of $\mathbb{R}^n$ in the plane $\mathbb{R}^{n-1}$ and ψ is the reflection of $\mathbb{R}^n$ in the sphere $S(e_n, \sqrt{2})$. Then ϕ is an isometry from the upper half-space U^n to B^n. Since $\phi = \psi\sigma$ is a bijective function, it has an inverse function which maps U^n to B^n.

Define a metric d_U on U^n by the formula

$$d_U(x,y) = d_B(\phi(x), \phi(y)).$$

The metric d_U is called the *Poincaré metric* on U^n. The metric space consisting of U^n together with the metric d_U is called the *upper half-space model* of hyperbolic n-space

Theorem 1.2.4. *The metric d_U on U^n is given by*

$$\cosh d_U(x,y) = 1 + \frac{|x-y|^2}{(2x_n y_n)}.$$

Henceforth, we use (H^n, d_H) instead of (U^n, d_U) to denote the upper half-space model. The conformal ball model B^n and H^n are the two basic models of the hyperbolic n-space we shall use in our discussion.

We end this section with a brief discussion of geodesics in $\mathbb{R}^n, S^n, H^n$ or B^n. Given any two points x, y in $\mathbb{R}^n, S^n, H^n$ or B^n (Note, $x \neq -y$ in S^n) , there exists a unique curve joining x, y and giving the shortest distance between them. Such a curve is a segment of a geodesic passing through these two points. A geodesic in $\mathbb{R}^n, S^n, H^n$ or B^n is defined as the following:

- *A geodesic of $\mathbb{R}^n$ is an Euclidean line.*
- *A geodesic of S^n is a great circle.*
- *A geodesic of H^n is the intersection of H^n with either a straight line orthogonal to $\mathbb{R}^{n-1}$ or a circle orthogonal to $\mathbb{R}^{n-1}$.*
- *A geodesic of B^n is either an open diameter of B^n or the intersection of B^n with a circle orthogonal to S^{n-1} .*

where $\mathbb{R}^{n-1}$ is the boundary of H^n and S^{n-1} is the boundary of B^n.

Note, given two distinct points b_1, b_2 in the spaces above, there is a unique geodesic L passing through both b_1 and b_2.

Chapter 2

Möbius transformations on $\hat{\mathbb{R}}^n$

The discrete elementary groups and Kleinian groups are subgroups of the group of Möbius transformations. We consider the Möbius transformations in this Chapter. All of our discussions are based on [1] except where otherwise specified.

2.1 The Möbius group on $\hat{\mathbb{R}}^n$

A sphere $S(a, r)$ in $\mathbb{R}^n$ is the set of points which satisfy

$$S(a,r) = \{x \in \mathbb{R}^n : |x - a| = r\}$$

where $a \in \mathbb{R}^n$ and $r > 0$. The reflection in $S(a, r)$ is the function ψ defined by

$$\psi(x) = a + \left(\frac{r}{|x - a|}\right)^2 (x - a)$$

Figure 4 is a geometrical interpretation of the reflection x in $S(a, r)$, where $a, x \in \hat{\mathbb{R}}^n$ and ψ is the reflection in $S(a, r)$.

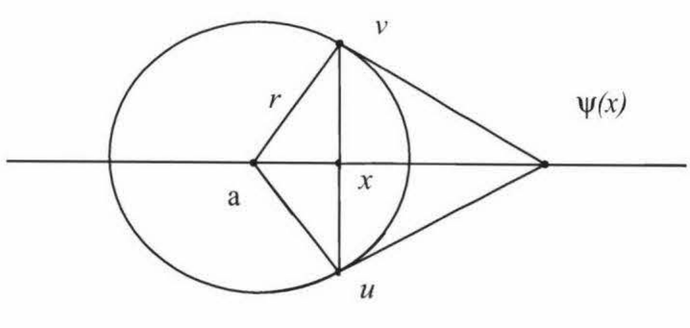


Figure 4 The reflection of a point x in a sphere $S(a, r)$

A plane $P(a, t)$ in $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \infty$, the Riemann sphere, is given by

$$P(a,t) = \{x \in \mathbb{R}^n : (x \cdot a) = t\} \cup \infty,$$

where $a \in \mathbb{R}^n, a \neq 0$, $(x \cdot a)$ is the usual dot product of vectors in $\mathbb{R}^n$ and t is a real number. The reflection in $P(a,t)$ is given by

$$\psi(x) = x - 2[(x \cdot a) - t]a^*$$

where $a^* = a/|x|^2$.
A geometrical interpretation is shown in Figure 5, where P is a plane in $\hat{\mathbb{R}}^n$ passing through 0, ψ is a reflection with respect to P.

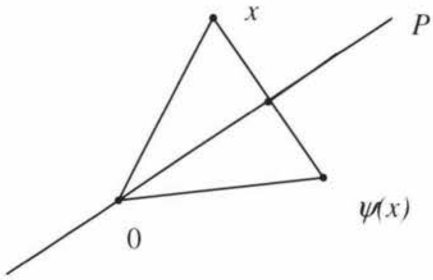


Figure 5 The reflection of a point x in the plane P

The *chordal metric* d on $\hat{\mathbb{R}}^n$ is defined by

$$d(x,y) = \begin{cases} \dfrac{2|x-y|}{(1+|x|^2)^{1/2}(1+|y|^2)^{1/2}} & \text{if} \quad x, y \neq \infty \\ \dfrac{2}{(1+|x|^2)^{1/2}} & \text{if} \quad y = \infty \end{cases} \tag{2.1}$$

where $x, y \in \hat{\mathbb{R}}^n$. Any reflection $\psi(x)$ (in a sphere or a plane) is continuous with respect to the *chordal metric* d throughout $\hat{\mathbb{R}}^n$.

A Möbius transformation acting in $\hat{\mathbb{R}}^n$ is a finite composition of reflections in spheres or planes as described above and is conformal, which means it preserves the angles. The set of Möbius transformations forms a group under function composition, called the *General Möbius group*, denoted by $GM(\hat{\mathbb{R}}^n)$. The *Möbius group* consists all orientation-preserving Möbius transformations, which are the composition of an even number these reflections. We are primarily interested in the geometric action of subgroups of $M(\hat{\mathbb{R}}^2)$. It can be shown that $M(\hat{\mathbb{R}}^2)$ is identical to $\mathcal{M}$ of form (1) on page 1. Henceforth we shall use the notation $\mathcal{M}$ in preference to $M(\hat{\mathbb{R}}^2)$.

2.2 Poincaré extension

Poincaré observed that each Möbius transformation ψ acting in $\hat{\mathbb{R}}^n$ has an extension to a Möbius transformation $\tilde{\psi}$ acting in $\hat{\mathbb{R}}^{n+1}$. The extension depends on the function:

$$x \mapsto \tilde{x} = (x_1, ..., x_n, 0), \quad x = (x_1, ..., x_n)$$

of $\hat{\mathbb{R}}^n$ into $\hat{\mathbb{R}}^{n+1}$. Let ψ be the reflection in $S(a,r)$ $(P(a,r))$,$a \in \mathbb{R}^n$, then $\tilde{\psi}$ is the reflection in $S(\tilde{a},r)$ $(P(\tilde{a},r)$), $\tilde{a} \in \hat{\mathbb{R}}^n$ Hence for $x \in \hat{\mathbb{R}}^n$, and $y = \psi(x)$ we have

$$\tilde{\psi}(x_1, ..., x_n, 0) = (y_1, ..., y_n, 0) \tag{2.2}$$

Alternatively, we can write (2.2) as

$$\tilde{\psi}(x, 0) = (\psi(x), 0)$$

Hence $\tilde{\psi}$ is regarded as an extension of ψ. It can be shown that $\tilde{\psi}$ leaves the plane $x_{n+1} = 0$ invariant and preserves each of the half-spaces $x_{n+1} > 0$ and $x_{n+1} < 0$.For every Möbius transformation ψ acting in $\hat{\mathbb{R}}^n$, there is at most one such extension $\tilde{\psi}$ that preserves

$$H^{n+1} = \{(x_1, ..., x_{n+1}) : x_{n+1} > 0\}$$

Definition 2.2.1. *The Poincaré extension of* ψ *in* $GM(\hat{\mathbb{R}}^n)$ *is the transformation defined as above.*

Let $x, y \in H^{n+1}$, Poincaré extension of ψ in $\mathrm{GM}(\hat{\mathbb{R}}^n)$ also leaves

$$\frac{|y - x|^2}{y_{n+1} x_{n+1}}$$

invariant. Notice this means that the Poincaré extension preserves the metric d_H of H^{n+1}. It is a direct consequence of this invariance that the Poincaré extension of any ψ in $\mathrm{GM}(\hat{\mathbb{R}}^n)$ is an isometry of hyperbolic $(n+1)$-space with metric d_H. In fact Klein has shown that the group of Möbius transformation of n-space is isomorphic to the group of isometries of hyperbolic $(n+1)$ space as mentioned at the beginning of this thesis. Hence $\mathcal{M}$ is isometric to the group of conformal isometries of H^3. We end this section with a theorem about the groups of isometries of H^2 and B^2 (see [1]).

Theorem 2.2.2. Let (H^2, d_H) and (B^2, d_B) be two models of hyperbolic 2 space, then

- The group of isometries of (H^2, d_H) is precisely the group of maps of the form

  $$z \mapsto \frac{az+b}{cz+d}, \qquad z \mapsto \frac{a(-\bar{z})+b}{c(-\bar{z})+d}$$

  where a, b, c and d are real and $ad - bc = 1$.

- The group of isometries of (B^2, d_B) is precisely the group of maps of the form

  $$z \mapsto \frac{az+\bar{c}}{cz+\bar{a}}, \qquad z \mapsto \frac{a\bar{z}+\bar{c}}{c\bar{z}+\bar{a}}$$

  where $|a|^2 - |c|^2 = 1$.

2.3 The Discrete elementary subgroups of $\mathcal{M}$

Define a metric on $\mathcal{M}$ by letting

$$D(f,g) = \sup\{d(f(z),g(z)), z \in \hat{\mathbb{C}}, f,g \in \mathcal{M}\};$$

a subgroup G of $\mathcal{M}$ is *discrete* if there exists a constant $\epsilon = \epsilon(G) > 0$ such that $D(f,g) \geq \epsilon$ for each distinct pair of elements f and g in $\mathcal{M}$ [5]. Alternatively we can say that a subgroup G of $\mathcal{M}$ is discrete if and only if for each positive k, the set

$$\{A \in G : \|A\| \leq k\},$$

where A is a matrix of form (2) on page 1, is finite. Let X be any non-empty set, and $x \in X$, $G(x)$, the G-orbit of x, is the subset of X defined by

$$G(x) = \{g(x) \in X : g \in G\};$$

a subgroup of $\mathcal{M}$ is *elementary* if and only if there exists a finite G-orbit in $\mathbb{R}^3$.

There are three basic types of discrete elementary groups. This classification is based on the types of elements in $\mathcal{M}$. There are four types of elements in $\mathcal{M}$. These can be defined by using either the fixed points or the traces of the elements. The following theorem concerns the classification of these elements by their traces.

Theorem 2.3.1. *Let $g(\neq I)$ be any Möbius transformation. Then*

(i) g is *parabolic* if and only if $tr^2(g) = 4$;
(ii) g is *elliptic* if and only if $tr^2(g) \in [0,4)$;
(iii) g is *hyperbolic* if and only if $tr^2(g) \in (4,+\infty)$;
(iv) g is *strictly loxodromic* if and only if $tr^2(g) \notin [0,+\infty)$.

Where $tr^2(g) = tr^2(A)$, A is the matrix representation of g. If $tr^2 \notin [0,4]$, we call g loxodromic. Thus a loxodromic element is either hyperbolic or strictly loxodromic.

A discrete elementary group G is one of the following three types:

- *Type 1:* G contains only elliptic elements, there are two subcases:

  (i) G is a finite cyclic group of rotations of $\hat{\mathbb{C}}$;

  (ii) G is a two-generator group which is isomorphic to one of the symmetry groups of a regular plane n-gon, the tetrahedron, the octahedron and the icosahedron; these symmetry groups correspond to the abstract groups D_n, A_4, S_4, A_5.

- *Type 2*: Every element of G is of the form

$$z \mapsto \omega^k z + n\lambda + m\mu, \tag{2.3}$$

  where k, m, n are integers, $0 \leq k \leq q$ and $q \leq 6, q \neq 5$

There are 5 types of two-generator groups that contain elements of form (2.3). Such a group is isomorphic to

(1) $\langle f(z) = z+1,\ \ g(z) = az\rangle$, where g is of order $2,3,4,6$;

(2) $\langle f(z) = z+1,\ \ g(z) = z+\tau\rangle$, where f, g are parabolic elements and $\tau \in \mathbb{C}$ and $\tau \notin \mathbb{R}$; or

(3) one of the Euclidean Triangle groups , which are discussed in Chapter 3.

- *Type 3:* every element of G is either of the form

$$z \mapsto \omega^k \alpha^n z$$

or of the form

$$z \mapsto \omega^k \alpha^n / z.$$

More specifically, there are three types of elements:

(i) loxodromic elements: $f(z) = az,\ \ |a| \neq 1$.

The elements of this type fix both 0 and ∞;

(ii) elliptic elements of the type: $g(z) = bz, |b| = 1$.
The elements of this type belong to a finite cyclic group and fix both 0 and ∞;

(iii) elliptic elements of the type $h(z) = c/z$ which exchange 0 and ∞.

A two-generator subgroup of G of this type is generated by a pair of elements described above. If one of the generators is of the form $h(z) = c/z$, then the resulting group is the dihedral group D_n. Unlike *Type 1*, here the order of dihedral group can be infinite. We denote this group by D_∞.

Chapter 3

Triangle groups

Triangle groups are important classes of discrete subgroups of $\mathcal{M}$. Indeed in the first two geometries (Euclidean and Spherical) , the triangle groups are the most interesting discrete groups and it is these groups which we will find the parameters for. This Chapter is devoted to these groups.

3.1 Triangle groups in three different geometries

We start with the definition of *a group of type* (α, β, γ) that does not mention discreteness.

Definition 3.1.1. *A group G of isometries of the Euclidean plane (the sphere or the hyperbolic plane) is said to be of type (α, β, γ) if and only if G is generated by the reflections across the sides of some triangle with angles α, β, γ.*

According to Theorem 7.1.3 in [17], if each of α, β, γ is of the form

$$\pi/p, \qquad 2 \leq p \leq +\infty$$

then the group G of *type* (α, β, γ) is discrete. Also, Each G has a distinguished subgroup G_0 of index two that contains the conformal elements of G: we call G_0 a conformal group of *type* (α, β, γ). We shall discuss discrete conformal groups and we adopt the definition in [1].

Definition 3.1.2. *A group G is a (p,q,r)-Triangle group if and only if G is a conformal group of type $(\pi/p, \pi/q, \pi/r)$: we call G a Triangle group if it is a (p,q,r)-Triangle group for some integers p, q, r.*

A Triangle group is necessarily discrete. There are three types of Triangle groups, they are:

- Hyperbolic Triangle groups if $\pi/p + \pi/q + \pi/r < \pi$;
- Euclidean Triangle groups if $\pi/p + \pi/q + \pi/r = \pi$;

- Spherical Triangle groups if $\pi/p + \pi/q + \pi/r > \pi$.

In the hyperbolic case, there are infinitely many integral solutions for (p, q, r). Each solution determines a hyperbolic triangle $T(\alpha, \beta, \gamma)$ and a corresponding group G of *type* (α, β, γ). It can be shown that hyperbolic Triangle groups are discrete but not elementary (see [1]). Hence we shall confine our attention to Euclidean and Spherical triangle groups.

The number of integral solutions for the Euclidean and spherical Triangle groups is finite. We consider them in detail in the next two sections. We state here some preliminary definitions for that discussion.

Definition 3.1.3. *A **topology** on a set X is a collection $\mathfrak{I}$ of subsets of X having the following properties:*

(1) *$\emptyset$ and X are in $\mathfrak{I}$.*

(2) *The union of the elements of any subcollection of $\mathfrak{I}$ is in $\mathfrak{I}$.*

(3) *The intersection of the elements of any finite subcollection of $\mathfrak{I}$ is in $\mathfrak{I}$*

A set X for which a topology $\mathfrak{I}$ has been specified is called a ***topological space*** (see [16], page 76).

Note $\emptyset$ means the empty set.

Definition 3.1.4. *A collection $\mathcal{S}$ of subsets of a topological space X is locally finite if and only if for each point x of X, there is an open neighborhood U of x in X such that U meets only finitely many members of $\mathcal{S}$ (see [17], page 166).*

Definition 3.1.5. *A polygon P is the interior of a Jordan curve*

$$[a_1, a_2] \cup [a_2, a_3] \cup \ldots \cup [a_{n-1}, a_n] \cup [a_n, a_1].$$

where $(a_i, a_{i+1}) \cap (a_j, a_{j+1}) = \emptyset$ for all i and j.

Definition 3.1.6. *A tessellation of $X (= H^2, B^2, S^2, \mathbb{R}^2)$ is a collection $\mathcal{P}$ of polygons in X such that*

(1) *the interiors of the polygons in $\mathcal{P}$ are mutually disjoint;*

(2) *the union of the polygons in $\mathcal{P}$ is X; and*

(3) *the collection $\mathcal{P}$ is locally finite.*

Locally finite in the Definition 3.1.6 means each compact subset of X meets only finitely many polygons in $\mathcal{P}$.

3.2 Euclidean Triangle groups

In the Euclidean case, the integral solutions for (p, q, r) are $(3, 3, 3), (2, 3, 6), (2, 4, 4)$. Figure 6 shows a ΔABC with sides a, b, c and angles $(\pi/3, \pi/3, \pi/3)$.

Reflecting the interior of ΔABC with respect to its sides respectively and repeatedly, the resulting reflection group is a group G of *type* $(\pi/3, \pi/3, \pi/3)$. Figure 7 shows the tessellation of a part the Euclidean plane by ΔABC and its reflecting images in its three sides.

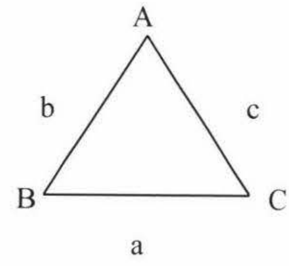


Figure 6 Euclidean triangle $(3, 3, 3)$

In general, we can define a group using its *presentation*. A *presentation* of a group is a set of elements that generate the group together with a set of relations that specify the conditions that these generators are to satisfy. If two groups have the same presentations, then they are the same group (see [15]). The group Γ_1 of *type* $(\pi/3, \pi/3, \pi/3)$ has the presentation

$$\Gamma_1 = \langle a, b, c | a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ac)^3 = identity \rangle .$$

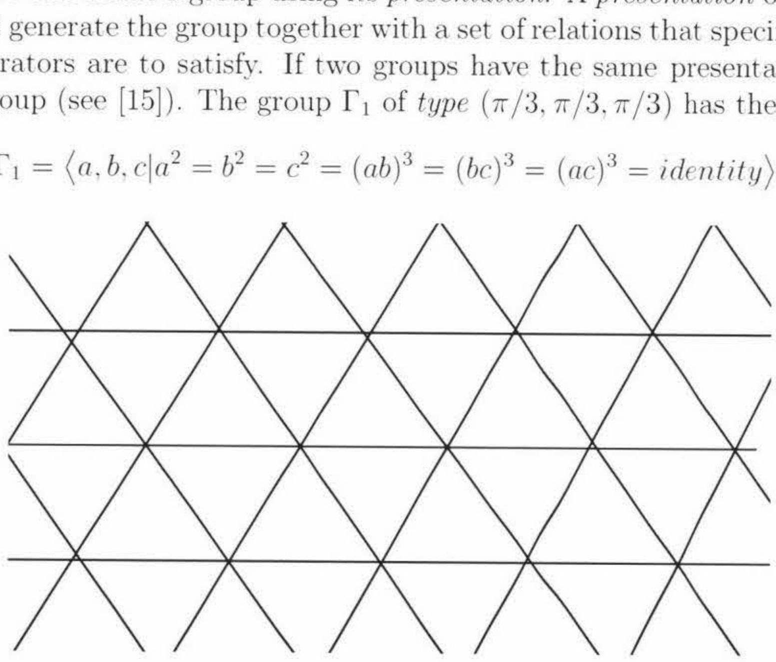


Figure 7 The tessellation of $\mathbb{R}^2$ corresponding to triangle $(3, 3, 3)$

Similarly, the group Γ_2 of type $(\pi/2, \pi/3, \pi/6)$ and the group Γ_3 of type $(\pi/2, \pi/4, \pi/4)$ are resulting reflection groups of two triangles with angles $(\pi/2, \pi/3, \pi/6)$ and $(\pi/2, \pi/4, \pi/4)$, shown in Figure 8

The tessellations of the Euclidean plane generated by reflecting in the sides of triangle $(\pi/2, \pi/3, \pi/6)$ and $(\pi/2, \pi/4, \pi/4)$ respectively are shown in Figure 9. The group Γ_2 of type $(\pi/2, \pi/3, \pi/6)$ and the group Γ_3 of type $(\pi/2, \pi/4, \pi/4)$ have presentations:

$$\Gamma_2 = \langle a, b, c | a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ac)^6 = identity \rangle , \quad \text{for } (2, 3, 6);$$

$$\Gamma_3 = \langle a, b, c | a^2 = b^2 = c^2 = (ab)^2 = (bc)^4 = (ac)^4 = identity \rangle , \quad \text{for } (2, 4, 4).$$

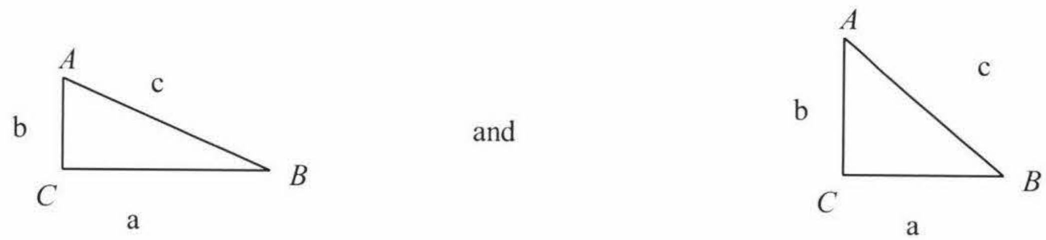


Figure 8 Euclidean triangles $(2,3,6)$ and $(2,4,4)$ respectively

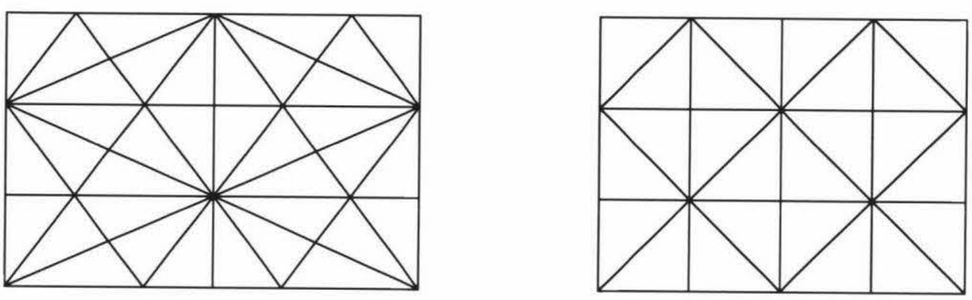


Figure 9 Tessellation of $\mathbb{R}^2$ corresponding to triangles $(2,3,6)$ and $(2,4,4)$ respectively

Let

$$\Gamma_{1e} = \langle ab, bc | (ab)^3 = (bc)^3 = (ac)^3 = identity \rangle \tag{3.1}$$

$$\Gamma_{2e} = \langle ab, bc | (ab)^2 = (bc)^3 = (ac)^6 = identity \rangle \tag{3.2}$$

$$\Gamma_{3e} = \langle ab, bc | (ab)^2 = (bc)^4 = (ac)^4 = identity \rangle \tag{3.3}$$

then Γ_{1e} is a $(\pi/3, \pi/3, \pi/3)$-Triangle group and all the elements in Γ_{1e} are the composition of an even number of reflections, The same is true for Γ_{2e}, and Γ_{3e}.

3.3 Spherical Triangle groups

In the spherical case, the integral solutions for (p,q,r) satisfying Definition 3.1.2 are $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$. Reflecting the triangle $(2,2,n)$ in its sides generates a group of type $(\pi/2, \pi/2, \pi/n)$ of order $4n$. This group has a representation

$$\Gamma_4 = \langle a, b, c | a^2 = b^2 = c^2 = (ab)^2 = (bc)^2 = (ac)^n = identity \rangle . \tag{3.4}$$

The spherical triangle $(\pi/2, \pi/2, \pi/3)$ and the tessellation of the sphere generated by reflecting in the sides of this triangle is shown in Figure 10. Similarly, the groups of *type* $(\pi/2, \pi/3, \pi/3)$, *type* $(\pi/2, \pi/3, \pi/4)$ and *type* $(\pi/2, \pi/3, \pi/5)$ are generated by reflecting the interiors of spherical triangles $(\pi/2, \pi/3, \pi/3)$, $(\pi/2, \pi/3, \pi/4)$ and $(\pi/2, \pi/3, \pi/5)$. These spherical triangles and their corresponding tessellations of S^2 are shown in Figure 11.

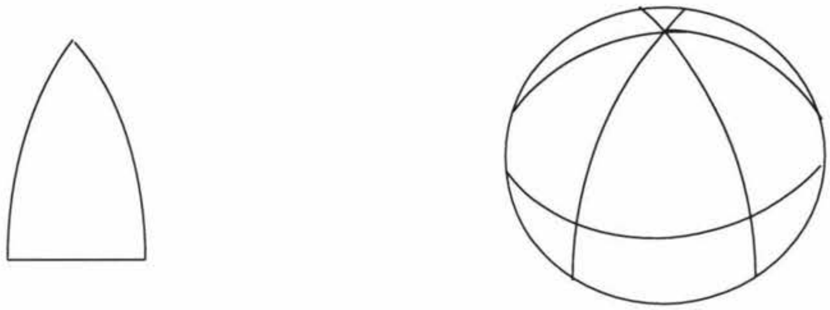


Figure 10 Spherical triangle $(2, 2, 3)$ and its tessellation of S^2

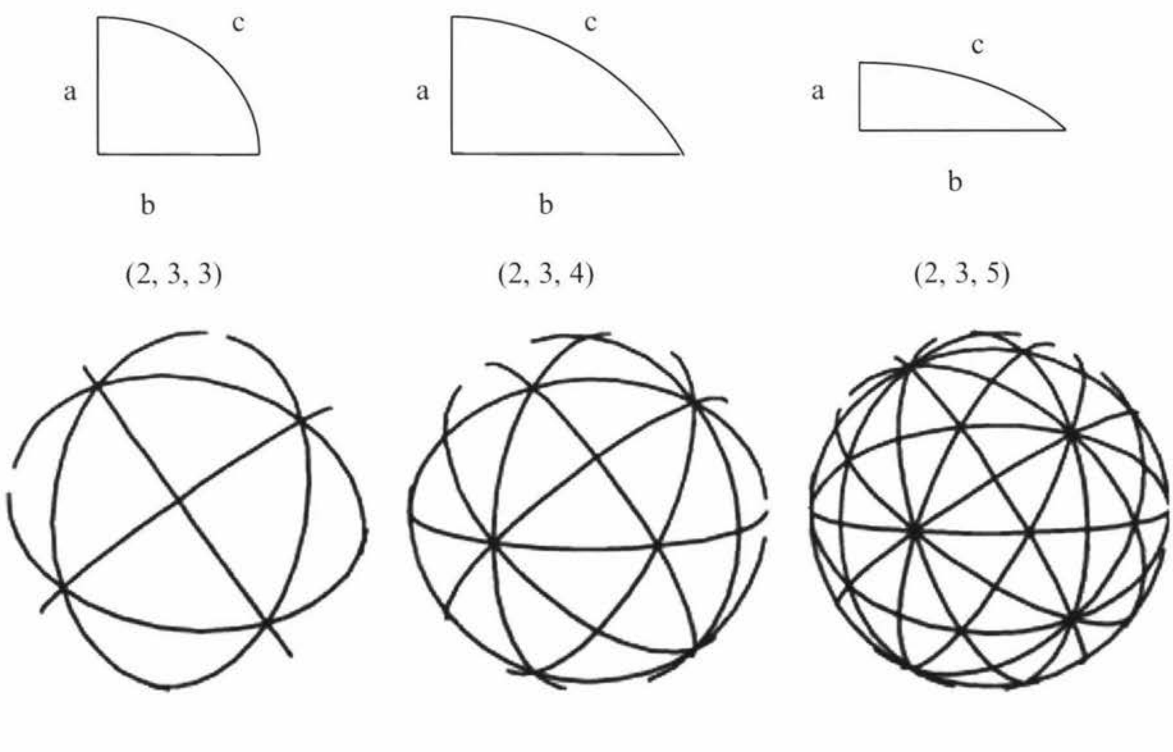


Figure 11 Spherical triangles $(2, 3, 3), (2, 3, 4), (2, 3, 5)$ and their corresponding tessellations of S^2

These groups can be presented as:

$$\Gamma_5 = \langle a, b, c | a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ac)^3 = identity \rangle ; \qquad (3.5)$$

$$\Gamma_6 = \langle a, b, c | a^2 = b^2 = c^2 = (ab)^2 = (bc)^4 = (ac)^3 = identity \rangle ; \qquad (3.6)$$

and

$$\Gamma_7 = \langle a, b, c | a^2 = b^2 = c^2 = (ab)^2 = (bc)^5 = (ac)^3 = identity \rangle. \tag{3.7}$$

Let

$$\Gamma_{5e} = \langle ab, bc | (ab)^2 = (bc)^3 = (ac)^3 = identity \rangle; \tag{3.8}$$

$$\Gamma_{6e} = \langle ab, bc | (ab)^2 = (bc)^4 = (ac)^3 = identity \rangle; \tag{3.9}$$

and

$$\Gamma_{7e} = \langle ab, bc | (ab)^2 = (bc)^4 = (ac)^3 = identity \rangle. \tag{3.10}$$

then Γ_{5e}, Γ_{6e}, and Γ_{7e} are $(\pi/2, \pi/3, \pi/3)$, $(\pi/2, \pi/3, \pi/4)$, $(\pi/2, \pi/3, \pi/5)$-Triangle groups respectively. They are indeed A_4, S_4, A_5 (see [1]). Recall that S_n is the group of all permutations of $\{1, 2, \ldots, n\}$, A_n is the group of even permutations of $\{1, 2, \ldots, n\}$. This make it possible to find the parameter of A_4, S_4, A_5 both geometrically and algebraically. The algebraic method will be discussed in detail in Chapter 4. A formula must be known for our geometric method. We now discuss this formula and the geometrical method.

Let $f, g \in \mathcal{M}$ be nonparabolic, we define (see [5])

- *axis(f)* as the geodesic in H^3 which has the fixed points of f in $\mathbb{C}$ as its endpoints and
- $\delta(f, g)$ as the hyperbolic distance in H^3 between the *axis(f)* and *axis(g)*.

The fixed points of f are the points $z \in \mathbb{C}$ such that $f(z) = z$. Note that a nonparabolic element in $\mathcal{M}$ has exactly two fixed points in $\mathbb{C}$ and $\delta(f, g) = 0$ whenever f, g have a common fixed point. If f is elliptic, the axis of f is the set of fixed points of f in in H^3.
Now we are ready to introduce the formula (see [5]), which relates the geometry of axes and the parameters introduced.

Lemma 3.3.1. *Suppose that $f, g \in \mathcal{M}$ have disjoint pairs of fixed points and that α is the geodesic in H^3 which is orthogonal to the axes of f and g. Then*

$$\sinh^2(\delta + i\theta) = \frac{4\gamma(f, g)}{\beta(f)\beta(g)} \tag{3.11}$$

Where $\delta = \delta(f, g)$ and $\theta = \theta(f, g)$ is the angle between the spheres or hyperplanes which contain $axis(f) \cup \alpha$ and $axis(g) \cup \alpha$ respectively.

Figure 12 is a geometric interpretation of Lemma 3.3.1.

According to Theorem 4.3.7 of [1], the elements of A_4 have a common fixed point in H^3. Therefore $\delta(f, g) = 0$ for any two elements f, g in A_4. The same is true for the elements in S_4, A_5. Since $\sinh^2(i\theta) = -\sin^2\theta$, so (3.11) becomes

$$\sin^2\theta = -\frac{4\gamma(f, g)}{\beta(f)\beta(g)}. \tag{3.12}$$

We can compute all the angles between the axes of an element of order n and an element of order m using the First and Second laws of Cosine for the spherical triangle. The computation process is lengthy but elementary, we omit this process and state our final results for the angles between the axes of generating pairs of spherical Triangle groups and their subgroups in Table 1, 2, 3 of Chapter 4. Once we know the orders of two generators and the angle between them, $\gamma(f, g)$ follows easily from (3.12).

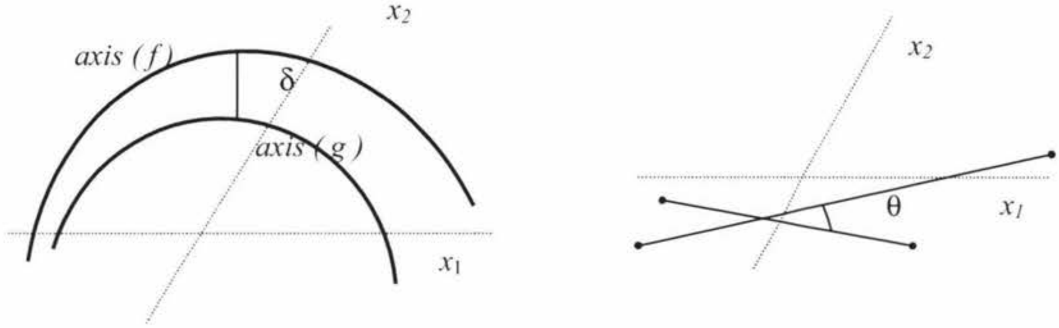


Figure 12 The distance and angle between the *axis(f)* and *axis(g)*

Chapter 4

The parameters of two-generator subgroups of $\mathcal{M}$

4.1 Determining $\beta(f)$ algebraically

Recall that the parameters of a two generator subgroup $\langle f, g\rangle$ of $\mathcal{M}$

$$(\gamma(f,g), \beta(f), \beta(g))$$

define it uniquely up to conjugacy if $\gamma(f,g) \neq 0$(see [5]). In this section we discuss how to determine $\beta(f)$ by its definition $\beta(f) = tr^2(f) - 4$. From Theorem 2.3.1 and the definition of $\beta(g)$ we have

- $\beta(g) = 0$ *if and only if g is parabolic;*
- $\beta(g) \in [-4, 0)$ *if and only if g is elliptic;*
- $\beta(g) > 0$ *if and only if g is hyperbolic;*
- $\beta(g) \notin [-4, +\infty)$ *if and only g is strictly loxodromic.*

If g is elliptic we can write it as

$$g = \begin{pmatrix} e^{ip\pi/n} & 0 \\ 0 & e^{-ip\pi/n} \end{pmatrix} \tag{4.1}$$

where n is the order of g, p is an integer and relatively prime to n and $1 \leq p < n$. A simple computation shows $\beta(g) = 4(\cos^2(p\pi/n) - 1)$. $\beta(g)$ can be computed if we know the order n of g. We state here $\beta(g)$ for $n = 2, 3, 4, 5, 6$

- $\beta(g) = -4$ if g is of order 2;
- $\beta(g) = -3$ if g is of order 3;
- $\beta(g) = -2$ if g is of order 4;

- $\beta(g) = \frac{\sqrt{5}-5}{2}$ or
  $\beta(g) = -\frac{\sqrt{5}+5}{2}$ if g is of order 5;
- $\beta(g) = -1$ if g is of order 6.

If g is hyperbolic or strictly loxodromic, we can write g as

$$B = \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}$$

where $u \neq \pm 1, 0$ and B is the matrix representation of g then $\beta(g) = (u - \frac{1}{u})^2$. We focus on the exact parameters of two generator discrete elementary subgroups of $\mathcal{M}$ groups which contain no loxodromic elements.

4.2 The parameters of A_4, S_4, A_5 and their two generator subgroups

$\gamma(f, g)$ follows easily from the definition once we know the order of the commutator $[f, g]$. Note that since the group in question is finite so is the order of $[f, g]$. The order of $[f, g]$ depends on the two generators, first we need to find out all pairs of generators for the group. Notice also that the parameters of $\langle f, g\rangle$ are related to their generators; different generators of an elementary group may give different parameters. Therefore we want to find out all the possible generating pairs. To start with we need to find out all the elements in A_4, S_4, A_5.

4.2.1 The elements in A_4 and their conjugacy classes

There are twelve elements in A_4, they are:
$\alpha_1 = (1)$,
$\alpha_2 = (12)(34)$, $\alpha_5 = (123)$, $\alpha_8 = (134)$, $\alpha_{11} = (234)$,
$\alpha_3 = (13)(24)$, $\alpha_6 = (243)$, $\alpha_9 = (132)$, $\alpha_{12} = (124)$.
$\alpha_4 = (14)(23)$, $\alpha_7 = (142)$, $\alpha_{10} = (143)$.
The conjugacy classes are
$cl(\alpha_1) = \{\alpha_1\}$,
$cl(\alpha_2) = \{\alpha_2, \alpha_3, \alpha_4\}$,
$cl(\alpha_5) = \{\alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
$cl(\alpha_9) = \{\alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}\}$.

4.2.2 The elements in S_4 and their conjugacy classes

There are 24 elements in S_4, they are
$\alpha_1 = (1)$, $\alpha_7 = (14)$, $\alpha_{13} = (142)$, $\alpha_{19} = (1234)$,
$\alpha_2 = (12)(34)$, $\alpha_8 = (23)$, $\alpha_{14} = (134)$, $\alpha_{20} = (1243)$,
$\alpha_3 = (13)(24)$, $\alpha_9 = (24)$, $\alpha_{15} = (132)$, $\alpha_{21} = (1423)$,

$\alpha_4 = (14)(23)$, $\alpha_{10} = (34)$, $\alpha_{16} = (143)$, $\alpha_{22} = (1342)$,
$\alpha_5 = (12)$, $\alpha_{11} = (123)$, $\alpha_{17} = (234)$, $\alpha_{23} = (1324)$,
$\alpha_6 = (13)$, $\alpha_{12} = (243)$, $\alpha_{18} = (124)$, $\alpha_{24} = (1432)$

The conjugacy classes are:
$cl(\alpha_1) = \{\alpha_1\}$
$cl(\alpha_2) = \{\alpha_2, \alpha_3, \alpha_4\}$
$cl(\alpha_5) = \{\alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}\}$
$cl(\alpha_{11}) = \{\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}\}$
$cl(\alpha_{19}) = \{\alpha_{19}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}\}$

4.2.3 The elements in A_5 and their conjugacy classes

There are 60 elements in A_5, they are
$\alpha_1 = (1)$,
$\alpha_2 = (12)(34)$, $\alpha_3 = (12)(35)$, $\alpha_4 = (12)(45)$, $\alpha_5 = (13)(24)$,
$\alpha_6 = (13)(25)$, $\alpha_7 = (13)(45)$, $\alpha_8 = (14)(23)$, $\alpha_9 = (14)(25)$,
$\alpha_{10} = (14)(35)$, $\alpha_{11} = (15)(23)$, $\alpha_{12} = (15)(24)$, $\alpha_{13} = (15)(34)$,
$\alpha_{14} = (23)(45)$, $\alpha_{15} = (24)(35)$, $\alpha_{16} = (25)(34)$, $\alpha_{17} = (123)$,
$\alpha_{18} = (124)$, $\alpha_{19} = (125)$, $\alpha_{20} = (132)$ $\alpha_{21} = (134)$,
$\alpha_{22} = (135)$, $\alpha_{23} = (142)$, $\alpha_{24} = (143)$, $\alpha_{25} = (145)$
$\alpha_{26} = (152)$, $\alpha_{27} = (153)$, $\alpha_{28} = (154)$, $\alpha_{29} = (234)$,
$\alpha_{30} = (235)$, $\alpha_{31} = (243)$, $\alpha_{32} = (245)$, $\alpha_{33} = (253)$,
$\alpha_{34} = (254)$, $\alpha_{35} = (345)$, $\alpha_{36} = (354)$, $\alpha_{37} = (12345)$,
$\alpha_{38} = (12354)$, $\alpha_{39} = (12435)$, $\alpha_{40} = (12453)$, $\alpha_{41} = (12534)$,
$\alpha_{42} = (12543)$, $\alpha_{43} = (13245)$, $\alpha_{44} = (13254)$, $\alpha_{45} = (13425)$,
$\alpha_{46} = (13452)$, $\alpha_{47} = (13524)$, $\alpha_{48} = (13542)$, $\alpha_{49} = (14235)$,
$\alpha_{50} = (14253)$, $\alpha_{51} = (14325)$, $\alpha_{52} = (14352)$, $\alpha_{53} = (14523)$,
$\alpha_{54} = (14532)$, $\alpha_{55} = (15234)$, $\alpha_{56} = (15243)$, $\alpha_{57} = (15324)$,
$\alpha_{58} = (15342)$, $\alpha_{62} = (15423)$, $\alpha_{60} = (15432)$.

The conjugacy classes are
$cl(\alpha_1) = \{\alpha_1\}$,
$cl(\alpha_2) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}\}$,
$cl(\alpha_{17}) = \{\alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{26}, \alpha_{27}, \alpha_{28}, \alpha_{29}, \alpha_{30}, \alpha_{31},$
$\alpha_{32}, \alpha_{33}, \alpha_{34}, \alpha_{35}, \alpha_{36}\}$,
$cl(\alpha_{37}) = \{\alpha_{37}, \alpha_{40}, \alpha_{41}, \alpha_{44}, \alpha_{45}, \alpha_{48}, \alpha_{49}, \alpha_{52}, \alpha_{53}, \alpha_{56}, \alpha_{57}, \alpha_{60}\}$,
$cl(\alpha_{38}) = \{\alpha_{38}, \alpha_{39}, \alpha_{42}, \alpha_{43}, \alpha_{46}, \alpha_{47}, \alpha_{50}, \alpha_{51}, \alpha_{54}, \alpha_{55}, \alpha_{58}, \alpha_{59}\}$.

Note that we have not only found out all the elements in each of the above groups, we have also found out the conjugacy classes of each of the elements. This is because to find out all two generator subgroups generated by two elements, in A_4 say, we need to pair up every two elements. The conjugacy classes above can make this process much easier. Taking $cl(\alpha_2)$ and $cl(\alpha_5)$ as an example; choosing one element in $cl(\alpha_2)$, say α_2, pairing it up with every other element in $cl(\alpha_5)$, the resulting two generator groups are $\langle\alpha_2, \alpha_i\rangle$ where $i = 5, 6, 7, 8$. Next, do the same to α_3, the resulting two generator groups are $\langle\alpha_3, \alpha_i\rangle$.Choose an arbitrary group from $\langle\alpha_3, \alpha_i\rangle$, say $\langle\alpha_3, \alpha_5\rangle$, let us look at its conjugacy class. Recall that α_3 is conjugate to α_2, that means there exists some element α_j in A_4 such that $\alpha_j\alpha_3(\alpha_j)^{-1} = \alpha_2$:

$$\alpha_j \langle\alpha_3, \alpha_5\rangle (\alpha_j)^{-1} = \langle\alpha_j\alpha_3(\alpha_j)^{-1}, \alpha_j\alpha_5(\alpha_j)^{-1}\rangle$$

$$= \langle\alpha_2, \alpha_i\rangle$$

where $i = 5, 6, 7, 8$. This means every group in $\langle\alpha_3, \alpha_i\rangle$is conjugate to one of $\langle\alpha_2, \alpha_i\rangle$. Since trace is a conjugacy invariant, to identify the range of parameters we just need to choose one element from each of the conjugacy classes, pair the chosen element with every other non-identity element including the elements which are conjugate to it. We do not need to repeat the same process for the elements which are conjugate to the chosen element, since the complex parameters are conjugacy invariant.

The final results of the parameters of A_4, S_4, A_5 and their two generator subgroups are illustrated in *Table 1*. Note in *Table 1*, we use D_2 to denote the Klein-4 group, and Orders means the orders of the generating pair, Angles means the angles between axes of the generating pair.

It is worth mentioning that the definition $\gamma(f,g) = tr[f,g] - 2$ can not be used to determine the values of $\gamma(A,B)$ alone. For example, let $\langle f,g\rangle$ represent the generating pair $\langle f,g\rangle$ of S_4 with orders 2 and 4 respectively. Then the order of $[f,g]$ is 3 and according to the definition, $\gamma(f,g) = -1$ or $\gamma(f,g) = -3$. But the angle between the axes of f and g and (3.12) shows that $\gamma(A,B) = -3$ is not a possibility. Nevertheless, a combination of geometrical and algebraical methods assure us the results we have in the table above are correct.

4.3 Parameters of two generator groups with parabolic elements

There are 5 types of two generator groups with parabolic elements. The simplest group contains only translations which is

$$\langle f(z) = z + 1, \;\; g(z) = z + \tau\rangle$$

Where f, g are parabolic elements and $\tau \in \mathbb{C}$ and $\tau \notin \mathbb{R}$. Observe that $[f,g]$ is also a parabolic element. Hence the parameters corresponding to $\langle f(z) = z + 1, \;\; g(z) = z + \tau\rangle$ are $0, 0, 0$.

Recall in (3.1), (3.2), (3.3), We choose (ab, bc) as generators for each of those groups. In fact if we can choose any pair say (ab, ac) or (ac, bc) as generators, they will also generate the Triangle groups. This can easily be seen by their geometrical results in $\mathbb{R}^2$. In any one of these groups, ab, bc, ac are of finite orders. Then from the definition of the elliptic elements that ab, bc, ac are necessarily elliptic. The fixed points of ab, bc, ac in $\mathbb{C} \cup \infty$ are $\{C, \infty\}$,

Table 1

Groups	Orders	Angles	$(\gamma(f,g),\beta(f),\beta(g))$
D_2	$(2,2)$	$\cos\theta = 0$	$(-4,-4,-4)$
D_3	$(2,2)$	$\cos\theta = \frac{1}{2}$	$(-3,-4,-4)$
	$(2,3)$	$\cos\theta = 0$	$(-3,-4,-3)$
D_4	$(2,2)$	$\cos\theta = \frac{1}{\sqrt{2}}$	$(-2,-4,-4)$
	$(2,4)$	$\cos\theta = 0$	$(-2,-4,-2)$
D_5	$(2,2)$	$\cos\theta = \frac{1}{4}(\sqrt{5}+1)$	$(\frac{1}{2}(\sqrt{5}-5),-4,-4)$
		$\cos\theta = \frac{1}{4}(\sqrt{5}-1)$	$(-\frac{1}{2}(\sqrt{5}+5),-4,-4)$
	$(2,5)$	$\cos\theta = 0$	$(\frac{1}{2}(\sqrt{5}-5),-4,\frac{1}{2}(\sqrt{5}-5))$
			$(-\frac{1}{2}(\sqrt{5}+5),-4,-\frac{1}{2}(\sqrt{5}+5))$
A_4	$(2,3)$	$\cos\theta = \frac{1}{\sqrt{3}}$	$(-2,-4,-3)$
	$(3,3)$	$\cos\theta = \frac{1}{3}$	$(-2,-3,-3)$
S_4	$(2,3)$	$\cos\theta = \sqrt{\frac{2}{3}}$	$(-1,-3,-4)$
	$(2,4)$	$\cos\theta = \frac{1}{\sqrt{2}}$	$(-1,-4,-2)$
	$(3,4)$	$\cos\theta = \frac{1}{\sqrt{3}}$	$(-1,-3,-2)$
	$(4,4)$	$\cos\theta = 0$	$(-1,-2,-2)$
A_5	$(2,3)$	$\cos\theta = \frac{8}{\sqrt{3}(\sqrt{5}-1)}$	$(-\frac{1}{2}(\sqrt{5}+3),-4,-3)$
		$\cos\theta = \frac{8}{\sqrt{3}(\sqrt{5}+1)}$	$(\frac{1}{2}(\sqrt{5}-3),-4,-3)$
	$(2,5)$	$\cos\theta = \sqrt{\frac{2}{(5-\sqrt{5})}}$	$(\frac{1}{2}(\sqrt{5}-3),-4,\frac{1}{2}(\sqrt{5}-5))$
			$(-1,-4,-\frac{1}{2}(\sqrt{5}+5))$
		$\cos\theta = \frac{\sqrt{5}-1}{2(5-\sqrt{5})}$	$(-1,-4,\frac{1}{2}(\sqrt{5}-5))$
			$(-\frac{1}{2}(\sqrt{5}+3),-4,-\frac{1}{2}(\sqrt{5}+5))$
	$(3,3)$	$\cos\theta = \frac{\sqrt{5}}{3}$	$(-1,-3,-3)$
	$(3,5)$	$\cos\theta = \frac{1+\sqrt{5}}{\sqrt{6(5-\sqrt{5})}}$	$(\frac{1}{2}(\sqrt{5}-3),-3,\frac{1}{2}(\sqrt{5}-5))$
			$(-1,-3,-\frac{1}{2}(\sqrt{5}+5))$
		$\cos\theta = \frac{3-\sqrt{5}}{\sqrt{6(5-\sqrt{5})}}$	$(-1,-3,\ \frac{1}{2}(\sqrt{5}-5))$
			$(-\frac{1}{2}(\sqrt{5}+3),-3,-\frac{1}{2}(\sqrt{5}+5))$
	$(5,5)$	$\cos\theta = \frac{\sqrt{5}-1}{5-\sqrt{5}}$	$(\frac{1}{2}(\sqrt{5}-3),\frac{1}{2}(\sqrt{5}-5),\frac{1}{2}(\sqrt{5}-5))$
			$(-1,\frac{1}{2}(\sqrt{5}-5),-\frac{1}{2}(\sqrt{5}+5))$
			$(-\frac{1}{2}(\sqrt{5}+3)],-\frac{1}{2}(\sqrt{5}+5),-\frac{1}{2}(\sqrt{5}+5))$

$\{A,\infty\},\{B,\infty\}$ respectively. A,B,C are the vertices of the triangles in Figure 7 and Figure 8 of Chapter 3. Hence they have ∞ as a common fixed point. According to Theorem 4.3.5 of [1], $tr[ab,bc] = tr[ab,ac] = tr[ac,bc] = 2$. The parameters of groups $\Gamma_{1e},\Gamma_{2e},\Gamma_{3e}$ and the group $\langle f(z) = z+1,\ \ g(z) = az\rangle$, g is of order $2,3,4,6$, are shown in *Table 2*. Note that the order of $f(z) = z+1$ is ∞, $n = 2,3,4,6$ and $a = -4\sin^2(p\pi/n)$

Observe all the γ's in *Table 2* are 0's. This means the parameters do not necessarily define the corresponding groups uniquely. In A_4 for example, $\alpha_5 = (123), \alpha_9 = (132)$, and $(\alpha_5)^2 = \alpha_9$, hence $< \alpha_5, \alpha_9 >$ is a cyclic group of order 3 and $\gamma(\alpha_5, \alpha_9) = 0$. The parameters corresponding to $< \alpha_5, \alpha_9 >$ are $\{0, -3, -3\}$ which are the same as γ_{1e}.

Table 2

Groups	Orders	$(\gamma(f,g), \beta(f), \beta(g))$
$\langle f, g\rangle$	(∞, n)	$(0,\ 0,\ a)$
Γ_{1e}	$(3,3)$	$(0,-3,-3)$
	$(2,3)$	$(0,-4,-3)$
Γ_{2e}	$(2,6)$	$(0,-4,-1)$
	$(3,6)$	$(0,-3,-1)$
	$(2,4)$	$(0,-4,-2)$
Γ_{3e}	$(4,4)$	$(0,-2,-2)$

4.4 The parameters of two generator discrete elementary groups containing only loxodromic elements

In general, the elements in Type 3 of section 2.3 can be describe as

- loxodromic elements: $f(z) = az$, $|a| \neq 1$. The elements of this type fix both 0 and ∞
- elliptic elements of the type :$g(z) = bz, |b| = 1$. The elements of this type belong to a finite cyclic group and fix both 0 and ∞;
- elliptic elements of the type $h(z) = c/z$ which exchange 0 and ∞

The two generators groups generated by a pair of elements above are shown in *Table 3*.

Table 3

Groups	Generators	$(\gamma(f,g), \beta(f), \beta(g))$
G_1	$\langle f_1(z) = a_1 z, f_2(z) = a_2 z\rangle$, $\|a_1\| \neq 1, \|a_2\| \neq 1$	$(0, \mathbb{C}\backslash[-4,0], \mathbb{C}\backslash[-4,0])$
G_2	$\langle f(z) = az, g(z) = bz\rangle$, $\|a\| \neq 1, \|b\| = 1$	$(0, \mathbb{C}\backslash[-4,0], -4\sin^2(p\pi/n))$
D_n	$\langle g(z) = bz, h = c/z\rangle$ $\|b\| = 1$	$(-4\sin^2(p\pi/n), -4\sin^2(p\pi/n), -4)$
	$\langle h_1(z) = c_1/z, h_2(z) = c_2/z\rangle$ $\|c_1/c_2\| = 1$	$(2(\cos(p\pi/n) - 1), -4, -4)$
D_∞	$\langle f(z) = az, h = c/z\rangle$ $\|a\| \neq 1$	$(\mathbb{C}\backslash[-4,0], \mathbb{C}\backslash[-4,0], -4)$
	$\langle h_3(z) = c_3/z, h_4(z) = c_4/z\rangle$ $\|c_3/c_4\| \neq 1$	$(\mathbb{C}\backslash[-4,0], -4, -4)$

where G_1 is a group containing only loxodromic elements, G_2 is a group containing loxodromic elements and elliptic elements which belong to a finite cyclic group, $a, a_1, a_2, b, c, c_1, c_2, c_3, c_4$ are complex numbers.

From the parameters we have found above, we can see that dihedral groups have parameters of form $(\beta, \beta, -4)$ or $(\gamma, -4, -4)$.

Chapter 5

The isolation of A_4, S_4, A_5

Recall that $D^3 \subset \mathbb{C}^3$ contains all the parameters of two generator discrete groups. In this Chapter, we prove that whenever $(-1, -2, \beta_1)$, $(-2, -3, \beta_2)$ and $(-1, -3, \beta_3)$ correspond to discrete elementary groups, they are isolated respectively in D^3 from the set of points corresponding to Kleinian groups. Recall that this set is closed in D^3. Our results shall be obtained by looking for a value of c in (5) where (a, b) is $(-1, -2)$, $(-2, -3)$,or $(-1, -3)$. Also in this Chapter, we show the dihedral groups with parameters $(\beta, \beta, -4)$ and $\beta \in \mathbb{C} \backslash [0, 4]$ are isolated in the space of two generator discrete groups. We start with some previous results.

5.1 previous results

5.1.1. *If f and g are in $\mathcal{M}$ with $\gamma(f, g) = \gamma$ and $\beta(f) = \beta$, then*

$$\gamma(f, gfg^{-1}) = \gamma(\gamma - \beta) \tag{5.1}$$

$$\gamma(f^3, gfg^{-1}) = (\beta + 3)^2 \gamma(-\beta + \gamma) \tag{5.2}$$

The proof of (5.1) depends on the trace identities (see [5])

$$tr(fg) + tr(fg^{-1}) = tr(f)tr(g), \tag{5.3}$$

and

$$tr[f, g] = tr^2(f) + tr^2(g) + tr^2(fg) - tr(f)tr(g)tr(fg) - 2. \tag{5.4}$$

(5.2) is proved by using trace identity (see [8])

$$\gamma(f^3, g) = \gamma(f, g)(\beta(f) + 3)^2$$

and (5.1).

We recall the following from Propostion 1 of [19] and Lemma 9 of [21].

Proposition 5.1.2. *Let G be a Möbius group and $(\psi_n)_{n\in \boldsymbol{N}}$ a sequence of mappings of G into the full group of Möbius transformations. Assume that for each $g \in G$, we have*

$$g = \lim_{n\to\infty} \psi_n(g)$$

Assume further that for each $n \in \boldsymbol{N}$, the transformations $\psi_n(g), g \in G$, generate a discrete and non-elementary group. Then G is discrete and non-elementary.

This is discussed also in [22] of section 5. It is false in dimension $n \geq 4$, but true with additional assumptions. This proposition shows that the discrete non-elementary groups with a fixed number of generators, say r, form a closed set in the space of r-generator groups. Hence if a sequence of non-elementary groups converges, it only converges to a non-elementary group (see [19]).

Note the isolation of a discrete elementary group (a, b, b') from the set of Kleinian groups only depends on a and b. The explanation lies in Proposition 5.1.2. Let (a, b, b') be a discrete elementary group. If (a, b, b') is not isolated from other discrete groups then there is a sequence of discrete groups (a_i, b_i, b'_i), $i = 1, 2, \ldots$ in D^3 converging to (a, b, b') as i tends to infinity. If both a and b are zeros, then Jørgensen's inequality shows that there is no sequence of Kleinian groups converging to (a, b, b'). Suppose a, b are not both zeros. since there are infinitely many elements in the sequence (a_i, b_i, b'_i) and all the parameters of discrete elementary groups are known, it is easy to see that (a_i, b_i, b'_i) are a sequence of discrete Kleinian groups. According to Proposition 5.1.2, (a, b, b') is discrete non-elementary, this is a contradiction. Thus we have proved

Lemma 5.1.3. *Let $(a, b, b') \in \mathbb{C}^3$ be a discrete elementary group. If a, b satisfy the inequality (5), then (a, b, b') is isolated from the set of Kleinian groups.*

According to this lemma, to prove the isolations of $(-1, -2, \beta_1)$, $(-2, -3, \beta_2)$, $(-1, -3, \beta_3)$, we need only to consider the inequalities of the form (5) satisfied by $(-1, -2)$, $(-2, -3)$, $(-1, -3)$. We shall need the following theorem to work with these inequalities:

Lemma 5.1.4. *Suppose that f and g_i are elements of a discrete subgroup G of $\mathcal{M}$, that $\beta = \beta(f) \neq -4$ and that $\gamma_0 = 0$ or $\gamma_0 = \beta + 1$. If $\gamma(f, g_j) \neq \gamma_0$ for all j, then*

$$\inf_j |\gamma(f, g_j) - \gamma_0| > 0.$$

We state here several other previous results we shall use in the next section.

Theorem 5.1.5. *Two Möbius transformations g and h have a common fixed point in $\mathbb{C}$ if and only if $tr[g, h] = 2$ (equivalently $\gamma(g, h) = 0$.*

This theorem says if $[g, h]$ is non-trivial, then $[g, h]$ is parabolic and $\gamma(g, h) = 0$. It also implies that if $\gamma(g, h) = 0$, then $\langle g, h\rangle$ is elementary (see [1]).

Theorem 5.1.6. *Let f be loxodromic and suppose that f and g have exactly one fixed point in common. Then $\langle f, g\rangle$ is not discrete.*(see [1])

Lemma 5.1.7. *If $\langle f, g\rangle$ is a discrete subgroup of $\mathcal{M}$ with $\gamma(f,g) = \beta(f) \neq 0$, then either f is elliptic of order $2, 3, 4, 6$ or g is of order 2*, see [5]

Lemma 5.1.8. *Suppose that $\langle f, h\rangle$ is a discrete subgroup of $\mathcal{M}$ with $\gamma(f,h) \neq 0$ and $\gamma(f,h) \neq \beta(f)$. Then there exist elliptic h_1 and h_2 of order 2 such that $\langle f, h_1\rangle$ and $\langle f, h_2\rangle$ are discrete with*

$$\gamma(f,h_1) = \gamma(f,h) \qquad \text{and} \qquad \gamma(f,h_2) = \beta(f) - \gamma(f,h).$$

5.2 Isolation of $(-1,-2,b_1), (-2,-3,b_2), (-1,-3,b_3)$

We start with an analysis of groups with $\gamma = \beta$. Let $\langle f, g\rangle$ be a two generator group with $\gamma(f,g) = \gamma = \beta = \beta(f)$. There are two cases:

Case 1: g is elliptic of order two. Then $\langle f, g\rangle$ is dihedral

Case 2: g is not of order 2. There are three subcases to consider

- f is loxodromic. According to (5.1), there is a subgroup $\langle f, gfg^{-1}\rangle$ of $\langle f, g\rangle$ with $\gamma(f, gfg^{-1}) = \gamma(\gamma - \beta) = 0$. Theorem 5.1.5 implies f and gfg^{-1} have a common fixed point and if $[f, gfg^{-1}]$ is non-trivial, then $[f, gfg^{-1}]$ is parabolic.f and $[f, gfg^{-1}]$ have exactly one common fixed point and $\langle f, [f, gfg^{-1}]\rangle$ is not discrete. Since every subgroup of a discrete group is also discrete, therefore $\langle f, g\rangle$ is not discrete.

- f is elliptic. According to Lemma 5.1.7, a necessary condition for $\langle f, g\rangle$ to be discrete is that f is of order $2, 3, 4, 6$; otherwise, $\langle f, g\rangle$ is non-discrete. When $\langle f, g\rangle$ is discrete, it can be elementary or non-elementary. For example if the parameters of $\langle f, g\rangle$ are $(-1,-1,-1)$, then $\langle f, g\rangle$ is discrete and non-elementary. The proof of this depends on Theorems 4.2, 5.13, 5.14 and Table 4 of [9]. In the case $\langle f, g\rangle$ is discrete and elementary, according the parameters we have found in Chapter 4, g has to be of order 2, and $\langle f, g\rangle$ is Klein-4 group, D_3, D_4,or D_6.

- f is parabolic, then $\beta(f) = \gamma(f,g) = 0$. According to Theorem 5.1.5, f and g have a common fixed point, hence $\langle f, g\rangle$ is elementary.

We now prove Jørgensen's inequality by using the iteration of trace commutators parameters.

Proof. Let $\langle f, g\rangle$ be a two generator group with $\gamma(f,g) = \gamma$, then $\gamma\left(f, gfg^{-1}\right) = P(\gamma) = \gamma(\gamma - \beta)$.

$$P(\gamma) \longmapsto P(P(\gamma)) \longmapsto P(P(P(\gamma)))\ldots \tag{5.5}$$

is a sequence of trace commutator parameters of $\langle f, g\rangle$. The polynomial $P(\gamma)$ has two fixed points: $\gamma = 0$ and $\gamma = \beta + 1$. We analyze what happens under iteration near $\gamma = 0$. Notice that the derivative of $P(\gamma)$ is $P'(\gamma) = 2\gamma - \beta$ and $P'(0) = -\beta$. According to the Lemma 5.1.4, if $\langle f, g\rangle$ is discrete with $\gamma \neq 0$, $\gamma \neq \beta$, then the sequence (5.5) does not converge to 0 in a neighborhood of 0. Now, $|P'(0)| = |\beta| < 1$ implies 0 is an attracting fixed point, if the sequence (5.5) does converge to 0. Then in a small neighborhood $B(0, r)$ of 0, where $0 < r << 1$, we have

$$P(B(0, r)) \subset B(0, r).$$

Hence we have

$$\left|re^{i\theta}(re^{i\theta} - \beta)\right| < r. \tag{5.6}$$

From this we deduce if $\langle f, g\rangle$ is discrete with $\gamma \neq \beta$ and $\gamma \neq 0$, then

$$|\gamma| + |\beta| \geq 1.$$

Because

$$\begin{aligned} & |\gamma| + |\beta| < 1, \\ \Longrightarrow \quad & \left|re^{i\theta} - \beta\right| < 1, \qquad (|\gamma| = r) \\ \Longrightarrow \quad & \left|re^{i\theta}\right| \left|re^{i\theta} - \beta\right| < r, \\ \Longrightarrow \quad & \left|re^{i\theta}(re^{i\theta} - \beta)\right| < r. \end{aligned}$$

As we know (5.6) violates the discreteness of $\langle f, g\rangle$, we have derived Jørgensen's inequality.

□

Similarly, by analyzing another fixed point $\gamma = \beta + 1$ of $P(\gamma) = \gamma(\gamma - \beta)$, we can prove the first part of Theorem 0.2, that is

If $\langle f, g\rangle$ is discrete, then

$$|\gamma + 1| + |\beta + 2| \geq 1 \tag{5.7}$$

unless $\gamma = -1$.

Proof. The derivative of $P(\gamma) = \gamma(\gamma - \beta)$ is $P'(\gamma) = 2\gamma - \beta$, and

$$P'(1 + \beta) = 2 + 2\beta - \beta = 2 + \beta.$$

$|2 + \beta| < 1$ implies $1 + \beta$ is an attracting fixed point. Then:

$$\begin{aligned} & \left|P(1 + \beta + re^{i\theta}) - (1 + \beta)\right| & (5.8) \\ = \; & \left|(1 + \beta + re^{i\theta})(1 + \beta + re^{i\theta} - \beta) - (1 + \beta)\right| & (5.9) \\ = \; & \left|1 + \beta + re^{i\theta} + re^{i\theta}(1 + \beta + re^{i\theta}) - (1 + \beta)\right| & (5.10) \\ = \; & |r| \left|2 + \beta + re^{i\theta}\right| & (5.11) \end{aligned}$$

So $P(D(1 + \beta, r)) \subset D(1 + \beta, r)$ if $\left|2 + \beta + re^{i\theta}\right| < 1$. Since $\gamma = 1 + \beta + re^{i\theta}$, this is implied by

$$|\gamma + 1| < 1, \tag{5.12}$$

The inequality (5.12) violates the discreteness of $\langle f, g\rangle$. Hence if γ is a trace commutator parameter for a discrete group, then

$$|\gamma + 1| \geq 1$$

unless $\gamma = -1$. Note that $P(-1) = 1 + \beta$. Now,

$$\begin{aligned} & |\gamma + 1| && (5.13)\\ = \ & |\gamma - \beta + \beta + 2 - 1| && (5.14)\\ \leq \ & |\gamma - \beta - 1| + |\beta + 2|. && (5.15)\end{aligned}$$

This implies

$$|\gamma - \beta - 1| + |\beta + 2| \geq 1$$

unless $\gamma = 1 + \beta$. Note that $|2 + \beta| < 1$. Then according to Lemma 5.1.8, we are done. □

When $\gamma = -1$, according to (5.1), $\gamma(f, gfg^{-1}) = \beta + 1$. $(\beta + 1, \beta, \beta)$ are the parameters of the subgroup $\langle f, gfg^{-1}\rangle$ of $\langle f, g\rangle$. We consider the parameters of this subgroup. We start with a discussion of *Nielsen Move* and *Nielsen equivalence*. Suppose (f, g) is a pair of generators of $\langle f, g\rangle$, it is natural to ask what are the other generating pairs for this group? The answer lies in the following three moves:

- change one generator to its inverse: $(f, g) \longmapsto (f^{-1}, g)$;
- swap f and g: $(f, g) \longmapsto (g, f)$;
- replace one generator by itself times a power of the other generator: $(f, g) \longmapsto (fg^n, g)$ where $n \in \mathbb{Z}$.

The above three moves are called *Nielsen moves* and the resulting relations are called *Nielsen equivalences*. So another generating pair for $\langle f, g\rangle$ is (f^{-1}, g), (g, f) or (fg^n, g).

Theorem 5.2.1. *A two generator subgroup G of $\mathcal{M}$ is Nielsen equivalent to a group with elliptic generators of orders 2 and 3 if and only if it is Nielsen equivalent to a group with parameters $(\beta + 1, \beta, \beta)$.*

Theorem 5.2.1 is discussed in [5], where it is proved by using *Nielsen moves* and the trace identities (5.3), (5.4) and

$$tr(fg^{-2}) = tr(f)(tr(fg^{-1}) - 1).$$

Lemma 5.2.2. *Suppose that $\langle f, g\rangle$ is a discrete subgroup of $\mathcal{M}$ and that either f and g are elliptics of order 3 and 2 respectively or that $\gamma(f, g) = \beta(f) + 1$. Then*

$$|\gamma(f, g) + 1| \geq \frac{1}{2}(\sqrt{5} - 1) \quad or \quad \gamma(f, g) = -1.$$

Proof. Let $h = gfg^{-1}$. If $\gamma(f,g) = \beta(f) + 1$, then $\gamma(f,h) = \gamma(f,g)$ by (5.1). Also, h is conjugate to f, so $\beta(h) = \beta(f)$ and $\langle f, h\rangle$ is Neilsen equivalent to a group with elliptic generators of order 2 and 3 by Theorem 5.2.1. Gehring and Martin have shown that in the disks

$$\{z : 0 < |z + \frac{1}{2}(3 \pm 1)| < \frac{1}{2}(\sqrt{5} - 1)\},$$

there is no complex number assumed by the commutator parameter of a discrete two generator group with a generator of order 3 and we obtain the results. □

Lemma 5.2.3. *Suppose $\langle f, g\rangle$ is discrete with parameters $(-1, \beta, \beta')$. Then*

$$\beta = -2$$

or

$$|\beta + 2| \geq \frac{1}{2}(\sqrt{5} - 1)$$

Proof. Recall that when $\gamma(f,g) = -1$, $\gamma(f, gfg^{-1}) = \beta + 1$. $(\beta + 1, \beta, \beta)$ are the parameters of the subgroup $\langle f, gfg^{-1}\rangle$ of $\langle f, g\rangle$. Then the results follow easily from Lemma 5.2.2. □

Combining Lemma 5.2.3 and (5.7), we have showed that

Lemma 5.2.4. *A discrete elementary group G with parameters $(-1, -2, \beta')$ is isolated from Kleinian groups.*

Lemma 5.2.5. *A discrete elementary group G with parameters $(-2, -3, \beta')$ is isolated from Kleinian groups .*

Proof. Suppose $\langle f, g\rangle$ is a discrete group with $\gamma(f,g) = \gamma$, $\beta(f) = \beta$. According to (5.2), $(P_1(\gamma), \beta, \beta')$ are the parameters of the subgroup $\langle f^3, gfg^{-1}\rangle$ of $\langle f, g\rangle$; where

$$P_1(\gamma) = (\beta + 3)^2\gamma(-\beta + \gamma).$$

The derivative of $P_1(\gamma)$ is

$$P_1'(\gamma) = (\beta + 3)^2(-\beta + 2\gamma).$$

And

$$P_1'(0) = (\beta + 3)^2(-\beta),$$

where $\gamma = 0$ is a fixed point of $P_1(\gamma)$. $|(\beta + 3)\beta| < 1$ implies that 0 is an attracting fixed point. When $\beta \neq -3, \gamma \neq 0$, and $\gamma \neq \beta$, we suppose the trace sequence

$$P_1(\gamma) \longmapsto P_1(P_1(\gamma)) \longmapsto P_1(P_1(P_1(\gamma)))\ldots$$

converges to 0. Hence in a small neighborhood $B(0, r_1)$ of 0, We have

$$P_1(B(0, r_1)) \subset B(0, r_1), \tag{5.16}$$

$$\begin{aligned}
&\Longrightarrow \quad \left|P_1(r_1e^{i\theta})\right| < r_1, \\
&\Longrightarrow \quad \left|(3+\beta)^2(r_1e^{i\theta})(-\beta+r_1e^{i\theta})\right| < r_1, \\
&\Longrightarrow \quad \left|r_1e^{i\theta}\right|\left|(3+\beta)^2(-\beta+r_1e^{i\theta})\right| < r_1, \\
&\Longrightarrow \quad \left|(3+\beta)^2(-\beta+r_1e^{i\theta})\right| < 1, \qquad \left(\left|r_1e^{i\theta}\right| = r_1\right) \\
&\Longrightarrow \quad |(3+\beta)^2(-\beta+\gamma)| < 1. \qquad \left(\gamma = r_1e^{i\theta}\right)
\end{aligned}$$

Now,

$$|\gamma-\beta| = |\gamma+2-(\beta+3)+1|, \tag{5.17}$$

$$\leq |\gamma+2| + |(\beta+3)| + 1. \tag{5.18}$$

Suppose

$$|\gamma+2| + |(\beta+3)| < \epsilon$$

where ϵ is a positive real number. Then (5.16) holds if

$$\epsilon^2(\epsilon+1) < 1. \tag{5.19}$$

Solve (5.19), we have

$$|\gamma+2| + |(\beta+3)| < 0.7539. \tag{5.20}$$

Since (5.20) violates the discreteness of $\langle f, g\rangle$, we conclude

$$|\gamma+2| + |\beta+3| \geq 0.7539. \tag{5.21}$$

Obviously the groups with $\gamma = 0$ and $\gamma = \beta$ satisfy (5.21). The remaining question is whether or not a discrete group with $\beta = -3$ is isolated from $(-2, -3, \beta')$. In [5], Gehring and Martin have shown that if $\beta = -3$, then in the punctured closed disk

$$D = \{z : |z+2| \leq 0.5574, \quad z \neq -2\},$$

there is no value which is assumed by a two generator discrete group with $\beta = -3$. Therefore we conclude that if $\langle f, g\rangle$ is discrete with $\beta = -3$, then

$$|\gamma+2| > 0.5574. \tag{5.22}$$

Hence $(-2, -3, \beta')$ is isolated from all other discrete groups. □

Similarly using $P_1(\gamma)$, we can show that if $\langle f, g\rangle$ is discrete with $\gamma(f,g) = \gamma, \beta(f) = \beta$ then

$$|\gamma+1| + |\beta+3| \geq 0.6180 \tag{5.23}$$

unless $\beta = -3$. Again in [5], Gehring and Martin have shown inside or on the boundary of the punctured disk

$$D_1 = \{z : |z+1| < 0.5574, \quad z \neq -1\}$$

there is no value assumed by a discrete two generator group with $\beta = -3$. We conclude for discrete $\langle f, g\rangle$ with $\beta = -3$:

$$|\gamma + 1| > 0.5574. \tag{5.24}$$

Hence we have shown that $(-1, -3, \beta')$ is isolated from all the other discrete two generator groups.

We have shown the isolations of $(-1, -2, \beta_1), (-2, -3, \beta_2), (-1, -3, \beta_3)$ from all other points in $\mathbb{C}^3$ corresponding to two generator discrete groups. In reference to the parameters we have found in Chapter 5 for A_4, S_4, A_5, we can say that A_4, S_4, A_5 are isolated respectively in the space of two generator discrete groups.

We end this section with a theorem that shows the isolation of a dihedral group $(\beta, \beta, -4)$, where $\beta \in \mathbb{C}\backslash[0, 4]$, in the space of two generator discrete groups. As we can see from the parameters we have found in Chapter 4, that such a group is indeed D_∞, the dihedral group with an infinite order. We need the following theorem for the proof, see [11].

Theorem 5.2.6. *Suppose that $\langle f, g\rangle$ is a discrete subgroup of $\mathcal{M}$ with $\gamma(f, g) \neq 0$ and $\beta(f) = \beta(g) \neq -4$. Then*

$$|\gamma(f, g)| > 0.193. \tag{5.25}$$

If, in addition, $\gamma(f, g)$ is real, then

$$|\gamma(f, g)| \geq 2 - 2\cos(\pi/7) = 0.198\ldots \ . \tag{5.26}$$

Inequality (5.26) is sharp.

Theorem 5.2.7. *There exists a real $\epsilon > 0$ such that if (x, y, z) are parameters of a discrete two generator group $\langle f, g\rangle$, then*

$$|x - \beta| + |y - \beta| + |z - 4| > \epsilon \tag{5.27}$$

where $\beta \in \mathbb{C}\backslash[-4\ \ 0]$

Proof. Suppose that (5.27) does not hold, then there is a sequence of discrete two generator groups $\langle f_i, g_i\rangle$ with parameters (x_i, y_i, z_i) such that

$$x_i \to \beta, \quad y_i \to \beta, \quad z_i \to -4$$

and

$$|x_i - \beta| + |y_i - \beta| + |z_i - 4| \to 0 \tag{5.28}$$

We now have two cases to consider:

case 1: $x_i \neq y_i$. According to Exercise 3 of Section 4.5 of [1], a sequence of elliptic elements cannot converge to a loxodromic element. Since $\beta \in \mathbb{C}\backslash[-4, 0]$, each f_i is necessarily loxodromic element and hence $y_i \in \mathbb{C}\backslash[-4, 0]$. There exists a subgroup $\langle f_i, g_i f_i g^{-1}\rangle$ of $\langle f_i, g_i\rangle$ with parameters $(x_i(x_i - y_i), y_i, y_i)$. Since $x_i \neq 0$ and $x_i \neq y_i$, it is easy to see that $\langle f_i, g_i f_i g_i^{-1}\rangle$ is also discrete and non-elementary. Hence $\langle f_i, g_i f_i g_i^{-1}\rangle$ satisfies the conditions in Theorem 5.2.6 and we have

$$|x_i(x_i - y_i)| > 0.193. \tag{5.29}$$

We also have

$$\begin{aligned} |x_i - y_i| &= |x_i - \beta + \beta - y_i| & (5.30)\\ &\leq |x_i - \beta| + |y_i - \beta| \,. & (5.31)\\ && (5.32) \end{aligned}$$

From (5.29) and the inequality above , we get

$$|x_i - \beta| + |y_i - \beta| > \epsilon$$

for all i. Thus we reach a contradiction.

case 2: $x_i = y_i$, then the parameters of $\langle f_i, g_i f_i g^{-1}\rangle$ are $(0, y_i, y_i)$, therefore f_i and $g_i f_i gi^{-1}$ have a common fixed point and $\langle f_i, g_i f_i g_{-1}\rangle$ is a discrete elementary group. The set of fixed points of f_i is $\{fix(f_i)\}$ and the set fixed points of $g_i f_i g_i^{-1}$ is $\{g_i(fix(f_i)\})$. In a discrete elementary group, these two sets having one common fixed point means they are equal, that is $\{g_i(fix(f_i)\}) = \{fix(f_i)\}$.This implies either $fix(g_i) = fix(f_i)$ or g_i interchanges the fixed points of f_i, in the latter case, g_i is of order 2. Hence we deduce that either $x_i = 0$ or $z_i = -4$. The first case is not possible since $x_i = y_i$ and $y_i \neq 0$ therefore every element in the sequence of discrete groups which converges to $(\beta, \beta, -4)$ has parameters $(x_i, x_i, -4)$. Such a sequence is a sequence of dihedral groups,thus we have $x_i = \beta$ for all sufficient large i. Therefore (5.28) holds. □

Jørgensen's inequality shows that elementary discrete groups $(0, 0, \beta')$ are isolated from Kleinian groups. Using the same technique we can show that dihedral groups $(\beta, \beta, -4)$, where $\beta \in [-4, 0)\backslash\{-1, -2, -3, -4\}$ are isolated from Kleinian groups. when $\beta \in \{-1, -2, -3, -4\}$, the group is D_6, D_4, D_3, or the Klein-4 group.

Chapter 6

Remarks and further research

Our main concern in this thesis has been all the generating pairs and the corresponding parameters of the elementary discrete two generator groups . To this end we have discussed some preliminary topics like hyperbolic and spherical geometry, Möbius transformations, Triangle groups. We have considered a few examples to show what we are able to do with the parameters we have found in this paper. By using inequalities such as (5.7), (5.21), (5.22), (5.23), we have shown the isolation of some parameters from the parameters of Kleinian groups. But not all the minimum values in the inequalities are sharp. The following theorem is proved by Tan in [2].

Theorem 6.1.1. *Suppose that the Möbius transformations f and g generate a discrete group. If $\beta \neq -3$, then*

$$|\gamma + 2| + |\beta + 3| \geq 1 \tag{6.1}$$

The minimum value 1 in (6.1) is better than 0.7539 in (5.21) in the sense that (6.1) is sharp for discrete non-elementary two generator groups. For example, $(-2, -2, -2)$ is a discrete non-elementary two generator group (see [9]). The inequality (5.7) is sharp for discrete non-elementary two generator groups. The inequality (5.23) is sharp for discrete elementary two generator groups. For example the parameters of A_5 are $(\frac{1}{2}(\sqrt{5}-3), -3, -4)$. In Lemma 5.2.3., we obtain a better estimation for β in the discrete group $(-1, \beta, \beta')$ than in Theorem 0.2. In fact $|\beta + 2| \geq \frac{1}{2}(\sqrt{5} - 1)$ is sharp for discrete elementary groups since $(-1, \frac{1}{2}(\sqrt{5} - 5), -4)$ is A_5. The estimations in (5.22) and (5.24) are not the best possible.

Our goal in studying the isolation of elementary discrete two generator groups is to learn more about the Kleinian groups. In my future research, I shall prove the isolation of all other parameters I have found in this paper, attempt to find the best minimum values and then use these results to study the geometry and topology of hyperbolic 3-manifolds and orbifolds.

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