A CLINICAL STUDY OF THE MATHEMATICAL INCOMPETENCE OF SOME UNIVERSITY STUDENTS

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GORDON HENRY KNIGHT

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ABSTRACT

The objectives of the study are to contribute to the clear identification and understanding of the factors which lie behind the severe mathematical difficulties experienced by some otherwise able university students.

A careful description of the phenomenon, which might lead to an explanation, is dependant on an understanding of the cognitive processes of the individuals concerned. Consequently a research method and theoretical perspective were chosen which would enable a study of these processes to be made as they were used in solving mathematical problems. The method was based on the Piagetian clinical interview and the theoretical background was essentially that of Skemp's (1979) model of intelligent behaviour. The principal advantages of this model were its structural rather than global features and the close relationship implied between the cognitive and affective determinants of behaviour.

Twenty six subjects were interviewed having a wide range of mathematical abilities and interest. Each subject was presented with the same sequence of tasks taken from the primary-secondary school arithmetic-algebra syllabus. The responses were probed in an unstructured manner.

The analysis of the interview data had two stages. Firstly, in order to provide an overview, a formal coding was undertaken in which the response to each item was classified according to the level of understanding indicated. The resulting data was analysed initially in an entirely descriptive manner and then was subjected to Latent Response Analysis. Following this statistical analysis a closer clinical analysis was made using a multiple-coding approach to build up a mosaic of evidence concerning the conceptual structures used by the subjects.

The principal conclusions of the study relate firstly to the vital importance of the availability of appropriate initial frameworks for the successful handling of mathematics.

It is argued that the absence of such frameworks, or schema, interpreted in the light of Skemp's theory, explains both the affective reaction of subjects having difficulty with mathematics, and the development of intelligent behaviour within one form of knowledge but not in another.

Secondly, the evidence of the study indicated that it was unlikely that the difficulties which the students were having with mathematics were due either to the abstract nature of the concepts involved or to the logical nature of the subject matter.

Thirdly, the topic of fractions emerged clearly as the most likely source of real difficulty. It is suggested that generations of curriculum designers have seriously under-estimated the difficulties associated with learning in this area.

Finally attention is drawn to the necessity for teachers to constantly monitor the development of the cognitive structures of their students and to be sensitive to signals in the affective domain which might indicate developing problems in the cognitive area. In this way the vicious interaction of cognitive and affective reactions to mathematics, which is the most distressing feature of the problem, might be avoided.

PREFACE AND ACKNOWLEDGEMENTS

My interest in the topic of this thesis arises from some twenty five years of teaching mathematics in school, technical college, teachers' college, and university. It would be impossible to acknowledge all of the formative influences during that time. However I wish to express my special debt to the following for their particular support over the period in which this thesis has taken shape.

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TABLE OF CONTENTS

TABLE OF CONTENTS CONTINUED

TABLE OF CONTENTS CONTINUED

LIST OF FIGURES

LIST OF FIGURES CONTINUED

LIST OF TABLES

CHAPTER ONE

THE NATURE OF THE PROBLEM

INTRODUCTION

Henri Poincaré, in a lecture delivered to the Psychological Society in Paris at the turn of the century, asked:

> How does it happen that there are people who do not understand mathematics? If mathematics invokes only the rules of logic, such as are accepted by all normal minds; if its evidence is based on principles common to all men, and that none could deny without being mad, how does it come about that so many persons are here refactory?
>
> (Byers, 1980: 3)

That there are many people who, in areas other than mathematics, show themselves to be very able, but who find mathematics extremely difficult is undeniable. Often they become convinced that they are incapable of understanding and success in the subject. Why should this be so? What is it about mathematics which escapes such people?

Richard Skemp, now Professor of Education at Warwick University, became intrigued by the problem as a mathematician. He writes:

> I began my professional career as a teacher of mathematics. As my task shifted from that of learning mathematics myself to that of teaching it to other people, I became increasingly concerned with the problem of those pupils who, although intelligent and hard working, 'couldn't do mathematics'. This did not seem to make sense. Surely the main ability required for mathematics was the ability to form and manipulate abstract ideas: and surely this ability coincided closely with what we mean by intelligence? So there seemed to

be a contradiction.

(Skemp, 1971: 14)

The objectives of this study are to contribute to the clear identification and understanding of the factors which lie behind this apparent contradiction, and consequently to provide a possible basis for preventative and remedial action.

The problems which such people have with mathematics are often manifest in both the cognitive and affective domains. The principal indicators are cognitive and consist of errors in performing mathematical tasks, or perhaps the apparent inability to handle such tasks at all. It will be shown, for example, that many students, when adding fractions, use the incorrect algorithm: $\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$, and that very few students can provide any explanation of the rule for multiplying fractions, even though they can use the rule confidently and accurately.

Associated with these errors there are often symptoms in the affective area relating to factors such as attitude, anxiety and motivation. For example, a reluctance to consider any task with mathematical content is very common in mature students with mathematical difficulties.

Such symptoms are well known to teachers and researchers, but the tendency has been to treat them in isolation. Observed failure to handle fractions has led to the search for more efficient teaching procedures for that topic. Observed lack of motivation in students has given rise to attempts to make mathematics learning more interesting and enjoyable. There is, of course, nothing wrong with these approaches, any more than there is something wrong with prescribing aspirin for the headache associated with a brain tumour. However, the major premise of this thesis is that real progress towards an understanding of the problem, and consequent remediation, is dependent upon a much deeper appreciation of the factors which lie behind the mosaic of cognitive and affective symptoms than is currently available.

THE SYMPTOMS

The essence of the problem considered is that the subject's mathematical behaviour is contrary to expectation. The question then arises as to what mathematical behaviour might be reasonably expected of an intelligent person. There are two aspects to this question. Firstly there is the character of the mathematical tasks which one might expect the subject to master, and secondly there is the nature, or degree, of mastery involved.

In New Zealand schools, children follow a common mathematics programme from Standard 1 (7 years) to Form 4 (14 years). Implicit in the existence of such a uniform programme is the assumption that the mathematical tasks included are within the capabilities of the majority of pupils, and certainly within the capabilities of the more able. The validity of this assumption will be brought into question later in this thesis, but the primary-secondary school syllabus does provide an appropriate pool of tasks for this study.

In relation to the ability to master such tasks, Werdelin offers the following description:

> Mathematical ability is the ability to understand the nature of mathematical (and similar) problems, symbols, methods and proofs: to learn them, to retain them in the memory and to reproduce them: to combine them with other problems, symbols, methods and proofs: and to use them when solving mathematical (and similar) tasks.
>
> (Werdelin, 1958: 13)

Within this description there are three different aspects of mathematical competence:

- the ability to understand,
- the ability to learn, retain and reproduce,
- the ability to combine and use.

These aspects are clearly not independent, and the relationships between them are likely to be of relevance to this study. Each of the aspects have been the subjects of previous comment and research.

The ability to learn, retain and reproduce mathematical material is perhaps the most straightforward of the three. In a review of research relating to the learning of skills in mathematics, Suydam and Dessart write:

> One of the most frequently stated goals of mathematics instruction involves the development of skills. Skills are comparatively easy to describe or specify, to teach, and to evaluate. A skill is what a learner should be able to do.
> .... Skills are generally characterized in terms of (a) proficiency or accuracy and (b) efficiency or speed. When mastered, skills require relatively little reflection. In effect, they become reflexes, to be exercised at a subconscious level.
>
> (Suydam and Dessart, 1980: 207).

The ability to combine mathematical techniques and apply them is similarly well defined and corresponds closely to problem solving ability. Lester (1980) presents a comprehensive review of the considerable volume of recent research in this area and comments:

> The fact that problem solving has been the object of so much research, a focal point for several curriculum development efforts and the subject of innumerable books, articles, and conference reports attests to its importance in the study of mathematics. Indeed, there is substantial support for the notion that the ultimate aim of learning mathematics at every level is to be able to solve problems.
>
> (Lester, 1980: 287)

The ability to understand mathematics is much less precise than the other two aspects, but seems to be of considerable importance to this study. The questions: What does it mean to understand mathematics? Are there different kinds of understanding, either qualitatively or quantitatively? Is understanding necessary for mathematical competence? have an obvious relevance.

Mathematical understanding has been the subject of considerable recent discussion in mathematics education literature. Articles have appeared by Skemp (1976, 1979b), Lehman (1977), Byers and Herscovics (1977), Buxton (1978), Backhouse (1978), Davis (1978), Gordon (1978), Shelley (1978), Michener (1978), Knight (1979), Thwaites (1979), Byers (1980) and Malaty (1980).

Much of this work has its origins in the paper by Skemp (1976) who developed an earlier suggestion by Mellin-Olsen that there are two distinct kinds of understanding of mathematics - instrumental understanding and relational understanding.

The main characteristic of instrumental understanding is that what has to be done next is determined purely by the local situation. The instrumental understander knows that in order to get from A to B a certain procedure must be followed. Instrumental understanding is roughly described as 'knowing how'. Relational understanding of mathematics consists of building up a conceptual structure from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within this structure to any finishing point. Because the relational understander 'knows why' as well as how, he is able to deviate from the narrow path of specific procedure to see and use relationships which pass unnoticed by the instrumental understander.

Another classification, with a little more precision, is proposed by Lehman (1977). He considers that understanding of statements or theories in mathematics consists of any of three distinguishable kinds of knowledge.

(a) Knowledge of <u>applications</u> of a statement or theory. The statement in question might be a mathematical rule such as the rule for the multiplication of decimal numbers - count the number of decimal places in each factor and add these to give the number of decimal places in the product. Ability to apply this rule indicates Lehman's first level of understanding.

(b) Knowledge of the meaning of a statement or theory. It is quite possible to apply the above rule without having any idea of either the significance of a decimal number or of the process of multiplication. Children can recite 2 + 2 = 4 long before it becomes meaningful.

(c) Knowledge of the logical relationships among the components of a statement or theory. In the example above this is the knowledge of the logical relationship between the position of the decimal point in the factors and its position in the product.

These models of Skemp and Lehman are, of course, not unrelated. It seems, in fact, that Lehman is describing some particular features of instrumental and relational understanding. Instrumental understanding is likely to include both knowledge of applications and knowledge of meaning, and knowledge of logical relationships might be a useful indicator of relational understanding. Both classifications were found to be useful in describing and analysing students' mathematical behaviour in this study.

Returning, then, to the expectations which might reasonably be made of an intelligent subject in relation to mathematics learning, it is suggested that such a subject should be able to achieve relational understanding of the mathematical concepts and procedures found in a normal school programme and be able to learn, retain and apply the associated techniques. The study is concerned with those otherwise able students who fail significantly in this regard.

Associated with such failure there are often difficulties in the affective area. The relationship between factors such as attitude, motivation and anxiety, and mathematics learning has been explored by many researchers. Reviews of this research are presented in Biggs (1962, 1967) and Aiken (1970, 1976). The more recent literature has concentrated on sex differences, particularly in relation to mathematical anxiety (Tobias 1978, Kogelman 1979, Stamp 1979, Luchins 1979). In fact, so much has been written on this matter that Luchins (1979) comments on the degree of 'anxiety about mathematical anxiety' and warns of the dangers of this adding to the stereotyping of women as being poor at mathematics.

From the point of view of this study, however, it is in the cognitive rather than the affective domain that the apparent contradiction in the behaviour of able students is evident. Indeed, it would be surprising if a well motivated, intelligent subject, having considerable success in other areas did not react anxiously to repeated failure in mathematics. It would be equally surprising if the presence of this anxiety did not induce negative attitudes and the development of interests away from mathematics.

This view of the affective symptoms as being, at least initially, a consequence of the cognitive difficulties is held by most researchers. Tobias, for example, states quite categorically:

> My principal purpose in writing this book is to convince women and men that their fear of mathematics is the result and not the cause of their negative experiences with mathematics, and to encourage them to give themselves one more chance.
>
> (Tobias, 1978: 5)

There is, however, considerable danger of creating a false dichotomy by this division between the cognitive and affective factors of learning. In this connection Skemp writes:

> In everyday human activity and interaction, feeling and cognition are combined in varying degrees; and this close association between cognitive and affective experiences will follow as a necessary consequence of the theoretical approach which is being developed in this book. The dissociation of the two is, I believe, an artifical one, which has led to one-sided approaches in both psychological and educational theory.
>
> (Skemp, 1979a: 11)

In Skemp's model the affective experiences such as pleasure, unpleasure, fear and relief are seen as signals of movement in the cognitive area towards, or away from goal or antigoal states. Feelings of confidence, frustration, security and anxiety are indications of the organism's knowledge of its ability to make such movement. (Fig. 1).

	State perceived as changing	Signals from comparator	Knowledge of ability to change state	Knowledge of inability to change state
Goal state	Towards	*Pleasure*	*Confidence*	*Frustration*
	Away from	*Unpleasure*	-	-
Anti-goal state	Towards	*Fear*	-	-
	Away from	*Relief*	*Security*	*Anxiety*
			Signals that organism is within prohabitat	Signals that organism is outside prohabitat

<u>Figure 1: Affective Symptoms</u> (Skemp, 1979a: 81)

This interpretation of the affective symptoms as indicators of difficulties in the cognitive area is particularly useful in this study. It will be shown, for example, that the emotional reaction expressed by subjects to tasks in different content areas within the school mathematics programme is quite variable. The indicated dislike of tasks involving fractions being particularly strong. The actual measurement of such emotional reaction, however, presents major methodological problems. McKeachie, for example, in discussing the measurement of state anxiety, writes:

> Perhaps we need to have one instrument sampling a range of potentially anxiety producing situations: another tapping the intensity of anxiety typically elicited: and perhaps still another dealing with the types of reacting to anxiety such as withdrawal, worry, or impulsive disorganisation. Some anxious persons may respond to anxiety by attending to their physical symptoms; others may perseverate on thoughts about future consequences of failure: still others may work harder.
>
> (McKeachie, 1977: 4)

These difficulties, and the views of Skemp, justify the consideration of the subjects' reported affective reaction to mathematical tasks as an indication of probable cognitive difficulties, and the concentration of the study on these cognitive difficulties rather than on the emotional reaction itself.

MATHEMATICAL HIERARCHIES

One obvious concern of this study will be to identify the timing of such cognitive difficulties within a subject's mathematical education. There may be sensitive or critical periods during which problems are most likely to arise. This possibility leads to a consideration of the many forms of hierarchy which influence the sequencing of tasks in mathematics education.

One of the major characteristics of mathematics as a form of knowledge (Hirst 1972) is its hierarchical nature. Apart from the most basic concepts, all mathematical objects are specified in terms of other, prerequisite objects. The fraction 3/4, for example, is defined in terms of the natural numbers 3 and 4. It may be that given two related objects, either may be taken as subordinate, but this only affects the form of the hierarchy, not its character.

Associated with this 'structural hierarchy' there is the possibility of a 'learning hierarchy' (Gagné 1965). The basic premise of such hierarchies is that the ability to perform a given class of tasks cannot be manifest unless all of a set of relevant subordinate skills, or elements of knowledge, are already possessed by the learner (White 1974). In establishing such hierarchies in mathematics learning, strong use is made of the corresponding structural hierarchy to suggest links between elements, the strengths of these links being tested empirically.

These two forms of hierarchy relate directly to mathematical tasks, but there are other, indirect, orderings which are implied by categorizations of subjects' abilities, such as Piaget's (1963) stages of cognitive development, or Scandura's (1971) process abilities in mathematics. Specific tasks can be grouped according to the ability required for their mastery (Collis 1975, Kuchemann 1978, Noeltung and Gagné 1980), and the proposed hierarchy of abilities imposes a ranking on the tasks.

The actual sequencing of tasks within a school programme reflects each of these forms of hierarchy to a greater or lesser extent, together with the influence of other societal or cultural variables (Griffiths and Howson, 1974). For example, the 'new mathematics' syllabuses of the 1960's were heavily influenced by the 'structural hierarchy' with its emphasis on logical development and mathematical structure.

Each of these sequences is of importance to this study. If an area of difficulty can be identified for a particular student, the position of that area in a mathematical sequence, in a learning hierarchy, in a developmental sequence, and in the school programme, may all be factors in understanding the difficulty.

In Chapter 2 the choice of the clinical method for this study is discussed and the problem is placed in a theoretical perspective.

Chapter 3 contains a description, and justification, of the choice of subjects made and the particular clinical procedures used.

In order to present the evidence from which conclusions are drawn in a manner which is accessible to subsequent critical assessment, Chapter 4 contains a discussion of the clinical analysis procedures developed and a detailed analysis of the interview with one particular student. The tentative conclusions formed on the basis of this evidence are then tested against the results of a more general clinical analysis in Chapter 5.

In order to complement this clinical analysis, the interview responses were coded according to the level of understanding indicated, and a statistical analysis of the data is presented, and discussed, in Chapter 6.

Finally, in Chapter 7, the study is summarized and the conclusions are drawn, together with their educational implications and suggestions for future research.

CHAPTER TWO

THEORETICAL PERSPECTIVE AND CHOICE OF METHOD

In this chapter an attempt will be made to relate the choice of method for this study to the research question discussed in the previous chapter and to place both of these in a theoretical framework.

The character of the relationship between research questions and methods is illustrated by a consideration of previous studies of mathematical ability.

RESEARCH METHODS

Krutetskii (1976) suggests that studies into possible typological differences in mathematical abilities may be considered as belonging to two major schools, the 'factorial' school and the 'introspective' school. The first is typified by the use of the method of correlation followed, often, by factor analysis of the test results. The second uses:

> .... psychological observation and experimental introspective analysis of the thought process when mathematical problems are being solved.
>
> (Krutetskii, 1976: 23)

This is, of course, not the only possible classification. Piaget (1929), for example, distinguishes between the 'clinical method', the 'test method', and the 'method of pure observation'. Clearly the factorial school studies use the test method and studies using the clinical method or the method of pure observation may be included amongst the introspective studies. In the case of studies into the nature of mathematical ability, however, the vast majority of studies have been either factorial or clinical in nature.

Studies in the two schools, although referring to the same basic question concerning the nature of mathematical ability, address different aspects of this question.

Krutetskii, for example tried:

> .... to clarify the features that characterize the mental activity of mathematically gifted pupils as they solve various mathematical problems.
>
> (Krutetskii, 1976: 78)

For this purpose he used a clinical method, although factor analysis was used as an auxiliary method.

The questions asked by researchers in the factorial school are different in character. Wrigley states the general question as:

> Is there a group factor for mathematical ability over and above 'g', the general factor?
>
> (Wrigley, 1958: 62)

The choice between the methods, then, depends as much on the appropriateness of the questions asked as on the appropriateness of the methods used to answer these questions.

FACTORIAL STUDIES

Comprehensive reviews of factorial studies of mathematics ability are available in Mitchell (1938), Wrigley (1958), Biggs (1962), Aiken (1973), Krutetskii (1976), Bishop (1980) and De Guire (1980) and it is not intended to reproduce this material here. However, some comment on the character of the results and their applicability to the research question of this study is appropriate.

Much of the evidence from these studies has been conflicting. Wrigley (1958) comments on this conflict in relation to the central question as to whether or not the separate abilities, which together make up mathematical ability, form a group factor over and above 'g', the general factor. He quotes eight studies which deny this and five which affirm it. Krutetskii (1976) identifies a number of factors which have been suggested as having a particular relationship to mathematical ability. The principal factors are:

(a) The general factor g.
(b) The numerical factor N.
(c) The spatial factor S (visual V_i).
(d) The verbal factors V and W.

There is, however, no consensus on the importance of each of these factors. In summary, Krutetskii writes:

> Our analysis of factor analytic studies of mathematical ability allows us to conclude that the attempt to reveal the essence and structure of mathematical ability by testing, followed by factor analysis, without incorporating a psychological analysis of process, has not proved its value. Such a one-sided analysis gives us no accurate, meaningful concept of the structure of mathematical ability. The hypothetical 'factorial' structure of mathematical ability has proved to be amorphous, schematic, and lacking in content; consequently its theoretical and practical value in no way corresponds to the effort spent on studying it.
>
> (Krutetskii, 1976: 36)

Not all the reviewers are so scathing. De Guire (1980) is more optimistic and believes that families of factors can be identified in the studies, each family representing a class of cognitive processes, possibly very broad and very divergent, but in some way correlated. She suggests:

> ... a hierarchical structure of mathematical abilities with the primary factor families being Reasoning, Spatial, Numerical and Verbal.
>
> (De Guire, 1980: 12)

From the point of view of the problem associated with this project, however, a careful study of the factor analysis literature has not proved to be helpful. Indeed the principal impression left is that any supposed structure of mathematical abilities identified in these studies is no more than the structure of the questions used to identify these abilities. Furthermore, a conclusion such as:

> All of this has suggested to investigators that mathematical ability constitutes the 'central part' of general intelligence.
>
> (Krutetskii, 1976: 31)

- merely adds to the apparent contradiction in the existence of intelligent subjects who cannot handle mathematics.

CLINICAL STUDIES

Skowronek in his summary of the discussion which took place during the session on research related to the mathematical learning processes at the 3rd International Congress on Mathematical Education, writes:

> Numerous studies of mathematics learning are mostly statistical in character and accordingly focus on average achievements and learning results, not on learning processes, thus tending to conceal the decisive recognition that mathematical 'insight' or mathematical understanding may be achieved in individually highly different learning processes.
>
> (Skowronek, 1977: 243)

There is a growing awareness in mathematics education circles of the importance of an understanding of these individual learning processes. Skowronek again writes:

> Not until a comprehensive description of the individual acquisition processes operating in teaching has been provided will it be possible to make the first attempts at a formulation of a comprehensive theory of mathematical learning.
>
> (Skowronek, 1977: 244)

Whether or not this very general statement is justified, it does seem that individual learning processes are at the very heart of the problem under consideration. We are concerned to determine why some intelligent individuals do not respond to mathematical experiences in the way we would expect, and clinical, or introspective, studies seem to have considerable potential here.

The method has its roots in the work of Piaget but may take various forms. Romberg and Uprichard (1977) suggest that there are five distinct methods which can be placed within the realm of 'clinical investigation' (Irons, 1979).

(a) The structured individual interview. In this method, the inteviews are highly structured and each subject is given the same set of questions. The responses are usually analysed in a formal manner. Within mathematics education the work of Brownell in the 1940's (Weaver, 1976) and that of Weaver (1955) and Lankford (1972) provide examples.

(b) The Piagetian clinical method. The method begins in the same way as the structured interview with each subject being given the same task or question, but responses are probed to delineate the cognitive processes involved. There are, of course, many examples from the work of Piaget himself. Examples in mathematics education are described in Rosskopf *et al* (1971), Easley (1977), and Booker (1980), and the method, applied to mathematics is described in detail in Opper (1977).

(c) Clinical intervention research. Here, rather than trying to simply discern the subject's knowledge, an intermediate factor of instruction is introduced after an initial interview, and subsequent performance is related to the initial assessment of the subject and an analysis of the task (Booker, 1980). The major source of examples here is the work of the Russian psychologists (Kilpatrick and Wirszup, 1969, Krutetskii, 1976) but this line of research is also being pursued in Australia (Booker, 1980).

(d) Ethnographic case study. In this case, the investigator does not start with control over the problem situation but describes and investigates the nature of events as they arise. Examples in mathematics education are scarce but Erlwanger's (1973) investigation of six children in the school setting comes close to this method, as does Ginsberg (1977).

(e) Process-development evaluation. This contains the same features as the ethnographic case study, but with the added feature of a time series design. There seem to be no reported examples of this method applied to the learning of mathematics.

The choice amongst these possible clinical methods is related both to the choice of subjects and to the nature of the problem under discussion. The structured individual interview has the advantages of replicability, validity testing, etc., which are associated with statistical methods, but does not provide sufficient flexibility to explore the individual differences which, it is suggested, are the key to the problem being studied. Clinical intervention research has considerable potential once specific problem areas have been clearly defined, but one of the major tasks of this study is to identify such problem areas. The same difficulty applies to both ethnographic case studies and process-development evaluation - the particular events to be described and investigated have to be chosen before the study can begin.

On the other hand Piaget's clinical method allows both the flexibility to explore individual differences, and the possibility of covering a wide content area - precisely what is required for this study. A series of mathematical tasks, taken from the school programme, can be selected and these can be used as the common starting points for interviews.

These initial tasks may identify the difficulties which can be explored in depth in an individual manner.

The choice of subjects is influenced by this choice of Piaget's method. There are two major possibilities, either the problem is studied as it arises, that is with children in the school situation, or after it has arisen with students at high school or beyond. There are advantages with either choice.

In relation to the affective domain, working with children has some advantages. As discussed previously, it would be surprising if mature students having problems with mathematics did not display anxiety, lack motivation, and have poor attitudes to mathematics as a result of their difficulties. Consequently, if mature subjects are used it is possible that any effect which affective factors may have had initially on the cognitive processes will be masked by the inevitable, consequential affective factors.

It is important to acknowledge that in the cognitive area too the recall of past experiences may be filtered by subsequent experiences and care is necessary in interpreting this particular aspect of reflection. However it does seem that there are considerable advantages in the cognitive area in using mature subjects. Davis, in analysing Erlwanger's interviews, comments on the difficulties of interpreting children's explanations of the processes they used in solving mathematical problems:

> Erlwanger's interviews contain two different kinds of data: on the one hand, the explicit mathematical responses of the children, but on the other hand, also the children's remarks, discussions and explanations of what they were thinking, why they selected specific answers, and so on. Do these explanation marks help us to understand how the children select mini-procedures? Someday they may, if anyone ever figures out how to interpret them, but for the moment we are unable to make very much use of them.
>
> (Davis, 1977: 376)

If this is the case, it indicates a very serious disadvantage of using children in this study. It is these very remarks, discussions and explanations which may suggest the key to the problem. More mature subjects, particularly university students, may be expected to be able both to reflect on their cognitive processes, their past experiences and their feelings, and to express the results of this reflection in a way which is most unlikely with younger children.

The other, more general advantage of using mature students is that they are subjects in which one can be more confident that the phenomenon under consideration actually exists. Many chldren have periods of difficulty with mathematics, caused perhaps by illness, lack of readiness, or poor teaching. The majority of able children overcome these difficulties later with a change of teacher or maturity. This study is concerned with those who never recover from their initial difficulties, and such subjects cannot be reliably identified until much later.

THEORETICAL BACKGROUND

As with the choice of method, it is possible to identify two different styles of theoretical background to research studies into mathematical learning and ability. On the one hand there are the general learning theories of Piaget (1963), Bruner (1960), Gagne (1965) and Ausubel (1968) which seek to provide a basis for the understanding of learning processes in terms of 'developmental theory', 'discovery learning', 'structured learning', or 'meaningful learning'. More recently there has developed a theoretical approach which seeks to explain learning at an individual rather than general level. Some proponents of this approach have been particularly critical of the more traditional theories.

For example, Bauersfeld writes:

> Only competing explanations for partial views of learning have been developed, such as Piaget's genetic theory of epistemology, Gagne's hierachical model, or the gestalt theories. These positions cannot be integrated; they represent generalizations of only limited value for explanation and projection.
>
> (Bauersfeld, 1979: 20)

Other writers have seen the approaches as being complementary rather than competing in character. Skowronek comments:

> The study of cognitive structures, as done by Piaget and his research team, is relevant to the elucidation of mathematical learning processes. But so far, these studies have only proven that corresponding structures exist in a static sense, as general potentialities. However, they obviously cannot be automatically applied to concrete mathematical problems. There is a need for a study of the conditions under which pupils learn how to productively transfer general cognitive schemata to specific mathematical objects - possibly in individually highly different learning processes.
>
> (Skowronek, 1977: 243)

This difficulty of applying a generalized theory to the behaviour of individuals is illustrated by an aspect of Piaget's developmental theory which highlights rather than resolves the apparent contradiction which was discussed in the previous chapter.

The difficulty arises in the universality of the suggested stages of development. For example, in discussing the stage of formal thought, Inhelder and Piaget assert:

> The most distinctive property of formal thought is this reversal of direction between <u>reality</u> and <u>possibility</u>; instead of deriving a rudimentary type of theory from the empirical data as was done in concrete inferences, formal thought begins with a theoretical synthesis implying that certain relations are necessary and thus proceeds in the opposite direction.
>
> (Inhelder and Piaget, 1958: 251)

That is, the content of a problem becomes subordinate to its structure.

This aspect of Piaget's theory has been criticised by a number of authors (Braine 1959, Berzonsky 1971, Wason 1977). Wohlwill, for example, writes:

> ... Piaget has repeatedly been taken to task for his inclination to see nothing but perfect logic and rationality in adult intelligence. His reliance on the principles of abstract logic as a model for human thinking has blinded him to the question of the breadth and stability of logic as <u>used</u> by the individual.
>
> (Wohlwill, 1968: 481)

It is this question which is at the centre of this study. The cognitive behaviour of the subjects of this study is quite different in different subject areas.

Collis and Biggs comment:

> When we began testing our own subjects on items from the various content areas we found that the decalages became very much the rule, not the exception to an even developmental growth. Not only were preoperational and early concrete responses very common among high school adolescents (and even among university students), the level of responding was highly unstable across subject areas, the same subject ranging from preoperational in English, say, to formal in mathematics. Worse, the same subject could vary three and even four stages in the same subject area.
>
> (Collis and Biggs, 1979: 2)

As a consequence, Collis and Biggs, in their study, shifted the emphasis away from classifying a particular subject as being at a certain developmental level to a consideration of the quality of individual responses per se.

Gagné, too, comments on the general rather than specific nature of his theory:

> A learning hierarchy, then, in the present state of our knowledge, cannot represent a unique or most efficient route for any given learner. Instead, what it represents is the most probable expectation of greatest positive transfer for an entire sample of learners concerning whom we know nothing more than what specifically relevant skills they start with.
>
> (Gagné, 1971: 115)

Consequently, a decision not to use these general theories as the principal basis for this study is, as in the decision to use clinical rather than statistical methods, a reflection of the nature of the research question asked, rather than a criticism of the theories themselves.

INDIVIDUAL DIFFERENCES

Until recently individual differences, which are the major focus of this thesis, received minimal attention in the study of cognitive development (Carpenter, 1980). Within mathematics education one approach to the problem has been in the area of aptitude-treatment interaction (ATI). This work is generally seen as growing out of that of Cronbach (1957) and reviews are available in Cronbach (1975), Hunt (1975), Tobias (1976) and Cronbach and Snow (1977). Early research in the area was marked by a lack of significant results (McLeod and Adams, 1977) but later work, using cognitive style as an aptitude variable was more encouraging. However the results are still too general to be of significant assistance with this project. Witken *et al*, for example, report:

> In a good majority of the large number of studies with college populations, relatively field-independent students were found to perform significantly better in the mathematics, sciences, engineering and architecture domains than field-dependent students.
>
> (Witkin *et al*, 1977: 45)

This suggests, of course, that able students who find mathematics difficult are likely to be relatively field-dependent. But this does not contribute significantly to the determination of what it is about mathematics which such people find difficult, nor does it imply any obvious remedial action.

More promising, it seems, are the growing number of models of mathematical behaviour which have an information-processing base. Davis and McKnight (1979), for example, suggest twelve hypothetical mechanisms in mathematical thought. Five of these refer to sequential processes used, six to Gestalt processes, and one to deeper level rules. They state:

> We would argue that some collection of human information-processing mechanisms must be hypothesised if discussions of mathematical thinking are to become fruitful.
>
> (Davis and McKnight, 1979: 111)

Such a model is, of course, not independent of the more general learning theories. There may be a very close relationship indeed. In an editors' comment on Witz's (1973) paper concerning the analysis of 'frameworks' in young children, Davis and Ginsburg write:

> Professor Wits' paper breaks new ground in cognitive psychology, describing new phenomena in children's thinking, making Piaget's theory more accessible, and introducing important methodological innovations.
>
> (Davis and Ginsburg, 1973: 44)

The major strength of such hypothesised structures in relation to the problem of this study is that they provide a means of analysing the particular behaviour of a subject in relation to a specific mathematics task. Examples of such analysis are to be found in Witz (1973), Brown and Burton (1978), Davis *et al* (1978, 1979).

Skemp's (1979a) model of intelligent behaviour, which was mentioned in the previous chapter in relation to the interaction of cognitive and affective symptoms, has this useful structural feature in common with the information-processing models. Skemp himself provides a 'thumbnail sketch' of the model.

> Our starting point is the observation that much, possibly most, human behaviour is goal-directed; together with the conjecture that, cumulatively, success in achieving our goals is a major factor in survival.
>
> For goal directed activity operating on the physical environment, we have a director system, delta-one, which receives information about the present state of the operand (what is being acted on), compares this with the goal state, and with the help of a plan which it constructs from its available schemas, takes the operand from its present state to its goal state and keeps it there. We may if we like call delta-one a sensori-motor system.
>
> Delta-two is another director system, with a difference. Its operands are not in the outside environment, but in delta-one. They are not physical objects but mental objects. The function of delta-two is to optimise the functioning of delta-one. I prefer not to call delta-two a reflective system for reasons which will appear later. In a nutshell: the job of delta-one is to direct physical actions. The job of delta-two is goal-directed mental activity, also of many kinds, including learning, but not only. Learning includes the construction and testing by delta-two within delta-one of the schemas and plans which delta-one must have to do its job.
>
> (Skemp, 1979b: 44)

Skemp compares this model to others in the same or closely related fields and in relation to Piaget's developmental psychology has this to say:

> There can now be seen to be very little resemblance between my own model of intelligence as a second-order goal-directed activity, and that of Piaget which rejects teleology and is based on equilibration. Another contrast is between Piaget's conception of intelligence as a stage of mental development, reached after a succession of earlier stages, and my own as that mental function which brings about these changes, from the earliest onwards. Yet paradoxically, it is here that the nearest to a relationship can be found between his theory and my own. For it is the investigation and description of these stages which form the core of Piaget's developmental psychology. ......
>
> .... Piaget's descriptions of these successive states are, however, global descriptions. He does not, for example, show how by reflective analysis of a concept, a teacher can choose and group together suitable examples by which to help a learner form the concept more quickly and reliably; nor how by further analysis, a teacher can find out what lower-order concepts a learner requires to have available before a particular higher-order concept can be formed.
>
> (Skemp, 1979a: 220)

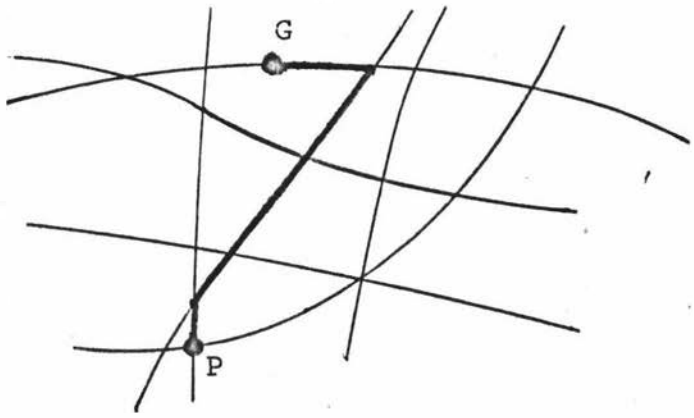
The strengths of Skemp's model, then, as a theoretical basis for the consideration of the problem of the thesis lie firstly in the hypothesised relationship between the cognitive and affective determinants of learning, and secondly in the structural rather than global nature of the theory. In view of Skemp's background as a mathematician and his long-standing interest in the problem, it is not surprising that his model is so apposite.

The model is very comprehensive being:

> .... concerned both with the subtle content that is to be learned, and also with the social motivation, and cognitive aspects of the environmental setting where learning will hopefully occur.
>
> (Davis, 1980: 357)

One of the major features of the model which has particular significance for this study concerns the representation of schemas or conceptual structures as 'cognitive maps'. Within the confines of such a cognitive map a 'director system' operates forming links between the present state P and a goal state G.

The achievement of the goal state is conditional on the realization of both the present state and the goal state within an appropriate schema, and the availability of a director system to form the path between them.



For example, faced with the problem of explaining why 2 + 3 = 5, a subject retrieves a schema which includes a representation of addition as the union of two sets and within this conceptual structure is able to see and explain the connection between the 2 and the 3 on the left-hand side of the equation and the 5 on the right. In this case, the schema has much in common with the 'framework' of Witz's (1973) model.

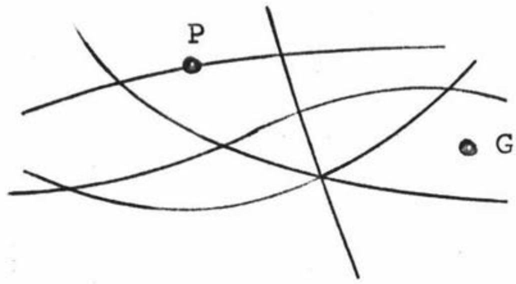
Failure to achieve the goal state may occur for a variety of reasons. Firstly the present state may not be realized in any existing schema. This would occur if a student were asked a question which was entirely outside his experience.

For example, a student who had no experience of calculus would not be able to place the question: Evaluate the integral

$$\int \text{Sin } x \, dx$$

in any appropriate schema - he would not know where to start.

Secondly, the schema activated by the present state may be inappropriate and may not contain the goal state.



For example, Kent (1978) has commented on the essential 'fiveness' of the solution to $\frac{2}{3} + \frac{3}{5} = ?$ The subject, seeing and addition sign, retrieves the same 'union of sets' schema as before and argues that two 'somethings' plus three 'somethings' must be five 'somethings'. In this case the achievement of the goal state would be dependent on extending and refining the schema to include the concept of equivalent fractions.

A third possibility is that although both the present state and the goal state are realized within an appropriate schema, the subject cannot make the necessary connections between them. For example, some students although able to add two fractions together correctly are quite unable to explain their reasons for doing so. In terms of Skemp's model, if the goal state is one of finding the answer to $\frac{2}{3} + \frac{3}{5}$ then their knowledge of the rules enables them to form an appropriate path within their cognitive map. However, if the goal is to justify their solution, clinical investigation may indicate that although the prerequisite concepts of equivalent fractions and the properties of addition are available to the students, the necessary connections are not made.

Skemp discusses in detail the character, construction, testing and use of such conceptual structures and their relationship to cognitive and affective experiences. In relation to this study, the identification of the conceptual structure used, and the links made within these structures when students are solving mathematical problems would clearly be of considerable value in determining the character of any difficulties present. Such structures are obviously not directly observable and can only be inferred from the student's behaviour. It is in this process of inference that the clinical method has its strength. Beginning with a mathematical task, the interviewer may probe various aspects of the subject's responses to the task, building up of a mosaic of information concerning the way the subject thinks about the task and its relationship to other tasks and experiences. This data may then be subjected to detailed analysis in an attempt to reconstruct some of the features of the cognitive map used by the subject.

In building up this many-sided, 'holistic' (Diesing, 1971) picture of a subject's mathematical behaviour, use can be made of a number of techniques and classifications used by other clinical researchers.

PIAGET'S CLINICAL METHOD

Piaget's own description of the method is found in The Child's Conception of the World (Piaget, 1929). Here, as was mentioned earlier, he distinguishes between the 'clinical method', the 'test method' and the 'method of pure observation'.

The major characteristic of the clinical method is the probing of responses and consequently the element of judgement as to the nature of a probe is crutial. Piaget warns:

> It is so hard not to talk too much when questioning a child, especially for a pedagogue! It is so hard not to be suggestive! And above all it is so hard to find the middle course between systematisation due to preconceived ideas and incoherence due to the absence of any directing hypothesis.
>
> (Piaget, 1929: 9)

The technique can, in Piaget's view only be learned by long practice.

> When students begin they either suggest to the child all they hope to find, or they suggest nothing at all, because they are not on the lookout for anything, in which case, to be sure, they will never find anything.
>
> (Piaget, 1929: 8)

The obvious non-suggestive probes (Codd, 1979) such as 'Why?' or 'How do you mean?' or 'What makes you think so?' need to be supplemented with carefully chosen, more suggestive probes which direct the responses towards the working hypothesis which the investigator is seeking to check. The balance of suggestive or non-suggestive probes is partly determined by the character of the phenomenon under investigation. For example, the probes in Inhelder and Piaget's The Growth of Logical Thinking (1958) tend to be more suggestive and specific than those in Piaget's The Child's Conception of the World (1929).

The interpretation of interview protocol also calls for judgement. Piaget distinguishes five kinds of answers which children might give to a clinical probe.

(i) Answer at random.

... when a child appears uninterested ... it replies at random and whatever first comes into its head.

(ii) Romancing.

... when the child, without further reflection, replies to the question by inventing an answer in which he does not really believe.

(iii) Suggested conviction.

... when the child makes an effort to reply to the question but either the question is suggestive or the child is simply trying to satisfy the examiner without attempting to think for himself.

(iv) Liberated conviction.

... when the child replies after reflection, drawing the answer from the stores of his own mind, without suggestion, although the question is new to him.

(v) Spontaneous conviction.

... when the child has no need of reasoning to answer the question, but can give an answer forthwith since already formulated.

(Piaget, 1929: 10)

The liberated conviction is, in Piaget's view of greatest interest. Again care is necessary in interpreting these convictions. Firstly, the influence of the question must not be discounted. For example, if the reply to the question 'How did the sun begin?' is 'Men made it', it may not be inferred that the child believes that the sun had a beginning - that was implied by the question - but only that there is some vague connection between the sun and men. Care must also be taken not to infer a logical coherence to an answer where the coherence is of an organic rather than logical character. Verbal responses are necessarily imperfect representations of the world of thought and care must be taken not to read too much into a particular verbal representation.

It should be remembered too that Piaget's method as described was developed for the investigation of the cognitive behaviour of children. In this study a similar method will be used on adults. The difference is important. Piaget comments:

> This research itself may be guided by the following principle. Observation shows that the child's thought has little systematisation, little coherence, is not in general deductive, is for the most part untroubled by the need for avoiding contradiction, juxtaposes statements rather than synthesises them and accepts syncratic schemas without feeling the need to analyse. In other words the child's thought more nearly resembles a sum total of inclinations resulting from both action and reverie (play combining these two processes, which are the simplest to yield organic satisfaction) than it resembles the self conscious and systematic thought of the adult.
>
> (Piaget, 1929: 25)

It is likely that this 'self conscious and systematic thought of the adult' will be a significant feature of the interview data. Consequently a smaller proportion of 'liberated conviction' responses may be expected since little of the material is entirely new to the subjects. The concern will be rather to examine carefully the 'spontaneous conviction' responses in order to determine the nature of the previously formulated ideas. A different character of probes might be appropriate for this purpose.

A consideration of Piaget's method as it has been applied to research questions in mathematics education indicates that some modifications have taken place. In the conduct of interviews, for example, Piaget describes the researcher as having constantly in mind a hypothesis which he is seeking to test.

Opper (1977) describes how the method has been modified so that beginning with an initial hypothesis, the interviewer may formulate new hypotheses with successive responses of the child.

There have also been a number of later suggestions concerning the interpretation of interview data which have inevitably affected the character of the interviews. The usual approach has been to classify the observed behaviour as falling into one of a number of predetermined categories, thereby giving a profile of the subject's behaviour. There have been a number of suggested classifications, each addressing a different aspect of the behaviour and consequently being complementary in building up a holistic view of the cognitive structures used by the subject.

CLASSIFICATIONS OF INTERVIEW RESPONSES

Piaget's five fold classification (answer at random, romancing, suggested conviction, liberated conviction, spontaneous conviction) which has just been described is qualitative in nature and intended to be applied to individual responses within an interview.

A slightly different approach is suggested by Tall (1979) who also proposes a qualitative classification, but one which takes account of the relationship of a particular response to the whole sequence of responses of which it is part. He suggests that the qualitative nature of the thinking processes which occur in a clinical interview may be considered in the context of:

(a) Initial responses: immediate responses to stimuli which occur without time for reflection.

(b) Ongoing schema: which may be -
   (i) resonance schema: carried on by the intense power of the thought process itself.

(ii) superimposed schema: where the initial response involves two disparate schemas which produce a novel train of thought superimposing two (or more) resonances.

(iii) conflict schema: the subject realises a conflict is present but has not, as yet, resolved it.

(iv) explanatory schema: an explanation of thought processes which have already taken place.

(c) Discontinuities.

(i) conflict: causing disruption of thought.

(ii) mental blocks: stoppages in thought, often preceded by conflict.

(iii) insights: sudden leaps in thought.

(iv) finish: end of schematic action.

Most Piagetian research (Sigel and Hooper, 1968; Rosskopf *et al*, 1971; Rosskopf, 1975; Collis, 1975) has employed a classification system which does not take into account individual responses per se, but identifies the Piagetian developmental stage of the subject as indicated by a whole sequence of responses. Closely associated with this developmental stage approach is the SOLO Taxonomy (Structure of the Observed Learning Outcome) of Collis and Biggs (1979). However this classification relates specifically to the responses themselves and not to any implied developmental stage of the subject. The five categories used are pre-structural, uni-structural, multi-structural, relational and extended abstract. These relate directly to the pre-operational, early concrete, middle concrete, concrete generalization, and formal operations developmental stages.

In this study the concern is to explore the conceptual structures used by students in working through certain mathematical tasks. For this purpose some rather more task specific classifications are appropriate. These classifications may again be related either to individual responses or to a whole sequence.

At an individual response level White and Mayer (1980) provide a classification system for analysing the types of knowledge associated with skills concerning functional relations. Each response is classified according to its Concepts (four categories), Arguments (twelve categories), and Contexts (eight categories).

In a more global assessment, total performance on a mathematical task, as indicated by a whole sequence of responses has been interpreted as indicating a particular class of understanding. For example in 1975, Skemp, working with the National Foundation for Educational Research in England, undertook a feasibility study into the possibility of testing the quality of children's mathematical thinking by means of a written test.

Children were given a written statement and then asked two multichoice questions on that statement. The first asked them to choose between a number of alternative explanations for the statement and the second asked them to identify, from a number of alternatives, a statement which was basically concerned with the same kind of mathematical thinking as the original statement. The answers were classified according to whether they showed relational understanding, instrumental understanding, or no understanding at all of the statement. Three weeks after the written test the children were interviewed on the same tasks and their understanding classified as before. The interview assessments were taken as the valid ones and the results compared with those from the written tests. In a personal communication, Skemp (1977) writes:

> ... the first part of the written test was quite unreliable in evaluating the children's thinking within the categories instrumental and relational understanding. Their choice of written reply were made for a variety of reasons, which emerged during interview. My present belief

> is, fairly strongly, that if you want to assess the thinking of an individual in this respect, the only way to do this is by talking to them, in what has come to be called a Piaget style interview.

The complementary nature of these classification systems is obvious and they all proved to be of value in building up a pattern of evidence for the conclusions which were drawn concerning the cognitive structures of the subjects interviewed. The use made of these classifications is discussed and illustrated in Chapter 4.

Mention should be made at this stage of a category-free approach to interviews which has been suggested by Witz. He describes a method which:

> ... aims to describe and document mental structures which the child has, and which are specific to the child, without adopting a preconceived system of behaviour categories and thereby limiting a priori the totality of structures that are possible in children.
>
> (Witz, 1973: 46)

The central concept of the method is that of frameworks which are sets of schemes (cognitive structure representations) having the properties of being;

(a) connected (to the child)

(b) dominant over a period of time

(c) related to a specific class of situations.

The aim of the analysis of the interview data in this method is:

> ... to discover the child's frameworks and their period(s) of dominance; to describe them precisely, to document their dominance in the periods in question; to exhibit the discontinuities in assimilation that occur, and to explain the special interpretations and misunderstand-

ings the child shows vis-a-vis the experimenter's questions.

(Witz, 1973: 51)

The close association between Witz's 'frameworks' and Skemp's 'cognitive maps' has been noted earlier and the aims of the clinical analysis in this study correspond closely to those described by Witz. However, the approach to interviewing which was used, rather than being category-free as suggested by Witz, was multi-category. The intention of the interviewer was to explore as many facets of the subjects' thinking about the mathematical tasks as possible so that the resulting data could be classified in a number of different ways.

The specific procedures used are discussed in the next two chapters, but before these are presented, it is appropriate to discuss the important features of reliability and validity as they apply to a study of this kind. The selection of subjects, production of inteview tasks, conduct of interviews, and the analysis of the data are all influenced by the position adopted with respect to reliability and validity.

RELIABILITY AND VALIDITY

Diesing (1971) discusses reliability and validity at some length, contrasting the requirements of a clinical method with those of experimental or survey methods of research. He defines the clinical method as being essentially 'holistic' and writes:

> The holist uses evidence to build up a many-sided complex picture of his subject matter. He accomplishes this by using several kinds of evidence, each providing a partial or limited description that supplements other partial descriptions.
>
> (Diesing, 1971: 147)

These descriptions then form the basis for an explanation of the phenomenon which Kaplan calls the 'pattern model of explanation' in contrast to the 'deductive model' used by the experimentalist seeking causal connections. Kaplan writes:

> According to the pattern model, then, something is explained when it is so related to a set of other elements that together they constitute a unified system. We understand something by identifying it as a specific part in an organised whole.
>
> (Kaplan, 1964: 333)

Diesing discusses a number of significantly different features of the two models of explanation. One of these differences concerns generalization. In the deductive model there is always a sharp distinction between the 'explanandum', the thing to be explained, and the 'explanans', that which does the explaining.

The explanandum may be a particular occurance or an empirical regularity, while the explanans is always a general law, or system of laws. For example, the observed regularity in the motion of the planets (explanandum) is explained in terms of Newton's Laws of Motion and gravitation (explanans). The explanans is always more general and abstract than the explanandum.

However:

> In the pattern model both explanandum and explanans are on the same level of generality, and the relation is that of part and whole. Both are equally particularized to the system being described, and no general laws appear anywhere.
>
> (Diesing, 1971: 160)

For example, the criminal lawyer sets out a pattern of evidence and argument which relate to guilt or innocence of a defendant. Both a particular piece of evidence (explanandum) and the verdict which it supports (explanans) relate only to the particular case, not to some general law. (Codd 1979).

By interview, we can hope to build up a mosaic of evidence concerning a particular student's mathematical concepts and procedures which may enable us to explain the mathematical difficulties encountered by that student. The drawing of more general conclusions concerning the mathematical behaviour of a particular class of subjects would require the super-imposition of a deductive model, drawing on evidence from a large number of randomly selected subjects and employing a battery of techniques to ensure the validity and reliability of the methods and conclusions.

These terms, validity and reliability, are well defined in relation to the deductive model, but, as Diesing points out:

> The holist is interested neither in reliability nor in validity in this sense. Reliability implies the ideal of an impersonal, automatic investigator; but in case studies, the personality of the investigator and his relations with the people he is studying are an essential source of understanding. Validity in all of its officially (American Psychological Association) approved senses is in the relationship between a test response, profile, or pattern and some real attribute or quality; but to the holist such isolated data are nearly meaningless because they have no context.
>
> (Diesing, 1971: 146)

For example, if this study were employing the deductive model of explanation then some measure of inter-scorer reliability would be appropriate. A high correlation between different observers descriptions and interpretations of the behaviour of subjects in relation to individual items would be expected. However, in a clinical study the subject responds to verbalizations by the interviewer with further verbalizations. Since these verbal responses are necessarily imperfect representations of the world of thought, different descriptions and interpretations by different scorers of the underlying processes involved are to be expected. These different interpretations would complement rather than contradict each other, each providing a partial perspective useful in building up the pattern of explanation. The implication of inter-scorer reliability measures is that there is a 'correct' interpretation on which independent observers should agree.

However, if reliability is not required, in that differing interpretations are acceptable, there is still the obvious requirement that such interpretations must be, in some sense, valid. To distinguish the appropriate form of validity from that used by the psychometrician and the survey researcher, Diesing defines it as 'dependability'.

> The dependability of a source of evidence is the extent to which its output can be taken at face value relative to other sources of evidence, in the process of interpreting manifold evidence. None of the evidence used by clinicians and participant observers is absolutely dependable; none is ever completely free from the need for cross-checking and reinterpretation.
>
> (Diesing, 1971: 149)

The hierarchical nature of mathematics and mathematics learning has advantages for the achievement of this kind of dependability. For example, in this study a subject's understanding of the concept of addition of whole numbers is assessed by the response to a specific item early in the interview. But the operation is used many times by the subject in responding to later items on fractions, decimals, variables and problems. Consequently the interpretation which is put on the initial response can be checked each time the operation is used in later responses. In this way either confidence in the initial response is enhanced or the need for reinterpretation of the initial response is indicated.

Not all writers agree with Diesing on the relationship of the clinical method to the concepts of reliability and validity. Mouly, for example, writes;

> The first step of the case study is obviously the selection of the cases which exemplify the problem area under consideration. There is especially a need for typical cases - that is, not a random sample of the general population but a random sample of cases considered representative of the problem under investigation. The sample should be large enough to permit the derivation of valid generalizations.
>
> (Mouly, 1963: 358)

In this study, however, the search for, and use of 'typical cases' seems to be methodologically unsound (Zelditch, 1969). This is a multi-dimensional study of individual differences and no subject, whether randomly selected or not may be considered as representative, in all respects, of the population. Furthermore, to make a randomly selected sample large enough to permit the derivation of valid generalizations is quite impractical given the character of the study and the variability of the population.

Consequently, every effort is made in this study to ensure that the evidence and the conclusions are 'dependable' in the sense defined by Diesing. However, no attempt is made to achieve reliability or validity in the deductive sense by means of such techniques as randomly selecting subjects or the use of inter-scorer reliability measures.

CHAPTER THREE

THE CHOICE OF SUBJECTS AND DEVELOPMENT OF INTERVIEW TASKS

SELECTION OF SUBJECTS

Two principal criteria were used in the selection of subjects for the study. Firstly it was necessary that subjects should have a positive attitude towards the study in order that they would be willing to reflect on their cognitive processes and feelings and to express the results of this reflection to the interviewer. As a consequence, the purpose and character of the study was explained to all potential subjects, and all the subjects used were volunteers.

Secondly, the intention of the study was to try to explore the cognitive structures of students having difficulty with mathematics. It was decided that in order to provide a comparison of these structures with those of students who do not have the same degree of difficulty, and indeed with those who have considerable success in mathematics, the sample should include subjects with a wide range of mathematical abilities and interests. Consequently, roughly equally sized groups were chosen from those subjects who had done well in mathematics at school and had continued with the study of the subject as a possible major at university; from those subjects who had a somewhat weaker, but still reasonably competent, school background and who were using mathematics at university to support study in other areas; and from those subjects who had a poor school performance in mathematics in relation to other subject areas and who had chosen degree courses with little or no mathematical content. With hindsight, it seems likely that the study would have been improved had the less able group been made larger at the expense of the other two groups. The clinical analysis of the data from the more able subjects, in fact, produced few surprises. They had, in the main, learned the content very much as it had been presented to them.

The subjects themselves were asked into which of the three groups they felt they belonged and a final sample of twenty six were interviewed. On their assessment nine considered themselves to have had much more difficulty with mathematics than with other subject areas, eight had coped reasonably at school but were not very confident, and the remaining nine had found mathematics one of their best subjects and were keen to continue its study. The subjects were later ranked according to their performance on the interview tasks and a grouping based on this ranking, which is reported in Chapter 6, corresponds closely to that made by the subjects themselves.

Of the 26 subjects, 15 were female and 11 male, there being both males and females in each group. 5 of the subjects were older students who had some work experience before undertaking, or returning to university study. Again there were representatives of these more mature students in each group. The other 21 subjects had come straight to university from school.

THE MATHEMATICAL TASKS

As discussed previously, the subjects were to be led through a sequence of tasks representative of the primary-secondary school mathematics programme. The character of these tasks was clearly of importance and considerable care was taken in their selection and testing.

Mention was made in Chapter 1 of the importance of the variety of hierarchical structures within mathematics education and an initial approach to the selection of tasks was made using these structures.

The school mathematics programme begins with the development of the concept of natural number and its operations. On this basis, further number systems involving fractions, decimals, and negative numbers are developed and these concepts are generalised in the development of numerical variables and algebra. Running parallel to this sequence is a relatively independent sequence involving geometrical ideas, but this is very much less well defined and structured, particularly at the primary school level. Consequently the arithmetic-algebra sequence was chosen as the source of the interview tasks.

The relationships between topics in this sequence can be considered at a number of levels. At a global level, for example, the work on numerical variables involves previous work on natural numbers, fractions and negative numbers. The interdependant relationships of these topics as taught in the schools is illustrated in Figure 2.

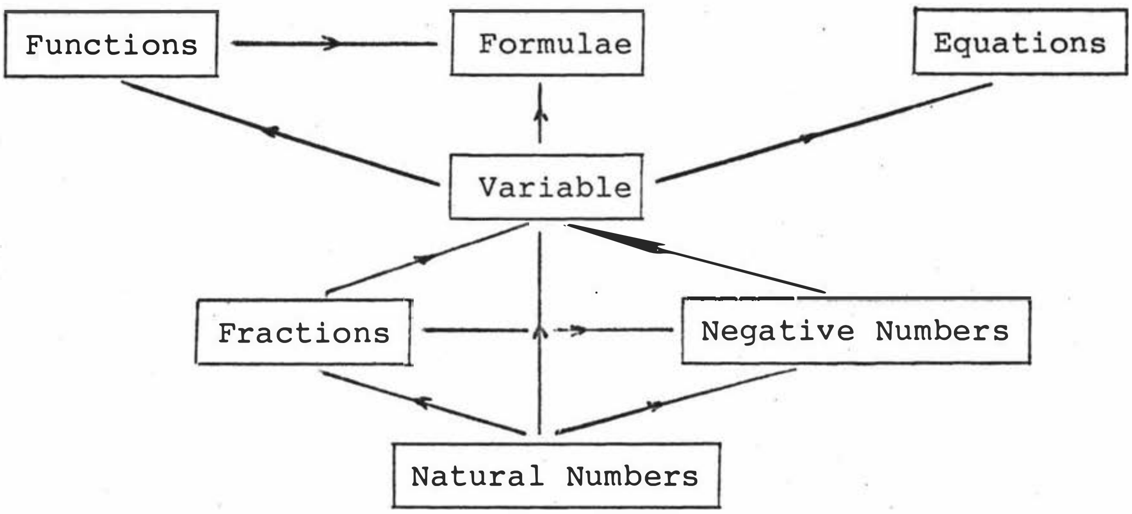


Figure 2: The relationship between content areas in the school arithmetic-algebra sequence

Topics within this sequence can be further subdivided and the relationships between the rather more precise areas considered. There are many such possible expansions, one attempt is provided in Figure 3.

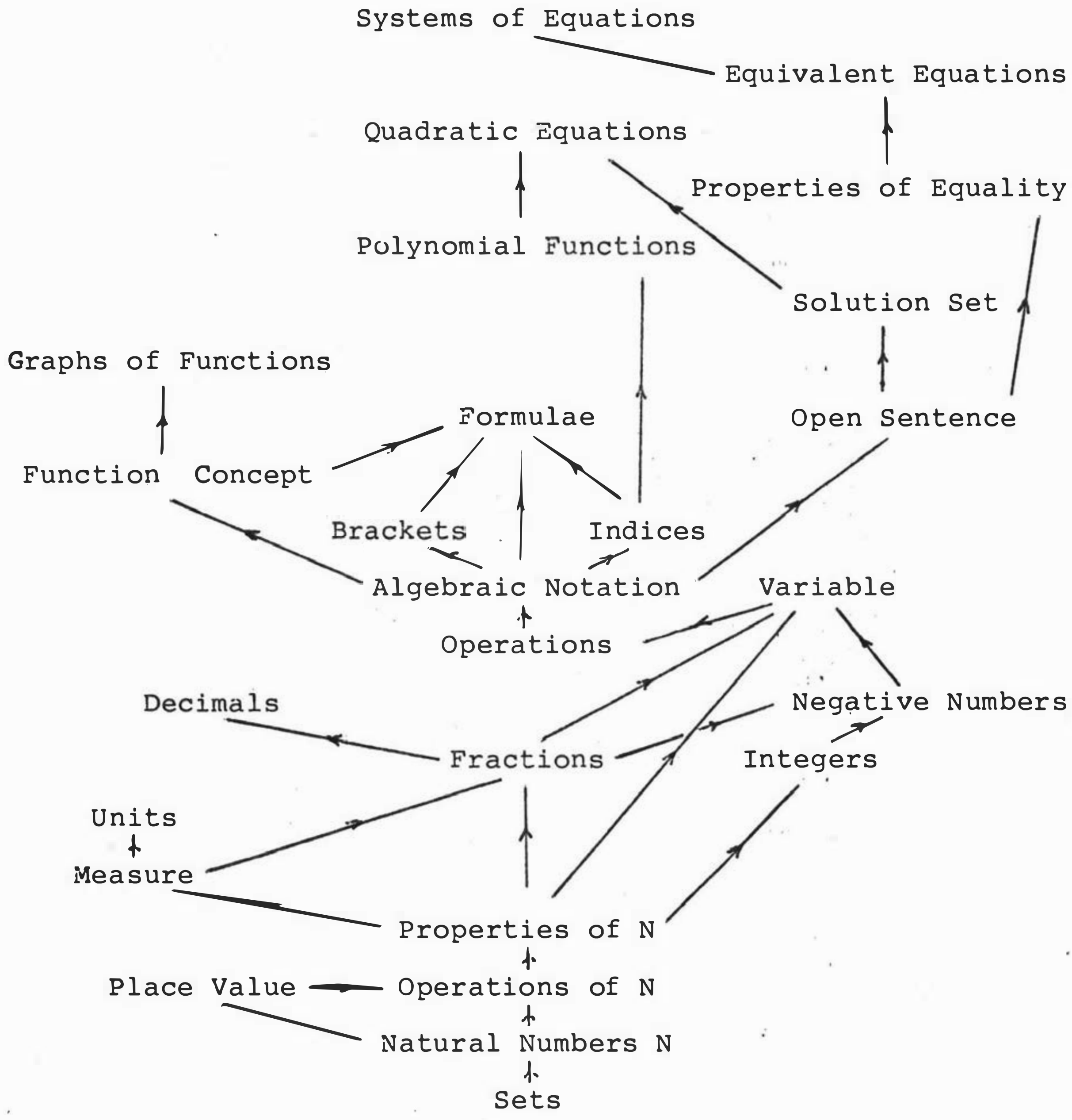


Figure 3: Hierarchy of Topics in the School Arithmetic - Algebra Sequence

Further division, relating more closely to conceptual structures, is possible. Skemp (1971), for example, provides the conceptual hierarchy presented in Figure 4 for fractions.

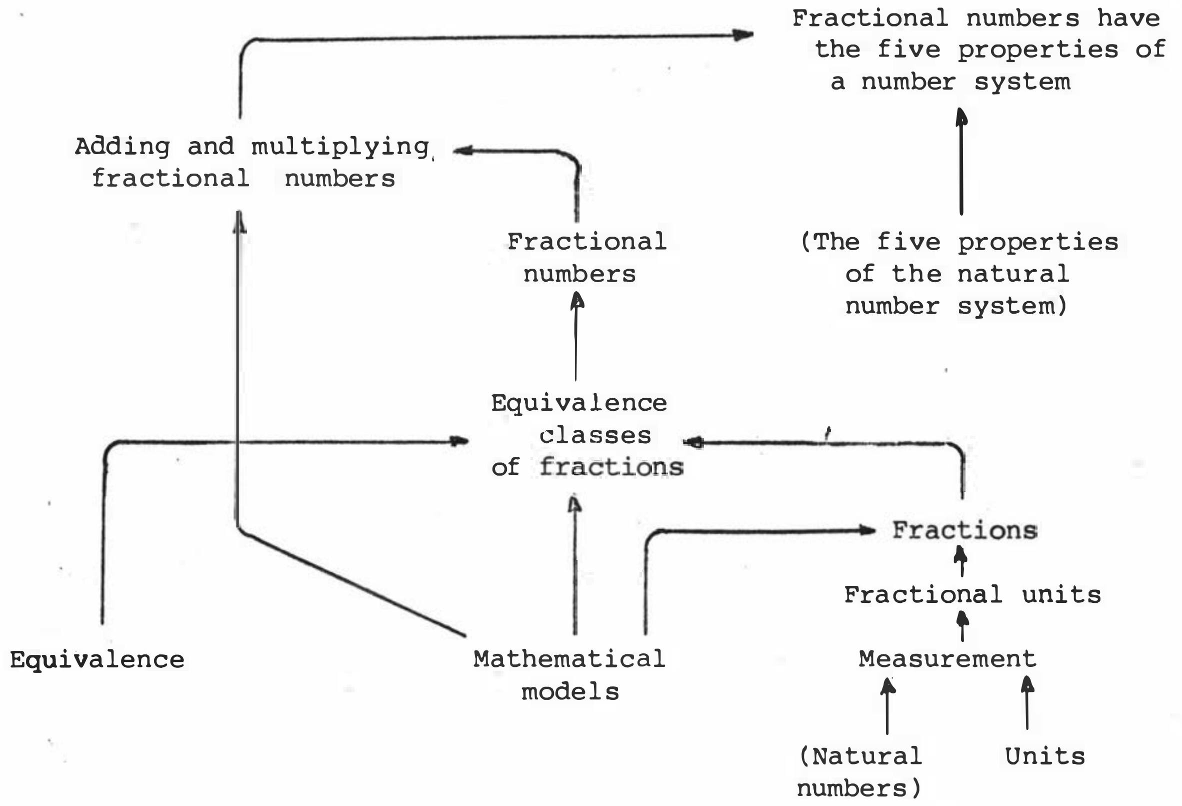


Figure 4: Conceptual hierarchy for fractions (Skemp, 1971: 310)

In relation to learning hierarchies, the elements of a conceptual hierarchy such as this must be broken down still further. A useful example of this is provided by Novilis (1976) who analysed the concept of equivalent fractions, producing a hierarchy of fifteen selected subconcepts, and tested the strength of the hierarchial dependences between them. This analysis proved useful in the consideration of the interview responses to questions relating to fractions and is reproduced in Figure 5.

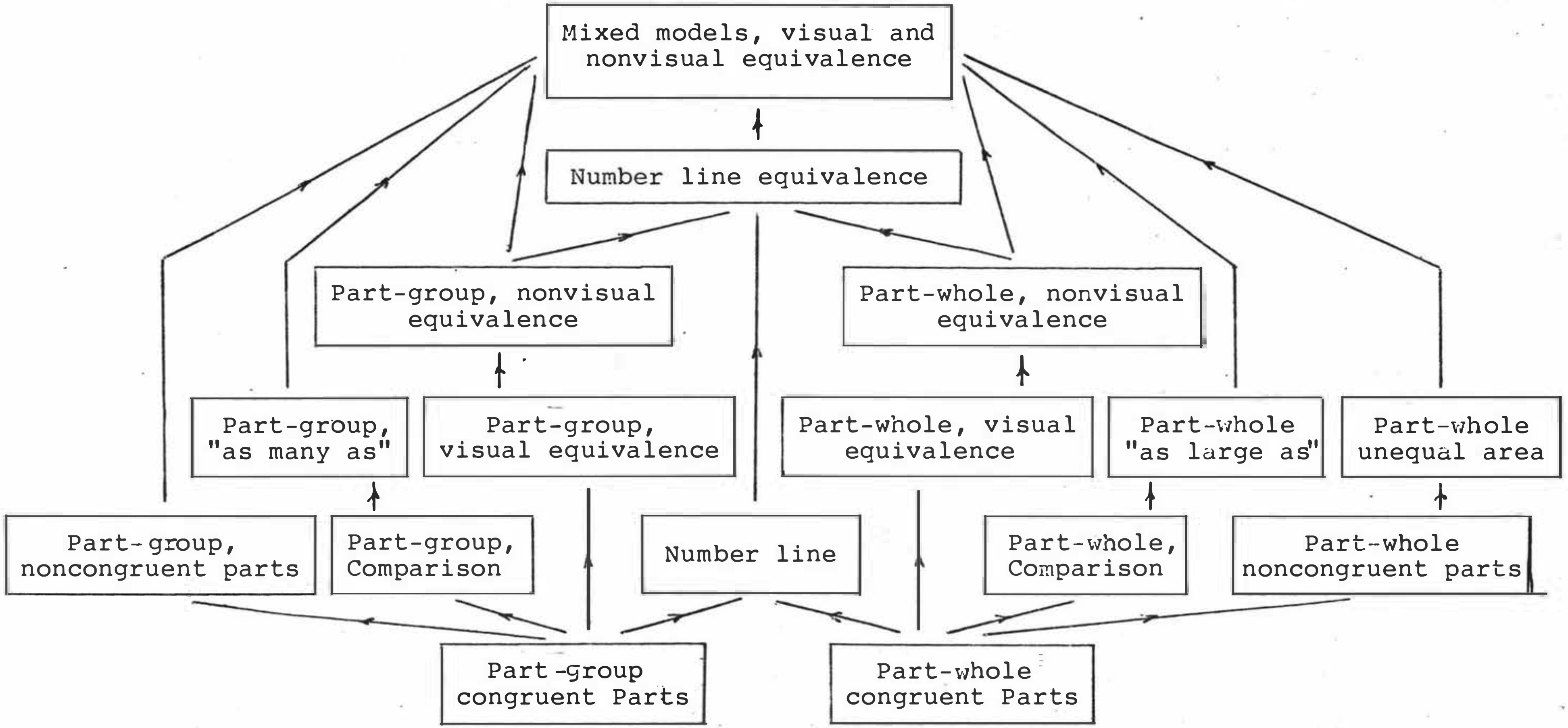


Figure 5: A Hierarchy of Selected Subconcepts of the Fraction Concept (Novilis, 1976: 132)

Given this progressively more precise view of the hierarchical structure of the school mathematics programme it is necessary to decide the levels within these hierarchies at which the interview tasks should be selected. It would clearly be impossible to explore a significant proportion of the arithmetic-algebra sequence if the tasks were chosen at the lower levels of Novilis. But even if it were possible it might not be appropriate. The essence of the Piagetian clinical method is that the initial responses to the task are probed to delineate the cognitive processes involved. This probing has the effect of exploring a hierarchy, such as that of Novilis, from the top downwards. An initial general question concerning equivalence of fractions might be asked and the response probed in order to explore the subjects understanding of the question in relation to the part-whole, part-group, and number line concepts of fractions.

This view was confirmed by pilot testing items. Questions such as $\frac{3}{4} = \frac{6}{8}$, Why? evoked much more thoughtful responses than more specific questions relating to individual sub-concepts.

The character of the questions had also to be determined. The choice seemed to be between a question relating to a specific example of a task and a question concerning the principles involved in the task. For example, a task involving multiplication of natural numbers might be either of the form:

$$2 \times 3 = 6 \quad \text{Explain}$$

or of the form:

> Tell me what you understand about the multiplication of whole numbers.

A further possibility is the type of question used by Brown and Kuchemann (1976) in assessing children's understanding in this area. The children were asked to write a 'story' to match the expression 9 x 3, perhaps in an attempt to identify the psychological structures being used.

In pilot testing it became clear that the explanation of a direct example was the most profitable approach with mature subjects. They all understood what was meant by an explanation in this context and their responses provided ample opportunity for probes.

From Figures 2 and 3 it can be seen that the key conceptual areas in the school progamme are those relating to natural numbers, fractions, negative number and variables. Lack of understanding in any one of these areas will affect later work to a much greater extent than will, for example, misunderstandings concerning decimals. Consequently it was decided that coverage of these four areas should be comprehensive. In regard to the number systems this implied testing each of the fundamental operations in turn, and questions were constructed and pilot-tested accordingly.

In relation to the questions concerning variables, Kuchemann (1978) identified six different ways in which letters are interpreted by children in questions involving variables. These he describes as:

Letter evaluated
Letter ignored
Letter as an object
Letter as a specific unknown
Letter as generalised number
Letter as variable

He postulates that items which can be solved at the first three of these levels require only concrete operations, while 'letter as a specific unknown' and 'letter as generalised number' require early or late formal operations depending on the complexity of the item. 'Letter as variable' is said to require late formal operations.

The tasks which Kuchemann used to assess, in a written test, children's understanding of each of these applications of variables in mathematics seemed ideal for the purpose of this study, and pilot-testing again confirmed that the questions elicited thoughtful responses.

The necessity to keep the interview within a reasonable time span, an hour was felt to be a satisfactory limit, meant that for areas other than number systems and variables the selection of tasks would have to be less comprehensive. Consequently in the areas of decimals, functions and graphs, algebraic manipulation , and solution of equations, items shown to have good discriminating power in written tests by other researchers were chosen.

After more pilot-testing, questions were retained from three sources. Skemp's (1975) written test of understanding was referred to in the previous chapter. Kerslake's (1977) test was of the understanding of graphs, and Galvin and Bell (1977) took some questions from Krutetskii (1976) and some from Kuchemann (1978) and used them in a study of some of the difficulties encountered in the solution of problems involving the formation of equations.

Although the emphasis of this study is, as was explained in Chapter 1, on understanding rather than on ability to retain or apply mathematics, it was felt that the clinical study of a subject's performance in solving a mathematical problem might well throw light on the understanding of the prerequisite mathematical principles. Consequently four problems from Krutetskii (1976) were included.

Similarly, since one of the major features of mathematics is its logical structure, Wason's (1977) two problems which present the same logical structure in two different contexts were used in an attempt to identify any difference in logical approach between the able and less able students. In connection with these questions it is interesting to note that the results of Wason's study:

> .... suggest that reasoning is radically affected by content in a systematic way, and that this is incompatible with the Piagetian view that in formal operational thought the content of a problem has at last been subordinated to the form of relations in it.
>
> (Wason, 1977: 132)

As a result of this analysis and pilot testing a battery of 38 cards were prepared, each card containing one or more items to be used as the basis for discussion. The content of these cards is presented in Table 1.

THE CONDUCT OF THE INTERVIEWS

The subjects sat at a table alongside the inteviewer. The interviews were tape recorded for later analysis and the recorder was placed on a chair on the opposite side of the table so as to be unobtrusive.

A pencil and paper were provided for the subject who was encouraged to write or sketch if this helped. The inteviewer did not take notes.

When the subject arrived, about five minutes was spent explaining the purpose of the study, asking permission to record the inteview on tape and trying to put the subject at ease.

The initial interview questions were designed to explore the subject's school background. Questions along the following lines were asked:

(a) Where did you go to school? - rural or urban?
- single sex or coeducational?
- Intermediate school?

(b) Describe your feelings towards mathematics as you progressed through the school system.

(c) Describe the attitudes and abilities of other members of your family in relation to mathematics.

(d) How would you rate your performance in mathematics against that in other subjects at different stages?

(e) How did you perform in examinations?

(f) Which subjects did you like most / least?

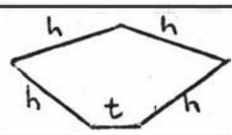
(g) Do you have any theories as to why you, or others, have difficulty with mathematics?

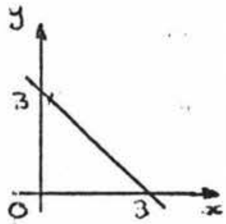
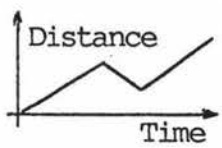
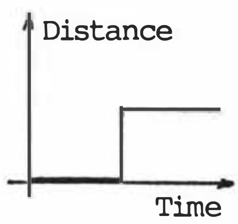
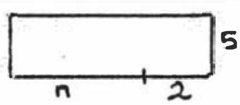
(h) What do you like or dislike about mathematics?

After this, attention was directed to the mathematical tasks. The character of the interviews is better indicated by example than by description and consequently a detailed discussion and analysis of one particular interview forms the basis of the next chapter.

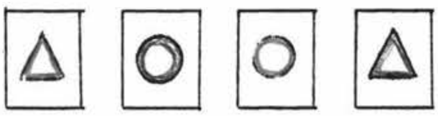
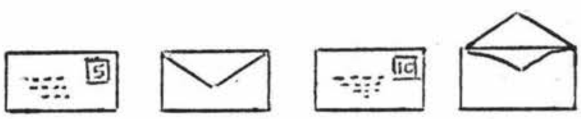
TABLE 1: THE CONTENT AREAS, QUESTIONS, AND SOURCES OF THE 38 INTERVIEW CARDS

Area	Card No	Question	Source
A Natural Numbers and Operations	1	Explain $2 + 3 = 5$ $2 \times 3 = 6$ $4 \div 2 = 2$ $6 - 3 = 3$	Knight
	2	$493 \times 256 = 256 \times \triangle$	Knight
	3	$12 \div 3 = \triangle \div 12$	Knight
	4	$6 - 3 = 3 - \triangle$	Knight
	5	$129 - 48 = \triangle - 50$	Knight
B Fractions	6	$\frac{6}{8} = \frac{3}{4}$ Why?	Knight
	7	$3 \times \frac{3}{11} = ?$ $3 \div \frac{3}{11} = ?$	Knight
	8	$\frac{3}{4} \times \frac{5}{8} = ?$ $\frac{3}{4} \div \frac{5}{8} = ?$	Knight
	9	$\frac{3}{5} + \frac{1}{3} = ?$ $\frac{4}{5} - ? = \frac{1}{10}$	Knight

C Decimals	10	0.3050 = 0.305 Why?	Skemp
	11	26 x 10 = ?	Skemp
D Integers	12	3 - 5 = △ (-3) + (-2) = △ (-3) x 2 = △ (-4) x (-2) = △ (-8) ÷ (-4) = △	Knight
E Variables	13	a + 5 = 8 a = ?	Kuchemann
	14	a + b = 43 a + b + 2 = ?	Kuchemann
	15	Perimeter = ? 	Kuchemann
	16	A figure has n sides all of length 2. What is the perimeter?	Kuchemann
	17	c + d = 10 c < d c = ?	Kuchemann
	18	Which is the larger? 2n or n + 2.	Kuchemann
	19	Cakes cost c cents and buns cost b cents each. What does 4c + 3b stand for?	Kuchemann

	20	Blue pencils cost 5 cents and red pencils 6 cents. If I buy b blue pencils and r red pencils and the cost is 90 cents, what can you say about b and r?	Kuchemann
F	21	$y = x + 2$ Find y if $x = 3$, $x = -2$ $x = n + 1$	Knight
Functions and Graphs	22	The equation of the line is $x + y = 3$. What does this mean? 	Skemp
	23	The graph shows a journey. Describe it. 	Kerslake
	24	Describe the journey 	Kerslake
G	25	Multiply n + 5 by 4	Kuchemann
Algebraic Manipulation	26	Find the area 	Kuchemann
	27	$x^3 . x^5 = x^8$ Why?	Skemp

	28	$\sqrt{4x^6} = ?$	Knight
	29	$(x + 3)(x + 4) = ?$	Skemp
H	30	Solve $3x + 17 = 44$	Skemp
Solutions of Equations	31	Solve $\frac{x}{4} = \frac{2}{3}$	Skemp
	32	Solve $(x - 2)(x - 3) = 0$	Knight
I	33	A jar of kerosene weighs 8 kg. Half of the kerosene is poured out of it, after which the jar weighs 4.5 kg. Determine the weight of the jar.	Krutetskii
Problems	34	A man climbs a mountain at 2 km per hour and returns along the same route at 6 km per hour. Find his average speed.	Krutetskii
	35	Two boys were playing draughts. There were three times as many empty squares on the board as squares occupied by counters. One player had 2 more counters than his opponent. How many did each have on the board?	Krutetskii

	36	A passenger who had travelled half his journey fell asleep. When he awoke, he still had to travel half the distance he had travelled while asleep. For what part of the entire journey had he been asleep?	Krutetskii
J Logic	37	All the cards have a triangle on one side and a circle on the other. Which cards do you <u>have to</u> turn over to see whether the following statement is true? "Every card which has a red triangle on one side has a blue circle on the other". 	Wason
	38	Which envelopes <u>must</u> you turn over to see whether the following rule is being obeyed? "If an envelope is sealed it must have a 10 cent stamp on it". 	Wason

CHAPTER FOUR

THE CLINICAL ANALYSIS - A CASE STUDY

THE METHOD OF ANALYSIS

A number of different approaches to the analysis of interview data were discussed in Chapter 2 and an indication was given that the approach in this study would be a multi-category one. Since this decision differs from that taken by other clinical researchers in the area, some further discussion and justification is appropriate.

The decision was based on the views of Diesing (1971) and Kaplan (1964) which were quoted at the beginning of the previous chapter. It was in an attempt to reflect the 'holistic' view of the research and the 'pattern model' of explanation that a multi-coding approach to the clinical analysis was developed.

Initially, a formal coding of the responses of each subject to each mathematical task was undertaken in which the level of understanding indicated by each sequence of responses was assessed. This coding and analysis is described and analysed statistically in Chapter 6. The interview data was not transcribed from the tapes at this stage, but the response to each mathematical task was played over several times before a decision concerning the level of understanding indicated was made. This initial formal coding provided a very useful overview of the data which prepared the way for a more searching analysis, particularly of the responses of those subjects having the most difficulty.

The second stage of the analysis involved concentrating on the responses of an individual subject, in some cases over a period of several weeks, in order to identify any patterns evident in the data. The method used was very similar to the Constant Comparative Method (Glaser, 1969) in which the analyst starts by coding each incident in the data in as many categories as possible, each incident being compared with previous incidents in the same category.

Glaser writes:

> This constant comparison of the incidents very soon starts to generate theoretical properties of the category. One starts thinking in terms of the full range of types or continua of the category, its dimensions, the conditions under which it is pronounced or minimised, its major consequences, the relation of the category to other categories, and other properties of the category.
>
> (Glaser, 1969: 225)

The categories used in this analysis, in addition to those in the initial coding, were:

(a) The mathematical nature of the responses:
- the character of the frameworks used.
- the nature of the algorithms employed.

(b) The logical nature of the responses.

(c) The developmental character of a sequence of responses:
- Solo Taxonomy (Collis and Biggs, 1979).

(d) The relationships of individual responses within a sequence:
- Tall (1979).

(e) The character of individual responses:
- Piaget's five fold classification.

(f) The character of the interviewer's contribution.

This procedure reflects the view of Diesing concerning the objectivity of the clinical method:

> The objectivity of this sort of model and of the explanations based on it lies not in any one component but in the whole. As I have indicated earlier, the sorts of evidence on which a holistic model are based are not highly reliable, considered in isolation. Consequently any particular interpretation of a theme is questionable, in the sense that plausible alternative intepretations can be developed

> using much the same evidence. A larger network based on a greater variety of evidence is not so readily questionable.
>
> (Diesing, 1971: 158)

The responses were not formally coded with respect to each category in the manner of the initial coding with respect to understanding. Rather each individual response and sequence of responses was considered in relation to each category and notes made of any features which seemed significant. These notes then formed the basis of summaries of the nature of the subject's responses to the tasks in different content areas.

There are considerable difficulties associated with the presentation of the results of such an analysis. The data consists of some thirty hours of tape recordings, and it is clear that this cannot be reproduced in full. It is, however, important that evidence from which conclusions are drawn is presented in such a way that it is accessible to subsequent critical assessment. With this in mind, the details of the interview with one particular subject and the subsequent analysis is presented in some detail in this chapter. For the other subjects, shorter extracts from the interviews will be presented in the next chapter to illustrate particular points.

THE INTERVIEW

General Background

The subject was a female, a student of Psychology and Education. She had attended school in America, a normal Junior School programme being followed by one year of Algebra and one of Geometry at High School.

Both parents were graduates. There was clear evidence of sex-role expectations on the part of the mother. The subject's brother, now an engineer, was always expected to do better at mathematics.

The responsiveness of the less able students to probes designed to explore the structures behind their answers to the interview questions varied considerably. One subject was particularly difficult in this regard, while another, whose interview is reported in this chapter, responded very openly. As a consequence the data from her interview was more revealing than that from any other interview.

The fact that this subject had attended school in America might be seen as a disadvantage in a New Zealand study. However, while there are certainly curriculum differences between countries, in mathematics there is a large body of basic knowledge which is intrinsic to the subject and which is to be found in every mathematics programme (Husen, 1967). Each of the content areas explored in this study are to be found in this 'common core'. Had the study, for example, included geometrical topics this might not have been the case.

This, together with the decision not to look for 'typical cases' which was discussed in Chapter 2, made the advantages of using this particularly revealing interview as the basis of this chapter outweigh any possible disadvantages.

Mother always said he was the good one at maths. He was younger than I was. He was the good one, I wasn't, and yet on an objective exam I scored much better than he did.

Her own assessment of the problem which she has with mathematics is that she began to feel insecure at High School.

Something happened in between Algebra and Geometry - I began I realise that I couldn't succeed unless I memorised all of it.

Her interview responses indicated that she had become extremely rule-dependent in dealing with mathematics.

I. *It didn't worry you whether you understood, as long as you got the right answer?*

S. *No, I couldn't care less why. I don't know why that is the case, because in everything else, in science and everything else, I'm very concerned, in fact I'm quite interested if you show me why.... In mathematics I don't care as long as it is right.*

The subject began to dislike mathematics when she found herself unable to get the right answer.

Where I began not to like it was probably fear of exams. We didn't get it back to know where I was wrong.

She now reacts to mathematical content in a very emotional way.

I have a daughter and she comes to me with a maths problem and I say 'Oh! go away, I don't know'. In fact it upsets me even to think about it.

The subject offered the opinion that she was highly achievement motivated and that she had always been considered as having an exceptional memory. The rule-dependence, which was such a feature of her responses to the mathematical tasks, makes it seem likely that she had learned by rote, relying on her memory to cover for her lack of understanding. Eventually the hierarchical nature of the subject matter caught up with her. Maternal expectations are also a likely factor in that they enabled her to rationalise her difficulties. There was no reason why she should expect to understand.

The Mathematical Tasks

In presenting the subject's reponses to the mathematical tasks, a choice had to be made between presenting both the responses and analysis together, perhaps with one on the left hand side of the page and the other on the right, and presenting the responses first and the analysis after. The latter was chosen in order that a reader might be able to react to the interview data without the influence of the analysis. Consequently, responses to individual items are presented, followed by an analysis of both the subject's and the interviewer's contribution. A general comment follows after the analysis of the responses in each content area.

Natural Numbers

Addition

I_1 (Card 1, first line only: Explain 2 + 3 = 5)
What I would like to know is this: Presumably you believe that 2 + 3 = 5 - but suppose you were asked to explain that to someone - how would you explain 2 + 3 = 5?

S_1 *Um... probably I'd say ... um ... here are two apples and here are three apples, put them together and how many apples have we got? One, two, three, four, five.*

I_2 *Good.*

S_2 *Is that right?*

I_3 *Yes, that's just the sort of thing I wanted.*

The hesitation in S_1 seemed to relate to uncertainty about the type of explanation required rather than indicating any lack of confidence in the explanation itself.

The use of specific objects, (apples, rather than 'things') indicates a uni-structural response (Collis and Biggs, 1979). The 'addition frame' indicated by the response is the most common one of the union of two disjoint sets.

Since the character of the subject's concept of addition of natural numbers seemed clear from the responses, the interviewer's comments were supportive in character rather than probing.

Multiplication

I_1 (Card 1, second line: 2 x 3 = 6)
What about two times three?

S_1 *.... Hm ... Well, I'd probably take three... er, two pencils ... two times ... two times ... hm .. Well, I'd probably take three pencils and another three pencils ... and I'd add them all up and it comes to six. Is that right?*

I_2 *Fine. I noticed that you had a job to decide whether you were going to have two lots or three.*

S_2 *Yes.*

I_3 *... or three lots of two. Which way do you think of two times three? Is it two lots of three or three lots of two?*

S_3 *... Well ...*

I_4 *I picked up from you ...*

S_4 *(interupting) When you first said to me "two times three", I was thinking that - I was looking out there (points to the window) - it would be three, but when I looked down here (points to the card), I saw the two first and began to think - hm, well - that's probably not right.*

I_5 *Would you read that (points to the card) as two times three?*

S_5 *Yes.*

The different reaction to the verbal 'two times three' stimulus and the verbal/symbolic '2 x 3' is interesting, particularly when the subject agrees with the verbal interpretation of the symbols.

The response was uni-structural again with the 'multiplication frame' being that of repeated addition.

The interviewer's responses I_2, I_3, and I_4 were non-suggestive probes.

Division

I_1 (Next line of card 1: $4 \div 2 = 2$)
What about four divided by two?

S_1 *Four divided by two - I have to 'explain that to someone. ... four somethings divided by two ... (long pause) ... (laugh) ... its difficult because I've never had to explain it - I just memorise it ... (long pause).*

I_2 *So you just remember, as a rule, that four divided by two is ....*

S_2 *... two, yes ... yes ... yes. If I have to think about it ... yes ... I guess it's memory, memorisation. Yes, I'm thinking that - that's an interesting thought - how would I go about showing four or something divided by two? Well, I suppose then I could put four pencils, one, two, three, four (draws four lines on the paper) and give one person two and give the other person two and then I'd add up and that gives two - Aw! that's not right, is it? (Pause)*

I_3 *Yes, that's fine, it's the process of sharing between two groups.*

S_3 *Yes, but it wouldn't be right would it? I still have four (pause)*

I_4 *But the two is the size of each group, isn't it? So if you have, say, twelve divided by three, you would have twelve pencils and share them up between three people and they get four each. (Pause)*

S_4 *I think a pie is easier - I wish I had thought of a pie, you know. (Pause)*

I_5 *But with a pie its hard though, because where does the four come in if you have a pie?*

S_5 *But when you asked me to divide, the first thing that popped into my head was an apple - because that's how we used to learn division - took an apple and divided it up, or a pie and divided it up - and you see I couldn't figure out about the four.*

No immediate explanation or model came to the subject's mind and this, in the terminology of Tall (1979), produced a conflict/mental block situation indicated by the nervous laugh and the rambling, playing-for-time, talking of responses S_1 and S_2. After this was brought under control, an appropriate 'sharing' model surfaced (S_2). However, the subject was still using the 'addition frame', which had been extended to cover multiplication, in which the answer was the union of the two sets - the total number of pencils (S_3). Even when the model was completed by the inteviewer (I_4) it was rejected and an inappropriate 'fraction' model substituted.

The responses indicated only instrumental understanding of division, and again the responses were uni-structural (pencils, apple, pie).

The interviewer's responses vary in character:

I_2 - encouragement to continue

I_3 - confirmation - confidence boost

I_4 - unsolicited explanation

I_5 - probe

<u>Subtraction</u>

I_1 (Last line of card 1: 6 - 3 = 3)
What about six minus three?

S_1 *That would be simple, I'd put six out there (makes a circle on the table with her hand) - minus three (indicates removal) - I'd put six out and minus three.*

No hesitation was shown here at all. The subject was clearly able to reverse the addition operation. There was no mention of specific objects this time, so perhaps it was a multi-structural response indicating relational understanding.

GENERAL COMMENTS

(i) It should not be inferred from the uni-structural responses that this is the level at which the subject operates with natural numbers - only that this is the type of explanation that is seen as most appropriate in this context.

(ii) The conflict between the visual and verbal presentations in the section on multiplication provides evidence for the Davis and McKnight (1979) concept of 'visually-moderated sequences'.

(iii) The sequence on division has an interesting pattern of qualitatively different thought processes (Tall 1979).

S_1	*Four divided ... someone. (long pause) (laugh)... ... memorise it.*	initial response. conflict due to failure to retrieve an appropriate schema leads to mental block.
S_2	*... something divided by two. Well, I suppose then... ... and that gives two.*	starts again with a superposed schema involving the two disparate 'sharing schema' of division and the 'union schema' of addition.
	Aw! That's not right is it?	conflict.
S_3	*Yes, but... I still have four.*	conflict schema.

S_4	*I think a pie is easier.. ... you know.*	<u>superposed schema</u> from fractions which the subject feels will resolve the conflict.
S_5	*But when you asked me .... four.*	<u>explanatory schema</u>, but it may be a rationalisation brought about by an inappropriate interviewer response.

(iv) The interviewer probes in the multiplication extract were successful, but in the division extract the probe concerning the representation of the four in the 'pie schema' was too specific and failed to produce anything but a rationalisation.

The unsolicited explanation concerning division (I_4) was inappropriate. With a less mature subject, an intervention of this kind would be likely to stop any further exploration by the subject.

<u>Fractions</u>

<u>Equivalent Fractions</u>

I_1 (Card 6: $\frac{6}{8} = \frac{3}{4}$ Why?)

S_1 *I don't know anything about fractions - now listen, remember I'm hopeless on this.*

I_2 *Fine, I'd just like to know why.*

S_2 *O.K. Now six over eight equals three over four, you want to know why. Well you want to know what I think about it?*

I_3 *Mm.*

S_3 *Well, the first thing I would do is take ... what would I do?... now you see that's my problem, 'cos this is my problem - all the time in recipes I seem to go - one quarter, one half, three quarters, one cup, and all the time you're asking me for two thirds, and I* *always think, now is two thirds more or less than three quarters? And when I first look at that (points to the card), I apply my rule. It's quite simple, its two into six and two into eight.*

Now you ask me why - I have to explain it. The first thing that pops into my mind - I'd take a pie, cut it all up into eight and then I'd count six (cuts an imaginary pie with her hand). Then I'd divide it into four and count three then that's the same thing. But then I'd stop and say no that's not right.

I_4 *Why isn't it right?*

S_4 *Because if I picture three quarters, I picture a half and a bit like this (draws ) that's three quarters of my pie. Now if you ask me to picture six eights it's like this (draws ) and this is going to contain - Oh! it is the same thing isn't it. Well, I was thinking no it would be more like that (draws ) - only two left over of the eight - so I thought that was wrong.*

The initial resonance had two representations of fractions in it:

(a) the rational number $\frac{p}{q}$ evidenced by the reading of $\frac{6}{8}$ as six over eight and by the rule 'two into six and two into eight'.

(b) the proportion of part of an object to the whole as indicated by the recipe illustration.

There is conflict between these representations until (a) is rejected as a basis for explanation. Representation (b) then forms the basis of an ongoing schema which is followed through very well by the subject without any intervention by the interviewer.

Multiplication

I_1 (Card 7; line 1: $3 \times \frac{3}{11} = ?$)

S_1 *I really don't know what to do with it ... the only thing I can think of ... I'll work on it. Three times three elevenths - you want me to work on it. Well, I need three elevenths of* *something - I divide something into eleven, I cut it up into eleven and then I add three of it, then add three more and then three more and that's what it would be. That's how I would have to do it.*

But my rule isn't very good - 'cos I know there is a rule there, see - I don't know whether I go three times the top and three times the bottom - nine divided by thirty three - um - I don't know what the rule is, but I can work it out like that - I don't know what the rule is.

The subject combines two previously established frameworks - multiplication as repeated addition, and the fraction as a proportion of the whole - very well, just failing to make the final addition to give the correct answer.

Again there is conflict between this logical schema and the rule, which this time has been forgotten.

Division

I_1' (Card 7, line 2: $3 \div \frac{3}{11} = ?$)

S_1 *Three divided by three elevenths ... I don't know. If you gave me one divided by a half, I know it would be a half, two divided by a half would be one. I know the rules. Three divided by a half would be one and a half, one point five. Yes, yes that's fair enough.*

I_2 *That's the rule?*

S_2 *Yes. Now three divided by three point eleven ... um ... I don't have a rule for that. I'm stuck.*

In view of the subject's failure to explain division of natural numbers it is not surprising that no explanation surfaced here.

The incorrect rule in S_1 is a common one, misinterpreting 'divided by a half' as 'divided into halves'.

The misreading of $\frac{3}{11}$ as three point eleven seems to be an isolated error without particular significance.

The subject was unable to offer any response at all to Card 8: $\frac{3}{4} \times \frac{5}{8} = ?$ and $\frac{3}{4} \div \frac{5}{8} = ?$

Addition

I_1 (Card 9. line 1: $\frac{3}{5} + \frac{1}{3} = ?$)

S_1 *This is fairly easy, isn't it?*

I_2 *If I say it's easy and you can't do it ...*

S_2 *O.K. That's easy, that's a half, isn't it?*

I_3 *What makes you think that?*

S_3 *Well, I'd add - I think I would - I'd add the top ones and the bottom ones, four eights, and that's a half, is that right? It's not right.*

I_4 *No, it's not right.*

S_4 *It's not. (pause)*

I_5 *What does three fifths look like?*

S_5 *(Draws*  *)*

I_6 *What does one third look like?*

S_6 *Draws*  *) It's going to be almost one.*

I_7 *It can't be a half anyway.*

S_7 *Oh no! There must be a rule, I don't know it.*

The incorrect rule of adding numerators and denominators is a very common one.

With some direction from the interviewer the subject was able again to combine the two well established fraction and addition frames to produce the good relational response 'it's going to be almost one'. However the most important thing for the subject was still to find the rule.

GENERAL COMMENT

In dealing with fractions the rule-dependence of the subject was particularly evident. This is not surprising since the 'proportion of the whole' framework is useful in understanding equivalent fractions, multiplication by a whole number, and to a lesser extent in addition of fractions, but is of no help at all in other fractional operations.

There are two good examples of Piaget's 'liberated conviction in this section. Working with $\frac{6}{8} = \frac{3}{4}$ the subject discovers for herself the equivalence in a way which she found surprising. The same thing happened in addition of fractions when, beginning with appropriate frameworks, the subject was able to develop, with very little help, an ongoing schema which led to the correct answer. These suggest that the problem which the subject has with mathematics is likely to be related more to the lack of appropriate conceptual frameworks, which act as premises, than to faulty logic in applying these frameworks.

Variables

Letter evaluated (Kuchemann 1978)

I_1 (Card 13: a + 5 = 8, a = ?)

S_1 *A plus five equals eight - a equals three.*

I_2 *Right. How do you know?*

S_2 *I just took five from eight to get three.*

I_3 *Why did you take the five from the eight?*

S_3 *Why did I do it? I don't know - because it says something plus five equals eight, so I know I have five of something and I've got to come up with more to give me eight.*

The variable is interpreted as 'something' - a place-holder in the addition/subtraction frame successfully used earlier. The response was judged to show relational understanding of the item.

Letter Ignored

I_1 (Card 14: a + b = 43, a + b + 2 = ?)

S_2 *O.K. a plus b equals fortythree, a plus b plus two ... (long pause) ... 45?*

I_2 *Mm (nods).*

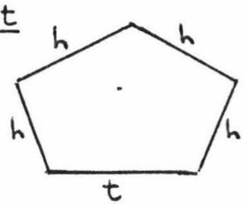
S_2 *(laughs) I thought it was more difficult. My thought processes were - I was looking at the a plus b - you see I don't know what the combination is that gets forty three, so I don't know what that a and b is, and then you tell me a and b, which I don't know what it is, and then add two - what the heck does that mean?*

The initial response was to try to evaluate a and b, as in the previous question. The resulting conflict was resolved with the help of the well established addition frame. It would have been interesting to explore this 'letter ignored' concept in a less well established operational framework. In view of the instrumental understanding of division shown by the subject, the response to:

$$a + b = 44, \ (a + b) \div 2 = ?$$

might well have been qualitatively different.
Again the response was judged as indicating relational understanding.

Letter as an object

I_1 (Card 15: 

perimeter = ?)

S_1 *What is the perimeter?*

I_2 *The perimeter is just the distance all the way round.*

S_2 *Yes, well, you can just add t plus h plus h plus h plus h.*

I_3 *Fine.*

S_3 *There is probably a cleverer way of doing it.*

I_4 *How would you write h plus h plus h plus h?*

S_4 *Four hs.*

No difficulty here with letter as an object. The response S_4 should have been followed up, it might mean 4h or perhaps hhhh.

Letter as a specific unknown

I_1 (Card 16: A figure has n sides all of length 2. What is its perimeter?)

S_1 *A figure has n sides all of length two, O.K. What is its perimeter? That means we don't know how many sides it has - is that right? - Oh! that has a length of two. (Points to the card).*

I_2 *Each side has a length two.*

S_2 *(pause) Two n.*

Again the response reflects the confidence the subject has in multiplication as repeated addition. She should have been asked to write S_2 down, but it is interesting to notice the difference between S_4 of the previous extract (Four hs) and S_2 of this section - 'two n', not 'two ns' or 'n twos'.

Letter as generalised number

I_1 (Card 17: $c + d = 10$, $c < d$, $c = ?$)

S_1 *c plus d equals ten, c ... (pause)*

I_2 *... is less than d.*

S_2 *Is that 'less than d?'*

I_3 *Yes, c is the smaller one.*

S_3 *O.K. ... (long pause) ... Four.*

I_4 *Fine.*

S_4 *Alright? (laughs since interviewer does not respond) - because if it was the same thing as d then it would be five - five plus five equals ten - but we know it is smaller than d - so the next one it could possibly be is four, and if we make it four then d is six. So it could be three and seven, or two and eight, or one and nine. So we don't know what c is.*

The initial response 'four' (S_3) is reconsidered in the light of the interviewer's lack of confirmation, which in fact acts as a probe. The conceptual framework of the response is again the confident 'addition of natural numbers'. The possibility of c being fractional, zero or negative was never considered.

The response was considered to indicate relational understanding of the item.

<u>Letter as a variable</u>

I_1 (Card 18: which is the larger, 2n or n + 2?)

S_1 *Which is the larger, two n or n plus two? ... ... two n, is that right?*

I_2 *Well, tell me why.*

S_2 *Which is the larger? Well because 2 is ... um ... two times any number ... O.K. ... two times ... O.K. ... which like four times four is sixteen ... O.K. ... otherwise you've got one number, like four, plus two which is six, so its bound to be more if it's times. Is that right?*

I_3 *(Interviewer does not respond verbally, but smiles with the intention of being encouraging without committing himself to saying whether or not the response is correct).*

S_3 *(laughs) - you think that's funny?*

I_4 *No, I don't think it is funny at all.*

S_4 *Is that right?*

I_5 *(Still no verbal response)*

S_5 It's not right - that's what I think.

I_6 It's not completely right, because what would happen if n was two?

S_6 ... (pause) ... yes, but I don't think it is.

I_7 You don't think n is two?

S_7 No.

I_8 What do you think n is?

S_8 Its probably something - not the same as two. Its probably bigger than two, because if it were two you would have put down two times two, wouldn't you? I always think of it as something a whole lot more - in fact you don't know how much more it is, you see. It refers to a whole lot more of something and whatever that whole lot more is, its two times that whole lot more, and that's a whole lot more than a lot more plus two. That's what I think.

I_9 You're right to this extent, that if n is a large number then multiplying by two will make it more than adding two onto it. But what made you think that n is a large number? It intrigues me to understand why you wont allow me to say n equals two, but you will let me say, for example, n equals ten.

S_9 I think because when I learned about letters they always stood for something more.

I_{10} n is always a large number?

S_{10} Yes, n is more.

I_{11} If I had said x?

S_{11} No, it would still be more. Letters always stand for a whole lot more than numbers, don't they?

I_{12} Do they?

S_{12} Yes, they mean more. You see if you have two of something, I can tell you what that is, one plus one corresponds to two. But a doesn't correspond to two does it?

I_{13} What if I write down: x plus two equals three, (writes $x + 2 = 3$) what is x?

S_{13} Well - one.

I_{14} *Come on (smiling) - it can't be one because letters are always big numbers.*

S_{14} *Well, I guess that's inconsistent - but that's different - I said that was O.K. - well no! (indignantly) you asked me two different questions.*

I_{15} *Yes, I know, I was just exploring this business of a letter always standing for a large number.*

S_{15} *Yes it does.*

This is one of the most surprising and interesting sections in the study.

The subject's initial responses (S_1, S_2), apart from the confusion between 4^2 and 4 x 2, are consistent with Kuchemann's experience with this question (Kuchemann, 1978), and with a written test this is all the information we would have.

The insistence on confirmation that this was in fact the right answer (S_2, S_4) seemed, in relation to the subject's behaviour in other parts of the interview, to reflect the confidence shown. She was sure the response was right and expected approbation. When this was not forthcoming (I_3, I_4, I_5), the reaction was to terminate the schema - "that's what I think" - a take-it-or-leave-it statement (S_5).

The interviewer responds with the n = 2 'torpedo' (Tall, 1979) designed to challenge this confidence. The response (S_6) indicates that the subject can see the implication of n = 2, but is unwilling to accept it. The logic is impeccable:

> If p (n = 2) implies q (2n is not greater than n + 2) and q is false (unacceptable) then p is false - n cannot be equal to two.

It seems that the rest of the extract may be interpreted in two conceptually quite different ways.

The interviewer assumed that in S_8 the subject was saying that n stands for a large number, and hence that 2n is larger than n + 2. In mathematics, it is common to talk of the nth term of a sequence and expressions such as 'n tends to infinity' occur frequently. Thinking that this might be the source of the subject's conception of n, the interviewer suggests x, but this is rejected. This presents the opportunity for the second 'torpedo' x + 2 = 3. The subject recognises the inconsistency, but believes this to be acceptable in view of the 'different questions' asked. This ability to live with apparent contradiction has been noted elsewhere in subjects with a rule-dominated approach to mathematics. (Erlwanger, 1973).

However, on careful consideration, a second, more satisfying, interpretation emerges.

There are two responses which do not fit into the pattern suggested above. In S_8, the subject says", ... because if it were two you would have put down two times two, wouldn't you", and in S_{12} "You see, if you have two of something, I can tell you what that is, one plus one corresponds to two. But a doesn't correspond to two, does it?"

This rejection of n as representing a specific number is inconsistent with the subjects own use of n = 4 in S_2 and was discounted by the interviewer as 'romancing' (Piaget, 1927). But there are other clues which indicate that the interviewer may have misinterpreted the responses. The consistent use of the term 'more' is interesting.

S_9 ... *when I learned about letters they always stood for something more.*

S_{10} *Yes, n is more*

S_{11} ... *Letters always stand for a whole lot more than numbers.*

It seems clear that the subject is using the word 'more' in a qualitative rather than a quantitative sense. With this interpretation, 'n is more than two' does not mean that n is numerically greater than two, but that, in the context of this question, n stands for more than just the number two or any other single number. Consequently the subject is expressing her belief that, in this question, n is a variable. Her indignation at the second 'torpedo' question; $x + 2 = 3$, is then quite justified. The questions are different, the 'torpedo' is an example of Kuchemann's 'letter evaluated'.

The possibility of 'romancing' should not be discounted in this extract. The subject had confidence in the initial response and this confidence was challenged. It is possible that there is a degree of rationalisation in the responses, aimed at restoring, or justifying, that initial confidence.

GENERAL COMMENT

Again the principal impression left by the subject's handling of variables is that any difficulties are likely to be due to initial misconceptions. No difficulty was experienced with the abstract nature of the questions. The previously established addition and multiplication of natural numbers frameworks were easily adapted to accomodate variables.

The subject had little or no experience of functions and graphs, algebraic manipulation, or solution of equations, so these questions were passed over quickly.

Problems

I_1 (Card 33)

S_1 *(Reads the card) A jar of kerosene weighs eight kilograms. Half the kerosene is poured out of it, after which the jar weighs four point five kilograms. Determine the weight of the jar.*

If you've got eight kilos, you're going to have four kilos. Oh! you want the weight of the jar. A jar of kerosene weighs eight kilos. Half the kerosene is poured out ... (pause)

I_2 *You can write something down if you want.*

S_2 *O.K. If the whole thing weighs eight kilos (writes 8). So then you subtract four point five and you get three point five, right? Then you divide that by two ... (pause) ... one point seven five. Is that right?*
(No response from the interviewer)
It's not right. One point seven five times five ...

I_3 *Are you looking for a rule?*

S_3 *Yes, I know I've got to ... no, I'm trying to figure it out - you see you have to ... (goes back to the division by two, which she had written down. Seems to think that this is incorrect).*

I_4 *Now tell me why you did that. You are right to there - (points to the 3.5) - you are not right when you divide by two. So tell me, what is that eight?*

S_4 *That represents the whole weight of the jar.*

I_5 *And what does this represent? (Points to the 4.5)*

S_5 *That weight represents the jar after its got everything out of it.*

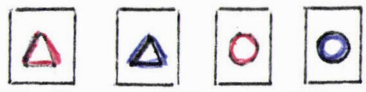
I_6 *So what does that represent? (points to the 3.5)*

S_6 *The total weight of what is in the jar.*

The problem here appears initially to be the subject's misreading of the question. But there is probably more to it than that. When the interviewer suggests (I_2) that the subject write something down, the obvious candidates are the numbers which appear on the card, 8 and 4.5. These immediately suggest a binary operation and the 'pouring out' action fits in well with the subject's 'subtraction frame' indicated in the section on natural number - hence the 3.5. The other piece of numerical information in the question is 'half is poured out', and it seems likely that this, taken out of context, is the reason for the division by two.

When asked to explain, the subject has to justify the steps taken and comes up with an interpretation of the numerical values which satisfies the proposed algorithm but not the original question. In other words, rather than reading the question incorrectly, the subject seems to have derived a question to fit her numerical calculations. These calculations were based on visually moderated sequences rather than on the problem set.

<u>Logic</u>

I_1 Card 37. All the cards  have a triangle on one side and a circle on the other. Which cards do you <u>have to</u> turn over to see whether the following statement is true?, 'Every card which has a red triangle on one side has a blue circle on the other).

S_1 *Which cards do I <u>have to</u> turn over ... I only have to turn over one ... Aw! perhaps that's not right.*

(After thinking and re-reading the question, the subject decides that this is right, she only needs to turn over the card showing the red triangle).

I_2 *Well you certainly need to turn that one over, I agree with that, but is that the only one?*

S_2 *Well yes, to be efficient, and that will tell me right away if the statement is true or false.*

I_3 *Right, you turn that over and suppose it has a blue circle on the othe side?*

S_3 *Then I'd say it is true, and if it doesn't then I can say it is false.*

I_4 *Certainly if it doesn't it is false, I will agree with that, but if it does, can you be quite sure?*

S_4 *Yes, because there is no other card with a red triangle on one side.*

I_5 *But you can't see all the triangles. There is a triangle under here (points to the blue circle) and under here as well (points to the red circle).*

S_5 *But you didn't ask me about those, you only asked me about these (points to the cards with the triangles on). ... (pause). Have you asked me about those?*

I_6 *Well, I asked you about all the cards. (Reads) Every card which has a red triangle on one side has a blue circle on the other. You see this one (points to the blue circle) could have a red triangle on the other side, so could this one (points to the red circle).*

(pause)

S_6 *No, I would still just take this one (red triangle) because you want to know if it is true or false and if I pick that one out, that triangle, then I know right away. Its either true or false on the red triangle - I don't know anything about blue circles.*

These responses are very typical of those given by all the subjects interviewed. In fact, only one of the twenty six could be persuaded to turn over the red circle - the other required card.

S_4 is the most important response in that it indicates that the subject has a good appreciation of the logical implication in the question and has not made the very common error of interpreting this as logical equivalence. Her error is less logical than it is conceptual, she seems unable to explore the hypothetical triangles on the other side of the cards.

In terms of Collis and Bigg's (1979) SOLO Taxonomy, the question demands an 'Extended Abstract' response involving generalization to situations not experienced, while the subject's response was only 'Relational'.

The response to card 38, which presents exactly the same logical problem in the more concrete context of envelopes and stamps, was answered in precisely the same way by the subject. She refused to be persuaded that it was necessary to turn over any other than the sealed envelope.

CONCLUSIONS

In terms of Skemp's model of intelligent behaviour which forms the theoretical basis of this study, the purpose of the clinical procedures is to provide evidence of the nature of the cognitive structures employed by the subjects.

There are three major features of this interview, which will be explored in greater depth in the next chapter, and which relate directly to Skemp's model.

Firstly, the interview indicates, quite clearly the importance of appropriate initial frameworks, or schemas, relating to arithmetic operations. The subject was confident in the 'union of disjoint sets' framework for addition and in the 'repeated addition' framework for multiplication. These schemas were employed successfully in the solution of items from natural number, fractions, decimals, negative number, variables and problem solving. The availability of these schemas were reflected in her willingness and ability to work on items involving these operations, in order to create the links between present and goal state, either when no 'built in' path was activated, or when a suggested link was challenged. In contrast no general schema for division of any kind, or multiplication of fractions was available and consequently there were no premises from which the subject could work on an item.

Secondly, any difficulty which the subject had in forming links within a schema seemed to relate much more to deficiencies in the schemas themselves than in her inability to use logical argument to form links. She was well able both to see the logical implications of the interviewer's comments and to draw impeccable conclusions from her own assumptions. There is certainly no evidence here of the failure at mathematics being due to an inability to think logically.

Thirdly, the subject's performance on the items relating to the use of variables indicates that her lack of some appropriate mathematical schemas is unlikely to be due to the abstract nature of these schemas. In each of the six different ways in which a letter is used in mathematical statements the subject approached the question appropriately and with confidence. There was no evidence of mental blocks here. In fact the only point in the whole interview when the subject showed an unwillingness to tackle material which she had seen before was in the section on fractions. Her initial response, "now remember, I'm hopeless on this", was followed by a complete unwillingness to even contemplate

$\frac{3}{4} \times \frac{5}{8}$ or $\frac{3}{4} \div \frac{5}{8}$.

These tentative conclusions are tested against the responses of the other subjects in the next chapter.

CHAPTER FIVE

GENERAL CLINICAL ANALYSIS

In this chapter a more general analysis is presented drawing on a number of different interviews. The coding of interview responses according to the level of understanding shown, which is presented in the next chapter, indicated (Figure 8) that there was a group of seven students whose understanding of mathematical concepts and procedures was very poor indeed. Their responses indicated, quite clearly, that they were the kind of subjects who are the focus of this study. One of them was N.I., whose interview formed the case study of the previous chapter. The interviews with the other six form the basis of this chapter, with comparisons being made with the responses of the more able subjects.

BACKGROUND INFORMATION

The school and family background of the subjects in the least able group is of obvious importance and is presented below.

Subject U.I. Female. Age - late forties.

The subject was very nervous and was the only one who asked that the interview should not be tape recorded. Her wishes were followed and the interviewer took notes which were amplified immediately after the interview.

Educated in New Zealand, the subject took mathematics to Fourth Form level. She took up nursing as a career but at the time of the interview was enrolled, full time, in a Business Studies degree course. In recent years she had twice attempted School Certificate Mathematics, studying by correspondence, but had failed both times.

She always found mathematics difficult, but was unable to indicate any particular time at which the problem arose or to give reasons for it.

U.I. was very tense during the interview, and the interviewer found it impossible to draw her with general questions. Consequently, the interview moved fairly quickly on to the specific items with which the subject was a little more comfortable. Even here, any attempt to explore the way in which the subject was thinking tended to be met with defensive, one word answers. The interviewer gained the impression that it upset her to have to think about her difficulties.

<u>Subject A.A.</u> Female. Age 18 years.

Educated in Northland at a primary school and then a Form 1 - 7 secondary school, she took mathematics to 5th Form level, but was having a great deal of trouble. Her parents obtained private coaching for her and, much to her surprise, she passed School Certificate (57%). She attempted University Entrance mathematics but performed very poorly. She was enrolled in a Bachelor of Social Work degree programme.

Her father was a primary school teacher and her mother a housewife. Her older sister did not find mathematics easy but was successful to Bursary level. Her younger brother was very good at mathematics.

I. *How do you feel about mathematics?*

S. *I try to avoid it.*

I. *Do you know why you try to avoid it?*

S. *I suppose it's a sort of mental block really. Anything maths I sort of think - Oh! I can't do it.*

I. *When did you first notice this feeling towards mathematics?*

S. *Probably about the third form - I think it was partly the teacher you had. If I have a good teacher, I quite enjoy it, I get pleasure out of it, and can do it.*

Although the subject has considerable problems with mathematics, her emotional reaction to the subject was much milder than many others in the group. Her ability to transfer at least some of the blame to the teacher may be a factor in this.

<u>Subject I.A.</u> Female. Age - early forties

Brought up in England during the Second World War, the subject was evacuated from London during the bombing. Both her parents were killed. Raised by an aunt and a grandmother, she attended 13 different primary schools but was sent to a boarding school for her secondary education. The mathematics programme was mainly arithmetic but with a little algebra - about fourth form level.

She has two sons, both very good at mathematics, and until recently, when she enrolled for a B.S.W. degree, she had been a housewife.

She had problems with mathematics from primary school.

I. *Can you identify some time when you realised that mathematics was not for you?,*

S. *When I was told that mathematics was not for me. I think after my mother died I sort of crashed at everything.*

I. *How old were you then?*

S. *Ten.*

Before that her achievement had been rather erratic.

S. *I remember going to one school and being bottom of the class for maths, and then I went to another school with a very small class and I was told that I had the brains to do maths, or sums, or whatever, and that I could easily come top of the class. That year I became top of the class. If someone in authority tells me that I can do it, I believe them.*

Like the other older subjects in the group, I.A. reacts very emotionally to mathematics. Undoubtedly the very disturbed primary school years contributed to the problem, together with the extreme sensitivity of the subject to extrinsic motivation in the form of teacher expectation.

<u>Subject A.R.</u> Male. Age - 18 years.

Went through a normal primary - intermediate - secondary school programme in Levin. He took mathematics in the School Certificate examination but failed (39%).

Father was a joiner and mother a housewife. Neither had indicated a positive or negative attitude to mathematics and all his brothers and sisters were much younger and consequently their mathematical ability was not yet evident.

His difficulty with mathematics was not evident until secondary school. He had always quite enjoyed and been successful at arithmetic. Not a very talkative subject, he could offer no explanation as to why he found secondary school mathematics difficult. He seemed to be quite resigned to the fact that he could not do mathematics and was not particularly upset by his difficulties.

<u>Subject A.N.</u> Female. Age - 19 years.

Primary - intermediate - secondary school sequence in Levin. She was surprised to get 45% in School Certificate mathematics, she had expected much less.

Her father admits to having had trouble with mathematics, but her mother, who is a primary school teacher, coped well. Two younger brothers show no sign of having difficulties.

She found mathematics difficult right from the start. In Standard 2 she can recall difficulty with whole number arithmetic. The teacher seemed unsure of the 'new maths' material at that time - although this did not seem to worry the rest of the class.

She reacted by withdrawal from many of the topics presented and said that her mind went blank.

In the discussion of decimals, a particularly difficult area for her, A.N.'s response was particularly significant.

I. *And you never understood at the time?*

S. *I think I understood it for a short period of time, but I forget it.*

I. *I see, you understand it when it is first being presented?*

S. *Yes, but I have to keep on being told, and that made me feel really stupid - after a while you get too embarrased to ask.*

Again a clear indication of the close association between emotional factors and extrinsic motivation. She clearly associates decimals with the acute embarrassment she felt at being unable to understand.

Subject N.E. Female. Age - 18 years

Attended a small country primary school up to Standard 3, then primary - intermediate - secondary school in Feilding. She dropped mathematics in the fourth form.

Her parents have never indicated their reaction to mathematics, but her older sister was quite good and her younger brother exceptionally good at mathematics.

She has had difficulty as long as she can remember - "Usually I just block my mind to it, pretend it is not there". She did not know why she had this reaction - thought it might be "teachers or something".

She had a very similar emotional block to that shown by A.N. but no real reason for this block emerged during the interview.

The important features of this background information seem to be:

(a) Difficulty in mathematics did not seem to run in families.

(b) Strong emotional reaction to mathematical content was common, but not universal, and usually manifested itself in some form of withdrawal from the situation.

(c) This emotional reaction was often associated with strong extrinsic motivational factors.

THE COGNITIVE SYMPTOMS

Frameworks

One of the tentative conclusions suggested by the interview with N.I. recorded in the previous chapter was of the importance of appropriate initial frameworks.

The term framework comes from Minsky (1975). It has a great deal in common both with Skemp's schemas and with concepts used by other writers (Schemata, script etc). There are differences (Davis and McKnight, 1979), but in the present context any one of the terms could be used.

Minsky describes frameworks in this way:

> When one encounters a new situation (or makes a substantial change in one's view of the present problem) one selects from memory a substantial structure called a frame. This is a remembered framework to be adapted to fit reality by changing details as necessary. A frame is a data-structure for representing a stereotyped situation, like being in a certain kind of living room, or going to a children's birthday party. Attached to each frame are several kinds of information. Some of this information is about how to use the frame. Some is about what one can expect to happen next. Some is about what to do if these expectations are not confirmed.
>
> (Minsky, 1975: 212)

In the context of this study, the stereotyped situations are those of meeting certain kinds of mathematical problem. When faced with Card 1: 2 + 3 = 5, Explain., the subject retrieves an addition frame. In the case of every subject interviewed this frame contained not only number facts but also some relational understanding concerning the addition of whole numbers. This was represented by the 'union of disjoint sets' concept.

There seem to be two reasons for the importance of this kind of information in a framework. Firstly, it enables the appropriate framework to be retrieved when it is not explicitly called by the situation. For example, in the jar of kerosene problem, N.I. selected the subtraction operation because of the 'pouring out' involved in the physical situation. Without some understanding it is not possible to solve anything other than standard mathematical problems for which the procedures have been learned. This corresponds very closely to Skemp's (1979a) concept of understanding as the making of connections with an existing schema.

Secondly, relational understanding provides the default procedures if the application of the framework does not immediately produce the expected, or desired, results. N.I., in handling Card 9: $\frac{3}{5} + \frac{1}{3} = ?$, initially used an incorrect rule to give $\frac{4}{8}$, but when this was challenged, she was able to use her relational understanding to come very close to the correct answer. Again Skemp's model is helpful. The subject had a well developed cognitive map by which she was able to get from her current state to, or in this case close to, her goal state.

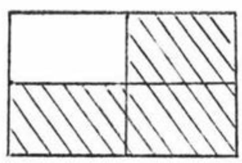
In exploring this idea further it seems best to concentrate on the section of the interviews on fractions. All the subjects were judged to have relational understanding of addition and subtraction of natural numbers, and almost all of multiplication and division. This was far from the case in the section on fractions. For every question on fractions, less than half the subjects indicated relational understanding by their responses.

The difficulty lies in the complexity of a successful fraction framework. The fraction concept has been analysed by a number of writers (Kieren 1976, Novilis 1976, Uprichard and Phillips 1977, Hiebert and Tonnessen 1978, Piaget *et al* 1960). The hierarchy of Novilis, containing fifteen subconcepts, was mentioned earlier and reproduced in Figure 5. Not all of these sub concepts are necessary for the questions used in the interviews, but the interview data illustrates how the unavailability of some of these subconcepts seriously affects the performance of the subject. For each of the interview questions the appropriate framework is discussed and related to the responses.

Equivalent Fractions

The most appropriate model for understanding the equivalence of $\frac{6}{8}$ and $\frac{3}{4}$ (Card 6) is the part-whole, congruent parts subconcept of Novilis (1976). The fraction $\frac{x}{y}$ is associated with the geometric region that has been separated into y congruent parts, x of which are considered.

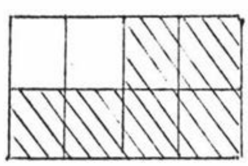
Example:



$\frac{3}{4}$ of the region is shaded.

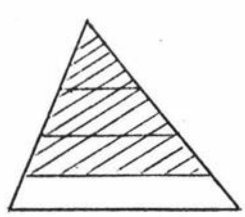
This is the subconcept analysed by Piaget *et al* (1960) and is the most commonly taught and understoood initial concept.

To illustrate equivalent fractions, each part of the diagram is subdivided into the same number of, again congruent, parts.

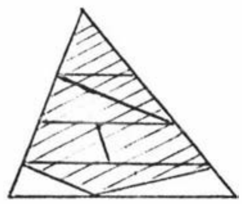


$\frac{6}{8}$ of the region is shaded.

This is not to say that the concept of equivalent fractions can only be understood in terms of this model. For example, the fraction $\frac{3}{4}$ may be interpreted as the solution of the equation $4x = 3$, which has, by using a 'balance' concept of equality, the same solution as the equation $6x = 8$. It is however true that some subconcepts, which are quite appropriate in other circumstances, can be a considerable hindrance to understanding equivalence. For example, the part-whole, non congruent parts concept of $\frac{3}{4}$ is illustrated below.



The triangle is divided into four non congruent parts, three of which are shaded. It is now true that 3/4 of the parts are shaded, although it is no longer true that 3/4 of the whole is shaded. Dividing the area into eight non congruent parts and shading six of them, the equivalence of $\frac{3}{4}$ and $\frac{6}{8}$ seems most unlikely.



In fact, the majority of subjects retrieved the appropriate model but had varied success in using it.

<u>Subject I.A.</u>

I_1 (Card 6: $\frac{6}{8} = \frac{3}{4}$, Why?) *Let's try fractions.*

S_1 *Ugh!*

I_2 *Don't you like these?*

S_2 *No I don't (most emphatically) Why? (long pause)*

I_3 *Well, do you believe that six eighths equals three quarters?*

S_3 *Yes, we've been taught this, haven't we? They go into one another so they are the same.*

I_4 *Do you have any picture of what $\frac{6}{8}$ and $\frac{3}{4}$ are? (pause) ... When we were talking about whole numbers you were looking at groups of objects.*

S_4 *Yes - the same thing - apples - chopping them up. Just chop that one (points to the $\frac{3}{4}$) up again. Chop the three and chop the four - double them and you get six eighths.*

One subject, having been taught the part-whole concept retrieved it successfully, but it was of no use since it was not appropriately connected to the fraction.

Subject N.E.

I_1 *Do you know what three quarters is? Does it mean anything to you?*

S_1 *Yes, that would, but that (points to the $\frac{6}{8}$ wouldn't.*

I_2 *Right. Well what is three quarters?*

S_2 *(Draws*  *) That bit out and it's that bit.*

I_3 *Right. Now in your picture, where is the three and where is the four? ... (pause) ... Why is that three (points to numerator) and that four (points to denominator)?*

S_3 *Would that be the three (points to the part cut out) and that the four (points to the part left over)?*

It is not surprising that the subject could not visualise six eighths! It was only a certain hesitancy in drawing the diagram in S_2 which prompted the probe I_3. Most teachers would accept the diagram as evidence of the understanding of three quarters.

The logic of S_3 is simple - there are two parts to the fraction and two to the diagram, they must correspond.

The availability of a model for certain, common fractions but not in general was found in other responses.

Subject A.R.

S. *I can envisage $\frac{1}{4}$ and $\frac{1}{2}$ but I can't envisage $\frac{6}{8}$. I think of a circle. I can envisage $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{2}$ but not $\frac{6}{8}$.*

Subject A.A.

I_1 *Are you happy working with fractions?*

S_1 *No, I don't mind - but when they get too small sort of thing. I mean like two over fifty eight or something.*

It transpired that large denominators were the problem rather than small fractions.

Two of the more able subjects did not retrieve the part-whole concept.

Subject E.E. (Discussing Card 3: $4 \div 2 = 2$)

S_1 *I see that as four over two.*

I_1 *You see it as a fraction. Do you always see division as a fraction?*

S_2 *Yes*

I_2 *When you see three quarters, what do you see then?*

S_3 *I try to say what times four equals three - that sort of thing.*

This is the more sophisticated concept of the fraction as the solution of an equation.

Subject E.U.

I_1 (Card 6: $\frac{6}{8} = \frac{3}{4}$, Why?)

S_1 *Well, that's two times that so to keep it even, that must be two times that.*

I_2 *Right, Now suppose I say that* $\frac{6}{8}$ *must also be the same as* $\frac{7}{9}$ *because I have added one to the top and the bottom.*

S_2 *No.*

I_3 *Why not?*

S_3 *Because that's twice that and you can't just - its got to be a multiple of it. It will cancel.*

I_4 *Why?*

S_4 *You can't just add ones there and there - it has to be multiply then it will cancel.*

I_5 *Do you visualise $\frac{3}{4}$ and $\frac{6}{8}$ at all?*

S_5 *No.*

The framework here was clearly rule dominated. With much further probing of the concept of 'cancel', the subject eventually explained that multiplying the top and bottom by two was equivalent to multiplying the fraction by one. In this case the subject has the 'rational number' concept of fractions as the quotient of two integers.

The concept of equivalent fractions is essential for understanding the addition of fractions so that will be considered next.

Addition of Fractions

The question $\frac{3}{5} + \frac{1}{3} = ?$ is solved by finding a common denominator, using equivalent fractions, and then applying the union of disjoint sets concept of addition. Since most of the subjects had both of these concepts available, it is perhaps surprising that so many found this question difficult.

The most common wrong answer $\frac{4}{8}$, in which N.I. in the previous chapter was confident, can be derived in at least two different ways. Firstly, it may be just the obvious thing to do. In multiplication of fractions it is appropriate to multiply the numerators and the denominators, and it is natural to do the same for addition. The problem is, though, that the incorrect result is entirely plausible if the part-group, congruent parts concept of a fraction is used instead of the part-whole concept.

$\frac{3}{5}$ may be represented by: 

and $\frac{1}{3}$ by: 

which, by the union of the two groups, quite clearly gives $\frac{4}{8}$. In this connection Kent (1978) comments on the essential fiveness of the solution to the problem two thirds plus three fifths.

The algorithm used $\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$ is, of course an entirely proper one in some circumstances. If, for example, a student achieves $\frac{6}{10}$ in one test and $\frac{15}{20}$ in a second, adding these marks gives $\frac{21}{30}$, not $\frac{27}{20}$.

The problem here, then, seems to be not so much the complexity of a proper justification of the correct, but rather difficult algorithm:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

but the plausibility of the much simpler algorithm. It feels right.

Only one subject had an alternative algorithm.

<u>Subject A.R.</u>

I_1 (Card 9: $\frac{3}{5} + \frac{1}{3} = ?$)

S_1 *Its sort of crossways.*

I_2 *That's right ... (long pause) ... Do you remember something like three times three and five times one and you put it over ... (long pause) ... fifteen?*

S_2 *I wouldn't have put it over fifteen, I'd have just gone six and six.*

I_3 *I see, and what would you have put it over?*

S_3 *Oh! I see. I thought you would just go like this, you get the top and the bottom, five plus one equals six and three plus three equals six.*

The algorithm, then, is $\frac{3}{5} + \frac{1}{3} = \frac{6}{6}$ or in general,

$$\frac{a}{b} + \frac{c}{d} = \frac{b + c}{a + d}$$

S_1 is presumably a vague recollection of the form of the correct rule and was interpreted in this way by the interviewer. The interviewer's response I_2 was really too suggestive and it was fortunate that the subject was able to reject the suggestion rather than just take it up. This is another illustration of the advantages of using Piaget's clinical method with mature subjects.

Multiplication of a fraction by a whole number

The question $3 \times \frac{3}{11} = ?$ has very much the same logical character as the previous one. Since all of the subjects viewed multiplication by three as repeated addition, the question becomes $\frac{3}{11} + \frac{3}{11} + \frac{3}{11}$. The response of N.I. in the previous chapter illustrates this.

Here the difficulties associated with the common denominator no longer exist, but still very few could provide any explanation even with prompting.

Subject A.R.

S_1 *Nine elevenths.*

I_1 *Have you any idea why?*

S_2 *I suppose something like you just times the top line and times the bottom line and that's it. Like three times three is nine and then just eleven because there is nothing there (points underneath the three).*

Often the rules were confused:

<u>Subject A.A.</u>

S_1 *Well, three times three is nine - no hang on - we've got to make them the same. Don't you need a common denominator?*

I_1 *Well, why would you need a common denominator?*

S_2 *To make them the same sort of base. You could say three elevens are thirty three, and thirty three over eleven times three over eleven.*

E.E., who had seen fractions as the solutions to equations, had some difficulty.

S_1 *Nine elevenths.*

I_1 *Right, and how did you do it?*

S_2 *Just three times three divided by eleven.*

I_2 *If you wanted to explain that to someone, why didn't you multiply the eleven as well?*

S_3 *Oh! ... (pause)*

I_3 *Some child says - you've only multiplied a bit of that number by three, you haven't multiplied it all, what about the eleven?*

S_4 *I think of three over eleven as a sort of decimal.*

I_4 *Right.*

S_5 *So if you multiply it by three you've got to get, you know, a number. So you think what it has got to be and then you work it out that way and it comes out to be what you would expect it to be.*

The confused response S_5 illustrates how difficult it is, even for the mathematically able students to retain and use more than one of the subconcepts relating to fractions in their frameworks. She had a fixed view of fractions as numbers - the solutions of equations, and it is extremely difficult to justify the above rule in terms of this subconcept.

Division of a whole number by a fraction

In solving $3 \div \frac{3}{11}$ the problems relate to the division concept as much as they do to fractions. Virtually all the subjects had seen $4 \div 2$ as separating four objects into two equal groups. Division is a process of sharing. However, with this concept $3 \div \frac{3}{11}$ is meaningless - how can you divide three into three elevenths equal groups? An alternative concept of division is required. One possibility is division as repeated subtraction: $6 \div 3 = 2$ because we can take two groups of three away from six. This works well enough with $3 \div \frac{3}{11}$ since we find that the answer is a whole number, we can take $\frac{3}{11}$ away from 3 eleven times, but $1 \div \frac{5}{2} = \frac{2}{5}$ is somewhat harder to explain.

Subject A.A., who wanted to use common denominators in every operation on fractions came close to using repeated subtraction appropriately.

S_1 *Let's have a look at what this is saying first ... (pause)*

I_1 *That's a good response - do you know what it is saying?*

S_2 *It's saying how many times does three elevenths go into thirty three elevenths?*

A general explanation of the rule 'invert and multiply' relies on the concept of division as the inverse of multiplication. $6 \div 3 = 2$ because $2 \times 3 = 6$. Consequently $3 \div \frac{3}{11} = 11$ because $11 \times \frac{3}{11} = 3$. An understanding of this clearly involves formal operations and at the time when the subjects first met division by a fraction these were unlikely to be available. Consequently virtually all the subjects relied on rules, often remembered incorrectly. This has obvious educational implications which will be discussed in the final chapter.

Subject A.R.

I_1 (Card 7: $3 \div \frac{3}{11} = ?$)

S_1 *Hm - I'm not sure - you divide three by three - zero or one I think.*

I_2 *What is three divided by three?*

S_2 *Zero, I think - I don't know - three divided by three - must be zero.*

I_3 *So you would divide that by that (points to the threes). What would you do with the eleven?*

S_3 *I'd just carry it through.*

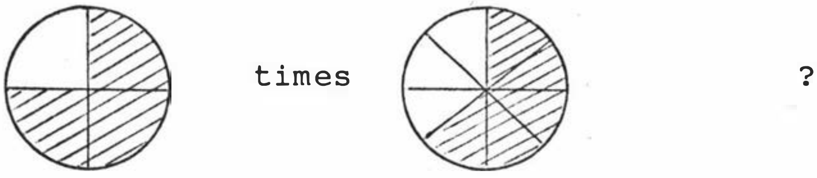
The rule here is just an extension of the multiplication rule

$$3 \times \frac{3}{11} = \frac{3 \times 3}{11}$$

$$\text{so} \quad 3 \div \frac{3}{11} = \frac{3 \div 3}{11}$$

Multiplication of fractions

An explanation of the rule for multiplying two fractions again involves the abandonment of a previously understood concept. Multiplication as repeated addition is no longer meaningful. The multiplication sign is now read as 'of'. We are asking for three quarters of five eighths rather than three quarters times five eighths. Conceptually this is quite difficult because the two fractions are interpreted differently. The second one is a quantity which may be modelled by the part-whole concept, but the first is an operation. It is conceptually difficult to use the part-whole concept for both. After all, what is



Fortunately, in this case the correct algorithm is also the most natural and almost all the subjects gave the right answer. However, there was not one subject who was able to give a really satisfactory explanation.

Division of one fraction by another

There is no conceptual difference between $\frac{3}{4} \div \frac{5}{8}$ and $3 \div \frac{3}{11}$ and the same problems were evident.

The response of A.R. again illustrates the choice of an easy algorithm.

I_1 (Card 8(i): $\frac{3}{4} \times \frac{5}{8} = ?$)

S_1 *I'd just go straight across. Three times five is fifteen, four times eight is thirty two.*

The correct algorithm. However:

I_1 (Card 8(ii): $\frac{3}{4} \div \frac{5}{8} = ?$)

S_1 *Hm. I'd divide the three into the five and the four into the eight to give two.*

Apart from the obvious similarity between the rules, it is interesting to note that he did not say $\frac{3 \div 5}{4 \div 8}$, but $\frac{5 \div 3}{8 \div 4}$ possibly because of a preference for dividing large numbers by smaller ones or perhaps because he knew the answer to $8 \div 4$.

The most dominant impression left by the many hours of interview and analysis involved in this study has been the utter confusion which was shown concerning fractions, and the educational implications of this will be discussed later. Perhaps this should not have been so surprising. The concept is an extremely complex one. No one subconcept will be useful in understanding all the operations. The operations on fractions behave quite differently from the same operations applied to natural numbers and previously understood. The reasons for the behaviour of fractions under these operations are largely abstract rather than concrete in character. With this complexity it seems inevitable that children will abandon understanding and rely on rules - clearly the route taken by almost all of those interviewed.

Unfortunately the rules too are extremely complex. Uprichard and Phillips (1977) identified 89 different specific tasks involved in the addition of fractions. Not all of these tasks are, of course, logically independent, but the use of the relationships between the tasks is dependent on some level of understanding. So successful handling of fractions entirely by rule seems to be out of the question.

It is not so surprising, then, that so many people cannot handle fractions. The surprising thing is that generations of educators have so drastically underestimated the learning difficulties in this area, and that so many children have succeeded in spite of their teachers.

Logic

The second tentative conclusion suggested in the last chapter was that N.I.'s difficulties were unlikely to be due to the logical nature of the subject matter. A careful study of the responses of the rest of the less able group tended to confirm this, but with some reservations.

One of the problems is to decide whether guessed wrong answers and the application of wrongly remembered rules are to be classified as illogical. If they are then of course the tentative conclusion is false. However, it seems more appropriate to classify these as alogical, providing the subject is aware that he is guessing and does not believe that he has a logical justification for the wrongly remembered rule.

Two responses from A.R. illustrate the point.

I_1 (Reads the kerosene problem)

S_1 *... (pause) ... Point five kilograms.*

I_2 *Right. How did you get that?*

S_2 *I don't know ... (long pause)*

I_1 (Card 3: $12 \div 3 = \Delta \div 12$)

S_1 *... (pause) ... four.*

I_2 *How did you come up with that?*

S_2 *Well, three times four equals twelve and four times three equals twelve.*

It seems that the first should be considered as being a response without logic, but the second, in view of the supposed explanation, should be considered illogical.

There were certainly plenty of examples of impeccable logic among the responses of the less able group. Subject A.A. was particularly good in this respect. In her work on fractions, reported earlier in this chapter, she began with the incorrect premise that it was necessary in all fractional operations to begin with common denominators. But, given this premise her deductions were without flaw. There were times when she was unable to go any further, but not once did she draw an invalid conclusion.

The following examples show her ability.

I_1 (Card 17: $c + d = 10$, $c < d$, $c = ?$)

S_1 *... (pause) ... Four or below it.*

I_2 *What did you do?*

S_2 *Well half way is five so c must be less.*

She had made the assumption that c and d were to be whole numbers, and given that premise S_1 is correct. S_2 indicates that if she had not made this assumption, the better answer, c is any number less than five, would probably have been given.

I_1 (Card 18: Which is the larger 2n or n + 2?)

S_1 (Writes 1 2 on the paper provided)

I_2 *What are you doing?*

S_2 *I'm letting n equal one, so that makes three, so therefore that one (points to n + 2) is bigger.*

I_3 *Always?*

S_3 *No, because if n equals six, two sixes are twelve, this equals nine ... (pause) ... hang on - when n is greater than two that's the larger (points to 2n).*

This is a good example of a liberated conviction response following her own logical deduction. This was repeated in the kerosene problem.

I_1 (Reads the problem)

S_1 *Half a kilogram - Oh! no. ... (long pause) ... One kilogram.*

I_2 *Good, now tell me what you did.*

S_2 *Er - subtract four point five from eight (writes 8 - 4.5)*

I_3 *Which gives three point five.*

S_3 *Yes.*

I_4 *What does that represent?*

S_4 *The liquid poured away - and then subtract that from there (points to the 4.5 on the card).*

Certainly A.A. was the most logical of the less able group and A.R. the least. But there were very few examples of invalid conclusions being drawn. The errors were much more likely to be due to incorrect premises, often in the form of the wrong rule or straight guesswork.

Abstraction

The third tentative conclusion of the previous chapter was that it was unlikely that N.I.'s problem lay in an inability to handle second order abstract ideas such as variables. A number is an abstraction from some physical situation and a variable is an abstraction from number. The view that this might lie at the heart of problems in mathematics learning was suggested by Skemp (1971). However his later view (Skemp, 1979a) is that mathematics learning is not different in character from other forms, it merely provides 'low noise' examples of a more general kind of intelligent behaviour. This 'low noise' makes mathematics an ideal medium for studying the learning process, but has some disadvantages for the learner. There is no subject area in which failure is more obvious.

In the interviews there was little evidence that the subjects were having difficulty in handling abstract ideas. The section on variables provided more examples of 'liberated conviction' among the less able than any other section. This might, of course, be due to the character of the particular questions asked rather than the character of the principles involved.

However, if we consider the response of A.R. to Card 31: Solve $\frac{x}{4} = \frac{2}{3}$

I_1 (Shows card)

S_1 ... (long pause) ...

I_2 *What are you thinking?*

S_2 *I'm thinking - how many of those can you get in there.*

I_3 *Yes.*

S_3 *I sort of can't see ... you can't divide three into four.*

The problem is once again the complexity of the fractions concept rather than an inability to handle the variable. The subject interprets $\frac{x}{4}$ as an unknown number of quarters and cannot see how any number of quarters can be made to be equal to two thirds. He needs to interpret $\frac{x}{4}$ as x divided by four if he is to make progress with the problem.

It might be argued that the reason that children have such difficulty with fractions is because of the abstract nature of the rules governing their manipulation. It is certainly true that the justifications of these rules are very sophisticated and require formal operations (Harrison *et al*, 1980). However the rule for multiplying two negative numbers is at least as difficult to explain and yet the evidence of these interviews is that subjects find this a much easier rule to handle. It is the relative simplicity of the negative number concept and its operations which accounts for this difference, not the level of abstraction.

Certainly, the emotional response to fractions was much more violent than to negative number.

Subject N.E.

I_1 (Card 6: $\frac{6}{8} = \frac{3}{4}$, Why?) *Fractions*

S_1 (Screws up her face) *Oh!* (Nervous laugh)

I_2 *Fractions worry you?*

S_2 *Yes, I could never understand them.*

The reaction to negative number was much calmer.

I_1 (Card 12 on negative numbers) *Did you ever do negative numbers.*

S_1 *Some time ago (quite calm response)*

I_2 *Let's just see if you can remember. What would three minus five be?*

S_2 *Negative two.*

⋮

I_3 (Points to (-3) x 2 on the card)

S_3 *I'm not sure. Positive five? No, six.*

⋮

I_4 *You've forgotten some of the rules, but you don't seem as worried about these as about fractions.*

S_4 *No.*

In this chapter the tentative conclusions indicated by the case study of chapter four were examined in the light of the other interviews, particularly those with the less able subjects. With some minor reservations these conclusions were supported.

In the next chapter the results of coding the interview responses of all the subjects and the corresponding statistical analysis are presented.

CHAPTER SIX

STATISTICAL ANALYSIS

In Chapter 4 mention was made of an initial formal coding of the interview data in relation to the levels of understanding shown by responses to the mathematical tasks. In this chapter this coding is discussed and the results analysed statistically.

There seem to be two advantages to this initial approach to interview data. Firstly, as mentioned previously, it provides a useful overview of the material as a background for a more searching analysis. Secondly, one of the dangers associated with a clinical analysis of interview data is that those parts of the data which support a particular conclusion will be selected in preference to those which do not. A formal coding procedure at the beginning of the analysis at least forces the investigator to view all the data in the same way. This is not to say, of course, that such a procedure will eliminate researcher bias. This may still be present in the selection of tasks and in the coding procedures. It does however provide for the possibility that any gross selection bias in the later clinical analysis will become apparent.

THE ASSESSMENT OF UNDERSTANDING

The concept of understanding in mathematics is most easily related to statements and theories (Lehman, 1977). Consequently, the coding of interview responses was restricted to Content Areas A to H (Table 1). Sections I and J, where the subjects were asked to solve mathematical problems, being quite different in character, were subjected to clinical analysis only.

No matter which of the many possible classifications of understanding referred to in Chapter 1 and discussed by Skemp (1979b) were to be used, some ambiguity in the responses was to be expected. Sometimes a response indicated quite clearly the type of understanding involved, but this was by no means always the case.

Initially an attempt was made to use Lehman's (1977) classification of understanding involving three levels:

(a) Knowledge of applications
(b) Knowledge of meaning
(c) Knowledge of logical relationships

All of the interviews were coded in this way but it was very difficult to distinguish, on the basis of the evidence of the interviews, between the first two categories. Consequently a second coding was made using Skemp's (1976) rather more inclusive categories of relational and instrumental understanding. Even then the difficulties of classification were greater than had been anticipated. When the interviews were conducted it was assumed that the boundary between instrumental and relational understanding would be the most difficult to delineate. Perhaps as a consequence of this anticipation, this was not the case.

It was, in fact, much more difficult to distinguish between instrumental understanding and no understanding.

Skemp describes instrumental understanding in this way:

> "Instumental understanding is the ability to apply an appropriately remembered rule to the solution of a problem without knowing why the rule works".
>
> (Skemp 1979b: 45)

In terms of Skemp's model the activities involved are delta-one activities, the operands being in the physical environment. The instrumental understander can retrieve an appropriate schema in which the present state and the goal state are linked by the possession of an appropriate rule.

The difficulty in assessing this condition arose from the fact that many children have, in relation to a topic such as the addition of fractions, a considerable supply of rules, many of them inappropriate, and the selection of a particular rule for a problem is more a matter of chance than understanding. The crucial question is; having selected an appropriate rule, do they know that it is appropriate and can they be relied upon to select the same rule if the problem is presented again at a later date? The lack of confidence which some of the subjects showed in some of their correct procedures and answers made this seem unlikely. This concept of confidence as a prerequisite of knowledge, and hence of understanding, is reflected in the view of many modern philosophers (Russell 1948, Ayer 1956, Scheffler 1965) that the essence of propositional knowledge can be expressed in three conditions of knowledge.

In order for person x to know a proposition p the following must be satisfied:

(a) x must believe p.
(b) p must be true.
(c) x must have adequate evidence for p.

The confidence factor is clearly contained in condition (a). It seems that, 'ability to apply an appropriately remembered rule' is likely to depend on belief in that rule.

Notwithstanding these difficulties, after two further attempts a coding with respect to the three states: relational understanding, instrumental understanding, and no understanding was completed and analysed.

METHOD OF ANALYSIS AND RESULTS

The initial approach to the analysis of the results of this coding was a purely descriptive one. The items were ranked in order of difficulty as indicated by the total number of positive responses to an item with respect to each of the criteria used. The subjects were also ranked according to the total number of items on which they were judged to show instrumental and relational understanding.

The results of this coding and ranking were then presented graphically to give a profile of the performance of each subject on progressively more difficult items, and a profile of individual items as handled by progressively more able students. This information is presented in Figures 6 and 7. For student identification the rankings on instrumental understanding criteria were used.

The total scores of individual students according to the two criteria are compared in Figure 8.

A number of features emerge from this presentation of the data.

Firstly, Figure 8 clearly identifies the group of seven subjects, mentioned in the previous chapter, whose understanding of and ability to use, school mathematics was very poor.

Secondly, the distributions shown in Figures 6 and 7 indicate an important difference in character between items. Figure 6, for example shows that item 14:

$a + b = 43, \quad a + b + 2 = ?$

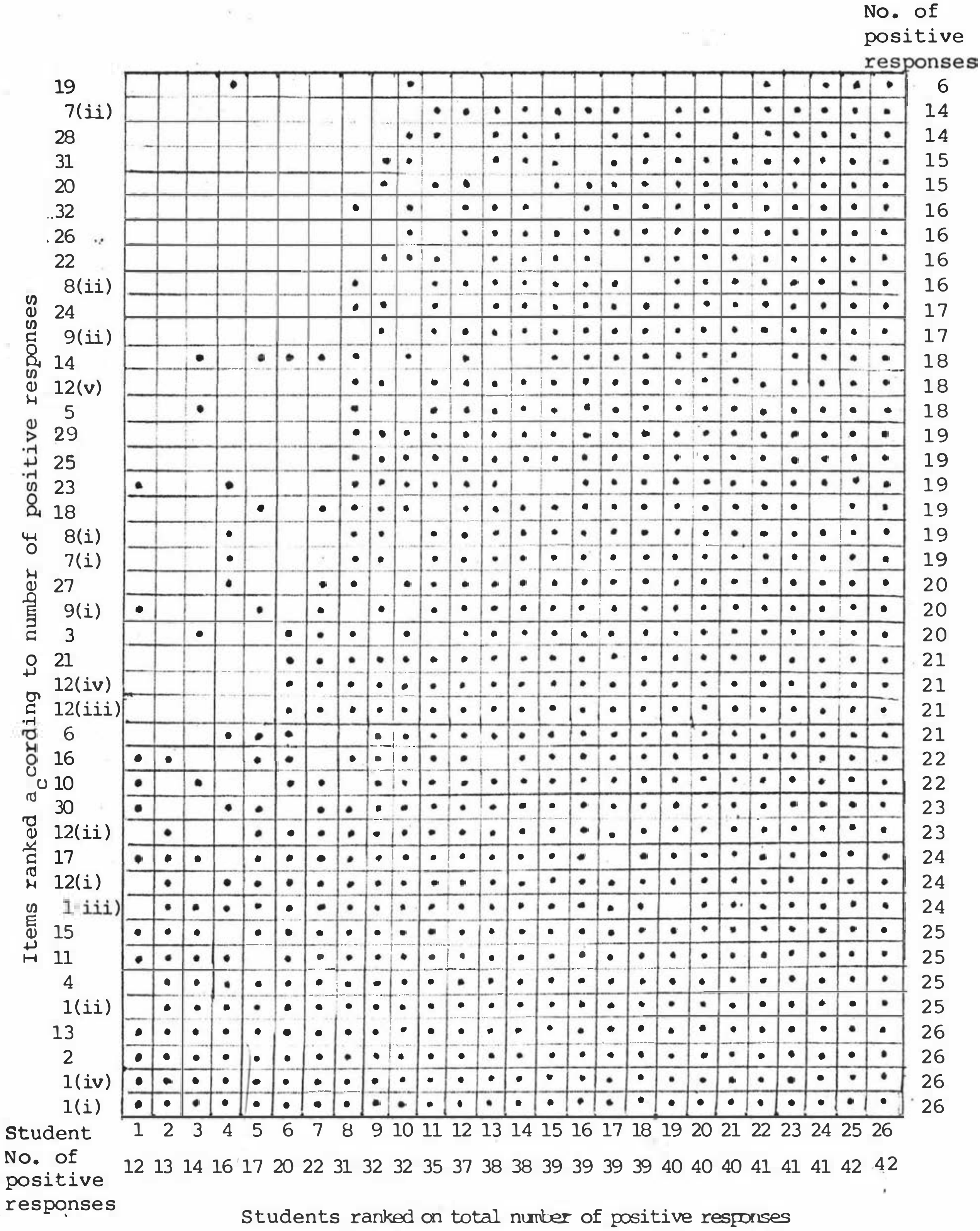
with 18 positive responses, had fewer positive responses by the sample as a whole than item 25:

Multiply $n + 5$ by 4

which had 19 positive responses.

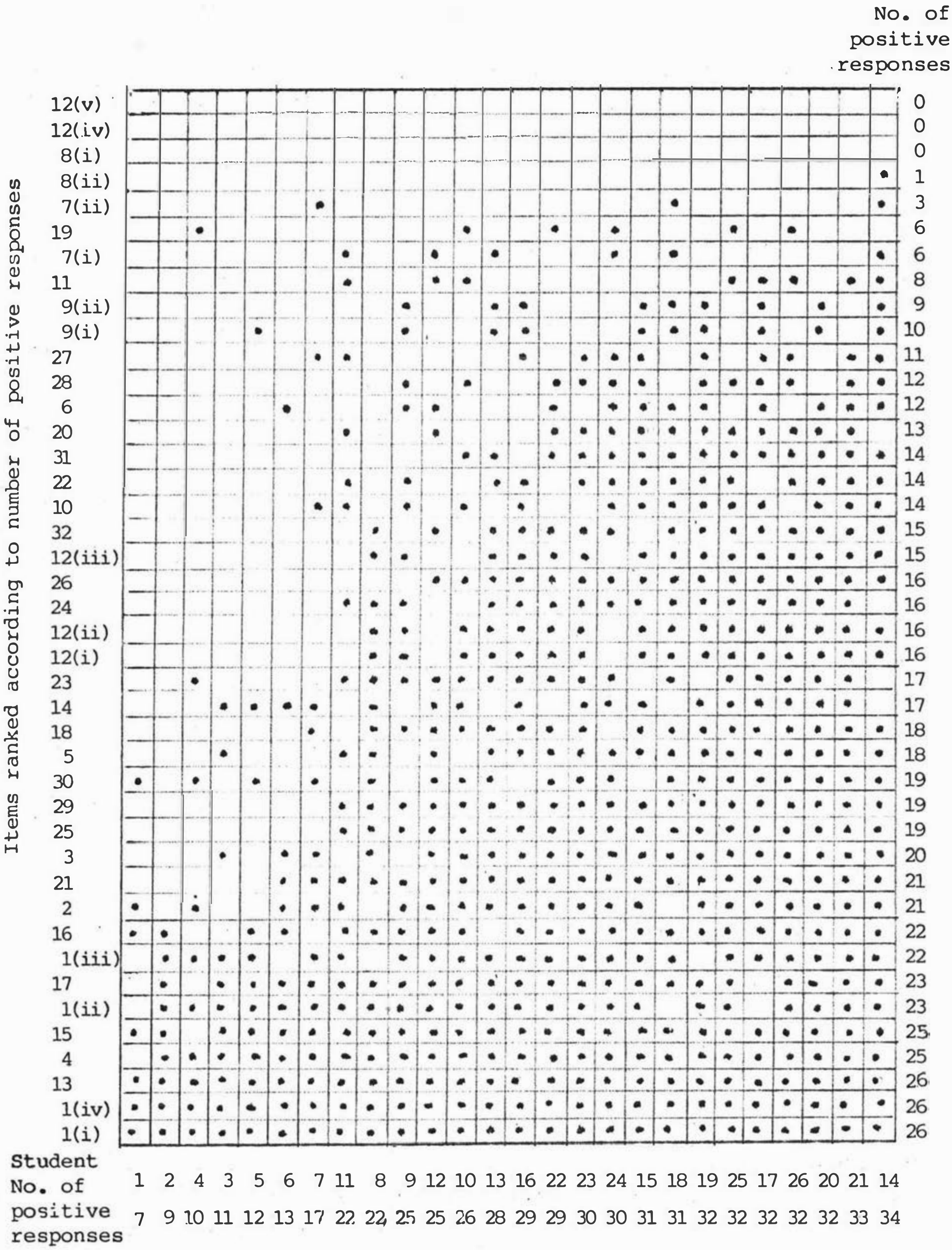
For the less able group, however, item 25 produced no positive responses while four out of the seven subjects in the group responded positively to item 14.

This difference in ranking between these items according to the ability of subjects can be illustrated by plotting either the <u>number</u> of positive responses by subjects below a particular rank, or the <u>proportion</u> of such responses. These comparisons for items 14 and 25 are presented in Figures 9 and 10.



Positive responses indicated by dots.

FIGURE 6: DISTRIBUTION OF POSITIVE RESPONSES - INSTRUMENTAL UNDERSTANDING



Students numbered according to Instrumental Understanding rank. Positive responses indicated by dots.

FIGURE 7: DISTRIBUTION OF POSITIVE RESPONSES - RELATIONAL UNDERSTANDING

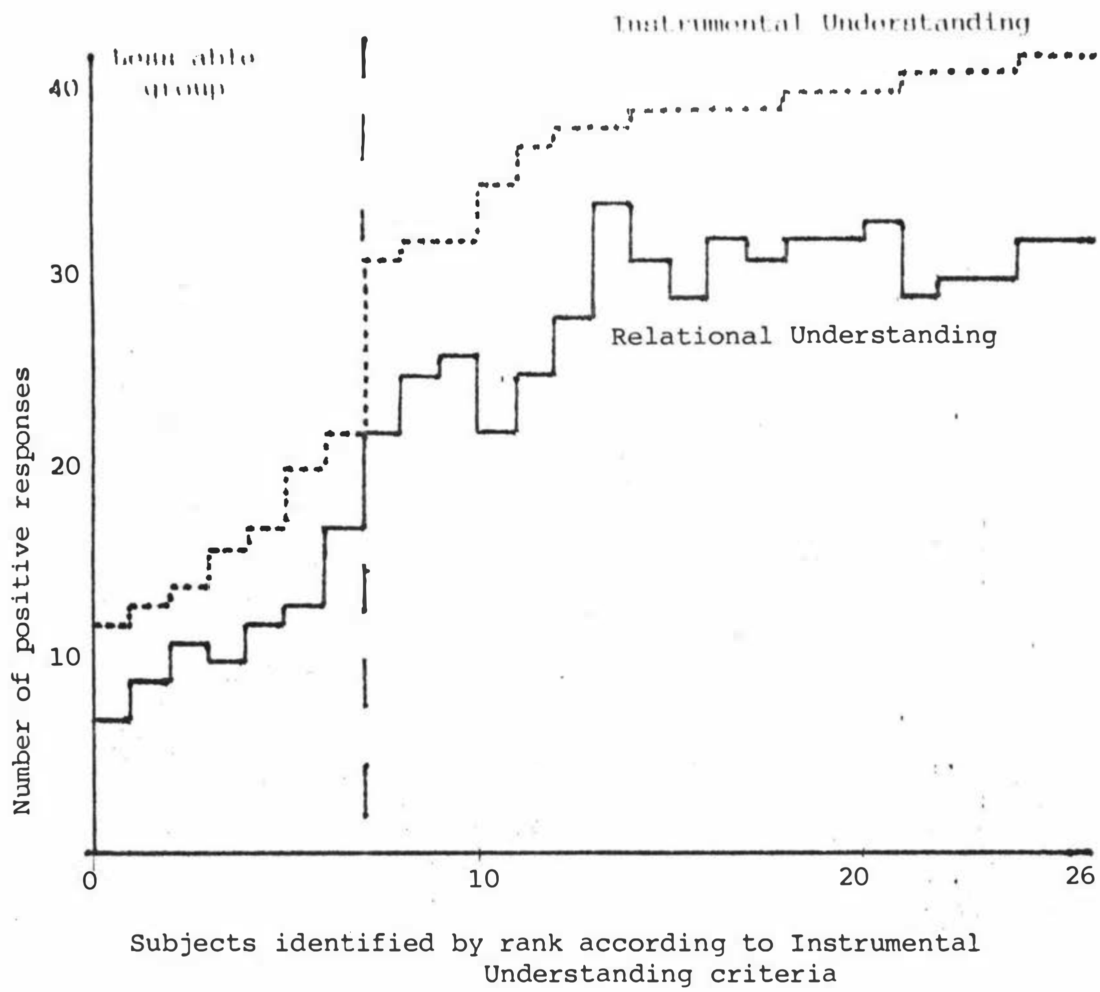


FIGURE 8: Comparison of scores according to Instrumental Understanding and Relational Understanding criteria.

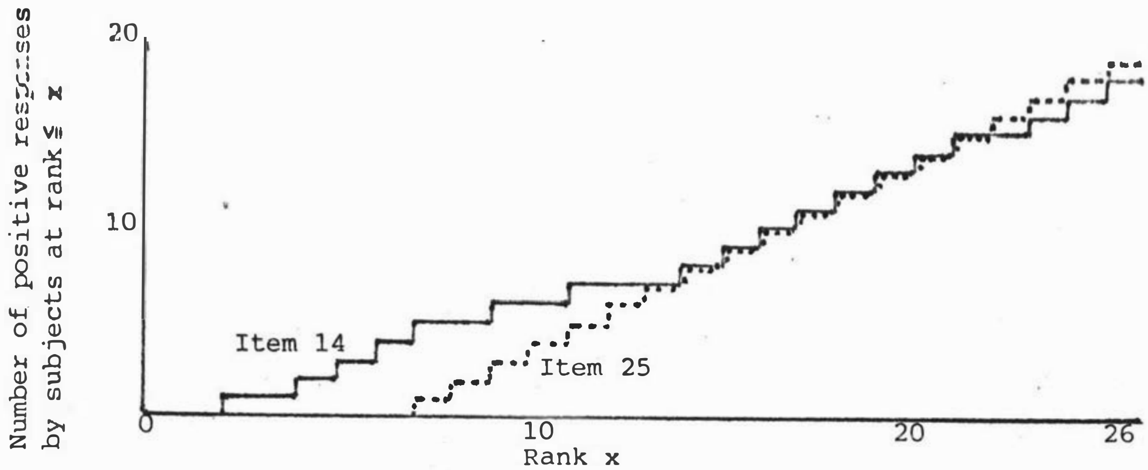


FIGURE 9: Comparison of rankings by Instrumental Understanding of Items 14 and 25. Number of positive responses by subjects at rank ≤ x.

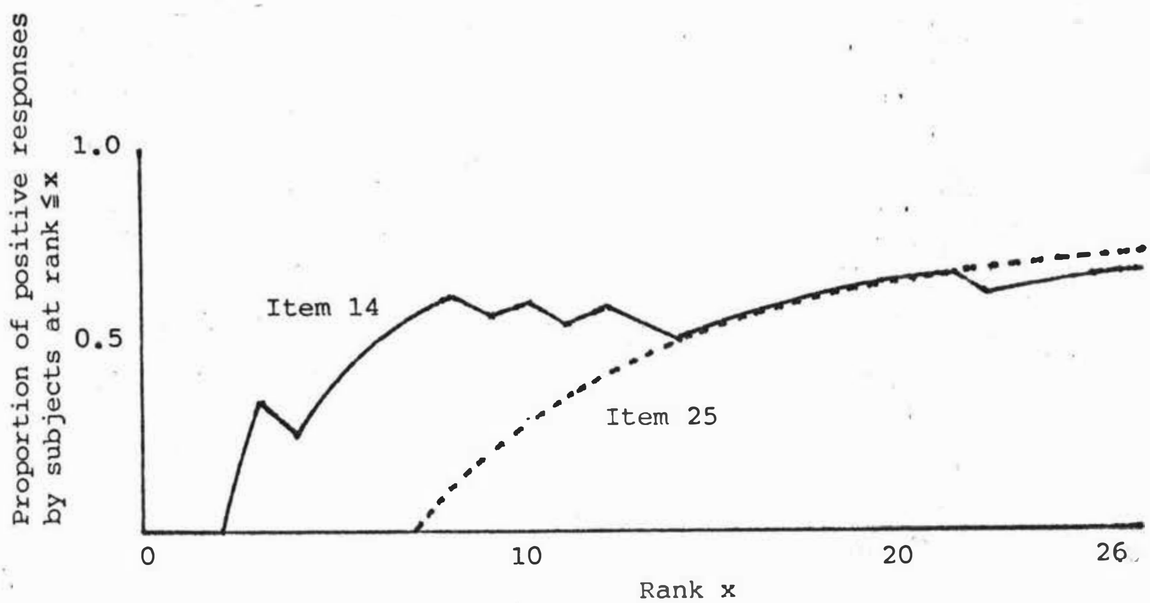


FIGURE 10: Comparison of rankings by Instrumental Understanding of Items 14 and 25. Proportion of positive responses by subjects at rank ≤ x.

It seems that these diagrams provide a useful profile of the items as seen by subjects at different ability levels, but comparisons between all 42 items in this way would be rather clumsy. Consequently, the possibility of using some statistical scaling technique, which might represent this character of the items parametrically, was considered.

The most popular such technique is Scalogram Analysis (Guttman Scaling). This has been used in other clinical studies (Thomas, 1975), but the major assumption underlying the method (Torgerson, 1958), is that the items being scaled are perfect and monotone. This means that all subjects at the same level of the underlying attribute against which the items are being scaled would be assessed in exactly the same way on each item. Since the assessment of a variable such as understanding by interview is necessarily somewhat subjective, this assumption seems unwarranted.

An alternative, which seems to be more appropriate, is to use Latent Structure Analysis. The method is described in Torgerson (1958), but mixed notation due to the method being described in a different chapter from its analysis, and an error in the worked example, make the method difficult to follow in this source. Consequently a description and justification of the method is presented in Appendix A.

LATENT RESPONSE ANALYSIS BY ITEM

The basic premise of the method is that the probability of a positive response to an item increases with the ability of the subject. Consequently a form of trace line for the items is assumed, usually either linear or in the form of the normal-ogive curve (Figure 11). The close relationship between this approach and that illustrated in Figure 10 made the method, using a normal-ogive trace line, seem particularly appropriate.

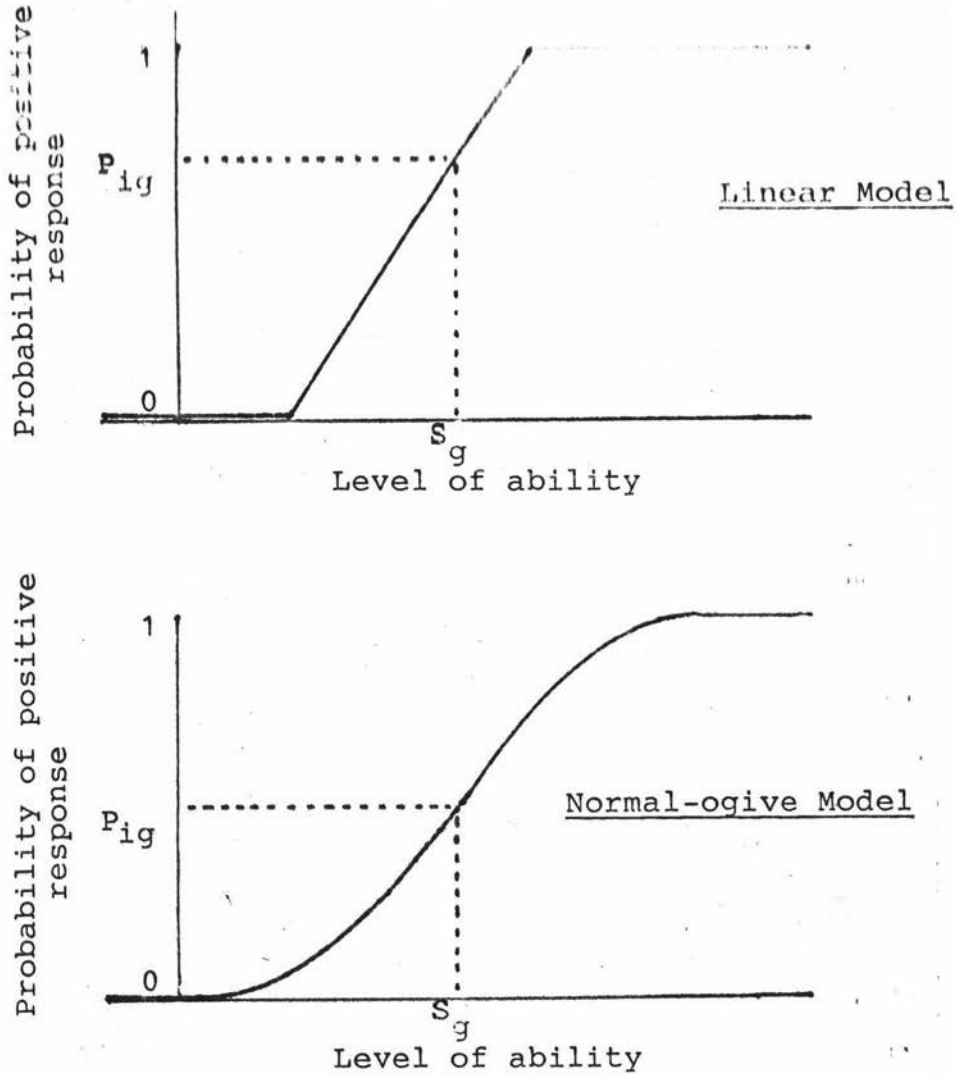


FIGURE 11: Latent Response Analysis. Forms of trace lines.

The major difficulty, in working with a small sample, is to derive reasonable estimates of P_{ig} - the probability of a positive response to item i by a subject at ability level g. The usual approach is to divide the subjects into reasonably homogeneous groups on the basis of the total number of positive responses recorded to all the items. The proportion of positive responses by each group to item i is then used to estimate P_{ig}. From these values estimates of S_g, the ability level of each group and of the location parameter M_i and the discrimination parameter σ_i of each item are calculated.

With only 26 subjects the difficulty obviously lies firstly in locating relatively homogeneous groups, and secondly in the size of such groups. It was decided, however, to continue with the analysis in the knowledge that the results could be compared with a descriptive presentation such as that of Figure 10.

The subjects were divided into four groups as shown in Table 2 and the analysis on all items completed. The values of M_i and σ_i for items other than those which produced all positive or no positive responses are given in Tables 3, 4.

Relational Understanding

Group	Mark Range	No. in group	Ability level S_g
I	7 - 17	7	-1.3
II	22 - 28	6	0.5
III	29 - 31	6	1.0
IV	32 - 34	7	1.4

Instrumental Understanding

Group	Mark Range	No. in group	Ability level S_g
I	12 - 22	7	-0.7
II	31 - 38	7	1.6
III	39 - 40	7	2.5
IV	41 - 42	5	2.9

TABLE 2: GROUPS OF SUBJECTS ACCORDING TO TOTAL SCORE

Item	1(ii)	1(iii)	2	3	4	5	6
σ_i	1.2	1.1	1.0	0.7	1.2	0.6	1.5
M_i	-1.4	-1.1	-0.9	-0.7	-2.6	-0.4	0.6

7(i)	7(ii)	8(ii)	9(i)	9(ii)	10	11	12(i)
0.9	1.2	1.2	0.9	0.8	1.3	0.6	0.5
1.4	2.2	3.5	0.7	1.0	0.3	1.2	0.2

12(ii)	12(iii)	14	15	16	17	18	19
0.5	0.6	2.5	1.2	0.8	0.9	0.7	3.8
0.2	0.6	-0.6	-2.6	-1.0	-1.3	-0.3	3.3

10	21	22	23	24	25	26	27
0.6	0.7	0.6	0.7	0.5	0.4	0.4	1.3
0.6	-1.0	0.6	-0.2	0.2	-0.2	0.1	0.8

28	29	30	31	32
0.7	0.4	0.9	0.5	0.5
0.7	-0.2	-0.5	0.4	0.3

TABLE 3: ANALYSIS BY ITEM. RELATIONAL UNDERSTANDING. LOCATION PARAMETER M_i AND DISCRIMINATION PARAMETER σ_i.

Item	1(ii)	1(iii)	3	4	5	6	7(i)
σ_i	1.6	1.4	1.0	1.6	0.8	1.0	0.8
M_i	-2.5	-1.3	0.0	-2.5	0.5	-0.2	0.4

Item	7(ii)	8(i)	8(ii)	9(i)	9(ii)	10	11	12(i)
σ_i	0.6	0.8	0.6	1.0	0.6	1.1	1.6	1.3
M_i	1.5	0.4	1.3	0.0	1.0	-0.4	-2.5	-1.5

Item	12(ii)	12(iii)	12(iv)	12(v)	14	15	16	17
σ_i	1.1	0.9	0.9	0.6	1.1	1.6	1.1	1.4
M_i	-1.0	-0.3	-0.3	1.0	0.5	-2.5	-0.4	-1.3

Item	18	19	20	21	22	23	24	25
σ_i	1.1	1.0	0.5	0.9	0.6	0.9	0.6	0.5
M_i	0.4	2.7	1.2	-0.3	1.3	0.1	1.0	0.8

Item	26	27	28	29	30	31	32
σ_i	0.6	0.9	0.6	0.5	1.1	0.6	0.6
M_i	1.1	0.1	1.5	0.8	-1.0	1.4	1.0

TABLE 4: ANALYSIS BY ITEM: INSTRUMENTAL UNDERSTANDING. LOCATION PARAMETER M_i AND DISCRIMINATION PARAMETER σ_i.

Using these values of M_i and σ_i a comparison was made between the Latent Response Analysis trace-lines of a number of items and the corresponding "proportion of positive responses" diagram (Fig. 10).

For items 14 and 25 the trace-lines are shown in Figure 12 and clearly, in spite of the small sample, these represent fairly the information presented in Figures 9 and 10.

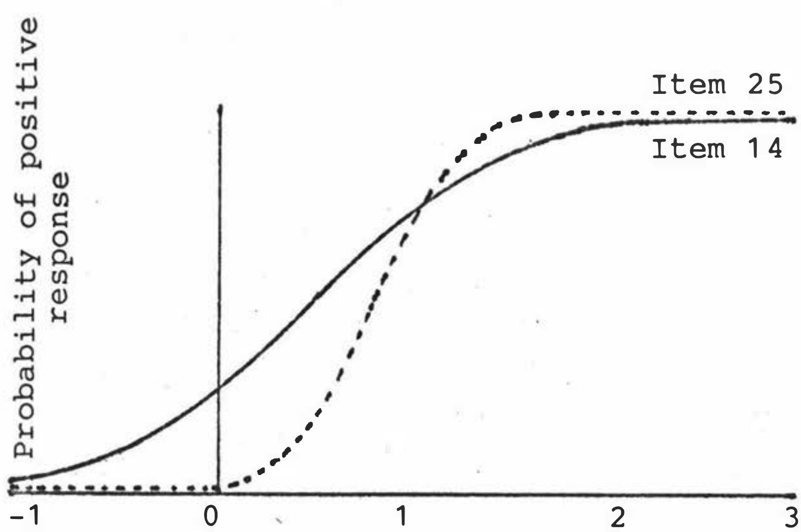


FIGURE 12: Comparison of rankings by Instrumental Understanding of Items 14 and 25. Latent Response Analysis trace lines

In Figure 12 the choice of ability levels from -1 to 3 reflects the fact that the analysis produced ability levels of -0.7 for the least able group and 2.9 for the most able on the Instrumental Understanding criteria.

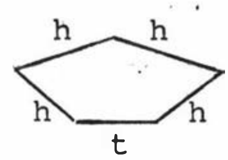
Alongside the obvious advantages of using Latent Response Analysis to produce a simple parametric and graphical comparison of items, the dangers of the representation must be kept in mind. There is, perhaps, an implied precision in a diagram such as Figure 12 which is quite unwarranted, at least in a small sample study. One might be tempted, for example, to deduce that at ability level 0.6, item 25, with a probability of positive response of about 0.3, is twice as difficult as item 14. Such a statement, even if it were meaningful, could not be justified on the basis of the data which produced the curves. The only deduction which is justified concerns the general way in which the ability of subjects affects the ranking of the items. That is that item 25 is more difficult for the less able subjects.

A further comparison between the representations of items 13-20 on variables with respect to Relational Understanding is presented in Figures 13 and 14. This comparison confirms that Latent Response Analysis retains the order characteristics of the items and provides a useful means of representing these characteristics. For the Relational Understanding curves an ability scale from -1.5 to 1.5 was used, reflecting again the calculated ability levels of the sample. The tracelines for the items on variables in relation to Instrumental Understanding are also presented in Figure 15.

From Figure 15, it can be seen that in relation to Instrumental Understanding the items maintain roughly the same ranking at all ability levels. From easiest to hardest the ranking is:

Item 13: $a + 5 = 8, \quad a = ?$

Item 15: perimeter = ?



Item 17: $c + d = 10, \quad c < d, \quad c = ?$

Item 16: Figure has n sides all of length 2. What is the perimeter?

Item 18: Which is larger 2n or n + 2?

Item 14: $a + b = 43, \quad a + b + 2 = ?$

Item 20: Red and blue pencils.

Item 19: Cakes and buns.

For Relational Understanding (Fig. 14), Items 13, 15, 17, 16, 18 and 20 have the same rank order over the ability range, but items 14 and 19 behave quite differently.

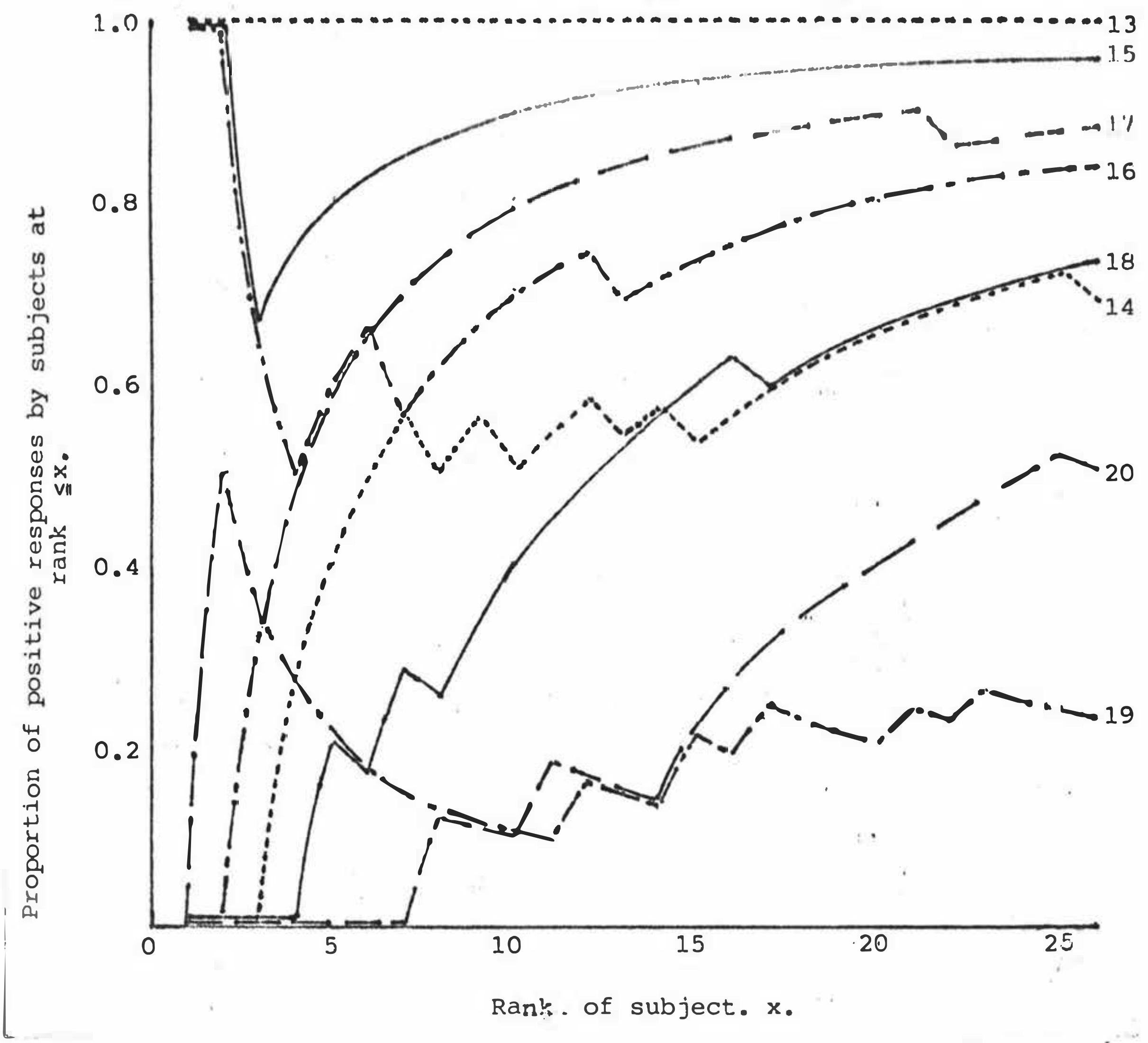


FIGURE 13: Proportion of positive responses by subjects at rank ≤x for items 13 - 20. Relational Understanding.

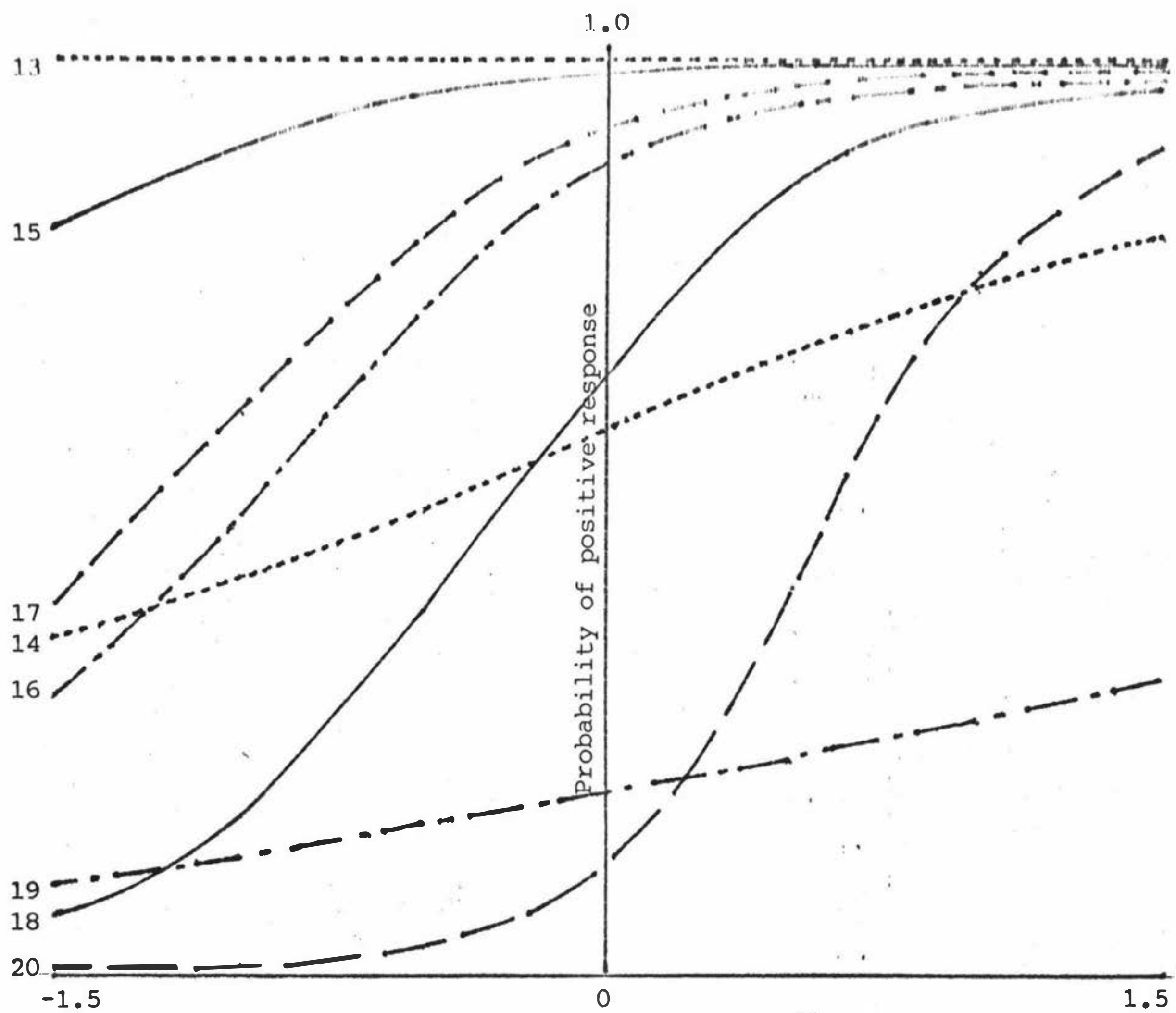


FIGURE 14: Latent Response Analysis trace lines for items 13 - 20. Relational Understanding.

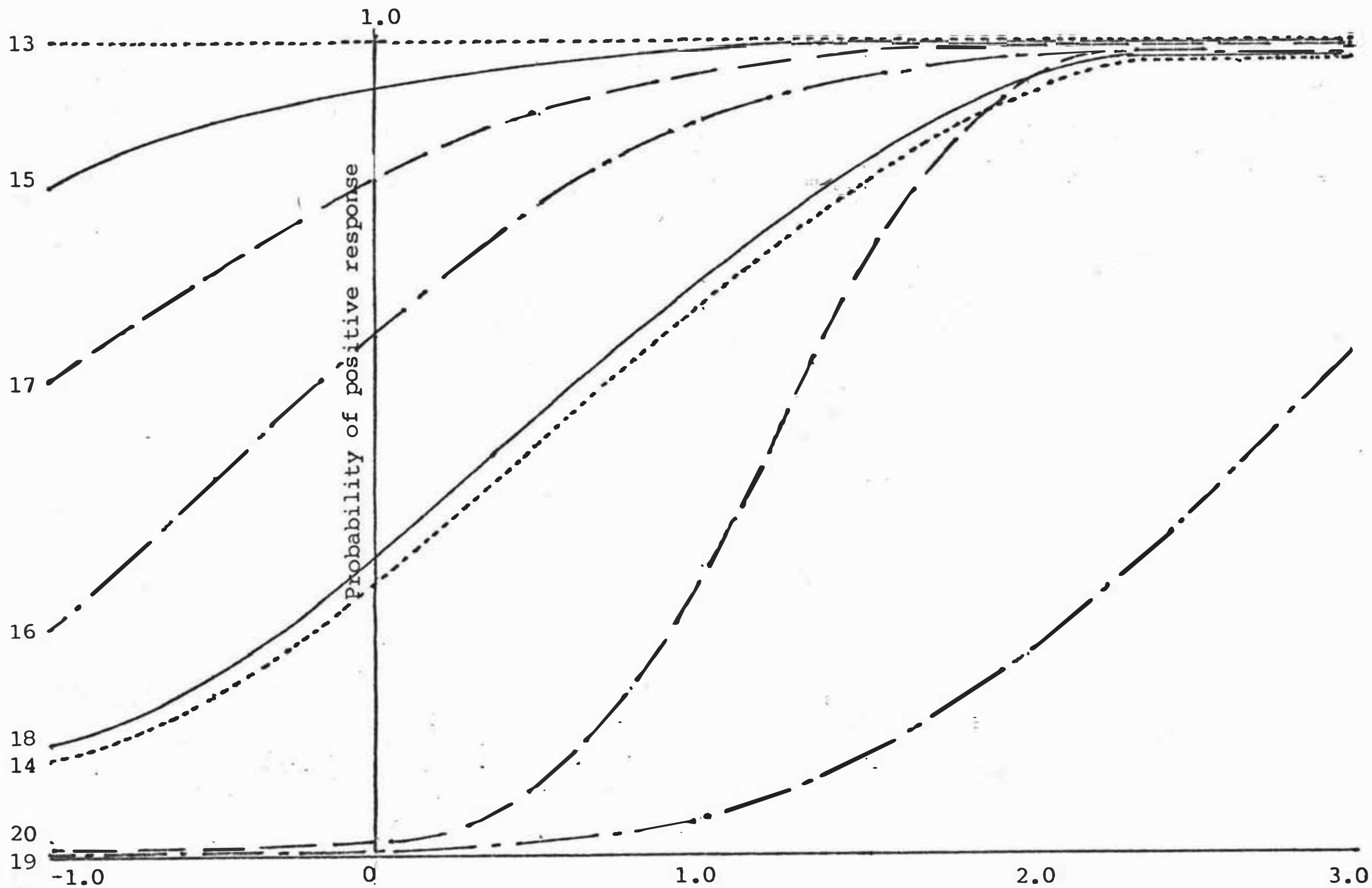


FIGURE 15: Latent Response Analysis trace lines for items 13 - 20 Instrumental Understanding.

The explanation in the case of item 14 seems to be that many able subjects, on seeing the two variables in the first equation, a + b = 43, retrieved a framework for the solution of a pair of simultaneous equations. In fact the solution requires that a + b be treated as a single variable. Those subjects with little experience of work with two variables were not distracted in this way.

For item 19, the question was: Cakes cost c cents and buns b cents each. What does 4c + 3b stand for? By far the most common answer, "four cakes and three buns", was given by able students as well as the less able. The response was remarkably resiliant to challenge and probes and a very direct suggestion by the interviewer was required to dislodge it in many cases. Galvin and Bell comment:

> "There appeared a strong tendency to attempt to read and write mathematical sentences in a manner too closely analogous to the reading and writing of English. In the sentence, "two rabbits were in the burrow", the word two is a numerical adjective and is placed before the noun it qualifies. In a mathematical situation, if r is the number of rabbits, "2r" is certainly not meant to be interpreted as "two rabbits". Such an interpretation occurred frequently during the interviews".
>
> (Galvin and Bell 1977: 10)

A similar comparison between items was made for the section on fractions and the corresponding trace-lines are presented in Figures 16 and 17.

In this case the ranking between items is not appreciably changed by ability level. The most significant difference in ranking between the two criteria is for item 8(i): $\frac{3}{4} \times \frac{5}{8} = ?$ No student was able to give a satisfactory explanation of the rule while many could use it accurately and confidently.

Comparisons between items from different content areas, and between items within areas other than fractions and variables did not produce any significant findings. It was, however, decided to compare the overall rankings of items according to the two criteria. The location parameters

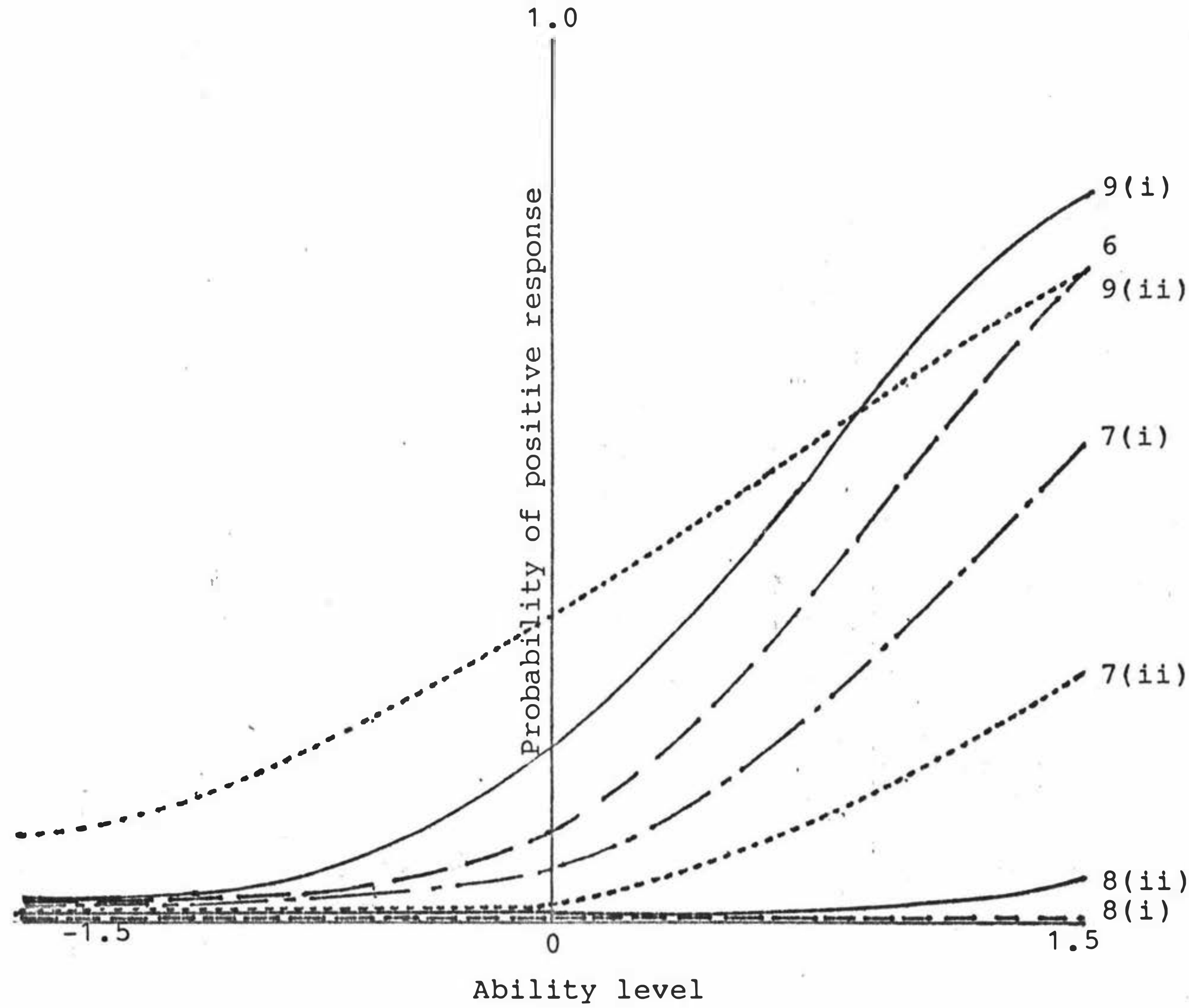


FIGURE 16: LATENT RESPONSE ANALYSIS TRACE LINES FOR ITEMS ON FRACTIONS. RELATIONAL UNDERSTANDING

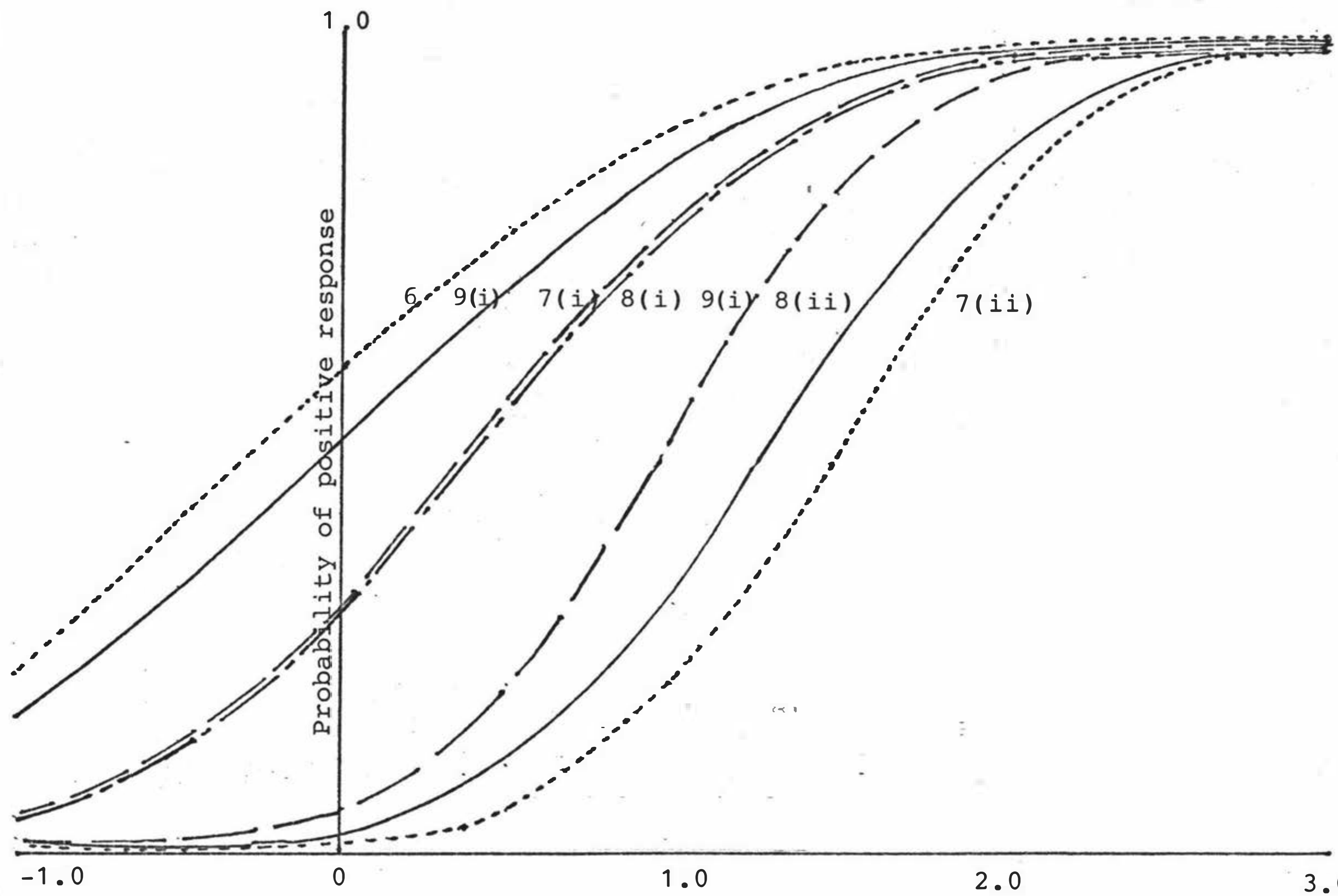


FIGURE 17: Latent Response Analysis trace lines for items on fractions. Instrumental Understanding

were used as indicating a general level of difficulty and the rankings are presented in Tables 5 and 6.

TABLE 5: RANKING OF ITEMS BY LOCATION PARAMETER FROM LEAST DIFFICULT TO MOST DIFFICULT - RELATIONAL UNDERSTANDING

Rank	1=	1=	1=	4=	4=	6	7	8
Item	1(i)	1(iv)	13	4	15	1(ii)	17	1(iii)

9=	9=	11	12	13	14	15	16	17=
16	21	2	3	14	30	5	18	23

17=	17=	20	21=	21=	21=	24=	24=	26
25	29	26	12(i)	12(ii)	24	10	32	31

27=	27=	27=	27=	31=	31=	33	34	35
6	12(iii)	20	22	9(i)	28	27	9(ii)	11

36	37	38	39	40=	40=	40=
7(i)	7(ii)	19	3(ii)	8(i)	12(iv)	12(v)

A positive relationship between the two rankings could be predicted, but a rank correlation coefficient of 0.62 indicates that the relationship is only moderate.

Looking at items which rank much higher on the instrumental than on the relational understanding scale, we find multiplication of fractions and most of the items on negative number having this characteristic. The most obvious interpretation of this is that the subjects have been successful in rote learning the content of these items, and their lack of relational understanding has not proved a handicap in using the material. This raises the interesting question of whether relational understanding is necessary for some topics or concepts to be operational but not for others. If this is the case, what are the characteristics of each group of concepts?

TABLE 6: RANKING OF ITEMS BY LOCATION PARAMETER FROM LEAST DIFFICULT TO MOST DIFFICULT - INSTRUMENTAL UNDERSTANDING.

Rank	1=	1=	1=	1=	5=	5=	5=	5=
Item	1(i)	1(iv)	2	13	1(ii)	4	11	15
9	10=	10=	12=	12=	14=	16=	16=	16=
12(i)	1(iii)	17	12(ii)	30	10	16	12(iii)	12(iv)
16=	19	20=	20=	22=	22=	24=	24=	24=
21	6	3	9(i)	23	27	7(i)	8(i)	18
27=	27=	29=	29=	31=	31=	31=	31=	35
5	14	25	29	9(ii)	12(v)	24	32	26
36	37=	37=	39	40=	40=	42		
20	8(ii)	22	31	7(ii)	28	19		

Since the sum of rank differences for all the items must be zero, the presence of the items mentioned above must be balanced by items for which the rank difference is in the opposite direction. That is they rank higher on the relational understanding scale. Inspection shows that the largest rank difference in this direction occurs for items in the algebraic manipulation - functions and graphs areas. The most likely explanation in this case seems to be that many of the less able subjects had little or no experience in these areas and consequently were unable to respond positively with respect to either criterion. This would inevitably bias the ranking on the less stringent instrumental understanding criterion rather more than for relational understanding.

COMPARISON BETWEEN CONTENT AREAS

As was explained in Chapter 3, it was decided that coverage of the four areas: natural numbers, fractions, negative number, and variables would be reasonably comprehensive in the interviews. Consequently it was possible to provide a comparison of the performance of subjects in these areas. The proportion of positive responses by a subject to the questions within a content area was taken as a measure of this performance. The data was again subjected to both the 'proportion of positive responses by subjects at rank ≤ x' analysis, and to Latent Response Analysis. Table 7 gives the location and discrimination parameters calculated and Figures 18, 19, 20, 21 show the corresponding graphs.

TABLE 7: ANALYSIS BY CONTENT AREA - LOCATION AND DISCRIMINATION PARAMETERS

		Area			
		Natural No.	Fractions	Negative No.	Variables
Relational Under-standing	σ_i M_i	1.3 -1.6	1.5 1.3	0.5 0.5	1.6 -0.9
Instrumental Under-standing	σ_i M_i	1.2 -0.8	0.7 0.6	0.7 0.1	2.0 -0.2

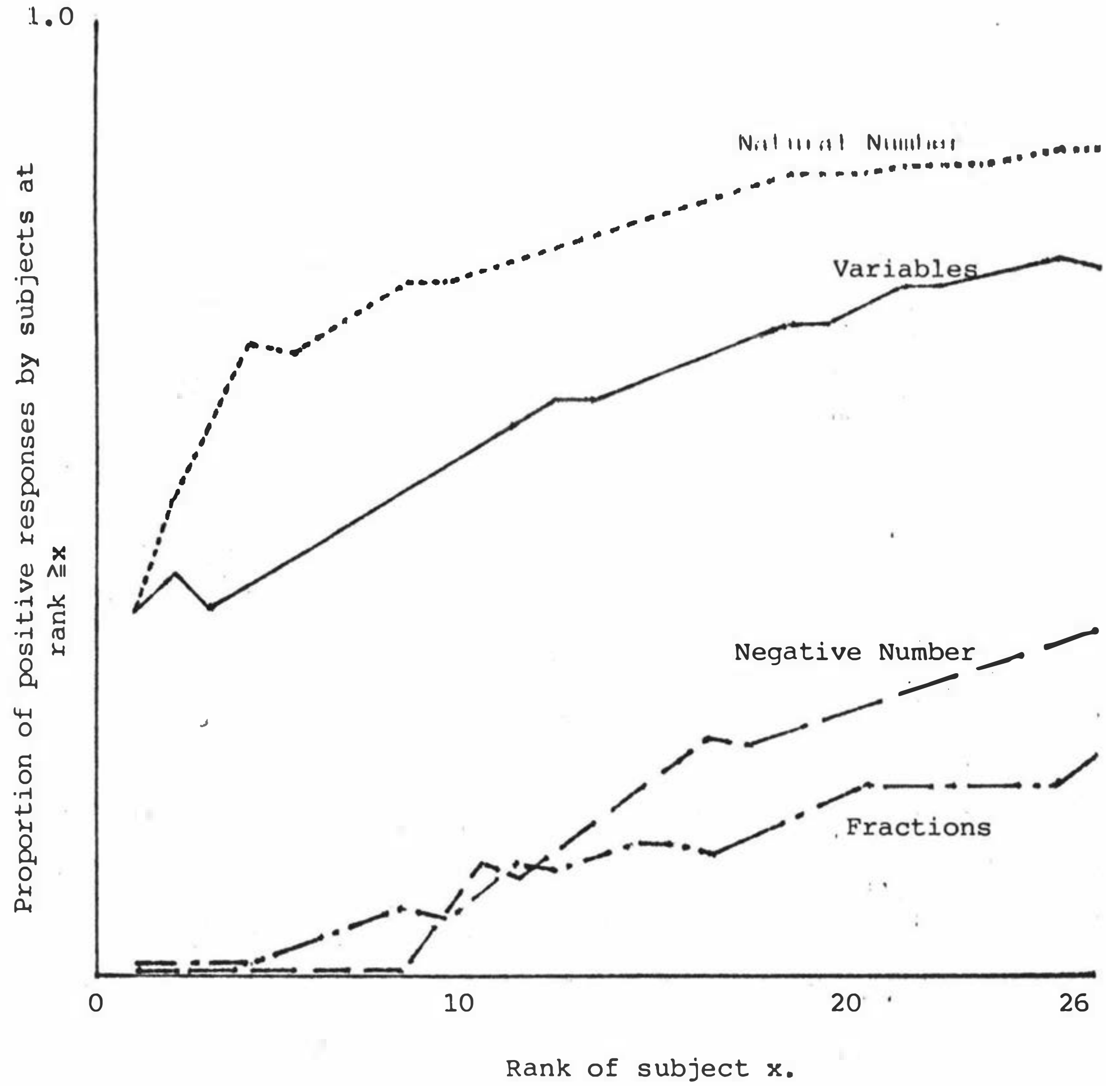


FIGURE 18: Proportion of positive responses by subjects at rank ≤x. Content Areas - Relational Understanding.

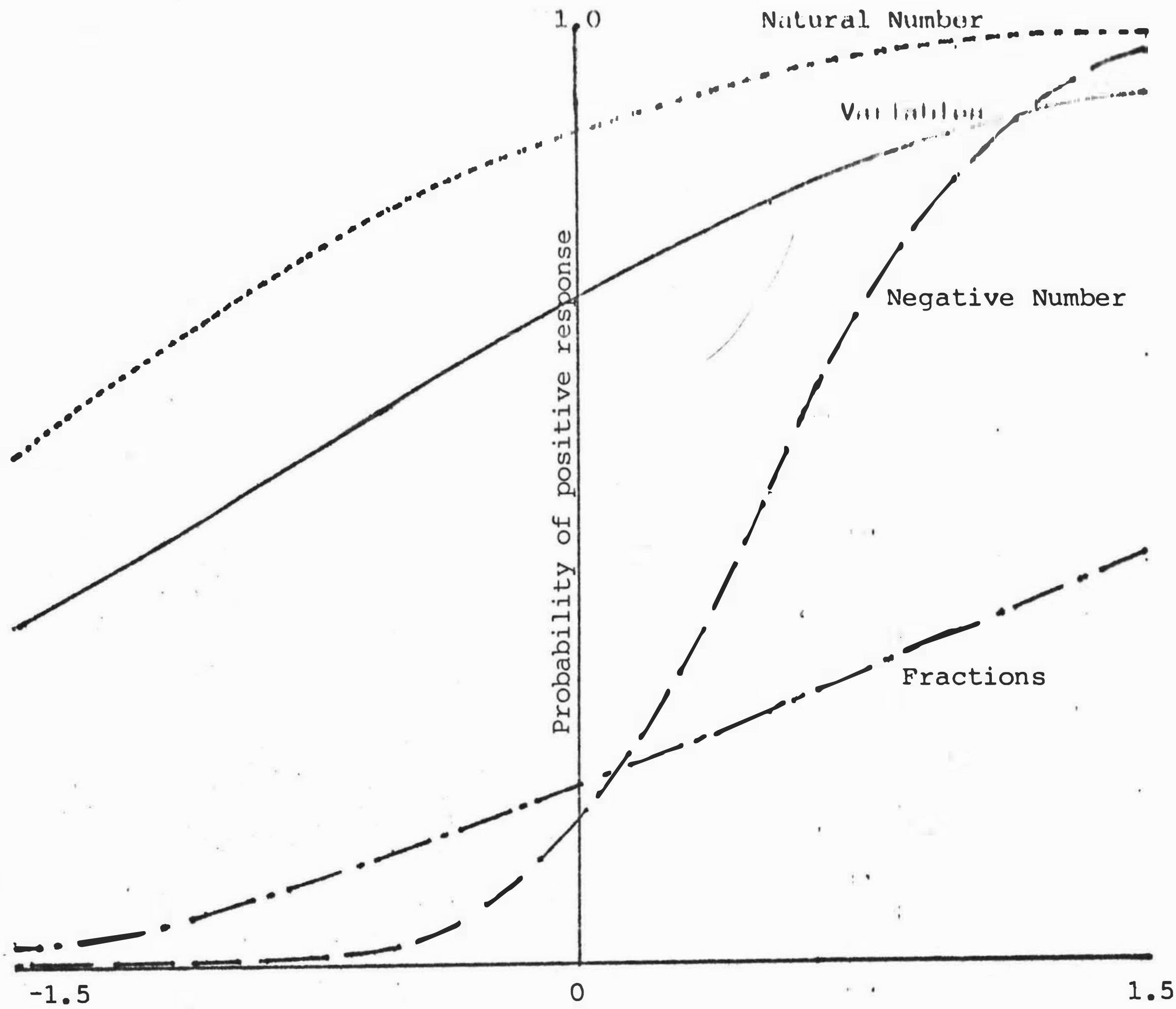


FIGURE 19: Latent Response Analysis trace lines.
Content Areas - Relational Understanding.

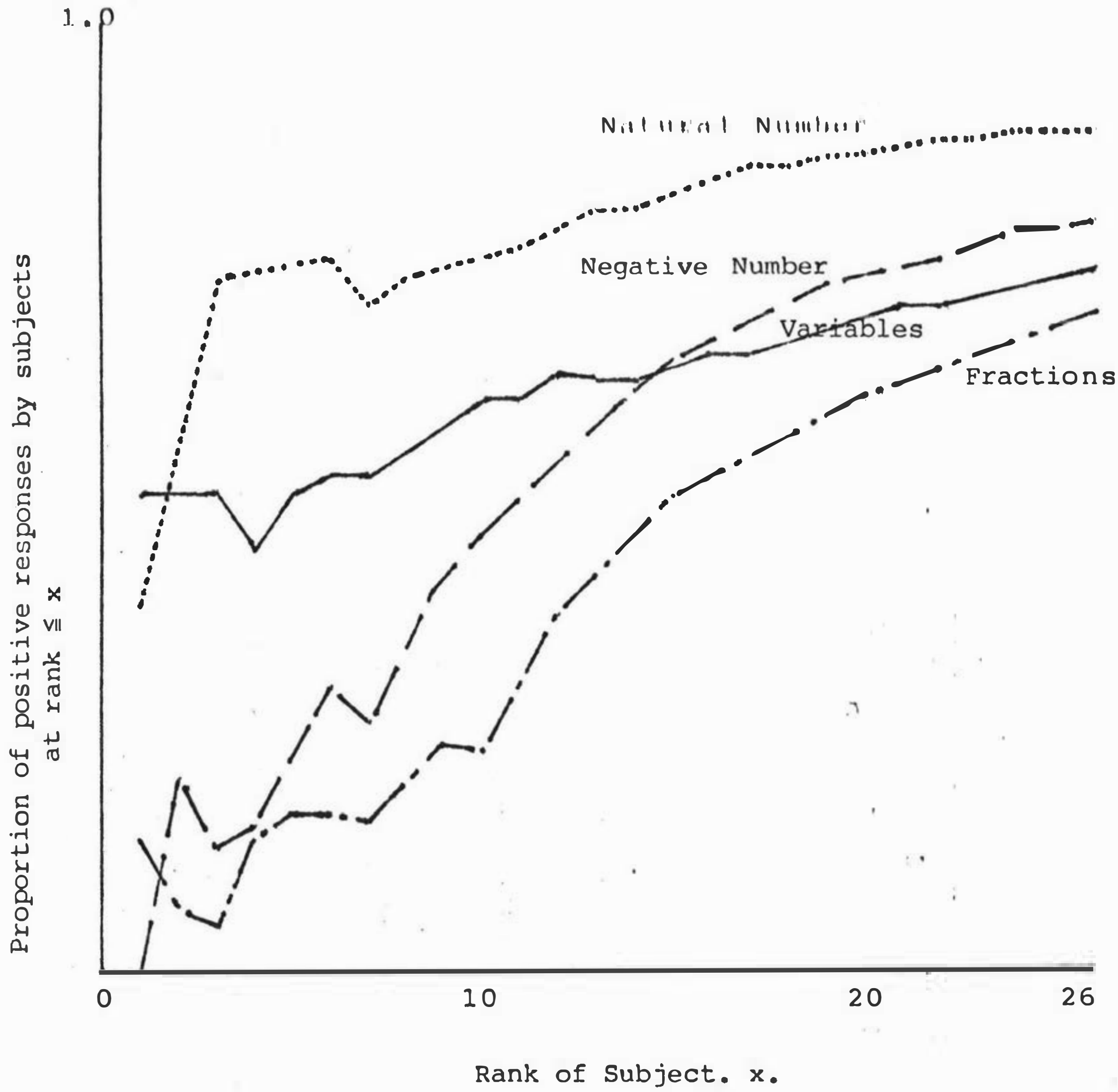


FIGURE 20: Proportion of positive responses by subjects at rank ≦ x. Content Areas - Instrumental Understanding.

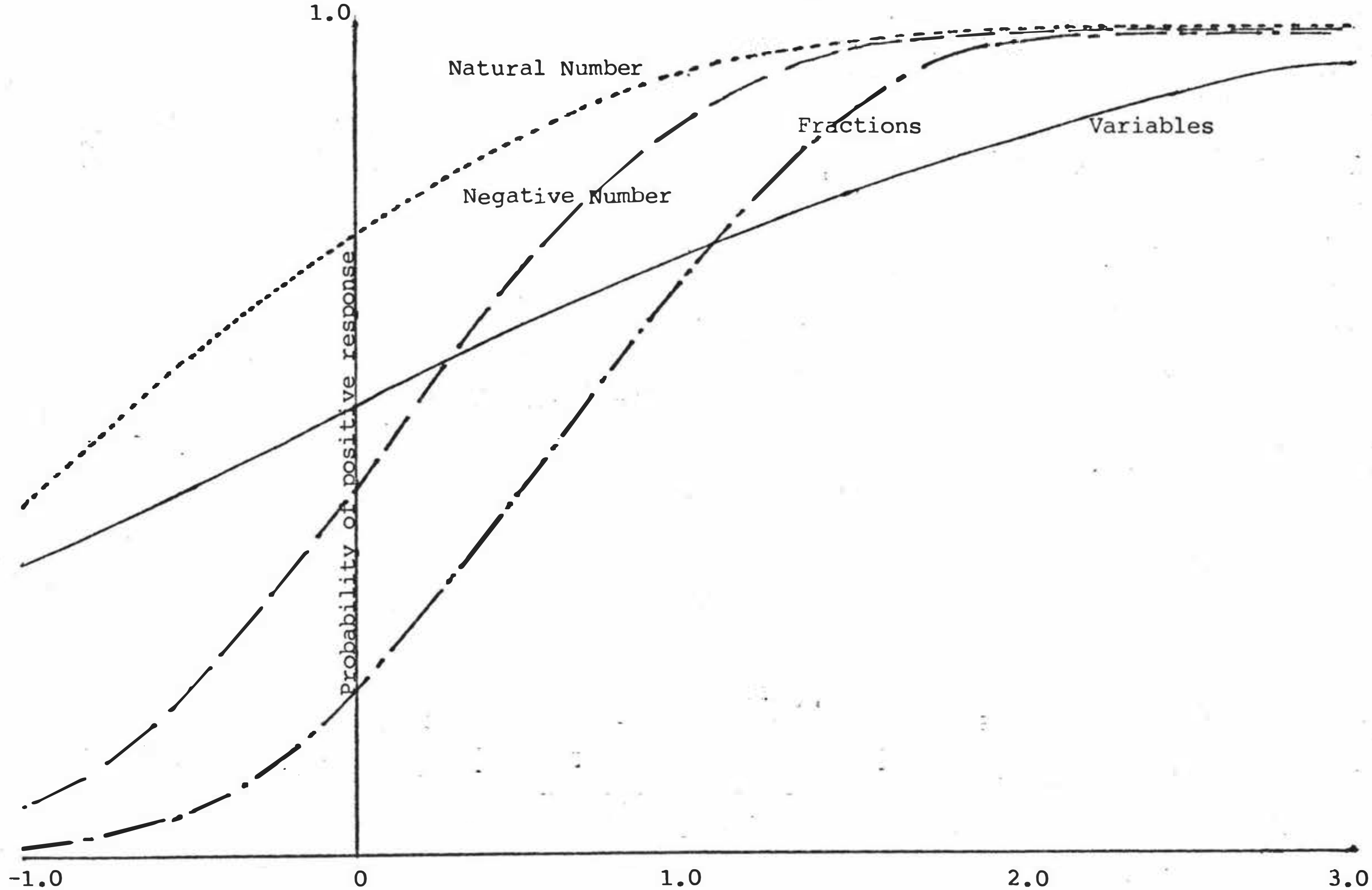


FIGURE 21: Latent Response Analysis trace lines. Content Areas - Instrumental Understanding

A major objective of this study, which was discussed in Chapter One, was to identify areas of particular difficulty within the school curriculum. The rankings shown in Figures 19 and 21 confirm the indication of the clinical analysis that fractions are just such a problem area.

A comparison of the two graphs indicates that with respect to Relational Understanding, subjects at all levels of ability found the topic difficult, although for the less able group relational understanding of negative number was less likely than for fractions. However, for instrumental understanding the identification of fractions as the most difficult area for the less able group is particularly marked.

The questions on variables, in relation to both criteria, appear as one of the easier topics for the less able group. This tends to confirm the suggestion from the clinical analysis that the abstract nature of the mathematics is not, in itself, likely to be a major factor in the difficulties which these students have with the subject.

The statistical analysis was intended to play an important supporting role to the principal clinical analysis of this study. The choice of Latent Response Analysis was made a little tentatively in that no previous uses of the method in a similar study could be found. However, comparisons with the raw data confirmed that the method is appropriate and provides a useful means of comparing performance on individual items and grouped items in a coded interview situation.

CHAPTER SEVEN

SUMMARY, CONCLUSIONS, EDUCATIONAL IMPLICATIONS AND SUGGESTIONS FOR FURTHER RESEARCH

SUMMARY OF THE RESEARCH

The very general questions which prompted this study were: 'Why are some otherwise able students apparently incapable of handling mathematics?' and 'What is it about mathematics which such students find so difficult?'

The essence of the problem is one of apparent contradiction. The abilities which seem to be necessary for success in mathematics are those of manipulating abstract ideas in a constructive way and this ability coincides closely with the usually accepted view of intelligence. It is necessary, then, to explain the undoubted existence of people whose success in other fields makes it impossible to label them unintelligent and yet who operate mathematically at a very low level.

A careful description of this phenomenon, which might lead to an explanation, is dependent on an understanding of the cognitive processes of the individuals concerned. Consequently a research method and theoretical perspective were chosen which would enable a study of these processes as they were used in solving mathematical problems. The method chosen was a clinical one, essentially the Piagetian clinical interview. The view taken of the method was the 'holistic' view of Diesing (1971) in which a mosaic of evidence is created from which a pattern model of explanation (Kaplan 1964) emerges.

As a theoretical background to the gathering and analysis of this evidence, the model of intelligent behaviour proposed by Skemp (1979a) proved useful. The structural features of this model, particularly the discussion of learning, understanding and action in relation to cognitive structures, and the close relationship between cognitive and affective determinants of behaviour were particularly appropriate for this study.

Twenty six subjects were interviewed. They were all university students and all had volunteered to be interviewed after the purposes of the study was explained. The intention was to compare the cognitive structures and strategies of students having considerable difficulty with mathematics both with those who had less difficulty and those who found mathematics one of their better subjects. Consequently, the subjects were initially allocated to three roughly equally sized groups representing low, average and high ability and interest in mathematics.

The interviews were semi-structured in that after an initial period of investigation of the subject's school and family background, each subject was led through the same sequence of mathematical tasks. After the initial presentation of the tasks, the responses of the subjects were probed in an unstructured way. The interviews were tape recorded for later analysis.

The interview tasks were taken from the arithmetic-algebra sequence of the primary-secondary school syllabus. They were chosen, after pilot testing, to reflect the many hierarchical structures (logical, developmental, pedagogical etc) which are a major feature of mathematics and mathematics learning.

The analysis of the interview data had two stages. Firstly, in order to provide an overview, a formal coding was undertaken in which each response was classified according to the level of understanding of the item indicated by the response. Skemp's (1976) classification of Relational and Instrumental understanding was used. This data was then analysed statistically. The initial analysis was purely a descriptive one in which the performance of subjects, ranked according to total performance on the items, was displayed in relation to progressively more difficult items. This information was then presented parametrically by the use of Latent Response Analysis. The analysis was used both to compare individual items within content areas (natural number, fractions, variables etc) and to compare the performance of subjects between these areas. The principal conclusion from this analysis, which was supported by the later clinical analysis, was that fractions are a major source of difficulty for all subjects but particularly for the less able.

Following this statistical analysis a closer clinical analysis of the data was made. In order to reflect the 'holistic' nature of the research, a multiple-coding approach was developed. With respect to each individual, every response, and sequence of responses, was considered in relation to six different categories:

The mathematical nature of the responses.
The logical nature of the responses.
The developmental character of a sequence of responses (Collis and Biggs, 1979)
The relationships of individual responses within a sequence (Tall, 1979)
The character of individual responses (Piaget 1929)
The character of the interviewer's contribution.

The responses were not formally coded with respect to each category, but notes were taken of significant features and these formed the basis of summaries of the character of the subject's responses to different items and in different content areas.

Comparisons between these multiple-coding analyses for different subjects enabled the following conclusions to be drawn. Firstly, the data indicated the importance of the availability of appropriate initial frameworks for the successful handling of mathematics. Secondly, neither the logical nor the abstract nature of mathematics seemed, in themselves, to be likely principal factors in the difficulties which these students had with the subject. The third conclusion related to the major difficulties experienced in working with fractions, indicating this as the principal area for profitable future research.

THE LIMITATIONS OF THE STUDY

Before considering these conclusions in more depth it is important to realise any limitations which may have to be placed on their interpretation.

The research reported in this thesis is not confirmative in the sense that it is designed to assess the truth of provable hypotheses, rather it is generative, intended to generate hypothesis with a priori possibility (Booker, 1980).

The evidence from which the conclusions of this chapter are drawn consists principally of interviews with seven otherwise able subjects who were having considerable difficulty with mathematics. This evidence was supported by interview data from other subjects having less difficulty, but even so the sample is small. It is possible that the study would have been improved had the sample of less able students been increased in size at the expense of the comparison group. There were few surprises in the responses of the higher ability group. By and large, their responses reflected directly the way in which the content is presented in textbooks and in the classroom.

That is not to say of course, that the small sample size invalidates the study. The very significant studies of Piaget were all small sample, and the more recent work of researchers such as Erlwanger 1973, Clement 1979, and Peck and Jencks

1979, has all been on very small samples - one or two subjects. Every effort was made in this study to ensure that the conclusions were 'dependable' (Diesing, 1971), but it is important to recognise that valid generalizations, such as might be expected from research employing the deductive or survey method, cannot be inferred from a study such as this.

In relation to the research method itself, although the clinical method has been available for many years, it is only comparatively recently that a significant number of such studies have been undertaken. Consequently, few well established procedures relating to the conduct, analysis and reporting of research based on this method are available. One significant feature, then, of this study was the development of techniques in this area and some comment is appropriate.

Firstly, in relation to the clinical analysis, a multiple-coding approach having much in common with the Constant Comparative Method (Glaser, 1969) was used and seems to have considerable potential. Its very close relationship with the 'holistic' view of clinical research makes it worthy of further consideration and development. It may be that the method would be improved by being rather more formalised. Glaser comments:

> "Another way to convey credibility of the theory along with the use of illustrations is to use a codified procedure for analysing data, such as presented here, which allows the reader to understand how the analyst obtained his theory from the data. In qualitative analyses the transition from data to theory is hard, if not impossible, to grasp when no codified procedure is used. Andin his turn the reader is likely to feel that the theory is somewhat impressionistic, even if the analyst strongly asserts he he has based it on hard study of data gathered during moths or years of field or library research".
>
> (Glaser, 1969: 225)

In this study, one formal coding procedure was used in that the data relating to the level of understanding indicated by responses to individual items was coded and analysed using Latent Response Analysis (Torgerson, 1958). A cautious approach to the use of the method was necessary in view of the small sample involved, and since no previous use of the method in a similar study could be found. However a careful comparison of the results of the Latent Response Analysis with the data displayed in an entirely descriptive manner indicated that the use of the statistical technique was justified. The method clearly provides a useful parametric representation of some important features of items used in a clinical study of this kind.

CONCLUSIONS

The two general questions, 'Why are some otherwise able students apparently incapable of handling mathematics?' and 'What is it about mathematics which such students find so difficult?', focus attention on two aspects of the problem - the students and the mathematics. Progress towards answering these questions is presented below, followed by a discussion of the resolution of the apparent contradiction implied by the existence of such students.

THE CHARACTERISTICS OF THE LESS ABLE STUDENTS

Each interview began with an exploration of the family and school background of the subject. When this material was included, it was considered possible that differences between the backgrounds of successful and unsuccessful students might have been indicated which were worthy of further investigation. This was not the case. Wide variation in the mathematical abilities and interests of parents and siblings was reported by subjects at all levels. There was some suggestion that parental and peer sex-role expectations had been a handicap for female subjects, but only in one case was this considered to have been a significant factor.

Similarly, in relation to school experience, subjects at all levels reported instances of what they considered to have been periods of good and bad teaching, and of positive and negative experiences with mathematics. This lack of evidence, of course, does not mean that family and school backgrounds are not a factor in the mathematical difficulties of some subjects. It is more a reflection of the inappropriateness of the small sample, clinical method, as a means of identifying any general factors of this type. A large sample, survey study would be more appropriate for this purpose.

If environmental differences were not evident between subjects then it is possible that the origins of the less able subjects' difficulties lie in 'nature' rather than 'nurture'. It is a commonly held view in the community that there exists a specific class of innately 'non-mathematical' individuals who are just made that way. This view is often reflected in schools when students, particularly girls, who are having difficulties with mathematics, are encouraged to accept their apparent lack of ability as one of their limitations. Consequently they withdraw from the study of mathematics at the earliest opportunity.

However, the evidence of this thesis suggests, quite strongly, that if any of the subjects who were having difficulty with mathematics could be persuaded to begin again, there is absolutely no reason why they could not succeed. Their difficulties seemed to stem from a lack of appropriate initial cognitive structures, or frameworks, rather than from an inability to form appropriate links within such structures. They were very well able both to draw logically valid conclusions from their own premises, whether or not these premises were true, and to see the implications of challenges to these premises presented by the interviewer.

There is no doubt that the poorer students were more rule-dependent than the successful ones and that many of the rules which they used were faulty. It did seem likely, however, that this rule-dependence was a result of their difficulties rather than a cause. It might be expected that potential university students would be highly extrinsically motivated, certainly this was the impression given by the subjects of this study. Consequently students who had failed to form the appropriate initial cognitive structures, which are a prerequisite for understanding, were forced into rote learning to avoid the immediate consequences of failure.

The influence of extrinsic motivational factors is illustrated by the subject who indicated that if she did not understand something in the mathematics classroom she would sometimes ask for an explanation. However, she would never admit to not understanding the explanation given, but would smile appreciatively and nod assent at the teacher's response, regardless of whether or not it was helpful. To do otherwise, she felt, would have given the impression to the teacher and her peers that she was 'stupid'.

It does seem likely that mathematics learning is more sensitive to factors such as this than other forms of learning. The reason lies in what Skemp (1979a) calls the 'low noise' nature of mathematics learning. The student of mathematics is constantly aware of what he knows and does not know, what he believes he understands and does not understand, and of what he can do and cannot do. There is no subject area for which failure is more obvious.

The educational implications of this are discussed at length in Buxton (1981) and relate, largely, to the way in which the particular nature of mathematics learning imposes on the teacher the need for extreme sensitivity to the relationship between the cognitive and affective factors of learning. The teacher must be aware that bright students may pretend to understand and that there is a constant danger of intelligent students using their intelligence to find ways of covering for their lack of understanding.

If, then, the principal characteristic of those who found mathematics difficult was their lack of appropriate cognitive structures, it follows that the character of these structures is of extreme importance. One of the more obvious features of mathematical concepts is their abstract nature. Skemp expresses this feature in this way:

> "Much of our everyday knowledge is learned directly from our environment, and the concepts involved are not very abstract. The particular problem (but also the power) of mathematics lies in its great abstractness and generality, achieved by successive generations of particularly intelligent individuals each of whom has been abstracting from, or generalizing, concepts of earlier generations. The present-day learner has to process, not raw data, but the data-processing systems of existing mathematics".
>
> (Skemp, 1971: 31)

This suggests that the failure of students to form appropriate frameworks might be due to the abstract nature of the concepts involved. Again this does not seem to be the case for the subjects interviewed. Their understanding of, for example, the concept of a variable was much better than their understanding of the apparently less abstract concepts associated with fractions.

Consequently the general conclusion drawn from the analysis of the interviews with the less able group, and the comparisons of this data with that taken from the other subjects, was that in terms of information-processing abilities there were no observable differences between the groups. The differences lay in the data to which these abilities were applied.

The educational implications of this conclusion are clear. If the acquisition of appropriate frameworks is of paramount importance in the learning of mathematics, it is essential that failure to form such frameworks should be diagnosed at an early stage. In this regard, there is a strong case for the use of the clinical interview technique as an integral part of mathematics teaching. It is difficult to see how the diagnosis of incorrectly formed frameworks is to be made by other means. Certainly the frameworks cannot be directly inferred from written responses to questions. A powerful reminder of this was given by the subject who illustrated the number 3/4 by the diagram 

In the context of a written test this might be taken as indicating an appropriate, part-whole, concept of fractions. The problem was that the subject thought that the three was represented by the part cut out and the four by the remainder.

It seems to be imperative that if such misunderstandings are to be identified and rectified the teacher must have access to the reasons behind students' choices of answers and methods, regardless of whether these answers and methods are correct or incorrect. The evidence of this thesis is that such access is available through the clinical interview technique. If such interviews could be incorporated into every mathematics classroom on a regular basis, early warning of the problems which able, and less able, students have with mathematics might well be available. Early treatment following this identification might prevent the vicious interaction between cognitive and affective symptoms which is the most distressing feature of the phenomenon.

THE CHARACTER OF THE MATHEMATICAL DIFFICULTIES

If the conclusions of the previous section are accepted, there remains the question of why the necessary initial schemas were not formed by some students. There is insufficient evidence in this thesis to answer this question, but there are indications of potential areas for further research.

The answer would seem to lie in an examination of the character of the schemas with which the students had most difficulty. The evidence points very clearly to fractions as the conceptually most difficult topic.

The nature of the fraction schema was discussed at length in Chapter 5 and was seen to be extremely complex, having a great deal of interiority. However it is appropriate now to consider the difficulties associated with acquisition of this schema. The problems which subjects had with fractions seemed to relate firstly to the many different representations of fractional quantities which are required, and secondly to the necessity to reinterpret the operations of addition, subtraction, multiplication and division which were well established in relation to natural numbers. In Skemp's terminology, the formation of the schema relies heavily on reconstruction. Skemp writes:

> Sometimes , however, we may encounter a situation for which we have a schema which is relevant, but not adequate. Attempts to realise this situation in terms this schema give such a faulty representation of actuality that our director systems cannot function. If possible we avoid such situations, for they are outside our prohabitats. But if we cannot, then we have no choice but to reconstruct our schema. Since this necessarily involves first taking it partially or completely to pieces this is disruptive, unwelcome and difficult: because while this is going on, we are unable to use our schemas effectively for directing our actions. Nevertheless, it is sometimes necessary.
>
> (Skemp, 1979: 126)

This is precisely the situation of the child learning, for example, multiplication of fractions. The existing schema for multiplication will include the concept of multiplication as repeated addition and knowledge that 'multiplication makes things larger'. This schema has to be dismantled before an understanding of $\frac{3}{4} \times \frac{5}{8}$ can take place. The multiplication sign has to be interpreted as 'of' instead of 'times' and the two fractions must have different representations, the first as a proportion and the second as a quantity. To compound the problem, multiplication now makes things smaller.

The hypothesized central role of reconstruction in the development of the fractions schema is entirely consistent with the high degree of emotional reaction to the topic shown by the subjects interviewed. In terms of Skemp's model, affective signals such as fear, relief, confidence, anxiety etc. are interpreted as results of the individuals knowledge of movements in relation to goal or anti-goal states. Reconstruction clearly creates an anti-goal state. Enforced movement towards such a state is signalled by fear and knowledge of one's inability to move away from such a state by anxiety.

In contrast, the development of the schemas relating to variables and negative number, with which the subjects had much less difficulty, rely much more on <u>expansion</u> and <u>differentiation</u> than they do on <u>reconstruction</u>.

It seems, then, that the role of reconstruction in the development of mathematical schemas, particularly the fraction schema, is well worth further research.

The educational implications of this hypothesised central role of fractions in the formation and development of mathematical difficulties are far reaching.

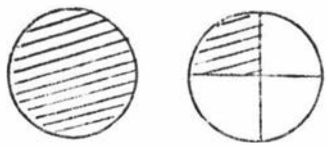
Firstly, in view of the lack of understanding in this area shown by even the most able university mathematics students, it seems highly likely that there is a good deal of confusion concerning fractions among primary school teachers. It would be relatively simple to conduct research to determine whether or not this is the case. If it was true then in-service training might be undertaken. However, the evidence of the interviews suggest that it is at least as much the character of the concepts themselves which cause the difficulties as it is the character of the teaching. Certainly the current New Zealand Primary School syllabus in this area (Appendix B), viewed in the light of the evidence of this thesis, makes the kind of confusion noted seem almost inevitable, no matter how the material is taught.

For example, by Standard 1 (7 years) a number of different interpretations of the simplest fractions have been introduced. These are:

> The part-whole congruent parts concept (halves and quarters of everyday objects and common shapes).
>
> The part-group congruent parts concept (half the cardinal number of a set).
>
> The part-group non congruent parts concept (half of the set of men wear hats).
>
> The fraction as a real number (showing 1/2, 1/4 and 3/4 on the number line).

The children are then expected to record practical discoveries using number sentences such as $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. This particular 'discovery' is, as discussed earlier, quite reasonable for the part-whole congruent parts concept of $\frac{1}{4}$, but is most unlikely for the part-group concept. The children are shown, in the syllabus, a set of four men, one of whom has a walking stick and are told that '$\frac{1}{4}$ of the set of men has a walking stick'. Given another similar set of men, the 'practical discovery' which an intelligent child should make is that $\frac{1}{4} + \frac{1}{4} = \frac{2}{8}$, two of the set of eight men have walking sticks. It would be extremely difficult for a teacher, no matter how well informed, to explain to a seven year old child why this is not so, except by saying 'that is the rule'.

This is not an isolated example. In the Standard 3 (9 years) syllabus the diagram shown below appears:



$\frac{5}{4}$ shaded

This is certainly confusing if not patently false. Fractions greater than one cannot sensibly be shown by the part-whole congruent parts concept. After all, how do you divide an area into four equal parts and shade five of them?

The most generous interpretation which can be placed on these and the many other difficulties inherent in this and previous syllabuses is that their constructors have seriously underestimated the complexity of the fraction schema.

The problem is, of course, not unique to New Zealand. Harrison *et al* (1980), considered the cognitive demands of curriculum material on fractions used in some Canadian Schools against the cognitive levels of the 7th. Grade students who were expected to use that material. Items from the material were classified as needing Concrete, Transitional or Formal Operations and the students were similarly classified according to the character of their work with fractions. Only 6% of the students were judged capable of formal operations, whereas the three sets of material tested had 41%, 90% and 62% respectively of their items requiring formal operations. A similar study using New Zealand material would be of value.

A very clear case, then, exists for a drastic revision of the place of fractions in the school syllabus. It may well be that apart from very simple work which formalises such fractional words such as half, quarter, three quarters and one third, which occur in everyday speech, all formal work with fractions should be delayed until secondary school. At this time the cognitive levels of the students may be a somewhat better match for the cognitive demands of the subject matter. Even then it is likely that considerable teacher retraining would be required.

THE CONTRADICTION

Finally, it is appropriate to consider the relationships of the conclusions drawn in this thesis to the resolution of the apparent contradiction which, it was suggested, is the essence of the problem being considered.

Again, Skemp's model is useful. In the final chapter of his book, Skemp discusses intelligence as being both creative and self-creative. He writes:

> "Another sense in which intelligence is self-creative results from the fact that each concept is a potential growing point for the schema of which it forms part: for it sensitizes its possessor to new regularities in actuality. Beyond this, a schema may suggest further questions to be put to activity, and thus be an agent of its own further expansion. This is not the case for sub-intelligent forms of learning, such as habit learning. A schema, itself the product of intelligent learning, thus increases a persons ability for further intelligent learning in that realm of thought. And by assimilation to itself, a schema extends to the newly learned knowldge those qualities of internal organization which have been distinguished as also characterizing intelligence.
>
> (Skemp, 1979a: 290)

This view of the central role of schema in the development of the ability for intelligent learning in a particular realm of thought corresponds precisely to the conclusions which have been drawn from this study.

The schema relating to different academic disciplines are quite different in character. Hirst (1972) highlights this by distinguishing between a number of different 'forms of knowledge' one of which is mathematics.

> "... by a form of knowledge is meant a distinct way in which our experience becomes structured round the use of public symbols"
>
> (Hirst 1972: 405)

The distinguishing features of such forms are: -

1. They each involve certain central concepts that are peculiar in character to the form.
2. These concepts form a network of possible relationships in which experience can be understood, giving the form a distincive logical structure.
3. Each form has distinctive expressions that are testable against experience in accordance with criteria that are particular to the form.
4. The forms have developed particular techniques and skills for exploring experience and testing their distinctive expressions.

It is quite reasonable to suggest, then, that the development of the ability for intelligent learning might take place within one form of knowledge but not, because of the lack of prerequisite schema, in another.

Thus the failure of some students to handle mathematics, whilst it must make them less 'educated' in the sense that it prevents them from being able to structure their experiences in one particular way, does not imply any general lack of intelligence. The tragedy is that many students, not understanding their failure, lose self-esteem and a vicious interaction of affective and cognitive reactions create a state in which any constructive approach to mathematics learning seems impossible.

There are two areas, suggested by the evidence of this thesis, in which action might prevent this unfortunate state of affairs. Firstly, curriculum planners need to be much more aware both of the nature of the schemas associated with topics they propose to include in the curriculum, and of the difficulties associated with the acquisition of those schemas. Secondly, teachers need to constantly monitor the development of such schemas and to be sensitive to signals in the affective domain which indicate developing problems in the cognitive area.

APPENDIX A

LATENT STRUCTURE ANALYSIS (TORGERSON (1958))

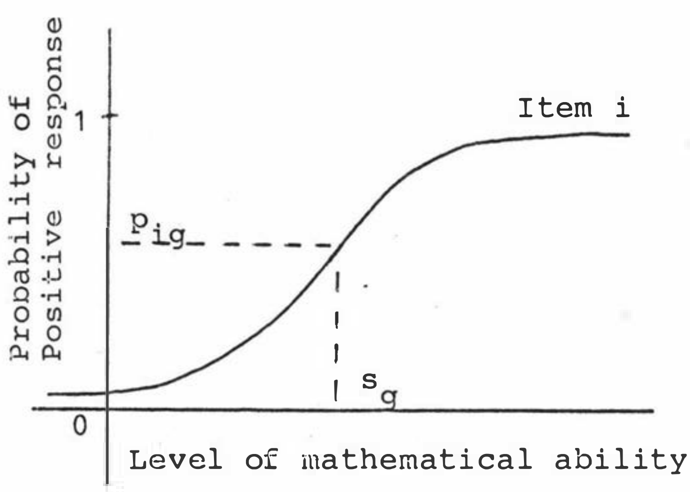
BASIC ASSUMPTIONS

(i) A unidimensional continuum of the variable of interest, in the case of this study - mathematical ability, is assumed to exist.

(ii) Subjects are assumed to be distributed along this continuum according to some unknown probability distribution.

(iii) A set of dichotomous items related to the variable are devised.

(iv) A form of trace lines for the items is assumed. The trace line for an item gives the probability of a positive response to the item from subjects located at any given point on the underlying continuum. In this study, a normal-ogive model (Tucker 1952, Lord 1953) was used.

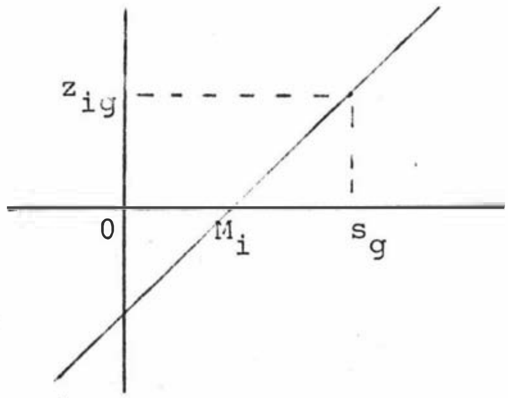


(v) Since items are related to the underlying variable, it follows that they are related to each other. It is however assumed that no two items have any relationship other than that which can be accounted for by their separate relationships to the underlying variable. In this study the items must only be related mathematically.

THE ANALYSIS

Consider a subject g having a scale value s_g on the scale of mathematical ability. p_{ig} is the probability that this subject will respond positively to item i.

If z_{ig} is the unit normal deviate corresponding to p_{ig}, the relationship between s_g and z_{ig} will be linear and the trace line is transformed to the straight line shown.



The equation of this line may be written.

$$z_{ig} = \frac{1}{\sigma_i} (s_g - M_i) \qquad \text{(A)}$$

M_i and σ_i are the mean and standard deviation of the normal curve which gives rise to the trace line. M_i can be considered as a location parameter for the item since it indicates the point on the attribute continuum at which the individual would have a probability of 0.50 of a positive response to the item. σ_i can be considered as a discrimination parameter. A large value for σ_i indicating that the item has not discriminated well between subjects of high and low ability.

If we take equation, (A) and sum over all of the n items, that is sum with respect to i, we have: -

$$\sum_{i=1}^{n} z_{ig} = s_g \sum_{i=1}^{n} \frac{1}{\sigma_i} - \sum_{i=1}^{n} \frac{M_i}{\sigma_i} \qquad \text{(B)}$$

The scale we use to measure the underlying attribute is arbitrary, since we are only ordering, so we may choose its origin and scale to suit the calculations.

Choose the origin so that $\sum_{i=1}^{n} \frac{M_i}{\sigma_i} = 0$

and the scale so that $\sum_{i=1}^{n} \frac{1}{\sigma_i} = n$

Equation (B) then becomes:

$$\sum_{i=1}^{n} z_{ig} = n\, s_g$$

which may be written $s_g = \bar{z}_{\cdot g}$ ——————— (C)

That is, given the values of z_{ig} we can calculate s_g. As an experimental procedure, (Tucker 1952) the total sample of subjects is divided into m relatively homogeneous subgroups on the basis of total score. The proportion of people in the gth subgroup who responded positively to the ith item is taken as an empirical estimate of the population proportion p_{ig}. This may then be converted to the unit normal deviate z_{ig}.

To estimate the values of M_i and σ_i we take equation (A) and find the standard deviation over subjects, that is over g.

We get $\tilde{z}_{i\cdot} = \frac{1}{\sigma_i} \tilde{s}_{\cdot}$ or $\sigma_i = \frac{\tilde{s}_{\cdot}}{\tilde{z}_{i\cdot}}$ ——————— (D)

and we can find σ_i from the experimentally determined z_{ig}.

Taking (A) again, we find the mean over subjects, that is over g.

$$\overline{z}_{i\cdot} = \frac{1}{\sigma_i} (\overline{s}_{\cdot} - M_i)$$

or $$M_i = \overline{s}_{\cdot} - \sigma_i \overline{z}_{i\cdot} \qquad \text{(E)}$$

Equations (C) (D) and (E) together with experimentally determined values of z_{ig} enable us to deduce the trace lines for all n items.

APPENDIX B

REVISION OF FRACTIONAL NUMBERS - A SUPPLEMENT TO THE SYLLABUS

DEPARTMENT OF EDUCATION, WELLINGTON, 1976

REVISION OF FRACTIONAL NUMBERS

CONTENTS

Aims of the Revision

1. To relate the teaching of fractional numbers to the ideas and skills which it is expected will be commonly used in our society after the changeover to metric measurement.

2. To ensure that the basic ideas and skills are firmly established through a variety of practical experiences provided regularly at all class levels.

3. To give greater emphasis to developing facility with decimal fractions, particularly their application to the recording and understanding of metric measurements, including money.

4. To equip children with a knowledge of principles of fractional numbers as a basis for later learning of computational skills when the need for these is met.

Infant Classes

1. Practical discoveries of halves and quarters of everyday objects, e.g., an apple, a chocolate bar.

2. Practical discoveries of halves and quarters of common shapes, e.g., a circle, a square.

3. Practical discoveries of half of the cardinal number of a set, e.g., half of six is three.

4. Discovery of the relationship of part to part and parts to whole, e.g., halves will match each other, and two halves will match one whole.

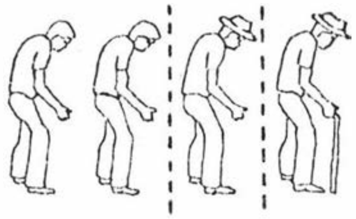
5. Recording practical discoveries, using the words *half, quarter,* and symbols $\frac{1}{2}, \frac{1}{4}$.

Standard 1

Extension and revision of work in infant classes.

1. Many and varied experiences with halves and quarters of continuous quantities, e.g., a region, a toffee bar, an apple.

2. Practical discoveries of halves and quarters of sets.



$\frac{1}{2}$ of the set of men wear hats.

$\frac{1}{4}$ of the set of men has a walking stick.

3. Arising from 1 and 2 above, practical and oral development of the ideas of two halves and of two, three and four quarters.

4. Finding half of an even natural number less than 21.

5. Showing $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{3}{4}$ on the number line.

6. Introduction, through paper cutting and other practical activities, of one third and the symbol $\frac{1}{3}$.

7. Recording practical discoveries using number sentences, e.g., $\frac{1}{4} < \frac{1}{2}$, $\frac{1}{4} + \frac{1}{4} = \frac{2}{4}$, $2 \times \frac{1}{4} = \frac{2}{4}$, $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, $2 \times \frac{1}{2} = 1$.

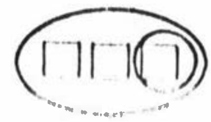
Standard 2

Extension and revision of work in Standard 1, with continued emphasis on working with a variety of materials.

1. Fractions ($\frac{1}{2}$, $\frac{1}{3}$, etc.) as numerals for fractional numbers.

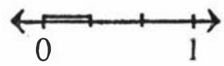
2. A fractional number can be represented:

   (i) By the relationship of a subset to its set.



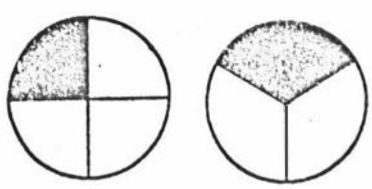
   (ii) By the relationship of a subregion to its region.

   (iii) On a number line.

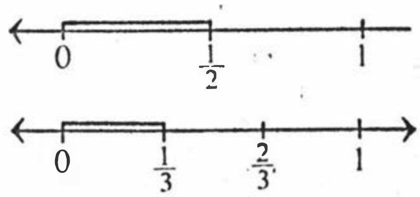


3. Introduction, through practical experiences, of fifths and tenths.
4. Discovering, through practical experiences, that fractional numbers can be renamed,

   e.g., $\frac{2}{4} = \frac{1}{2}$, $\frac{1}{5} = \frac{2}{10}$.
5. Comparison of fractional numbers, as represented by sets and regions and on the number line, and recording in number sentences:



$\frac{1}{4} < \frac{1}{3}$



$\frac{1}{2} > \frac{1}{3}$

6. Practical activities involving joining equivalent parts, leading to recording, e.g., $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3 \times \frac{1}{4} = \frac{3}{4}$.
7. Interpretation of simple sentences containing fractions through practical and oral problems.
8. The whole number *one* can be expressed as a fraction: $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, etc.

Standard 3

Extension and [illegible] of work in Standard 2, with continued emphasis on working with a variety of materials. Recording of results where appropriate only.

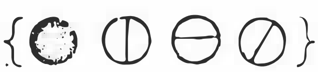
1. Further understanding of the idea of fractional numbers used to show:

   (i) Comparison of part of a continuous quantity with the whole:

$\frac{1}{4}$ of the circular region is shaded.

$\frac{1}{4}$ of the line segment is shown thicker.

   (ii) Comparison of the number of a subset with the number of the whole set:



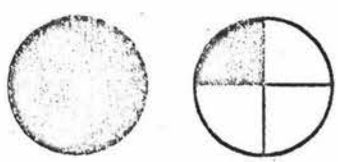
$\frac{1}{4}$ of the members of the set is shaded.

2. Introduction, through practical experience, of eighths.

3. Introduction of the terms *numerator* and *denominator.*

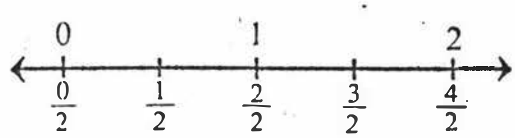
4. (a) Fractional numbers greater than 1 shown with:

   (i) Regions:



$\frac{5}{4}$ shaded.

   (ii) The number line:



   (b) A fractional number greater than 1 can be expressed as a mixed numeral, such as $1\frac{1}{2}$.

5. Further practice of renaming fractional numbers using equivalent fractions, by activities with sets, regions, cuisenaire rods and number lines,

   e.g., $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$, $\frac{2}{5} = \frac{4}{10}$, and vice versa.

6. Comparison of fractional numbers as a result of practical experiences.

   e.g., $\frac{1}{2} > \frac{1}{4}$, $\frac{2}{5} < \frac{3}{5}$, $\frac{4}{5} > \frac{4}{8}$.

7. Further practical activities involving the joining of equivalent parts, and recording,

   $\frac{1}{10} + \frac{1}{10} = \frac{1}{5}$, $2 \times \frac{1}{4} = \frac{1}{2}$, $2 \times \frac{1}{2} = 1$, etc.

8. (a) Introduction of one-place and two-place decimal fractions (including the decimal point) as an extension of the decimal numeration system.

   (b) Showing tenths as decimal fractions on the number line.

$\frac{0}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$	$\frac{10}{10}$
0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0

   (c) Application to:

   (i) metres and centimetres,

   (ii) dollars and cents.

Standard 4

Extension and revision of work in Standard 3, with the continued provision of practical experiences.

1. Further study of the nature of fractional numbers as represented:

   (a) by subsets and sets,

   (b) by subregions and regions,

   (c) on the number line.

2. Introduction of the idea of the bar between numerator and denominator as an indication of division,

   e.g., $\frac{3}{5}$ means $3 \div 5$.

3. Continued finding of equivalent fractions through practical activities, and study of their patterns.

4. Continued study of relationships of tenths and hundredths and of one- and two-place decimals. In particular:

(a) $\frac{1}{10}$ $\frac{10}{100}$, $\frac{2}{10}$ $\frac{20}{100}$, etc.

Therefore 0.1 0.10 0.2 0.20.

(b) 0.17 $\frac{1}{10}$ + $\frac{7}{100}$ = $\frac{17}{100}$, etc.

5. (a) Study of decimal fractions to three places as an extension of the decimal numeration system.

(b) Study of the relationship of decimal fractions and common fractions,

e.g., $0.1 = \frac{1}{10}$, $0.01 = \frac{1}{100}$, $0.001 = \frac{1}{1000}$.

(c) Addition and subtraction of two-place decimal fractions.

6. Recording experiences of joining, separating, and comparing parts as addition and subtraction sentences with related fractions,

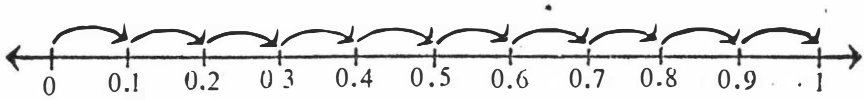
e.g., $\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$, $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $\frac{7}{8} - \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$.

7. (a) Multiplication of a one-place decimal related to addition, and represented on a number line,

e.g., $0.1 + 0.1 + 0.1 = 3 \times 0.1 = 0.3$.

(b) Multiplication of a one-place decimal by 10 shown by addition,

e.g., $0.1 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1 +$
$0.1 + 0.1 = 10 \times 0.1 = 1.0$.



Form I

1. Further study and practice of the basic ideas that fractional numbers:

(a) show comparison,

(b) show division ($\frac{2}{3} = 2 \div 3$),

(c) can be shown as a fraction or in decimal form,

(d) can be ordered.

2. (a) Introduction of ratio as a comparison of the cardinal numbers of two sets of like objects, which can be expressed by a fraction,

   e.g., the ratio of the number of vowels to consonants in the alphabet is 5 : 21, or $\frac{5}{21}$.

   (b) Ratio as a way of expressing probability.

3. Practical activities and patterning leading to the use of the identity element for multiplication to form equivalent fractions,

   e.g., $\frac{1}{2} \times \frac{4}{4} = \frac{4}{8}$, $\frac{6}{10} \div \frac{2}{2} = \frac{3}{5}$.

4. Simple examples of addition and subtraction with two fractions, with the renaming of one addend.

   e.g., $\frac{1}{5} + \frac{1}{10} = \frac{2}{10} + \frac{1}{10} = \frac{3}{10}$, $\frac{3}{4} - \frac{1}{2} = \frac{3}{4} - \frac{2}{4} = \frac{1}{4}$.

5. Exploring the properties of addition through the study of addition with related fractions.

   (a) Commutative: $\frac{1}{2} + \frac{3}{4} = \frac{3}{4} + \frac{1}{2}$.

   (b) Associative: $(\frac{1}{2} + \frac{1}{4}) + \frac{3}{4} = \frac{1}{2} + (\frac{1}{4} + \frac{3}{4})$.

   (c) Inverse operations: Adding $\frac{1}{4}$ is undone by subtracting $\frac{1}{4}$.

6. Exploring the properties of multiplication through the study of multiplying a whole number by a fractional number.

   (a) Commutative: $\frac{1}{4} \times 2 = 2 \times \frac{1}{4}$.

   (b) Associative: $(3 \times 2) \times \frac{1}{2} = 3 \times (2 \times \frac{1}{2})$.

   (c) Identity element: $\frac{2}{2}, \frac{3}{3}$, etc.

   (d) Reciprocals: $3 \times \frac{1}{3} = 1$.

   (e) Inverse operations: Multiplying by $\frac{1}{2}$ is undone by dividing by $\frac{1}{2}$.

   Inverse operations can be observed in patterns such as the following:

   $8 \div 2 = 4$ $\quad 8 \times \frac{1}{2} = 4$ $\quad 16 \times 2 = 32$

   $8 \div 1 = 8$ $\quad 8 \times 1 = 8$ $\quad 16 \times 1 = 16$

$$8 \div \frac{1}{2} = 16 \quad 8 \times 2 = 16 \quad 16 \times \frac{1}{2} = 8.$$

$$\text{Hence, } 16 \times \frac{1}{2} = 8, \quad 8 \div \frac{1}{2} = 16.$$

7. Addition and subtraction of three place decimal fractions.

8. (a) Multiplication of three-place decimal fractions by 10, 100, 1000.

   (b) Multiplication of a two-place decimal fraction by a one-place decimal fraction, or by a whole number less than 10,

   e.g., 0.27×0.8, 5.49×0.5, 0.26×7

9. (a) Division of up to three-place decimal fractions by whole numbers less than 10,

   e.g., $0.275 \div 5$, $24.16 \div 4$.

   (b) Division of whole numbers by 10, 100 and 1000; of one-place decimal fractions by 10 and 100; of two-place decimal fractions by 10,

   e.g., $4835 \div 10$, $376 \div 1000$, $10.4 \div 100$, $0.51 \div 10$.

10. Introduction of percent as a form of a fraction with the denominator 100,

    e.g., $10\% = \frac{10}{100} = \frac{1}{10} = 0.1$.

Form 2

Revision and extension of Form 1 work

1. Further practice of the addition and subtraction of three-place decimal fractions, including applications to weight, length, capacity and volume.

2. Extension of multiplication to include multiplying a three-place decimal fraction by a two-place decimal fraction, or by a whole number less than 100.

3. (a) Introduction of division of a three-place decimal fraction by a one-place decimal fraction.

   (b) Extension to division of a three-place decimal fraction by a two-digit number,

   e.g., $0.375 \div 25$, $0.375 \div 2.5$.

4. Rounding off to one decimal place.

5. Calculation of simple percents of numbers and measures, including money, e.g., 10% of 50, 25% of \$20.

   (a) 10% of 50 $= \frac{10}{100} \times 50 = \frac{1}{10} \times 50 = 5$.

   (b) 10% of 67 $= 0.1 \times 67 = 6.7$.

6. Discount and mark-up expressed as percents.

7. Problems involving ratio and simple proportion.

8. Addition and subtraction with two fractions, with the renaming of one addend.

9. Multiplication of a fractional number by a whole number.

10. Division of a fractional number by a whole number where the numerator is a multiple of the whole number,

    e.g., $\frac{4}{5} \div 2 = \frac{4 \div 2}{5} = \frac{2}{5}$, $1\frac{1}{2} \div 3 = \frac{3}{2} \div 3 = \frac{1}{2}$,

    $\frac{2}{3} \div 3 = \frac{2 \times 3}{3 \times 3} \div 3 = \frac{6}{9} \div 3 = \frac{2}{9}$.

11. (a) Two numbers are reciprocals if their product is 1,

    e.g., $4 \times \frac{1}{4} = 1$.

    (b) Division by a number is equivalent to multiplication by its reciprocal,

    e.g., $8 \div 4 = 2$, $8 \times \frac{1}{4} = 2$, $8 \div \frac{1}{2} = 16$, $8 \times 2 = 16$.

    (See patterning under Form 1, Section 6.)

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