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Topological Regularity for Solutions to the Generalised Hopf Equation

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Abstract

The generalised Hopf equation is the first order nonlinear equation defined on a planar domain $\Omega \subset \mathbb{C}$, with data Φ a holomorphic function and $\eta \geq 1$ a positive weight on Ω,

$$h_w \, \overline{h_{\overline{w}}} \, \eta(w) = \Phi.$$

The Hopf equation is the special case $\eta(w) = \tilde{\eta}(h(w))$ and reflects that h is harmonic with respect to the conformal metric $\sqrt{\tilde{\eta}(z)}|dz|$, usually η is the hyperbolic metric. This article obtains conditions on the data to ensure that a solution is open and discrete. We also prove a strong uniqueness result.

Keywords Harmonic mapping · Hopf equation · Finite distortion · Partial differential equations · Topological regularity

Mathematics Subject Classification 35B65 · 30C62

1 Introduction

The well known Hopf equation is the first order equation

$$h_w \overline{h_{\overline{w}}} \, \eta(h) = \Phi(w), \quad \text{almost every} z \in \Omega. \tag{1}$$

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Here Ω is a planar domain, and

1. $h:\Omega\to\mathbb{C}$ lies in $W^{1,1}_{loc}(\Omega)$,
2. Φ is holomorphic in Ω,
3. $\eta(w)\geq 1$ is locally Lipschitz (most often smooth), and
4. the Jacobian determinant $J(w,h)\geq 0$ at almost every $w\in\Omega$.

For a $W^{2,1}_{loc}$ solution h, differentiating (1) quickly reveals that h satisfies the *tension equation*,

$$\Delta h + 2(\log\eta)_z(h)h_w\, h_{\overline{w}} = 0, \tag{2}$$

expressing the fact that

$$h:(\Omega,|dw|)\to(\tilde{\Omega},\sqrt{\eta(\tilde{w})}\,|d\tilde{w}|)$$

is a harmonic mapping, see Jost [12] and Daskalopoulos and Wentworth [3] as general references for planar harmonic mappings, or Duren [4] in the case of $\eta\equiv 1$. Elliptic regularity subsequently implies a higher degree of smoothness of h. However no topological regularity can be guaranteed despite the assumption (4) above. Some aspects of the converse are well understood via Lewy's Theorem [13], subsequently Schoen and Yau for the hyperbolic metric [20] and finally its generalisation to all metrics and degree 1 mappings [17]. There topological regularity (e.g. a homeomorphism) implies a harmonic mapping is a diffeomorphism.

The purpose of this article is to consider the question of topological regularity in a more general setting.

2 The Generalised Hopf Equation

We consider the generalised Hopf equation:

$$h_w\overline{h_{\overline{w}}}\;\eta(w)=\Phi(w),\quad \text{almost every} z\in\Omega. \tag{3}$$

where h, together with the data Φ and η, satisfy (1–4) above. Of course if h is harmonic to the metric $\sqrt{\tilde{\eta}}\,|d\tilde{w}|$, then h solves (3) with $\eta(w)=\tilde{\eta}(h(w))$. Then h should be locally Lipschitz to satisfy our assumptions. This follows for the Hopf equation (1) by the result of Iwaniec et al. [7] who prove a much more general result which in fact shows solutions to (3) are locally Lipschitz.

There are natural extremal mapping problems where equations such as (3) occur, see [15, 19] in the case $p=1$.

We have assumed that $J(w,h)\geq 0$ to remove certain pathologies which can occur by piecing together reflections and so forth. In particular this implies $|h_{\overline{z}}|\leq|h_z|$. It follows that away from the discrete set

$$Z_\Phi=\{w:\Phi(w)=0\}\subset\Omega$$

we have $h_{\overline{w}}$ locally bounded and vanishing continuously on Z_Φ, and hence $h \in W^{1,p}_{loc}(\Omega)$ for all $1 \le p < \infty$, and as mentioned in fact h is locally Lipschitz. This now guarantees the local integrability of the Jacobian. We have established

Lemma 1 *Let $h : \Omega \to \mathbb{C}$ be a $W^{1,1}_{loc}(\Omega)$ solution to (3) with $J(w,h) > 0$ for almost every $w \in \Omega$. Then h is a mapping of finite distortion.*

The theory of mappings of finite distortion can be found in [1, 5, 8, 9] which we use as basic references. The hypothesis $J(w,h) > 0$ almost everywhere is necessary. If $h(z) = a(x)$ and $\Phi = c^2 > 0$ is constant, $z = x + iy$, the equation becomes

$$a'(x)\sqrt{\eta(x)} = c, \quad a(x) = a(x_0) + c\int_{x_0}^{x} \frac{dx}{\sqrt{\eta(x)}} \tag{4}$$

which is certainly not a mapping of finite distortion. We will see in a moment that the case Φ constant is (at least locally) the general case.

A mapping h is discrete if for every $\tilde{w} \in h(\Omega)$ the set $\{w \in \Omega : h(w) = \tilde{w}\}$ is a discrete subset of Ω. We will repeatedly use the following classical result whose proof we briefly sketch.

Lemma 2 *Suppose that $h : \Omega \to \mathbb{C}$ is continuous in Ω, open and discrete in $\Omega \setminus Z$, where Z is a discrete subset of Ω. Then h is open and discrete in Ω.*

Proof The Stoilow factorisation theorem implies that $h|\Omega \setminus Z = \Psi \circ f$ where $f : \Omega \setminus Z \to \Omega \setminus Z$ is a homeomorphism and $\Psi : \Omega \setminus Z \to \mathbb{C}$ is holomorphic. As h is continuous, Ψ is locally bounded and so the points of Z are removable singularities. Hence Ψ may be assumed to be holomorphic in Ω. Then f is continuous on Ω and one can show f extends to a self homeomorphism of Ω. From this the result follows. □

The *Beltrami coefficient* of a mapping of finite distortion h is

$$\mu_h = h_{\overline{z}}/h_z$$

and is defined almost everywhere. If G is open and $\|\mu_h\|_{L^\infty(G)} = k < 1$, then h is *quasiregular* in G or *quasiconformal* if $h|G$ is a homeomorphism. Quasiregular mappings are open and discrete.

3 Examples

Here we give some examples to motivate the hypotheses we make on the data Φ and η in Eq. (3).

3.1 η Constant on a Simply Connected Region Ω

Then $h_w \overline{h_{\overline{w}}} = \Phi$, a $W^{2,1}_{loc}$ solution is smooth and harmonic, and so we may find holomorphic $U, V : \Omega \to \mathbb{C}$ so that $h = U + \bar{V}$. That $|h_{\overline{z}}| \le |h_z|$ gives $|V'| \le$

$|U'|$. Away from the discrete set of points points where $\{U' = 0\}$ we have V'/U' holomorphic and $|V'/U'| \le 1$, so V'/U' is holomorphic in Ω. The maximum principle implies that ether $|\mu_h| < 1$ on Ω and h is locally quasiregular - hence open and discrete - or $|\mu_h| \equiv 1$, $|U'| = |V'|$, $U' = \bar{\zeta}^2 V' + \bar{c}$, $|\zeta| = 1$, and hence

$$h(z) = U + \bar{V} = 2\bar{\zeta}\,\Re e(\zeta U) = \bar{\zeta} u(z) + c$$

where u is a real valued harmonic function. In this case h cannot be open and discrete.

There is no way to avoid this dichotomy without making further assumptions on h such as $J(z,h) > 0$ almost everywhere, or assuming that η is nonconstant.

3.2 $\Phi\overline{\eta_w} = |\Phi|\eta_w$

This is a condition about the alignment of the arguments of Φ and η_w^2. We first examine this condition in the case $\Phi \equiv 1$, so h is a solution to

$$h_z \overline{h_{\bar{z}}}\, \eta(z) \equiv 1 \tag{5}$$

Then the condition above is $\overline{\eta_w} = \eta_w$ which implies $\eta_y = 0$, so that $\eta(z) = \eta(x)$. Then choose $a'(x) = 1/\sqrt{\eta(x)} \le 1$ and $h(z) = a(x)$ is a solution to (5) as discussed at (4).

More generally we make the following calculation. Suppose that G is open and we can find a holomorphic solution to

$$\phi'(z)^2\, \Phi(\phi) \equiv 1, \quad \phi : G \to \Omega. \tag{6}$$

Then if $h_w \overline{h_{\overline{w}}}\, \hat{\eta}(w) = \Phi(w)$ we see that $h \circ \phi$ solves (5) with $\eta(z) = \hat{\eta}(\phi(z))$;

$$(h\circ\phi)_z \overline{(h\circ\phi)_{\bar{z}}}\, \hat{\eta}(\phi(z)) = h_w(\phi)\overline{h_{\overline{w}}(\phi)}\, \hat{\eta}(\phi)\phi'(z)^2 = \Phi(\phi)\phi'(z)^2 = 1.$$

Actually we can always solve (6) as soon as we can define a single valued branch of $\sqrt{\Phi}$, which we can do in a simply connected region where $\Phi \neq 0$. Then simply solve

$$\psi'(w) = \frac{1}{\sqrt{\Phi(w)}} \neq 0$$

and put $\phi = \psi^{-1}$ on a maximal subdomain of definition. Next,

$$\eta_w = \hat{\eta}_z(\phi(w))\phi'(w)$$

Then $\eta_z = \overline{\eta_z}$ implies

$$\hat{\eta}_w(\phi(z))\phi'(z) = \overline{\hat{\eta}_w(\phi(z))\phi'(z)}$$

and hence

$$\hat{\eta}_w(\phi(z))\Phi(\phi) = \overline{\hat{\eta}_w(\phi(z))}|\Phi(\phi)| \tag{7}$$

and, writing $w = \phi(z)$ we have $\hat{\eta}_w\Phi = \overline{\hat{\eta}_w}|\Phi(w)|$ and this is the condition we are considering.

By way of example let us consider the hyperbolic metric. Then $\eta(w) = (1-|w|^2)^{-2}$ and $\Phi\overline{\eta_w} = |\Phi|\eta_w$ gives $\Phi w^2 = |\Phi w^2|$, that is $\Phi w^2 \geq 0$. Away from this set we will find that solutions are open and discrete, see Theorem 3.

We can rewrite the condition at Sect. 3.2 away from the zeros of Φ as follows.

$$\Phi\overline{\eta_w} = |\Phi|\eta_w \Rightarrow \sqrt{\Phi}\overline{\eta_w} = \sqrt{\bar{\Phi}}\eta_w$$

Thus $\sqrt{\Phi}\eta_{\overline{w}}$ is real and that is if and only if $\Phi\eta_{\overline{w}}^2 \geq 0$.

4 Main Results

As we have seen, away from Z_Φ we can arrange things locally so that $\Phi \equiv 1$. We next examine that case.

4.1 The Special Case $\Phi \equiv 1$

We have

$$h_z\,\overline{h_{\bar{z}}}\,\eta(z) \equiv 1. \tag{8}$$

We immediately note $0 \leq \mu_h \leq 1$. Let us remove the locally uniformly elliptic case so as to focus on the non-classical degenerate case.

Theorem 1 *Suppose that h is a $W^{1,1}_{loc}(\Omega)$ solution to (8) which is locally quasiregular; $\|\mu\|_{L^\infty(G)} \leq k < 1$ for any $G \Subset \Omega$. Then on G we have $h = \phi(g)$ where $g : G \to G$ is a quasiconformal diffeomorphism and $\phi : G \to \mathbb{C}$ is holomorphic.*

Proof On G we have $|h_{\bar{z}}| \leq k|h_z|$. Thus $|h_z| \geq 1/\sqrt{k\eta} > 2/\sqrt{(1+k)\eta}$. Define $A(z,\zeta)$, real valued, as follows.

$$A(z,\zeta) = \begin{cases} \frac{1+k}{2}|\zeta| & |\zeta| \leq 1/\sqrt{(1+k)\eta/2} \\ 1/(\eta|\zeta|) & |\zeta| \geq 1/\sqrt{(1+k)\eta/2} \end{cases} \tag{9}$$

Since $k < (1+k)/2$ and $\eta(z)$ is Lipschitz, we may smooth A along the set $|\zeta| = (1+k)/(2\eta)$ keeping the Lipschitz bound $|A(z,\zeta) - A(z,\xi)| \leq k'|\zeta - \xi|$, $k' < 1$.

Now, simply because $|h_{\bar{z}}| = \frac{1}{\eta(z)|h_z|}$ and so $|h_z|$ lies beyond where we have made any modification, h is a solution to the following Beltrami equation with Lipschitz coefficients:

$$h_{\bar{z}} = A(z,h_z)\frac{h_z}{|h_z|}$$

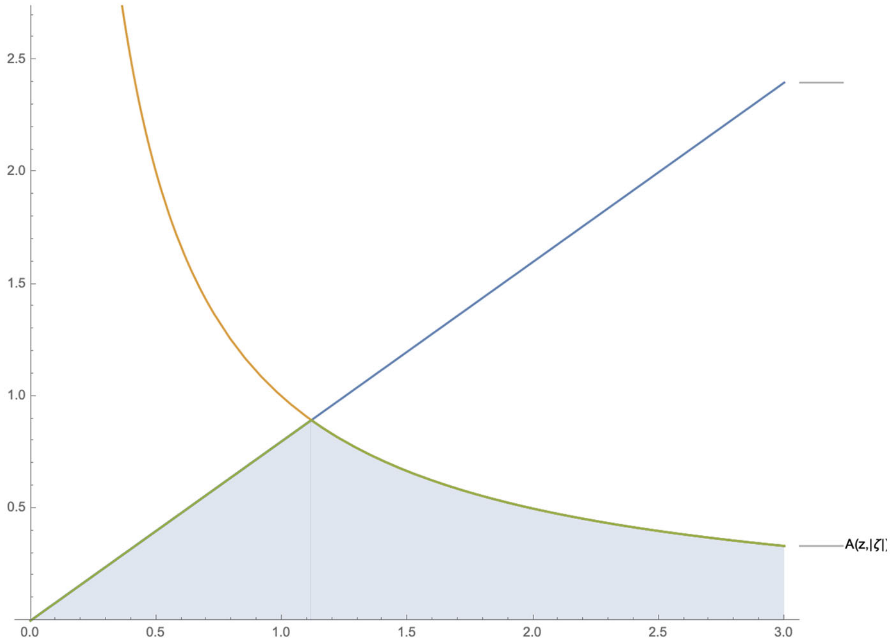

Fig. 1 The graphs $y = \frac{1+k}{2}x$ and $y = \frac{1}{\eta x}$. $A(z,\zeta)$ is obtained by smoothing the minimum of these two functions defined for $x \geq 0$

The Schauder estimates [1], or other more general approaches [6] show that $h \in W^{2,p}(G)$, $p < \infty$, or even $C^\infty(G)$ if η is. □

We now proceed under the additional assumption that $h \in W^{2,1}_{loc}(\Omega \backslash Z_\Phi)$ (and note that in our special case $Z_\Phi = \emptyset$) (Fig. 1).

Differentiating (8) yields

$$0 = h_{z\bar z}\,\mu\overline{h_z} + h_z\,\overline{h_{z\bar z}} + \frac{\eta_{\bar z}}{\eta^2} = h_{z\bar z}\,\mu\overline{h_z} + \mu h_z\,\overline{h_{z\bar z}} + (1-\mu)h_z\,\overline{h_{z\bar z}} + \frac{\eta_{\bar z}}{\eta^2}$$

which simplifies to

$$0 = 2\mu\Re e(h_{z\bar z}\overline{h_z}) + (1-\mu)h_{z\bar z}\overline{h_z} + \frac{\eta_z}{\eta^2}$$

and the conjugate equation

$$0 = 2\mu\Re e(h_{z\bar z}\overline{h_z}) + (1-\mu)\overline{h_{z\bar z}}h_z + \frac{\eta_{\bar z}}{\eta^2}$$

Subtracting the first of these equations from the second gives

$$0 = (1-\mu)2i\Im m\big[h_{z\bar z}\overline{h_z}\big] + \frac{\eta_z - \eta_{\bar z}}{\eta^2}$$

and hence

$$\Im m\big[h_{z\bar{z}}\,\overline{h_z}\,\eta^2\big] = \frac{\eta_y}{1-\mu} \tag{10}$$

While adding the same two equations gives

$$0 = (1+\mu)\Re e[h_{z\bar{z}}\,\overline{h_z}\,\eta^2] + \eta_x$$

This gives

$$\Im m\left[\frac{h_{z\bar{z}}}{h_{\bar{z}}}\right] = \frac{\eta_y}{(1-\mu)\eta^2}, \quad \Re e\left[\frac{h_{z\bar{z}}}{h_{\bar{z}}}\right] = -\frac{\eta_x}{(1+\mu)\eta^2} \tag{11}$$

This calculation has established the following lemma.

Lemma 3

$$h_{z\bar{z}} = h_{\bar{z}}\left(-\frac{\eta_x}{(1+\mu)\eta^2} + i\,\frac{\eta_y}{(1-\mu)\eta^2}\right)$$

As a consequence of this formula we also have the following theorem.

Theorem 2 *Suppose we have $|\eta_y| \geq \epsilon$ and $|\eta_x| \leq M < \infty$ in Ω. Then a solution $h \in W^{2,1}_{loc}(\Omega)$ to Eq. (8) is an open and discrete mapping.*

Proof As $|h_z|$ and $|h_{\bar{z}}|$ are locally bounded above and below (by a positive constant) the announced smoothness is assured by (3). Further, we immediately see

$$\mathbb{K}(z,h) \approx \frac{1}{1-\mu} \in L^1_{loc}(G).$$

and so the Iwaniec–Sverak version of the Stoilow factorisation theorem [11] applies. □

We record the following corollary related to the boundary value problem. The proof is simply to note by topological degree theory, or a simple application of the Stoilow theorem which reduces the question to the case of mappings holomorphic in G and continuous over the boundary, that if $h : \overline{G} \to \mathbb{C}$ is continuous, discrete and open in G, and degree ± 1 on the boundary $h : \partial G \to h(\partial G)$, then it is a homeomorphism in G.

Corollary 1 *Suppose that $\eta \geq 1$ is smooth and $\eta_y \neq 0$ in $\mathbb{D}$. If $h : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a $W^{2,1}_{loc}(\mathbb{D})$ solution to (8) and $h|\partial\mathbb{D} \to \partial\mathbb{D}$ has degree 1, then h is a homeomorphism.*

More generally

Corollary 2 *Suppose that $\eta \geq 1$ is smooth in $\mathbb{D}$. If $h : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a continuous $W^{2,1}_{loc}(\mathbb{D})$ solution to (8), if $h|\partial\mathbb{D} \to \partial\mathbb{D}$ has degree 1, and if $h|\{\eta_y = 0\}$ is injective, then h is a homeomorphism.*

Proof The set $\{\eta_y = 0\}$ is closed and $h|\{\eta_y = 0\}$ is a homeomorphism. On $\mathbb{D}\backslash\{\eta_y = 0\}$ h is open and discrete. The continuity of h now implies the result. □

As a specific consequence we note the following.

Corollary 3 *Suppose that $\eta(w) = (1-|w|^2)^{-2}$ is the hyperbolic area measure in $\mathbb{D}$. If $h : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a continuous $W^{2,1}_{loc}(\mathbb{D})$ solution to (8), if $h|\partial\mathbb{D} \to \partial\mathbb{D}$ has degree 1, and if $h|\{y = 0\}$ is injective, then h is a homeomorphism.*

We need only observe that $\eta_y = 2y(1-|w|^2)^{-4}$. Actually the same will be true for any radially symmetric metric $\eta(r)$ whose derivative vanishes only at $r = 0$, since $\eta(w) = \eta(\sqrt{x^2+y^2})$ so $\eta_y = y\eta'(\sqrt{x^2+y^2})/\sqrt{x^2+y^2}$.

4.2 The General Case

We have seen at (6) how the general case we are considering,

$$h_w\overline{h_{\overline{w}}}\,\hat{\eta} = \Phi,$$

can be reduced to the special case locally in $\Omega\backslash Z_\Phi$. Locally we have $\phi'(w)^2\,\Phi(\phi) \equiv 1$ and

$$\eta(z) = \hat{\eta}(\phi(z)). \tag{12}$$

Then away from Z_Φ,

$$\begin{aligned} 2i\eta_y = \eta_w - \eta_{\overline{w}} &= 2\Im m\big(\hat{\eta}_z(\phi(z))\phi'(z)\big) \\ &= 2\Im m\Big(\frac{\hat{\eta}_z(\phi(z))}{\sqrt{\Phi(\phi(z))}}\Big) = 2\Im m\Big(\frac{\hat{\eta}_z(w)}{\sqrt{\Phi(w)}}\Big) \end{aligned}$$

Thus the problematic set is the set where $\hat{\eta}_z(w)\overline{\sqrt{\Phi(w)}}$ is real. We can express this condition more easily as $\hat{\eta}_z(w)^2\overline{\Phi(w)} \geq 0$.

Theorem 3 *Let $\Omega \subset \mathbb{C}$ be a domain, Φ holomorphic on Ω and $Z_\Phi = \{\Phi = 0\}$. Suppose that $h : \Omega \to \mathbb{C}$ is a $W^{2,1}_{loc}(\Omega \setminus Z_\Phi)$ solution to the equation*

$$h_w\overline{h_{\overline{w}}}\,\eta(w) = \Phi \quad \textit{almost every } z \in \Omega. \tag{13}$$

If for every $G \Subset \Omega \setminus Z_\Phi$

$$\operatorname{ess\,inf}\{\Im m(\eta_w^2\bar{\Phi}) : w \in G\} > 0,$$

then h is open and discrete.

4.3 Harmonic Mappings

It is worthwhile now looking at what these conditions require in the harmonic case where $\eta(w)=\alpha(h(w))$.

$$\begin{aligned}\eta_w^2\bar\Phi &= (\alpha_z(h)h_w+\alpha_{\bar z}(h)\overline{h_{\overline{w}}})^2\bar\Phi\\ &= \alpha_z(h)^2h_w^2\bar\Phi+\overline{\alpha_z(h)^2h_{\overline{w}}^2}\bar\Phi+2\frac{|\alpha_z(h)|^2}{\alpha(h)}|\Phi|^2\\ \Im m(\eta_w^2\bar\Phi) &= \Im m\Big((\alpha_z(h)^2h_w^2+\overline{\alpha_z(h)^2h_{\overline{w}}^2})\bar\Phi\Big)\end{aligned}$$

Now from (13) we see $\bar\mu=|\mu|\frac{\Phi}{|\Phi|}$ and so

$$\Im m(\eta_w^2\bar\Phi)=\Im m\Big(\alpha_z(h)^2h_w^2\bar\Phi+|\mu|^2\Phi\overline{\alpha_z(h)^2h_w^2}\Big)$$

If we suppose that h is continuously differentiable, then μ is continuous. In a neighbourhood of a point where $|\mu|<1$ we obviously have open and discreteness. The only interest is in the case $|\mu|=1$. However,

$$\Im m(\eta_w^2\bar\Phi)=\Im m\Big(\alpha_z(h)^2h_w^2\bar\Phi+\Phi\overline{\alpha_z(h)^2h_w^2}\Big)=0$$

5 Uniqueness

Here we follow the ideas of [18].

Theorem 4 *Let Φ be holomorphic and $\eta\geq 1$ and continuous on $\mathbb{D}$. Let $g,h:\overline{\mathbb{D}}\to\overline{\mathbb{D}}$ be continuous mappings $g,h\in W^{1,1}_{loc}(\mathbb{D})$ with $g(0)=h(0)$ and $g(1)=h(1)$. Suppose that $g,h|\mathbb{S}:\mathbb{S}\to\mathbb{S}$ are monotone mappings of degree 1, and that g,h both solve the equation*

$$f_w\,\overline{f_{\overline{w}}}\,\eta(w)=\Phi,\quad \textit{almost everywhere in}\,\mathbb{D} \tag{14}$$

If

$$\frac{1}{|\eta g_w h_w|-|\Phi|}\in L^1_{loc}(\mathbb{D}\setminus Z_\Phi),$$

then $g\equiv h$.

Notice that $|\eta(w)h_wg_w|\geq|\Phi|$ almost everywhere, with equality holding if and only if $J(w,h)=J(w,g)=0$.

Proof The hypotheses quickly imply that g,h are mappings of finite distortion in $\mathbb{D}$. Let $F=g-h:\overline{\mathbb{D}}\to\mathbb{C}$. Then $F\in W^{1,2}_{loc}(\mathbb{D})$. Also, as a difference of monotone maps, the total variation of the argument of F on $\mathbb{S}$ is at most 2π, see [18]. Away from Z_Φ we compute that

$$\begin{aligned}\overline{F_{\overline{z}}} &= \overline{g_{\overline{z}}} - \overline{h_{\overline{z}}} = \frac{\Phi}{\eta}\Big(\frac{1}{g_z} - \frac{1}{h_z}\Big) \\ &= \frac{\Phi}{\eta g_z h_z} F_z = \nu F_z.\end{aligned}$$

Our hypothesis is that locally

$$K(z,F) = \frac{1+|\nu|}{1+|\nu|} \approx \frac{1}{1-\left|\frac{\Phi}{\eta g_z h_z}\right|} \approx \frac{1}{|\eta g_z h_z| - |\Phi|}$$

is locally integrable away from Z_Φ. Thus F is a discrete open map [11]. We apply the Stoilow factorisation theorem to write $F = \psi \circ f$, where $f : \mathbb{D} \to \mathbb{D}$ is a homeomorphism and ψ is holomorphic. As F is continuous in $\overline{\mathbb{D}}$ the homeomorphism f extends continuously. As $F(0) = 0$ and $F(1) = 1$ the open mapping ψ has degree at least two and total variation at least 4π. This contradiction establishes the result. □

Corollary 4 *With the hypotheses of Theorem 4, if for each $z \in \mathbb{D}$ either g or h is locally quasiregular at z, then $g \equiv h$.*

Corollary 5 *A locally quasiconformal solution to (14) which is continuous in $\overline{\mathbb{D}}$ is unique.*

One should compare this with Li and Tam's uniqueness theorem for quasiconformal mappings which are harmonic in the hyperbolic metric [14].

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Conflict of interest The authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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