The Writing of Explanations and Justifications in Mathematics

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ABSTRACT

This study reports on the writing of explanations and justifications in mathematics. A variety of approaches including a document analysis, teacher survey, students' responses to problem solving tasks, and student interviews were used to examine the complexities and interpretations of writing explanations and justifications in mathematics. The study involved six teachers and 36 Year 11 students from a provincial co-educational secondary school; 14 of the students were interviewed.

An analysis of the Year 11 national mathematics examination, School Certificate, revealed a significant increase in emphasis on the writing of explanations; from 2.7% of the total marks in 1992, to 16% of the total marks in 1997. It was not until 1997 that students were specifically asked to write justifications. In this study students experienced some difficulties writing explanations and had concerns about whether their explanations were satisfactory; a variety of modes of representation were used by students. Most students surveyed were unable to write justifications; they lacked knowledge and confidence in justifying their solutions. The teachers believed that the writing of explanations and justifications was an important process but expressed a number of concerns. These concerns were the class time needed, and the lack of resources and professional development. Both students and teachers were concerned about not knowing what makes a quality response.

The writing of explanations and justification should be a valued and regular part of the mathematics programme so that students are able to develop and evaluate mathematical arguments and proofs and effectively communicate their findings to others. The study suggests that students and teachers need to work together in negotiating an understanding of what is meant by an explanation, and a justification, and what makes a quality response.

PREFACE AND ACKNOWLEDGEMENTS

This thesis is essentially about the interface of mathematics, language, and the communication process. This study began to form in my mind when I became aware of the impact of a new mathematics syllabus which specifies mathematical processes as a separate and identifiable strand. Teachers and students were expressing interest in, and concern about the writing process in mathematics and so I decided to take the opportunity, in the final stages of my degree, to explore a topic that appeared important and timely. It also gave me an opportunity to mesh two personal interest areas, language and mathematics, and more specifically to try and answer key questions about the writing of explanations and justifications in mathematics. I felt confident that a qualitative methodology was appropriate in order to answer these questions and so my journey began...

I would like to acknowledge and thank the many people who made this study possible. Firstly I would like to acknowledge Dr. Glenda Anthony, my main supervisor, who provided continuing interest and invaluable professional support. My thanks are extended to Brian Finch, my second supervisor who gave such positive and encouraging feedback throughout the study.

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Lastly, I must acknowledge my family for their patience and tolerance in having a mother and wife whose attention at times seemed more focused on this study than on family interests and commitments. I have returned!

Good company in a journey makes
the way to seem the shorter.

Izaak Walton, *The Compleat Angler*

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CHAPTER 1: INTRODUCTION

1.1 BACKGROUND TO THE STUDY

In recent years there has been a growing interest in the importance of language in mathematics learning. Research of language in mathematics has been a major area of research interest in Australia throughout the period 1975-1996 (Ellerton & Clements, 1996). However, limited research has been conducted in this field in New Zealand.

In 1992 a new mathematics curriculum document, 'Mathematics in the New Zealand Curriculum', was written by the Ministry of Education and the implementation process into New Zealand schools began in 1993. The curriculum document embraces, implicitly, both constructivist and behaviourist notions (Neyland, 1995). 'Mathematics in the New Zealand Curriculum' is organized into six strands: five discrete content strands and a mathematical processes strand. The 'Mathematical Processes' strand focuses specifically on problem solving, developing logic and reasoning, and communicating mathematical ideas. These mathematical processes skills *are learned and assessed within the context of the more specific knowledge and skills of number, measurement, geometry, algebra, and statistics* (Ministry of Education, 1992, p. 31). It would be expected that the acknowledgement and focus on mathematical processes in the curriculum would contribute to significant changes in the learning and teaching of mathematics. Interest in, and concern about, the way in which this focus has affected the development of students' communications and understandings in mathematics determined the topic for this study.

Specifically in the area of communication, 'Mathematics in the New Zealand Curriculum' states that:

> *The mathematics curriculum intended by this statement will provide opportunities for students to:*

- *develop the skills of presentation and critical appraisal of a mathematical argument or calculation, use mathematics to explore and conjecture, and learn from mistakes as well as successes;*
- *develop the characteristics of logical and systematic thinking, and apply these in mathematical and other contexts, including other subjects of the curriculum;*
- *develop the skills and confidence to use their own language, and the language of mathematics, to express mathematical ideas.*

(Ministry of Education, 1992, p. 9)

There are many facets of language factors in mathematics education and the term 'communication' can refer to a variety of aspects of language. This study will focus particularly on the written language genres of explanations and justifications. These writing genres are an important part of the mathematical processes and give teachers a perspective on students' mathematical perceptions (Mousley & Marks, 1991).

The importance of the writing of explanations and justifications is recognized by both language and mathematics educators. This ability is also acknowledged by two influential national authorities, namely the Ministry of Education and the New Zealand Qualifications Authority (NZQA). The act of writing in mathematics involves processes that are fundamental to learning, and the process of writing mirrors and supports the process of learning (Clarke, Waywood, & Stephens, 1993; Halliday, 1978; Pimm, 1987). Mathematics instruction programmes should include the writing of explanations and justifications as part of the problem solving process (Niemi, 1996). The writing of explanations and justifications can then lead students to the next fundamentally important step in mathematics; developing and evaluating mathematical arguments and proof. *The act of communicating ideas within the culture of mathematics creates both the need for and the value of mathematical proofs* (Silver, Kilpatrick, & Schlesinger, 1990, p. 23).

The selection of this topic has been influenced by a number of key factors: the role which language plays in the formation of mathematical concepts, the influence of constructivism, student and teacher roles in the communication process, and the implementation of the relatively new syllabus 'Mathematics in the New Zealand

Curriculum' (Ministry of Education, 1992). It is timely for a study such as this to be undertaken. Recent curriculum reforms are being reviewed and any changes or modifications to curriculum documents should be based on current research in the domain.

1.2 RESEARCH OBJECTIVES

The primary aim of this study is to find out how students respond, in writing, to questions requiring explanations and justifications. The study also seeks to ascertain student and teacher views about the writing of explanations and justifications in mathematics. A related objective is to determine how well the examining authority, New Zealand Qualifications Authority, has addressed the requirement for students to be able to write explanations and justifications in the School Certificate mathematics examination. Specifically, the following research questions have been addressed:

1. What emphasis does the School Certificate mathematics examination give to the writing of explanations and justifications?
2. What are teachers' views about the writing of explanations and justifications?
3. How well do students answer questions requiring explanations and justifications?
4. How do students feel about the writing of explanations and justifications?

1.3 DEFINITION OF TERMS

The two main terms used in this study are 'explanation' and 'justification'.

- An **explanation** can be defined as making clear or telling why a state of affairs or an occurrence exists or happens;
- A **justification** provides grounds, evidence, or reasons to convince others (or to persuade ourselves) that a claim or assertion is true (Thomas, 1973).

In this study, the terms are sometimes used separately as prompts alerting students to write either an explanation or a justification and at other times they have been used conjointly.

1.4 OVERVIEW

Chapter 2 reviews the literature in the field and provides the necessary background and framework from which this project can be viewed. It provides the context for this study by summarizing relevant and essential findings on the issues of mathematics and language, problem solving, and assessment. In Chapter 3, the methodology for the study is presented, data collection instruments discussed, and a project schedule outlined.

Chapter 4 provides an analysis of School Certificate papers in mathematics for the years 1992-1997. This analysis focuses specifically on examination questions which required a candidate to write explanations and justifications.

Student responses to a set of problem solving tasks are summarized in Chapter 5. The following chapter reports on student views about the topic of study and the final results chapter, Chapter 7, presents and discusses teacher views about the writing of explanations and justifications in mathematics.

In Chapter 8, the collective results are discussed and conclusions drawn. Implications for teachers and assessment practice are presented as well as some suggestions for further research.

CHAPTER 2: LITERATURE REVIEW

2.1 INTRODUCTION

The role and importance of the writing of explanations and justifications in mathematics cannot be ascertained without first establishing the links between two inherently important learning domains: mathematics and language. Mathematicians communicate using language. They use language to make and clarify meaning, and to share their understandings. This study focuses on communication in the form of writing, more specifically the writing of explanations and justifications. The writing of explanations and justifications is described from a genre-based perspective to language. Specific research citing the use of explanatory and expository writing in mathematics will be described. This leads the researcher to consider the relationship of the writing of explanations and justifications to the higher-order writing of proof in mathematics.

If students are required to write explanations and justifications then it is in response to problems posed. The nature of problem solving is subsequently examined as it provides the context and purpose for the writing of explanations and justifications. There are differing kinds of problems which need to be defined as they are interpreted and implemented in mathematics classrooms in response to current curriculum reforms.

Students' ability to write explanations and justifications in response to problems posed is assessed in a variety of ways. No learning, teaching, or curriculum change occurs without assessment being made and so finally the place of assessment in relation to this study is discussed.

2.2 LANGUAGE AND MATHEMATICS

Part of learning mathematics is learning to speak like a mathematician.
(Pimm, 1987, p. 76)

There are many differing relationships between language and mathematics. Most theories view language as being very strongly connected to thought processes. Piaget (1959) found it necessary to study linguistics in order to understand cognitive development. He viewed thought as existing prior to language even though language is the vehicle for thought. Vygotsky (1986), on the other hand, advocated that formal thought is a product of language development. Zepp (1989) suggests that instead of being concerned about whether thought precedes language or language precedes thought, modern linguistics has *moved in the direction of viewing not only all language, but all meaning as social* (p. 34). The learning of the child is viewed as simply the acceptance of society's conventions. Meaning (according to the linguists) can only exist in a social context; by refining knowledge of situational contexts language is manipulated more effectively.

Language and mathematics learning cannot be separated. Language serves as a medium through which mathematical ideas are shared. It also plays a major role in the formation of mathematical concepts and functions in organizing mental activity. *Language provides some sort of medium for creating, preserving, and communicating mathematical thinking* (Brown, 1997, p. 214). Language is fundamental to the social formation and individual construction of mathematical ideas.

Mathematics can be viewed as a language, a tool for making sense of, describing and operating on aspects of our environment (Mousley & Marks, 1991). Like all languages it is both oral and written, formal and informal. It not only describes concepts but helps shape them in the mind of the user and like all languages it has its own characteristics (Usiskin, 1996). There is considerable debate about whether mathematics is a language and although Usiskin writes persuasively that mathematics is a language, it is more commonly described as making use of a special language, having some functions as a language but not being defined as a language in its own right. The social context of language means that rather than viewing mathematics as a separate language, mathematics can be defined as a register (Pimm, 1987).

The Mathematics Register

Register is a technical linguistic term described by Halliday (1975) *as a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings* (as cited in Pimm, 1987, p. 76). It is not just the use of technical terms and symbols but the phrases and characteristic modes of arguing that constitute a register (Pimm, 1987). This is evident in the mathematics register by the special way in which it uses particular vocabulary, symbols, and diagrams. Other special features are its abstract nature and emphasis on deductive reasoning. Real-life concepts and mathematical concepts may be very different and students may learn them in different ways. It is important therefore that students clearly understand in which register (mathematical or otherwise) the term is being used because in mathematics every word and symbol is important to the meaning (Chapman & Lee, 1990).

Many words have been appropriated from natural language and symbols appropriated from written language. Different levels of technicality or abstraction are presented leading to variations in meaning. For example the use of the word 'takeaway' assumes a totally different meaning from that of a real-life context when it is read in a mathematics register. Consider also the impact of brackets which in a mathematical statement clearly defines the meaning and value of a mathematical statement or equation. In prose, the bracketed phrase adds information or support for a statement. The mathematics register is not static; it responds to the needs of a changing society by introducing new words and terms as mathematical thinking is extended.

What is the Role of the Teacher?

The teacher's role in developing understanding will involve the negotiation of meanings for words and symbols. The reconciliation of new meanings with the student's existing understandings is part of this negotiation process. The teacher's role is to take account of the interpretation that is brought to the problem by the student as well as the mathematical significance of the language involved (Anghileri, 1995). Teachers need to devise strategies for classroom interaction that are both responsive to student's existing understandings and pro-active in helping students negotiate new meanings All too frequently classroom discourse is a one-way communication of ideas. Marks and Mousley (1990) found that teachers were generally involved in the expression of mathematical ideas, while students assumed the role of processors of information and ideas. According to these researchers it is through activities and discussion that teachers can help students develop an understanding of the role of language in specific mathematical contexts. Teachers can empower students in mathematics by helping them to make connections between the language used to teach mathematics and their construction of mathematical knowledge (Miller, 1993). They have a responsibility to alert students to the patterns of language construction in mathematics and to help students develop skills in making mathematical arguments and explanations. It cannot be assumed that students will independently develop the necessary language skills needed in mathematics.

Each mode of language has two aspects: receptive (processing someone else's communication) and expressive (communicating your own thoughts). According to Ellerton and Clements (1991) mathematics classrooms have traditionally emphasized receptive language. Expressive language has been limited mainly to copying forms and procedures demonstrated by the teacher. If students are to make connections between language and mathematical empowerment then strategies should be used that give students an opportunity to construct, in both receptive and expressive modes, the formal language of mathematics (Miller, 1993).

2.3 COMMUNICATION

> *Communication in and about mathematics serves many functions. It helps to (1) enhance understanding, (2) establish some shared understandings, (3) empower students as learners, (4) promote a comfortable learning environment and (5) assist the teacher in gaining insight into the students' thinking so as to guide the direction of instruction.*
>
> (Mumme & Shepherd, 1990, p. 18)

Communication is an integral feature of current curriculum reforms in mathematics (Australian Educational Council, 1991; Department for Education, 1995; Ministry of Education, 1992; National Council of Teachers of Mathematics, 1989). Communication (defined as the act of imparting information) has both cognitive and social significance in the classroom (Hiebert, 1992). It is through classroom discourse that students become engaged in 'doing' mathematics. *Doing mathematics means agreeing on assumptions, making assertions about relationships, and checking if the assertions are reasonable* (Hiebert, 1992, p. 444). Classroom discussions mean that students have opportunities to express their ideas and defend their findings. This requires the students to elaborate, clarify, refine, and reorganize their own thinking. Discussions give opportunities for social interaction so that shared meanings can be negotiated and developed. It is only through the ongoing process of negotiation that students construct meanings that are conversant with the mathematical community (Hiebert, 1992). Understanding is enhanced by communication and conversely, communication is enhanced by understanding (Greenes, Schulman, & Spungin, 1992). The communication and clarification of ideas through discourse can be of two forms, either speech or writing. This study focuses on discourse through writing.

Writing in Mathematics

Written mathematical records are an important aspect of the mathematical process (Pimm, 1987). They are visible, permanent, and accessible. Mathematical ideas become tangible when words and symbols are used to record them (Silver, Kilpatrick, & Schlesinger, 1990). Mathematics has traditionally been considered a subject in which the only writing is of the kind that uses symbols. The use of words is more commonly associated with reading, not mathematics where the emphasis is placed upon the numerical answer. Writing at secondary school has generally involved the reiteration of learned symbolic processes and it is this approach which has traditionally dominated mathematics learning (Baroody & Ginsburg, 1990). In recent years the representation and recording of mathematics has diversified from the traditional writing of digits, algorithms, and proofs to a greater variety of written representations.

It is recognized that *the child's language, written and spoken, is a window into the child's mind* (Robinson, 1990, p. 90). Written language externalizes thinking even more than speech by demanding a more accurate expression of ideas (Pimm, 1987). Students' writings can be informative; providing access to how they think, illustrating misconceptions, patterns of thoughts and beliefs, and indicating to the teacher students' variant conception of some notion (Pimm, 1987; Burns, 1995). Smith (1994) explains that writing is not simply speech written down but that writing *separates our ideas from ourselves* (p. 16). Compared to speech this makes it easier for us to examine, explore, and develop our ideas. Our ideas can be examined more objectively as writing is a tangible construction.

Pirie (1989) distinguishes between **recording** and **writing up.** 'Recording' is viewed as a personal process, not necessarily neat, with the purpose of trying out ideas, sketching diagrams, and recording hunches. 'Writing up' on the other hand is primarily for communication and so needs to be presented so that the learner's thoughts are clearly conveyed to the reader. 'Writing up' is different from penmanship. It is not so much how much or how neatly students write mathematics but rather what they write that is valued (Krussell, 1998). It is important however that students write with understanding. Hersh (1993) stresses the importance of mathematicians being able to write in natural language (prose). He sees it as advantageous so that mathematics is made more accessible and comprehensible for all. If students can write clearly about mathematical concepts then they probably understand them (Johnson, 1983). The challenge according to

Ellerton and Clarkson (1996) *is to create learning environments in which writing mathematics becomes a natural, regular and creative form of expressive communication* (p. 1013).

There is considerable interest in students' written work in mathematics. Typically the writing has involved mathematical symbols but the drive from the current mathematics curriculum is to promote writing as a technique that can be used to help develop and consolidate new concepts for students. Increased attention has been paid to the 'writing to learn' movement which embraces the idea of using writing in mathematics classes to enhance mathematics learning (Ellerton & Clarkson, 1996). One of the key forms of 'writing mathematics' has been journal writing. The aim of journal writing is to create a new form of dialogue between the teacher and the student (Borasi & Rose, 1989). Comprehensive studies (Borasi & Rose, 1989; Clarke, Waywood, & Stephens, 1993; Davison & Pearce, 1990; Waywood, 1992) concerning the effectiveness and benefits of using journal writing in mathematics classes concluded that students benefited cognitively and affectively from regular journal writing.

Writing and speaking involve different levels or forms of cognitive activity (MacGregor, 1990). The focused articulation of one's thoughts is a higher order ability and one of the goals of mathematics education. Students may be used to operating at this higher cognitive level in verbal interactions by making judgements, justifying, and evaluating. However, these skills are not commonly practised in written form, especially in mathematics (Ernest, 1989). Writing is a powerful process which can help students to clarify their thinking, reflect on, analyze, and synthesize the material being studied in a thoughtful and precise way (Davison & Pearce, 1988). A new idea makes sense if students are able to link with a network of mental representations. Writing encourages students to forge new links and think reflectively about links already made (King, 1982; Waters & Montgomery, 1993; Masingila & Prus- Wisniowska, 1996).

Ehrich (1994) describes four distinct cognitive processes that can be stimulated through writing: cognitive dissonance, affirmation, exploration, and the 'aha' experience. Cognitive dissonance results when students' knowledge is challenged by new information and if this conflict occurs during the writing process the likelihood that learning will occur is increased. Writing, according to Ehrich's findings, allows students to affirm and strengthen their mathematical understandings, even after they have arrived at a solution. The third process is that of exploring new problems and drawing on existing mathematics to create new levels of understanding. This written exploration sometimes produces what is described as the 'aha' experience - spontaneous discovery by the learner.

The **audience** and **purpose** of writing in mathematics are two key components of any attempts at communication (Krussell, 1998). At the secondary level students are required to write regularly and are judged by peers, teachers, and outside authorities on the purported quality of what they write. Students may be unaware of any purpose as to why they should make such a written record or for whom the recording is intended. Going from the oral situation where communication is to a live audience, to decontextualized written form where the audience is an abstraction, requires considerable effort and causes difficulty for some students (Zepp, 1989). In mathematics, the student's formulations are usually meant for the teacher who knows the solution to the problem being solved (Laborde, 1990). The usual aim in the classroom or examination is to convince the teacher or marker that the student knows how to solve the problem. Miller's (1990) study which used writing as a technique to improve communication between students and teachers, found that the best results came from students when they were directed to address their comments to someone such as a friend or a teacher.

A Genre Approach to Writing

Often students and teachers are unaware of the range of possible types and uses of writing that can be experienced in the mathematics classroom (Marks & Mousley, 1990). Writing structure and formats vary according to the purpose and function of the text. These differing structures or genres can be classified as either narrative or factual. The factual strand includes:

> *Procedure* *'how something is done'*
> *Description* *'what some particular thing is like'*
> *Report* *'what an entire class of things is like'*
> *Explanation* *'a reason why a judgement has been made'*
> *Exposition* *'arguments why a thesis has been proposed'.*
> (Martin, 1985, p. 15)

These conventionalized forms of texts or genres can be used in mathematics along with other genres which may exist in mathematics. However, Mousley and Marks (1991) found that a limited range of genres was actually taught or practised in mathematics classrooms. In the mathematics classrooms they observed, recounts (categorized under the narrative strand), a simple reconstruction of events, was the genre most commonly used. In both primary and secondary classes numerical recounts are ubiquitous and this is viewed as a matter of concern as mathematics makes little use of this genre in the real world (Marks & Mousley, 1990). Instead, Marks and Mousley argue that it is important that mathematics teachers become aware of and make use of genres from the factual strand as they are essential to the work of mathematicians.

The **explanation** genre is seen to be predominantly the domain of the teacher. The teacher provides extensive explanations, constructs proofs, and draws logical conclusions. Only occasionally are opportunities given to students to perform similar linguistic demands (Marks & Mousley, 1990).

> *Students should always be encouraged to think things through carefully, to understand, and to be able to explain. As students' arguments grow more sophisticated, the explanations should increasingly be conveyed in the formal language of mathematics.*
> (National Council of Teachers of Mathematics, 1998, p. 84)

Why should teachers be interested in developing students' ability to write explanations? Writing is viewed as a reflective process, one which helps students clarify their thinking as they try to explain processes and demonstrate understandings in their own words. It is from students' mathematical explanations, both oral and written, that teachers endeavour to gain a perception of students' mathematical understandings; to examine students' processes as well as their products. Having students explain in their own words can clearly indicate whether they understand the topic (Clarke, Waywood, & Stephens, 1993). As one teacher wrote in a student's journal: *"Unless you can explain it to me, you don't really understand"* (Clarke et al., 1993, p. 249).

Miller's (1990) action research investigated the use of writing as a means to improve the channels of communication between students and teachers at a secondary high school. A variety of prompts were selected including some defined as reflective prompts. These were written with the aim of soliciting an analytical response or promoting clarification and understanding. For example, the students were instructed to list every thought (or step) that had occurred in working towards the final answer of a problem. Those who attempted an explanation generally started with an example and proceeded with narrative which explained what they had done. The teachers found that students did not necessarily understand what they said they understood and it was only through examining the written explanations that the teacher discovered misconceptions. It was found however that students produced more and better writing if they were directed to address their explanations to a friend who had been absent from class. Writing was viewed by both teachers and students as an effective tool for communicating about the processes of mathematics.

In a study, conducted by Silver, Shapiro and Deutsch (1993) involving 200 middle school pupils, the researchers found that many students were neither accustomed nor comfortable with explaining their thinking and reasoning. Objections were raised by students about having to explain their answers. These came in two forms:

> *(a) objections based on never having been taught how to explain their work and that it was a difficult thing to do, and*
>
> *(b) objections based on the apparent belief that correct computations always produce correct answers, thereby obviating the need for further explanation. ... The requirements of a written response in the form of an explanation or justification clearly made the task difficult for students* (p. 130).

These researchers believed that some students may have been more capable of explaining their thinking and reasoning orally than in writing. Silver et al. (1993) recommended that written explanations become a more prevalent feature of school mathematics instruction. This is certainly consistent with the increased emphasis on communication advocated by the curriculum reforms in mathematics.

The factual genre seen as being most useful to adults in their daily lives and to mathematicians in their work are those from the **expository** strand. The exposition genre is concerned with analysis, interpretation, and evaluation (Derewianka, 1990). Justification belongs to this genre group where explanation is combined with judgements, and logical reasoning as the main focus. Mousley and Marks (1991) found in the classrooms they observed that students were asked to make mathematical judgements but were rarely asked for an oral or written justification for their answers. They believe that much might be gained by asking students for expository answers to mathematical tasks.

> *Students who cannot effectively explain the meaning of, and justify the use of, mathematical symbols, concepts, and operations are not yet full-fledged members of the community of discourse.* (Niemi, 1996, p. 361)

Expository writing has proven to be an effective and practical tool for the teaching of problem solving. Bell and Bell (1985) found in a study using two ninth-grade general mathematics classes that the process of expository writing can actually reinforce the mathematical concepts taught in the classroom. An experimental group was taught problem solving using a method which combined traditional mathematics techniques with expository writing and a control group was taught using only traditional methods. Both classes were given the same assessment tasks. The experimental group was assigned additional writing opportunities. These students were required to not only formally record their problem solving steps but to give justification or some degree of support for their judgement. The measurable results of the study positively supported the claims made for the value of expository writing in the mathematics class. The writing component was perceived as an integral element of the teaching process and not merely as an enrichment exercise.

More recently, Shield and Galbraith (1998) reported on a study concerning expository writing with Grade 8 (12 to 13 years) mathematics students. The data for the study came from two types of expository writing. The first was a letter to a friend using the prompt 'explain all about', and the second type required students to write about how they would explain a mathematical idea to someone who expressed a difficulty. Two hundred and ninety samples were used in order to develop the coding system and formulate the general model of student writing (Figure 2.1).

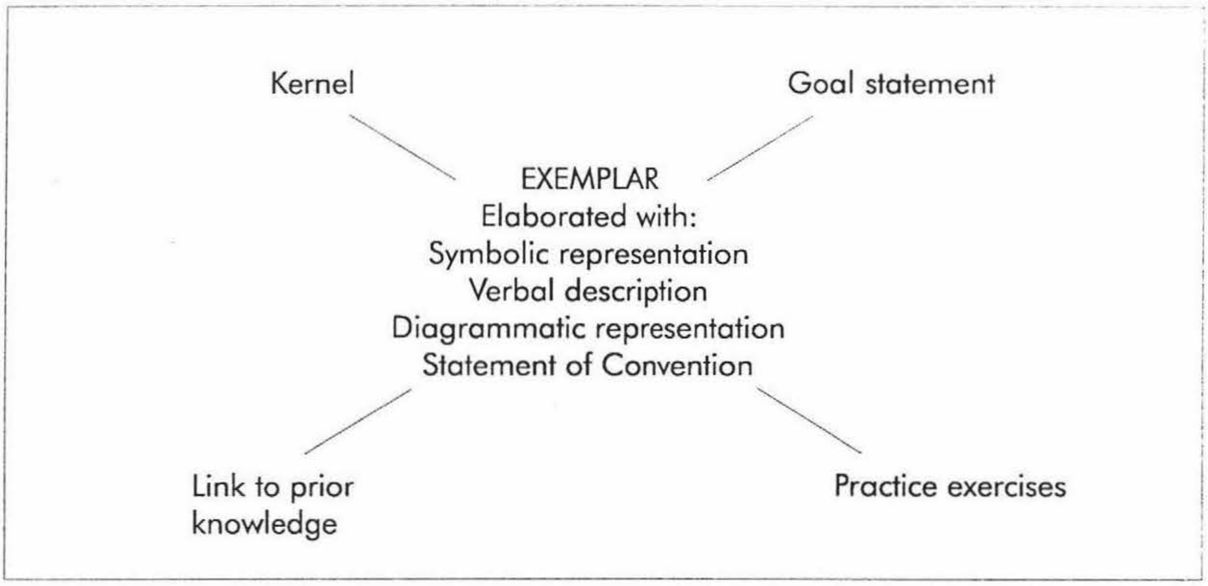

Figure 2.1 Elaborations in student expository writing (Shield & Galbraith, 1998, p. 42)

The exemplar was the main feature and was described as the symbolic worked example. The kernel was in the form of a definition, rule, or procedure; the goal statement identified the concept or procedure that is the subject; and the link a description of a prior mathematical skill needed for the new procedure. The focus of explanations analysed in this study was on recording a procedure as an algorithm, few other characteristics of an effective explanation were evident. The researchers felt that it was not possible to know just how closely the writing expresses understanding but the elaborations described may be indicative of the level of understanding of the student.

Exposition allows students to become more aware of their thinking processes and alerts them to choices that they are making and their analyses (Bell & Bell, 1985). If asked orally for explanations and justifications students generally do not have the chance to reflect upon what is said and then make changes to those statements (Miller, 1991). Not all students in a class have the chance to respond to expository questions orally whereas every student can take the opportunity to respond in writing. If students are required to explain and justify their ideas in writing, then this should gradually lead them to understand the limitations of visual and empirical justifications. Proof should then be viewed as a logical necessity (Galindo, 1998).

The Nature of Proof

In the mathematical community communication is in the form of discussion, argument, justification, and proof. Justifying demands an explanation which convinces oneself and is communicated to others, whereas proof is presented in a more formal sense of logical argument based on premises. Traditional proof is more formal, rigorous, and yet explanatory. Proof is concerned not only with the formal presentation of arguments, but with the student's process of verification or justification, reaching conviction, and communicating convictions about results to others (Bell, 1976). In setting about writing a justification or proof *students are challenged to think and reason about mathematics, and to communicate the results of their thinking to others* (Silver, Kilpatrick, & Schlesinger, 1990, p. 23). This thinking should be communicated clearly and convincingly. According to Silver et al. (1990) much can be learned by providing justification and constructing a proof. Proving an assertion can lead to insights into further statements, to refinement of ideas, and modifications to improve the clarity and precision of justification.

There are varying levels and types of proof (Balacheff, 1988). The simplest form is more informal and one of direct showing, a pragmatic or empirical proof. In contrast, a conceptual or deductive proof does not involve action but rests on formulations of the properties in question and relations between them. Studies document students' preference for empirical arguments over deductive arguments when presented with mathematical problems (Balacheff, 1988; Bell, 1976; Chazan, 1993; Porteous, 1994). Bell analysed responses from a group of 14-15 year old students to a series of items requiring explanations and justifications. They were classified according to whether they were empirical or deductive responses. He found that the highest level was rarely achieved and confirmed that students will not use formal proof with appreciation of its purpose until they are aware of the public status of knowledge and the value of public verification. As a consequence classroom explanations need to have meaning for the student rather than be formal rituals.

It is well documented that students have difficulty learning to write mathematical proof (Silver et al., 1990). According to Hoyles (1997) this may be partly because of its ambiguous meaning but also because of the need to co-ordinate a range of competencies, such as identifying assumptions and organizing logical arguments. Most secondary students show a preference for empirical argument (Bell, 1976; Galbraith, 1981; Porteous, 1990). More recent studies by Healey and Hoyles (1998) conducted in Britain have also highlighted students' difficulties in engaging with formally presented, analytical arguments. They found that high-attaining Year 10 students showed a consistent pattern of poor performance in constructing proofs. Empirical verification was the most popular form of argumentation used by students when attempting to construct proofs. In problems where empirical examples were not easily generated, the majority of students were not able to engage in the process of proving.

There has been a growing change of emphasis from teaching the form of proof to encouraging the process (Hoyles, 1997; Tall, 1992). Hanna (1995) has argued for an approach based upon what she calls explanatory proof. These are proofs that are acceptable from a mathematical point of view but whose focus is in understanding rather than on syntax requirements and formal deductive methods. She argues that students will gain a greater understanding of the mathematical topic and of proof by concentrating on the communication of meaning rather than on formal derivation. Writing explanations and justifications can help students in communicating the products and lead to the development of the more formal approach to mathematical proof (Healey & Hoyles, 1998).

There are differences of opinion as to when proof should be introduced in the mathematics curriculum because of the nature of the difficulties that students experience (O'Daffer & Thornquist, 1993). There has been a clear shift (as evidenced in syllabuses of New Zealand, Britain, Australia, and United States of America) to move away from the relatively meaningless routines that traditionally characterized geometric proof. Recently developed syllabuses acknowledge that students need to progress through the early stages of reasoning empirically and explaining their conjectures. They should engage in *age-level appropriate ways - in the kind of systematic thinking, conjecturing, and marshalling of evidence that are the precursors to formal mathematical argumentation* (National Council of Teachers of Mathematics, 1998, p. 80).

However, Hoyles (1997) believes that because these national curriculum documents appear to be only paying lip-service to proof, many students pursuing a study of mathematics after 16 years have failed to grasp the essence of the subject. Since the reforms students seem to have *little sense of mathematics; they think it is about measuring, estimating, induction from individual cases, rather than rational scientific process* (p. 10). Hoyles goes on to suggest that rather than point out what students lack it would be more fruitful and constructive to find out what students can do and understand as a result of these reforms. It is important to identify what students see as the nature of proof, its purposes, and whether they see it as verifying cases or as convincing and explaining.

There are aspects of proof and the proving process that can help promote students' understandings in mathematics. The studies made by educators interested in students' efforts to write proof indicates that it is an important aspect of the writing process in mathematics. The writing of explanations and justifications can be viewed as a precursor to the more formally recognized definition of proof. Another viewpoint promulgated by Hersh (1993) is that proving is convincing and explaining. This he believes is the purpose of proof in the classroom. *Enlightened use of proofs in the mathematics classroom aims to stimulate the students' understanding, not to meet abstract standards of "rigor" or "honesty"* (p. 389). The communication of mathematical findings, whether it be explaining, justifying, or proving, is a process that is learned and assessed within the context of problem solving.

2.4 PROBLEM SOLVING

> *Good writing is a part of the problem solving process.... students need the opportunity to use and refine writing skills in a mathematical context.*
> (Johnson, 1983, p. 117)

Current reforms have emphasized a change in the way that mathematics is taught and learned. Recently developed curricula, being implemented in New Zealand and abroad, define the need and importance for all students to have a mathematics education that is rich in opportunities for developing logical thinking and reasoning, communication, and problem solving. These current reforms are based on constructivist principles.

Teachers adopting a constructivist view embrace the ideas pioneered by Piaget and assert that *knowledge is the result of a learner's activity rather than of the passive reception of information or instruction* (von Glasersfeld, 1991, p. xiv). The student's development of new knowledge is through active construction processes in which new knowledge is linked to old knowledge along with an understanding of situations in which it can be used. This conception of learning changes the role of the teacher from one who dispenses knowledge to one who facilitates the construction and verification of knowledge and generates understanding. Attention is focused *on the dynamic nature of words and how meaning alters as the learner meets new situations and new use* (Gibbs & Orton, 1994, p. 100). Learning new mathematical ideas is perceived as a dynamic process.

A constructivist view suggests that students need to participate in activities that allow them to construct knowledge. Problem solving is one such activity in which students are encouraged to explore problems and discuss their methods and solutions (Jaworski, 1994). If students are writing explanations and justifications as part of the problem solving process they will be actively constructing mathematical knowledge for themselves. Love (1988, p. 260) suggests that students need to be allowed to engage in activities such as:

> *Identifying and expressing their own problems for investigation, expressing their own ideas and developing them in problems, testing their ideas and hypotheses against relevant experience, and rationally defending their own ideas and conclusions and submitting the ideas of others to a reasoned conclusion.*

Most mathematics educators seem to agree that problem solving is a complex form of human endeavour that involves more than the simple recall of facts or the application of well-learned procedures.

> *Successful problem solving involves the process of co-ordinating previous experiences, knowledge, and intuition in an effort to determine an outcome of a situation for which a procedure for determining the outcome is not known.* (Lester & Kroll, 1990, p. 56)

Why Teach Problem Solving?

Problem solving has claimed special attention here and overseas. *Mathematical problem solving, in its broadest sense, is nearly synonymous with doing mathematics* (National Council of Teachers of Mathematics, 1989, p. 137). *Problem solving should be a well-integrated part of the curriculum that supports the development of mathematical understanding* (National Council of Teachers of Mathematics, 1998, p. 76). It has been reified as a separate substrand of the 'Mathematical Processes' in the New Zealand curriculum document. Supporting material, 'Implementing Mathematical Processes in Mathematics in the New Zealand Curriculum' published by the Ministry Of Education (1995), states that: *unless the ability to solve problems is developed, there is little point in studying mathematics* (p. 20). Hiebert et al. (1996, p. 12) argue that *rather than mastering skills and applying them, students should be engaged in resolving problems.* They suggest the use of the term 'problematic', that students should be allowed to make the subject problematic and in doing so problematize what they are doing. *Appropriately designed problem situations provide a context within which students can solidify and extend what they know* (National Council of Teachers of Mathematics, 1998, p. 76). It is the process of problem solving that most likely leads to the construction of understanding.

Numerous advantages for teaching problem solving have been articulated by researchers. Holton and Lovitt, (1998, p. 4) suggest that solving problems:

- *teaches general problem-solving ability;*
- *teaches thinking, flexibility, creativity;*
- *is mathematics discovery;*
- *is interesting and enjoyable;*
- *relates to real life;*
- *provides a greater understanding of mathematics;*
- *is a useful way to practise skills;*
- *encourages co-operative skills;*
- *is similar to the way that other subjects are taught;*
- *gives confidence to students.*

Types of Problems

There are a variety of types of problems that can be solved by students. They may be defined as closed problems which follow a well-known pattern of solution or open-ended problems. 'Open-ended' or 'goal-free' problems designed to stimulate students thinking about contexts, constraints, and meanings of task situations (Ellerton & Clarkson, 1996) are advocated by the current mathematics curriculum.

> *Students need frequent opportunities to work with open-ended problems. The solutions to problems which are worth solving seldom involve only one item of mathematical understanding or one skill. Rather than remembering the single correct method, problem solving requires students to search the information for clues and to make connections to the various pieces of mathematics and other knowledge and skills which they have learned. Such problems encourage thinking rather than mere recall.*
>
> (Ministry of Education, 1992, p. 11)

From an information-processing point of view Owen and Sweller (1989) claim that open-ended problems facilitate learning because the learner is not addressing directed goals but processing different aspects of the problem.

Many open-ended problems are in word problem format. Most word problems according to Gerofsky (1996, p. 37) follow a three-component compositional structure:

> *(1) A 'set-up' component, establishing the characters and location of the putative story. (This component is often not essential to the solution of the problem itself.)*
> *(2) An 'information' component, which gives the information needed to solve the problem (and sometimes extraneous information as a decoy for the unwary).*
> *(3) A question.*

Mathematical word problems are commonly perceived to be difficult for students. In order to understand what is to be solved the written (or oral) problem statement needs to be understood. The student has to transform the problem into a task that can be solved mathematically. In a metaphorical sense, the student's job is to 'undress' the problem which has been 'dressed up in words', transform the words back into the required mathematics and so solve the problem. A major difficulty is that the language of mathematics and the language of common English usage often differ radically (Ellerton, 1989). The wording of problems appears to influence students' problem representations and strategies of solution (Laborde, 1990). Students' reading abilities can affect their interpretation of a problem as some linguistic features of mathematical word problems can provide difficulties. As Laborde (1990, p. 69) warns:

> *The teaching of mathematics is faced with the apparent contradiction that language is needed to introduce students to new notions and that language may turn out to be an obstacle to students' understanding.*

Newman (1983) has proposed methods for identifying these difficulties and for assisting students to read mathematical prose. The 'Newman Procedure' identifies a hierarchy of five performance strategies: reading recognition, comprehension, transformation, process skills, and encoding to help diagnose inadequate problem solving performance. Newman's (1977) original research involved the analysis of 124 sixth-grade pupils' errors on written mathematical tasks using diagnostic interviews. The analyses revealed that for 13% of the errors the students had not been able to read the questions accurately enough to gain meanings of the questions. For another 22% of errors the students failed to understand the meanings of the questions even though they could read them. For yet another 26% of errors the students had read and understood the items but had not been able to carry out the required mathematical processes successfully and finally 25% of errors were diagnosed as being due to carelessness. *It would seem therefore, that when pupils attempt to solve mathematical tasks they are influenced by factors other than "mathematical factors"* (p. 240). The implications from this study suggest that the focus should not only be on the mathematical processes involved but also whether students can read and understand the meaning of the question.

The degree to which a student is able to construct meaning from the reading process can also be dependent upon **context**. The mathematics curriculum stresses that mathematics should be *taught and learned within the context of problems which are meaningful to students and which lead to understanding of the way mathematics is applied in the world beyond school* (Ministry of Education, 1992, p. 5). According to Smith and Silver (1995), the challenge to teachers is to do this in contexts that are sensitive to the very different everyday experiences that students bring with them as these experiences may lead to diverse interpretations of problem situations. Teachers can do this by selecting tasks that reflect students' interests, dispositions, and experiences, and by integrating cultural backgrounds into their learning. *The context in which mathematics is placed seems to be a factor in determining mathematical procedure and therefore performance* (Boaler, 1993, p. 13).

Many word problems being posed to students are written to incorporate **real-life contexts**. The claim is made that word problems are used for practising real-life problem solving skills yet *their stories are hypothetical, their referential value is non-existent* (Gerofsky, 1996, p. 41). 'Meaningful' is supposedly meaningful to the student yet the tasks are often more 'real' to the adults who teach them (Boaler, 1993). Smith and Silver (1995) provide interesting examples of problems which although not having a real-world context can still have **relevance** to students' lives, interests, and cultures. The term relevance is sometimes narrowly equated with real world applications. However, the matter of relevance has other dimensions that do not all involve realistic or applied problems. Teachers in the Quantitative Understanding: Amplifying Student Achievement and Reasoning (QUASAR) Project, a national American project, found that in order to determine what was relevant to students they had to simply ask about their interests, listen to them explain their thinking, give opportunities to generate their own problems, collect their own data, and select relevant contexts (Smith & Silver, 1995). The nature of context can also be relevant to students in terms of affect and willingness to solve the problem (Pimm, 1995).

> *The reasons offered for learning in context seem to fall into two broad categories, one concerning the motivation and interest of students through an enriched and vivid curriculum, the other concerning the enhanced transfer of learning through a demonstration of the links between school mathematics and real world problems.* (Boaler, 1993, p. 14)

2.5 ASSESSMENT

> *If our assessment procedures are to encourage, rather than inhibit, good classroom practice then our assessment tasks should be designed to reflect those achievements that we value.* (Swan, 1993, p. 214)

A greater emphasis in actively involving students in problem solving creates new challenges, especially in assessment. Assessment in mathematics education is usually taken to concern the judging of the mathematical capability, performance, and achievement of students (Niss, 1993). Ways of judging student performance such as tests and examinations are therefore subsumed under the assessment category. Tests and examinations can be used as part of the mode of operation of an educational system, as in the case of School Certificate, or as indicators of the quality of such a system as in the case with international performance comparisons such as the 'International Mathematics and Science Studies'.

While there have been significant reforms in mathematics education these developments have not been matched by parallel developments in assessment (Niss, 1993). Despite the advocated change towards more authentic methods for assessment of mathematics learning in the 1990s, the lower level factual recall items such as multiple-choice and short answer questions, found in traditional tests, still dominate assessment methods (Garet & Mills, 1995).

Traditional Methods of Assessment

Traditional tests which force students to answer artificial questions under artificial circumstances, impose strict time constraints and teach the doctrine that mathematical problems have a single right answer and that all other answers are equally wrong. If teachers simply have to check multiple-choice responses or look for wrong or right answers, then benefits for the student are minimal. Students may choose a correct response yet it cannot be inferred from that response that they used correct reasoning processes. Conversely, they may have chosen an answer defined as incorrect in the multiple-choice setting when in fact their reasoning processes may have been valid. The inference made from correct responses to multiple-choice items is that the students have a good understanding of the problem posed when in fact their selection may have been based on an inappropriate reason or misconception (Santel-Parke & Cai, 1997). Moreover, multiple-choice items do not allow students the opportunity to produce their own answers, to display the processes involved in acquiring an answer, nor to explain the thinking or reasoning associated with an answer.

> *A reliance solely on the sleek efficiency of multiple-choice (and other short-answer) formats will severely hinder efforts to help students develop a reflective and interrogatory stance toward their learning.*
>
> (Silver & Kilpatrick, 1988, p. 181)

Conventional achievement tests produce hard substantial data but the substance of these results depend on their importance as viewed by teachers, students, and society. Tests have a symbolic function to signal to students, teachers, and the general public those aspects of learning that are valued (Silver & Kilpatrick, 1988). However, traditional testing practices can also undermine the processes of teaching and influence the quality of mathematics instruction (Silver, 1992). This means that a conflict is likely to exist between traditional assessment practices and the objectives of a mathematics curriculum oriented toward higher level thinking, reasoning, and problem solving. Assessment procedures should do justice to the goals of the curriculum and to the students but one needs to ask: *Does assessment reflect the theory of instruction and learning represented by the curriculum?* (De Lange, 1995, p. 93) or does assessment drive the curriculum delivery?

The 'Top-down' Approach

Curriculum planning and assessment can be viewed as a 'top-down' approach (Blane, 1992). Pressures to change and to initiate new mathematics curricula come from many sources: political, societal, mathematical, technological, psychological, and educational. National and international assessments have drawn the attention of policy makers who use them as leverage for mandating change in education in the hope of bettering the nation's competitive edge (Webb, 1993).

Teachers are influenced by their understanding of the content of externally-mandated tests especially when they are viewed as having important consequences for students, teachers, and schools (Romberg, Zarinnia, & Williams, 1989, cited in Silver, 1992). Research by Romberg et al. suggests that teachers tend to narrow their instruction, giving a disproportionate amount of time and attention to teaching the specific content that is most commonly tested. When writing their own tests, using the format and content prescribed for externally-mandated examinations, teachers will include items such as multiple-choice and short answers, thereby demanding and evaluating performance identical to that assessed in external examinations (Silver, 1992).

WYTIWYG ('What you test is what you get') an acronym suggested by British educators (Nottingham University) suggests this phenomenon is present in many countries. Teachers and schools wishing to have their students to do well in external examinations will prepare them carefully for such assessments. Many teachers resort to 'teaching to test' with an emphasis placed on what might appear on a test paper rather than exploring relationships within mathematics. The temptation may be to rush through the syllabus and thereby consider that they have taught the material, as opposed to students having learned it. The teacher's aim is to prepare students for examinations by ensuring correct formats and styles are adopted, sometimes at the expense of genuine understanding and application.

The 'top-down' approach is based on the belief that appropriate instructional change will occur in response to changes made in the content and form of tests. If externally-mandated assessments incorporate alternative methods then it is believed that teachers will broaden their instructional programmes to include parallel diversity in content and form (Kulm, 1990). It is a concern that the results from external examinations actually offer limited information on which to base instructional decisions. In many instances they are far removed from the classroom environment. Feedback is often delayed and the examination results used in a summative way.

> *Despite the presumption that externally mandated testing should be useful in providing information to improve instruction and learning, experience and research suggests that this is not often the result.*
> (Silver & Kenny, 1995, p. 54)

The results are therefore of minimal utility for detailed instructional guidance. At best they could be used for general, long-range planning at school and state level (Silver & Kenny, 1995).

For teachers keen to embrace the reforms advocated by the curriculum the need for changes in assessment is a paramount and rational expectation. For some, it is changes in the assessments that force the changes in the curriculum to be implemented. This is assessment-driven reform.

Reform in Assessment

The Australian Association of Mathematics firmly asserts that assessment should follow the curriculum, not lead it (Joffe, 1990). The 'Curriculum and Evaluation Standards for School Mathematics' (1989), 'Mathematics in the National Curriculum' (1991), and 'Mathematics in the New Zealand Curriculum' (1992) all argue for the use of multiple sources of assessment information. Authentic assessment of mathematical proficiency needs to address areas such as problem solving, communication, and logic and reasoning.

Students should find undertaking an assessment task a learning experience and teachers should discover what their students know or do not know as a result of the assessment task. Unfortunately,

> *current tests place greater emphasis on those aspects of the curriculum that are relatively easy to assess than on those aspects that are highly valued by professionals in the field of mathematics education.*
> (Silver & Kilpatrick, 1988, p. 70)

It is acknowledged that in order for assessment to be a more valuable source of information then the range of what and how we assess has to be considered and altered. Efforts are underway to develop alternatives to multiple-choice and short answer items for assessment of mathematical proficiency.

> *As a definition of mathematical proficiency has expanded to include frequent use of terms like reasoning, communication, problem solving, conceptual understanding, and mathematical power, the reform discussion has lead to consideration of how to assess students' attainment with respect to this new vision of mathematical proficiency and how to assess improvements that may result from curricular and instructional reforms that might be undertaken.* (Silver, 1992, p. 490)

Hence the drive for more authentic forms of assessment.

The constructivist view is that problem solving, modelling, and investigative work must be evaluated developmentally and holistically in context (Galbraith, 1993). Effective assessment of problem solving in mathematics requires more than looking at the answers students give. As students are prone to making calculations without explanations *we need to devise problem situations and questions that encourage and motivate students to communicate and explain their thinking* (Szetala & Nicol, 1992, p. 44). According to Szetela and Nicol the most natural and common method for assessing performance in problem solving is to gain general impressions about the quality of a solution while scanning students' work. The 'proximity of correctness' strongly influences these general impressions. Marking schedules and assessment procedures that focus attention on solution procedures need to be developed and used by students and teachers (Pandey, 1990; Clarke, 1995). It means that instead of scoring solutions only, teachers can analyse the responses to problems. If teachers provide continuing experiences for students to critically analyze solutions and communicate their observations and responses to problems, it can help students to engage in reasoning, evaluation, and communication. Essentially, teachers can assess problem solving processes more effectively and find out about students' thinking.

> *With better awareness about students' knowledge and thinking, teachers can plan more effective instruction, and the outcome is more likely to be better learning of higher-order skills essential to success in problem solving.*
> (Szetala & Nicol, 1992, p. 45)

Assessing the complex processes for solving problems is not an easy task. Difficulty in assessing written work is exacerbated by the failure of students to communicate clearly what they have done or what they are thinking (Szetela & Nicol, 1992). One of the difficulties when trying to assess students' mathematical understandings is the need to decide the students' meanings for the words that they use. (Pirie, 1991). Wrong assumptions of students' meanings can lead to misinterpretation of students' understandings and concepts links. Another difficulty is that students tend to make calculations without explanation and it is not usually possible to gain from calculations alone insight into the problem solver's work and thinking. Students should be required to explain the steps taken and to justify their results. From the student's viewpoint, another difficulty is not knowing what makes a quality response.

What Makes a Quality Response?

Many students are confused or do not know what makes a quality response; they have little idea of the characteristics of quality mathematics (Clarke, 1995). Constructive assessment should involve the students in the assessment process (Izard, 1993; Clarke, 1995).

> *At all grade levels, students benefit from opportunities to examine and discuss problematic and exemplary pieces of writing. By studying exemplary writing, students can develop an understanding of the features of clear mathematical communication. By examining and revising writing that needs improvement, students can incorporate the relevant standards and styles into their own efforts.*
>
> (National Council of Teachers of Mathematics, 1998, p. 87)

This process can help students understand and if necessary renegotiate the didactic contract in the mathematics classroom.

Within the classroom, the didactic contract is a set of shared understandings by which the teacher and class know what is expected. These include the types of questions which may be asked, the kinds of assistance that may be given, and the forms of response that may be considered satisfactory. Different classrooms may operate under different didactic contracts. Many of the tasks now found in mathematics classes today challenge the existing didactic contract by the recent commitment to task diversity in assessment (Clarke, 1995). Clarke identifies these task diversities as: diversity of content, from relatively abstract problems to elaborately contextualized word problems; diversity of types of tasks from closed, routine procedural tasks to open-ended problems and investigations; and diversity in communication modes.

Changes in the language of assessment policy and changes in the mathematics classroom are occurring so that open-ended tasks, problem solving, and investigations are more frequently found in assessment tasks and these tasks are located in contexts meaningful to the student (Clarke, 1993). Not all contexts are appropriate (refer Section 2.4) as some may interfere with students' understandings of the mathematics being assessed. Students may interpret the task differently depending on their individual backgrounds and experiences. Responses, because of the influence of context, may therefore not give a valid indication of students' mathematical knowledge. All students should be responding to a task that elicits their best performance and not be disadvantaged because of the context. Denvir (1988) suggests that in order to overcome this difficulty students should be assessed in a range of contexts and the contexts in which students are most likely to succeed should be sought. Tasks must also be selected not only on their appeal and appropriateness for students but be representative of tasks that the curriculum sets out as appropriate (Izard, 1993).

Appropriate Assessment Tasks

Problem situations may be of different kinds (refer Section 2.4). Some are 'closed' whereas others can be defined as 'open-ended' problems or tasks. In addition to providing answers, open-ended tasks require students to show their solution processes and justify their results. They are designed to incorporate specific mathematical content and to elicit a variety of appropriate explanations and solution strategies. Such open-ended tasks should also be designed to assess students' capacity to use higher level thinking and reasoning processes: understanding and representing mathematical problems, discerning mathematical relationships, organizing information, formulating conjectures, evaluating reasonableness of answers, and generalizing results. Open-ended tasks that prompt students to explain or justify their answers allow these cognitive processes to be developed.

Tasks may vary considerably by the nature of the **prompt**. Prompts should be written in such a way that students clearly understand what is required of them. The wording of the prompt should be familiar to them and may if necessary contain sufficient detail to give an indication of the criteria that will be used to evaluate the response.

> *To facilitate the assessment of students' thinking and reasoning processes, the prompt or directions should encourage students to communicate their solution strategies and reasoning in written words, mathematical expressions, diagrams, or some combination of these representations.*
> (Santel-Parke & Cai, 1997, p. 78)

According to the recently released draft 'Principles and Standards for School Mathematics' (National Council of Teachers of Mathematics, 1998, p. 87) *since written assessments of students' mathematical knowledge are becoming increasingly typical, students will need practice responding to typical assessment prompts.*

Communication Modes

Teachers can use information from qualitative analysis of students' responses to improve students' proficiency in communicating mathematically (Cai, Magone, Wang, & Lane, 1996). Two interrelated perspectives of communication that can be considered are the quality and the mode of communication. The mode of presentation and response is likely to influence students' thinking (Denvir, 1988). Written tests are commonly used in secondary schools as the primary method of assessment. They are more easily administered and data collected is viewed as more objective and reliable than other sources such as interviews and assignments. However, unless a follow-up interview is conducted, it is not possible to assess students' written responses in terms of the way in which they have perceived it. There is an assumption that once the question has been posed, the problem, as perceived by the teacher or examiner, becomes the student's problem and will be perceived in a similar way by all students. An 'inappropriate' response is sometimes the result of a student's inability to 'see' something from the teacher's point of view.

The manner in which the information about the problem is represented can be a factor that influences student performance. Would the task be more easily understood if represented through written text, diagrams, tables, charts, or mathematical expressions? The wording of tasks needs to be examined closely. Could the wording mean different things to children from varying cultural and experiential backgrounds? It is also important that the vocabulary is at the student's level so that the task does not assess reading ability.

By presenting students with different levels and modes of responses for evaluation, criteria for judging mathematical communication can be established and in turn help students generate their own high quality descriptions and responses. These responses may consist of written words, mathematical expressions, or diagrams. The chosen mode of representation may be determined by the nature of the problem or whichever mode of representation the student feels most comfortable with. Some tasks are better than others at getting students to communicate their solution processes and reasoning. Students need to develop an awareness and understanding of what constitutes a quality response in their own work and their peers (Clarke, 1995).

Developing a Rubric

Students and teachers can learn together about assessing problem solving and establishing a common understanding of what constitutes and how to assess 'good work' in problem solving. Petit and Zawojewski (1997) reported on the development and implementation, in Vermont, of a rubric for assessing problem solving, related staff development, and student self-assessment. Criteria have been developed to address both problem solving and communication. Complex problem solving tasks were selected that gave students opportunities to develop or select their own approach. Meaningful contexts were chosen which encouraged students to use rich mathematical vocabulary and representation. Students were also encouraged to be involved in identifying and selecting tasks based on their potential to produce high level responses of reasoning and communication. *Bringing students and teachers together as partners in assessment supports the development of a problem-rich experience in mathematics education* (Petit & Zawojewski, 1997, p. 477).

Clarke (1995) also advocates a strategy for involving students in identifying the characteristics of quality mathematics and for developing rubrics for assessing mathematical performance. The quality of responses may be determined by how difficult the problem is for the student. If it is perceived as being a relatively easy task then the required solution processes might be so familiar to the student that they perform them with very little thought and therefore do not feel it is necessary in their response to articulate all the required steps. Alternatively, the task might be so difficult, or include unfamiliar contexts, or too many conditions that the student may fail to express clearly the expectations for a complete response (Santel-Parke & Cai, 1997).

In order to gain an insight into students' thinking processes Galindo (1998) suggests the use of an analytic-scoring method which breaks down a task into parts and awards a separate score for each part. Such methods would be useful to give students feedback about their performance in important categories associated with mathematical problem solving. Additionally, the teacher could gain diagnostic information about students' strengths and weaknesses, and be able to identify specific aspects of mathematics that may require additional instructional time.

Looking Ahead

New developments in assessments will be closely linked to the reform efforts in mathematics education. The incorporation of open-ended assessments, performance assessments, portfolios, self-assessments, co-operative group assessments, and other alternative methods will no doubt continue to be developed in a response for more integration of assessment and instruction. However, there is still considerable work to be done on both the theoretical and practical problems of creating practical, reliable, and scientifically credible assessment alternatives. If tests are revised to reflect a broader range of content and format, then teaching to a test will not be viewed as an undesirable outcome (Silver, 1992).

Modes of assessment are being expanded and developed to include conceptual understanding, problem solving, the processes of reasoning, and the communication of mathematical ideas. It is important to consider the cognitive complexity of the processes students employ in solving problems and the meaningfulness of the problems. A basis for judging both the content quality and the comprehensiveness of the content coverage needs to be provided. The appropriateness and importance of the purposes to which assessments are put must also be considered along with the interpretations that are made of results. A measure that may be highly valid for one use may not be valid for another. If assessments are to contribute to the fundamental purpose of evaluation, the improvement of learning and teaching, then alternative forms of assessment must be considered.

> *An important outcome of the alternative assessment movement is that it challenges the education community at large to reconsider just what are valid interpretations of* ***any*** *kind of assessment information.*
> (Linn, Baker, & Dunbar, 1991, p. 20)

2.6 SUMMARY

Learning mathematics involves learning its characteristic patterns of language use, its register, and its genre forms. The literature supports the need for students to be able to use the various mathematical genres when communicating their mathematical findings. This includes the ability to be able to explain and justify mathematical reasoning and findings.

Communication in mathematics is inherently linked to problem solving. Problem solving has received increased attention as recent curriculum development reflects incorporation of a constructivisit view of learning. The solving of problems provides a primary context for students to learn mathematics in an active participatory way. It is a way which also stresses the importance and value of reasoning and communication. The problems to be solved by students should be both challenging and based on 'meaningful contexts'. Problem situations should encourage and motivate students to communicate and explain their thinking.

These communications, if assessed, can provide comprehensive insights into students' thinking. Examining written responses is one way in which students' responses can be assessed. This is the method adopted by external authorities. These externally-mandated examinations can influence and be influenced by curriculum reforms. More authentic methods of assessment are advocated by mathematics educators.

> *Teachers can provide continuing experiences for students to critically analyze solutions and communicate their observations and responses to relevant questions. Such practice can help students engage in reasoning, evaluating, and communicating, and can enable teachers to assess these problem solving processes more effectively.* (Szetela & Nicol, 1992, p. 44)

CHAPTER 3: RESEARCH DESIGN

3.1 INTRODUCTION

> *The goal for research in mathematics education should be to produce new knowledge about the teaching and learning of mathematics.*
>
> (Romberg, 1998, p. 379)

Research in education can be approached in a variety of different ways and deciding on which approach rests on careful consideration of the various philosophical viewpoints. These philosophical viewpoints can also inform and guide research in mathematics education. It is important that researchers have an understanding of which paradigm informs and guides their approach (Guba & Lincoln, 1994). There are three fundamentally different and competing philosophical perspectives of educational research: positivist, interpretive, and critical theory.

The positivist believes that the key to the gaining of knowledge is through scientific and experimental research. Educational research, from a positivist perspective, must therefore deal exclusively with factual matters, and focus on the observable and measurable. Positivism was embraced by many educational researchers who believed that the natural scientific methods could be redefined and applied to social research (Clark, 1997). The researcher is seen as being a rational observer whose values are independent from the research. Positivists adopt a limited view of science in that they fail to take heed of subjective and interpretive meaning.

Interpretivism places much importance on values. To understand human behaviour, the interpretive researcher focuses on interpreting the meanings that research participants give to their experience. An interpretivist holds that there is no social existence nor social reality outside of meanings and interpretations in the everyday world of commonsense. In contrast to the positivist focus on observable facts, the interpretive paradigm interprets the subjective meanings of the research participants and takes into account their values and interpretations.

Critical theory, as a philosophical framework, attempts to *retain the strengths of a scientific (positivist) and interpretive inquiry, transcend their limitations, and go beyond what they individually and jointly are able to offer* (Clark, 1997, p. 43). Applying a critical theory approach to educational research recognizes that research is not *on* or *about* education but research *in* and *for* education (Gitlin, Siegal, & Boru, 1993). Critical theory includes an emancipatory ideology and undertakes to critically evaluate and change the conditions that the participants find themselves in (Carr & Kemmis, 1986). Critical theorists hold that social scientists should engage in critiques of ideology, and in doing so examine social phenomena critically. This examination is claimed to liberate people from domination and repression and hence result in enlightened observations (Clark, 1997).

These research paradigms have been defined in relatively simplistic terms but provide a basis from which the chosen methodology can be viewed. While Hiebert (1998) does not believe that there are incommensurable differences between current paradigms of research in mathematics education it is important to be aware of and understand the methodological paradigms in order to appreciate why particular research methods are chosen. Research can be viewed as a sense-making activity with the aim of understanding important phenomena in mathematics education. Both researchers and students are sense-makers and so research should consider *how children learn mathematics, how they construct understandings, and how they show us that they understand* (Hiebert, 1998, p. 141). In order to gain an understanding from the study, and make sense of students' writing of explanations and justifications, a qualitative research design has been chosen as the most appropriate methodology.

A **qualitative** research design is influenced by both the interpretive and critical theory philosophical perspectives of inquiry. It is based on the philosophical view that reality is constructed by individuals interacting with their social world. The individual's view of the world develops cumulatively and is contributed to by particular life experiences (Bryman, 1988). Qualitative research is an umbrella term for a family of research methods covering several forms of inquiry. These include field study, case study, ethnography, participant observation, naturalistic study, and inductive research. The primary concern of qualitative research is in understanding human behaviour from the participant's perspective, to 'get inside' and uncover the thoughts, perceptions, and feelings experienced by the participant. The researcher is the key instrument for data collection and analysis and so the subjectivity and identity of the researcher is acknowledged (Stanley, 1990).

3.2 DATA COLLECTION METHODS

The mode of data collection and analysis used in this research was essentially qualitative. Qualitative methods allow the researcher to gain access to the understandings, responses, interpretations, and perceptions of people in the context of their everyday lives. This requires the use of data collection instruments that are sensitive to underlying meaning when gathering and interpreting data (Merriam, 1998). Data collection instruments selected as being appropriate for this study, in order to gain information about students' writing of explanations and justifications, were the questionnaire, interview, document analysis, and the researcher.

The Questionnaire

The questionnaire is a highly structured data collection technique which requires respondents to answer the same set of questions in order to access 'what is inside a person's head'; to measure what a person knows about the topic, their views, attitudes, and beliefs. This technique measures, not necessarily what they actually believe, but what they say they believe. Therefore, the researcher relies on the respondent's honesty to report what is, rather than what they think ought to be. The respondent also needs to know what they think in order to be able to report it.

A questionnaire has some distinct advantages over other data collection instruments. It is relatively easy to administer and is standardized (Burns,1997). Each respondent receives the identical set of questions phrased in exactly the same way. There are no difficulties regarding the personal relationship between researcher and respondent to be addressed. A questionnaire can guarantee confidentiality and thus may elicit more truthful responses than from an interview. The respondent is also free to answer the questionnaire in their own time and at their own pace.

> *Questionnaires are a good way of collecting certain types of information quickly and relatively cheaply as long as subjects are sufficiently literate and as long as the researcher is sufficiently disciplined to abandon questions that are superfluous to the main task.* (Bell, 1987, p. 58)

A questionnaire (Appendix D) was used in this study to gain an understanding of the teachers' views and perceptions of the writing of explanations and justifications in mathematics. A series of pre-determined questions was formulated and the questionnaire was self-administered. The questionnaire was designed to incorporate three different kinds of items: closed, open-ended, and scale items. The closed items allowed the respondent to choose from two (yes/no) or three (never/sometimes/often) fixed alternatives. The open-ended items included a frame of reference for the response, a prompt ('please comment') and a restraint on the expression in terms of the number of lines provided. It was hoped that the incorporation of the open-ended item would facilitate *a richness and intensity of response* (Burns, 1997, p. 473). Rating scales were also incorporated: both a scale of fixed alternatives, with five clearly defined options, and a graphic rating scale. Care was taken in selecting question types, writing the questions, and subsequent piloting and modification.

There are, however, limitations in the use of a questionnaire as a data gathering technique. Essentially, responses to a questionnaire have to be accepted as given and so a major limitation is that there is no opportunity to follow up responses. This method does not allow the opportunity for further probing. Questions have to be very carefully worded to ensure that appropriate language is used and the potential for misinterpretation is reduced. There may be poor response rates and a biased sampling as a result of non-returns.

In an attempt to address these limitations personal contact was made with the potential respondents as a group so that the purpose of the study could be explained 'firsthand' and an acceptable time-frame could be negotiated. The questionnaire was trialled with some teachers from another school and subsequent modifications made to ensure that instructions were clear, language appropriate, and to ensure that the potential data would assist in addressing the research questions.

Interviews

One of the most effective modes of gathering data in any inquiry is through the interview method (McKernan, 1996, p. 128). The interview method is a specialized form of communication; a dialogue between researcher and respondent in which the former seeks information about how the latter thinks. It is a personal contact situation in which carefully chosen questions are asked in order to elicit information pertinent to the research problem. An attempt is made to ascertain what the issue looks like from another person's vantage point.

The interview has some clear advantages over other data gathering instruments (Bell, 1987; Burns, 1997; McKernan, 1996). One of the most important is its flexibility. The interviewer is able to observe the respondent, repeat questions, and elicit explanatory detail to ensure that questions are understood and relevant. According to Burns (1997) there is usually a higher level of motivation among respondents and more complex responses are elicited. It is a useful method when the size of the representative sample is small and extensive data is required on a small number of topics.

Despite having distinct advantages, some disadvantages in the interview process have been identified. Interviews are more expensive and time-consuming than questionnaires (Burns, 1997) and the interaction between interviewer and respondent is prone to bias. In particular, scheduling is a problem which researchers need to be sensitive to when involving students as participants. In this study student interviews were scheduled at a time deemed appropriate by both the teacher and student. The researcher had ensured that students were willing, informed participants and a conscious effort was made to put the students at ease.

There are three main types of interviews, defined in terms of their content and organization: structured, semi-structured, and unstructured (McKernan, 1996). In the **structured** interview the researcher has written a specific set of questions from which they will not deviate. Questions are usually of the 'fixed-response' type allowing the interviewer little opportunity to gain clarification from the respondent and limited insights. Essentially, the structured interview is little more than a questionnaire in oral form (Anderson, 1990).

The **unstructured** interview centres around a topic and has the potential to produce a wealth of material. The interviewer converses with the respondent about the topic and uses a general plan as a guide. The primary objective is to give the respondent the opportunity to talk freely about those aspects of the topic considered to be crucial to the study. There may be an initial question posed but subsequent dialogue is determined by the respondent's comments. However, unstructured interviews allow for greater bias and subjectivity. The nature and amount of judgement made by the researcher increases significantly in unstructured interviews and contributes to the 'response effect' (Borg, 1987).

The **semi-structured** interview is an appealing compromise to many educational researchers. It is clearly important to allow the respondent to talk freely about those aspects of central importance to the study. However, having some structure identified beforehand ensures that those aspects of the topics considered crucial to the study are addressed. This semi-structured approach rectifies those difficulties posed by an exclusive structured or unstructured interview. The interviewer establishes a clear framework beforehand, identifying specific questions to be posed. Allowance is made for some degree of flexibility in which the interviewer can explore further any relevant issues.

This study utilized a semi-structured interview as one of the data gathering methods because of the potential to provide more valuable information than the other interview types. The semi-structured interview was used to obtain information about students' attitudes, perceptions, and beliefs, and also to question students about responses to problems that had been posed in an independent problem solving worksheet. The interview enabled students' views and beliefs about writing explanations and justifications to be measured in greater detail than would have been possible through a questionnaire. The interview also gave the students a chance to recall feelings and to explain their written responses. It allowed the researcher an opportunity to try and 'get inside the student's head'.

Document Analysis

Documents are a ready-made source of data that are easily accessed and can be used in qualitative research. *Documentary data are particularly good sources for qualitative case studies because they ground an investigation in the context of the problem being investigated* (Merriam, 1998, p. 126). The data collection from documents is guided by questions, educated hunches, and emerging findings and as such it is the skills and intuition of the researcher that determine the nature of the data to be gained. Documents are analysed by content analysis, a systematic procedure for describing the content of the communications. The goal of content analysis is to uncover themes, concepts, and indicators of the communication (McKernan, 1996).

The documents used in this study are Ministry of Education School Certificate examination papers in mathematics and researcher-generated documents. The former are public documents and provide raw data for the basis of analysis and are also used to support tests prepared by the researcher. The document written by the researcher (Appendix B) consists of a set of word problems designed to generate students' written explanations and justifications. There are five problem solving tasks covering each of the content strands of the curriculum: number, measurement, geometry, algebra, and statistics. Four of the problems are based on problems found in the School Certificate external examinations, 1996 and 1997. The other, Question 2, is based on the 1996 moderation test for the internally assessed students sitting School Certificate that year. The problems were carefully selected to incorporate a variety of prompts and to elicit a range of responses. The researcher checked with the class teachers to ensure that the material presented to the students would have been previously 'covered' in class. The researcher did not want to present students with material that had not yet been 'covered' as this would be a significant factor in explaining a non-response.

The Researcher

The researcher is the primary instrument in qualitative research (Merriam, 1998; Patton, 1990). The qualitative paradigm is based on minimizing the distance between researcher and the informant (Lincoln & Guba, 1985). It is the researcher who comes to the research project with certain experiences, expectations, and values. The researcher is the one who selects the design, the data gathering instruments, collects the data, responds to the situation, and makes meaning from the information gained. Therefore, it is important to acknowledge the role and influence of the researcher in qualitative research. The concept of reflexivity is used by Hammersley and Atkinson (1983) to describe the recognition of the inter-relationship between researcher and respondent. They suggest that reflexivity should be deployed at all stages of the research from design to writing up. The biases, values, and judgements should be stated explicitly in the research project as such openness is considered useful and positive.

The researcher is an experienced teacher who has taught at all levels in the school system. Currently, the researcher is teaching in a pre-service primary teacher education programme specialising in mathematics education. This project reflects an interest in the relatively new mathematics curriculum, especially those aspects which involve communication in the mathematics classroom.

As a result of experience in mathematics education, the researcher brings a number of assumptions to the research project which need to be articulated, namely:

(i) that there exists an important link between the domains of mathematics and language

(ii) that communication is an important part of the Mathematical Processes strand and therefore worthy of research, more specifically that the writing of explanations and justifications is an essential aspect of being able to communicate in mathematics

(iii) that the writing of explanations and justifications is possibly problematic for both students and teachers

(iv) that Year 11 is a significant year for addressing students' ability to write explanations and justifications

(v) that the national School Certificate examination in mathematics is a valid indicator of those aspects of mathematics to be taught and assessed

(vi) that the aspects of mathematics assessed in the School Certificate reflect the aims and objectives of the curriculum document

(vii) that students and teachers have concerns about the writing process and its place in the learning and teaching of mathematics

3.3 THE PROJECT: SETTING, SAMPLE, AND SCHEDULE

This section initially outlines the setting for the study. Relevant details about the sample of teachers and students who particicpated in the study are then provided. Finally, the four phases of the study and data analysis methods are described.

The Setting and the Sample

The research was conducted at a large provincial co-educational secondary school. The school is the only secondary school in the town and has students from a full range of socio-economic backgrounds. There is an urban-rural mix and about 14%

of the students are Māori. The school has a decile rating of 6.[1] The teachers involved in the study were members of the Mathematics Department who had initially expressed an interest in being involved in the research project. The students who participated in the study came from the two highest ability grouped mathematics classes in the school. The teachers expected the majority of these students to pass the School Certificate mathematics examination later that year. The interviewees were students aged 15 and 16 years in their third year at secondary school. Fourteen students were selected, seven from each class. They were selected according to the following criteria: gender and the type of responses given to the problem solving tasks.

The Project Schedule

The project consisted of four phases conducted over an 11 month period (January to November).

The first phase included a preliminary literature review, consultation with mathematics teachers and colleagues, and a preliminary analysis of School Certificate mathematics papers.

During Phase Two the Teacher Questionnaire was piloted with a group of teachers from another co-educational school. The set of written problems was trialled by a Year 11 class at this same school. Subsequent modifications were made to both the questionnaire and the set of written problems. Minor changes were made to items in the Teacher Questionnaire to enhance clarity in interpretation, layout, and the use of more appropriate vocabulary. More space was also provided so that teachers could qualify their answers. After piloting the Student Problem Solving Task Sheet, two questions were deleted because of the time factor. On the geometry question the statement: *This drawing is not to scale* had to be added as some students had used the diagram to measure the angle.

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1 Each state and integrated school is ranked into deciles, low to high, on the basis of an indicator. The indicator used measures the extent to which schools draw from low socio-economic communities.

Phase Three involved the completion of the teacher questionnaire, student problem solving task sheet, and follow-up interviews. To facilitate this process the researcher addressed the participating teachers. The objectives of the study were outlined, assurance of confidentiality was given, and each staff member received the Information Sheet (Appendix A) and a questionnaire (Appendix D) to be completed in their own time. The questionnaire was to be completed with no identifiable codings and returned to the researcher by a negotiated date. The two teachers, who had agreed to have their classes involved in the study, disseminated the Information Sheets to their students. Three documents were issued: the Information Sheet outlining the research project, a consent form giving permission for the researcher to use the work samples, and another consent form giving permission to conduct follow-up interviews.

The written tasks were completed by the two classes of students. The students from one class (defined as the most able students in mathematics in the school) all managed to complete the set of problems during one mathematics period (one hour). The students in the second class were allowed to complete the problems over two consecutive mathematics periods. The students handed in their solutions when they were satisfied with their responses. The teachers ensured that the problems were solved independently and felt that the students were not pressured by any time constraints. These responses were then analysed by the researcher in order to select students for individual interviews.

Students selected from the two classes involved in the study were then interviewed. The interview process began within a week of the students completing the problems; the intervening period had to be kept to a minimum so that students' recall was based on a reasonably fresh experience. Semi-structured interviews were held in a nearby office during the mathematics period. The student sat at a table alongside the researcher and the interviews were taped. After thanking the student for agreeing to participate and reminding them of the purpose for the interview, the first-stage scheduled questions were posed

(Appendix C). For the second stage, the researcher wanted to specifically address certain individual responses. Students were able to refer to their response sheet whilst explaining their thought processes and recalling feelings that they had experienced when writing responses to the problems. The interviews lasted an average of 25 minutes. This phase of the study was conducted during the month of August. Finally, data from documents, questionnaires, and interviews were analysed.

Data Analysis

The data was collected by various means: questionnaires, interviews, and documents. Categories and themes were identified and the information sorted according to these categories. Details of the categories are given in the subsequent chapters of findings and reflect the purpose of the research. Some of the results obtained from the analysis could be described as 'descriptive statistics'. Descriptive statistics methods, according to Bell (1987), provide 'pictures' of the topic or group under investigation. These 'pictures' may be in the form of charts, tables, percentages, and averages. These techniques were used to assist in reducing the data to simpler and more understandable terms.

3.4 QUALITY CRITERIA

Any data gathering method should be examined critically to assess its reliability and validity (Bell, 1987). Some qualitative researchers prefer to discuss quality criteria such as 'trustworthiness' and 'authenticity'. What is important is to address the concepts of validity and reliability in a qualitative plan and to frame these concepts within the procedures that have emerged from qualitative writings (Creswell, 1994).

Reliability

Reliability is the extent to which the procedure produces similar results under consistent conditions on all occasions. For example, a question which produces one type of answer on one occasion and a different answer on another occasion is deemed unreliable. For qualitative researchers, achieving reliability in the traditional sense is impossible (Merriam, 1998). Replication of a qualitative study will not produce the same result as there is no benchmark from which to take repeated measures. Instead, Lincoln and Guba (1985) suggest that the researcher thinks about the dependability and consistency of results obtained from the data. It is not a question of whether the findings will be found again *but whether the results are consistent with the data collected* (Merriam, 1998, p. 206). Reliability can be enhanced in the following ways:

> *(a)* *Investigators outline the reason for the research and the major question they want to address.*
> *(b)* *They explicate their perspectives on the question, stating their research assumptions and biases.*
> *(c)* *They explain their data-gathering procedures, including timing and time-lines of observations, spatial arrangements of interviews, relationships with subjects and categories developed for analysis.*
> (Burns, 1997, p. 323)

The uniqueness of this study mitigates against the testing of reliability. However, by addressing those aspects outlined by Burns, the chances of the study being replicated in another setting are enhanced.

Validity

Validity is a complement to reliability. It is a more complex concept and tells us whether an item measures or describes what it is supposed to measure or describe (Bell, 1987). There are two key issues that validity addresses (Burns, 1997). The first, internal validity, is whether the researcher actually observes or measures what they think they are observing and measuring. The second, is the problem of external validity or generalizability and answers the question: To what extent are the findings able to be generalized, or held up beyond the setting or individuals under study?

Internal validity relates to issues of truthfulness of responses and accuracy of reports (Anderson, 1990). We require internal validity to be confident that the results are true for those participating in the study. This study attempts to incorporate a chain of evidence so that the reader can follow the data collection, analysis, understand what is going on in the study, and how the researcher reached the stated conclusion. The multi-method approach used in this study allows not only for breadth of coverage but also a check on the validity of individual methods (Sommer & Sommer, 1991).

Triangulation is the use of two or more methods of data collection in the study of some aspect of human behaviour (Cohen & Manion, 1994). According to Delamont (1992) there are three main types of triangulation: between method, between investigators, and within method. This study has used a 'between methods' approach to triangulation as a means of finding convergence among the sources of information and the different methods of data collection. The use of triangular methods can help overcome the problem of 'method boundedness' (Merriam, 1998). This is when a researcher chooses one particular method due to either familiarity or a belief that it is a superior method. Such exclusive reliance on one method may bias or distort the researcher's picture of the particular slice of reality that is under investigation.

External validity refers to the generalizability of obtained results. The traditional view reflects the scientific goal of generalizing findings to diverse populations. The concept of generalizability has been reconsidered by those involved in qualitative research so that it is more useful, although, as Bell (1987) points out, not all qualitative researchers are concerned with the question of generalizability. The intent of external validity in qualitative research, according to Merriam (1998), is not to generalize findings but to form a unique interpretation of events. Guba and Lincoln (1982) suggest that the concept of generalizability be replaced with the concept of 'fittingness'. The degree to which the situation studied matches other situations is perceived as a more realistic and workable way of thinking about the generalizability of research results. Only with detailed information can an informed judgement be made about whether the conclusions drawn from one study can be useful in understanding another site.

It is openly acknowledged that this study uses limited generalizability or what Ebbutt (1988) describes as 'low order generalization' or simple common sense meaning. The important features of such generalizations are that they are descriptive and delivered in hindsight. The categories or themes that emerged from the data analyses of this study were teased out as a descriptive generalization and delivered retrospectively.

Ethical Considerations

Educational researchers conduct their studies within a framework of ethical deliberation (Clark, 1997). The practice of research is subject to ethical principles, rules, and conventions (Anderson, 1990). In qualitative studies, ethical dilemmas naturally emerge with regard to the collection of data and the dissemination of findings. While individual students and teachers often assist the researcher by agreeing to help, those involved have a right to know exactly what they will be required to do, how much time they will be required to give, and what will happen to the information that they provide. The researcher has to consider how much detail is revealed to participants, how informed the consent can actually be, and how much protection and privacy the participant can be guaranteed. As with many ethical issues in other areas of life, being thoughtful and considerate of the needs and feelings of others guides the researcher (Bouma, 1993).

The researcher, at all stages of the research process needs to continually focus on the following questions: How would the researcher react if she was in the place of the person or group being studied? Would the researcher be able to respond well to the questions posed or the procedures employed? It is imperative that the researcher is well prepared so that the participant's time and effort is not wasted and the opportunity for further research is not jeopardised. Students, teachers, administrators, and parents will need to be convinced of the researcher's integrity and the value of the research before they decide to participate or not. A researcher who ignores the courtesies or oversteps ethical bounds can do great harm (Anderson, 1990). Considerations must therefore include:

> *informed consent (of the participants), confidentiality (of the data and the individuals providing it), minimizing of harm (to participants, researchers, technician etc.), truthfulness (the avoidance of unnecessary deception), and social sensitivity (to the age, gender, culture, religion, social class of the subjects)* (Massey University, 1997, p. 1)

Additionally, a researcher must be considerate, do nothing to injure, harm, or disturb the subjects of the research, keep data collected on individuals and groups secure, accurately record information, and report the findings of the research in a public manner (Bouma, 1993).

In conducting the research the following steps were taken to ensure that these ethical principles were applied:

(i) Approval was given by Human Ethics Committee, Massey University.

(ii) Approval was given by the school principal to conduct the research study.

(iii) Informed consent was gained in writing from the participants after they had been given an information sheet about the project and had time to consider the implications of granting consent. Participants were informed about the researcher's credentials and why the study was being conducted. They were given a comprehensive explanation of the nature and purpose of the activities, their rights to decline participation, to withdraw from the activity at any time, to have privacy and confidentiality protected, to turn off a recording device at any time, and to receive information about the outcome of the activity in an appropriate from. Care was taken to ensure that students were provided with this information in a manner and form which they could understand. Under the terms of the Privacy Act all that is required is consent from the students themselves, however as schools operate in partnership with parents, the parents or caregivers were required by the school and Ethics Committee to give consent based on the information provided.

(iv) To ensure confidentiality the information was handled in a way which protected the confidentiality of the participants and safe custody of the data was maintained. The taped interviews were transcribed by the researcher only.

(v) Assurance was given that it should not be possible for others to identify any participant or school from the research report.

3.5 SUMMARY

A qualitative research design was adopted to explore the writing of explanations and justifications in mathematics by Year 11 students in a secondary school. Three major data collection techniques were used: questionnaires, interviews, and document analysis. Information, mediated through the researcher, was gained from teachers, students, and documents. The results of this data collection are presented in Chapters 4, 5, 6, and 7.

CHAPTER 4: SCHOOL CERTIFICATE ANALYSIS

4.1 INTRODUCTION

School Certificate is the first nationally prescribed examination sat by students under the present national assessment schedule. The majority of New Zealand students sit this examination at the end of their eleventh year at school. Students are assessed either by a three hour written examination or an approved system of internal assessment. Many students, teachers, and other members of society recognize School Certificate as a valid national indicator of a student's achievement level.

The examination assesses the general and specific objectives as officially prescribed by the New Zealand Qualifications Authority (NZQA). The prescription, which defines the requirements for School Certificate mathematics, is based on and read in conjunction with 'Mathematics in the New Zealand Curriculum' (Ministry of Education, 1992). Specific objectives from the content and mathematical processes strands of the curriculum are identified, up to and including Level 5.

The basic structure of the examination paper has not changed during the period 1992-1997. The paper is comprised of two sections; Section A is a multiple-choice section and is worth 30 marks; Section B, the written section, is worth 120 marks. The paper is formatted so that students write on the examination script. Where students are required to write an explanation or show a number of steps in a problem, a series of lines are provided.

A content analysis of School Certificate examination papers in mathematics from 1992 to 1997 was chosen in order to ascertain if any changes have occurred in this six year period. During this period of time the new curriculum in mathematics was introduced. Specifically the analysis was conducted in order to answer the following questions:

- What emphasis is given to the writing of explanations and justifications?

- Are there any trends in the assessment of students' writing of explanations and justifications?
- Which content strands (number, measurement, geometry, algebra, statistics) are used for the problems which require written explanations and justifications?
- What prompts are used for students to respond to with a written explanation and justification?

4.2 ANALYSIS CRITERIA

Questions included in the analysis were those which required students to write an explanation or justification as defined in Chapter 1. Any question which required either an explanation or justification, or both an explanation and justification as part of the response, was included in the data collection. Graph interpretations which required a purely numerical response were excluded from the data.

In order to determine the exact weightings given to responses that required the writing of explanations and justifications the New Zealand Qualifications Authority Marker's schedule was used. This meant looking at the 'breakdown' of marks within a particular question, as illustrated by the following example:

> *Phil and Ken are gathering data for a traffic survey.*
> *Phil and Ken wish to find the percentage of drivers who are wearing seatbelts.*
> *State the TWO items of data they need to gather, and explain what calculation must be done with these two items to find the percentage of drivers who are wearing seatbelts.* (School Certificate, 1997, p. 14)

A total of three marks was allocated for the complete question. According to the Marking Schedule, one mark was allocated for the idea of quantity, one mark for reference to drivers and drivers wearing seatbelts, and one mark for the explanation of the necessary calculation. Consequently, for the analysis, only the weighting of one mark for the explanation was counted.

In deciding which content strand a question should be placed some professional judgement was used. The context of the initial problem was used to determine the content strand.

4.3 WHAT EMPHASIS IS GIVEN TO THE WRITING OF EXPLANATIONS AND JUSTIFICATIONS?

The following table represents the number of marks allocated in each examination paper for the writing of explanations and justifications.

Table 4.1 Explanations and Justifications in School Certificate Mathematics

Year	No. of Marks	Percentage of Section B	Percentage of Total Marks for the Exam
1992	4	3.3	2.7
1993	2	1.7	1.3
1994	8	6.7	5.3
1995	9	7.5	6.7
1996	21	17.5	14
1997	24	20	16

It is evident from Table 4.1 that there has been an increased emphasis placed on the writing of explanations and justifications in School Certificate papers during the period 1992-1997. The first most significant increase occurred in 1994. This was the first year that the new prescription was implemented and questions were set to reflect the prescribed weightings:

Key Skills	*25%*
Applications	*50%*
Information Processing	*15%*
Communication	*10%*

(NZQA, 1994, p. 3)

The result of this was a greater emphasis being placed on students' ability to write explanations in the School Certificate examination. The Chief Marker suggested that teachers should *insist on the writing of accurate English statements in communication. Complete sentences, correct spelling and correct punctuation should be encouraged* (NZQA, 1994, p. 3) . The advice was also given that *written responses* *should be both concise and precise. Candidates do not have to use all the lines that are provided* (p. 4). According to the Chief Marker it was apparent that students had some difficulty with words and phrases such as *show that*, *describe*, and *explain*.

In 1995, an investigation question was introduced as an innovation to provide an opportunity for investigative and creative thinking; aspects of the mathematical processes that are viewed as important parts of the new curriculum. Space for a written response spanned two pages with five out of the possible seven marks being awarded for the explanation. Proofs were no longer required in the School Certificate examination but clear and simple explanations (of up to two or three steps) as to why a particular geometric relationship or value exists were. The Chief Marker suggested that the ability to make and justify geometric connections needed to be given special attention by teachers.

Another significant increase in emphasis occurred in 1996 in an attempt to further consolidate some of the changes in the national curriculum and the School Certificate prescription. There was a continued requirement for students to interpret information, to explain, describe, or justify their mathematics clearly and concisely. It was commented by the Chief Marker that students coped better than in 1995 with questions which required written answers and that there was overall improvement in sorting through information, selecting processes that would reach a solution, and writing clearly about what had been done or what the solution meant. A general improvement was noted in the way in which students handled the amount of reading involved in the context-based questions which required written answers. Additionally, it was found that:

> *Geometric reasoning and explanation seemed to have improved this year, possibly because candidates were told that three statements with reasons were expected. Nevertheless, many candidates passed over this question, and others showed an inability to set out their thinking in a clear and reasoned manner. Candidates should be able to present a full solution on just three lines. This continues to need attention.* (NZQA, 1996, p. 5)

Although spaces provided a guide as to how much working or writing was expected, limited space was deliberately provided for the written explanations in order to encourage conciseness.[2] Many candidates in this particular year made no effort with questions that asked them to *explain your working so that someone else can understand you* and markers had difficulty in following candidates' thinking and thus had to award marks accordingly.

By 1997 the percentage of marks being allocated for written explanations and justifications had reached 20% of the total marks allocated for Part B, the written section. This confirms the importance given to this aspect of the communication process in mathematics. Improvement in communicating mathematical ideas was noted by the Chief Marker: *Many students understood what was required when asked to "describe", "explain" or "justify" and were able to communicate in a precise and clear manner* (NZQA, 1997, p. 1).

The increase during this seven year period of marks allocated to the writing of explanation and justifications in School Certificate mathematics examinations has been from 2.7% of the total marks in 1992 to 16% of the total marks in 1997. As a percentage of the written section it represents an increase from 3.3% to 20% in the same period. During this time a new prescription had been introduced and a new national curriculum implemented. The papers were written with the purpose of consolidating both of these initiatives (NZQA, 1996). It is evident that the requirement for students to interpret information, explain, describe, and justify their mathematics clearly and concisely had received significant attention in both examination questions and the classroom. The Chief Marker commented that by 1997 candidates had gained increasing competence in responding to questions requiring written explanations and justifications.

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2 The issue of student expectations and interpretation of the number of designated lines is addressed in Chapter 6.

4.4 IN WHICH CONTENT STRANDS DO THE QUESTIONS APPEAR?

Table 4.2 gives the breakdown of marks allocated for the writing of explanations and justifications in each content strand. The total for each strand, as a percentage of the total marks allocated for writing explanations and justifications, is also provided in Figure 4.1.

Table 4.2 Marks Allocated to Explanations and Justifications in School Certificate Mathematics by Content Strand

Year	Number	Measurement	Geometry	Algebra	Statistics	Total Marks
1992	0	0	2	1	1	4
1993	0	0	1	0	1	2
1994	0	2	4	1	1	8
1995	0	1	7	1	0	9
1996	0	11	6	2	2	21
1997	5	7	4	3	5	24

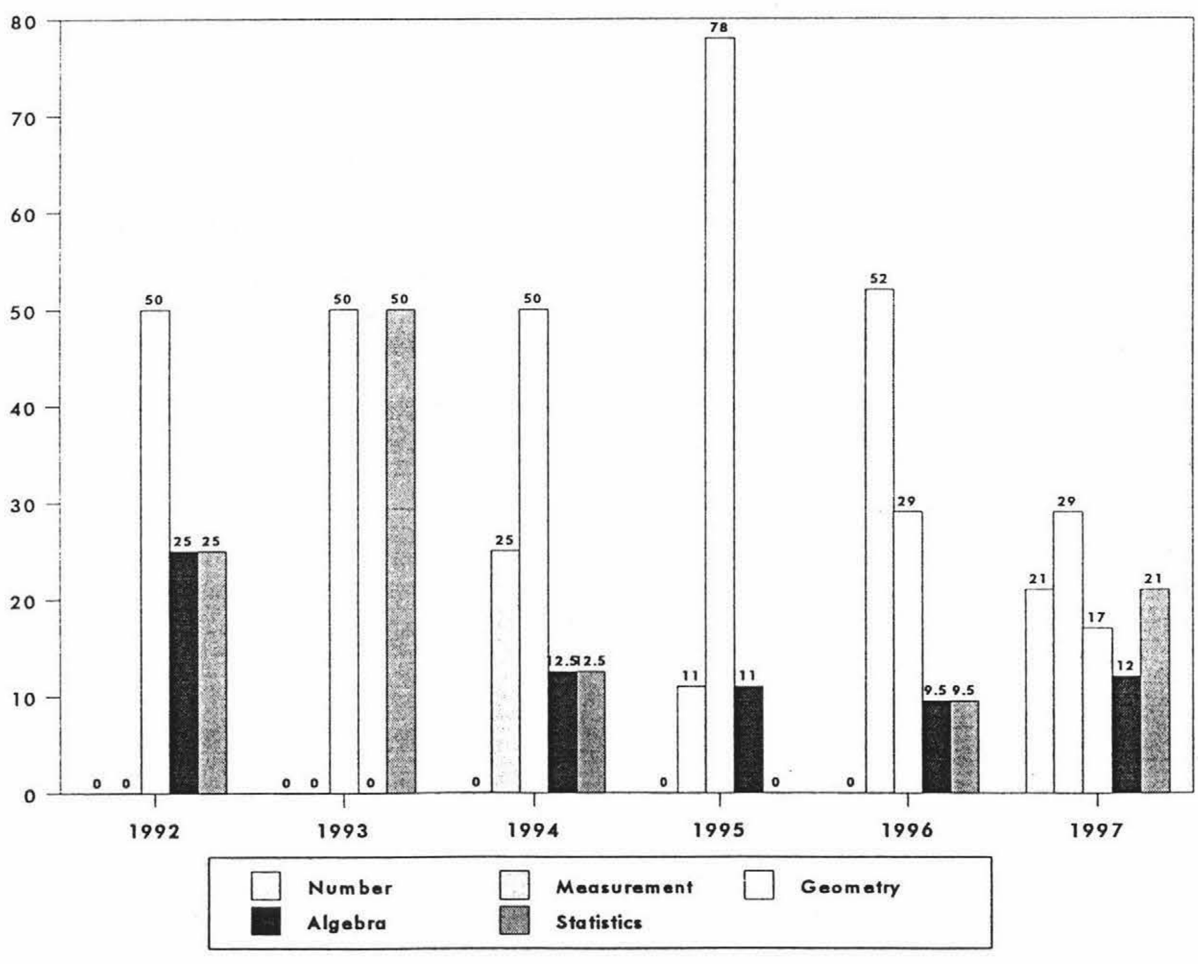


Figure 4.1 Percentage of total marks for each content strand, 1992-1997

In 1995 an 'investigation' task was introduced as an innovation. It was felt that: *investigative and creative thinking is an important part of the new curriculum, and it is important to provide an opportunity for such to occur in an examination* (NZQA, 1995, p. 1). The Chief Marker stated that: *An investigation can draw on any strand, can involve any one of many techniques, and could often be open-ended with no simple clear-cut conclusion* (NZQA, 1996, p. 2). The investigation question in 1995 drew mainly on the geometry strand, in 1996, the measurement strand, and in 1997, the statistics strand. Within these 'investigation' tasks a significant proportion of marks were allocated for the writing of explanations and justifications.

While in the past there was a greater proportion of the specific communication marks allocated to one particular strand, it would appear that a relative balance across the five content strands of number, measurement, geometry, algebra, and statistics was achieved in the most recent examination script.

4.5 WHAT PROMPTS ARE USED AND HOW WELL DID STUDENTS RESPOND TO THEM?

The objective of this aspect of the analysis was to collate information about the range and type of instruction used to signify the writing of an explanation or justification. Instructions appeared to be consistently clear and added emphasis was provided with the use of bold type and upper case script as exemplified. In Table 4.3 the various prompts used are listed with the frequency given in parentheses.

Table 4.3 Writing Prompts Used in School Certificate Mathematics for the Writing of Explanations and Justifications

1992:	*Explain why* *Give a reason why*
1993:	*Explain why (2)*
1994:	*Explain, using geometric reasons, why* *Explain why (2)* *Explain how* *Explain what your solution means* *Give a reason why*
1995:	*Clearly explain what* *Clearly explain why* *Give a reason why* ***Explain why*** ***CLEARLY EXPLAIN EVERYTHING THAT YOU DO** so that* ***someone else can understand you***
1996:	***CLEARLY EXPLAIN YOUR WORKING*** *Explain why (2)* ***Clearly explain** your answer* *Explain your answer* *Give a reason as to why* *Give two possible reasons* ***CLEARLY EXPLAIN EVERYTHING THAT YOU DO** so that* ***someone else can understand you*** *Explain what*
1997:	*Justify your answer* *Explain why (3)* *Explain what calculation must be done* *Clearly explain your working so that someone else can understand it* *Give a reason for your decision* *Justify*

It is evident that a greater number and variety of prompts have been introduced during this period of time. The earlier papers used the relatively simplistic prompts such as:

- *Explain why a histogram would not have been appropriate for showing these results (1992).*
- *Explain why x must be 72 (1993).*
- *Explain why Trudy is wrong (1993).*

In 1994, when the first significant increase in problems requiring written explanations appeared there was an increase, not only in weightings as previously shown, but also in the frequency and variety of prompts. A corresponding increase in the space allocated for responses occurred. The following example from the 1994 paper was the first communication problem to be allocated more than the traditional one mark.

13

Possible Mark	Leave Blank
3	

4. The pilot notices that on the map Bryton is the same distance from Deebridge as it is from Ahi.

   This is modelled in the diagram.

   Length BD = Length BA
   Lines SN and AD are parallel.

   Explain, using geometrical reasons, why angle p = angle r

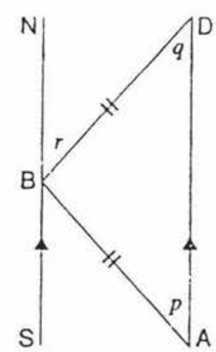


Figure 4.2 School Certificate, 1994, Question Two

It was noted in the Chief Marker's Report that very few students actually gained the three marks for this question.

In another question from the same paper (Figure 4.3) it was found that many candidates described how they did their calculation rather than interpret their solution.

30

QUESTION ELEVEN

Possible Mark	Leave Blank

(a) The north side of Rata St has odd-numbered houses.

House	Number
1st	1
2nd	3
3rd	5
.	.
.	.
.	.
n-th	$2n-1$
.	.

1. What is the number of the tenth house on the north side of Rata St?

   ______________________________ 1

2. The last odd-numbered house is 87.
   Find the solution of the equation $2n - 1 = 87$

   ____

   ____

   ______________________________ $n =$ ______________ 1

   Explain what your solution means.

   ____

   ____

   ______________________________ 1

Figure 4.3 School Certificate, 1994, Question Eleven

In the following year, 1995, there was increased use of bold print for key words and upper case lettering was used for the question prompting a more lengthy explanation. Photographs were included and a greater number of diagrams were used to help students understand or clarify the written text. It was hoped that these strategies, combined with the more contextual nature of the problems, would be *a lot more user friendly and inviting compared with those that are starkly mathematical in appearance and language* (NZQA, 1995, p. 1). The prompt: *Clearly explain everything that you do so that someone else can understand you* was introduced for the first time. This gave students the sense of writing for an audience. This type of problem certainly required more reading than the simple skill and straightforward application questions of preceding years. Despite the use of more detailed prompts and bold type to alert students to important aspects of the instructions, 15% of candidates still avoided the following question shown at Figure 4.4.

Explanations, according to the report, were vague and the quality often poor. For example *inside sheet* was given as an explanation of what was being calculated and *sheet equals* was given rather than *area of one inside sheet equals*. It was suggested that students use the format of writing the explanation before the calculation.

30

Possible Mark	Leave Blank
7	

QUESTION ELEVEN

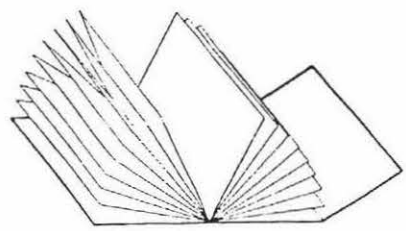
Sheets of paper have a **g.s.m.** rating, where **g.s.m.** stands for grams per square metre. This means that one square metre of 45 g.s.m. paper will weigh 45 grams.

This Mathematics examination booklet is made from 2 types of paper.

- The **cover** uses 85 g.s.m. paper, and folds out to measure 0.30 m by 0.60 m.

  This means that the cover has an area of 0.18 m^2, and weighs about 15.3 g.

- The **8 inside sheets** are all 70 g.s.m. paper, and each sheet measures 0.30 m by 0.42 m.



The number of people sitting this examination is approximately 41 000 (2 s.f.).

Calculate the approximate weight of paper used to produce enough copies of this Mathematics examination booklet for this number of people.

To gain full marks, you must

- **State what you are calculating at each step, and show your working.** [4 marks]
- **Use sensible units in your answer.** [1 mark]
- **Round your answer appropriately, stating your degree of accuracy.** [1 mark]
- **Explain why you have chosen this degree of accuracy.** [1 mark]

Figure 4.4 School Certificate, 1995, Question Eleven

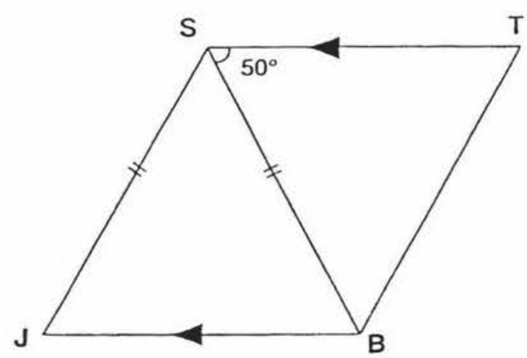
Generally, the students did not respond well to the problems which required an explanation as in this example (Figure 4.5) from the same examination paper:

25

Possible Mark	Leave Blank
3	

(b) This diagram (which is not drawn accurately) shows Sam's position at 2 pm relative to the 3 landmarks beside the lake.

Sam is the same distance from the Beacon as he is from the Jetty, and he is due west of the TV Tower.
The Jetty is due west of the Beacon.
The angle BST is 50°.



1. Explain, with geometrical reasons, why angle BSJ must equal 80°.

Figure 4.5 School Certificate, 1995, Question Eight

According to the Chief Marker's Report this was one of the most poorly attempted questions in the examination. Students displayed poor geometric reasoning and explanation:

> *The examination did not require candidates to produce intricate geometric proofs, but it did require clear and simple explanations (of up to 2 or 3 steps) as to why a particular geometric relationship or value exists.*
>
> (NZQA, 1995, p. 3)

Common errors were:

- *Poor notation (eg. Angle AJ).*
- *Vague reasons (e.g. Angle B with parallel lines).*
- *Circular arguments.*
- *Wordy explanations rather than concise reasoning (good clear answers only used 3 lines for 3 marks).*
- *Referring to every angle rule that might be relevant in the vain hope of picking up a mark.*

(NZQA, 1995, p. 13)

While it was noted that there is no requirement (according to the prescription) to produce a formal proof, the ability to reason clearly and logically is important. It was suggested that the ability to make and justify geometric connections needed more attention in class.

In 1996, with an increase in the number of questions requiring written explanations and justifications, came another increase in the number and range of prompts. The use of bold type and upper case letters in the prompts were retained. Overall it was noted by the Chief Marker that there was an improvement in the thinking displayed and the presentation of answers. However, the Chief Marker commented that many students made no effort when asked to *explain your working so that someone else can understand you*. The use of this prompt did not achieve the desired result of inspiring clearly written responses. While there was an expectation that students had to provide brief rather than detailed responses, many students still displayed an inability to set out their thinking in a clear and reasoned manner. It was noted that: *Communication skills are improving but students need to have more practice at writing about their mathematics in a concise, precise and unambiguous manner* (NZQA, 1996, p. 5). It was suggested that students need to write answers that are less vague, wordy or shallow and practise making right answers better.

In 1997, bold type and capitalization was no longer used for the key writing prompts. Another significant change was the introduction of the prompt *justify your answer*. This was the first time students had been asked specifically to write a justification. It was used in the first question of Section B, the written section of the paper, and is reproduced below (Figure 4.6).

(b) The *Talking Points* that you earn can be saved for 3 years.
Sharon wants to get the *Audiovox 405 mobile phone* reward.
In her first 4 months, Sharon earned:
1780 points, 1550 points, 1600 points and 1650 points.

Do you think Sharon will earn the mobile phone within the 3 years?
Justify your answer.

4

Figure 4.6 School Certificate, 1997, Question One

Many students, according to the Chief Marker's Report, did not answer this well; omitting important detail in their justification. However, it was noted that overall, students were showing greater understanding of the prompts *explain* and *justify* and were able to communicate in a more precise and clear manner.

4.6 SUMMARY

A content analysis of School Certificate mathematics papers for the six year period, 1992-1997, revealed that an increased emphasis was placed on the communication of mathematical findings, specifically the writing of explanations and justifications. During this period the percentage of total examination marks allocated to the writing of explanations and justifications increased from 2.7% to 16%.

The questions requiring candidates to write explanations and justifications had been based primarily on either the measurement or geometry strand although the most recent examination achieved a relative balance across the five strands: number, measurement, geometry, algebra, and statistics.

There was an increase in the frequency and variety of prompts used in the examination papers during this period. The initial direction to *explain why* was replaced in subsequent years by the more elaborate direction to *clearly explain everything that you do so that someone else can understand you.* In 1995, bold type and upper case lettering was used to add emphasis and to alert students to important information and requirements of the task. This strategy was no longer used in most of the writing prompts of the 1997 examination script.

By 1994 emphasis on the writing of explanations had increased significantly. However, the Chief Marker found that *many otherwise well prepared candidates were unable to score marks in the problem solving questions* (NZQA, 1994, p. 3). Students continued to provide poorly written explanations in the 1995 and 1996 examination scripts. It was not until 1997 that significant improvement in the writing of explanations was noted. However, a new requirement had appeared! This was the need to write justifications; a requirement that students did not respond well to.

CHAPTER 5: STUDENT RESPONSES

5.1 INTRODUCTION

This chapter provides a qualitative analysis of students' responses to the problem solving tasks. There are 36 student responses in the sample, 18 from each of the two classes involved in the study. The objective of the tasks was to give students problems to solve so that their abilities to write explanations and justifications in a range of mathematical contexts could be examined. Relatively open-ended problems were posed, so that instead of focusing on factual knowledge and routine application of procedures, students had an opportunity to read, interpret, and communicate their understandings.

Five tasks were chosen, each based on one of the content strands: measurement, number, statistics, algebra, and geometry. The material had been taught in class and the problems were deemed by the teachers to be at an appropriate level for the students in the study. The problems were based on School Certificate problems and therefore covered material that had been selected by the examining authority as important. A range of prompts was carefully chosen so that each problem would generate a different type of response. The cognitive complexity of each task is examined and the features of each problem that relate to the research topic are outlined.

5.2 PROBLEM ANALYSIS

The students' solutions were analysed to see if the problems were interpreted in the intended ways; if the intended content, representations, and processes were evoked. This qualitative analysis of student responses was modelled on that used in the QUASAR project. This project uses the QUASAR Cognitive Assessment Instrument (QCAI) to measure cognitive skills by having students produce answers, show their working, and explain or justify their answers. The following categories: mode of representation, solution strategies, quality of mathematical arguments, and mathematical errors are used to qualitatively analyse students responses to assessment tasks (Magone, Cai, Silver & Wang, 1994). Not all of these categories have been used for the solutions to each problem. However, it provided a useful framework for considering the quality of students' responses.

Also included in the analysis is information obtained from follow-up interviews. Fourteen students were interviewed. Their selection was based on the nature and variety of their responses (see Chapter 6.1). The students' solutions were available for reference during the interview so that students could retrospectively elaborate on certain aspects of their solutions if they wished to. These reflective comments are included in the following discussion in which each problem is addressed separately. Common issues which arise are presented in the following chapter.

PROBLEM 1

Derryn investigated the packaging of snack bars. She measured the packet with a ruler and found that it was 16.5 cm long, 9.2cm high, and 4.6 cm wide.

She calculated the volume to be 698 cm^3 (3sf).

The packet had 6 muesli bars in it.
Each muesli bar was 7.5 cm long, 4 cm wide but the thickness of the bars varied between 2.6 cm and 2.8 cm.

Find the volume of the 6 muesli bars as a percentage of the volume of the packet.

- ***Explain what you are calculating at each step and show your working.***
- ***Round your answer appropriately, stating the degree of accuracy.***
- ***Justify why you have chosen this degree of accuracy.***

Figure: 5.1 The measurement problem

This problem was based on Question Five of the 1996 School Certificate examination. Mathematically, the problem first of all required students to deal with the varying thickness of the muesli bars. Students needed to state or imply an average or typical thickness. Explanations about the necessary calculations were expected, including their method of calculation of thickness, volume, and percentage. Finally, a justification for the choice of degree of accuracy was required.

The responses were analysed for the overall quality of the mathematical argument; the mathematical correctness, and whether an explanation and justification had been provided.

Table 5.1 Students' Mathematical Arguments for Problem 1

Quality of the Mathematical Argument	Frequency
Mathematically correct, explained, justified	7
Mathematically correct, explained, not justified	19
Mathematically correct, not explained, justified	0
Mathematically correct, not explained, not justified	2
Mathematically incorrect, explained, justified	4
Mathematically incorrect, explained, not justified	2
Mathematically incorrect, not explained, justified	0
Mathematically incorrect, not explained, not justified	2

A significant proportion (72%) of the students were able to provide a mathematically correct and well explained answer. However many (69%) either failed to attempt to write a justification or were unable to provide a quality justification.

The responses were also analysed according to the mode of representation in the explanation and the mathematical basis of the justification.

The students' modes of written representations were categorised as: symbols only, symbols and words, or a combination of symbols, words, and diagrams. The quality of the explanations varied considerably. Only three of the students provided a response using symbols only and devoid of any verbal explanation of what they were actually calculating. The remainder all included verbal explanations with 56% of the students writing an explanation using symbols and words. A combination of symbols, words, and diagrams was the mode of representation chosen by 36% of the students.

Kim's response (Figure 5.2) is an example of an explanation using symbols only.

$7.5 \times 4 \times 2.6 = 78$

$7.5 \times 4 \times 2.8 = \underline{+84}$

(average) $162 \div 2 = 81 \times 6$

$\frac{486}{698} = 69.6\%$ is muesli.

Figure: 5.2 An example of an explanation using symbols

In the follow-up interview Kim reflects on her answer: *I'm not quite sure whether I've explained it. It's a bit messy and I'm missing the labels so people won't know what I'm doing. I don't explain at the start what I'm doing.*

Jessica's answer (Figure 5.3) is an example of a comprehensively written explanation using symbols and words.[3]

Firstly, the volume of the 6 muesli bars have to be found. As the thickness of the bars varied between 2.6cm and 2.8cm the average thickness has to be found. To do this you minus the smaller thickness from the larger thickness and divide by two, the add the smaller thickness as seen below

$$\frac{\text{larger thickness} - \text{smaller thickness}}{2} + \text{smaller thickness} = \text{average thickness}$$

$$\frac{2.8\text{cm} - 2.6\text{cm}}{2} + 2.6\text{cm} = \text{average thickness}$$

$$\frac{0.2}{2} + 2.6 = 2.7\text{cm}$$

Now that an average thickness has been found the volume of one muesli bar can be found, then multiplied by six to get the total volume of the 6 muesli bars

volume of 1 muesli bar = length × width × thickness

= 7.5cm × 4cm × 2.7cm

= 81 cm^3

To find the volume of the 6 muesli bars you now multiply the answer from the volume of 1 muesli bar by 6

Volume of 6 muesli bars = volume of 1 muesli bar × 6

Figure: 5.3 An example of an explanation using symbols and words

3 Only part of Jessica's explanation is included.

However, the interview with Jessica revealed that she was unsure whether this is what was expected:

> *I don't know if this is a good explanation. I probably write too much but I want to get it all across. I could simplify it. We've been told to write out more so it explains it. I don't want to leave anything out.*

Many of the students included a diagram in their response. Jan's is one example (Figure 5.4):

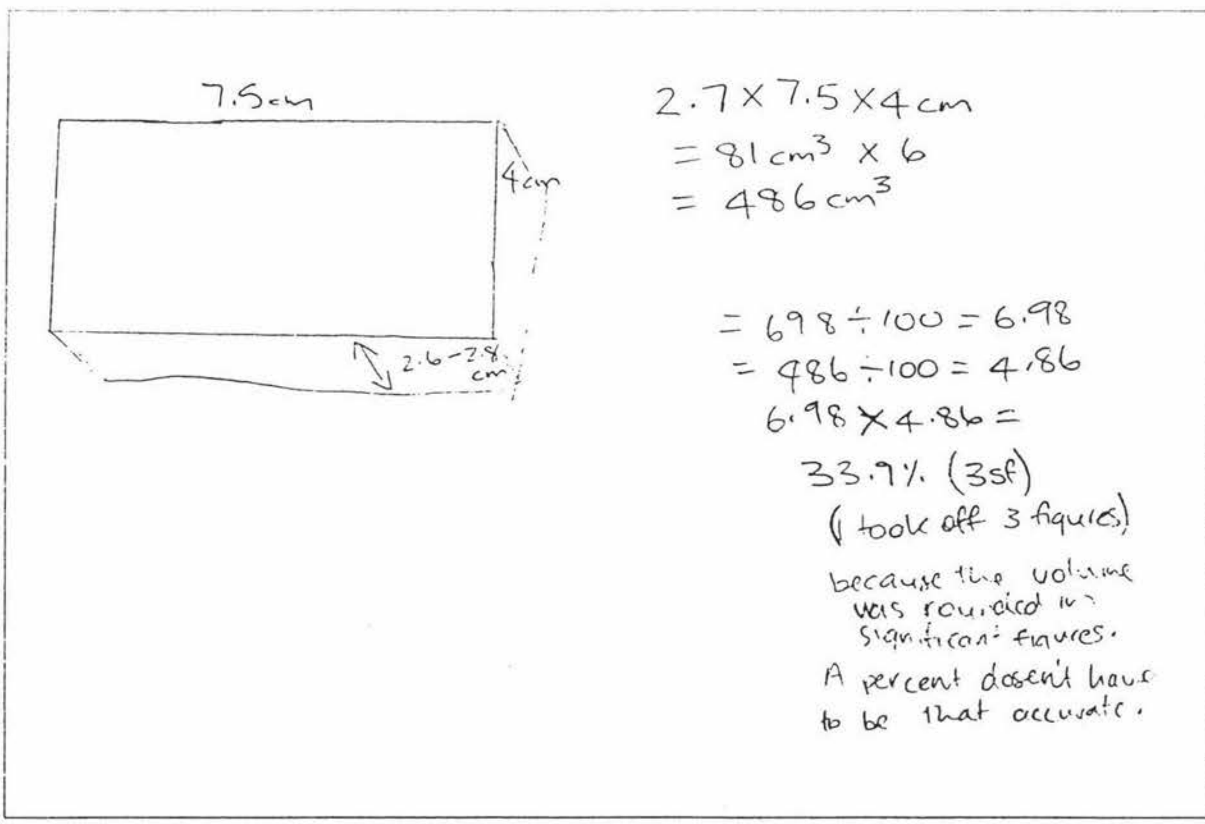


Figure: 5.4 An example of an explanation which includes a diagram

In the interview Jan is clearly able to justify her use of diagrams: *I like using diagrams, it helps you know which bit you're talking about.* Anne also comments positively on the use of diagrams: *A scale-type drawing gives you a better picture of what you are doing. It shows you how to find things and if you are thinking appropriately.*

There were common concerns about not knowing whether an explanation was satisfactory nor whether enough had been written in the explanation.

Amy: *I don't feel confident about it.*

Jan: *I probably could write more about what I was actually doing - explain more about what I was doing.*

Richard: *I think that I've done it all right but I wouldn't be certain if I've done it how it's supposed to be done.*

The written responses were analysed as to whether a justification had been provided. Only 11 students provided a justification and their justifications included a range of mathematical understandings. The main reasons included: percentages are best recorded in whole numbers (two responses), other measurements are to one decimal place (d.p.) therefore results should be to one decimal place (four responses), and three significant figures are used for other measurements in the problem so the result should be to three significant figures (two responses). Other mathematical reasons were:

Louise: *I would round this up to 70% because of the variable thickness of the bars.*

Jane: *I have the degree of accuracy of one decimal place because it's easier to tell what or how close it is to a whole number.*

Martin: *2 d.p. is a standard degree of accuracy.*

The majority of students did not provide a justification. Interviewed students who had not written a justification expressed a lack of understanding of what it means to justify, and of the mathematical term, 'degree of accuracy'. Either of these misunderstandings could have been contributing factors to the majority of students failing to justify their choice for a particular degree of accuracy. Even among students who attempted the justification there was evidence of a lack of understanding of the concept of 'degree of accuracy'.

Jan: *'Degree of accuracy' means significant figures, how scientific you're going to be.*

John: *It means to explain why I put one significant figure and I don't know how to do that kind of thing.*

Kim: *You can't really get .689712% sort of thing, that's a bit hard. It's easier with one decimal place to put it into a pie or bar graph.*

When interviewed, students gave the following reasons for not writing a justification:

Amy: *I read it but skipped the justification as I probably didn't see it as important. We probably haven't done that in class when we've practised problems. I'm just not used to justifying in the written form.*

Kim: *I haven't got a clue what it means to justify.*

John: *I don't know how to do that kind of thing.*

Anthony: *I'm just not used to justifying in the written form.*

PROBLEM 2

Cash Flow is selling new CD players for $695. They are offering customers two deals.

(i) Trade in your old stereo for $200, or
(ii) get a 20% discount for cash.

Paul can sell his stereo for $75 to a friend. Should Paul trade in his old stereo to buy the new one or sell it and pay cash for the new one?

In order to get the best deal which decision should Paul make? **Justify your answer.**

Figure: 5.5 The number problem

This problem was based on a question from the 1996 moderation test for the internally assessed candidates sitting School Certificate. A similar problem is also found in 'National Curriculum Mathematics: Level 6' (Catley & Tipler, 1997, p. 130). The story context required students to interpret the two options available to Paul.

The students' solutions were analysed to determine whether a justification was given and the bases for the justifications. Justifications were provided by 28 students (78%). However, many of these students (17) based this justification on incorrect calculations. Some of the justifications were based on reasons other than a mathematical argument. For example:

Nancy: *If I was him, I would trade in my old one for $200, get a new one and I could even get to pay it off on hire-purchase.*

Clare: *It means he doesn't have to pay cash if he trades in his old stereo.*

Paula: *If he sells it to a friend he is only receiving $75 and no longer has a stereo. If he pays cash and receives a 20% discount he is only getting $139 off the price, whereas if he trades in his old one the price falls to $495 and he still has a stereo.*

Some of the interviewed students who did not justify their answer for this problem explain why:

Kim: *I didn't really think about what justify meant. I thought I'd just better get on.*

John: *I thought justify was just to put down what he would do.*

Merryn: *I haven't really justified it. I have to say why I did it and prove that it's cheaper to trade.*

However, some students when interviewed, appeared more knowledgeable and critical of their approaches to the writing of justifications:

Mark: *A good justification would be saying there would be three different options, having all the options. It's the conclusion - saying which is the best choice and why. I didn't do that completely.*

Charles: *It said justify your answer so I thought that I had to write everything out again. Looking at it,it's probably not a very good setting out... The justification is basically telling them how you got it and what would be the best one. I didn't really justify my answer, I just wrote down what I did.*

The students' justifications highlighted a lack of confidence and understanding of what is meant by a justification in this context. There were also misunderstandings and differing interpretations of the problem. For example, some simplified the problem, including either two or three options while others muddled the alternatives. Two students showed no calculations. It would seem that the context or setting of the problem and the language used encouraged students to use real-life criteria in favour of the anticipated mathematical calculations.

Kim: *I had a question. Could he sell it to his friend and then get the 20% off? It didn't really say that.*

Mark: *I got mixed up in this one. He could either trade in his old one for $200, or get a 20% discount, or give it to his friend and get a cash discount. Oh, there's three options.*

Richard: *A lot of stuff gets in the way. The way they word things is stupid. They just word it really difficult to make it harder for you to understand. That's the worst part of it. It's the final question that makes it confusing. I think that they put in a whole lot of contexts to wind your brain around and then they ask you the final question and you just think what's* **that** *got to do with everything else.*

PROBLEM 3

James and Richard are gathering data for a survey on students' lunchtime eating habits. They wish to find the percentage of students who buy their lunch from the school canteen.

State the two items of data they need to gather and **explain** what calculation they need to make in order to be able to calculate the percentage of students who buy their lunch from the school canteen.

Figure: 5.6 The statistics problem

This problem was based on Question Three of the 1997 School Certificate examination. In order to answer the question the students firstly had to extract the two items of data needed and in their response convey the idea of quantity. Secondly, they had to give a process description explaining the calculation.

The majority of students (89%) were able to clearly state the two items of data. The difficulty of defining and interpreting the scope of the explanation was illustrated by the number of students (7) who elaborated on how the data was to be collected. These ideas related to how to conduct a survey, data gathering techniques, and finding averages:

Louise: *They would have to find out how many people were in the school and then find out on each day (M,T,W, T, F) for a week how many people bought their lunch from the school canteen and average this number.*

Other suggestions were to randomly choose 100 or 200 students on a particular day and from that sample calculate how many bought their lunch. These students described quite clearly the process of conducting a survey and how a large sample should be used to get *more accuracy*.

Some of the students complicated the problem by focusing on issues related to context:

Anne: *They need to find out if they provide their own money in order to buy their lunch - if they are in a hostel situation type thing or flatting, or if they receive their money from parents in order to buy their lunch.*

Nancy: *Over a few weeks they need to keep receipts of what the students bought and work out how many of them actually bought their lunch or did they buy just lollies, chocolate bars etc. They need to find out if people buy on a regular basis (e.g. like every Friday) or whether they just buy when they have the money.*

When the explanations were categorised according to the mode of representation, namely: the use of words, symbols, algebraic expressions, and whether a worked example was used, the following distribution resulted:

Table 5.2 Students' Modes of Representation for Problem 3

Mode of Representation	Frequency
Explanation using words only	11
Explanation using words and symbols	15
Explanation using an algebraic statement	2
Explanation using words and an algebraic statement	2
Use of a worked example	1
No explanation	5

Many of the students (30%) provided a verbal explanation but the preferred mode of representation was to combine both words and symbols.

Divide the number of people who buy their lunch buy the total number of students and multiply by a hundred to give you a percentage of people who buy their lunch

Figure: 5.7 An example of a verbal explanation

Richard explains why he wrote his explanation exclusively in words: *I thought that I'd get more marks writing it down like that, but again I probably didn't really know. I saw it as a 50% chance of getting it right.*

A group of students (42%) gave a verbal explanation which also incorporated symbols. An example is shown below:

The two items of data James and Richard need are
① the number of students who buy their lunch from the school canteen
AND
② the total number of students in the survey

Once they have collected their data they need to divide the number of students who buy their lunch from the school canteen by the total number of students in the survey then multiply that answer by 100

$$\text{Percentage of students who buy their lunch from the school canteen} = \frac{\text{no. of students who buy lunch from school canteen}}{\text{Total no. of students surveyed}} \times 100$$

Figure: 5.8 An example of an explanation combining words and symbols

Paula explains why she combined the two approaches:

> *I remember that I had so much to say and I didn't know how to put it properly so I just wrote down the whole lot. I replaced what I said with an equation. Often when you read a text like that you try to imagine what the equation is going to look like. I probably wrote a bit too much but I think that in order to make sure that you've got what the marker requires you need to write it all down.*

Two students used only algebraic statements but explained the meaning of each variable, whilst five of the students did not attempt to explain the required calculation.

The students interviewed expressed concern about not really knowing if they had provided an adequate explanation for this problem. Amy had provided a comprehensive verbal and algebraic answer yet expressed some doubt about whether it was what was expected.

Amy: *I'm not sure that I did it right because it seemed too simple when I wrote it down. I think that it was a good explanation.*

In contrast, Mark believed quite confidently that his solution (Figure 5.9) was appropriate.

> They need to know the total number of people at
> the school and then find out how many student
> buy from the school canteen.
> Then they will have to find the perctage of students
> buying lunch ~~[illegible]~~ and percentage of students
> not buying from canteen.

Figure: 5.9 An example of an incomplete verbal explanation

Mark: *I answered the question because it didn't say solve it, it just says explain your calculations and I've done that and I've written it in sentence form.*

PROBLEM 4

The tickets for a Show at the Regent Theatre cost \$48 for an adult and \$30 for a child.

(a) What would it cost a group of x adults and y children to get in to the Show?

Cost = ______ x + _______ y

(b) A total of 500 tickets were sold altogether totalling \$21 840. If **n** represents the number of adults at the Show, then we can write the equation

$$48n + 30(500 - n) = 21\,840$$

Explain what the expression (500 - n) represents.

Figure: 5.10 The algebra problem

This problem was modelled on Question Twelve from School Certificate, 1996. The task was made complex by the use of differing variables. Initially x and y are used to represent adults and children respectively, and then in the second part the variable n is introduced to represent the number of adults at the show. A further problem for the students is the interpretation of the instruction. The prompt *explain* is not in itself, viewed as problematic by the students, but the whole expression: *Explain what the expression represents* is. The key word *represents* implies that students need to correctly interpret the expression.

Despite the complexity of the problem, 11 students (31%) gave a correct mathematical interpretation with a well-explained response. However, nearly half of the students merely translated the expression (500-n) into *500 minus the number of adults* thereby making no interpretation. These findings parallel those of the 1996 School Certificate examination results in which *fewer than half correctly **interpreted** the expression* (NZQA,1996, p. 18). When further questioned about this problem these students did not recognize any shortcomings in their responses. Charles, however, retrospectively registered an understanding of the instruction: *It's simply writing down what that means, converting it into words. It's 500 minus the number of adults, oh you mean what's left - the children!*

A group of eight students either misinterpreted the problem or made no attempt to answer the question. Of those interviewed, some felt that they should have solved an equation and others were genuinely confused by the use of different variables and the wording of the question:

Anne: *I had to get my head around what it was saying. I had to remember what x was and 'number of', and x and y which is probably the number of people. Then it talked about n, it didn't say anything about children, it just says the number of adults.*

Kim: *It's a trick question.*

Megan: *That was a hard question. It took me a while to work it out. It was harder to understand. We had to explain what* ***that*** (the expression) *means.*

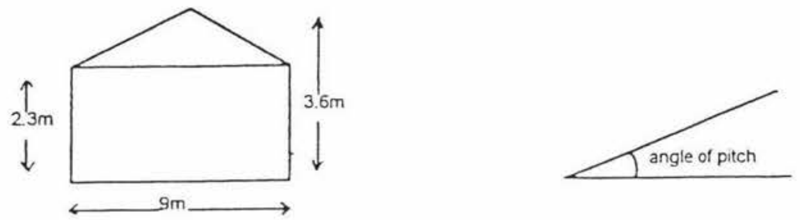
Fay: *I've never had to deal with 'explain what the expression represents'. I just couldn't think of what it meant. I remember reading it through about ten times and I just couldn't get it. I've never had to explain anything like that before, you just solve it.*

Mark: *It was my reading skills. I went ahead and did what I thought was wanted but I was confused.*

This problem provided difficulties for the students in interpreting the prompt; *explain what the expression represents*. Instead of simply giving an explanation, an interpretation was required. Students who were interviewed explained that they felt inexperienced at having to explain algebraic **expressions** in this way; their more common experience was to solve an algebraic **equation**.

PROBLEM 5

A new sports shed is to be built at the school. The walls are to be 9m wide, have 2.3m high walls, and be symmetrical in shape with the greatest height of 3.6m. The Council will give approval for the building as long as the angle of pitch of the roof is greater than 16°.



(Diagram is not to scale)

Use trigonometry to help you decide if the shed will be given Council approval.

Clearly explain your working so that someone else can understand it.
Justify your decision.

Figure: 5.11 The geometry problem

This problem was based on Question Five of the 1997 School Certificate examination; a question that *was disappointingly omitted by many candidates* (NZQA, 1997, p. 9). In this problem the students were guided by the instruction; *Use trigonometry to help you,* and were required to give a detailed explanation and justification. Bold type was used to alert students to the requirements. A supporting diagram showing what is meant by 'angle of pitch' was provided, but the required angle had to be extracted from the text, and height of the roof calculated using information provided both verbally and diagrammatically.

Initially students' responses to this problem were analysed for the quality of the mathematical argument.

Table 5.3 Students' Mathematical Arguments for Problem 5

Quality of the Mathematical Argument	Frequency
Mathematically correct, explained, justified	11
Mathematically correct, explained, not justified	8
Mathematically correct, not explained, justified	0
Mathematically correct, not explained, not justified	0
Mathematically incorrect, explained, justified	5
Mathematically incorrect, explained, not justified	9
Mathematically incorrect, not explained, justified	0
Mathematically incorrect, not explained, not justified	3

Only 11 (31%) of the students managed to correctly provide a comprehensive and correct response which included both an explanation and a justification. Nearly half of the students had difficulty with the mathematical content of the problem. They were confused about which trigonometry ratio to use and some students failed to halve the base length. Some students created more work for themselves, using Pythagoras' Theorem to find the hypoteneuse so that they could use the sine or cosine ratio.

Kim: *I used cos cause it's the only one that I could think of at the time. I'd forgotten the others.*

Jan: *I don't know what angle I was trying to find. The angles are both the same because it's symmetrical. I think I was trying to find that distance* (hypoteneuse) *so that I could then find the angle.*

Some mathematical errors were caused by incorrect use of the calculator and a few students rounded solutions inappropriately, leading to different answers from what was intended.

The students' responses were then analysed according to the mode of representation. All students used a diagram in order to extract the required information to help solve the problem and so the following two categories were used: explanation using symbols, and explanation using symbols and words. Responses with verbal labels were included in the second category. More students (58%) provided an explanation using symbols exclusively than an explanation using words and symbols. Helen's explanation (Figure 5.12) is an example of a typical explanation using symbols only.

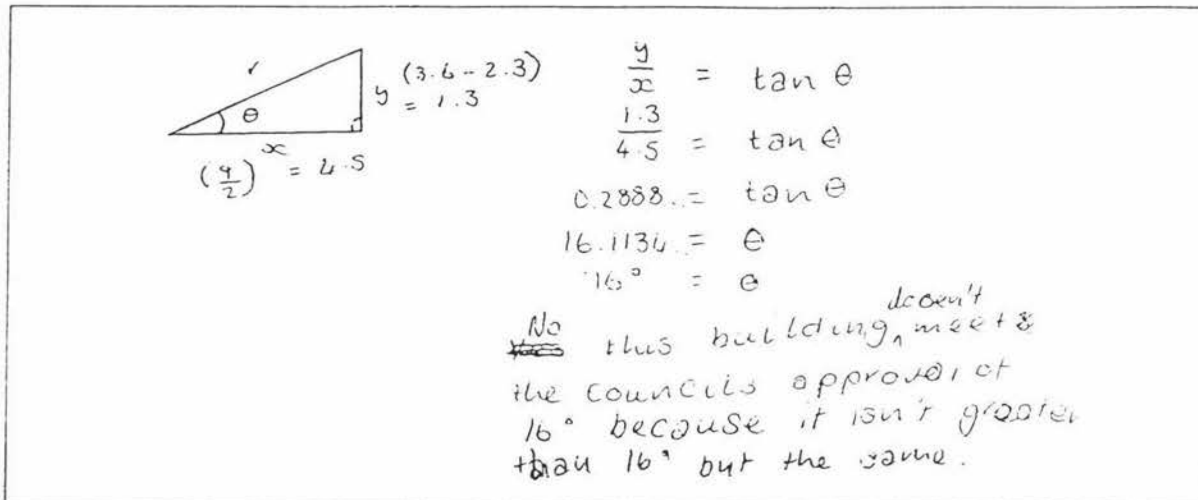


Figure: 5.12 An example of an explanation using symbols

Anne's explanation (Figure 5.13) shows how some students were prepared to offer a verbal explanation supporting their equations and calculations.

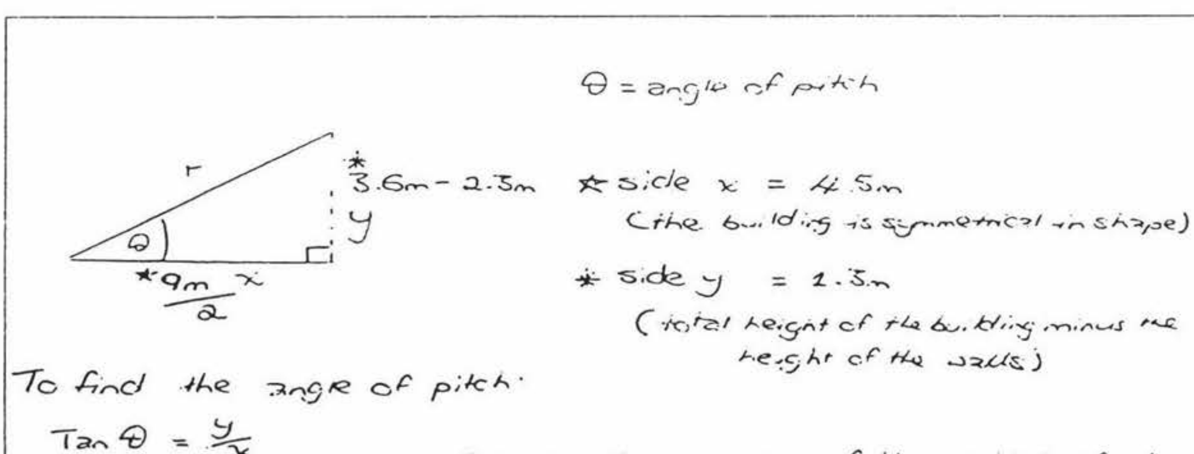


Figure: 5.13 An example of an explanation using symbols, words, and a diagram

Charles explains his strategy which results in a response using symbols and words. *I do this first* (the calculations) *because you can adjust this easier than you can the words. I check that I've got it right and then I write the words.*

Less than half of the students were able to provide a justification for this problem. The justification had to include a clearly stated comparison with the required pitch (of 16°). Of the 16 justifications provided by students, three were incorrect. Instead of providing a justification, many students simply stated their decision: *The council will give approval.* These decisions were not accepted as a justification. Retrospectively, the interviewed students explained how they felt about the question and expressed their concern about not knowing what is required by a justification.

Anthony: *I've got all the working there but I'm not sure whether this is a justification or not. If it's a justification, I'd need to say whether it's over 16 degrees which it is, so maybe I did justify it.*

Merryn: *I've explained what I've done but I haven't justified it. I haven't said why, I've just said 'yes they can'.*

Richard: *I think it would be a justification but I wouldn't be certain. I really don't know how far you should go when justifying.*

Those who justified their decision provided some very interesting differing interpretations about council approval. Once again, context appeared to influence students' interpretations:

Jessica: *The council will probably not give approval for the building because it is only 0.1 greater than 16 degrees.*

Jessica was not alone in believing that the council would not approve the building because the calculation had shown the angle to be **only** 0.1° greater than the required 16 degrees. Another misunderstanding, shown by four of the students, was interpreting the required angle as less than 16 degrees. Two students commented that the council would **just** give approval as it was so close to the required 16 degrees. If the calculation had given a result, not as close to the required 16 degrees, than these differing interpretations may not have surfaced.

5.3 SUMMARY

Five problems based on School Certificate questions were solved by the students. Solving the problems required use of knowledge and skills from each of the content strands of the curriculum: measurement, number, statistics, algebra, and geometry. The students' responses were analysed by the researcher in a variety of ways based on the prompts given in the questions and the quality of the students' mathematical arguments.

The first problem, using knowledge and skills from the measurement strand, required students to write both an explanation and a justification. The majority of students used diagrams, symbols, and words in their explanations. However, most students did not justify the chosen degree of accuracy for their answer.

The solutions to the second problem, based on the number strand, confirmed that students have difficulty in writing justifications. The context of this problem presented difficulties for students and lead to a variety of interpretations.

The statistics problem, Problem 3, generated a range of written responses. Students perceived it to be a relatively simple problem, but those interviewed expressed doubt about whether they had provided an adequate explanation. Providing a correct response to the algebra problem, Problem 4, caused difficulty for many of the students. The majority of the students had difficulty interpreting the prompt *explain what the expression represents.*

The last question which required students to use trigonometry skills and to write an explanation and a justification, generated a range of mathematical arguments. Students found this a reasonably challenging problem mathematically. Most presented their solutions using diagrams and symbols, but failed to provide a detailed explanation as prompted. The majority of students did not provide the necessary justification for their answer.

The common themes identified from these results and the results of Chapters 6 and 7 are discussed in Chapter 8.

CHAPTER 6: STUDENT VIEWS

6.1 INTRODUCTION

This chapter reports on findings ascertained from interviewing a sample of 14 students selected from those who answered the set of problems given in class. Seven students, from each of the classes that participated in the study, were selected based on the nature of their responses to the written problems. The range of students' responses included elaborate or skeletal responses, interesting or unusual interpretations, and non-responses. Gender differences were not being explored, although a gender balance relative to the number of female and male participants in the whole study was achieved.

The general structure of the interview was in two parts. The first part consisted of a set of structured questions designed to encourage students to articulate their attitudes towards, and beliefs about, writing in mathematics. The first part of the interview was based on the following questions:

- How do students feel about having to write explanations and justifications?
- What difficulties, if any, do the students perceive in having to write explanations and justifications?
- For what purpose do the students think they are required to write explanations and justifications?
- How well prepared do they feel they are to answer this type of question?
- What do they think makes a quality response to a question requiring on explanation or justification?
- How do they think writing in mathematics differs from writing in English?[4]

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[4] English is taken to mean the school curriculum subject.

6.2 STUDENTS' FEELINGS ABOUT THE WRITING OF EXPLANATIONS AND JUSTIFICATIONS

The majority of the students interviewed (86%) felt that the writing of explanations and justifications in mathematics was important. The students collectively identified three distinct reasons supporting their belief. These were: to help develop their thinking skills in mathematics; to acquire skills useful in 'real-life'; and to convey to the teacher or examiner knowledge, skills, and understandings that they had acquired.

In examining students' comments it became apparent that they believed that the writing of explanations and justifications helped increase their mathematical understanding and metacognition.

Paula: *It's a little unexpected having to write it down but it is important because it personally helps me understand it more. I know what I'm doing if I write it down. It's easier to look back and say: 'I know why I put that number there'.*

Anne: *It makes you have to think about what you're actually doing. That, sometimes helps you understand and puts you back on track.*

Paula explains what the process means for her:

> *I think it's good because if you can go through the processes and write down what you did, it probably makes it easier for you to come out with an answer at the end because you're able to say: 'You have to do this and you have to do that.'. Then it comes together and if you realize that you've done something wrong then you can read back through what you've written to see if it's what you've written down or if it's a mistake you've made with your numbers. So yeah, it helps a lot!*

This viewpoint was supported by other students who also acknowledged that it helped them identify possible errors and track down specifically where they went wrong in the process of explaining an answer.

Kim: *If you get things wrong you can go back through and catch on to where you've gone wrong. It helps you to go through the steps. If you just write down an answer you don't know where you've gone wrong. It's better than just writing down an answer.*

For Jessica, having to write a detailed explanation helped confirm the correctness of an answer:

> *You know that you understand it if you're able to write out an explanation.*

Many students acknowledged that writing out an explanation also helped them remember what they had learned.

Paula: *It's revision, too, it jogs my memory about how to do it rather than just writing down the answer.*

Some of the students clearly identified benefits, outside of the mathematics classroom, from being able to write explanations and justifications.

Mark: *It's pretty important as it shows that you understand the question and in the workforce they're wouldn't give it to you as a question just with the numbers. They'd want you to show some understanding and put all the pieces together.*

Jessica: *If you are putting it to work in daily life you can't just write out answers.*

John: *You need to be able to write explanations and justifications for jobs and work these days, they don't just want the straight answer. They try to find out if you know what you've got to do with the answer. It makes you think about it more.*

There was an awareness of writing for an audience; that audience being the teacher, or the marker of a test or examination. From this, according to the students, teachers are able to evaluate the effectiveness of their teaching and possibly identify difficulties that students may be having.

Amy: *It's so the teacher knows that you understand what they've taught you.*

Mark: *You have to put in more information so that the teacher can see where you've gone wrong.*

Paula: *It helps the teacher know how much the student understands. They can pick up where a student has made a mistake and go back over and help correct it - help the student fix up where they are confused, or what they've got wrong.*

In the context of examinations, students are aware of writing for a marker and how more marks can be gained by giving detailed explanations.

As Richard explains:

> *That's what they're looking for in an exam, you get marks for that as well. So if you can learn how to write it out and justify what you're doing at each stage, you'll get a lot more marks for it. So, in that sense, it's probably very important that you know how to do it properly.*

Despite most of the students identifying positive benefits of the writing process, two of the students expressed a dislike for writing explanations and justifications in mathematics. Being expected to write explanations and justifications was perceived as a nuisance, something that helped fill in examination time, and really was quite a tedious exercise.

Anthony: *It helps you a bit in the beginning when you're becoming familiar with something but after you've been doing it for a while it becomes boring and in the exams you spend too much time writing.*

Charles: *I find it a real pain. We know what we're doing and you can look and see where we got the numbers from or you can see it on a diagram, but writing it down, 'sux! '.*

Charles finds writing explanations a frustrating process. First, he carries out the numerical calculation and then he returns to write out the explanation. He explains:

> *I find it a pain doing the question twice. You work it all out and then you have to read through and think 'what did I do there and what do I do here?' You have to write out all that you did, yet what you did is all in the answer. It's just putting it into words which is a pain! I find it heaps easier just doing the maths; I get stuck sometimes writing it out. You end up writing half a page. I find it easier just doing the numerical side.*

Unlike the two students who perceived the writing of explanations and justifications as a pointless learning and assessing process, one student, Richard expressed ambivalence:

> *If I could I'd prefer to write the answer but if I have to explain it well, that's just what I have to do. It takes me a while to understand the problem. I'd rather they just gave me the all the numbers and the stuff that I have to calculate. It takes me a little bit longer. I have to read over it quite a bit just to make sure that I haven't missed anything out.*

Student Difficulties

Many of the students expressed a variety of reasons why they found the writing of explanations and justifications difficult. For some students this difficulty was not essentially because of the writing process, but being able to read and comprehend the problem.

Jan: *I don't like some problems which are complicated ones with heaps of writing and problems where they put in heaps of other stuff. It's really confusing where you have to sort out what they actually want you to do, what formula they want you to use.*

Fay: *I don't like big long word problems. I don't like the proof reading, reading and trying to work everything out.*

Charles: *The wording of the questions, it's like geography, some of them are pretty ambiguous, they take you a while to figure out what they want.*

Amy: *I had to get my head around what it was saying.*

Students also expressed concern about the time that had to be spent on reading and re-reading questions.

Anne: *You read the question and then you end up having to read them over and over again. Then you try and start working things out. You get worried about time, you're just sitting there and everyone else is writing down pages.*

Mark: *I normally go too fast and have to stop and read the question a couple of times to work out what they are asking, sort out something in my mind and then go ahead and write it down.*

Another difficulty students experienced was not knowing how much to write. Some of the students appeared to have a perception that a marker wants quantity, or else are 'put off' if a large space or numerous lines are provided.

Amy: *If there's quite a big gap on the answer sheet I think they are obviously expecting me to do quite a bit of writing. I'm a bit confined because my writing is kind of messy. It's a hassle for me ... I have to squish it in.*

Faye: *I get put off if they have pages and pages of lines like some of the School Certificate papers.*

A further concern was expressed about the relatively limited use of symbols when writing explanations. Charles was one of the students who did not feel very well prepared

> *because in some cases it's hard to write down what you're actually doing and they don't let you use symbols so writing down takes heaps of room.*

The real-life context actually got in the way of the problem solving process for some students. As Merryn explains:.

> *I get put off by heaps of writing and problem solving especially if you put in real-life. If you put in a diagram it's easier. I hate having to sort everything out.*

The degree to which the context of a task affected students' performance was evident in the results presented in the preceding chapter.

Moreover, there was a common concern that they really did not know what was required when asked to justify an answer. As Kim exclaims:

> *I really don't know what they mean. Justify is not really my sort of word, like, it's a **big** word. Too big, usually you're asked to explain, not justify.*

Merryn alluded to having some understanding. As she explains:

> *I'm not sure but I think it's like proving, like showing that it is the answer.*

Amy suggested that it meant that you had say:

> *I did this because....*

Anthony responded with a reciprocal question:

> *Well what would they be looking for if they asked you to justify?*

6.3 CLASSROOM PRACTICE

Eight of the students acknowledged having worked on similar problems in class and one student recalled having completed problem solving activities of this type from a textbook.

Mark: *In class situations we do a lot of them.*

Richard: *Yes, we do lots, we have to do it quite a lot because you have to explain it all in the exam, explain what you have to do.*

Five of these students came from the same class and felt well prepared to write explanations and justifications because of the in-class work. The students commonly felt that practice came from solving problems of this type in preparation for a test or examination. The problems themselves usually came from 'old papers' in which they were required to *clearly explain* an answer.

Paula: *We were told we had to write things down or else we would lose marks. We had one particular test which wasn't based on your answer but what you wrote down and how you got to that answer.*

In contrast, the requirement to explain and justify answers did not, according to six students, appear to be part of the culture of their classroom. They were not aware of having regularly practised the writing of explanations and justifications in class and did not feel well prepared to write in these genres.

Anne: *You just don't know what to expect. They blow you away with some of the stuff they give you. You don't know what makes a good answer.*

6.4 KNOWING WHAT MAKES A QUALITY RESPONSE

The students offered a variety of suggestions about what they thought made a quality response to a question requiring the writing of an explanation. They identified features relating to communication, layout, and types of writing.

All students articulated the need for clarity in writing. For the students this meant explaining clearly all the steps taken to solve a problem in a form that can be understood by the teacher or examiner.

Merryn: *Communicating it well. Clearly showing that you understand it and putting down everything that you do and not just the answer but putting in the steps.*

Jessica: *Not putting just the right answer but the way that you've gone about it. You need to write enough so that someone else can follow it, can understand what it's about. Even though it's for Maths people to mark, you want other people to understand it too.*

The need for a tidy layout and legible writing was expressed by a number of students. The students did not feel that there was a need to write in sentences but note-form was viewed as being acceptable.

Mark: *Setting out, the layout of what you are doing, logical, labels to show what you are doing and finally having the right answer. Also overall tidiness, it does count if it's easy to read.*

Two students believed that a quality response was made up of calculations supported by words.

Charles: *Having the numbers there and the words right next to it so that when you're reading it you can look across and see exactly what you are doing - seeing that it all makes sense, that the words match the numbers.*

Paula: *You replace the words with the numbers so that everything matches up. You should have the equations and the words.*

For most of the students there appeared to be a real concern about not knowing what is required in a justification and then could not therefore articulate what makes a quality response to a question that asks them to justify their answer.

6.5 HOW DOES WRITING IN MATHEMATICS DIFFER FROM WRITING IN ENGLISH?

The students accepted that writing was now a part of mathematics and not exclusive to the English curriculum. As Amy explains:

> *I used to think why are we doing this* (writing) *in Maths, we've got English for this. It's probably become quite an important part of Maths. People were just remembering stuff and just writing down without actually knowing what they were saying. Now they're going to tell you how they know it.*

Most of the students were able to identify differences in the registers of writing for English as opposed to writing in mathematics. There was an awareness that

Mark: *In Maths you can just put down short meanings, wordings, and headings. In English you have to do it all properly. You have to write out a big long answer and there's usually more complicated sentences that you have to try and put together.*

The students were also aware that the writing of mathematics came from the factual genres and included reporting as well as explanatory and expository writing.

Paula: *You're explaining what you're doing, it's very short and brief whereas in English you're writing about something. I suppose you are explaining at the same time but in Maths it's short explanations of what you did.*

Anthony: *It's just a different type of communication. English is more writing about something, this is just explaining something and doing an answer to something which is pretty easy.*

Other students recognized that there was a significant difference between writing in mathematics and writing in English and suggested that the ability to write well in English did not necessarily mean that you would be successful at writing in mathematics.

Charles: *Writing in Maths is just converting symbols into words but in English it's a lot different. Anyone can probably do this Maths* (the writing of explanations and justifications) *whether they are good at English or not. I don't think English results have got much to do with the writing down in Maths. You haven't got hyperboles or those other strange English things in Maths.*

However, for some students an ability to write well in English was perceived as helpful when writing in mathematics. As Anne explains:

> *I don't have a problem with writing the explanation, justification type problems because I'm all right at English so I suppose that helps. I don't have any difficulty with writing down explanations and making them make sense.*

A few students who lack confidence in writing in English found writing in mathematics a more positive experience.

Mark: *These questions* (referring to the Problem Solving Tasks) *are simpler and quite well explained with all the numbers whereas in English there's usually a more complicated sentence which you have to try and put together. The hardest thing is when you have to write out a big long answer, whereas in Maths you can just put down short meanings, workings, and headings. In English you have to write it all out properly.*

6.6 SUMMARY

The majority of students felt positive about the writing of explanations in mathematics. They believed that it helped develop their thinking skills by having to communicate their findings and clarify their understandings. They found that it helped them remember and confirm new mathematical understandings. The students believed that being able to explain mathematical findings is a worthwhile life-skill and that it is important to be able to convey to a teacher or examiner the understandings or misunderstandings that they may have in mathematics. Only two students expressed a dislike for the process of writing explanations. They preferred to focus on the numerical aspect of solving problems and found the need to write a nuisance. One student was ambivalent about the process and accepted it as a necessary part of mathematics.

The students expressed concerns about the problem solving process. They experienced difficulties reading and comprehending problems, knowing how much to write, and interpreting the context problems. There was also a common concern that the students did not know what it meant to justify and therefore could not describe a quality justification.

The writing of explanations and justifications was practised more in one of the classes than the other. The students from one of the classes felt well prepared to answer questions of this type but the focus came from a need to prepare for tests or examinations. The students were not confident about knowing what makes a quality response but were prepared to offer suggestions. They felt that clear communication was important and that this meant clearly identifying and explaining all steps in the problem solving process. Layout, tidiness, and the use of labels was viewed as being advantageous when writing an explanation. Note-form as opposed to correctly constructed sentences was viewed as an appropriate writing style. Students used differing strategies such as writing calculations first then corresponding explanatory notes, and explanatory notes with the calculations written beside.

Students had varying views about the relationship between writing in mathematics and writing in English. They all accepted that it was now a feature of the mathematics curriculum but felt that writing in mathematics differed from writing in English. Success in English was not viewed as a precursor for successful writing in mathematics although it was realized that if you could write well in English then it certainly helped when writing in mathematics. Some students who did not perform well in English felt that they could still write good explanations in mathematics because of the nature of the genre.

CHAPTER 7: TEACHER RESPONSES

7.1 INTRODUCTION

This chapter provides a summation of findings from the Teacher Questionnaire. The purpose of the questionnaire was to learn about teachers' views regarding the writing of explanations and justifications, the assessment of this aspect of the mathematical processes, and to find out if teachers had any concerns about this aspect of communication.

The questionnaire was completed by all six of the full-time mathematics teachers teaching in the school at the time of the study. It was administered prior to the students completing the written problem solving tasks. A summary of the number of years of teaching experience for each teacher is given below:

Table 7.1 Years of Teaching Experience

Teacher:	A	B	C	D	E	F
Years in Teaching:	3	9	13	15	20	23

The first section of this chapter summarises the teachers' views on the importance of the writing of explanations and justifications in mathematics. The reasons why teachers believe it should be taught and their role in developing this aspect of communication is discussed. Secondly, the influence of the mathematics curriculum is described from the teachers' viewpoints. The teachers' beliefs as to when the writing of explanations and justifications should be taught, appropriate contexts, and opinions about the assessment process are described. Finally, problems associated with the learning and teaching of this aspect of communication in mathematics are explained.

7.2 THE IMPORTANCE OF WRITING EXPLANATIONS AND JUSTIFICATIONS

The teachers all agreed that the writing of explanations and justifications is an important aspect of the mathematical processes and recognized advantages to students' learning in mathematics. It was articulated that the writing process helped students to sort out and clarify understanding of concepts in their own minds and thereby understand the work that they were doing. The teachers believed that the process of writing explanations and justifications helped students *develop their reasoning and analytical ability and their communication skills.* More specifically, it was viewed as helping students *crystallize the task and a way to encourage alternative ideas and methods.* Teachers felt that it will *often show up other areas for developing and exploring while they are working through it.* Having students write justifications can reduce the 'guess and leave it at that' approach. Written explanations were seen as particularly useful for helping diagnose students' difficulties or misconceptions.

The teachers all believed that the writing of explanations and justifications should be specifically taught. The teacher's role was seen as important in helping students develop skills in efficient and systematic recording methods, and to help improve the clarity and specificity of explanations. To assist students in developing these skills the teachers felt that regular practice should be provided, especially through investigations and assignments. The teachers all acknowledged giving their students opportunities in class to develop skills in the writing of explanations and justifications. Students were regularly encouraged to explain, although for most of the teachers this was in oral situations.

Teacher A: *I insist that they should explain what they did to get the answer.*

Teacher C: *At every opportunity - it helps students to clarify their understanding of a concept.*

One teacher uses a strategy of having students write out their explanations first and then picks 'volunteers' to read out their answers to the rest of the class. Another teacher uses group work as an opportunity to encourage students to provide oral justifications.

The teachers all expressed the opinion that students acquire an understanding of what constitutes a good explanation from being given opportunities to write explanations to problems posed. In order to encourage students' awareness and development of quality responses, the teachers felt that students should be encouraged to attempt explanations, share responses, and consider the range of answers to a problem. It was considered important that students be given examples of good explanations so that they can evaluate them and make comparisons with their own responses. Modelling by the teacher, both orally and in writing, was also considered a useful strategy for helping students develop an understanding and awareness of what makes a quality response.

7.3 USING 'MATHEMATICS IN THE NEW ZEALAND CURRICULUM'

'Mathematics in the New Zealand Curriculum' was implemented into New Zealand schools in 1993 and provides the basis for the development of mathematics programmes in schools for students working at all levels of the primary and secondary school. As such, it is reasonable to assume that all teachers in New Zealand schools now have a 'working knowledge' of the curriculum statement and use it for planning, teaching, and assessing mathematics.

Most of the teachers acknowledged that they referred to 'Mathematics in the New Zealand Curriculum' when planning their mathematics programme. It was only the least experienced teacher, Teacher A who never used it. The two most experienced teachers used it occasionally, whilst the rest referred to it often. According to the teachers who had been teaching prior to the introduction of the new syllabus, the emphasis placed on communication had impacted on their teaching and assessment of mathematics.

Teacher C: *A greater understanding of context-type questions is required. Students at all levels need to be able to communicate their thought processes - i.e. not just be able to give an answer but also to be able to justify and explain the process needed to find that answer. Exams and assessments reflect this increased emphasis.*

When Should Students Begin Writing Explanations and Justifications?

According to the curriculum statement, students should be able to explain mathematical ideas using their own language, and mathematical language and diagrams from Level 2 (approximately Year Two) onwards. However, the teachers offered a range of suggested years in which students should be introduced to the writing of explanations and justifications, from Year 1 to Year 7.

Table 7.2 **Suggested Year for Introducing Writing Explanations and Justifications**

Teacher:	A	B	C	D	E	F
Suggested Year:	7	1	1	3	4	7

Essentially, the teachers all acknowledged that the writing of explanations and justifications should begin to be addressed in the primary school. Those teachers who advocated the later years accepted that younger children would be explaining in other more appropriate forms.

In What Contexts Should the Writing of Explanations and Justifications Occur?

The majority of the teachers felt that students should be writing explanations and justifications in the context of all the content strands of the curriculum: number, measurement, geometry, algebra, and statistics.

Teacher F: *They have to be an integral part of the course - they cannot be taught in isolation.*

Three teachers identified strands that they thought were more likely to be appropriate. Teacher B considered geometry to be an especially appropriate strand in which students should be developing skills in writing explanations and justifications. Teacher D identified statistics and measurement as being more appropriate than the other strands, whilst Teacher F, the most experienced teacher, stated that geometry and algebra were the more appropriate strands.

Assessing the Writing of Explanations and Justifications

There was agreement by five of the teachers that the writing of explanations and justification should be assessed. Only Teacher E, one of the more experienced teachers, was undecided about whether assessment was appropriate. The other teachers felt that it should be assessed at all levels of the secondary school and that the assessment should be used in a formative way.

Two teachers commented specifically on the writing of explanations and justifications in School Certificate examinations. Changes to recent papers, with respect to the writing of explanations and justifications, had been observed:

Teacher C: *There are many more examples of this type of question. More communication is required overall, both in reading and giving written responses to questions.*

Teacher F: *There is increased use but it is still in 'artificial' situations. Explanations and justifications are best incorporated in holistic assessment whereby students have to link and use a number of related concepts - School Certificate does not allow this type of use.*

Both teachers felt that a reasonable emphasis was presently given to the writing of explanations and justifications in the School Certificate mathematics examination but would like to see some changes:

Teacher C: *I would like the questions to be very clear about what is required in the response.*

Teacher F: *In the long term I'd like to move to more open-ended questions rather than broken up as is the case at the moment.*

7.4 PROBLEMS IDENTIFIED BY THE TEACHERS

The problems identified by the teachers related to barriers for successful student learning and barriers for effective teaching of the writing of explanations and justifications in mathematics.

Concerns were expressed that there are distinct disadvantages for students weak in language skills:

Teacher F: *Communicating ideas requires a good level of language skills and a lot of the students do not have language skills well enough developed...they do not have the vocabulary to explain and link ideas together coherently.*

It was felt that students for whom English is a second language, and those students who do not have well-developed writing skills, can feel threatened by the writing process. For some students the emphasis on writing in mathematics could also lead to feelings of inadequacy. It was suggested, by Teacher D, that having students recognize that the writing of explanations and justifications can be a combination of equations and explanatory statements, could help the students who are uncomfortable with the writing process.

Despite both the curriculum and external assessments emphasising the importance of students being able to communicate mathematical findings, there appeared to be teacher concerns about whether this was happening in practice.

Teacher D *It's in the too hard basket and is promised but shelved. A sudden burst of 'have to do Mathematical Processes' and so questions are put into the test. However, it's good to see thinking, not rote learning, more in balance.*

This teacher wondered whether the writing of explanations and justifications was really taken into account in our system yet. She posed the question: *Is the focus still on the correct answer?*

There was a common concern about the time factor involved when incorporating this approach to writing in the mathematics class. While time is a problem Teacher C contends that *the value in the learning process far outweighs this disadvantage.*

Teacher F: *The curriculum needs changing with some content eliminated - this sort of work requires hands-on, investigative work which takes time. As Maths teachers some decisions need to be made as to what is worthwhile and then worked on with greater time for reflective thinking, writing, and analysing.*

A concern raised by one teacher was the need to mark the writing in terms of grammar and spelling. It was an aspect addressed by the Chief Marker of the School Certificate examination in 1994 who stated that teachers should *insist on the writing of accurate English statements in communication. Complete sentences, correct spelling and correct punctuation should be encouraged* (New Zealand Qualifications Authority, 1994, p. 3). However, specific attention has not been given to spelling and grammar in recent Marking Schedules and Reports for external mathematics assessments.

The teachers expressed a common concern about the lack of support given to them to develop expertise in the teaching of the mathematical processes. The limited support that they had received had come from both internal and external sources. Internally, some support had come from departmental meetings and discussions. This was viewed as one way of providing collegial support in learning how to teach the writing of explanations and justifications. However, Teacher C commented that *basically there's been trial and lots of error. The students themselves have probably had the most input into my work.* The teachers had also addressed the issue by trying to find exemplars. An effort was made to provide exemplary explanations to support test and examination marking schedules. Examiner's comments and examples also helped to ensure consistency of marking.

It would seem apparent that little external support has been provided for teachers to gain knowledge and expertise in teaching students to write explanations and justifications in mathematics. Three of the teachers (Teachers A, D and E) had attended brief (half-day) workshops. These were focused on the Mathematical Processes strand of the curriculum and not specifically on the writing process. As Teacher C commented: I attended *a couple of workshops 'touching' on it.* The most experienced teacher felt that the Ministry had provided no support. Not one of the teachers identified specific resource material that they had available to help with the teaching of writing explanations and justifications. Teacher C summed the situation up by stating cynically: *What a joke*!

Officially, however there is the curriculum statement 'Mathematics in the New Zealand Curriculum' and a supporting document 'Implementing Mathematical Processes in Mathematics in the New Zealand Curriculum' (Ministry of Education, 1995). Surprisingly, not one of the teachers mentioned the second document as a form of support. One chapter of 'Implementing Processes' is devoted specifically to 'Communicating Mathematical Ideas'. Case studies are used to exemplify opportunities for communication, co-operative and collaborative problem solving activities. The actual writing of explanations and justifications, although identified in the introduction as an important component of mathematical communication, is not explicitly modelled in the cases described. Apart from these two Ministry releases limited new national resource material is available. A new series of texts, the 'National Curriculum Mathematics' series (Catley & Tipler, 1997) provides some problems which require the writing of explanations. Local associations of mathematics teachers, and school advisers have attempted to provide some support by offering workshops. A biennial national mathematics conference is held to offer teachers knowledge and support in the teaching of mathematics.

7.5 SUMMARY

The participating teachers recognized the importance of writing explanations and justifications in mathematics. The teachers believed that this process should be specifically taught and all felt reasonably comfortable about their abilities to teach it. They felt that the writing process would help students develop skills in thinking and communicating. Examining students' written explanations and justifications, gave teachers an opportunity to diagnose students' difficulties or misconceptions. They reported that opportunities are provided in class for students to practise and develop skills in writing explanations and justifications.

Not all teachers use the curriculum in their planning although there is an awareness that changes have occurred since its implementation. Changes in the teaching and assessing of students' ability to communicate mathematical findings were recognized by the teachers. They felt that the writing of explanations and justifications should occur early in a child's schooling. There were mixed opinions about whether this aspect of communication should be taught within all the content strands of the curriculum. However, most of the teachers felt that the writing of explanations and justifications should be assessed, and this assessment should occur at all levels of the secondary school.

The teachers viewed it as important that students acquire an understanding of what makes a quality response. They believed that they could help students by modelling good examples, encouraging class discussion, and by involving students in the evaluation of responses.

The teachers identified a number of barriers when endeavouring to incorporate this aspect of communication into their classes. They expressed concerns that the writing process could cause difficulties for some students who are not strong linguistically. Writing in mathematics, for these students, could be a threatening process. Teachers expressed concerns about the time factor and the lack of teacher resource material. According to these teachers there also appears to be limited opportunity for professional development in this area.

CHAPTER 8: DISCUSSION AND CONCLUSION

8.1 INTRODUCTION

The major goal of this study was to examine the writing of explanations and justifications in mathematics. Particular focus was on both the students' responses to word problems requiring explanations and justifications, and students' views about the process. A further focus was to determine the extent to which teachers viewed this aspect of mathematics as being important and how a nationally mandated examination was used to measure students' abilities to write explanations and justifications. In this chapter the role of writing explanations and justifications in mathematics is examined and then teacher and student concerns about the topic are discussed. Implications of this study and suggestions for further research are outlined. Finally conclusions from this study are presented.

8.2 THE ROLE OF WRITING EXPLANATIONS AND JUSTIFICATIONS

The writing of explanations and justifications is an important aspect of communication according to the teachers who participated in this study. They believe that students should be developing skills in writing explanations and justifications early in their mathematics learning. The importance of being able to write explanations and justifications was also supported by the majority of students involved in this research. These views are in accord with the evidence provided by the content analysis of the national assessment, School Certificate mathematics, which shows that the writing of explanations and justifications has received increased attention since 1994.

The Teacher's Role

The teachers involved in the study acknowledged that they play a key role in providing opportunities for students to develop skills in writing explanations and justifications, and in developing awareness of what constitutes a quality response. Many of these teachers acknowledged that it is more common for students to use an oral mode of communicating explanations and justifications in the classroom than it is to use a written mode. Classroom discourse and group work were recognized as being important activities for fostering communication in the mathematics classroom. The teachers felt that they have a responsibility to present students with mathematical problems that provoke explanation and justification of mathematical reasoning. Having students share responses and evaluate a variety of responses was considered a useful strategy to enhance awareness of what makes a good response. The teachers believed that their role was also to model quality explanations and justifications for the students.

Students' Views

Students supported the teachers in believing that writing explanations is an important process in mathematics. They clearly identified benefits to the development of their understanding and metacognition gained from writing explanations of their mathematical reasoning. The students also identified writing explanations as a useful skill to develop for use outside of the classroom. Interestingly, the teachers had not articulated this link to the 'real world'.

The writing of explanations and justifications was not recognized by all of the students as a feature of their mathematics learning experiences. Some of the students felt that while their teachers encouraged explanations in the classroom they usually practised writing explanations and justifications in preparation for tests and examinations. The students identified differences between writing for the subject English and writing in mathematics and although they saw a relationship between these two areas of writing, they did not feel disadvantaged in mathematics because of poor writing skills.

8.3 ISSUES OF CONCERN

A number of specific concerns have emerged from this study. The teachers expressed concerns about practical support and time. The students' concerns were about understanding the problems, written prompts, context, knowing how much to write, and more importantly what it means to justify. Common concerns were about developing an awareness of what makes a quality response and the assessment process.

Teachers' Concerns

A concern expressed by the teachers was the lack of support provided in the area of communication in mathematics. 'Communicating mathematical ideas' is a substrand of the Mathematical Processes strand of the curriculum and the one which the teachers felt the least confident about teaching and assessing. Three of the teachers had attended half-day workshops on the teaching of the process skills but considered this to be inadequate professional development. Teachers felt that limited resource material was available for them to help students develop skills in writing explanations and justifications, and for the teachers themselves to develop expertise in assessing these process skills.

Teachers were also concerned about the lack of classroom time available to spend on developing skills in explaining and justifying. They felt that the best contexts for developing these skills were investigations and open-ended problems but noted that these required more time than the traditional approaches. They felt pressured to 'cover' specific content material and were concerned that incorporating such problem solving activities would mean less time available for developing content knowledge.

Students' Concerns

The students expressed concern about not understanding what it means to justify an answer. They felt ill-prepared to answer questions requiring justifications and the majority usually avoided providing a justification, despite being specifically prompted to do so. Those interviewed who had attempted to write justifications, expressed a lack of confidence in answering 'why' when asked to justify an answer. However, it is the 'why' question that provides the opportunity for developing skills in reasoning and proof construction. They were unsure as to whether their answer was a valid justification. Richard makes a comparison between the prompts *explain* and *justify*, and articulates how he feels when he meets these two instructions in a problem:

Richard: *It's quite hard to understand what it is that they want you to write. You don't meet the word justify that often.... Explain and justify, they're sort of the same thing. If it was explain I could probably do it a lot easier. When you come across justify you think that they want a really long answer or why you have chosen this. If it was explain you could do it in a sentence or so, but justify you have to think more about what it is they want you to write down.*

Being asked to justify was only one of the prompts that presented difficulty for the students. Some students expressed concern about not understanding some of the other prompts used in the problems for this study. These included prompts such as *justify why you have chosen this degree of accuracy*, *explain what calculations you need to make*, and *explain what the expression ... represents.* Despite the use of comprehensive prompts such as *explain what you are calculating at each step,* and *clearly explain your working so that someone else can understand it,* students still expressed concern about not knowing how to write an explanation and what was expected. Students used varying modes of representation for an explanation which ranged from the purely symbolic to a combination of symbols, diagrams, and words. Regardless of whether they wrote detailed answers or only recorded calculations, many of these students expressed doubt about whether they had responded to the problems as intended.

Contexts, specifically the use of real-life contexts in some of the problems, caused concern for a number of students. Real-life context means that students are required to integrate mathematical and non-mathematical skills. The problems in this case did not contain superfluous or insufficient information, but the real-life context did 'get in the way' for some students, causing concern and confusion. Problem 2, for example, caused difficulties for students who did not understand the procedure of 'trading in'. Problem 5 used the building of a shed as a way of 'dressing up' trigonometry in a realistic situation. The intention of real-life contexts is to primarily assess facility and understanding of mathematics, and its application to real life. The focus is essentially still on a mathematical idea but it is embodied in a realistic application, the purpose being to motivate and interest students, and make mathematics more meaningful to students. However, it is no use contriving word problems which in effect isolate students from being able to interpret the problem as intended by the writer.

Not only the context, but also the language used in the problems appeared to affect students' success in solving the problems:

Paula: *A lot of subjects and setting gets in the way. The way that they word things is stupid. They just word it really difficult to make it harder for you to understand. That's the worst part of it.*

Several students commented on the 'wordiness' of problems and having to spend time reading, interpreting, and often re-reading a problem posed. Time consequently became an issue for the students who could not read and comprehend problems as quickly as others.

In mathematics it is not how much or how neatly students write but rather what they write that is important. However, students expressed concerns about not knowing how much they should write. For the problems presented in this study, no lines were provided for the students to write on but a reasonable space was allocated. For some students the amount of space influenced how much they thought it was intended that they should write.

Jessica: *I sometimes think about the space and if I do just the calculations I know it won't take up the space. That's why I write but then it goes over.*[5]

Paula: *You feel like you need to fill up the whole space. The space provided makes you think that you have to write heaps.*

Students need to convey an understanding of their mathematical thinking in their explanations and justifications; the 'how' and the 'why' can be conveyed using clear, concise and unambiguous language. The common misconception of the students was that they were expected to 'fill' the allocated space on a 'write-on' script.

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5 Jessica writes prolifically in all her responses and frequently uses the blank back of a page to continue her response.

Common Concerns

A key concern for the students and some teachers was not knowing what makes a quality response. There is a clear need to develop a common understanding of what is meant by a quality response. Students realize that their intended audience is usually the teacher or examiner and are therefore usually motivated to do their best. However, it is knowing what students should be aiming to achieve that is problematic when there are few suitable exemplars available to students and teachers. A summary of comments from an examiner's report is insufficient feedback to develop teacher knowledge and expertise of what makes a quality response. As schools incorporate more internal assessment and greater use of open-response items, informed reliable professional judgement becomes paramount.

8.4 IMPLICATIONS

It is important to consider the issues raised by the preceding discussion and the implications they have for the learning and teaching of mathematics. There are implications for teachers, students, and examining authorities. Tasks requiring students to write explanations and justifications are consistent with the aims of the curriculum focus on communication in mathematics. Students need to be presented with a range of mathematical problems to be solved and encouraged to write explanations and justifications as part of the problem solving process. A learning environment should be established in which such writing is a regular and natural part of classroom discourse so that students become more confident at writing explanations and justifications. It would be most useful for teachers if more could be found out about the nature of classroom discourse which contributes positively to the development of student confidence and expertise in the writing of explanations and justifications.

Generally, the students showed a reluctance to write justifications and had a poor understanding of what it means to justify. Secondary students *should be able to approach a problem or mathematical situation systematically, and suggest why what they* *think is true* (National Council of Teachers of Mathematics, 1998, p. 80). For students to be able to write justifications and develop thinking skills which will lead them to the more formal writing of proofs, a major shift in teaching practice is required. Students need to be encouraged to not only develop mathematical arguments and proofs but to be able to evaluate mathematical arguments.

> *Classrooms where students are encouraged to present their thinking and in which everyone is encouraged to contribute by evaluating other students' line of thinking provide rich opportunities for developing and evaluating mathematical arguments.*
>
> (National Council of Teachers of Mathematics, 1998, p. 84)

It is important that teachers endeavour to ensure that written material is not a barrier to mathematical achievement. Problems should be carefully written so that the reading level is appropriate for the intended students. Reading ability affects achievement in a variety of learning areas including mathematics (Ellerton, 1989). An assumption is often made that material written for a particular class level can be read and understood by all students at that class level. The readability and comprehensibility of problems is most important, particularly if students are then going to be assessed on their written responses. It is also important that students are able to interpret written prompts. They should be exposed to a variety of prompts and be given opportunities to develop understanding and confidence in knowing what is expected when responding to a particular prompt.

When choosing problems for students to solve contexts also need to be carefully selected so that they do not act as distracters or barriers to mathematical understanding (Boaler, 1993). Finding and using contexts which are appropriate, authentic, and readily assimilated by students is not an easy task. Teachers and examiners need to collaborate and trial problems to ensure that contexts are not a prohibitive factor but enhance students' mathematics learning by making mathematics meaningful for them.

The writing products of students are most likely to be constrained by the models of mathematical representation to which they have become accustomed. To increase student awareness of the range of modes of representation available and the appropriateness of each of these, is a task for teachers.

If students and teachers are to learn what makes a quality response then criteria for evaluating students' writing should be developed. A generalized scoring rubric containing criteria that promote a common understanding of what constitutes a quality response in problem solving is needed. The criteria should include aspects such as the clarity of calculations, desired modes of communication including diagrams and explanatory text, and clarification of how conclusions are reached. As students gain more sophistication in writing solutions to problems, including written explanations and justifications, exemplary responses will need to be shared.

Teachers and examining authorities communicate to students what is important and what is unimportant. Generally, students internalize as important those aspects of instruction which are emphasized and regularly assessed (Lester & Kroll, 1990). If students' ability to explain and justify their mathematical thinking is to be encouraged and valued, then it is important that both teachers and examining authorities continue to give emphasis to communication and problem solving in mathematics. Students should be working together with the teachers in this process so that information gained from assessment is both a formative and summative evaluation of student learning. *Constructive assessment seeks to optimize students' involvement in the assessment process* (Clarke, 1995, p. 327).

An additional related area is the use of mathematical texts. Mathematical texts are still used regularly in secondary school mathematics classes. Since the implementation of a new mathematics curriculum, some texts have been modified and a limited number of new series have appeared. Many schools struggle with making a decision as to which is the best text for their students. It is a decision that has to be made carefully as the opportunity to replace texts does not occur regularly. It is important therefore that texts be examined to measure the extent to which problem solving and communication is encouraged and modelled. The prompts should be examined to see if there has been a move from the traditional *solve, simplify,* and *calculate* to *clearly explain your working* and *justify your answer.* Students experience of being asked to write explanations and justifications should not come exclusively from teachers and tests but should also be a feature of their mathematics text.

Teachers want and require appropriate professional development in the teaching of the mathematical process skills. Limited provision has been made at a regional and national level for teachers to develop knowledge and expertise in this area. Reports from national examinations should be more comprehensively and constructively written; student exemplars should be provided, and appropriate resource material developed so that teachers can learn how to confidently and competently teach students how to write explanations and justifications in mathematics.

8.5 FURTHER RESEARCH

It is suggested that the following issues identified from the results and implications of this study warrant further research.

1. Students in this study expressed differing interpretations of their classroom experience which encouraged the writing of explanations and justifications. Further research which examines classroom discourse and the way in which students' ability to explain and justify is developed through oral discourse would be valuable. A related objective would be to examine how students' ability to **write** explanations and justifications compares with students' **oral** explanations and justifications.
2. Two research questions which also relate to the writing of explanations and justifications in the mathematics classroom are:
   - Is there a teaching style that positively supports students in developing skill in writing explanations and justifications?

- What type of classroom activities are conducive to the development of this skill?

3. This study revealed the difficulties that students have in writing justifications. It would be most useful to find out more about how teachers can best help students to progress from the writing of explanations and justifications to the more formal writing of proofs.
4. Some students in this study experienced difficulties with the context of the word problems. Continued New Zealand research is needed in order to examine the effect that context has on New Zealand students' abilities to solve problems. Many New Zealand schools have students from a variety of cultures and so it is important that the role of contexts in our mathematics classrooms is examined.
5. Research as to whether students' writing of explanations and justifications in mathematics are used formatively in schools should be conducted. It is important to find out how effective formative evaluation of students' writings might be for student learning and how such writing could be used most effectively for the benefit of both students and teachers.
6. While the curriculum statement warns *that there are dangers in adhering too closely to any particular textbook* (Ministry of Education, 1992, p.13) textbooks are commonly used in New Zealand secondary schools. It would be useful if the texts that are presently being used in schools, both primary and secondary, could be evaluated to find out if, and how, they support students in being able to provide explanations and justifications orally and in writing. How texts and more specifically, which type of questions are most conducive to positively supporting students in this aspect of communication in mathematics should be researched. This research could lead to guidelines and suggestions being formulated to support the use of current texts and the writing of new texts.
7. Both the students and teachers articulated a concern about not knowing what makes a quality response. Research which examines the most effective way of developing, evaluating, and using exemplary responses would have worthwhile practical implications for students and teachers.

8. The teachers in this study expressed the need for more effective and comprehensive professional development focusing specifically on the communication process. Research should be conducted to find out what the most effective form of professional development would be so that ultimately the students become confident and competent at writing explanations and justifications in mathematics.

8.6 CONCLUSION

The methodology used in this research incorporated qualitative approaches using interviews, questionnaires, and documents to find out about the writing of explanations and justifications in mathematics. A sample of 36 students from two classes at a co-educational school and six full-time mathematics teachers from the same school were included in the study.

The findings of this study contribute to our understanding of students' abilities to write explanations and justifications in mathematics and more generally to our understanding of the relationship between language and mathematics. The results obtained illustrate the teachers' and students' views about the writing of explanations and justifications in mathematics. More specifically, the study highlights the difficulties students experience in knowing what makes a quality explanation and how to write a justification.

The results of this study suggest that unless, and until, explanations and justifications become a regular feature of the mathematics classroom, it is unlikely that students will spontaneously and confidently engage in such activity even when it is appropriate to do so. Students need to be given opportunities to develop an understanding of what makes a quality explanation. Some students may be more capable of explaining their thinking and reasoning processes orally than in writing. Teachers need to be aware of those students who lack confidence in writing in mathematics.

Problem solving and communicating are significant features of teaching and learning mathematics and support a constructivist approach to mathematics education. Conceptual development is an individual process in which learners construct or build knowledge for themselves within the mathematics classroom. The students' own language in explaining and justifying their thinking must therefore be acknowledged and encouraged. This study has shown that there are positive signs of change; communication in mathematics has received increasing attention both in assessment and classroom practice. Students are interested in, and value, the process of communicating mathematical findings. However, there is still a need for Year 11 students in New Zealand to develop skill and confidence in writing explanations and justifications in mathematics. These skills must be explicitly taught and students given time and opportunity to practise, develop, and perfect them. Essentially, students must be challenged to fully engage in making mathematical sense by writing explanations and justifications.

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APPENDIX A: Information Sheet

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Education Research Project

The Writing of Explanations and Justifications in Mathematics

My name is Brenda Bicknell and I am a lecturer in mathematics education employed by Massey University. I am a registered teacher and work in the Department of Learning and Teaching at the College of Education. I have considerable teaching experience in both primary and secondary education. I am currently studying towards a Masterate in Education and am conducting this research as part of the requirements for this degree.

The topic which I am studying is the writing of explanations and justifications by students when solving mathematics problems. I am focusing on this skill at the Form 5 (Year 11) level because this is an important year in terms of the School Certificate national assessment. Since the implementation of the new mathematics curriculum a greater emphasis has been placed on problem solving skills and the ability to communicate mathematical findings.

As part of this research I wish to examine students' solution processes to class mathematics problems which require explanations/justifications. I will examine the responses to these problems only for those students who agree to participate in the study. These written responses will be treated confidentially and kept in a secure place.

In order to gain more detailed information about the writing process I would also like to interview some of the students. The interviews will be arranged at a time suitable for the students and approved by the teacher concerned. It is expected that the interviews will take approximately 30 minutes and will be taped for subsequent analysis. The student may at any time refuse to answer a question or ask to have the tape recorder turned off. The tape recordings will be treated confidentially and stored securely. After completion of the report they will be destroyed.

The students' names will not be used in any written reports and neither the school, teacher or students will be able to be identified. At the completion of the project a summary of the findings will be made available to the school and can be requested by students and parents.

The teachers of Form 5 (Year 11) classes are also invited to participate by completing a questionnaire to gain an understanding of their views on the writing of explanations/justifications in mathematics and the teaching and assessment of this process.

If you have any questions regarding this research please don't hesitate to contact me at Massey University College of Education (Ph 357 9104, ext 8862) or contact my research supervisor, Dr Glenda Anthony, Mathematics Department (Ph 356 9099, ext 5336).

APPENDIX B: Problem Solving Tasks

Problems: Name:..

Instructions: Read the following problems and answer them in the spaces provided. You may answer the problems in any order. Read the instructions for each problem carefully.

1. Derryn investigated the packaging of snack bars. She measured the packet with a ruler and found that it was 16.5 cm long, 9.2cm high, and 4.6 cm wide.
She calculated the volume to be 698 cm^3 (3sf).
The packet had 6 muesli bars in it.
Each muesli bar was 7.5 cm long, 4 cm wide but the thickness of the bars varied between 2.6cm and 2.8cm.

Find the volume of the 6 muesli bars as a percentage of the volume of the packet.

- **Explain what you are calculating at each step and show your working.**
- **Round your answer appropriately, stating the degree of accuracy.**
- **Justify why you have chosen this degree of accuracy.**

2. *Cash Flow* is selling new CD players for \$695. They are offering customers two deals.
(i) Trade in your old stereo for \$200 or
(ii) get a 20% discount for cash.

Paul can sell his stereo for \$75 to a friend. Should Paul trade in his old stereo to buy the new one or sell it and pay cash for the new one?
In order to get the best deal which decision should Paul make? **Justify your answer.**

3. James and Richard are gathering data for a survey on students' lunchtime eating habits. They wish to find the percentage of students who buy their lunch from the school canteen.
State the two items of data they need to gather and **explain** what calculation they need to make in order to be able to calculate the percentage of students who buy their lunch from the school canteen.

Name: ....

4. The tickets for a Show at the Regent Theatre cost \$48 for an adult and \$30 for a child.

(a) What would it cost a group of x adults and y children to get in to the Show?

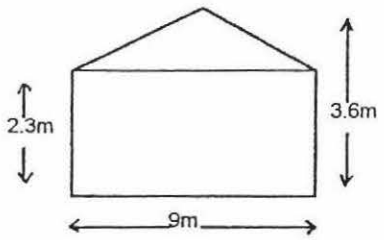
Cost = ______ x + ________ y

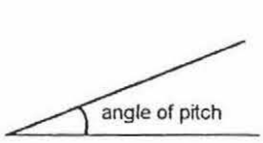
(b) A total of 500 tickets were sold altogether totalling \$21 840. If n represents the number of adults at the Show, then we can write the equation

$$48n + 30(500 - n) = 21\ 840$$

Explain what the expression (500 - n) represents.

5. A new sports shed is to be built at the school. The walls are to be 9m wide, have 2.3m high walls, and be symmetrical in shape with the greatest height of 3.6m. The Council will give approval for the building as long as the angle of pitch of the roof is greater than 16^0.





(Diagram is not to scale)

Use trigonometry to help you decide if the shed will be given Council approval.

Clearly explain your working so that someone else can understand it.

Justify your decision.

APPENDIX C: Student Interview Schedule

Student Interview: Questions

Part A: (For all students)

1. Do you have a preference for the type of question you have to answer in maths exam? (eg multiple-choice, short answer, investigation) Why?

2. Have you practised this type of problem in class? Does your teacher ask you this type of question in discussion/ tests/ textbooks?

3. How well prepared do you feel you are to write explanations/justifications in mathematics?
4. How do you feel about writing answers to this type of question?

(Follow up on this response to try and determine the nature of the difficulties experienced)

5. What do you think makes an excellent response to this type of question?

6. How important do you think the writing of explanations/justifications in mathematics is?

7. Why do you think you are required to write explanations/justifications in mathematics? Do you think that it helps your mathematics learning and understanding? (If so, in what way?)

8. How do you feel writing in Mathematics differs from writing in English? Do you have any preferences? Why/why not?

Part B: (focused on individual responses to specific problems solved)

1. How well do you think you answered these problems? Do you wish to comment on any one in particular?

2. Were any questions easier than others. Why/why not?.

3. What can you remember about trying to answer this question?

4. Do you think that this is a good answer? Why/why not?

(For students with incomplete response)
- Why did you not complete this question?

(For students with calculation only, no language component in response)
- Do you think that you answered the question fully? Why/why not?

(For students with a non-answer)
- Why did you not attempt to answer this question?

APPENDIX D: Teacher Questionnaire

Teacher Questionnaire

The document 'Mathematics in the New Zealand Curriculum'(MiNZC) includes as a substrand of Mathematical Processes: 'Communicating Mathematical Ideas'. It is stated that students should be able to use their own language and mathematical language to explain mathematical ideas.

The purpose of this questionnaire is to learn about teacher views and perceptions about this aspect of the curriculum, in particular the teaching and assessment of the writing of explanations and justifications in mathematics. In order to find out more about this aspect of communication in mathematics I would appreciate your assistance by completing this questionnaire.

Years in Teaching: ....

Part A

1. How often do you refer to 'MiNZC' in your planning? Never Sometimes Often

2. Since the introduction of 'MiNZC' what changes in teaching and assessment have you noticed with respect to the communication substrand?

....

....

....

....

3. *For each of the following statements please tick the box which best indicates the extent to which you* ***agree*** *or* ***disagree.***

	Strongly Agree	Agree	Undecided	Disagree	Strongly Disagree
(i) The writing of explanations and justifications is an important aspect of mathematical processes.	☐	☐	☐	☐	☐
(ii) The writing of explanations and justifications should be specifically taught.	☐	☐	☐	☐	☐
(iii)The writing of explanations and justifications should be assessed.	☐	☐	☐	☐	☐
(iv)The writing of explanations and justifications should be assessed at all levels of the secondary school.	☐	☐	☐	☐	☐

4. When do you think students should be introduced to the writing of explanations and justifications in mathematics? Year........

5. How do you think students get an understanding of what constitutes a good explanation?

....

....

....

6. Do you specifically teach or give opportunities for students to develop skills in the writing of explanations and justifications? Yes/No

Please comment..

....

7. How competent do you feel to teach the writing of explanations and justifications?

Indicate on the scale. ________________________________

Not at all competent	reasonably competent	very competent

8. Do you believe that some strands are more appropriate for the writing of explanations and justifications than others. Yes/No

If Yes, please state which strand(s)..

9. What support have you received to help you learn how to teach the writing of explanations and justifications ? *Please comment.*

....

....

....

10. What advantages/disadvantages do you believe the writing of explanations and justifications has for students?

Advantages:...

....

....

....

Disadvantages:...

....

....

....

Part B

Students are required to write explanations and justifications as part of their answers to questions in School Certificate Mathematics examinations.

11. What changes have you noticed in the School Certificate mathematics examinations regarding the writing of explanations and justifications in mathematics?

....

....

....

....

....

12. *Tick one of the following statements which best indicates your belief about the writing of explanations and justifications in the S.C. Mathematics examination.*

☐ I believe that there is too much emphasis given to the writing of explanations and justifications?

☐ I believe that there is a reasonable emphasis given to the writing of explanations and justifications?

☐ I believe that there is not enough emphasis given to the writing of explanations and justifications?

13. Are there any changes that you would like to see made in the School Certificate Mathematics examination with respect to the assessment of the writing of explanations and justifications in mathematics? *Please comment.*

....

....

....

....

Any further comments?

....

....

....

....

Thank you very much for your co-operation in completing this questionnaire.

Brenda Bicknell